**American Mathematical Monthly Problem 11392 by Omran Kouba, Damascus, Syria.**

Let $P$ be a regular $n$-gon. We label the consecutive vertices of this $n$-gon $P$ by $A_0$, $A_1$, ..., $A_{n-1}$, and we let $A_n = A_0$.

Let $M$ be a point in the plane, and let $B_k$ be the orthogonal projection of this point $M$ on the line $A_kA_{k+1}$ for each $k \in \{0, 1, ..., n-1\}$. Assume that this projection $B_k$ lies inside the segment $A_kA_{k+1}$ for each $k \in \{0, 1, ..., n-1\}$. Prove that

$$\sum_{k=0}^{n-1} \operatorname{Area}\left(\Delta\left(MA_kB_k\right)\right) = \frac{1}{2} \operatorname{Area}\left(P\right).$$

**Solution by Darij Grinberg.**

We denote the area of any triangle $XYZ$ by $|XYZ|$ (instead of the lengthy notation $\operatorname{Area}(\Delta(XYZ))$).

We set $A_{n+1} = A_1$ just as the problem author set $A_n = A_0$.

We WLOG assume that the $n$-gon $P$ is directed counter-clockwise.

For every $k \in \{0, 1, ..., n-1\}$, let $C_k$ denote the midpoint of the side $A_kA_{k+1}$ of $P$. Also, let $O$ be the center of $P$. Due to the symmetry of $P$, we have $OC_k \perp A_kA_{k+1}$ for every $k \in \{0, 1, ..., n-1\}$. Let $2a$ be the sidelength of $P$, and let $d$ be the distance from $O$ to every side of $P$.

Let $D_k$ be the foot of the perpendicular from $M$ to $OC_k$ for every $k \in \{0, 1, ..., n-1\}$. Then, $MB_kC_kD_k$ is a rectangle for every $k \in \{0, 1, ..., n-1\}$ (due to right angles at $B_k$, $C_k$ and $D_k$).

We will use directed segments, denoting the directed length of any segment $XY$ by $\overline{XY}$. Of course, this directed length is well-defined only if the points $X$ and $Y$ lie on some directed line. For every $k \in \{0, 1, ..., n-1\}$,

- we direct the line $A_kA_{k+1}$ in such a way that $\overline{A_kA_{k+1}} > 0$ (so that $\overline{A_kA_{k+1}} = 2a$),
- we direct the line $MB_k$ in such a way that $\overline{MB_k} > 0$,
- we direct the line $OC_k$ in such a way that $\overline{OC_k} > 0$ (so that $\overline{OC_k} = d$),
- we direct the line $MD_k$ in the same way as the line $A_kA_{k+1}$ (to which it is parallel, because $MB_kC_kD_k$ is a rectangle)[1],
- we direct the line $OM$ in such a way that $\overline{OM} > 0$.

For any two directed lines $g$ and $h$, we can not only endow segments along these lines with signs (what leads to directed segments), but also define a *directed angle* $\measuredangle(g, h)$ between the directed lines $g$ and $h$; this is the angle about which $g$ must be rotated in order to end up parallel and equidirected to $h$. This angle $\measuredangle(g, h)$ is an element of the group $\mathbb{R}/(2\pi\mathbb{Z})$ (in other words, it is an angle defined up to integral multiples of $2\pi$).

[1] If $M$ coincides with $D_k$, the line $MD_k$ has to be understood as the perpendicular from $M$ to $OC_k$ (remember the definition of $D_k$).

Define an angle $\rho \in \mathbb{R}/(2\pi\mathbb{Z})$ by $\rho = \measuredangle(A_kA_{k+1}, A_{k+1}A_{k+2})$ for every $k \in \{0, 1, ..., n-1\}$ (this is possible since all angles $\measuredangle(A_kA_{k+1}, A_{k+1}A_{k+2})$ are equal, because $P$ is a regular $n$-gon). Then, $n\rho = 0$ (since $n\rho = \sum\limits_{k=0}^{n-1} \rho = \sum\limits_{k=0}^{n-1} \measuredangle(A_kA_{k+1}, A_{k+1}A_{k+2}) = \measuredangle(A_0A_1, A_nA_{n+1}) = \measuredangle(A_0A_1, A_0A_1) = 0$) and thus $n \cdot 2\rho = 0$, but $\rho \neq 0$ and $2\rho \neq 0$ (since the lines $A_kA_{k+1}$ and $A_{k+1}A_{k+2}$ are not parallel). Let $\phi = \measuredangle(OM, A_0A_1)$. Then,

