American Mathematical Monthly Problem 11403 by Yaming Yu, Irvine, CA.

For every integer $n \geq 0$, define a polynomial $f_n \in \mathbb{Q}[x]$ by

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j).$$

Find $\deg f_n$ for every $n > 1$.

Solution by Darij Grinberg.

We claim that $\deg f_n = \left\lfloor \frac{n}{2} \right\rfloor$ for every integer $n > 1$. The key to the proof is the following recurrence relation for our polynomials:

$$f_n(x) = (n-1)\left(f_{n-1}(x) + x f_{n-2}(x)\right) \tag{11403.1}$$

for every integer $n > 1$.

Proof of (11403.1). Let $n > 1$. Then,

$$\begin{aligned}
f_n(x) &= \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n} \left(\binom{n-1}{i-1} + \binom{n-1}{i} \right) (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) \\
&= \sum_{i=0}^{n} \binom{n-1}{i-1} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) \\
&= \sum_{i=-1}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \prod_{j=0}^{i} (x+j) + \sum_{i=0}^{n} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) \\
&\qquad \text{(here we substituted } i+1 \text{ for } i \text{ in the first sum)} \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \prod_{j=0}^{i} (x+j) + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j) \\
&\qquad \text{(here we removed a zero addend from each of the two sums)} \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} (x+i) \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} (-x) \prod_{j=0}^{i-1} (x+j)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{n-i-1} \underbrace{((x+i)+(-x))}_{=i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n-1} i \cdot \binom{n-1}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \\
&= \sum_{i=0}^{n-1} (n-1) \cdot \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \qquad \left(\text{since } i \cdot \binom{n-1}{i} = (n-1) \cdot \binom{n-2}{i-1} \right) \\
&= (n-1) \cdot \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j). \qquad (11403.2)
\end{aligned}$$

But adding up the equalities

$$\begin{aligned} f_{n-1}(x) &= \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^{(n-1)-i} \prod_{j=0}^{i-1} (x+j) = \sum_{i=0}^{n-1} \left(\binom{n-2}{i} + \binom{n-2}{i-1} \right) (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \\ &= \sum_{i=0}^{n-1} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) + \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \end{aligned}$$

and

$$\begin{aligned} x f_{n-2}(x) &= x \sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{(n-2)-i} \prod_{j=0}^{i-1} (x+j) = -\sum_{i=0}^{n-2} \binom{n-2}{i} \underbrace{(-x)(-x)^{(n-2)-i}}_{=(-x)^{n-i-1}} \prod_{j=0}^{i-1} (x+j) \\ &= -\sum_{i=0}^{n-2} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) = -\sum_{i=0}^{n-1} \binom{n-2}{i} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j) \end{aligned}$$

(here we introduced a zero addend into our sum)

yields

$$f_{n-1}(x) + x f_{n-2}(x) = \sum_{i=0}^{n-1} \binom{n-2}{i-1} (-x)^{n-i-1} \prod_{j=0}^{i-1} (x+j)$$

(since two of the three sums cancel out), and thus (11403.2) becomes $f_n(x) = (n-1) \cdot (f_{n-1}(x) + x f_{n-2}(x))$. This proves (11403.1).

Next, we introduce a *notation*: For any polynomial $p \in \mathbb{Q}[x]$, and for any integer $k \geq 0$, we denote by $\operatorname{coeff}(p, k)$ the coefficient of p before x^k. Then, every polynomial $p \in \mathbb{Q}[x]$ satisfies $p(x) = \sum_{k \geq 0} \operatorname{coeff}(p, k) \cdot x^k$.

The recurrence (11403.2) immediately yields the relations

$$\deg f_n \leq \max \{\deg f_{n-1}, 1 + \deg f_{n-2}\} \tag{11403.3}$$

and

$$\operatorname{coeff}(f_n, s) = (n-1) (\operatorname{coeff}(f_{n-1}, s) + \operatorname{coeff}(f_{n-2}, s-1)) \tag{11403.4}$$

for every positive integers $n > 3$ and s. Now, a straightforward induction (using $f_2(x) = x$ and $f_3(x) = 2x$ as the induction base, and (11403.2) and (11403.4) for the induction step) shows the three relations

$$\deg f_{2u} = \deg f_{2u+1} = u, \tag{11403.5}$$

$$\operatorname{coeff}(f_{2u}, u) > 0, \tag{11403.6}$$

$$\operatorname{coeff}(f_{2u+1}, u) > 0 \tag{11403.7}$$

for every integer $u \geq 1$ (we won't use the relations (11403.6) and (11403.7) anymore, but we need them to survive in the induction step, since they ensure that the leading terms of the polynomials $f_{n-1}(x)$ and $x f_{n-2}(x)$ don't cancel out each other in (11403.1)). In particular, (11403.5) yields our assertion that $\deg f_n = \left\lfloor \frac{n}{2} \right\rfloor$ for every integer $n > 1$.