The order of birational rowmotion

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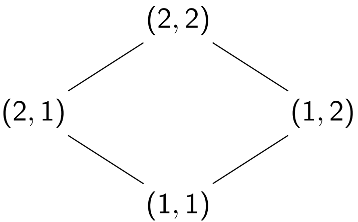
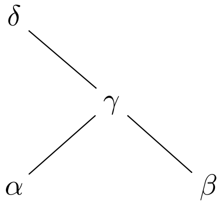
slides:
http://mit.edu/~darij/www/algebra/vienna2014.pdf
paper: http://mit.edu/~darij/www/algebra/skeletal.pdf
or arXiv:1402.6178v3

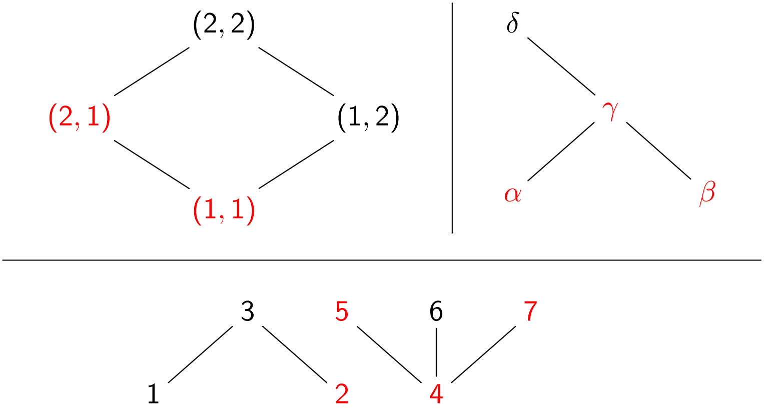
- A **poset** (= partially ordered set) is a set P with a reflexive, transitive and antisymmetric relation.
- We use the symbols $<$, $\leq$, $>$ and $\geq$ accordingly.
- We draw posets as Hasse diagrams:





- We only care about finite posets here.
- We say that $u \in P$ **is covered by** $v \in P$ (written $u \lessdot v$) if we have $u < v$ and there is no $w \in P$ satisfying $u < w < v$.
- We say that $u \in P$ **covers** $v \in P$ (written $u \gtrdot v$) if we have $u > v$ and there is no $w \in P$ satisfying $u > w > v$.

- An **order ideal** of a poset P is a subset S of P such that if $v \in S$ and $w \leq v$, then $w \in S$.
- Examples (the elements of the order ideal are marked in red):



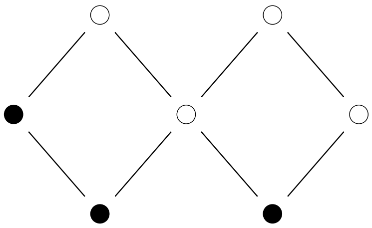
- We let $J(P)$ denote the set of all order ideals of P.

- **Classical rowmotion** is the rowmotion studied by Striker-Williams (`arXiv:1108.1172`). It has appeared many times before, under different guises:
  - Brouwer-Schrijver (1974) (as a permutation of the antichains),
  - Fon-der-Flaass (1993) (as a permutation of the antichains),
  - Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
  - Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).

- Let P be a finite poset.
  Classical rowmotion is the map $\mathbf{r} : J(P) \to J(P)$ which sends every order ideal S to the order ideal obtained as follows:
  Let M be the set of minimal elements of the complement $P \setminus S$.
  Then, $\mathbf{r}(S)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

Example:

Let S be the following order ideal ($\bullet$ = inside order ideal):



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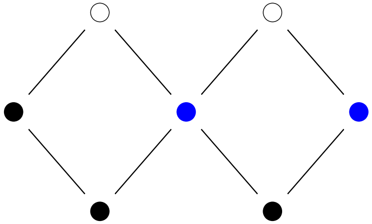
Example:
Mark M (= minimal elements of complement) blue.