$$\measuredangle(OM, A_kA_{k+1}) = \measuredangle(OM, A_0A_1) + \sum_{i=0}^{k-1} \measuredangle(A_iA_{i+1}, A_{i+1}A_{i+2}) = \phi + \sum_{i=0}^{k-1} \rho = \phi + k\rho$$

for every $k \in \{0, 1, ..., n-1\}$. Besides, $\measuredangle(A_kA_{k+1}, OC_k) = \frac{\pi}{2}$ (in fact, $OC_k \perp A_kA_{k+1}$ yields $\measuredangle(A_kA_{k+1}, OC_k) = \pm\frac{\pi}{2}$, and the $\pm$ becomes a $+$ since the $n$-gon $P$ is directed counter-clockwise), so that

$$\measuredangle(OM, OC_k) = \underbrace{\measuredangle(OM, A_kA_{k+1})}_{=\phi+k\rho} + \underbrace{\measuredangle(A_kA_{k+1}, OC_k)}_{=\frac{\pi}{2}} = \phi + k\rho + \frac{\pi}{2}$$

for every $k \in \{0, 1, ..., n-1\}$.

Since triangle $MA_kB_k$ is right-angled at $B_k$, we have $|MA_kB_k| = \frac{1}{2} \cdot \overline{MB_k} \cdot \overline{A_kB_k}$. Since triangle $OA_kC_k$ is right-angled at $C_k$, we have $|OA_kC_k| = \frac{1}{2} \cdot \overline{OC_k} \cdot \overline{A_kC_k}$. Notice that $\overline{A_kC_k} = a$ (since $C_k$ is the midpoint of $A_kA_{k+1}$, and $\overline{A_kA_{k+1}} = 2a$) and $\overline{OC_k} = d$, so this becomes $|OA_kC_k| = \frac{1}{2} \cdot d \cdot a$.

The rectangle $MB_kC_kD_k$ yields $\overline{D_kC_k} = \overline{MB_k}$. On the other hand, $C_kB_k$ is the orthogonal projection of the segment $OM$ onto the line $A_kA_{k+1}$, so that $\overline{C_kB_k} = \overline{OM} \cdot \cos\measuredangle(OM, A_kA_{k+1})$. Besides, $OD_k$ is the orthogonal projection of the segment

$OM$ onto the line $OC_k$, so that $\overline{OD_k} = \overline{OM} \cdot \cos \measuredangle (OM, OC_k)$. Thus,

$$\sum_{k=0}^{n-1} |MA_kB_k| = \sum_{k=0}^{n-1} \frac{1}{2} \cdot \overline{MB_k} \cdot \overline{A_kB_k} = \frac{1}{2} \cdot \sum_{k=0}^{n-1} \underbrace{\overline{MB_k}}_{\substack{=\overline{D_kC_k} \\ =\overline{OC_k}-\overline{OD_k} \\ =d-\overline{OD_k}}} \cdot \underbrace{\overline{A_kB_k}}_{\substack{=\overline{A_kC_k}+\overline{C_kB_k} \\ =a+\overline{C_kB_k}}} = \frac{1}{2} \cdot \sum_{k=0}^{n-1} \left(d - \overline{OD_k}\right) \cdot \left(a + \overline{C_kB_k}\right)$$

$$= \frac{1}{2} \cdot \sum_{k=0}^{n-1} \left(d - \overline{OM} \cdot \cos \measuredangle (OM, OC_k)\right) \cdot \left(a + \overline{OM} \cdot \cos \measuredangle (OM, A_kA_{k+1})\right)$$

