Witt vectors. Part 1

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Witt#4: Some computations with symmetric functions

[version 1.2 (1 May 2013), not completed, not proofread]

In this note, we will prove some of the formulae from section 9 of [1] which remain unproven in [1]. First, some definitions:

Definition 1. We denote by $\mathbb{N}$ the set $\{0, 1, 2, ...\}$ (and not the set $\{1, 2, 3, ...\}$, as Hazewinkel does in [1]).

Definition 2. Let I be an arbitrary countable set. (Note that throughout most of section 9 of [1], it is silently assumed that $I = \{1, 2, 3, ...\}$.) Every element $\alpha \in \mathbb{N}^I$ is a family of nonnegative integers, indexed by elements of I. For every $\alpha \in \mathbb{N}^I$ and every $i \in I$, we denote by α_i the i-th member of the family α. Then, of course, every element $\alpha \in \mathbb{N}^I$ satisfies $\alpha = (\alpha_i)_{i\in I}$.

We denote by $\mathbb{N}^I_{\text{fin}}$ the subset

$$\left\{\alpha \in \mathbb{N}^I \ \mid \ \text{only finitely many } i \in I \text{ satisfy } \alpha_i \neq 0\right\}$$

of $\mathbb{N}^I$. [1] Obviously, for every element $\alpha \in \mathbb{N}^I_{\text{fin}}$, the sum $\sum\limits_{i\in I} \alpha_i$ is a well-defined nonnegative integer (since only finitely many addends of this sum are nonzero), so that we can define a function $\operatorname{wt} : \mathbb{N}^I_{\text{fin}} \to \mathbb{N}$ by

$$\left(\operatorname{wt} \alpha = \sum_{i\in I} \alpha_i \qquad \text{for every } \alpha \in \mathbb{N}^I_{\text{fin}}\right).$$

Consider this function wt.

We consider the polynomial ring $\mathbb{Z}[\xi_i \mid i \in I]$ and the power series ring $\mathbb{Z}[[\xi_i \mid i \in I]]$, where $(\xi_i)_{i\in I}$ is a family of pairwise distinct symbols indexed by elements of I.

For every element $\alpha \in \mathbb{N}^I_{\text{fin}}$, we can define a polynomial $\xi^\alpha \in \mathbb{Z}[\xi_i \mid i \in I]$ by $\xi^\alpha = \prod\limits_{i\in I} \xi_i^{\alpha_i}$ (this product is well-defined, since only finitely many of its factors are $\neq 1$). Such a polynomial ξ^α is called a monomial. We consider

[1] More generally, if A is any subset of $\mathbb{N}$, then we will denote by A^I_{fin} the subset

$$\left\{\alpha \in A^I \ \mid \ \text{only finitely many } i \in I \text{ satisfy } \alpha_i \neq 0\right\}$$

of A^I.

the polynomial ring $\mathbb{Z}[\xi_i \mid i \in I]$ as a graded ring with unity[2], with the n-th graded component being the $\mathbb{Z}$-module

$$\left\langle \xi^\alpha \mid \alpha \in \mathbb{N}_{\text{fin}}^I \text{ such that } \operatorname{wt}\alpha = n \right\rangle .$$

An element of $\mathbb{Z}[\xi_i \mid i \in I]$ is said to be *n-homogeneous* (or *homogeneous of degree n*) if it lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]$.

We consider the polynomial ring $\mathbb{Z}[\xi_i \mid i \in I]$ as a subring of the ring $\mathbb{Z}[[\xi_i \mid i \in I]]$ of power series in the indeterminates ξ_i. We will now define a ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ that lies "between these two rings" (i. e., that contains $\mathbb{Z}[\xi_i \mid i \in I]$ as a subring, but is a subring of $\mathbb{Z}[[\xi_i \mid i \in I]]$):

For every power series $P \in \mathbb{Z}[[\xi_i \mid i \in I]]$ and every $\alpha \in \mathbb{N}_{\text{fin}}^I$, we denote by $\operatorname{coeff}_\alpha P$ the coefficient of the power series P before the monomial ξ^α. We denote by $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ the subring

$$\left\{ P \in \mathbb{Z}[[\xi_i \mid i \in I]] \mid \left(\begin{array}{c} \text{there exists some } n \in \mathbb{N} \text{ such that every } \alpha \in \mathbb{N}_{\text{fin}}^I \\ \text{with } \operatorname{wt}\alpha \geq n \text{ satisfies } \operatorname{coeff}_\alpha P = 0 \end{array} \right) \right\}$$

of $\mathbb{Z}[[\xi_i \mid i \in I]]$. In other words, we define $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ as the ring of all power series $P \in \mathbb{Z}[[\xi_i \mid i \in I]]$ where all monomials of sufficiently high degree appear with zero coefficient.

This ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ is obviously a subring of $\mathbb{Z}[[\xi_i \mid i \in I]]$ (even a proper subring, if $I \neq \varnothing$), but contains the ring $\mathbb{Z}[\xi_i \mid i \in I]$ as a subring (and is larger than $\mathbb{Z}[\xi_i \mid i \in I]$ if I is an infinite set).

The difference between the rings $\mathbb{Z}[[\xi_i \mid i \in I]]$ and $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ is that the ring $\mathbb{Z}[[\xi_i \mid i \in I]]$ contains power series like $1 + \xi_\iota + \xi_\iota^2 + \xi_\iota^3 + ...$ (where ι is some element of I), while the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ does not (since the power series $1 + \xi_\iota + \xi_\iota^2 + \xi_\iota^3 + ...$ contains monomials ξ^α with arbitrarily large degree $\operatorname{wt}\alpha$). The difference between the rings $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ and $\mathbb{Z}[\xi_i \mid i \in I]$ is that the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ contains power series like $\sum_{i \in I} \xi_i$, while the ring $\mathbb{Z}[\xi_i \mid i \in I]$ does not, unless I is a finite set. The moral of the story is that the elements of the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ are something between power series and polynomials: They may contain infinitely many monomials, but all these monomials must have bounded (from above) degree. Of course, if I is a finite set, then $\mathbb{Z}[\xi_i \mid i \in I]_\infty = \mathbb{Z}[\xi_i \mid i \in I]$ (since there are only finitely

[2] *Remark.* Different authors sometimes use different (and non-equivalent!) notions of a "graded ring with unity". The one that we are using here is defined as follows:

Definition. A "*graded ring with unity*" means a ring A with unity equipped with a family $(A_n)_{n \in \mathbb{N}}$ of subgroups of the additive group A satisfying $1 \in A_0$ and

$$(A_n A_m \subseteq A_{n+m} \text{ for every } n \in \mathbb{N} \text{ and } m \in \mathbb{N}) .$$

Also, we use the following notation:

Definition. If a ring A, equipped with a family $(A_n)_{n \in \mathbb{N}}$, is a graded ring, then the family $(A_n)_{n \in \mathbb{N}}$ is said to be the *grading* of this graded ring A.

Definition. If a ring A, equipped with a family $(A_n)_{n \in \mathbb{N}}$, is a graded ring, then, for each $n \in \mathbb{N}$, the group A_n is called the *n-th graded component* of the graded ring A.

many monomials of each degree if I is a finite set). But if I is infinite, then $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ is truly larger than $\mathbb{Z}[\xi_i \mid i \in I]$.

We consider the polynomial ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ as a graded ring with unity, with the n-th graded component being the $\mathbb{Z}$-module

$$\left\{P \in \mathbb{Z}[[\xi_i \mid i \in I]] \mid \text{every } \alpha \in \mathbb{N}_{\text{fin}}^I \text{ with } \operatorname{wt} \alpha \neq n \text{ satisfies } \operatorname{coeff}_\alpha P = 0\right\}.$$

An element of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ is said to be *n-homogeneous* (or *homogeneous of degree n*) if it lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$.

(Note that, unlike $\mathbb{Z}[\xi_i \mid i \in I]$ or $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, the ring of power series $\mathbb{Z}[[\xi_i \mid i \in I]]$ does not naturally have a grading in our sense of this word.)

Definition 3. I define a *partition* as a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ of nonnegative integers satisfying $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ....$ This definition of a partition is slightly different from the one given in [1], 9.30 - but these two definitions are easily seen to be equivalent. In fact, in [1], 9.30, Hazewinkel defines a partition as a finite sequence $(\lambda_1, \lambda_2, ..., \lambda_n)$ of nonnegative integers satisfying $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, and identifies any two such partitions which only differ in the number of trailing zeroes[3]. But any partition $(\lambda_1, \lambda_2, ..., \lambda_n)$ in Hazewinkel's sense can be extended to a partition in my sense - i. e., to a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ of nonnegative integers satisfying $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ...$ - by adding trailing zeroes (i. e. by setting $\lambda_i = 0$ for all $i > n$), and conversely, any partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ in my sense is an extension of a partition in Hazewinkel's sense by trailing zeroes (in fact, there exists some $\nu \in \mathbb{N}$ such that $\lambda_\nu = \lambda_{\nu+1} = \lambda_{\nu+2} = ... = 0$ [4], so that the sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$ is the extension of the finite sequence $(\lambda_1, \lambda_2, ..., \lambda_{\nu-1})$ by trailing zeroes). This yields a one-to-one correspondence between partitions in my sense and partitions in Hazewinkel's sense, so these two notions of partition can be regarded as equivalent.

We denote the set of all partitions by Par.

Definition 4. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$ be a partition (in my sense).

(a) Let $\alpha \in \mathbb{N}_{\text{fin}}^I$. Then, we write $\lambda \sim \alpha$ (and we say that the family α is a *permutation* of the partition λ) if and only if there exist

- a subset I' of I such that $\alpha_i = 0$ for every $i \in I \setminus I'$,
- a subset N of $\{1, 2, 3, ...\}$ such that $\lambda_n = 0$ for every $n \in \{1, 2, 3, ...\} \setminus N$,
- and a bijection $\Phi : N \to I'$ such that $\alpha_{\Phi(n)} = \lambda_n$ for every $n \in N$.

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[3] I. e., he identifies any partition $(\lambda_1, \lambda_2, ..., \lambda_n)$ with the partition $\left(\lambda_1, \lambda_2, ..., \lambda_n, \underbrace{0, 0, ..., 0}_{m \text{ zeroes}}\right)$ for every $m \in \mathbb{N}$.

[4] In fact, since $\lambda \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$, there exists some $\nu \in \mathbb{N}$ such that $\lambda_\nu = 0$, and thus this ν satisfies $\lambda_\nu = \lambda_{\nu+1} = \lambda_{\nu+2} = ... = 0$ (since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ...$).

[5] In other words, we write $\lambda \sim \alpha$ if and only if the multiset $[\alpha_i \mid i \in I]$ and the multiset $[\lambda_n \mid n \in \{1,2,3,...\}]$ are "equal up to the element 0 " (this means that they contain every element $k \neq 0$ the same number of times, but may contain the element 0 differently often).[6]

In other words, we write $\lambda \sim \alpha$ if and only if

$$|\{i \in I \mid \alpha_i = k\}| = |\{n \in \{1,2,3,...\} \mid \lambda_n = k\}| \qquad \text{for every } k \in \{1,2,3,...\}$$

(but not necessarily for $k = 0$).

Clearly,

$$\text{if } \lambda \sim \alpha, \text{ then } \operatorname{wt}\lambda = \operatorname{wt}\alpha. \tag{1}$$

Besides,

$$\text{for every } \alpha \in \mathbb{N}^I_{\text{fin}}, \text{ there exists one and only one partition } \lambda \text{ satisfying } \lambda \sim \alpha . \tag{2}$$

(b) Let $\alpha \in \mathbb{N}^I_{\text{fin}}$. Then, we write $\lambda \nsim \alpha$ if and only if $\lambda \sim \alpha$ is false.

(c) We define a power series $m_\lambda \in \mathbb{Z}[[\xi_i \mid i \in I]]$ by

$$m_\lambda = \sum_{\substack{\alpha \in \mathbb{N}^I_{\text{fin}};\\ \lambda \sim \alpha}} \xi^\alpha.$$

Clearly, for every $\alpha \in \mathbb{N}^I_{\text{fin}}$, we have $\operatorname{coeff}_\alpha(m_\lambda) = \begin{cases} 1, & \text{if } \lambda \sim \alpha; \\ 0, & \text{if } \lambda \nsim \alpha \end{cases}$. Thus, the power series m_λ lies in $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ [7]. This power series m_λ is called the *monomial symmetric function associated to the partition* λ. Actually, this power series m_λ is a polynomial (i. e., an element of $\mathbb{Z}[\xi_i \mid i \in I]$) if I is a finite set, but in the case of I being infinite, m_λ is only a "symmetric function" (i. e., an element of **Symm**, as defined in the Appendix of [1]).

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[5] If the set I is infinite, this definition is equivalent to the following simpler definition: We write $\lambda \sim \alpha$ if there exists a bijection $\widetilde{\Phi} : \{1,2,3,...\} \to I$ such that $\alpha_{\widetilde{\Phi}(n)} = \lambda_n$ for every $n \in \{1,2,3,...\}$.

However, if the set I is finite, then this simpler definition makes no sense (because there can never be a bijection $\widetilde{\Phi} : \{1,2,3,...\} \to I$).

[6] Here, we denote by $[\alpha_i \mid i \in I]$ the multiset formed by writing down α_i for every $i \in I$, and we denote by $[\lambda_n \mid n \in \{1,2,3,...\}]$ the multiset formed by writing down λ_n for every $n \in \{1,2,3,...\}$.

[7] *Proof.* Every $\alpha \in \mathbb{N}^I_{\text{fin}}$ such that $\operatorname{wt}\alpha \geq \operatorname{wt}\lambda + 1$ satisfies $\lambda \nsim \alpha$ (because otherwise, it would satisfy $\lambda \sim \alpha$, so that

$$\begin{aligned} \operatorname{wt}\lambda &= \operatorname{wt}\alpha \qquad (\text{by } (1)) \\ &\geq \operatorname{wt}\lambda + 1 > \operatorname{wt}\lambda, \end{aligned}$$

which is absurd). Hence, every $\alpha \in \mathbb{N}^I_{\text{fin}}$ such that $\operatorname{wt}\alpha \geq \operatorname{wt}\lambda + 1$ satisfies $\operatorname{coeff}_\alpha(m_\lambda) = \begin{cases} 1, & \text{if } \lambda \sim \alpha; \\ 0, & \text{if } \lambda \nsim \alpha \end{cases} = 0$ (since $\lambda \nsim \alpha$). Thus, there exists some $n \in \mathbb{N}$ such that every $\alpha \in \mathbb{N}^I_{\text{fin}}$ with $\operatorname{wt}\alpha \geq n$ satisfies $\operatorname{coeff}_\alpha(m_\lambda) = 0$ (in fact, take $n = \operatorname{wt}\lambda + 1$). In other words, $m_\lambda \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ (by the definition of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$), qed.

Here are some explicit examples for m_λ where $I = \{1, 2, 3, ...\}$:

$$\begin{aligned}
m_{(0)} &= 1 \qquad \text{(note that } (0) = (0, 0, 0, ...) \text{ is the zero partition)};\\
m_{(1)} &= \xi_1 + \xi_2 + \xi_3 + \xi_4 + ...;\\
m_{(2)} &= \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + ...;\\
m_{(1,1)} &= \xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 + \xi_1\xi_4 + \xi_2\xi_4 + \xi_3\xi_4 + ...;\\
m_{(2,1)} &= \xi_1^2\xi_2 + \xi_1\xi_2^2 + \xi_1^2\xi_3 + \xi_1\xi_3^2 + \xi_2^2\xi_3 + \xi_2\xi_3^2 + \xi_1^2\xi_4 + \xi_1\xi_4^2 + \xi_2^2\xi_4 + \xi_2\xi_4^2 + \xi_3^2\xi_4 + \xi_3\xi_4^2 + ...;\\
m_{(1,1,1)} &= \xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4 + \xi_1\xi_3\xi_4 + \xi_2\xi_3\xi_4 + ...
\end{aligned}$$

We note that for every partition λ, the power series $m_\lambda \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ is $\operatorname{wt}\lambda$-homogeneous. This is because every $\alpha \in \mathbb{N}_{\text{fin}}^I$ with $\operatorname{wt}\alpha \neq \operatorname{wt}\lambda$ satisfies $\operatorname{coeff}_\alpha(m_\lambda) = 0$ (since $\operatorname{wt}\alpha \neq \operatorname{wt}\lambda$ yields $\lambda \nsim \alpha$ [8] and thus $\operatorname{coeff}_\alpha(m_\lambda) = \begin{cases} 1, & \text{if } \lambda \sim \alpha; \\ 0, & \text{if } \lambda \nsim \alpha \end{cases} = 0$).

Definition 5. Let λ be a partition. Let n be a positive integer. We define a nonnegative integer $m_n(\lambda)$ by

$$m_n(\lambda) = |\{i \in \{1, 2, 3, ...\} \mid \lambda_i = n\}|,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$. This integer $m_n(\lambda)$ is the number of all blocks of size n in the block representation of the partition λ.

We can define a map $m : \operatorname{Par} \to \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ by

$$m(\lambda) = (m_1(\lambda), m_2(\lambda), m_3(\lambda), ...) \qquad \text{for all } \lambda \in \operatorname{Par}.$$

It is easy to see that this map m is a bijection. The inverse map $m^{-1} : \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}} \to \operatorname{Par}$ is given by

$$m^{-1}(a_1, a_2, a_3, ...) = (1^{a_1}, 2^{a_2}, 3^{a_3}, ...) \qquad \text{for every } (a_1, a_2, a_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}.$$

Here, $(1^{a_1}, 2^{a_2}, 3^{a_3}, ...)$ denotes the partition

$$\left(\underbrace{\nu, \nu, ..., \nu}_{a_\nu \text{ times}}, \underbrace{\nu - 1, \nu - 1, ..., \nu - 1}_{a_{\nu-1} \text{ times}}, ..., \underbrace{2, 2, ..., 2}_{a_2 \text{ times}}, \underbrace{1, 1, ..., 1}_{a_1 \text{ times}}\right),$$

where ν is the maximal element of $\{1, 2, 3, ...\}$ satisfying $a_\nu \neq 0$.

Note that every partition $\lambda \in \operatorname{Par}$ satisfies

$$\operatorname{wt}\lambda = \sum_{k=1}^{\infty} k m_k(\lambda), \tag{3}$$

since

$$\begin{aligned}
\operatorname{wt}\lambda &= \sum_{n\in\{1,2,3,...\}} \lambda_n = \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{n\in\{1,2,3,...\};\\ \lambda_n = k}} k}_{=|\{n\in\{1,2,3,...\} \mid \lambda_n = k\}|\cdot k} = \sum_{k=0}^{\infty} \underbrace{|\{n \in \{1, 2, 3, ...\} \mid \lambda_n = k\}|}_{\substack{=|\{i\in\{1,2,3,...\} \mid \lambda_i = k\}| \\ = m_k(\lambda)}} \cdot k \\
&= \sum_{k=0}^{\infty} m_k(\lambda) \cdot k = \underbrace{m_0(\lambda) \cdot 0}_{=0} + \sum_{k=1}^{\infty} \underbrace{m_k(\lambda) \cdot k}_{= k m_k(\lambda)} = \sum_{k=1}^{\infty} k m_k(\lambda).
\end{aligned}$$

[8] since if $\lambda \sim \alpha$, then $\operatorname{wt}\lambda = \operatorname{wt}\alpha$, contradicting $\operatorname{wt}\alpha \neq \operatorname{wt}\lambda$

Definition 6. For every $n \in \mathbb{N}$, we define a power series $h_n \in \mathbb{Z}[[\xi_i \mid i \in I]]$ by

$$h_n = \sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \lambda = n}} m_\lambda.$$

The sum $\sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \lambda = n}} m_\lambda$ is a finite sum (since there are only finitely many partitions λ satisfying $\operatorname{wt} \lambda = n$). Consequently, $h_n = \sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \lambda = n}} m_\lambda$ is a sum of finitely many m_λ, and hence is an element of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ (because each m_λ is an element of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$). Besides, the power series h_n is n-homogeneous [9]. Again, this h_n is a polynomial if I is a finite set, but in the case of general I, this h_n is solely a symmetric function.