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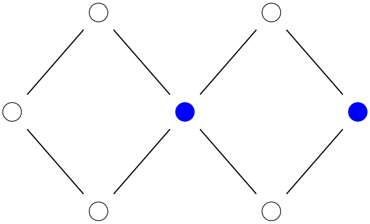
Example:

Forget about the old order ideal:



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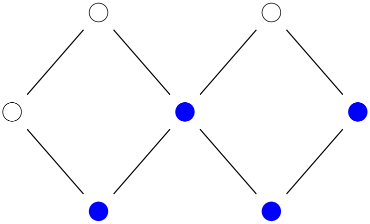
Example:
$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):



Classical rowmotion: properties

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

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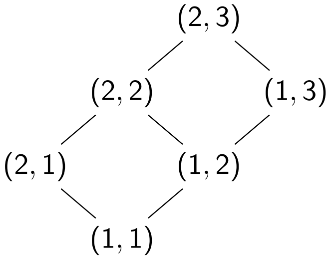
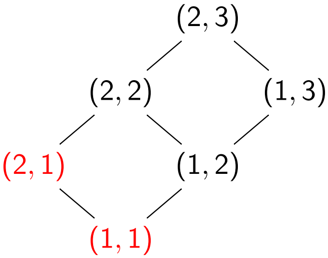
However, **for some types of** P, the order can be explicitly computed or bounded from above.
See Striker-Williams for an exposition of known results.

- If P is a $p \times q$-rectangle:

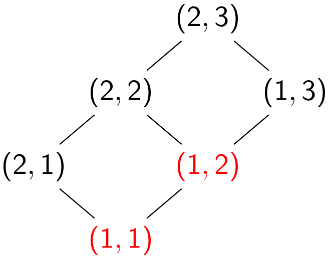


(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

Example:
Let S be the order ideal of the 2×3-rectangle given by:

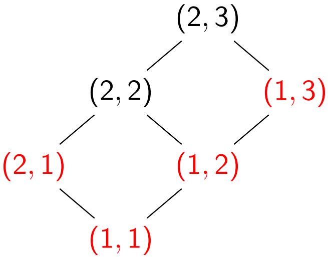


Example:
$\mathbf{r}(S)$ is



Example:

$\mathbf{r}^2(S)$ is



Example:

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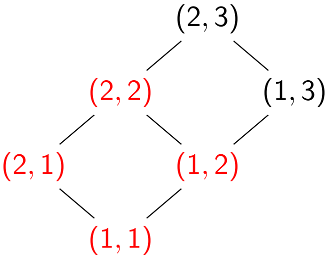
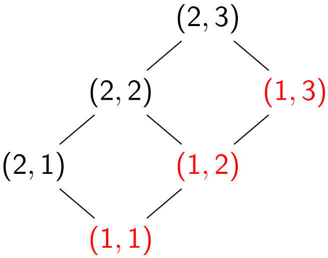
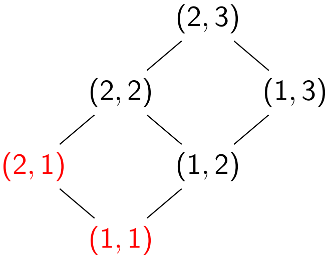


Example:
$\mathbf{r}^4(S)$ is



Example:

$\mathbf{r}^5(S)$ is

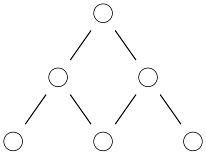


which is precisely the S we started with.

$\operatorname{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$

Further posets for which classical rowmotion has small order:
(Still see Striker-Williams for references.)

- If P is a Δ-shaped triangle with sidelength $p-1$:



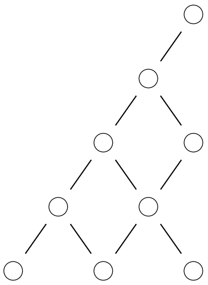
(shown here for $p = 4$), then $\operatorname{ord}(\mathbf{r}) = 2p$ (if $p > 2$).

- In this case, $\mathbf{r}^p$ is "reflection in the y-axis" (i.e., the central vertical axis).

Classical rowmotion: properties

Yet further posets for which classical rowmotion has small order: (Still see Striker-Williams for references.)

- If P is the poset of all positive roots of a finite Weyl group W, then $\mathbf{r}^{2h} = \text{id}$, where h is the Coxeter number of W. (Armstrong-Stump-Thomas, `arXiv:1101.1277v2`.)
- This includes the triangles from previous slide, but also these kind of beasts:



(for B_3).

There is an alternative definition of classical rowmotion, which splits it into many little steps.

