Combinatorial proof of Chio Pivotal Condensation

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In this note, we shall give a direct proof of the *Chio pivotal condensation theorem*. This theorem is one of many that relate the determinant of a matrix to the determinant of a smaller matrix (and, in many cases, reduce the computation of the former to that of the latter – thus "condensing" the determinant). It can be stated as follows:

Theorem 0.1. Let $\mathbb{K}$ be a commutative ring with unity. Let $n \geq 2$ be an integer. Let $A = \left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} \in \mathbb{K}^{n\times n}$ be a matrix. Then,

$$\det\left(\left(a_{i,j}a_{n,n} - a_{i,n}a_{n,j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right) = a_{n,n}^{n-2} \cdot \det\left(\left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}\right).$$

We refer to [Grinbe15] for undefined notations used here (though they should all be standard).

Classically, Theorem 0.1 is proven using a trick. Namely, it is first proven under the assumption that $a_{n,n}$ be invertible (see, e.g., [Grinbe15, Exercise 6.19] and the reference therein); then, the "universality of polynomial identities" [Conrad09] shows that it holds in the general case as well (since it is an identity between two fixed polynomials in the n^2 variables $a_{1,1}, a_{1,2}, \ldots, a_{n,n}$).

In this note, we shall give a different proof of Theorem 0.1 which proceeds directly (and, to some extent, combinatorially, using bijections and sign-reversing involutions).

We fix $\mathbb{K}$, n, A and $a_{i,j}$ as in Theorem 0.1.

We start with a computation:

The definition of a determinant yields

$$\det\left(\left(a_{i,j}a_{n,n}-a_{i,n}a_{n,j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right)$$
$$=\sum_{\sigma\in S_{n-1}}(-1)^{\sigma}\prod_{i=1}^{n-1}\left(a_{i,\sigma(i)}a_{n,n}-a_{i,n}a_{n,\sigma(i)}\right)$$
$$=\sum_{\sigma\in S_{n-1}}(-1)^{\sigma}\sum_{(p_1,p_2,\ldots,p_{n-1})\in\{0,1\}^{n-1}}\prod_{i=1}^{n-1}\begin{cases}a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i=0;\\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i=1\end{cases}$$
$$\left(\begin{array}{c}\text{since every } \sigma\in S_n \text{ satisfies}\\ \prod_{i=1}^{n-1}\left(a_{i,\sigma(i)}a_{n,n}-a_{i,n}a_{n,\sigma(i)}\right)=\sum_{(p_1,p_2,\ldots,p_{n-1})\in\{0,1\}^{n-1}}\prod_{i=1}^{n-1}\begin{cases}a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i=0;\\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i=1\end{cases}\end{array}\right)$$
$$=\sum_{(p_1,p_2,\ldots,p_{n-1})\in\{0,1\}^{n-1}}\sum_{\sigma\in S_{n-1}}(-1)^{\sigma}\prod_{i=1}^{n-1}\begin{cases}a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i=0;\\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i=1\end{cases}. \tag{1}$$

We shall need three further notations:

- For any $p=(p_1,p_2,\ldots,p_{n-1})\in\{0,1\}^{n-1}$, set $|p|=p_1+p_2+\cdots+p_{n-1}\in\mathbb{N}$. (Here and further below, $\mathbb{N}$ means the set $\{0,1,2,\ldots\}$.)
- We set $[m]=\{1,2,\ldots,m\}$ for every $m\in\mathbb{N}$.
- For every $k\in[n]$, we set $T_{n,k}=\{\tau\in S_n\mid\tau(k)=n\}$. It is clear that the sets $T_{n,1},T_{n,2},\ldots,T_{n,n}$ are pairwise disjoint, and their union is S_n.

Now, we state a lemma:

Lemma 0.2. Let $p=(p_1,p_2,\ldots,p_{n-1})\in\{0,1\}^{n-1}$.