It is easy to see that every $n \in \mathbb{N}$ satisfies

$$h_n = \sum_{\substack{\alpha \in \mathbb{N}^I_{\text{fin}}; \\ \operatorname{wt} \alpha = n}} \xi^\alpha \tag{4}$$

(since

$$\begin{aligned}
h_n &= \sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \lambda = n}} m_\lambda = \sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \lambda = n}} \sum_{\substack{\alpha \in \mathbb{N}^I_{\text{fin}}; \\ \lambda \sim \alpha}} \xi^\alpha \qquad \text{(by the definition of } m_\lambda\text{)} \\
&= \sum_{\alpha \in \mathbb{N}^I_{\text{fin}}} \sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \lambda = n; \\ \lambda \sim \alpha}} \xi^\alpha = \sum_{\alpha \in \mathbb{N}^I_{\text{fin}}} \sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \alpha = n; \\ \lambda \sim \alpha}} \xi^\alpha \qquad \text{(since } \operatorname{wt} \lambda = \operatorname{wt} \alpha \text{ if } \lambda \sim \alpha\text{)} \\
&= \sum_{\substack{\alpha \in \mathbb{N}^I_{\text{fin}}; \\ \operatorname{wt} \alpha = n}} \underbrace{\sum_{\substack{\lambda \text{ partition;} \\ \lambda \sim \alpha}} \xi^\alpha}_{= \xi^\alpha \text{ (by (2))}} = \sum_{\substack{\alpha \in \mathbb{N}^I_{\text{fin}}; \\ \operatorname{wt} \alpha = n}} \xi^\alpha
\end{aligned}$$

).

Definition 7. For every $n \in \mathbb{N}$, we define a power series $e_n \in \mathbb{Z}[[\xi_i \mid i \in I]]$ by $e_n = m_{\underbrace{(1, 1, ..., 1)}_{n \text{ ones}}}$. Then,

$$e_n = m_{\underbrace{(1, 1, ..., 1)}_{n \text{ ones}}} = \sum_{\substack{\alpha \in \mathbb{N}^I_{\text{fin}}; \\ \underbrace{(1, 1, ..., 1)}_{n \text{ ones}} \sim \alpha}} \xi^\alpha. \tag{5}$$

[9] This is because h_n is a finite sum of n-homogeneous power series (in fact, h_n is the finite sum $\sum_{\substack{\lambda \text{ partition;} \\ \operatorname{wt} \lambda = n}} m_\lambda$, and for every partition λ satisfying $\operatorname{wt} \lambda = n$, the power series m_λ is n-homogeneous (since m_λ is $\operatorname{wt} \lambda$-homogeneous, and $\operatorname{wt} \lambda = n$)).

We notice that $e_n \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ (since $e_n = m_{\underbrace{(1,1,...,1)}_{n \text{ ones}}} \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$, because $m_\lambda \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ for every partition λ), and that the power series e_n is n-homogeneous (in fact, $e_n = m_{\underbrace{(1,1,...,1)}_{n \text{ ones}}}$ is $\operatorname{wt}\underbrace{(1,1,...,1)}_{n \text{ ones}}$-homogeneous, but $\operatorname{wt}\underbrace{(1,1,...,1)}_{n \text{ ones}} = \underbrace{1+1+...+1}_{n \text{ ones}} = n$).

Now, if $\mathcal{P}_n(I)$ denotes the set of all n-element subsets of I, then there exists a bijection

$$R : \mathcal{P}_n(I) \to \left\{ \alpha \in \mathbb{N}_{\text{fin}}^I \mid \underbrace{(1,1,...,1)}_{n \text{ ones}} \sim \alpha \right\},$$

defined by

$$R(D) = \left(\begin{cases} 1, & \text{if } i \in D; \\ 0, & \text{if } i \notin D \end{cases} \right)_{i \in I} \qquad \text{for every } D \in \mathcal{P}_n(I).$$

This bijection satisfies

$$\xi^{R(D)} = \prod_{i \in I} \xi_i^{(R(D))_i} = \prod_{i \in D} \underbrace{\xi_i^{(R(D))_i}}_{\substack{=\xi_i, \text{ since} \\ (R(D))_i = \begin{cases} 1, & \text{if } i \in D; \\ 0, & \text{if } i \notin D \end{cases} = 1 \\ (\text{since } i \in D)}} \cdot \prod_{i \in I \setminus D} \underbrace{\xi_i^{(R(D))_i}}_{\substack{=1, \text{ since} \\ (R(D))_i = \begin{cases} 1, & \text{if } i \in D; \\ 0, & \text{if } i \notin D \end{cases} = 0 \\ (\text{since } i \notin D)}}$$

$$= \prod_{i \in D} \xi_i \cdot \underbrace{\prod_{i \in I \setminus D} 1}_{=1} = \prod_{i \in D} \xi_i$$

for every $D \in \mathcal{P}_n(I)$. Thus, (5) becomes

$$e_n = \sum_{\substack{\alpha \in \mathbb{N}_{\text{fin}}^I; \\ \underbrace{(1,1,...,1)}_{n \text{ ones}} \sim \alpha}} \xi^\alpha = \sum_{D \in \mathcal{P}_n(I)} \underbrace{\xi^{R(D)}}_{= \prod_{i \in D} \xi_i} \qquad \left(\begin{array}{c} \text{here, we substituted } R(D) \text{ for } \alpha, \\ \text{since } R \text{ is a bijection} \end{array} \right)$$

$$= \sum_{D \in \mathcal{P}_n(I)} \prod_{i \in D} \xi_i. \tag{6}$$

In other words, e_n is the sum of all possible products of n pairwise different variables among the ξ_i (with each such product being taken only once).

Now, in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$, we have

$$\prod_{i \in I} (1 - \xi_i T) = \sum_{d=0}^{\infty} (-1)^d e_d T^d, \tag{7}$$

since

$$\prod_{i\in I}\underbrace{(1-\xi_i T)}_{=\sum_{j=0}^{1}(-\xi_i T)^j}=\prod_{i\in I}\sum_{j=0}^{1}(-\xi_i T)^j=\sum_{\alpha\in\{0,1\}^I_{\text{fin}}}\underbrace{\prod_{i\in I}(-\xi_i T)^{\alpha_i}}_{=\prod_{\substack{i\in I;\\ \alpha_i=0}}(-\xi_i T)^{\alpha_i}\cdot\prod_{\substack{i\in I;\\ \alpha_i=1}}(-\xi_i T)^{\alpha_i}}\qquad\text{(by the product rule)}$$

$$=\sum_{\alpha\in\{0,1\}^I_{\text{fin}}}\prod_{\substack{i\in I;\\ \alpha_i=0}}\underbrace{(-\xi_i T)^{\alpha_i}}_{\substack{=1,\text{ since}\\ \alpha_i=0}}\cdot\prod_{\substack{i\in I;\\ \alpha_i=1}}\underbrace{(-\xi_i T)^{\alpha_i}}_{\substack{=-\xi_i T,\\ \text{since }\alpha_i=1}}$$

$$=\sum_{\alpha\in\{0,1\}^I_{\text{fin}}}\underbrace{\prod_{\substack{i\in I;\\ \alpha_i=1}}(-\xi_i T)}_{=(-T)^{|\{i\in I|\alpha_i=1\}|}\prod_{\substack{i\in I;\\ \alpha_i=1}}\xi_i}=\sum_{\alpha\in\{0,1\}^I_{\text{fin}}}(-T)^{|\{i\in I|\alpha_i=1\}|}\prod_{\substack{i\in I;\\ \alpha_i=1}}\xi_i$$

$$=\sum_{d=0}^{\infty}\sum_{\substack{\alpha\in\{0,1\}^I_{\text{fin}};\\ |\{i\in I|\alpha_i=1\}|=d}}(-T)^d\prod_{\substack{i\in I;\\ \alpha_i=1}}\xi_i=\sum_{d=0}^{\infty}(-T)^d\sum_{\substack{\alpha\in\{0,1\}^I_{\text{fin}};\\ |\{i\in I|\alpha_i=1\}|=d}}\prod_{\substack{i\in I;\\ \alpha_i=1}}\xi_i=\sum_{d=0}^{\infty}(-T)^d\underbrace{\sum_{\substack{\alpha\in\mathbb{N}^I_{\text{fin}};\\ \underbrace{(1,1,...,1)}_{d\text{ ones}}\sim\alpha}}\prod_{\substack{i\in I;\\ \alpha_i=1}}\xi_i}_{=e_d}$$

$$\left(\text{since }\left\{\alpha\in\{0,1\}^I_{\text{fin}}\ \mid\ |\{i\in I\mid\alpha_i=1\}|=d\right\}=\left\{\alpha\in\mathbb{N}^I_{\text{fin}}\ \mid\ \underbrace{(1,1,...,1)}_{d\text{ ones}}\sim\alpha\right\}\right)$$

$$=\sum_{d=0}^{\infty}(-T)^d e_d=\sum_{d=0}^{\infty}(-1)^d e_d T^d.$$

Now we will prove a very easy identity - (9.37) in [1]:

Theorem 1 (the Wronski relations). (a) In the ring $(\mathbb{Z}[\xi_i \mid i\in I]_\infty)[[T]]$ of formal power series, we have

$$\prod_{i\in I}\frac{1}{1-\xi_i T}=\sum_{d=0}^{\infty}h_d T^d. \qquad (8)$$

(b) In the ring $\mathbb{Z}[\xi_i \mid i\in I]_\infty$, we have

$$\sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=n}}(-1)^i h_i e_j=\begin{cases}0,\text{ if }n\geq 1;\\ 1,\text{ if }n=0\end{cases} \qquad (9)$$

for every $n\in\mathbb{N}$.

Proof of Theorem 1. **(a)** We have

$$\prod_{i\in I}\frac{1}{1-\xi_i T}=\prod_{i\in I}\sum_{j=0}^{\infty}(\xi_i T)^j \qquad \left(\text{since } \frac{1}{1-\xi_i T}=\sum_{j=0}^{\infty}(\xi_i T)^j\right)$$
$$=\sum_{\alpha\in\mathbb{N}^I_{\text{fin}}}\prod_{i\in I}\underbrace{(\xi_i T)^{\alpha_i}}_{=\xi_i^{\alpha_i}T^{\alpha_i}} \qquad \text{(by the product rule)}$$
$$=\sum_{\alpha\in\mathbb{N}^I_{\text{fin}}}\prod_{i\in I}\left(\xi_i^{\alpha_i}T^{\alpha_i}\right)=\sum_{\alpha\in\mathbb{N}^I_{\text{fin}}}\underbrace{\left(\prod_{i\in I}\xi_i^{\alpha_i}\right)}_{=\xi^\alpha}\underbrace{\left(\prod_{i\in I}T^{\alpha_i}\right)}_{=T^{\sum_{i\in I}\alpha_i}=T^{\operatorname{wt}\alpha}}$$
$$=\sum_{\alpha\in\mathbb{N}^I_{\text{fin}}}\xi^\alpha\cdot T^{\operatorname{wt}\alpha}=\sum_{d=0}^{\infty}\underbrace{\sum_{\substack{\alpha\in\mathbb{N}^I_{\text{fin}};\\ \operatorname{wt}\alpha=d}}\xi^\alpha}_{=h_d \text{ (by (4))}}\cdot T^d=\sum_{d=0}^{\infty}h_d T^d,$$

and (8) is proven.

(b) In the ring $(\mathbb{Z}[\xi_i \mid i\in I]_\infty)[[T]]$, we have

$$1=\prod_{i\in I}\frac{1}{1-\xi_i T}\cdot\prod_{i\in I}(1-\xi_i T)=\left(\sum_{d=0}^{\infty}h_d T^d\right)\cdot\left(\sum_{d=0}^{\infty}(-1)^d e_d T^d\right)$$
$$\text{(by (8) and (7))}$$
$$=\sum_{d=0}^{\infty}\sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=d}}h_i\underbrace{(-1)^j}_{\substack{=(-1)^{2i+j}=(-1)^i(-1)^{i+j}\\ =(-1)^i(-1)^d,\\ \text{since } i+j=d}}e_j T^d=\sum_{d=0}^{\infty}(-1)^d\sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=d}}(-1)^i h_i e_j T^d.$$

Comparing the coefficients before T^n of the power series on the left and on the right hand side of this equation, we obtain

$$\begin{cases}0, & \text{if } n\geq 1;\\ 1, & \text{if } n=0\end{cases}=(-1)^n\sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=n}}(-1)^i h_i e_j.$$

Thus,

$$\sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=n}}(-1)^i h_i e_j=(-1)^n\cdot\begin{cases}0, & \text{if } n\geq 1;\\ 1, & \text{if } n=0\end{cases}=\begin{cases}0, & \text{if } n\geq 1;\\ 1, & \text{if } n=0\end{cases},$$

and therefore, (9) is proven. This completes the proof of Theorem 1.

The next formula that we want to prove is (9.44) in [1]. First, we need two more definitions:

Definition 8. Let λ be a partition. Then, we define a power series $h_\lambda\in\mathbb{Z}[\xi_i \mid i\in I]_\infty$ by

$$h_\lambda=\prod_{n=1}^{\infty}h_n^{m_n(\lambda)}.$$

(This is actually a finite product, since only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $h_n^{m_n(\lambda)} \neq 1$, because only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $m_n(\lambda) \neq 0$.) This power series h_λ can be written in a simpler way if we write our partition λ in the form $(\lambda_1, \lambda_2, ..., \lambda_m)$ for some $m \in \mathbb{N}$; namely,

$$\text{if } \lambda = (\lambda_1, \lambda_2, ..., \lambda_m), \text{ then } h_\lambda = h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_m} \tag{10}$$

(since if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$, then

$$h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_m} = \prod_{i \in \{1,2,...,m\}} h_{\lambda_i} = \prod_{n=0}^{\infty} \prod_{\substack{i \in \{1,2,...,m\}; \\ \lambda_i = n}} h_n = \left(\prod_{\substack{i \in \{1,2,...,m\}; \\ \lambda_i = 0}} \underbrace{h_0}_{=1} \right) \cdot \left(\prod_{n=1}^{\infty} \prod_{\substack{i \in \{1,2,...,m\}; \\ \lambda_i = n}} h_n \right)$$
$$= \prod_{n=1}^{\infty} \underbrace{\prod_{\substack{i \in \{1,2,...,m\}; \\ \lambda_i = n}} h_n}_{\substack{= h_n^{\left|\{ i \in \{1,2,...,m\} \mid \lambda_i = n \}\right|} \\ = h_n^{m_n(\lambda)}}} = \prod_{n=1}^{\infty} h_n^{m_n(\lambda)} = h_\lambda$$

). Hence, our definition of h_λ agrees with the definition of h_λ given by Hazewinkel in [1], (9.36).

Note that the power series h_λ is $\operatorname{wt} \lambda$-homogeneous.[10]

Similarly to how we defined h_λ using the already-defined symmetric functions h_n, we can define e_λ using the e_n. Namely, for every partition λ, we define a power series $e_\lambda \in \mathbb{Z} [\xi_i \mid i \in I]_\infty$ by

$$e_\lambda = \prod_{n=1}^{\infty} e_n^{m_n(\lambda)}.$$

Again, this is actually a finite product. We can prove that

$$\text{if } \lambda = (\lambda_1, \lambda_2, ..., \lambda_m), \text{ then } e_\lambda = e_{\lambda_1} e_{\lambda_2} ... e_{\lambda_m} \tag{11}$$

(this is proven in exactly the same way as (10)). Hence, our definition of e_λ agrees with the definition of e_λ given by Hazewinkel in [1], (9.36). The power series e_λ is $\operatorname{wt} \lambda$-homogeneous[11].

We notice that for every $\alpha \in \mathbb{N}_{\text{fin}}^I$, we have

$$\sum_{\substack{\lambda \in \text{Par}; \\ \lambda \sim \alpha}} h_\lambda = \prod_{i \in I} h_{\alpha_i} \qquad \text{and} \tag{12}$$

$$\sum_{\substack{\lambda \in \text{Par}; \\ \lambda \sim \alpha}} e_\lambda = \prod_{i \in I} e_{\alpha_i}. \tag{13}$$

[10] In fact, if we write our partition λ in the form $(\lambda_1, \lambda_2, ..., \lambda_m)$, then (10) yields $h_\lambda = h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_m}$. Since the power series h_{λ_i} is λ_i-homogeneous for every $i \in \{1, 2, ..., m\}$, the product $h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_m}$ must be a $(\lambda_1 + \lambda_2 + ... + \lambda_m)$-homogeneous power series. But $h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_m} = h_\lambda$ and $\lambda_1 + \lambda_2 + ... + \lambda_m = \operatorname{wt} \lambda$. Thus, h_λ is a $\operatorname{wt} \lambda$-homogeneous power series, qed.

[11] This is proven in the same way as we showed that h_λ is $\operatorname{wt} \lambda$-homogeneous.

Proof. Fix some $\alpha \in \mathbb{N}_{\text{fin}}^I$. Let λ be a partition satisfying $\lambda \sim \alpha$. Let us write our partition λ in the form $(\lambda_1, \lambda_2, \lambda_3, ...)$. Since λ is a partition, there exists some $\nu \in \mathbb{N}$ such that $\lambda_\nu = 0$, and thus $\lambda_\nu = \lambda_{\nu+1} = \lambda_{\nu+2} = ... = 0$ (since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ...$), so that $\lambda_n = 0$ for every integer $n \geq \nu$. But $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...) = (\lambda_1, \lambda_2, ..., \lambda_{\nu-1})$ (since $\lambda_n = 0$ for every integer $n \geq \nu$). Thus, (10) yields $h_\lambda = h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_{\nu-1}}$. On the other hand, $\lambda \sim \alpha$ yields that there exist:

- a subset I' of I such that $\alpha_i = 0$ for every $i \in I \setminus I'$,
- a subset N of $\{1, 2, 3, ...\}$ such that $\lambda_n = 0$ for every $n \in \{1, 2, 3, ...\} \setminus N$,
- and a bijection $\Phi : N \to I'$ such that $\alpha_{\Phi(n)} = \lambda_n$ for every $n \in N$.

Consider this I', this N and this Φ. Since $I' \subseteq I$, we have

$$\prod_{i \in I} h_{\alpha_i} = \left(\prod_{i \in I'} h_{\alpha_i} \right) \cdot \left(\prod_{i \in I \setminus I'} \underbrace{h_{\alpha_i}}_{\substack{=1 \\ \text{(since } i \in I \setminus I' \text{, so that} \\ \alpha_i = 0 \text{, thus } h_{\alpha_i} = h_0 = 1)}} \right) = \prod_{i \in I'} h_{\alpha_i} = \prod_{n \in N} h_{\alpha_{\Phi(n)}}$$
$$\left(\begin{array}{c} \text{here, we substituted } \Phi(n) \text{ for } i \text{ in the product, since} \\ \Phi : N \to I' \text{ is a bijection} \end{array} \right)$$
$$= \prod_{n \in N} h_{\lambda_n} \qquad \left(\text{since } a_{\Phi(n)} = \lambda_n \text{ for every } n \in N \right).$$

Since $N \subseteq \{1, 2, 3, ...\}$, we have

$$\prod_{n \in \{1,2,3,...\}} h_{\lambda_n} = \left(\prod_{n \in N} h_{\lambda_n} \right) \cdot \left(\prod_{n \in \{1,2,3,...\} \setminus N} \underbrace{h_{\lambda_n}}_{\substack{=1 \\ \text{(since } n \in \{1,2,3,...\} \setminus N \text{, thus} \\ \lambda_n = 0 \text{ and hence } h_{\lambda_n} = h_0 = 1)}} \right) = \prod_{n \in N} h_{\lambda_n} = \prod_{i \in I} h_{\alpha_i},$$

so that

$$\prod_{i \in I} h_{\alpha_i} = \prod_{n \in \{1,2,3,...\}} h_{\lambda_n} = \prod_{n=1}^{\infty} h_{\lambda_n}$$
$$= h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_{\nu-1}} \prod_{n=\nu}^{\infty} \underbrace{h_{\lambda_n}}_{\substack{=1 \\ \text{(since } \lambda_n = 0 \text{ for } n \geq \nu)}} = h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_{\nu-1}} = h_\lambda.$$

Now forget that we fixed λ. We thus have shown that $\prod_{i \in I} h_{\alpha_i} = h_\lambda$ for every partition λ satisfying $\lambda \sim \alpha$. Thus,

$$\sum_{\substack{\lambda \in \text{Par}; \\ \lambda \sim \alpha}} \prod_{i \in I} h_{\alpha_i} = \sum_{\substack{\lambda \in \text{Par}; \\ \lambda \sim \alpha}} h_\lambda. \tag{14}$$

But for any fixed $\alpha \in \mathbb{N}_{\text{fin}}^I$, there exists one and only one partition λ satisfying $\lambda \sim \alpha$ (by (2)), and therefore we have $\sum\limits_{\substack{\lambda \in \text{Par};\\ \lambda \sim \alpha}} \prod\limits_{i \in I} h_{\alpha_i} = \prod\limits_{i \in I} h_{\alpha_i}$. Hence, (14) rewrites as

$$\prod_{i \in I} h_{\alpha_i} = \sum_{\substack{\lambda \in \text{Par};\\ \lambda \sim \alpha}} h_\lambda,$$

and this proves (12). The proof for (13) is exactly similar (we just have to replace h by e). This completes the proofs of (12) and (13).