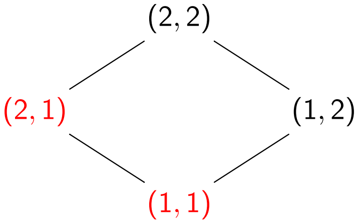
- If P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Simpler way to state this: $\mathbf{t}_v(S)$ is:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - S otherwise.

  ("Try to add or remove v from S; if this breaks the order ideal axiom, leave S fixed.")

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

Example:
Start with this order ideal S:



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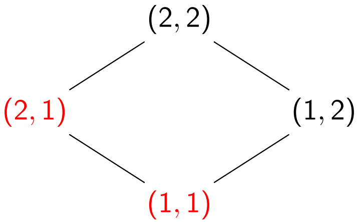
Example:
First apply $\mathbf{t}_{(2,2)}$, which changes nothing:



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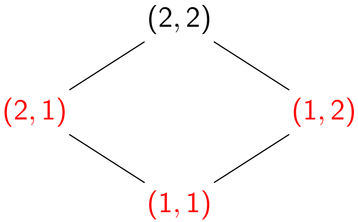
Then apply $\mathbf{t}_{(1,2)}$, which adds $(1,2)$ to the order ideal:



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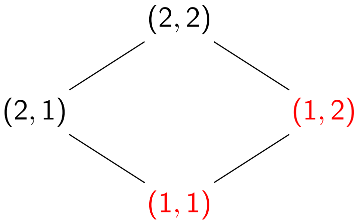
Example:

Then apply $\mathbf{t}_{(2,1)}$, which removes $(2,1)$ from the order ideal:



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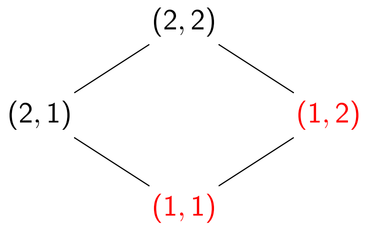

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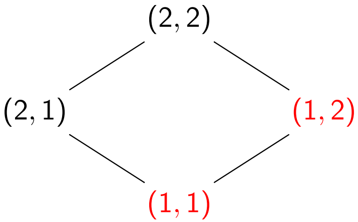
Example:
So this is $\mathbf{r}(S)$:

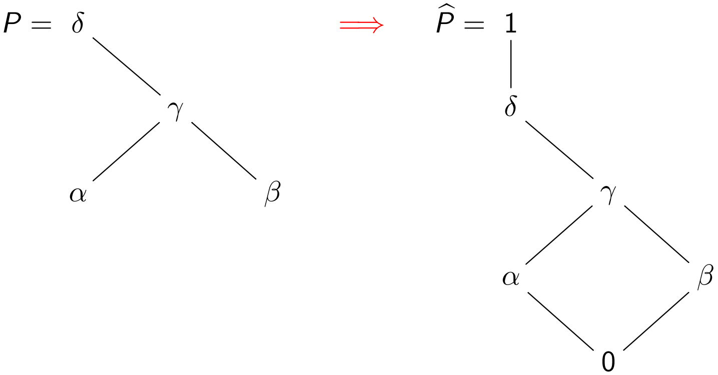


Goals of the talk

- define **birational rowmotion** (a generalization of classical rowmotion introduced by David Einstein and James Propp, based on ideas of Anatol Kirillov and Arkady Berenstein).
- show how some properties of classical rowmotion generalize to birational rowmotion.
- ask some questions and state some conjectures.

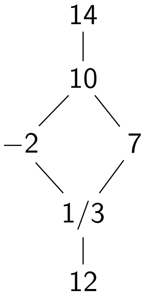
- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to P and forcing
  - 0 to be less than every other element, and
  - 1 to be greater than every other element.

Example:



- Let $\mathbb{K}$ be a semifield (i.e., a field minus "minus").
- A $\mathbb{K}$-**labelling of** P will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

Example: This is a $\mathbb{Q}$-labelling of the 2×2-rectangle:



- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

  for all $w \in \widehat{P}$.

- That is,
  - **invert** the label at v,
  - **multiply** it with the **sum** of the labels at vertices **covered by** v,
  - **multiply** it with the **harmonic sum** of the labels at vertices **covering** v.

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  for all $w \in \widehat{P}$.

- Notice that this is a **local change** to the label at v; all other labels stay the same.
- We have $T_v^2 = \text{id}$ (on the range of T_v), and T_v is a birational map.