(a) If $|p|=0$, then

$$\sum_{\sigma\in S_{n-1}}(-1)^{\sigma}\prod_{i=1}^{n-1}\begin{cases}a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i=0;\\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i=1\end{cases}=a_{n,n}^{n-2}\sum_{\tau\in T_{n,n}}(-1)^{\tau}\prod_{i=1}^{n}a_{i,\tau(i)}.$$

(b) If $|p|=1$, then $p=(0,0,\ldots,0,1,0,0,\ldots,0)$ with the 1 being at position k for some $k\in[n-1]$. This k further satisfies

$$\sum_{\sigma\in S_{n-1}}(-1)^{\sigma}\prod_{i=1}^{n-1}\begin{cases}a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i=0;\\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i=1\end{cases}=a_{n,n}^{n-2}\sum_{\tau\in T_{n,k}}(-1)^{\tau}\prod_{i=1}^{n}a_{i,\tau(i)}.$$

(c) If $|p|>1$, then

$$\sum_{\sigma\in S_{n-1}}(-1)^{\sigma}\prod_{i=1}^{n-1}\begin{cases}a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i=0;\\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i=1\end{cases}=0.$$

We shall prove Lemma 0.2 further below; let us first see how we can derive Theorem 0.1 from it:

Proof of Theorem 0.1 using Lemma 0.2. Among all the $(n-1)$-tuples $p = (p_1, p_2, \ldots, p_{n-1}) \in \{0,1\}^{n-1}$, exactly one satisfies $|p| = 0$ (namely, $p = (0, 0, \ldots, 0)$), and exactly $n-1$ satisfy $|p| = 1$ (namely, the $(n-1)$-tuples $p = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$, with the 1 being at position k for some $k \in [n-1]$); all other $(n-1)$-tuples p satisfy $|p| > 1$. Using this observation, and using Lemma 0.2, we may simplify the right hand side of (1) as follows:

$$\begin{aligned}
&\sum_{(p_1, p_2, \ldots, p_{n-1}) \in \{0,1\}^{n-1}} \sum_{\sigma \in S_{n-1}} (-1)^{\sigma} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)} a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n} a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&= a_{n,n}^{n-2} \sum_{\tau \in T_{n,n}} (-1)^{\tau} \prod_{i=1}^{n} a_{i,\tau(i)} + \sum_{k=1}^{n-1} a_{n,n}^{n-2} \sum_{\tau \in T_{n,k}} (-1)^{\tau} \prod_{i=1}^{n} a_{i,\tau(i)} \\
&= \sum_{k=1}^{n} a_{n,n}^{n-2} \sum_{\tau \in T_{n,k}} (-1)^{\tau} \prod_{i=1}^{n} a_{i,\tau(i)} = a_{n,n}^{n-2} \sum_{k=1}^{n} \sum_{\tau \in T_{n,k}} (-1)^{\tau} \prod_{i=1}^{n} a_{i,\tau(i)}.
\end{aligned}$$

Since $T_{n,1}, T_{n,2}, \ldots, T_{n,n}$ are pairwise disjoint and their union is S_n, this reduces to

$$a_{n,n}^{n-2} \sum_{\tau \in S_n} (-1)^{\tau} \prod_{i=1}^{n} a_{i,\tau(i)} = a_{n,n}^{n-2} \cdot \det \left(\left(a_{i,j} \right)_{1 \leq i \leq n,\ 1 \leq j \leq n} \right).$$

Thus, (1) becomes

$$\det \left(\left(a_{i,j} a_{n,n} - a_{i,n} a_{n,j} \right)_{1 \leq i \leq n-1,\ 1 \leq j \leq n-1} \right) = a_{n,n}^{n-2} \cdot \det \left(\left(a_{i,j} \right)_{1 \leq i \leq n,\ 1 \leq j \leq n} \right).$$

Hence, Theorem 0.1 is proven. □

It remains to prove Lemma 0.2.

Before we do so, we shall introduce one further notation. Namely, for any $\sigma \in S_{n-1}$, we let $\widehat{\sigma}$ be the permutation of $[n]$ defined by

$$\widehat{\sigma}(i) = \begin{cases} \sigma(i), & \text{if } i < n; \\ n, & \text{if } i = n \end{cases} \qquad \text{for every } i \in \{1, 2, \ldots, n\}.$$

It is well-known that the map $S_{n-1} \to T_{n,n}$, $\sigma \mapsto \widehat{\sigma}$ is well-defined (i.e., we have $\widehat{\sigma} \in T_{n,n}$ for every $\sigma \in S_{n-1}$) and a bijection.[1] Furthermore, it is well-known[2] that

$$(-1)^{\widehat{\sigma}} = (-1)^{\sigma} \qquad \text{for every } \sigma \in S_n. \tag{2}$$

[1]See [Grinbe15, proof of Lemma 6.44] (where this map has been denoted by Φ, and where $T_{n,n}$ has been denoted by T) for a proof.