Before we proceed further, we must introduce a simple notation relating to power series. In fact, we will often want to apply one and the same power series to different sets of variables. Here is our notation for that:

Definition 9. For every partition $\lambda \in \text{Par}$, we denote by $m_\lambda(\xi)$ the element m_λ of the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, and by $m_\lambda(\eta)$ the "corresponding" element of the ring $\mathbb{Z}[\eta_j \mid j \in J]_\infty$ (that is, the power series we would obtain if we would replace the set I by the set J and the indeterminates ξ_i by the indeterminates η_j in the definition of m_λ). [12] Similarly, we denote by $h_\lambda(\xi)$ the element h_λ of the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, and by $h_\lambda(\eta)$ the "corresponding" element of the ring $\mathbb{Z}[\eta_j \mid j \in J]_\infty$ (that is, the power series we would obtain if we would replace the set I by the set J and the indeterminates ξ_i by the indeterminates η_j in the definition of h_λ).

Also, for every $n \in \mathbb{N}$, we denote by $h_n(\xi)$ the element h_n of the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, and by $h_n(\eta)$ the "corresponding" element of the ring $\mathbb{Z}[\eta_j \mid j \in J]_\infty$ (that is, the power series we would obtain if we would replace the set I by the set J and the indeterminates ξ_i by the indeterminates η_j in the definition of h_n).

Now, we are approaching a proof of formula (9.44) in [1]. First, we need one remark about power series:

Let A be a commutative ring with unity. Assume that for every partition λ, we have given some element α_λ of A. Then, in the ring $A[[T]]$ of power series in the indeterminate T over A,

$$\text{the infinite sum } \sum_{\lambda \in \text{Par}} \alpha_\lambda T^{\text{wt}\,\lambda} \text{ is convergent} \tag{15}$$

(with respect to the (T)-adic topology on the ring $A[[T]]$). This is because this infinite

[12] Explicitly, this means that

$$m_\lambda(\xi) = m_\lambda = \sum_{\substack{\alpha \in \mathbb{N}_{\text{fin}}^I;\\ \lambda \sim \alpha}} \xi^\alpha, \qquad \text{while} \qquad m_\lambda(\eta) = \sum_{\substack{\beta \in \mathbb{N}_{\text{fin}}^J;\\ \lambda \sim \beta}} \eta^\beta,$$

where η^β stands for $\prod\limits_{j \in J} \eta_j^{\beta_j}$ (just as ξ^α stands for $\prod\limits_{i \in I} \xi_i^{\alpha_i}$).

sum $\sum\limits_{\lambda \in \operatorname{Par}} \alpha_\lambda T^{\operatorname{wt} \lambda}$ rewrites as

$$\sum_{\lambda \in \operatorname{Par}} \alpha_\lambda T^{\operatorname{wt} \lambda} = \sum_{n=0}^{\infty} \underbrace{\sum_{\substack{\lambda \in \operatorname{Par}; \\ \operatorname{wt} \lambda = n}} \alpha_\lambda}_{\substack{\text{this is a finite sum of elements} \\ \text{of } A \text{, since there are only finitely} \\ \text{many partitions } \lambda \text{ such that } \operatorname{wt} \lambda = n}} T^n .$$

Now, we present the formula (9.44) from [1] in a slightly generalized form[13]:

Theorem 2. Let I and J be two countable sets. In the ring $\left(\left(\mathbb{Z} \left[\xi_i \mid i \in I \right]_\infty \right) \left[\eta_j \mid j \in J \right]_\infty \right) [[T]]$, we have

$$\sum_{\lambda \in \operatorname{Par}} h_\lambda (\xi) \, m_\lambda (\eta) \, T^{\operatorname{wt} \lambda} = \prod_{(i,j) \in I \times J} \frac{1}{1 - \xi_i \eta_j T} .$$

[14]

Proof of Theorem 2. For every $\lambda \in \operatorname{Par}$, the power series $m_\lambda (\eta) \in \mathbb{Z} \left[\eta_j \mid j \in J \right]_\infty$ is defined as the power series we would obtain if we would replace the set I by the set J and the indeterminates ξ_i by the indeterminates η_j in the definition of m_λ. But the definition of m_λ is

$$m_\lambda = \sum_{\substack{\alpha \in \mathbb{N}^I_{\operatorname{fin}}; \\ \lambda \sim \alpha}} \xi^\alpha ,$$

and thus, replacing I by J and ξ_i by η_j in this definition, we obtain

$$m_\lambda (\eta) = \sum_{\substack{\alpha \in \mathbb{N}^J_{\operatorname{fin}}; \\ \lambda \sim \alpha}} \eta^\alpha ,$$

where the polynomial $\eta^\alpha \in \mathbb{Z} \left[\eta_j \mid j \in J \right]$ is defined by $\eta^\alpha = \prod\limits_{j \in J} \eta_j^{\alpha_j}$. Hence,

$$\sum_{\lambda \in \operatorname{Par}} h_\lambda (\xi) \, m_\lambda (\eta) \, T^{\operatorname{wt} \lambda} = \sum_{\lambda \in \operatorname{Par}} h_\lambda (\xi) \sum_{\substack{\alpha \in \mathbb{N}^J_{\operatorname{fin}}; \\ \lambda \sim \alpha}} \eta^\alpha \underbrace{T^{\operatorname{wt} \lambda}}_{\substack{= T^{\operatorname{wt} \alpha}, \\ \text{since } \lambda \sim \alpha \\ \text{yields } \operatorname{wt} \lambda = \operatorname{wt} \alpha}} = \sum_{\lambda \in \operatorname{Par}} h_\lambda (\xi) \sum_{\substack{\alpha \in \mathbb{N}^J_{\operatorname{fin}}; \\ \lambda \sim \alpha}} \eta^\alpha T^{\operatorname{wt} \alpha}$$

$$= \sum_{\alpha \in \mathbb{N}^J_{\operatorname{fin}}} \eta^\alpha T^{\operatorname{wt} \alpha} \sum_{\substack{\lambda \in \operatorname{Par}; \\ \lambda \sim \alpha}} h_\lambda (\xi) . \qquad (16)$$

[13] Actually, our Theorem 2 is slightly more general than formula (9.44) in [1], since formula (9.44) in [1] follows from our Theorem 2 by setting T equal to 1. However, in turn, our Theorem 2 follows from formula in [1] by replacing η_j by $T\eta_j$, so we do not win much generality by introducing the variable T. The main reason for the introduction of the variable T in Theorem 2 is to make the convergence of the sum $\sum\limits_{\lambda \in \operatorname{Par}} h_\lambda (\xi) \, m_\lambda (\eta) \, T^{\operatorname{wt} \lambda}$ more obvious.

[14] The sum $\sum\limits_{\lambda \in \operatorname{Par}} h_\lambda (\xi) \, m_\lambda (\eta) \, T^{\operatorname{wt} \lambda}$ is convergent according to (15).

But $\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ \lambda\sim\alpha}} h_\lambda = \prod\limits_{j\in J} h_{\alpha_j}$ for every $\alpha \in \mathbb{N}^J_{\operatorname{fin}}$ (this is simply the equation (12), with I replaced by J). In other words, $\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ \lambda\sim\alpha}} h_\lambda\left(\xi\right) = \prod\limits_{j\in J} h_{\alpha_j}\left(\xi\right)$ (since $h_\lambda = h_\lambda\left(\xi\right)$ and $h_{\alpha_j} = h_{\alpha_j}\left(\xi\right)$). Hence, (16) becomes

$$\sum_{\lambda\in\operatorname{Par}} h_\lambda\left(\xi\right) m_\lambda\left(\eta\right) T^{\operatorname{wt}\lambda} = \sum_{\alpha\in\mathbb{N}^J_{\operatorname{fin}}} \underbrace{\eta^\alpha}_{=\prod\limits_{j\in J}\eta_j^{\alpha_j}} \underbrace{T^{\operatorname{wt}\alpha}}_{=T^{\sum\limits_{j\in J}\alpha_j}=\prod\limits_{j\in J}T^{\alpha_j}} \prod_{j\in J} h_{\alpha_j}\left(\xi\right) = \sum_{\alpha\in\mathbb{N}^J_{\operatorname{fin}}} \prod_{j\in J}\eta_j^{\alpha_j}\prod_{j\in J}T^{\alpha_j}\prod_{j\in J}h_{\alpha_j}\left(\xi\right)$$

$$= \sum_{\alpha\in\mathbb{N}^J_{\operatorname{fin}}}\prod_{j\in J}\eta_j^{\alpha_j}T^{\alpha_j}h_{\alpha_j}\left(\xi\right) = \prod_{j\in J}\left(\sum_{a\in\mathbb{N}}\eta_j^a T^a h_a\left(\xi\right)\right) \qquad (17)$$

(by the product rule). But for every $j\in J$, we have

$$\sum_{a\in\mathbb{N}}\eta_j^a T^a h_a\left(\xi\right) = \sum_{a\in\mathbb{N}}\underbrace{h_a\left(\xi\right)}_{=h_a}\underbrace{\eta_j^a T^a}_{=\left(\eta_j T\right)^a} = \sum_{a\in\mathbb{N}} h_a\left(\eta_j T\right)^a = \sum_{d\in\mathbb{N}} h_d\left(\eta_j T\right)^d$$

$$= \sum_{d=0}^\infty h_d\left(\eta_j T\right)^d = \prod_{i\in I}\frac{1}{1-\xi_i\eta_j T}$$

(since $\prod\limits_{i\in I}\frac{1}{1-\xi_i\eta_j T} = \sum\limits_{d=0}^\infty h_d\left(\eta_j T\right)^d$, which follows from substituting $\eta_j T$ for T in (8)), and thus (17) becomes

$$\sum_{\lambda\in\operatorname{Par}} h_\lambda\left(\xi\right) m_\lambda\left(\eta\right) T^{\operatorname{wt}\lambda} = \prod_{j\in J}\prod_{i\in I}\frac{1}{1-\xi_i\eta_j T} = \prod_{(i,j)\in I\times J}\frac{1}{1-\xi_i\eta_j T}.$$

Thus, Theorem 2 is proven.

We will prove some more identities later, but first we recall the definition and basic properties of the "power sum" symmetric functions p_n:

Definition 10. For every $n\in\mathbb{N}$, we define a power series $p_n \in \mathbb{Z}\left[\left[\xi_i \mid i\in I\right]\right]$ by $p_n = m_{(n)}$.

We notice that $p_n \in \mathbb{Z}\left[\xi_i \mid i\in I\right]_\infty$ (since $p_n = m_{(n)} \in \mathbb{Z}\left[\xi_i \mid i\in I\right]_\infty$, because $m_\lambda \in \mathbb{Z}\left[\xi_i \mid i\in I\right]_\infty$ for every partition λ), and that the power series p_n is n-homogeneous (in fact, $p_n = m_{(n)}$ is $\operatorname{wt}(n)$-homogeneous, but $\operatorname{wt}(n) = n$).

It is easy to see that

$$p_n = \sum_{i\in I}\xi_i^n \qquad \text{for every } n\in\{1,2,3,...\} \qquad (18)$$

(but not for $n=0$, unless $|I|=1$ [15]).

[15] This is because $p_0 = m_{(0)} = 1$, whereas $\sum\limits_{i\in I}\xi_i^0$ is undefined for infinite sets I (and distinct from 1 even when I is finite, unless $|I|=1$). This is a reason why most authors prefer not to define p_0 at all. However, we define p_0 to be 1 here, since this makes Definition 11 a little bit simpler. But let us remember that (18) does not hold for $n=0$, and that our convention $p_0 = 1$ is *not* compatible with the convention that Hazewinkel uses in [1] (in fact, Hazewinkel sets p_0 to be 0 in [1], (9.58)).

Proof. Fix $n \in \{1, 2, 3, ...\}$. For every $j \in I$, we define a family $e_j^n \in \mathbb{N}_{\text{fin}}^I$ by $e_j^n = \left(\begin{cases} n, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases} \right)_{i \in I}$. In other words, we let e_j^n be the family whose j-th component is n and whose other components are all 0. It is clear that these families e_j^n for different j (but fixed $n \in \{1, 2, 3, ...\}$) are all pairwise different, and that these families e_j^n are the only families $\alpha \in \mathbb{N}_{\text{fin}}^I$ satisfying $(n) \sim \alpha$. Hence,

$$\sum_{\substack{\alpha \in \mathbb{N}_{\text{fin}}^I; \\ (n) \sim \alpha}} \xi^\alpha = \sum_{j \in I} \xi^{e_j^n} = \sum_{j \in I} \xi_j^n \qquad \left(\text{since } \xi^{e_j^n} = \prod_{i \in I} \xi_i^{\left(e_j^n\right)_i} = \prod_{i \in I} \xi_i^{\begin{cases} n, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}} = \xi_j^n \right).$$

Thus,

$$p_n = m_{(n)} = \sum_{\substack{\alpha \in \mathbb{N}_{\text{fin}}^I; \\ (n) \sim \alpha}} \xi^\alpha = \sum_{j \in I} \xi_j^n = \sum_{i \in I} \xi_i^n,$$

and consequently, (18) is proven.

Let us now verify the so-called Newton relations (formulae (9.59) and (9.57) in [1]):

Theorem 3 (the Newton relations). (a) In the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$ of formal power series, we have[16]

$$\sum_{n=1}^{\infty} p_n T^n = T \frac{d}{dT} \log(H(T)) = \frac{T H'(T)}{H(T)}, \tag{19}$$

where the power series $H(T) \in (\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$ is defined by

$$H(T) = \prod_{i \in I} \frac{1}{1 - \xi_i T} = \sum_{d=0}^{\infty} h_d T^d$$

(where we are using $\prod_{i \in I} \frac{1}{1 - \xi_i T} = \sum_{d=0}^{\infty} h_d T^d$, which holds because of Theorem 1 **(a)**).

(b) In the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, we have

$$n h_n = \sum_{i=0}^{n-1} h_i p_{n-i} \tag{20}$$

for every $n \in \mathbb{N}$.

[16]It should be remarked that the logarithmic derivative $\frac{d}{dT} \log(H(T))$ is well-defined in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$ even though the logarithm $\log(H(T))$ itself is not defined in this ring. In general, if A is a commutative ring with unity, and $f \in A[[T]]$ is a formal power series with constant term 1, then the logarithmic derivative $\frac{d}{dT} \log f$ of f is defined as the formal power series $\frac{f'}{f}$, no matter whether the logarithm $\log f$ is well-defined in $A[[T]]$ or not.

Proof of Theorem 3. Let us work in the ring $(\mathbb{Q}[[\xi_i \mid i \in I]])[[T]]$. (In this ring, logarithms like $\log(H(T))$ are well-defined, and not just logarithmic derivatives like $\frac{d}{dT}\log(H(T))$.)

Since $H(T) = \prod_{i\in I} \frac{1}{1-\xi_i T}$, we have

$$T\frac{d}{dT}\log(H(T)) = T\frac{d}{dT}\underbrace{\log\left(\prod_{i\in I}\frac{1}{1-\xi_i T}\right)}_{=\sum_{i\in I}\log\frac{1}{1-\xi_i T}} = T\frac{d}{dT}\sum_{i\in I}\underbrace{\log\frac{1}{1-\xi_i T}}_{=-\log(1-\xi_i T)}$$

$$= T\frac{d}{dT}\sum_{i\in I}\underbrace{(-\log(1-\xi_i T))}_{\substack{=\sum_{n=1}^{\infty}\frac{1}{n}(\xi_i T)^n \text{ due to the}\\ \text{power series } -\log(1-X)=\sum_{n=1}^{\infty}\frac{1}{n}X^n}} = T\frac{d}{dT}\sum_{i\in I}\sum_{n=1}^{\infty}\frac{1}{n}(\xi_i T)^n$$

$$= T\sum_{i\in I}\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\frac{d}{dT}(\xi_i T)^n}_{=n\xi_i^n T^{n-1}} = T\sum_{i\in I}\sum_{n=1}^{\infty}\xi_i^n T^{n-1} = \sum_{i\in I}\sum_{n=1}^{\infty}\xi_i^n T^n$$

$$= \sum_{n=1}^{\infty}\underbrace{\sum_{i\in I}\xi_i^n}_{=p_n \text{ by } (18)} T^n = \sum_{n=1}^{\infty} p_n T^n.$$

Besides, $T\frac{d}{dT}\log(H(T)) = \frac{TH'(T)}{H(T)}$, since the well-known formula for the logarithmic derivative yields $\frac{d}{dT}\log(H(T)) = \frac{H'(T)}{H(T)}$. Thus, Theorem 3 **(a)** is proven.

(b) We have

$$H'(T) = \frac{d}{dT}H(T) = \frac{d}{dT}\sum_{d=0}^{\infty}h_d T^d \qquad \left(\text{since } H(T) = \sum_{d=0}^{\infty}h_d T^d\right)$$
$$= \sum_{d=0}^{\infty}h_d d T^{d-1}$$

(where dT^{d-1} is considered to be 0 for $d = 0$) and thus

$$TH'(T) = T\sum_{d=0}^{\infty}h_d d T^{d-1} = \sum_{d=0}^{\infty}h_d d T^d = \sum_{n=0}^{\infty}h_n n T^n. \tag{21}$$

Now, (19) yields $\sum_{n=1}^{\infty}p_n T^n = \frac{TH'(T)}{H(T)}$, so that

$$TH'(T) = \underbrace{H(T)}_{=\sum_{d=0}^{\infty}h_d T^d}\cdot\underbrace{\sum_{n=1}^{\infty}p_n T^n}_{=\sum_{u=1}^{\infty}p_u T^u} = \sum_{d=0}^{\infty}h_d T^d\cdot\sum_{u=1}^{\infty}p_u T^u = \sum_{n=0}^{\infty}\sum_{i=0}^{n-1}h_i p_{n-i}T^n$$

(by the definition of the product of two power series). Comparing this with (21), we see that

$$\sum_{n=0}^{\infty} h_n n T^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} h_i p_{n-i} T^n.$$

Thus, every $n \in \mathbb{N}$ satisfies

$$h_n n = \sum_{i=0}^{n-1} h_i p_{n-i}.$$

This proves Theorem 3 **(b)**.