- We define **birational rowmotion** as the rational map

$$R := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent on the linear extension, because:

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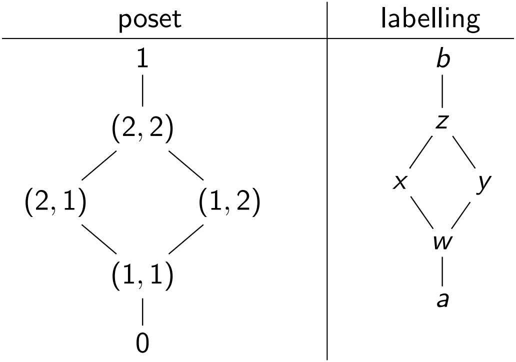
where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.

- This is indeed independent on the linear extension, because:
  - T_v and T_w commute whenever v and w are incomparable (or just don't cover each other);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



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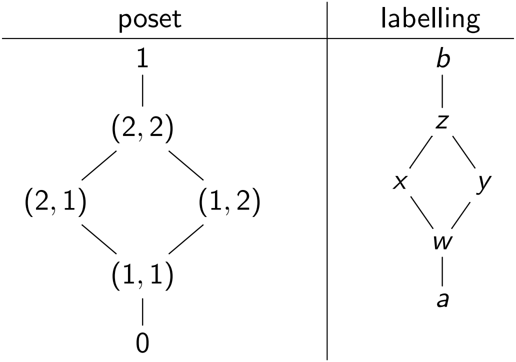
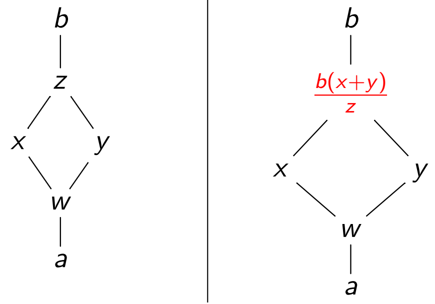


We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1),(1,2),(2,1),(2,2))$).
That is, toggle in the order "top, left, right, bottom".

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(2,2)} f$



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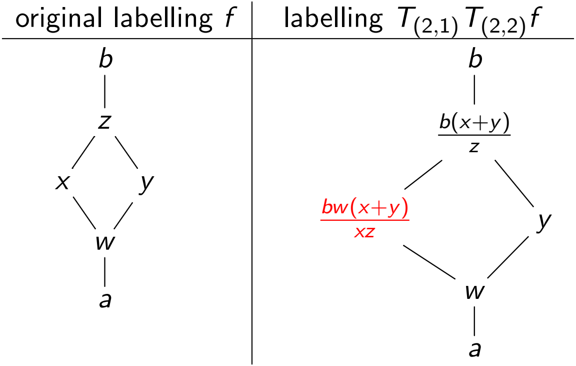

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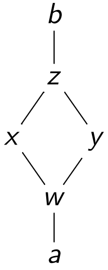	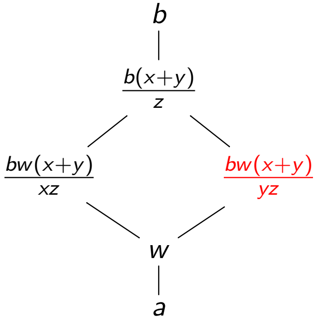

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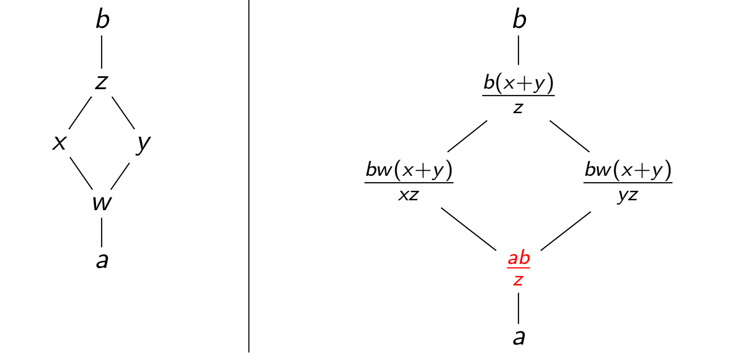

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Birational rowmotion: motivation

- Why is this called birational rowmotion?
- Indeed, it generalizes classical rowmotion:
  - Let $\operatorname{Trop}\mathbb{Z}$ be the **tropical semiring** over $\mathbb{Z}$. This is the set $\mathbb{Z} \cup \{-\infty\}$ with "addition" $(a, b) \mapsto \max\{a, b\}$ and "multiplication" $(a, b) \mapsto a + b$. This is a semifield.