$$= \frac{1}{2} \cdot \sum_{k=0}^{n-1} \left(d - \overline{OM} \cdot \underbrace{\cos\left(\phi + k\rho + \frac{\pi}{2}\right)}_{=-\sin(\phi+k\rho)}\right) \cdot \left(a + \overline{OM} \cdot \cos(\phi + k\rho)\right)$$

$$= \frac{1}{2} \cdot \sum_{k=0}^{n-1} \left(da + d \cdot \overline{OM} \cdot \cos(\phi + k\rho) + a \cdot \overline{OM} \cdot \sin(\phi + k\rho) + \overline{OM}^2 \cdot \underbrace{\sin(\phi + k\rho)\cos(\phi + k\rho)}_{\substack{=\frac{1}{2}\sin(2(\phi+k\rho)) \\ =\frac{1}{2}\sin(2\phi+k\cdot 2\rho)}}\right)$$

$$= \frac{1}{2} \cdot \sum_{k=0}^{n-1} da + \frac{1}{2} d \cdot \overline{OM} \cdot \sum_{k=0}^{n-1} \cos(\phi + k\rho) + \frac{1}{2} a \cdot \overline{OM} \cdot \sum_{k=0}^{n-1} \sin(\phi + k\rho) + \frac{1}{4} \overline{OM}^2 \cdot \sum_{k=0}^{n-1} \sin(2\phi + k \cdot 2\rho). \tag{11392.1}$$

Now, we will show that any two angles $\phi$ and $\rho$ such that $n\rho = 0$ and $\rho \neq 0$ satisfy

$$\sum_{k=0}^{n-1} \cos(\phi + k\rho) = 0; \tag{11392.2}$$

$$\sum_{k=0}^{n-1} \sin(\phi + k\rho) = 0, \tag{11392.3}$$

and that any two angles $\phi$ and $\rho$ such that $n \cdot 2\rho = 0$ and $2\rho \neq 0$ satisfy

$$\sum_{k=0}^{n-1} \sin(2\phi + k \cdot 2\rho) = 0. \tag{11392.4}$$

In fact, $n\rho = 0$ yields $e^{i \cdot n\rho} = 1$, but $\rho \neq 0$ yields $e^{i\rho} \neq 1$. Thus,

$$0 = e^{i\phi} \frac{1 - 1}{e^{i\rho} - 1} = e^{i\phi} \frac{e^{i \cdot n\rho} - 1}{e^{i\rho} - 1} = e^{i\phi} \sum_{k=0}^{n-1} e^{i \cdot k\rho} = \sum_{k=0}^{n-1} e^{i \cdot (\phi + k\rho)} = \sum_{k=0}^{n-1} \left(\cos(\phi + k\rho) + i \sin(\phi + k\rho)\right).$$

Taking the real part of this equation, we obtain (11392.2); the imaginary part yields (11392.3). The identity (11392.4) is nothing but (11392.3) applied to the angles $2\phi$ and $2\rho$ instead of $\phi$ and $\rho$.

Using (11392.2)-(11392.4), our equation (11392.1) simplifies to

$$\sum_{k=0}^{n-1} |MA_kB_k| = \frac{1}{2} \cdot \sum_{k=0}^{n-1} da = \sum_{k=0}^{n-1} \frac{1}{2} \cdot d \cdot a = \sum_{k=0}^{n-1} |OA_kC_k|. \tag{11392.5}$$

By the symmetry of the regular $n$-gon $P$, we have $\sum_{k=0}^{n-1} |OA_k C_k| = \sum_{k=0}^{n-1} |OC_k B_k|$, while obviously $\sum_{k=0}^{n-1} |OA_k C_k| + \sum_{k=0}^{n-1} |OC_k B_k| = \text{Area}\, P$. Thus, $\sum_{k=0}^{n-1} |OA_k C_k| = \frac{1}{2} \text{Area}\, P$, so that (11392.5) becomes $\sum_{k=0}^{n-1} |MA_k B_k| = \frac{1}{2} \text{Area}\, P$, qed.

*Remark.* A user of the MathLinks webforum called Myth (Mikhail Leptchinski in real life) found this problem in 2005:
`http://www.mathlinks.ro/viewtopic.php?p=207173#207173`