The Wronski relations (Theorem 1) relate the sequences $(h_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$, and the Newton relations (Theorem 3) relate the sequences $(h_n)_{n\in\mathbb{N}}$ and $(p_n)_{p\in\{1,2,3,\ldots\}}$. Now we shall prove the so-called summed Viete relations, which relate the sequences $(e_n)_{n\in\mathbb{N}}$ and $(p_n)_{p\in\{1,2,3,\ldots\}}$, thus completing the circle.

Theorem 4 (the summed Viete relations). (a) In the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$ of formal power series, we have[17]

$$\sum_{n=1}^{\infty} p_n T^n = -T \frac{d}{dT} \log(E(T)) = -\frac{TE'(T)}{E(T)}, \tag{22}$$

where the power series $E(T) \in (\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$ is defined by

$$E(T) = \prod_{i\in I} (1 - \xi_i T) = \sum_{d=0}^{\infty} (-1)^d e_d T^d$$

(where we are using $\prod_{i\in I} (1 - \xi_i T) = \sum_{d=0}^{\infty} (-1)^d e_d T^d$, which holds because of (7)).

(b) In the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, we have

$$n e_n = \sum_{i=0}^{n-1} (-1)^{n-i+1} e_i p_{n-i}. \tag{23}$$

for every $n \in \mathbb{N}$.

Proof of Theorem 4. Let us work in the ring $(\mathbb{Q}[[\xi_i \mid i \in I]])[[T]]$. (In this ring, logarithms like $\log(E(T))$ are well-defined, and not just logarithmic derivatives like $\frac{d}{dT}\log(E(T))$.)

[17] It should be remarked that the logarithmic derivative $\frac{d}{dT}\log(E(T))$ is well-defined in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$ even though the logarithm $\log(E(T))$ itself is not defined in this ring. In general, if A is a commutative ring with unity, and $f \in A[[T]]$ is a formal power series with constant term 1, then the logarithmic derivative $\frac{d}{dT}\log f$ of f is defined as the formal power series $\frac{f'}{f}$, no matter whether the logarithm $\log f$ is well-defined in $A[[T]]$ or not.

(a) We have

$$E(T) = \prod_{i\in I}(1-\xi_i T) = \left(\underbrace{\prod_{i\in I}\frac{1}{1-\xi_i T}}_{=H(T)}\right)^{-1} = (H(T))^{-1},$$

and thus $\log(E(T)) = \log\left((H(T))^{-1}\right) = -\log(H(T))$. Consequently,

$$-T\frac{d}{dT}\log(E(T)) = -T\frac{d}{dT}(-\log(H(T))) = T\frac{d}{dT}\log(H(T)) = \sum_{n=1}^{\infty} p_n T^n$$

(by (19)). Besides, $-T\frac{d}{dT}\log(E(T)) = -\frac{TE'(T)}{E(T)}$, since the well-known formula for the logarithmic derivative yields $\frac{d}{dT}\log(E(T)) = \frac{E'(T)}{E(T)}$. Thus, Theorem 4 **(a)** is proven.

(b) We have

$$\begin{aligned} E'(T) &= \frac{d}{dT}E(T) = \frac{d}{dT}\sum_{d=0}^{\infty}(-1)^d e_d T^d \qquad \left(\text{since } E(T) = \sum_{d=0}^{\infty}(-1)^d e_d T^d\right) \\ &= \sum_{d=0}^{\infty}(-1)^d e_d d T^{d-1} \end{aligned}$$

(where dT^{d-1} is considered to be 0 for $d = 0$) and thus

$$\begin{aligned} -TE'(T) &= -T\sum_{d=0}^{\infty}(-1)^d e_d d T^{d-1} = -\sum_{d=0}^{\infty}(-1)^d e_d d T^d = -\sum_{n=0}^{\infty}(-1)^n e_n n T^n \\ &= \sum_{n=0}^{\infty}(-1)^{n+1} e_n n T^n. \end{aligned} \tag{24}$$

Now, (22) yields $\sum\limits_{n=1}^{\infty} p_n T^n = -\frac{TE'(T)}{E(T)}$, so that

$$-TE'(T) = \underbrace{E(T)}_{=\sum\limits_{d=0}^{\infty}(-1)^d e_d T^d} \cdot \underbrace{\sum_{n=1}^{\infty} p_n T^n}_{=\sum\limits_{u=1}^{\infty} p_u T^u} = \sum_{d=0}^{\infty}(-1)^d e_d T^d \cdot \sum_{u=1}^{\infty} p_u T^u = \sum_{n=0}^{\infty}\sum_{i=0}^{n-1}(-1)^i e_i p_{n-i} T^n$$

(by the definition of the product of two power series). Comparing this with (24), we see that

$$\sum_{n=0}^{\infty}(-1)^{n+1} e_n n T^n = \sum_{n=0}^{\infty}\sum_{i=0}^{n-1}(-1)^i e_i p_{n-i} T^n.$$

Thus, every $n \in \mathbb{N}$ satisfies

$$(-1)^{n+1} e_n n = \sum_{i=0}^{n-1} (-1)^i e_i p_{n-i}.$$

Upon multiplication by $(-1)^{n+1}$, this becomes

$$e_n n = \sum_{i=0}^{n-1} (-1)^{n-i+1} e_i p_{n-i}.$$

This proves Theorem 4 **(b)**.

We need some more definitions now:

Definition 11. Let λ be a partition. Then, we define a power series $p_\lambda \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ by

$$p_\lambda = \prod_{n=1}^{\infty} p_n^{m_n(\lambda)}.$$

(This is actually a finite product, since only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $p_n^{m_n(\lambda)} \neq 1$, because only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $m_n(\lambda) \neq 0$.) This power series p_λ can be written in a simpler way if we write our partition λ in the form $(\lambda_1, \lambda_2, ..., \lambda_m)$ for some $m \in \mathbb{N}$; namely,

$$\text{if } \lambda = (\lambda_1, \lambda_2, ..., \lambda_m), \text{ then } p_\lambda = p_{\lambda_1} p_{\lambda_2} ... p_{\lambda_m} \qquad (25)$$

(we recall that p_0 is supposed to mean 1). This is proven in the same way as we showed (10). Hence, our definition of p_λ agrees with the definition of p_λ given by Hazewinkel in [1], (9.61).

The power series p_λ is wt λ-homogeneous[18].

We notice that for every $\alpha \in \mathbb{N}_{\text{fin}}^I$, we have

$$\sum_{\substack{\lambda \in \text{Par}; \\ \lambda \sim \alpha}} p_\lambda = \prod_{i \in I} p_{\alpha_i} \qquad (26)$$

(again, remembering that p_0 was defined as 1). The proof of this equation is exactly the same as that of (12) (but with h replaced by p throughout the proof).

Definition 12. For every partition $\lambda \in \text{Par}$, we denote by $p_\lambda(\xi)$ the element p_λ of the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, and by $p_\lambda(\eta)$ the "corresponding" element of the ring $\mathbb{Z}[\eta_j \mid j \in J]_\infty$ (that is, the power series we would obtain if we would replace the set I by the set J and the indeterminates ξ_i by the indeterminates η_j in the definition of p_λ).

Besides, for every $n \in \mathbb{N}$, we denote by $p_n(\xi)$ the element p_n of the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, and by $p_n(\eta)$ the "corresponding" element of the ring

[18]This is proven in the same way as we showed that h_λ is wt λ-homogeneous.

$\mathbb{Z}[\eta_j \mid j \in J]_\infty$ (that is, the power series we would obtain if we would replace the set I by the set J and the indeterminates ξ_i by the indeterminates η_j in the definition of p_n).

Definition 13. For every partition $\lambda \in \operatorname{Par}$, we denote by z_λ the nonnegative integer defined by

$$z_\lambda = \prod_{n=1}^{\infty} n^{m_n(\lambda)} (m_n(\lambda))!.$$

This product is actually finite, because only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $n^{m_n(\lambda)} (m_n(\lambda))! \neq 1$ (since $n^{m_n(\lambda)} (m_n(\lambda))! \neq 1$ yields $m_n(\lambda) \neq 0$, and only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $m_n(\lambda) \neq 0$).

We now come to another formula from [1] - with a generalization:

Theorem 5. Let I and J be two countable sets.

(a) In the ring $\left((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]]$, we have

$$\sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} p_\lambda(\xi) p_\lambda(\eta) T^{\operatorname{wt}\lambda} = \prod_{(i,j) \in I \times J} \frac{1}{1 - \xi_i \eta_j T}. \tag{27}$$

[19]

(b) In the ring $\left(\left((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]]\right)[[S]]$, we have

$$\sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi) p_\lambda(\eta) T^{\operatorname{wt}\lambda} = \prod_{(i,j) \in I \times J} \left(\frac{1}{1 - \xi_i \eta_j T}\right)^S \tag{28}$$

[20], where the function $\operatorname{msum} : \operatorname{Par} \to \mathbb{N}$ is defined by

$$\operatorname{msum}\lambda = m_1(\lambda) + m_2(\lambda) + m_3(\lambda) + ... = \sum_{k=1}^{\infty} m_k(\lambda) \qquad \text{for every partition } \lambda.$$

Here, for any power series $P \in \left(\left((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]]\right)[[S]]$ with constant term 1, the power series $P^S \in \left(\left((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]]\right)[[S]]$ is defined by $P^S = \exp(S \log P)$ (where $\log P$ is computed using the $\log(1+X) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} X^k$ formula).

Note that Theorem 5 **(a)**, upon substitution of 1 for T, becomes the formula (9.62) in [1], while Theorem 5 **(b)** is a generalization which doesn't occur in [1].

Proof of Theorem 5. **(b)** We have

$$\begin{aligned} &\sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi) p_\lambda(\eta) T^{\operatorname{wt}\lambda} \\ &= \sum_{a \in \mathbb{N}_{\operatorname{fin}}^{\{1,2,3,...\}}} z_{m^{-1}(a)}^{-1} S^{\operatorname{msum}\left(m^{-1}(a)\right)} p_{m^{-1}(a)}(\xi) p_{m^{-1}(a)}(\eta) T^{\operatorname{wt}\left(m^{-1}(a)\right)} \end{aligned} \tag{29}$$

[19] The sum $\sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} p_\lambda(\xi) p_\lambda(\eta) T^{\operatorname{wt}\lambda}$ is convergent according to (15).

[20] The sum $\sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi) p_\lambda(\eta) T^{\operatorname{wt}\lambda}$ is convergent according to (15).

(here, we substituted $m^{-1}(a)$ for λ, since $m : \operatorname{Par} \to \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ is a bijection). Now, every $a \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ satisfies

$$m_n\left(m^{-1}(a)\right) = a_n \qquad \text{for every } n \in \{1,2,3,...\} \tag{30}$$

(since the definition of the map m yields that

$$\left(m_1\left(m^{-1}(a)\right), m_2\left(m^{-1}(a)\right), m_3\left(m^{-1}(a)\right), ...\right) = m\left(m^{-1}(a)\right) = a = (a_1, a_2, a_3, ...)$$

), and thus

$$z_{m^{-1}(a)} = \prod_{n=1}^{\infty} k^{m_n\left(m^{-1}(a)\right)} \left(m_n\left(m^{-1}(a)\right)\right)! = \prod_{n=1}^{\infty} n^{a_n} a_n!$$

(by (30)), further

$$\operatorname{msum}\left(m^{-1}(a)\right) = \sum_{k=1}^{\infty} \underbrace{m_k\left(m^{-1}(a)\right)}_{=a_k \text{ by } (30)} = \sum_{k=1}^{\infty} a_k, \qquad \text{so that}$$

$$S^{\operatorname{msum}\left(m^{-1}(a)\right)} = S^{\sum\limits_{k=1}^{\infty} a_k} = \prod_{k=1}^{\infty} S^{a_k} = \prod_{n=1}^{\infty} S^{a_n},$$

furthermore

$$\begin{aligned}\operatorname{wt}\left(m^{-1}(a)\right) &= \sum_{k=1}^{\infty} k \underbrace{m_k\left(m^{-1}(a)\right)}_{=a_k \text{ by } (30)} \qquad (\text{by } (3)) \\ &= \sum_{k=1}^{\infty} k a_k, \qquad \text{so that}\end{aligned}$$

$$T^{\operatorname{wt}\left(m^{-1}(a)\right)} = T^{\sum\limits_{k=1}^{\infty} k a_k} = \prod_{k=1}^{\infty} T^{k a_k} = \prod_{k=1}^{\infty} \left(T^k\right)^{a_k} = \prod_{n=1}^{\infty} \left(T^n\right)^{a_n},$$

and finally

$$\begin{aligned}p_{m^{-1}(a)} &= \prod_{n=1}^{\infty} p_n^{m_n\left(m^{-1}(a)\right)} \qquad (\text{by the definition of } p_\lambda \text{ for } \lambda \in \operatorname{Par}) \\ &= \prod_{n=1}^{\infty} p_n^{a_n} \qquad (\text{by } (30)).\end{aligned} \tag{31}$$

This rewrites as

$$p_{m^{-1}(a)}(\xi) = \prod_{n=1}^{\infty} \left(p_n(\xi)\right)^{a_n}.$$

On the other hand, replacing the set I by the set J and the variables ξ_i by the variables η_j in (31), we obtain

$$p_{m^{-1}(a)}(\eta) = \prod_{n=1}^{\infty} \left(p_n(\eta)\right)^{a_n}.$$

Thus, (29) transforms into

$$\sum_{\lambda\in\mathrm{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi)\, p_\lambda(\eta)\, T^{\operatorname{wt}\lambda}$$

$$= \sum_{a\in\mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}} \Bigg(\underbrace{z_{m^{-1}(a)}}_{=\prod_{n=1}^{\infty} n^{a_n} a_n!} \Bigg)^{-1} \underbrace{S^{\operatorname{msum}\left(m^{-1}(a)\right)}}_{=\prod_{n=1}^{\infty} S^{a_n}} \underbrace{p_{m^{-1}(a)}(\xi)}_{=\prod_{n=1}^{\infty}(p_n(\xi))^{a_n}} \underbrace{p_{m^{-1}(a)}(\eta)}_{=\prod_{n=1}^{\infty}(p_n(\eta))^{a_n}} \underbrace{T^{\operatorname{wt}\left(m^{-1}(a)\right)}}_{=\prod_{n=1}^{\infty}(T^n)^{a_n}}$$

$$= \sum_{a\in\mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}} \left(\prod_{n=1}^{\infty} n^{a_n} a_n!\right)^{-1} \prod_{n=1}^{\infty} S^{a_n} \prod_{n=1}^{\infty} (p_n(\xi))^{a_n} \prod_{n=1}^{\infty} (p_n(\eta))^{a_n} \prod_{n=1}^{\infty} (T^n)^{a_n}$$

$$= \sum_{a\in\mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}} \prod_{n=1}^{\infty} (n^{a_n} a_n!)^{-1} S^{a_n} (p_n(\xi))^{a_n} (p_n(\eta))^{a_n} (T^n)^{a_n}$$

$$= \prod_{n=1}^{\infty} \Bigg(\sum_{a\in\mathbb{N}} \underbrace{(n^a a!)^{-1} S^a (p_n(\xi))^a (p_n(\eta))^a (T^n)^a}_{=\frac{1}{a!}\left(ST^n\cdot\frac{1}{n}p_n(\xi)p_n(\eta)\right)^a} \Bigg) \qquad \text{(by the product rule)}$$

$$= \prod_{n=1}^{\infty} \Bigg(\underbrace{\sum_{a\in\mathbb{N}} \frac{1}{a!}\left(ST^n\cdot\frac{1}{n}p_n(\xi)\,p_n(\eta)\right)^a}_{=\exp\left(ST^n\cdot\frac{1}{n}p_n(\xi)p_n(\eta)\right)} \Bigg) = \prod_{n=1}^{\infty} \exp\left(ST^n\cdot\frac{1}{n}p_n(\xi)\,p_n(\eta)\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} ST^n\cdot\frac{1}{n}p_n(\xi)\,p_n(\eta)\right) = \exp\left(S\cdot\sum_{n=1}^{\infty}\frac{1}{n}T^n p_n(\xi)\,p_n(\eta)\right). \tag{32}$$

But for every $n\in\{1,2,3,\ldots\}$, we know that $p_n = \sum_{i\in I}\xi_i^n$ (by (18)), which rewrites as $p_n(\xi) = \sum_{i\in I}\xi_i^n$. If we replace the set I by J and the variables ξ_i by η_j in this formula, we obtain $p_n(\eta) = \sum_{j \in J} \eta_j^n$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n} T^n p_n(\xi) p_n(\eta) = \sum_{n=1}^{\infty} \frac{1}{n} T^n \underbrace{\sum_{i \in I} \xi_i^n \cdot \sum_{j \in J} \eta_j^n}_{= \sum_{(i,j) \in I \times J} \xi_i^n \eta_j^n} = \sum_{n=1}^{\infty} \frac{1}{n} T^n \sum_{(i,j) \in I \times J} \xi_i^n \eta_j^n$$

$$= \sum_{(i,j) \in I \times J} \sum_{n=1}^{\infty} \frac{1}{n} \underbrace{\xi_i^n \eta_j^n T^n}_{= (\xi_i \eta_j T)^n} = \sum_{(i,j) \in I \times J} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n} (\xi_i \eta_j T)^n}_{\substack{= -\log(1 - \xi_i \eta_j T) \text{ due to the} \\ \text{formula } \sum_{n=1}^{\infty} \frac{1}{n} X^n = -\log(1 - X)}}$$

$$= \sum_{(i,j) \in I \times J} \underbrace{(-\log(1 - \xi_i \eta_j T))}_{= \log \frac{1}{1 - \xi_i \eta_j T}} = \sum_{(i,j) \in I \times J} \log \frac{1}{1 - \xi_i \eta_j T}.$$

Therefore, (32) becomes

$$\sum_{\lambda \in \mathrm{Par}} z_\lambda^{-1} p_\lambda(\xi) p_\lambda(\eta) T^{\mathrm{wt}\,\lambda} = \exp\left(S \cdot \sum_{(i,j) \in I \times J} \log \frac{1}{1 - \xi_i \eta_j T}\right) = \prod_{(i,j) \in I \times J} \underbrace{\exp\left(S \cdot \log \frac{1}{1 - \xi_i \eta_j T}\right)}_{= \left(\frac{1}{1 - \xi_i \eta_j T}\right)^S}$$

$$= \prod_{(i,j) \in I \times J} \left(\frac{1}{1 - \xi_i \eta_j T}\right)^S.$$

This proves Theorem 5 **(b)**.

Theorem 5 **(a)** trivially follows from Theorem 5 **(b)** by evaluating at $S = 1$.

A useful corollary from Theorem 5 is the following fact:

Theorem 6. Let I be a countable set. In the ring $\mathbb{Q}[\xi_i \mid i \in I]_\infty$, we have

$$\sum_{\substack{\lambda \in \mathrm{Par}; \\ \mathrm{wt}\,\lambda = n}} z_\lambda^{-1} p_\lambda = h_n \qquad \text{for every } n \in \mathbb{N} \tag{33}$$

and

$$\sum_{\substack{\lambda \in \mathrm{Par}; \\ \mathrm{wt}\,\lambda = n}} z_\lambda^{-1} (-1)^{\mathrm{msum}\,\lambda} p_\lambda = (-1)^n e_n \qquad \text{for every } n \in \mathbb{N}, \tag{34}$$

where the map msum : Par $\to \mathbb{N}$ is defined as in Theorem 5 **(b)**.