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  - To every order ideal $S \in J(P)$, assign a $\operatorname{Trop}\mathbb{Z}$-labelling $\operatorname{tlab} S$ defined by

$$(\operatorname{tlab} S)(v) = \begin{cases} 1, & \text{if } v \notin S \cup \{0\}; \\ 0, & \text{if } v \in S \cup \{0\} \end{cases}.$$

  - Easy to see:

$$T_v \circ \operatorname{tlab} = \operatorname{tlab} \circ \mathbf{t}_v, \qquad R \circ \operatorname{tlab} = \operatorname{tlab} \circ \mathbf{r}.$$

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    (And tlab is injective.)
  - If you don't like semifields, use $\mathbb{Q}$ and take the "tropical limit".

- Let $\operatorname{ord}\phi$ denote the order of a map or rational map ϕ. This is the smallest positive integer k such that $\phi^k = \operatorname{id}$ (on the range of ϕ^k), or ∞ if no such k exists.
- The above shows that $\operatorname{ord}(\mathbf{r}) \mid \operatorname{ord}(R)$ for every finite poset P.
- Do we have equality?

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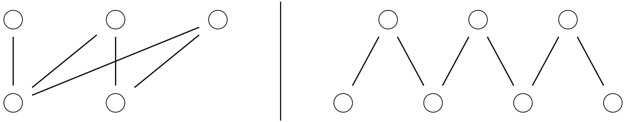
- Do we have equality?
  No! Here are two posets with $\operatorname{ord}(R) = \infty$:



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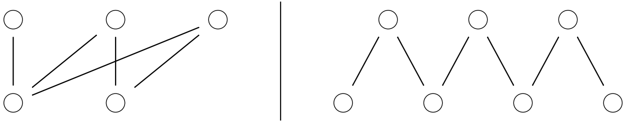
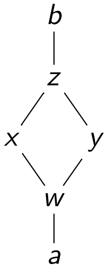


- **Nevertheless**, equality holds for many special types of P.

Example:
Iteratively apply R to a labelling of the 2×2-rectangle.
$R^0 f =$



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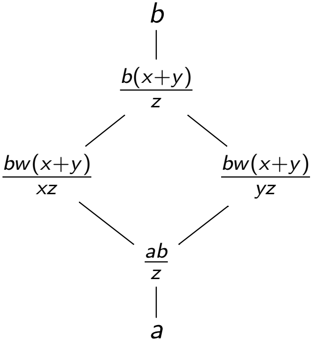

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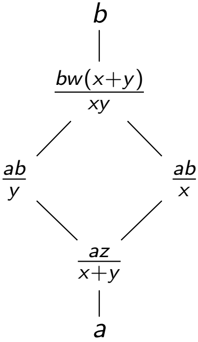

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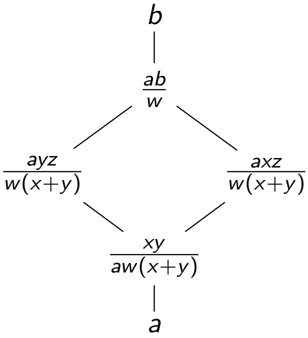

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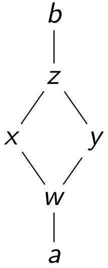

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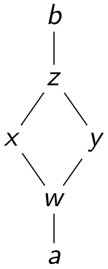


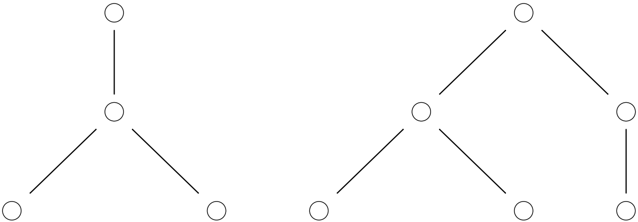
So we are back where we started.