[2]See, for example, [Grinbe15, Section 6.6, (395)].

Proof of Lemma 0.2. **(a)** Assume that $|p| = 0$. In other words, $p_1 + p_2 + \cdots + p_{n-1} = 0$. Since $(p_1, p_2, \ldots, p_{n-1}) \in \{0,1\}^{n-1}$, this yields $p_1 = p_2 = \cdots = p_{n-1} = 0$.

Now,

$$\begin{aligned}
&\sum_{\sigma \in S_{n-1}} (-1)^\sigma \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)} a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n} a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&= \sum_{\sigma \in S_{n-1}} (-1)^\sigma \prod_{i=1}^{n-1} \left(a_{i,\sigma(i)} a_{n,n} \right) \qquad (\text{since } p_i = 0) \\
&= a_{n,n}^{n-1} \sum_{\sigma \in S_{n-1}} (-1)^\sigma \prod_{i=1}^{n-1} a_{i,\sigma(i)} = a_{n,n}^{n-2} a_{n,n} \sum_{\sigma \in S_{n-1}} (-1)^\sigma \prod_{i=1}^{n-1} a_{i,\sigma(i)} \\
&= a_{n,n}^{n-2} \sum_{\sigma \in S_{n-1}} (-1)^\sigma a_{n,n} \prod_{i=1}^{n-1} a_{i,\sigma(i)} = a_{n,n}^{n-2} \sum_{\sigma \in S_{n-1}} (-1)^{\widehat{\sigma}} \prod_{i=1}^{n} a_{i,\widehat{\sigma}(i)} \\
&\qquad \left(\begin{array}{c} \text{since the definition of } \widehat{\sigma} \text{ shows that } a_{n,n} \prod_{i=1}^{n-1} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\widehat{\sigma}(i)}, \\ \text{and since (2) shows that } (-1)^\sigma = (-1)^{\widehat{\sigma}} \text{ for every } \sigma \in S_{n-1} \end{array} \right) \\
&= a_{n,n}^{n-2} \sum_{\tau \in T_{n,n}} (-1)^\tau \prod_{i=1}^{n} a_{i,\tau(i)}
\end{aligned}$$

(here, we have substituted τ for $\widehat{\sigma}$, since the map $S_{n-1} \to T_{n,n}$, $\sigma \mapsto \widehat{\sigma}$ is a bijection). This proves Lemma 0.2 **(a)**.

(b) Assume that $|p| = 1$. In other words, $p_1 + p_2 + \cdots + p_{n-1} = 1$. Since $(p_1, p_2, \ldots, p_{n-1}) \in \{0,1\}^{n-1}$, this yields that exactly one of the p_i's must be equal to 1. In other words, $p_k = 1$ for a unique $k \in \{1, 2, \ldots, n-1\}$. Consider this k.

Then we have

$$\begin{aligned}
&\sum_{\sigma \in S_{n-1}} (-1)^\sigma \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)} a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n} a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&= \sum_{\sigma \in S_{n-1}} (-1)^\sigma \left(-a_{k,n} a_{n,\sigma(k)} \right) \prod_{i \in [n-1] \setminus \{k\}} \left(a_{i,\sigma(i)} a_{n,n} \right) \\
&= a_{n,n}^{n-2} \sum_{\sigma \in S_{n-1}} (-1)^\sigma \left(-a_{k,n} a_{n,\sigma(k)} \right) \prod_{i \in [n-1] \setminus \{k\}} a_{i,\sigma(i)}. \qquad (3)
\end{aligned}$$

Now, let $t_{n,k} \in S_n$ be the transposition which switches n with k while leaving all other elements of $\{1, 2, \ldots, n\}$ unchanged. Since $t_{n,k}$ is a transposition, we have $(-1)^{t_{n,k}} = -1$. Furthermore, it is easy to see that the map $T_{n,n} \to T_{n,k}$, $\tau \mapsto \tau \circ t_{n,k}$ is well-defined and a bijection[3].