Proof of Theorem 6. Let $J = \{1\}$. Theorem 5 **(b)** yields that (28) holds in $((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty)[[S]]$. Now, we have

$$\prod_{(i,j) \in I \times J} \left(\frac{1}{1 - \xi_i \eta_j T}\right)^S = \prod_{i \in I} \underbrace{\prod_{j \in J} \left(\frac{1}{1 - \xi_i \eta_j T}\right)^S}_{\substack{= \left(\frac{1}{1 - \xi_i \eta_1 T}\right)^S, \\ \text{since } J = \{1\}}} = \prod_{i \in I} \left(\frac{1}{1 - \xi_i \eta_1 T}\right)^S. \tag{35}$$

Besides, for every $n \in \{1, 2, 3, ...\}$, we have

$$p_n(\eta) = \sum_{j\in J} \eta_j^n$$

(by (18), with the set I replaced by J and the indeterminates ξ_i replaced by η_j)

$$= \eta_1^n$$

(since $J = \{1\}$). Hence, every $\lambda \in \operatorname{Par}$ satisfies

$$p_\lambda(\eta) = \prod_{n=1}^{\infty} (p_n(\eta))^{m_n(\lambda)} = \prod_{n=1}^{\infty} (\eta_1^n)^{m_n(\lambda)} = \prod_{n=1}^{\infty} \eta_1^{nm_n(\lambda)} = \eta_1^{\sum_{n=1}^{\infty} nm_n(\lambda)} = \eta_1^{\operatorname{wt}\lambda}$$

(since $\sum_{n=1}^{\infty} nm_n(\lambda) = \sum_{k=1}^{\infty} km_k(\lambda) = \operatorname{wt}\lambda$ by (3)). Using this equation and (35), we can rewrite (27) as

$$\sum_{\lambda\in\operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi)\, \eta_1^{\operatorname{wt}\lambda} T^{\operatorname{wt}\lambda} = \prod_{i\in I} \left(\frac{1}{1-\xi_i\eta_1 T}\right)^S. \tag{36}$$

This holds in the ring $\left(\left(\left(\mathbb{Q}[\xi_i \mid i \in I]_\infty\right)[\eta_j \mid j \in J]_\infty\right)[[T]]\right)[[S]]$. But

$$\begin{aligned}(\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty &= (\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J] \qquad \text{(since } J = \{1\} \text{ is a finite set)} \\ &= (\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_1] \qquad \text{(since } J = \{1\}\text{)},\end{aligned}$$

and therefore, (36) holds in the ring $(((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_1])[[T]])[[S]]$.

By the universal property of a polynomial ring, there exists a ring homomorphism

$$(\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_1] \to \mathbb{Q}[\xi_i \mid i \in I]_\infty$$

that leaves each element of $\mathbb{Q}[\xi_i \mid i \in I]_\infty$ invariant and maps η_1 to 1. This homomorphism extends to a continuous[21] ring homomorphism

$$(((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[\eta_1])[[T]])[[S]] \to ((\mathbb{Q}[\xi_i \mid i \in I]_\infty)[[T]])[[S]]$$

that leaves each element of $\mathbb{Q}[\xi_i \mid i \in I]_\infty$ invariant and maps η_1, T and S to 1, T and S, respectively. This homomorphism respects infinite sums and infinite products (since it is continuous), and thus it maps $\sum_{\lambda\in\operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi)\, \eta_1^{\operatorname{wt}\lambda} T^{\operatorname{wt}\lambda}$ to $\sum_{\lambda\in\operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi)\, 1^{\operatorname{wt}\lambda} T^{\operatorname{wt}\lambda}$ and maps $\prod_{i\in I}\left(\frac{1}{1-\xi_i\eta_1 T}\right)^S$ to $\prod_{i\in I}\left(\frac{1}{1-\xi_i\cdot 1T}\right)^S$. Therefore, upon applying this homomorphism to the equation (36), we obtain

$$\sum_{\lambda\in\operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi)\, 1^{\operatorname{wt}\lambda} T^{\operatorname{wt}\lambda} = \prod_{i\in I} \left(\frac{1}{1-\xi_i\cdot 1T}\right)^S.$$

This simplifies to

$$\sum_{\lambda\in\operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi)\, T^{\operatorname{wt}\lambda} = \prod_{i\in I} \left(\frac{1}{1-\xi_i T}\right)^S.$$

[21] Here, "continuous" means "continuous with respect to the (T, S)-adic topologies on the two rings".

Since

$$\sum_{\lambda\in\operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi) T^{\operatorname{wt}\lambda} = \sum_{n=0}^{\infty} \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} \underbrace{p_\lambda(\xi)}_{=p_\lambda} \underbrace{T^{\operatorname{wt}\lambda}}_{\substack{=T^n\\ (\text{since } \operatorname{wt}\lambda=n)}}$$
$$= \sum_{n=0}^{\infty} \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda T^n,$$

this rewrites as

$$\sum_{n=0}^{\infty} \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda T^n = \prod_{i\in I}\left(\frac{1}{1-\xi_i T}\right)^S. \qquad (37)$$

Evaluating this identity at $S=1$ yields[22]

$$\sum_{n=0}^{\infty} \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} p_\lambda T^n = \prod_{i\in I}\frac{1}{1-\xi_i T}.$$

Since

$$\prod_{i\in I}\frac{1}{1-\xi_i T} = \sum_{d=0}^{\infty} h_d T^d \qquad (\text{by } (8))$$
$$= \sum_{n=0}^{\infty} h_n T^n,$$

this rewrites as

$$\sum_{n=0}^{\infty} \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} p_\lambda T^n = \sum_{n=0}^{\infty} h_n T^n.$$

Comparing coefficients in this equation, we obtain

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} p_\lambda = h_n \qquad \text{for every } n\in\mathbb{N}.$$

Thus, (33) is proven.

On the other hand, evaluating the identity (37) at $S=-1$, we get

$$\sum_{n=0}^{\infty} \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} (-1)^{\operatorname{msum}\lambda} p_\lambda T^n = \prod_{i\in I}\left(\frac{1}{1-\xi_i T}\right)^{-1}.$$

[22] This is a bit sloppy formulation - in fact, (37) is not even a polynomial identity in S, so it is not really clear what "evaluating it at $S=1$" means. But what I mean is: if we replace S by 1 throughout the proof of (37), we arrive at

$$\sum_{n=0}^{\infty} \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} p_\lambda T^n = \prod_{i\in I}\frac{1}{1-\xi_i T}.$$

Since

$$\prod_{i\in I}\left(\frac{1}{1-\xi_i T}\right)^{-1} = \prod_{i\in I}(1-\xi_i T) = \sum_{d=0}^{\infty}(-1)^d e_d T^d \qquad \text{(by (7))}$$
$$= \sum_{n=0}^{\infty}(-1)^n e_n T^n,$$

this rewrites as

$$\sum_{n=0}^{\infty}\sum_{\substack{\lambda\in\mathrm{Par};\\ \mathrm{wt}\,\lambda=n}} z_\lambda^{-1}(-1)^{\mathrm{msum}\,\lambda} p_\lambda T^n = \sum_{n=0}^{\infty}(-1)^n e_n T^n.$$

Comparing coefficients in this equation, we obtain

$$\sum_{\substack{\lambda\in\mathrm{Par};\\ \mathrm{wt}\,\lambda=n}} z_\lambda^{-1}(-1)^{\mathrm{msum}\,\lambda} p_\lambda = (-1)^n e_n \qquad \text{for every } n\in\mathbb{N}.$$

Thus, (34) is proven.

Now it is time to introduce some more elements of $\mathbb{Z}[\xi_i \mid i\in I]_\infty$, namely the power series x_1, x_2, ... They are rather difficult to define directly, so we define them by means of a theorem:

Theorem 7. Let A be a commutative ring with unity. Let $(\rho_0,\rho_1,\rho_2,...)\in A^{\mathbb{N}}$ be a sequence of elements of A such that $\rho_0 = 1$.

(a) There exists one and only one sequence $(X_1,X_2,X_3,...)\in A^{\{1,2,3,...\}}$ of elements of A that satisfies the equation

$$\prod_{d=1}^{\infty}\left(1-X_d T^d\right)^{-1} = \sum_{n=0}^{\infty}\rho_n T^n \tag{38}$$

in the ring $A[[T]]$. [23]

[23] Note that the infinite product $\prod\limits_{d=1}^{\infty}\left(1-x_d T^d\right)^{-1}$ converges (with respect to the (T)-adic topology on the ring $A[[T]]$) for every sequence $(x_1,x_2,x_3,...)\in A^{\{1,2,3,...\}}$. In fact, the sequence $\left(\prod\limits_{d=1}^{m}\left(1-x_d T^d\right)^{-1}\right)_{m\in\mathbb{N}}$ is a Cauchy sequence.

Proof. Let $n\in\mathbb{N}$. Let a and b be two integers such that $a\geq n$ and $b\geq n$. Then, since $a\geq n$, we have

$$\prod_{d=1}^{a}\left(1-x_d T^d\right)^{-1} = \left(\prod_{d=1}^{n-1}\left(1-x_d T^d\right)^{-1}\right)\cdot\left(\prod_{d=n}^{a}\left(\underbrace{1-x_d T^d}_{\substack{\equiv 1 \bmod (T^n)\\ (\text{since } d\geq n)}}\right)^{-1}\right)$$
$$\equiv \left(\prod_{d=1}^{n-1}\left(1-x_d T^d\right)^{-1}\right)\cdot\underbrace{\left(\prod_{d=n}^{a}1^{-1}\right)}_{=1} = \prod_{d=1}^{n-1}\left(1-x_d T^d\right)^{-1} \bmod (T^n).$$

(b) Assume that the ring A is graded, and that

$$\left(\text{for every } n \in \mathbb{N}, \text{ the element } \rho_n \text{ lies in the } n\text{-th graded component of } A\right). \tag{39}$$

Then, the unique sequence $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$ that satisfies (38) has the property that

$$\left(\text{for every } n \in \{1, 2, 3, ...\}, \text{ the element } X_n \text{ lies in the } n\text{-th graded component of } A\right). \tag{40}$$

Proof of Theorem 7. **(a)** In order to establish Theorem 7 **(a)**, we must prove two assertions:

Assertion 1: There exists a sequence $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$ of elements of A that satisfies the equation

$$\prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \sum_{n=0}^{\infty} \rho_n T^n$$

in the ring $A[[T]]$.

Similarly, $\prod\limits_{d=1}^{b} \left(1 - x_d T^d\right)^{-1} \equiv \prod\limits_{d=1}^{n-1} \left(1 - x_d T^d\right)^{-1} \operatorname{mod} (T^n)$. Hence,

$$\prod_{d=1}^{a} \left(1 - x_d T^d\right)^{-1} \equiv \prod_{d=1}^{n-1} \left(1 - x_d T^d\right)^{-1} \equiv \prod_{d=1}^{b} \left(1 - x_d T^d\right)^{-1} \operatorname{mod} (T^n).$$

Now, forget that we fixed a and b. We thus have proven that any two integers a and b such that $a \geq n$ and $b \geq n$ satisfy

$$\prod_{d=1}^{a} \left(1 - x_d T^d\right)^{-1} \equiv \prod_{d=1}^{b} \left(1 - x_d T^d\right)^{-1} \operatorname{mod} (T^n).$$

Hence, there exists an $N \in \mathbb{N}$ such that any two integers a and b such that $a \geq N$ and $b \geq N$ satisfy

$$\prod_{d=1}^{a} \left(1 - x_d T^d\right)^{-1} \equiv \prod_{d=1}^{b} \left(1 - x_d T^d\right)^{-1} \operatorname{mod} (T^n)$$

(namely, $N = n$).

Now, forget that we fixed n. We thus have shown that for every $n \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that any two integers a and b such that $a \geq N$ and $b \geq N$ satisfy

$$\prod_{d=1}^{a} \left(1 - x_d T^d\right)^{-1} \equiv \prod_{d=1}^{b} \left(1 - x_d T^d\right)^{-1} \operatorname{mod} (T^n)$$

In other words, the sequence $\left(\prod\limits_{d=1}^{m} \left(1 - x_d T^d\right)^{-1}\right)_{m \in \mathbb{N}}$ is a Cauchy sequence (with respect to the (T)-adic topology on the ring $A[[T]]$). Hence, this sequence converges (since $A[[T]]$ is complete with respect to the (T)-adic topology). In other words, the infinite product $\prod\limits_{d=1}^{\infty} \left(1 - x_d T^d\right)^{-1}$ converges, qed.

Assertion 2: If $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$ and $(Y_1, Y_2, Y_3, ...) \in A^{\{1,2,3,...\}}$ are two sequences of elements of A that satisfy the equations

$$\prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \sum_{n=0}^{\infty} \rho_n T^n \qquad \text{and} \tag{41}$$

$$\prod_{d=1}^{\infty} \left(1 - Y_d T^d\right)^{-1} = \sum_{n=0}^{\infty} \rho_n T^n \tag{42}$$

in the ring $A[[T]]$, then $(X_1, X_2, X_3, ...) = (Y_1, Y_2, Y_3, ...)$.

Once both Assertions 1 and 2 are proven, Theorem 7 **(a)** will ensue (since Assertion 1 yields the existence of the required sequence $(X_1, X_2, X_3, ...)$, while Assertion 2 yields the uniqueness thereof). So it remains to prove Assertions 1 and 2.

Proof of Assertion 1. We construct the required sequence $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$ by recursion: Let $m \in \{1, 2, 3, ...\}$ be given. We want to define an element $X_m \in A$, assuming that the elements X_1, X_2, ..., X_{m-1} are already defined.

We define X_m as the coefficient before T^m of the power series

$$\sum_{n=0}^{\infty} \rho_n T^n \cdot \prod_{d=1}^{m-1} \left(1 - X_d T^d\right). \tag{43}$$

This way, we have recursively defined a sequence $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$.

We now will show that every $m \in \mathbb{N}$ satisfies

$$\prod_{d=1}^{m} \left(1 - X_d T^d\right)^{-1} \equiv \sum_{n=0}^{\infty} \rho_n T^n \bmod \left(T^{m+1}\right), \tag{44}$$

where (T^{m+1}) means the ideal $T^{m+1} \cdot A[[T]]$ of the ring $A[[T]]$ (so that the congruence of two power series modulo T^{m+1} simply means that they are equal in all of their terms in which T occurs in a power less than $m+1$).

We will prove (44) by induction over m. First, the induction base is clear, since for $m = 0$, the congruence (44) is true (because the left hand side, $\prod\limits_{d=1}^{m} \left(1 - X_d T^d\right)^{-1}$, is an empty product and therefore $= 1$, while the right hand side is $\sum\limits_{n=0}^{\infty} \rho_n T^n$ and thus congruent to $\rho_0 = 1$ modulo (T^1)). Now we come to the induction step: Let $m \in \mathbb{N}$ be such that $m > 0$. We want to prove (44), assuming that (44) holds with m replaced by $m - 1$.

We have assumed that (44) holds with m replaced by $m - 1$; in other words, we have assumed that

$$\prod_{d=1}^{m-1} \left(1 - X_d T^d\right)^{-1} \equiv \sum_{n=0}^{\infty} \rho_n T^n \bmod \left(T^m\right).$$

Multiplication by $\prod\limits_{d=1}^{m-1} \left(1 - X_d T^d\right)$ yields

$$1 \equiv \sum_{n=0}^{\infty} \rho_n T^n \cdot \prod_{d=1}^{m-1} \left(1 - X_d T^d\right) \bmod \left(T^m\right).$$

Since $1 \equiv (1 - X_m T^m)^{-1} \mod (T^m)$ (because $1 - X_m T^m \equiv 1 \mod (T^m)$), this becomes

$$(1 - X_m T^m)^{-1} \equiv \sum_{n=0}^{\infty} \rho_n T^n \cdot \prod_{d=1}^{m-1} \left(1 - X_d T^d\right) \mod (T^m) .$$

In other words, the power series (43) is congruent to the power series $(1 - X_m T^m)^{-1}$ modulo the ideal (T^m). This means that the coefficients of the power series (43) before T^0, T^1, ..., T^{m-1} are equal to the corresponding coefficients of the power series $(1 - X_m T^m)^{-1}$. But the coefficient of the power series (43) before T^m is also equal to the corresponding coefficient of the power series $(1 - X_m T^m)^{-1}$ (because the coefficient of the power series (43) before T^m is X_m (by our definition of X_m), and the coefficient of the power series $(1 - X_m T^m)^{-1}$ before T^m is also X_m (since $(1 - X_m T^m)^{-1} = \sum\limits_{k=0}^{\infty} (X_m T^m)^k$)). Hence, the coefficients of the power series (43) before T^0, T^1, ..., T^m are equal to the corresponding coefficients of the power series $(1 - X_m T^m)^{-1}$. In other words, the power series (43) is congruent to $(1 - X_m T^m)^{-1}$ modulo the ideal (T^{m+1}). This means that

$$(1 - X_m T^m)^{-1} \equiv \sum_{n=0}^{\infty} \rho_n T^n \cdot \prod_{d=1}^{m-1} \left(1 - X_d T^d\right) \mod \left(T^{m+1}\right) .$$

Multiplying this congruence by $\prod\limits_{d=1}^{m-1} \left(1 - X_d T^d\right)^{-1}$ yields (44) (since $(1 - X_m T^m)^{-1} \cdot \prod\limits_{d=1}^{m-1} \left(1 - X_d T^d\right)^{-1} = \prod\limits_{d=1}^{m} \left(1 - X_d T^d\right)^{-1}$). Hence, (44) is proven, and our induction is complete.

Now, we have $\lim\limits_{m \to \infty} \prod\limits_{d=1}^{m} \left(1 - X_d T^d\right)^{-1} = \sum\limits_{n=0}^{\infty} \rho_n T^n$ (where the limit is taken with respect to the (T)-adic topology on the ring $A[[T]]$), since for every $N \in \mathbb{N}$, there exists some $\nu \in \mathbb{N}$ such that

$$\prod_{d=1}^{m} \left(1 - X_d T^d\right)^{-1} \equiv \sum_{n=0}^{\infty} \rho_n T^n \mod \left(T^N\right)$$

for every $m \geq \nu$ (in fact, this holds for $\nu = N - 1$ [24]). Hence, the sequence $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$ that we constructed satisfies

$$\prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \lim_{m \to \infty} \prod_{d=1}^{m} \left(1 - X_d T^d\right)^{-1} = \sum_{n=0}^{\infty} \rho_n T^n.$$

Consequently, Assertion 1 is proven.

Proof of Assertion 2. Let $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$ and $(Y_1, Y_2, Y_3, ...) \in A^{\{1,2,3,...\}}$ be two sequences of elements of A that satisfy the equations (41) and (42). We are now going to prove that $X_n = Y_n$ for every $n \in \{1, 2, 3, ...\}$.

[24] because of (44), and since any two elements that are congruent to each other modulo $\left(T^{m+1}\right)$ must automatically be congruent to each other modulo $\left(T^N\right)$ (since $m + 1 \geq \nu + 1 = (N - 1) + 1 = N$)

In fact, we are going to prove this by strong induction over n. So we fix some $n \in \{1,2,3,...\}$, and we try to prove that $X_n = Y_n$, assuming that we have already proved that $X_d = Y_d$ for every $d \in \{1,2,3,...\}$ satisfying $d < n$.