$$\operatorname{ord}(R) = 4.$$

- **Theorem.** Assume that $n \in \mathbb{N}$, and P is a poset which is a forest (made into a poset using the "descendant" relation) having all leaves on the same level n (i.e., each maximal chain of P has n vertices). Then,

$$\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, ..., n+1) .$$

Example:
This poset



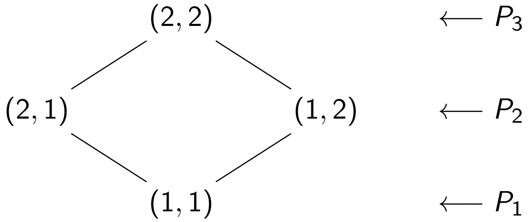
has $\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, 3, 4) = 12$.

- Even the $\operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, ..., n+1)$ part of this result seems to be new.
- We will very roughly sketch a proof of $\operatorname{ord}(R) \mid \operatorname{lcm}(1, 2, ..., n+1)$. Details are in the "Skeletal posets" section of our paper, where we also generalize the result to a wider class of posets we call "skeletal posets". (These can be regarded as a generalization of forests where we are allowed to graft existing forests on roots on the top and on the bottom, and to use antichains instead of roots. An example is the 2×2-rectangle.)

Birational rowmotion: n-graded posets

- Consider any n-**graded** finite poset P. This means that P is partitioned into nonempty subsets P_1, P_2, ..., P_n such that:
  - If $u \in P_i$ and $u \lessdot v$, then $v \in P_{i+1}$.
  - All minimal elements of P are in P_1.
  - All maximal elements of P are in P_n.

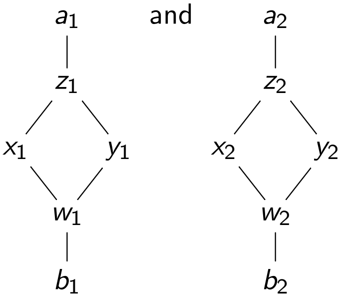
Example: The 2×2-rectangle is a 3-graded poset:



- Two $\mathbb{K}$-labellings f and g of P are said to be **homogeneously equivalent** if there is a $(\lambda_1, \lambda_2, ..., \lambda_n) \in (\mathbb{K} \setminus 0)^n$ such that

$$g(v) = \lambda_i f(v) \qquad \text{for all } i \text{ and all } v \in P_i.$$

Example: These two labellings:



are homogeneously equivalent if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$.

Birational rowmotion: homogeneous equivalence and R

- Let $\overline{\mathbb{K}^{\widehat{P}}}$ denote the set of all $\mathbb{K}$-labellings of P (with no zero labels) modulo homogeneous equivalence.
  Let $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ be the canonical projection.
- There exists a rational map $\overline{R} : \overline{\mathbb{K}^{\widehat{P}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{K}^{\widehat{P}} & \overset{R}{\dashrightarrow} & \mathbb{K}^{\widehat{P}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \overline{\mathbb{K}^{\widehat{P}}} & \underset{\overline{R}}{\dashrightarrow} & \overline{\mathbb{K}^{\widehat{P}}} \end{array}$$

  commutes.
- Hence $\operatorname{ord}\left(\overline{R}\right) \mid \operatorname{ord}(R)$.

Birational rowmotion: interplay between R and $\overline{R}$

- But in fact, any n-graded poset P satisfies

$$\operatorname{ord}(R) = \operatorname{lcm}\left(n+1, \operatorname{ord}\left(\overline{R}\right)\right).$$

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- Furthermore, if P and Q are both n-graded, then the disjoint union PQ of P and Q satisfies

$$\operatorname{ord}\left(R_{PQ}\right) = \operatorname{ord}\left(\overline{R}_{PQ}\right) = \operatorname{lcm}\left(\operatorname{ord}\left(R_P\right), \operatorname{ord}\left(R_Q\right)\right)$$

(where R_S means the R defined for a poset S).