But recall that the map $S_{n-1} \to T_{n,n}$, $\sigma \mapsto \widehat{\sigma}$ is a bijection. Composing this bijection with the bijection $T_{n,n} \to T_{n,k}$, $\tau \mapsto \tau \circ t_{n,k}$, we obtain a bijection $S_{n-1} \to$

[3] Its inverse is the map $T_{n,k} \to T_{n,n}$, $\tau \mapsto \tau \circ t_{n,k}$.

$T_{n,k}$, $\sigma \mapsto \widehat{\sigma} \circ t_{n,k}$. This bijection satisfies

$$(-1)^{\widehat{\sigma}\circ t_{n,k}} = \underbrace{(-1)^{\widehat{\sigma}}}_{=(-1)^{\sigma}} \cdot \underbrace{(-1)^{t_{n,k}}}_{=-1} = -(-1)^{\sigma} \qquad \text{for every } \sigma \in S_{n-1}. \tag{4}$$

Moreover, for any $\sigma \in S_{n-1}$, it is easy to see that $\left(\widehat{\sigma} \circ t_{n,k}\right)(i) = \begin{cases} \sigma(i), & \text{if } i \notin \{k,n\}; \\ n, & \text{if } i = k; \\ \sigma(k), & \text{if } i = n \end{cases}$ for all $i \in [n]$. Thus, for any $\sigma \in S_{n-1}$, we have

$$a_{k,n} a_{n,\sigma(k)} \prod_{i \in [n-1] \setminus \{k\}} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\left(\widehat{\sigma}\circ t_{n,k}\right)(i)}. \tag{5}$$

Now, (3) becomes

$$\begin{aligned}
&\sum_{\sigma \in S_{n-1}} (-1)^{\sigma} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)} a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n} a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&= a_{n,n}^{n-2} \sum_{\sigma \in S_{n-1}} (-1)^{\sigma} \left(-a_{k,n} a_{n,\sigma(k)}\right) \prod_{i \in [n-1] \setminus \{k\}} a_{i,\sigma(i)} \\
&= a_{n,n}^{n-2} \sum_{\sigma \in S_{n-1}} \underbrace{\left(-(-1)^{\sigma}\right)}_{\substack{=(-1)^{\widehat{\sigma}\circ t_{n,k}} \\ \text{(by (4))}}} \underbrace{a_{k,n} a_{n,\sigma(k)} \prod_{i \in [n-1] \setminus \{k\}} a_{i,\sigma(i)}}_{\substack{=\prod_{i=1}^{n} a_{i,\left(\widehat{\sigma}\circ t_{n,k}\right)(i)} \\ \text{(by (5))}}} \\
&= a_{n,n}^{n-2} \sum_{\sigma \in S_{n-1}} (-1)^{\widehat{\sigma}\circ t_{n,k}} \prod_{i=1}^{n} a_{i,\left(\widehat{\sigma}\circ t_{n,k}\right)(i)} = a_{n,n}^{n-2} \sum_{\tau \in T_{n,k}} (-1)^{\tau} \prod_{i=1}^{n} a_{i,\tau(i)}
\end{aligned}$$

(here, we have substituted τ for $\widehat{\sigma} \circ t_{n,k}$ in the sum, since the map $S_{n-1} \to T_{n,k}$, $\sigma \mapsto \widehat{\sigma} \circ t_{n,k}$ is a bijection). This proves Lemma 0.2 **(b)**.

(c) Assume that $|p| > 1$. In other words, $p_1 + p_2 + \cdots + p_{n-1} > 1$. Thus there exist two distinct elements u and v of $[n-1]$ such that $p_u = p_v = 1$ (since $(p_1, p_2, \ldots, p_{n-1}) \in \{0,1\}^{n-1}$). We choose such u and v.

We now consider the sum

$$\sum_{\sigma \in S_{n-1}} (-1)^{\sigma} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)} a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n} a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \tag{6}$$

To show that this sum is equal to 0, we use the following claim.