The equation (41) yields

$$\sum_{n=0}^{\infty} \rho_n T^n = \prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \prod_{d=1}^{n-1} \left(1 - X_d T^d\right)^{-1} \cdot (1 - X_n T^n)^{-1} \cdot \prod_{d=n+1}^{\infty} \left(\underbrace{1 - X_d T^d}_{\substack{\equiv 1 \bmod \left(T^{n+1}\right), \\ \text{since } T^d \equiv 0 \bmod \left(T^{n+1}\right) \\ \text{because of } d \geq n+1}} \right)^{-1}$$

$$\equiv \prod_{d=1}^{n-1} \left(1 - X_d T^d\right)^{-1} \cdot (1 - X_n T^n)^{-1} \bmod \left(T^{n+1}\right).$$

Multiplying this congruence with $\left(\sum\limits_{n=0}^{\infty} \rho_n T^n\right)^{-1} \cdot (1 - X_n T^n)$ (the power series $\sum\limits_{n=0}^{\infty} \rho_n T^n$ is indeed invertible, since its coefficient before T^0 is $\rho_0 = 1$), we obtain

$$1 - X_n T^n \equiv \left(\sum_{n=0}^{\infty} \rho_n T^n\right)^{-1} \cdot \prod_{d=1}^{n-1} \left(1 - X_d T^d\right)^{-1} \bmod \left(T^{n+1}\right),$$

so that

$$X_n T^n \equiv 1 - \left(\sum_{n=0}^{\infty} \rho_n T^n\right)^{-1} \cdot \prod_{d=1}^{n-1} \left(1 - X_d T^d\right)^{-1} \bmod \left(T^{n+1}\right). \tag{45}$$

Similarly,

$$Y_n T^n \equiv 1 - \left(\sum_{n=0}^{\infty} \rho_n T^n\right)^{-1} \cdot \prod_{d=1}^{n-1} \left(1 - Y_d T^d\right)^{-1} \bmod \left(T^{n+1}\right). \tag{46}$$

Thus,

$$\begin{aligned} X_n T^n &\equiv 1 - \left(\sum_{n=0}^{\infty} \rho_n T^n\right)^{-1} \cdot \prod_{d=1}^{n-1} \left(1 - X_d T^d\right)^{-1} \qquad (\text{by } (45)) \\ &= 1 - \left(\sum_{n=0}^{\infty} \rho_n T^n\right)^{-1} \cdot \prod_{d=1}^{n-1} \left(1 - Y_d T^d\right)^{-1} \\ &\qquad (\text{since } X_d = Y_d \text{ for every } d \in \{1,2,3,...\} \text{ satisfying } d < n) \\ &\equiv Y_n T^n \bmod \left(T^{n+1}\right) \qquad (\text{by } (46)). \end{aligned}$$

In other words, the power series $X_n T^n - Y_n T^n$ must belong to the ideal (T^{n+1}). But a power series belonging to the ideal (T^{n+1}) must have its coefficient before T^n equal to 0. Thus, the power series $X_n T^n - Y_n T^n$ has its coefficient before T^n equal to 0. In other words, $X_n - Y_n = 0$ (since $X_n - Y_n$ is the coefficient of the power series $X_n T^n - Y_n T^n$ before T^n), and therefore $X_n = Y_n$. This completes our induction step.

We have therefore shown that $X_n = Y_n$ for every $n \in \{1,2,3,...\}$. Thus, $(X_1, X_2, X_3, ...) = (Y_1, Y_2, Y_3, ...)$. This proves Assertion 2.

As both Assertions 1 and 2 are verified now, Theorem 7 **(a)** is proven.

(b) Let us introduce a notation: A power series $\alpha \in A[[T]]$ is said to be *equigraded* if and only if

$$\left(\text{for every } n \in \mathbb{N}, \text{ the coefficient of } \alpha \text{ before } T^n \text{ lies in the } n\text{-th graded component of } A\right).$$

It is easy to see that

$$\{\alpha \in A[[T]] \mid \text{ the power series } \alpha \text{ is equigraded}\}$$

is a subring of $A[[T]]$ (for a proof of this, see [2], Theorem 1 **(a)**). In other words, the sum, the difference and the product of finitely many equigraded power series are equigraded as well, and the two power series 0 and 1 are both equigraded.

Now, let us prove Theorem 7 **(b)**. The unique sequence $(X_1, X_2, X_3, ...) \in A^{\{1,2,3,...\}}$ that satisfies (38) was recursively constructed in the proof of Assertion 1 above; according to that construction, this sequence satisfies

$$\left(X_m \text{ is the coefficient before } T^m \text{ of the power series (43)}\right) \tag{47}$$

for every $m \in \{1,2,3,...\}$.

Now, we are going to prove (40) by strong induction over n. That is, we fix some $n \in \{1,2,3,...\}$, and we want to show that X_n lies in the n-th graded component of A, assuming that X_d lies in the d-th graded component of A for every $d \in \{1,2,3,...\}$ satisfying $d < n$.

For every $d \in \{1,2,3,...\}$ satisfying $d < n$, the power series $X_d T^d$ is equigraded (since X_d lies in the d-th graded component of A, according to our assumption), and thus the power series $1 - X_d T^d$ is equigraded, too (because it is the difference of the two equigraded power series 1 and $X_d T^d$). Hence, the power series $\prod\limits_{d=1}^{m-1} \left(1 - X_d T^d\right)$ is the product of finitely many equigraded power series, and thus it is equigraded as well. Besides, the power series $\sum\limits_{n=0}^{\infty} \rho_n T^n$ is equigraded (by (39)). Therefore, the power series (43) is the product of two equigraded power series, and therefore equigraded as well. Consequently, the coefficient before T^n of the power series (43) lies in the n-th graded component of A. But the coefficient before T^n of the power series (43) is X_n (due to (47), applied to $m = n$). Thus, X_n lies in the n-th graded component of A. This completes our induction, and thus (40) is proven. In other words, Theorem 7 **(b)** is proven.

Theorem 7 **(a)** makes the following definition possible:

Definition 14. There exists one and only one sequence $(X_1, X_2, X_3, ...) \in \left(\mathbb{Z}[\xi_i \mid i \in I]_\infty\right)^{\{1,2,3,...\}}$ of elements of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ that satisfies the equation

$$\prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \sum_{n=0}^{\infty} h_n T^n$$

in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$. [25] This sequence will be denoted by $(x_1, x_2, x_3, ...)$ from now on until the end of this note. Hence, this sequence $(x_1, x_2, x_3, ...)$ satisfies

$$\prod_{d=1}^{\infty} \left(1 - x_d T^d\right)^{-1} = \sum_{n=0}^{\infty} h_n T^n \tag{48}$$

This way, we have defined a sequence $(x_1, x_2, x_3, ...)$ of power series. Note that this definition agrees with the definition of $(x_1, x_2, x_3, ...)$ given in [1], (9.64).

Besides, we define a power series $x_0 \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ by $x_0 = 1$.

We notice a first property of the power series x_0, x_1, x_2, ...: For every $n \in \mathbb{N}$, the power series $x_n \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ is homogeneous of degree n. [26]

Now, we are going to define a power series x_λ for every partition λ as a product of x_n's in the same way as we defined h_λ as a product of h_n's, as we defined e_λ as a product of e_n's, and as we defined p_λ as a product of p_n's.[27]

Definition 15. Let λ be a partition. Then, we define a power series $x_\lambda \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ by

$$x_\lambda = \prod_{n=1}^{\infty} x_n^{m_n(\lambda)}.$$

(This is actually a finite product, since only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $x_n^{m_n(\lambda)} \neq 1$, because only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $m_n(\lambda) \neq 0$.) This power series x_λ can be written in a simpler way if we write our partition λ in the form $(\lambda_1, \lambda_2, ..., \lambda_m)$ for some $m \in \mathbb{N}$; namely,

$$\text{if } \lambda = (\lambda_1, \lambda_2, ..., \lambda_m), \text{ then } x_\lambda = x_{\lambda_1} x_{\lambda_2} ... x_{\lambda_m} \tag{49}$$

(we recall that x_0 is supposed to mean 1). This is proven in the same way as we showed (10). Hence, our definition of x_λ agrees with the definition of x_λ given by Hazewinkel in [1], (9.66).

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[25] This follows from Theorem 7 **(a)**, applied to $A = \mathbb{Z}[\xi_i \mid i \in I]_\infty$ and $(\rho_0, \rho_1, \rho_2, ...) = (h_0, h_1, h_2, ...)$.

[26] *Proof.* Recall that $(x_1, x_2, x_3, ...)$ is the unique sequence $(X_1, X_2, X_3, ...) \in (\mathbb{Z}[\xi_i \mid i \in I]_\infty)^{\{1,2,3,...\}}$ of elements of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ that satisfies the equation

$$\prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \sum_{n=0}^{\infty} h_n T^n$$

in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$. Thus, Theorem 7 **(b)**, applied to $A = \mathbb{Z}[\xi_i \mid i \in I]_\infty$ and $(\rho_0, \rho_1, \rho_2, ...) = (h_0, h_1, h_2, ...)$, yields that for every $n \in \{1, 2, 3, ...\}$, the element x_n lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ (because for every $n \in \mathbb{N}$, the element h_n lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, since h_n is a homogeneous power series of degree n). In other words, for every $n \in \{1, 2, 3, ...\}$, the power series x_n is homogeneous of degree n. This holds for $n = 0$, as well (since $x_0 = 1$ is clearly homogeneous of degree 0), so we can conclude that for every $n \in \mathbb{N}$, the power series x_n is homogeneous of degree n, qed.

[27] Actually we are copying the definition of p_λ verbatim, just replacing every p by x and changing the reference to [1].

The power series x_λ is wt λ-homogeneous[28].

We notice that for every $\alpha \in \mathbb{N}^I_{\text{fin}}$, we have

$$\sum_{\substack{\lambda \in \text{Par};\\ \lambda \sim \alpha}} x_\lambda = \prod_{i \in I} x_{\alpha_i} \tag{50}$$

(again, remembering that x_0 was defined as 1). The proof of this equation is exactly the same as that of (12) (but with h replaced by x throughout the proof).

Our next definition will be a simple notation:

Definition 16. Let $P \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ be a power series. Let $n \in \{1, 2, 3, ...\}$. Then, we define a power series $P(\xi^n) \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ as follows: If we write the power series P in the form $P = \sum\limits_{\alpha \in \mathbb{N}^I_{\text{fin}}} P_\alpha \cdot \xi^\alpha$ (with P_α being an element of $\mathbb{Z}$ for every $\alpha \in \mathbb{N}^I_{\text{fin}}$), then the power series $P(\xi^n)$ is defined as $\sum\limits_{\alpha \in \mathbb{N}^I_{\text{fin}}} P_\alpha \cdot \xi^{n\alpha}$. Here, $n\alpha$ means the family $(n\alpha_i)_{i \in I} \in \mathbb{N}^I_{\text{fin}}$.

Informally speaking, the power series $P(\xi^n)$ is what we obtain if we replace every variable ξ_i by its n-th power ξ_i^n in the power series P.

Note that $P(\xi^1) = P$ for every power series $P \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$.

Next, we are going to define yet some more power series (but this time, they are not defined in the same way as h_λ, e_λ, p_λ and x_λ):

Definition 17. Let λ be a partition. Then, we define a power series $r_\lambda \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ by

$$r_\lambda = \prod_{n=1}^{\infty} h_{m_n(\lambda)}(\xi^n).$$

(This product $\prod\limits_{n=1}^{\infty} h_{m_n(\lambda)}(\xi^n)$ is a finite product, since $h_{m_n(\lambda)}(\xi^n) = 1$ for all but finitely many $n \in \{1, 2, 3, ...\}$, since $m_n(\lambda) = 0$ for all but finitely many n, and if $m_n(\lambda) = 0$, then $\underbrace{h_{m_n(\lambda)}}_{=h_0=1}(\xi^n) = 1$.) Note that this definition of r_λ is the same as the one given by Hazewinkel in [1], 9.63.

For every partition λ, the power series r_λ is wt λ-homogeneous.[29]

Now, we will show an identity relating the power series x_λ and r_λ:

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[28] This is proven in the same way as we showed that h_λ is wt λ-homogeneous.

[29] In fact, for every $n \in \{1, 2, 3, ...\}$, the power series $h_{m_n(\lambda)}$ is $m_n(\lambda)$-homogeneous, and thus the power series $h_{m_n(\lambda)}(\xi^n)$ is $nm_n(\lambda)$-homogeneous (since for every $k \in \mathbb{N}$ and every k-homogeneous power series $\alpha \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$, the power series $\alpha(\xi^n)$ is nk-homogeneous). Hence, the product $\prod\limits_{n=1}^{\infty} h_{m_n(\lambda)}(\xi^n)$ is $\sum\limits_{n=1}^{\infty} nm_n(\lambda)$-homogeneous. Since $\prod\limits_{n=1}^{\infty} h_{m_n(\lambda)}(\xi^n) = r_\lambda$ and $\sum\limits_{n=1}^{\infty} nm_n(\lambda) = \sum\limits_{k=1}^{\infty} km_k(\lambda) = \text{wt}\,\lambda$, this means that r_λ is wt λ-homogeneous, qed.

Theorem 8. Let I and J be two countable sets. In the ring $\left(\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]]$, we have

$$\sum_{\lambda \in \operatorname{Par}} x_\lambda(\xi)\, r_\lambda(\eta)\, T^{\operatorname{wt}\lambda} = \prod_{(i,j) \in I \times J} \frac{1}{1 - \xi_i \eta_j T}.$$

[30]

Before we start proving this theorem, let us recall a standard fact from the theory of formal power series:

Power series substitution rule. If A is a commutative ring with unity, and $P \in A[[T]]$ is a power series with constant term 0, then there exists a continuous[31] ring homomorphism $\operatorname{ev}_{T,P} : A[[T]] \to A[[T]]$ which maps T to P and is the identity on the ring A.

(In fact, this homomorphism $\operatorname{ev}_{T,P}$ is defined by $\operatorname{ev}_{T,P}\left(\sum\limits_{n=0}^{\infty} a_n T^n\right) = \sum\limits_{n=0}^{\infty} a_n P^n$ for every power series $\sum\limits_{n=0}^{\infty} a_n T^n \in A[[T]]$ with $a_i \in A$ for all $i \in \mathbb{N}$. The infinite sum $\sum\limits_{n=0}^{\infty} a_n P^n$ is convergent, because for any $n \in \mathbb{N}$, the power series $a_n P^n$ has no monomial of degree $< n$.)

Proof of Theorem 8. We have

$$\sum_{\lambda \in \operatorname{Par}} x_\lambda(\xi)\, r_\lambda(\eta)\, T^{\operatorname{wt}\lambda} = \sum_{a \in \mathbb{N}_{\operatorname{fin}}^{\{1,2,3,\ldots\}}} x_{m^{-1}(a)}(\xi)\, r_{m^{-1}(a)}(\eta)\, T^{\operatorname{wt}\left(m^{-1}(a)\right)} \tag{51}$$

(here, we substituted $m^{-1}(a)$ for λ, since $m : \operatorname{Par} \to \mathbb{N}_{\operatorname{fin}}^{\{1,2,3,\ldots\}}$ is a bijection). Now, every $a \in \mathbb{N}_{\operatorname{fin}}^{\{1,2,3,\ldots\}}$ satisfies $T^{\operatorname{wt}\left(m^{-1}(a)\right)} = \prod\limits_{n=1}^{\infty} (T^n)^{a_n}$ (as we have seen during the proof of Theorem 5) and furthermore

$$\begin{aligned} x_{m^{-1}(a)} &= \prod_{n=1}^{\infty} x_n^{m_n\left(m^{-1}(a)\right)} \qquad \text{(by the definition of } x_\lambda \text{ for } \lambda \in \operatorname{Par}) \\ &= \prod_{n=1}^{\infty} x_n^{a_n} \qquad \text{(by (30))}. \end{aligned}$$

In other words,

$$x_{m^{-1}(a)}(\xi) = \prod_{n=1}^{\infty} \left(x_n(\xi)\right)^{a_n}.$$

Besides,

$$\begin{aligned} r_{m^{-1}(a)} &= \prod_{n=1}^{\infty} h_{m_n(m^{-1}(a))}(\xi^n) \qquad \text{(by the definition of } r_\lambda \text{ for } \lambda \in \operatorname{Par}) \\ &= \prod_{n=1}^{\infty} h_{a_n}(\xi^n) \qquad \text{(by (30))}. \end{aligned}$$

[30] The sum $\sum\limits_{\lambda \in \operatorname{Par}} x_\lambda(\xi)\, r_\lambda(\eta)\, T^{\operatorname{wt}\lambda}$ is convergent according to (15).

[31] Here, "continuous" means "continuous with respect to the (T)-adic topology on the ring $A[[T]]$".

Replacing the set I by the set J and the variables ξ_i by the variables η_j in this equation, we obtain

$$r_{m^{-1}(a)}(\eta) = \prod_{n=1}^{\infty} h_{a_n}(\eta^n)$$

(where $h_{a_n}(\eta^n)$ is defined in the same way as $h_{a_n}(\xi^n)$, but with the set I replaced by J and the variables ξ_i replaced by η_j). Thus, (51) transforms into

$$\begin{aligned}
&\sum_{\lambda \in \mathrm{Par}} x_\lambda(\xi) r_\lambda(\eta) T^{\mathrm{wt}\,\lambda} \\
&= \sum_{a \in \mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}} \underbrace{x_{m^{-1}(a)}(\xi)}_{=\prod_{n=1}^{\infty}(x_n(\xi))^{a_n}} \underbrace{r_{m^{-1}(a)}(\eta)}_{=\prod_{n=1}^{\infty} h_{a_n}(\eta^n)} \underbrace{T^{\mathrm{wt}\left(m^{-1}(a)\right)}}_{=\prod_{n=1}^{\infty}(T^n)^{a_n}} \\
&= \sum_{a \in \mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}} \prod_{n=1}^{\infty}(x_n(\xi))^{a_n} \prod_{n=1}^{\infty} h_{a_n}(\eta^n) \prod_{n=1}^{\infty}(T^n)^{a_n} \\
&= \sum_{a \in \mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}} \prod_{n=1}^{\infty}(x_n(\xi))^{a_n} h_{a_n}(\eta^n)(T^n)^{a_n} \\
&= \prod_{n=1}^{\infty}\left(\sum_{a \in \mathbb{N}} \underbrace{(x_n(\xi))^a h_a(\eta^n)(T^n)^a}_{=h_a(\eta^n)(x_n(\xi)T^n)^a}\right) \qquad \text{(by the product rule)} \\
&= \prod_{n=1}^{\infty}\left(\sum_{a \in \mathbb{N}} h_a(\eta^n)(x_n(\xi)T^n)^a\right).
\end{aligned} \tag{52}$$

Now, fix $n \in \{1,2,3,\ldots\}$. We are going to simplify the term $\sum_{a \in \mathbb{N}} h_a(\eta^n)(x_n(\xi)T^n)^a$. First, we remember that (8) yields

$$\prod_{i \in I} \frac{1}{1-\xi_i T} = \sum_{d=0}^{\infty} h_d T^d = \sum_{a=0}^{\infty} h_a T^a = \sum_{a \in \mathbb{N}} h_a T^a.$$

Replacing the variables ξ_i by the variables ξ_i^n in this equation, we obtain

$$\prod_{i \in I} \frac{1}{1-\xi_i^n T} = \sum_{a \in \mathbb{N}} h_a(\xi^n) T^a.$$

Replacing the set I by the set J and the variables ξ_i by the variables η_j in this equation, we obtain

$$\prod_{j \in J} \frac{1}{1-\eta_j^n T} = \sum_{a \in \mathbb{N}} h_a(\eta^n) T^a. \tag{53}$$

This is an equality in the ring $\left((\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]]$. According to the power series substitution rule (applied to $A = (\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty$ and $P = x_n(\xi) T^n$), there exists a continuous[32] ring homomorphism

$$\left((\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]] \to \left((\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]]$$

[32]Here, "continuous" means "continuous with respect to the (T)-adic topology on the ring $\left((\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty\right)[[T]]$".

which maps T to $x_n(\xi) T^n$ and is the identity on the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty$. This homomorphism respects infinite sums and infinite products (since it is continuous), and thus it maps $\prod_{j\in J} \dfrac{1}{1-\eta_j^n T}$ to $\prod_{j\in J} \dfrac{1}{1-\eta_j^n x_n(\xi) T^n}$ and maps $\sum_{a\in\mathbb{N}} h_a(\eta^n) T^a$ to $\sum_{a\in\mathbb{N}} h_a(\eta^n)(x_n(\xi) T^n)^a$. Therefore, upon applying this homomorphism to the equation (53), we obtain

$$\prod_{j\in J} \frac{1}{1-\eta_j^n x_n(\xi) T^n} = \sum_{a\in\mathbb{N}} h_a(\eta^n)(x_n(\xi) T^n)^a . \tag{54}$$

Now forget that we fixed n. The equality (52) becomes

$$\sum_{\lambda\in\mathrm{Par}} x_\lambda(\xi) r_\lambda(\eta) T^{\mathrm{wt}\,\lambda} = \prod_{n=1}^{\infty} \left(\underbrace{\sum_{a\in\mathbb{N}} h_a(\eta^n)(x_n(\xi) T^n)^a}_{\substack{=\prod_{j\in J} \frac{1}{1-\eta_j^n x_n(\xi) T^n} \\ \text{by (54)}}} \right)$$

$$= \prod_{n=1}^{\infty}\prod_{j\in J} \frac{1}{1-\eta_j^n x_n(\xi) T^n} = \prod_{j\in J}\prod_{n=1}^{\infty} \frac{1}{1-\eta_j^n x_n(\xi) T^n} = \prod_{j\in J}\prod_{n=1}^{\infty} \Bigg(1-\underbrace{\eta_j^n x_n(\xi) T^n}_{=x_n(\xi)(\eta_j T)^n}\Bigg)^{-1}$$

$$= \prod_{j\in J}\prod_{n=1}^{\infty} (1-x_n(\xi)(\eta_j T)^n)^{-1} = \prod_{j\in J}\prod_{d=1}^{\infty} \left(1-x_d(\xi)(\eta_j T)^d\right)^{-1} \tag{55}$$

(here we substituted d for n in the second product).