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- Finally, if P is n-graded, and $B_1'P$ denotes the $(n+1)$-graded poset obtained by adding a new element on top of P (such that it is greater than all existing elements of P), then

$$\operatorname{ord}\left(\overline{R}_{B_1'P}\right) = \operatorname{ord}\left(\overline{R}_P\right).$$

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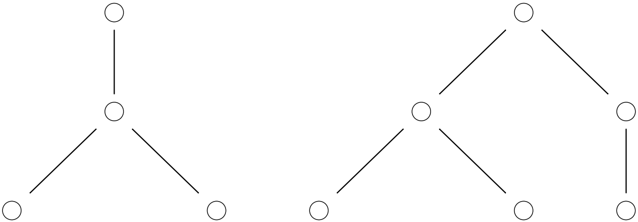
- Combining these, we can inductively compute $\operatorname{ord}\left(R_P\right)$ and $\operatorname{ord}\left(\overline{R}_P\right)$ for any n-graded forest P, and prove $\operatorname{ord}(R) \mid \operatorname{lcm}\left(1, 2, ..., n+1\right)$.

Example:

Here is how we can get our forest poset using the PQ and $B_1'P$ constructions from $\varnothing$:



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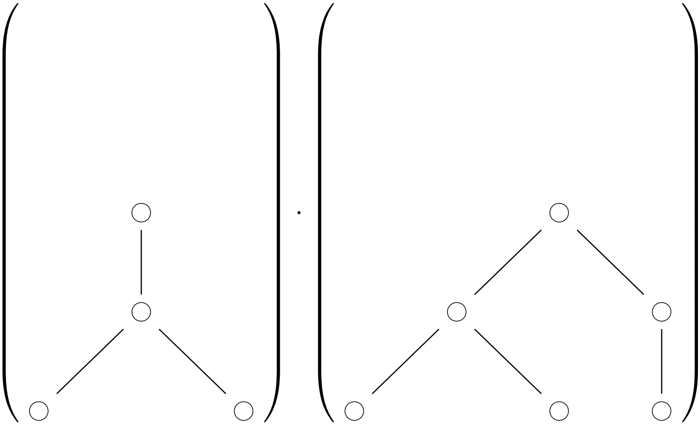

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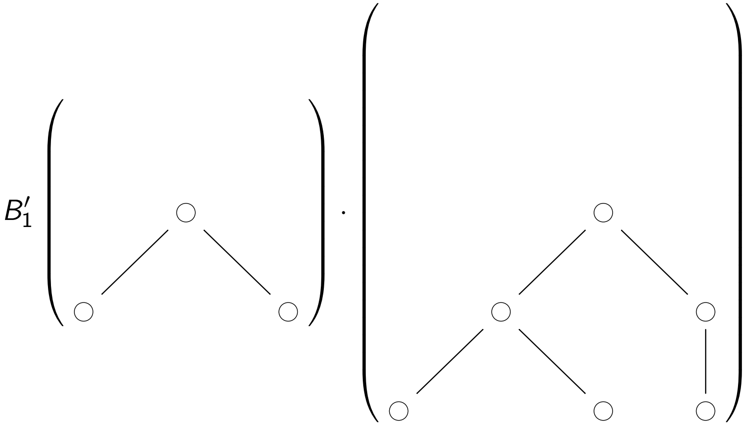

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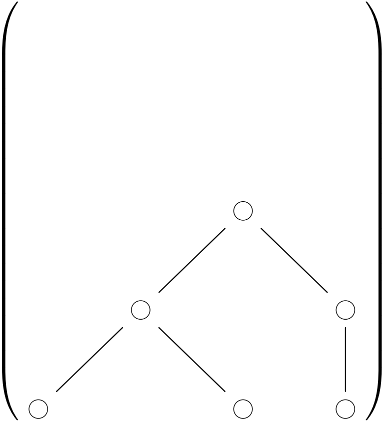
Here is how we can get our forest poset using the PQ and $B_1'P$ constructions from $\varnothing$:

$$B_1' \left(B_1' \left(\circ \qquad \circ \right) \right) \cdot$$



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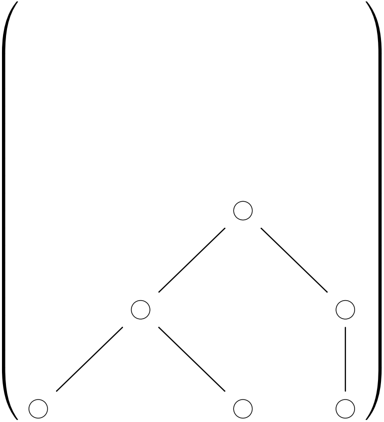