Claim 1: For each $\sigma \in S_{n-1}$, we have

$$\prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)} a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n} a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} = \prod_{i=1}^{n-1} \begin{cases} a_{i,(\sigma\circ t_{u,v})(i)} a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n} a_{n,(\sigma\circ t_{u,v})(i)}, & \text{if } p_i = 1 \end{cases}. \tag{7}$$

Proof of Claim 1: Let $\sigma \in S_{n-1}$. We must prove the equation (7). We note that the products on the left and the right hand sides of this equation are identical for each factor except for the u^{th} factor and the v^{th} factor (because for any $i \in [n-1] \setminus \{u, v\}$, we have $(\sigma \circ t_{u,v})(i) = \sigma\left(\underbrace{t_{u,v}(i)}_{=i}\right) = \sigma(i)$). Thus it remains to show that

$$\left(-a_{u,n}a_{n,\sigma(u)}\right)\left(-a_{v,n}a_{n,\sigma(v)}\right) = \left(-a_{u,n}a_{n,(\sigma\circ t_{u,v})(u)}\right)\left(-a_{v,n}a_{n,(\sigma\circ t_{u,v})(v)}\right)$$

(here we are using the fact that $p_u = p_v = 1$). But this follows from $(\sigma \circ t_{u,v})(u) = \sigma(v)$ and $(\sigma \circ t_{u,v})(v) = \sigma(u)$. Hence, Claim 1 is proven. ■

Let

$$A_{n-1} = \left\{\sigma \in S_{n-1} \mid (-1)^{\sigma} = 1\right\}$$

and

$$C_{n-1} = \left\{\sigma \in S_{n-1} \mid (-1)^{\sigma} = -1\right\}.$$

The sets A_{n-1} and C_{n-1} are disjoint, and satisfy $A_{n-1} \cup C_{n-1} = S_{n-1}$. Furthermore, every $\sigma \in A_{n-1}$ satisfies $(-1)^{\sigma\circ t_{u,v}} = \underbrace{(-1)^{\sigma}}_{\substack{=1 \\ (\text{since } \sigma\in A_{n-1})}} \cdot \underbrace{(-1)^{t_{u,v}}}_{=-1} = -1$ and thus $\sigma \circ t_{u,v} \in C_{n-1}$. Hence, the map $A_{n-1} \to C_{n-1}$, $\sigma \mapsto \sigma \circ t_{u,v}$ is well-defined. It is easy to see that this map is also a bijection[4]. Now,

$$\begin{aligned}
&\sum_{\sigma\in S_{n-1}} (-1)^{\sigma} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&= \sum_{\sigma\in A_{n-1}} \underbrace{(-1)^{\sigma}}_{\substack{=1 \\ (\text{since } \sigma\in A_{n-1})}} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&\qquad + \sum_{\sigma\in C_{n-1}} \underbrace{(-1)^{\sigma}}_{\substack{=-1 \\ (\text{since } \sigma\in C_{n-1})}} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&= \sum_{\sigma\in A_{n-1}} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} - \sum_{\sigma\in C_{n-1}} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} \\
&= \sum_{\sigma\in A_{n-1}} \prod_{i=1}^{n-1} \begin{cases} a_{i,\sigma(i)}a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n}a_{n,\sigma(i)}, & \text{if } p_i = 1 \end{cases} - \sum_{\sigma\in A_{n-1}} \prod_{i=1}^{n-1} \begin{cases} a_{i,(\sigma\circ t_{u,v})(i)}a_{n,n}, & \text{if } p_i = 0; \\ -a_{i,n}a_{n,(\sigma\circ t_{u,v})(i)}, & \text{if } p_i = 1 \end{cases}
\end{aligned}$$

(here, we have substituted $\sigma \circ t_{u,v}$ for σ in the second sum, since the map $A_{n-1} \to C_{n-1}$, $\sigma \mapsto \sigma \circ t_{u,v}$ is a bijection). But Claim 1 shows that the two sums on the right hand side of this equation are equal to each other term by term, and thus the right hand side is 0. Therefore, so is the left hand side.

This proves Lemma 0.2 **(c)** and completes the proof of Theorem 0.1. □

[4] Its inverse is the map $C_{n-1} \to A_{n-1}$, $\sigma \mapsto \sigma \circ t_{u,v}$.

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