Now, fix some $j \in J$. Note that

$$\prod_{d=1}^{\infty} \left(1-x_d T^d\right)^{-1} = \prod_{i\in I} \frac{1}{1-\xi_i T} \tag{56}$$

(since

$$\prod_{d=1}^{\infty} \left(1-x_d T^d\right)^{-1} = \sum_{n=0}^{\infty} h_n T^n \qquad \text{(by (48))}$$
$$= \sum_{d=0}^{\infty} h_d T^d = \prod_{i\in I} \frac{1}{1-\xi_i T} \qquad \text{(by (8))}$$

). According to the power series substitution rule (applied to $A = (\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty$ and $P = \eta_j T$), there exists a continuous[33] ring homomorphism

$$((\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty)[[T]] \to ((\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty)[[T]]$$

[33] Here, "continuous" means "continuous with respect to the (T)-adic topology on the ring $((\mathbb{Z}[\xi_i \mid i \in I]_\infty)[\eta_j \mid j \in J]_\infty)[[T]]$".

which maps T to $\eta_j T$ and is the identity on the ring $\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty$. This homomorphism respects infinite products (since it is continuous), and thus it maps $\prod\limits_{d=1}^{\infty}\left(1-x_d T^d\right)^{-1}$ to $\prod\limits_{d=1}^{\infty}\left(1-x_d\cdot\left(\eta_j T\right)^d\right)^{-1}$ and maps $\prod\limits_{i\in I}\frac{1}{1-\xi_i T}$ to $\prod\limits_{i\in I}\frac{1}{1-\xi_i\eta_j T}$. Therefore, upon applying this homomorphism to the equation (56), we obtain

$$\prod_{d=1}^{\infty}\left(1-x_d\cdot\left(\eta_j T\right)^d\right)^{-1}=\prod_{i\in I}\frac{1}{1-\xi_i\eta_j T}.$$

In other words,

$$\prod_{d=1}^{\infty}\left(1-x_d\left(\xi\right)\left(\eta_j T\right)^d\right)^{-1}=\prod_{i\in I}\frac{1}{1-\xi_i\eta_j T}$$

(since $x_d = x_d\left(\xi\right)$). Thus, (55) becomes

$$\sum_{\lambda\in\operatorname{Par}} x_\lambda\left(\xi\right) r_\lambda\left(\eta\right) T^{\operatorname{wt}\lambda} = \prod_{j\in J}\underbrace{\prod_{d=1}^{\infty}\left(1-x_d\left(\xi\right)\left(\eta_j T\right)^d\right)^{-1}}_{=\prod_{i\in I}\frac{1}{1-\xi_i\eta_j T}} = \prod_{j\in J}\prod_{i\in I}\frac{1}{1-\xi_i\eta_j T}$$
$$=\prod_{(i,j)\in I\times J}\frac{1}{1-\xi_i\eta_j T}.$$

This proves Theorem 8.

Combining Theorem 8 with Theorem 9.42 in [1] yields the relations (9.70) in [1].

Theorem 8 can be generalized. In order to formulate this generalization, we will have to generalize Definitions 14, 15 and 17. But first, we generalize the sequence of power series $\left(h_0, h_1, h_2, ...\right)$ in such a way that we get a sequence of power series $\left(h_0^{[\alpha]}, h_1^{[\alpha]}, h_2^{[\alpha]}, ...\right)$ defined for every $\alpha\in\mathbb{Z}$ which coincides with $\left(h_0, h_1, h_2, ...\right)$ if $\alpha = 1$ and coincides with $\left((-1)^0 e_0, (-1)^1 e_1, (-1)^2 e_2, ...\right)$ if $\alpha=-1$.

Before we define this sequence $\left(h_0^{[\alpha]}, h_1^{[\alpha]}, h_2^{[\alpha]}, ...\right)$, we notice that the power series $\sum\limits_{n=0}^{\infty} h_n T^n \in \left(\mathbb{Z}\left[\xi_i \mid i\in I\right]_\infty\right)[[T]]$ is invertible (because its coefficient before T^0 is $h_0 = 1$). Hence, it makes sense to speak of $\left(\sum\limits_{n=0}^{\infty} h_n T^n\right)^{\alpha}$ for every $\alpha\in\mathbb{Z}$.

Definition 18. Let $\alpha\in\mathbb{Z}$. There exists one and only one sequence $\left(h_0^{[\alpha]}, h_1^{[\alpha]}, h_2^{[\alpha]}, ...\right)\in\left(\mathbb{Z}\left[\xi_i \mid i\in I\right]_\infty\right)^{\mathbb{N}}$ [34] of elements of $\mathbb{Z}\left[\xi_i \mid i\in I\right]_\infty$ that satisfies the equation

$$\sum_{n=0}^{\infty} h_n^{[\alpha]} T^n = \left(\sum_{n=0}^{\infty} h_n T^n\right)^{\alpha} \tag{57}$$

[34] Note that the upper index $[\alpha]$ in $h_0^{[\alpha]}$, $h_1^{[\alpha]}$, $h_2^{[\alpha]}$, ... is not an exponent. It is just an index that reminds us that the power series $h_0^{[\alpha]}$, $h_1^{[\alpha]}$, $h_2^{[\alpha]}$, ... depend upon α.

in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$. [35] This sequence will be denoted by $\left(h_0^{[\alpha]}, h_1^{[\alpha]}, h_2^{[\alpha]}, ...\right)$ henceforth until the end of this note.

Examples: **1)** The sequence $\left(h_0^{[1]}, h_1^{[1]}, h_2^{[1]}, ...\right)$ is identical to the sequence $(h_0, h_1, h_2, ...)$. This is because (57) yields

$$\sum_{n=0}^{\infty} h_n^{[1]} T^n = \left(\sum_{n=0}^{\infty} h_n T^n\right)^1 = \sum_{n=0}^{\infty} h_n T^n,$$

and comparing coefficients, we obtain that $h_n^{[1]} = h_n$ for every $n \in \mathbb{N}$, and therefore $\left(h_0^{[1]}, h_1^{[1]}, h_2^{[1]}, ...\right) = (h_0, h_1, h_2, ...)$.

2) The sequence $\left(h_0^{[-1]}, h_1^{[-1]}, h_2^{[-1]}, ...\right)$ is identical to the sequence $\left((-1)^0 e_0, (-1)^1 e_1, (-1)^2 e_2, ...\right)$. This is because (57) yields

$$\sum_{n=0}^{\infty} h_n^{[-1]} T^n = \left(\sum_{n=0}^{\infty} h_n T^n\right)^{-1} = \left(\sum_{d=0}^{\infty} h_d T^d\right)^{-1} = \left(\prod_{i \in I} \frac{1}{1-\xi_i T}\right)^{-1}$$
$$\left(\text{since (8) yields } \sum_{d=0}^{\infty} h_d T^d = \prod_{i \in I} \frac{1}{1-\xi_i T}\right)$$
$$= \prod_{i \in I} (1-\xi_i T) = \sum_{d=0}^{\infty} (-1)^d e_d T^d \qquad \text{(by (7))}$$
$$= \sum_{n=0}^{\infty} (-1)^n e_n T^n,$$

and comparing coefficients, we obtain that $h_n^{[-1]} = (-1)^n e_n$ for every $n \in \mathbb{N}$, and therefore $\left(h_0^{[-1]}, h_1^{[-1]}, h_2^{[-1]}, ...\right) = \left((-1)^0 e_0, (-1)^1 e_1, (-1)^2 e_2, ...\right)$.

3) The sequence $\left(h_0^{[0]}, h_1^{[0]}, h_2^{[0]}, ...\right)$ is identical to the sequence $\left(1, \underbrace{0, 0, ...}_{\text{only zeroes}}\right)$. This is because (57) yields

$$\sum_{n=0}^{\infty} h_n^{[0]} T^n = \left(\sum_{n=0}^{\infty} h_n T^n\right)^0 = 1,$$

and comparing coefficients, we obtain $h_n^{[0]} = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases}$ for every $n \in \mathbb{N}$, and consequently, $\left(h_0^{[0]}, h_1^{[0]}, h_2^{[0]}, ...\right) = \left(1, \underbrace{0, 0, ...}_{\text{only zeroes}}\right)$.

[35] This is clear, because $\left(\sum_{n=0}^{\infty} h_n T^n\right)^\alpha$ is a power series in the indeterminate T over the ring $\mathbb{Z}[\xi_i \mid i \in I]_\infty$.

We notice that for every $\alpha \in \mathbb{Z}$ and every $n \in \mathbb{N}$, the power series $h_n^{[\alpha]}$ is n-homogeneous.[36] Also, for every $\alpha \in \mathbb{Z}$, we have $h_0^{[\alpha]} = 1$ (since $h_0^{[\alpha]}$ is the coefficient of the power series $\left(\sum_{n=0}^{\infty} h_n T^n\right)^{\alpha}$ before T^0 (according to (57)), and the coefficient of the power series $\left(\sum_{n=0}^{\infty} h_n T^n\right)^{\alpha}$ before T^0 is 1 [37]).

Now comes a generalization of Definition 14:

Definition 19. Let $\alpha \in \mathbb{Z}$. There exists one and only one sequence $(X_1, X_2, X_3, ...) \in (\mathbb{Z}[\xi_i \mid i \in I]_\infty)^{\{1,2,3,...\}}$ of elements of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ that satisfies the equation

$$\prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \sum_{n=0}^{\infty} h_n^{[\alpha]} T^n \qquad (58)$$

in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$. [38] This sequence will be denoted by $\left(x_1^{[\alpha]}, x_2^{[\alpha]}, x_3^{[\alpha]}, ...\right)$ [39] from now on until the end of this note. Hence, this sequence $\left(x_1^{[\alpha]}, x_2^{[\alpha]}, x_3^{[\alpha]}, ...\right)$ satisfies

$$\prod_{d=1}^{\infty} \left(1 - x_d^{[\alpha]} T^d\right)^{-1} = \sum_{n=0}^{\infty} h_n^{[\alpha]} T^n \qquad (59)$$

This way, for every $\alpha \in \mathbb{Z}$, we have defined a sequence $\left(x_1^{[\alpha]}, x_2^{[\alpha]}, x_3^{[\alpha]}, ...\right)$ of power series.

Besides, we define a power series $x_0^{[\alpha]} \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ by $x_0^{[\alpha]} = 1$.

[36] In fact, using the notion of "equigraded power series" that we have introduced in the proof of Theorem 7 **(b)**, we notice that if P is an invertible equigraded power series, then P^α is an equigraded power series for every $\alpha \in \mathbb{Z}$. (For a proof of this fact, see [2], Theorem 1 **(d)**). Hence, since we know that the power series $\sum_{n=0}^{\infty} h_n T^n \in (\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$ is equigraded (because h_n lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ for every $n \in \mathbb{N}$, since h_n is an n-homogeneous power series for every $n \in \mathbb{N}$), we can conclude that the power series $\left(\sum_{n=0}^{\infty} h_n T^n\right)^{\alpha}$ is equigraded as well, and therefore $h_n^{[\alpha]}$ lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ for every $n \in \mathbb{N}$ (since (57) yields that $h_n^{[\alpha]}$ is the coefficient of the power series $\left(\sum_{n=0}^{\infty} h_n T^n\right)^{\alpha}$ before T^n). In other words, the power series $h_n^{[\alpha]}$ is n-homogeneous for every $n \in \mathbb{N}$.

[37] because in general, the coefficient of a power series $\left(\sum_{n=0}^{\infty} u_n T^n\right)^{\alpha}$ before T^0 is u_0^α, and thus the coefficient of the power series $\left(\sum_{n=0}^{\infty} h_n T^n\right)^{\alpha}$ before T^0 is $h_0^\alpha = 1^\alpha = 1$

[38] This follows from Theorem 7 **(a)**, applied to $A = \mathbb{Z}[\xi_i \mid i \in I]_\infty$ and $(\rho_0, \rho_1, \rho_2, ...) = \left(h_0^{[\alpha]}, h_1^{[\alpha]}, h_2^{[\alpha]}, ...\right)$.

[39] Note that the upper index $[\alpha]$ in $x_1^{[\alpha]}$, $x_2^{[\alpha]}$, $x_3^{[\alpha]}$, ... is not an exponent. It is just an index that reminds us that the power series $x_1^{[\alpha]}$, $x_2^{[\alpha]}$, $x_3^{[\alpha]}$, ... depend upon α.

Examples: **1)** The sequence $\left(x_1^{[0]}, x_2^{[0]}, x_3^{[0]}, ...\right)$ is identical with $(0, 0, 0, ...)$. This is because the sequence $\left(x_1^{[0]}, x_2^{[0]}, x_3^{[0]}, ...\right)$ was defined as the only sequence $(X_1, X_2, X_3, ...) \in (\mathbb{Z}[\xi_i \mid i \in I]_\infty)^{\{1,2,3,...\}}$ of elements of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ that satisfies the equation (58), but the sequence $(X_1, X_2, X_3, ...) = (0, 0, 0, ...)$ does satisfy this equation[40].

2) The sequence $\left(x_1^{[1]}, x_2^{[1]}, x_3^{[1]}, ...\right)$ is identical with the sequence $(x_1, x_2, x_3, ...)$ defined in Definition 14.[41] In other words, $x_n^{[1]} = x_n$ for every $n \in \{1, 2, 3, ...\}$. Since it is also clear that $x_0^{[1]} = x_0$, we can therefore conclude that $x_n^{[1]} = x_n$ for every $n \in \mathbb{N}$.

We notice a first property of the power series $x_0^{[\alpha]}$, $x_1^{[\alpha]}$, $x_2^{[\alpha]}$, ...: For every $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}$, the power series $x_n^{[\alpha]} \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ is homogeneous of degree n. [42]

Now, we are going to generalize Definition 15 in order to define a power series $x_\lambda^{[\alpha]}$ for every partition λ:

Definition 20. Let $\alpha \in \mathbb{Z}$. Let λ be a partition. Then, we define a power series $x_\lambda^{[\alpha]} \in \mathbb{Z}[\xi_i \mid i \in I]_\infty$ by

$$x_\lambda^{[\alpha]} = \prod_{n=1}^{\infty} \left(x_n^{[\alpha]}\right)^{m_n(\lambda)}.$$

(This is actually a finite product, since only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $\left(x_n^{[\alpha]}\right)^{m_n(\lambda)} \neq 1$, because only finitely many $n \in \{1, 2, 3, ...\}$ satisfy $m_n(\lambda) \neq 0$.) This power series $x_\lambda^{[\alpha]}$ can be written in a simpler way if we write our partition λ in the form $(\lambda_1, \lambda_2, ..., \lambda_m)$ for some $m \in \mathbb{N}$; namely,

$$\text{if } \lambda = (\lambda_1, \lambda_2, ..., \lambda_m), \text{ then } x_\lambda^{[\alpha]} = x_{\lambda_1}^{[\alpha]} x_{\lambda_2}^{[\alpha]} ... x_{\lambda_m}^{[\alpha]} \tag{60}$$

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[40]since

$$\prod_{d=1}^{\infty} \Bigg(\underbrace{1 - 0T^d}_{=1}\Bigg)^{-1} = \prod_{d=1}^{\infty} 1^{-1} = 1 = \sum_{n=0}^{\infty} h_n^{[0]} T^n$$

[41]This is because Definition 14 is the particular case of Definition 19 for $\alpha = 1$ (since $\left(h_0^{[1]}, h_1^{[1]}, h_2^{[1]}, ...\right) = (h_0, h_1, h_2, ...)$).

[42]*Proof.* Recall that $\left(x_1^{[\alpha]}, x_2^{[\alpha]}, x_3^{[\alpha]}, ...\right)$ is the unique sequence $(X_1, X_2, X_3, ...) \in (\mathbb{Z}[\xi_i \mid i \in I]_\infty)^{\{1,2,3,...\}}$ of elements of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ that satisfies the equation

$$\prod_{d=1}^{\infty} \left(1 - X_d T^d\right)^{-1} = \sum_{n=0}^{\infty} h_n^{[\alpha]} T^n$$

in the ring $(\mathbb{Z}[\xi_i \mid i \in I]_\infty)[[T]]$. Thus, Theorem 7 **(b)**, applied to $A = \mathbb{Z}[\xi_i \mid i \in I]_\infty$ and $(\rho_0, \rho_1, \rho_2, ...) = \left(h_0^{[\alpha]}, h_1^{[\alpha]}, h_2^{[\alpha]}, ...\right)$, yields that for every $n \in \{1, 2, 3, ...\}$, the element $x_n^{[\alpha]}$ lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$ (because for every $n \in \mathbb{N}$, the element $h_n^{[\alpha]}$ lies in the n-th graded component of $\mathbb{Z}[\xi_i \mid i \in I]_\infty$, since $h_n^{[\alpha]}$ is a homogeneous power series of degree n). In other words, for every $n \in \{1, 2, 3, ...\}$, the power series $x_n^{[\alpha]}$ is homogeneous of degree n. This holds for $n = 0$, as well (since $x_0^{[\alpha]} = 1$ is clearly homogeneous of degree 0), so we can conclude that for every $n \in \mathbb{N}$, the power series $x_n^{[\alpha]}$ is homogeneous of degree n, qed.

(we recall that $x_0^{[\alpha]}$ is supposed to mean 1). This is proven in the same way as we showed (10).

In particular, $x_\lambda^{[1]} = x_\lambda$ for every partition λ, because the definition of x_λ (Definition 15) is the particular case of the definition of $x_\lambda^{[\alpha]}$ for $\alpha = 1$ (since $x_n^{[1]} = x_n$ for every $n \in \mathbb{N}$).

The power series $x_\lambda^{[\alpha]}$ is wt λ-homogeneous[43].

So we have generalized Definition 15. Next, we generalize Definition 17:

Definition 21. Let $\alpha \in \mathbb{Z}$. Let λ be a partition. Then, we define a power series $r_\lambda^{[\alpha]} \in \mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty$ by

$$r_\lambda^{[\alpha]} = \prod_{n=1}^{\infty} h_{m_n(\lambda)}^{[\alpha]}\left(\xi^n\right).$$

(This product $\prod\limits_{n=1}^{\infty} h_{m_n(\lambda)}^{[\alpha]}\left(\xi^n\right)$ is a finite product, since $h_{m_n(\lambda)}^{[\alpha]}\left(\xi^n\right) = 1$ for all but finitely many $n \in \{1, 2, 3, ...\}$, since $m_n(\lambda) = 0$ for all but finitely many n, and if $m_n(\lambda) = 0$, then $\underbrace{h_{m_n(\lambda)}^{[\alpha]}}_{=h_0^{[\alpha]}=1}\left(\xi^n\right) = 1$.)

Note that $r_\lambda^{[1]} = r_\lambda$ for every partition λ. This is because the definition of r_λ (Definition 17) is the particular case of the definition of $r_\lambda^{[\alpha]}$ for $\alpha = 1$ (since $\left(h_0^{[1]}, h_1^{[1]}, h_2^{[1]}, ...\right) = \left(h_0, h_1, h_2, ...\right)$).

For every partition λ, the power series r_λ is wt λ-homogeneous.[44]

Finally, let us introduce a trivial notation:

Definition 22. Let $\alpha \in \mathbb{Z}$. For every partition $\lambda \in \text{Par}$, we denote by $h_\lambda^{[\alpha]}(\xi)$ the element $h_\lambda^{[\alpha]}$ of the ring $\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty$, and by $h_\lambda^{[\alpha]}(\eta)$ the "corresponding" element of the ring $\mathbb{Z}\left[\eta_j \mid j \in J\right]_\infty$ (that is, the power series we would obtain if we would replace the set I by the set J and the indeterminates ξ_i by the indeterminates η_j in the definition of $h_\lambda^{[\alpha]}$). Similarly, we define the power series $h_n^{[\alpha]}(\xi)$, $h_n^{[\alpha]}(\eta)$, $x_n^{[\alpha]}(\xi)$, $x_n^{[\alpha]}(\eta)$, $x_\lambda^{[\alpha]}(\xi)$, $x_\lambda^{[\alpha]}(\eta)$, $r_\lambda^{[\alpha]}(\xi)$ and $r_\lambda^{[\alpha]}(\eta)$.

Now, we will show an identity relating the power series $x_\lambda^{[\alpha]}$ and $r_\lambda^{[\alpha]}$, generalizing Theorem 8:

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[43]This is proven in the same way as we showed that h_λ is wt λ-homogeneous.

[44]In fact, for every $n \in \{1, 2, 3, ...\}$, the power series $h_{m_n(\lambda)}^{[\alpha]}$ is $m_n(\lambda)$-homogeneous, and thus the power series $h_{m_n(\lambda)}^{[\alpha]}\left(\xi^n\right)$ is $nm_n(\lambda)$-homogeneous (since for every $k \in \mathbb{N}$ and every k-homogeneous power series $\gamma \in \mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty$, the power series $\gamma\left(\xi^n\right)$ is nk-homogeneous). Hence, the product $\prod\limits_{n=1}^{\infty} h_{m_n(\lambda)}^{[\alpha]}\left(\xi^n\right)$ is $\sum\limits_{n=1}^{\infty} nm_n(\lambda)$-homogeneous. Since $\prod\limits_{n=1}^{\infty} h_{m_n(\lambda)}^{[\alpha]}\left(\xi^n\right) = r_\lambda^{[\alpha]}$ and $\sum\limits_{n=1}^{\infty} nm_n(\lambda) = \sum\limits_{k=1}^{\infty} km_k(\lambda) = \text{wt}\,\lambda$, this means that $r_\lambda^{[\alpha]}$ is wt λ-homogeneous, qed.

Theorem 9. Let $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}$. Let I and J be two countable sets. In the ring $\left(\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]]$, we have

$$\sum_{\lambda \in \mathrm{Par}} x_\lambda^{[\alpha]}(\xi)\, r_\lambda^{[\beta]}(\eta)\, T^{\mathrm{wt}\,\lambda} = \left(\prod_{(i,j) \in I \times J} \frac{1}{1 - \xi_i \eta_j T}\right)^{\alpha\beta}.$$

[45]

The proof of this theorem is completely analogous to that of Theorem 8; it was mostly copy-pasted from the latter. Again, we will use the power series substitution rule.

Proof of Theorem 9. We have

$$\sum_{\lambda \in \mathrm{Par}} x_\lambda^{[\alpha]}(\xi)\, r_\lambda^{[\beta]}(\eta)\, T^{\mathrm{wt}\,\lambda} = \sum_{a \in \mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}} x_{m^{-1}(a)}^{[\alpha]}(\xi)\, r_{m^{-1}(a)}^{[\beta]}(\eta)\, T^{\mathrm{wt}\left(m^{-1}(a)\right)} \tag{61}$$

(here, we substituted $m^{-1}(a)$ for λ, since $m : \mathrm{Par} \to \mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}$ is a bijection). Now, every $a \in \mathbb{N}_{\mathrm{fin}}^{\{1,2,3,\ldots\}}$ satisfies $T^{\mathrm{wt}\left(m^{-1}(a)\right)} = \prod_{n=1}^{\infty} (T^n)^{a_n}$ (as we have seen during the proof of Theorem 5) and furthermore

$$\begin{aligned} x_{m^{-1}(a)}^{[\alpha]} &= \prod_{n=1}^{\infty} \left(x_n^{[\alpha]}\right)^{m_n\left(m^{-1}(a)\right)} && \left(\text{by the definition of } x_\lambda^{[\alpha]} \text{ for } \lambda \in \mathrm{Par}\right) \\ &= \prod_{n=1}^{\infty} \left(x_n^{[\alpha]}\right)^{a_n} && (\text{by } (30)). \end{aligned}$$

In other words,

$$x_{m^{-1}(a)}^{[\alpha]}(\xi) = \prod_{n=1}^{\infty} \left(x_n^{[\alpha]}(\xi)\right)^{a_n}.$$

Besides,

$$\begin{aligned} r_{m^{-1}(a)}^{[\beta]} &= \prod_{n=1}^{\infty} h_{m_n(m^{-1}(a))}^{[\beta]}(\xi^n) && \left(\begin{array}{c}\text{in fact, we have } r_\lambda^{[\beta]} = \prod_{n=1}^{\infty} h_{m_n(\lambda)}^{[\beta]}(\xi^n) \text{ for every } \lambda \in \mathrm{Par}, \\ \text{according to the definition of } r_\lambda^{[\alpha]} \text{ (with } \alpha \text{ replaced by } \beta) \end{array}\right) \\ &= \prod_{n=1}^{\infty} h_{a_n}^{[\beta]}(\xi^n) && (\text{by } (30)). \end{aligned}$$

Replacing the set I by the set J and the variables ξ_i by the variables η_j in this equation, we obtain

$$r_{m^{-1}(a)}^{[\beta]}(\eta) = \prod_{n=1}^{\infty} h_{a_n}^{[\beta]}(\eta^n)$$

[45] The sum $\sum_{\lambda \in \mathrm{Par}} x_\lambda^{[\alpha]}(\xi)\, r_\lambda^{[\beta]}(\eta)\, T^{\mathrm{wt}\,\lambda}$ is convergent according to (15).

(where $h_{a_n}^{[\beta]}(\eta^n)$ is defined in the same way as $h_{a_n}^{[\beta]}(\xi^n)$, but with the set I replaced by J and the variables ξ_i replaced by η_j). Thus, (61) transforms into

$$\begin{aligned}
&\sum_{\lambda\in\operatorname{Par}} x_\lambda^{[\alpha]}(\xi)\, r_\lambda^{[\beta]}(\eta)\, T^{\operatorname{wt}\lambda} \\
&= \sum_{a\in\mathbb{N}_{\text{fin}}^{\{1,2,3,\ldots\}}} \underbrace{x_{m^{-1}(a)}^{[\alpha]}(\xi)}_{=\prod_{n=1}^{\infty}\left(x_n^{[\alpha]}(\xi)\right)^{a_n}} \underbrace{r_{m^{-1}(a)}^{[\beta]}(\eta)}_{=\prod_{n=1}^{\infty} h_{a_n}^{[\beta]}(\eta^n)} \underbrace{T^{\operatorname{wt}\left(m^{-1}(a)\right)}}_{=\prod_{n=1}^{\infty}(T^n)^{a_n}} \\
&= \sum_{a\in\mathbb{N}_{\text{fin}}^{\{1,2,3,\ldots\}}} \prod_{n=1}^{\infty}\left(x_n^{[\alpha]}(\xi)\right)^{a_n} \prod_{n=1}^{\infty} h_{a_n}^{[\beta]}(\eta^n) \prod_{n=1}^{\infty}(T^n)^{a_n} \\
&= \sum_{a\in\mathbb{N}_{\text{fin}}^{\{1,2,3,\ldots\}}} \prod_{n=1}^{\infty}\left(x_n^{[\alpha]}(\xi)\right)^{a_n} h_{a_n}^{[\beta]}(\eta^n)\,(T^n)^{a_n} \\
&= \prod_{n=1}^{\infty}\left(\sum_{a\in\mathbb{N}} \underbrace{\left(x_n^{[\alpha]}(\xi)\right)^{a} h_a^{[\beta]}(\eta^n)\,(T^n)^{a}}_{=h_a^{[\beta]}(\eta^n)\left(x_n^{[\alpha]}(\xi)T^n\right)^a}\right) \qquad \text{(by the product rule)} \\
&= \prod_{n=1}^{\infty}\left(\sum_{a\in\mathbb{N}} h_a^{[\beta]}(\eta^n)\left(x_n^{[\alpha]}(\xi)\,T^n\right)^a\right). \qquad (62)
\end{aligned}$$

Now, fix $n\in\{1,2,3,\ldots\}$. We will simplify the term $\sum_{a\in\mathbb{N}} h_a^{[\beta]}(\eta^n)\left(x_n^{[\alpha]}(\xi)\,T^n\right)^a$. First,

$$\begin{aligned}
&\prod_{i\in I}\left(\frac{1}{1-\xi_iT}\right)^{\beta} = \left(\prod_{i\in I}\frac{1}{1-\xi_iT}\right)^{\beta} = \left(\sum_{d=0}^{\infty} h_dT^d\right)^{\beta} \qquad \text{(by (8))} \\
&= \left(\sum_{n=0}^{\infty} h_nT^n\right)^{\beta} = \sum_{n=0}^{\infty} h_n^{[\beta]}T^n \\
&\qquad \left(\text{since } \sum_{n=0}^{\infty} h_n^{[\beta]}T^n = \left(\sum_{n=0}^{\infty} h_nT^n\right)^{\beta} \text{ by (57), applied to } \beta \text{ instead of } \alpha\right) \\
&= \sum_{a=0}^{\infty} h_a^{[\beta]}T^a = \sum_{a\in\mathbb{N}} h_a^{[\beta]}T^a.
\end{aligned}$$

Replacing the variables ξ_i by the variables ξ_i^n in this equation, we obtain

$$\prod_{i\in I}\left(\frac{1}{1-\xi_i^nT}\right)^{\beta} = \sum_{a\in\mathbb{N}} h_a^{[\beta]}(\xi^n)\,T^a.$$

Replacing the set I by the set J and the variables ξ_i by the variables η_j in this equation, we obtain

$$\prod_{j\in J}\left(\frac{1}{1-\eta_j^nT}\right)^{\beta} = \sum_{a\in\mathbb{N}} h_a^{[\beta]}(\eta^n)\,T^a. \qquad (63)$$

This is an equality in the ring $\left(\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]]$. According to the power series substitution rule (applied to $A = \left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty$ and $P = x_n^{[\alpha]}(\xi) T^n$), there exists a continuous[46] ring homomorphism

$$\left(\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]] \to \left(\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]]$$

which maps T to $x_n^{[\alpha]}(\xi) T^n$ and is the identity on the ring $\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty$. This homomorphism respects infinite sums and infinite products (since it is continuous), and thus it maps $\prod\limits_{j\in J}\left(\dfrac{1}{1-\eta_j^n T}\right)^\beta$ to $\prod\limits_{j\in J}\left(\dfrac{1}{1-\eta_j^n x_n^{[\alpha]}(\xi) T^n}\right)^\beta$ and maps $\sum\limits_{a\in\mathbb{N}} h_a^{[\beta]}(\eta^n) T^a$ to $\sum\limits_{a\in\mathbb{N}} h_a^{[\beta]}(\eta^n)\left(x_n^{[\alpha]}(\xi) T^n\right)^a$. Therefore, upon applying this homomorphism to the equation (63), we obtain

$$\prod_{j\in J}\left(\frac{1}{1-\eta_j^n x_n^{[\alpha]}(\xi) T^n}\right)^\beta = \sum_{a\in\mathbb{N}} h_a^{[\beta]}(\eta^n)\left(x_n^{[\alpha]}(\xi) T^n\right)^a. \tag{64}$$

Now, forget that we fixed n. The equality (62) becomes

$$\begin{aligned}
\sum_{\lambda\in\mathrm{Par}} x_\lambda^{[\alpha]}(\xi) r_\lambda^{[\beta]}(\eta) T^{\mathrm{wt}\,\lambda} &= \prod_{n=1}^{\infty}\left(\underbrace{\sum_{a\in\mathbb{N}} h_a^{[\beta]}(\eta^n)\left(x_n^{[\alpha]}(\xi) T^n\right)^a}_{\substack{=\prod_{j\in J}\left(\frac{1}{1-\eta_j^n x_n^{[\alpha]}(\xi) T^n}\right)^\beta \\ \text{by (64)}}}\right) \\
&= \prod_{n=1}^{\infty}\prod_{j\in J}\left(\frac{1}{1-\eta_j^n x_n^{[\alpha]}(\xi) T^n}\right)^\beta = \prod_{j\in J}\prod_{n=1}^{\infty}\left(\frac{1}{1-\eta_j^n x_n^{[\alpha]}(\xi) T^n}\right)^\beta = \prod_{j\in J}\left(\prod_{n=1}^{\infty}\frac{1}{1-\eta_j^n x_n^{[\alpha]}(\xi) T^n}\right)^\beta \\
&= \prod_{j\in J}\left(\prod_{n=1}^{\infty}\left(1-\underbrace{\eta_j^n x_n^{[\alpha]}(\xi) T^n}_{=x_n^{[\alpha]}(\xi)(\eta_j T)^n}\right)^{-1}\right)^\beta \\
&= \prod_{j\in J}\left(\prod_{n=1}^{\infty}\left(1-x_n^{[\alpha]}(\xi)(\eta_j T)^n\right)^{-1}\right)^\beta = \prod_{j\in J}\left(\prod_{d=1}^{\infty}\left(1-x_d^{[\alpha]}(\xi)(\eta_j T)^d\right)^{-1}\right)^\beta \\
&\qquad \text{(here we substituted } d \text{ for } n \text{ in the second product)}.
\end{aligned} \tag{65}$$

Now, fix some $j \in J$. Note that

$$\prod_{d=1}^{\infty}\left(1-x_d^{[\alpha]} T^d\right)^{-1} = \prod_{i\in I}\left(\frac{1}{1-\xi_i T}\right)^\alpha \tag{66}$$

[46] Here, "continuous" means "continuous with respect to the (T)-adic topology on the ring $\left(\left(\mathbb{Z}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]]$".

(since

$$\prod_{d=1}^{\infty}\left(1-x_d^{[\alpha]}T^d\right)^{-1}=\sum_{n=0}^{\infty}h_n^{[\alpha]}T^n \qquad \text{(by (59))}$$
$$=\left(\sum_{n=0}^{\infty}h_nT^n\right)^{\alpha} \qquad \text{(by (57))}$$
$$=\left(\sum_{d=0}^{\infty}h_dT^d\right)^{\alpha}=\left(\prod_{i\in I}\frac{1}{1-\xi_iT}\right)^{\alpha} \qquad \left(\text{since } \sum_{d=0}^{\infty}h_dT^d=\prod_{i\in I}\frac{1}{1-\xi_iT} \text{ by (8)}\right)$$
$$=\prod_{i\in I}\left(\frac{1}{1-\xi_iT}\right)^{\alpha}$$

). According to the power series substitution rule (applied to $A=\left(\mathbb{Z}\left[\xi_i \mid i\in I\right]_{\infty}\right)\left[\eta_j \mid j\in J\right]_{\infty}$ and $P=\eta_jT$), there exists a continuous[47] ring homomorphism

$$\left(\left(\mathbb{Z}\left[\xi_i \mid i\in I\right]_{\infty}\right)\left[\eta_j \mid j\in J\right]_{\infty}\right)[[T]]\to\left(\left(\mathbb{Z}\left[\xi_i \mid i\in I\right]_{\infty}\right)\left[\eta_j \mid j\in J\right]_{\infty}\right)[[T]]$$

which maps T to η_jT and is the identity on the ring $\left(\mathbb{Z}\left[\xi_i \mid i\in I\right]_{\infty}\right)\left[\eta_j \mid j\in J\right]_{\infty}$. This homomorphism respects infinite products (since it is continuous), and thus it maps $\prod\limits_{d=1}^{\infty}\left(1-x_d^{[\alpha]}T^d\right)^{-1}$ to $\prod\limits_{d=1}^{\infty}\left(1-x_d^{[\alpha]}\cdot\left(\eta_jT\right)^d\right)^{-1}$ and maps $\prod\limits_{i\in I}\left(\frac{1}{1-\xi_iT}\right)^{\alpha}$ to $\prod\limits_{i\in I}\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha}$. Therefore, upon applying this homomorphism to the equation (66), we obtain

$$\prod_{d=1}^{\infty}\left(1-x_d^{[\alpha]}\cdot\left(\eta_jT\right)^d\right)^{-1}=\prod_{i\in I}\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha}.$$

In other words,

$$\prod_{d=1}^{\infty}\left(1-x_d^{[\alpha]}\left(\xi\right)\left(\eta_jT\right)^d\right)^{-1}=\prod_{i\in I}\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha} \qquad (67)$$

(since $x_d^{[\alpha]}=x_d^{[\alpha]}\left(\xi\right)$). Thus, (65) becomes

$$\sum_{\lambda\in\operatorname{Par}}x_{\lambda}^{[\alpha]}\left(\xi\right)r_{\lambda}^{[\beta]}\left(\eta\right)T^{\operatorname{wt}\lambda}=\prod_{j\in J}\left(\underbrace{\prod_{d=1}^{\infty}\left(1-x_d^{[\alpha]}\left(\xi\right)\left(\eta_jT\right)^d\right)^{-1}}_{\substack{=\prod_{i\in I}\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha} \\ \text{(by (67))}}}\right)^{\beta}=\prod_{j\in J}\left(\prod_{i\in I}\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha}\right)^{\beta}$$
$$=\prod_{j\in J}\prod_{i\in I}\left(\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha}\right)^{\beta}=\prod_{(i,j)\in I\times J}\left(\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha}\right)^{\beta}$$
$$=\prod_{(i,j)\in I\times J}\left(\frac{1}{1-\xi_i\eta_jT}\right)^{\alpha\beta}.$$

[47]Here, "continuous" means "continuous with respect to the (T)-adic topology on the ring $\left(\left(\mathbb{Z}\left[\xi_i \mid i\in I\right]_{\infty}\right)\left[\eta_j \mid j\in J\right]_{\infty}\right)[[T]]$".

This proves Theorem 9.

Theorem 8 is the particular case of Theorem 9 for $\alpha = \beta = 1$ (since $x_\lambda^{[1]} = x_\lambda$ and $r_\lambda^{[1]} = r_\lambda$ for every partition λ). Using Theorem 9.42 in [1], we can use Theorem 8 to conclude that $\langle x_\lambda, r_\kappa \rangle = \delta_{\lambda,\kappa}$ for any two partitions λ and κ (where $\langle \cdot, \cdot \rangle$ denotes the Hall inner product, defined in [1], 9.40). In the same way, we can use Theorem 9 to conclude that $\left\langle x_\lambda^{[-1]}, r_\kappa^{[-1]} \right\rangle = \delta_{\lambda,\kappa}$ for any two partitions λ and κ.

We can also generalize Theorem 9 by replacing $\mathbb{Z}$ by any binomial ring (see [1], 17.19 for the definition of a binomial ring). The reason why we need the ring to be binomial is that otherwise, $\left(\sum\limits_{n=0}^{\infty} h_n T^n\right)^\alpha$ would not be well-defined (we can define the α-th power of a power series only if the binomial coefficients $\binom{\alpha}{k}$ exist in our ring), and thus $h_n^{[\alpha]}$ and $x_n^{[\alpha]}$ would not be well-defined either.

References

[1] Michiel Hazewinkel, *Witt vectors. Part 1*, revised version: 20 April 2008.
`http://arxiv.org/abs/0804.3888v1`

[2] Darij Grinberg: *Witt#4a: Equigraded power series.*
`http://www.cip.ifi.lmu.de/~grinberg/algebra/witt4a.pdf`