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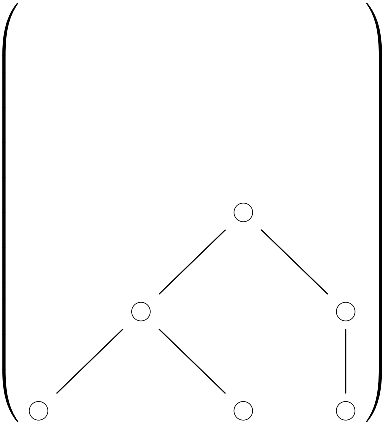

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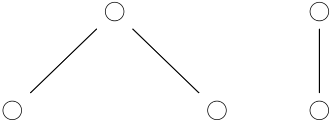

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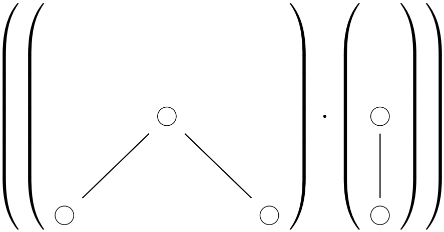
$B_1'\left(B_1'\left(\left(B_1'\varnothing\right)\cdot\left(B_1'\varnothing\right)\right)\right)\cdot B_1'$ 

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Classical rowmotion: the graded forest case

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- This can be done by "tropicalizing" the notions of homogeneous equivalence, π and $\overline{R}$. Details in the "Interlude" section of our paper.
- Actually, not as much tropicalizing as booleanizing: we only use the boolean semiring $\{0, 1\}$ to get classical rowmotion. With the full force of the tropical semiring we get **more** (see later)!

- **Theorem.** Assume that $n \in \mathbb{N}$, and P is a poset which is a forest (made into a poset using the "descendant" relation) having all leaves on the same level n (i.e., each maximal chain of P has n vertices). Then,

$$\begin{aligned}\operatorname{ord}(\overline{R}) &= \operatorname{ord}(\bar{\mathbf{r}}) \\ &= \operatorname{lcm}\left\{ n - i \mid i \in \{0, 1, \ldots, n-1\} ; \ \left|\widehat{P}_i\right| < \left|\widehat{P}_{i+1}\right| \right\},\end{aligned}$$

where $\widehat{P}_k$ denotes the set of elements of $\widehat{P}$ which are a distance of k away from 0.

Birational rowmotion: the rectangle case

- **Theorem (periodicity):** If P is the $p \times q$-rectangle (i.e., the poset $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$ with coordinatewise order), then
$$\operatorname{ord}(R) = p + q.$$

Example: For the 2×2-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

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Example: For the 2×2-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

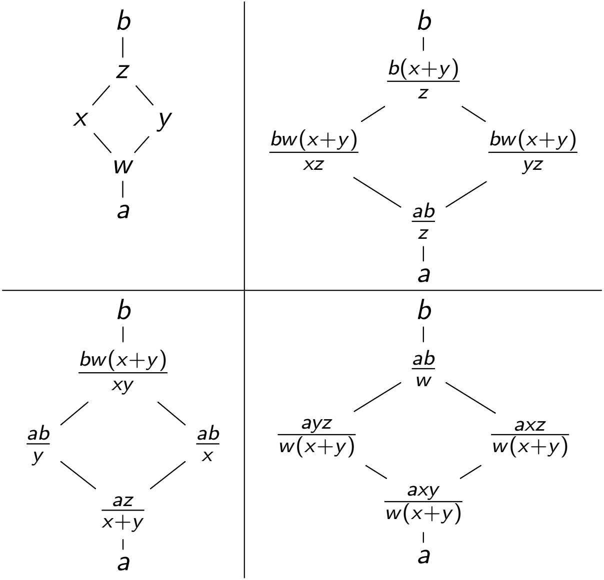
- **Theorem (reciprocity):** If P is the $p \times q$-rectangle, and $(i, k) \in P$ and $f \in \mathbb{K}^{\widehat{P}}$, then

$$f\left(\underbrace{(p+1-i, q+1-k)}_{\substack{=\text{antipode of } (i,k) \\ \text{in the rectangle}}}\right) = \frac{f(0) f(1)}{\left(R^{i+k-1} f\right)((i, k))}.$$

- These were conjectured by James Propp and Tom Roby.

Birational rowmotion: the rectangle case, example

Example: Here is the generic R-orbit on the 2×2-rectangle again:



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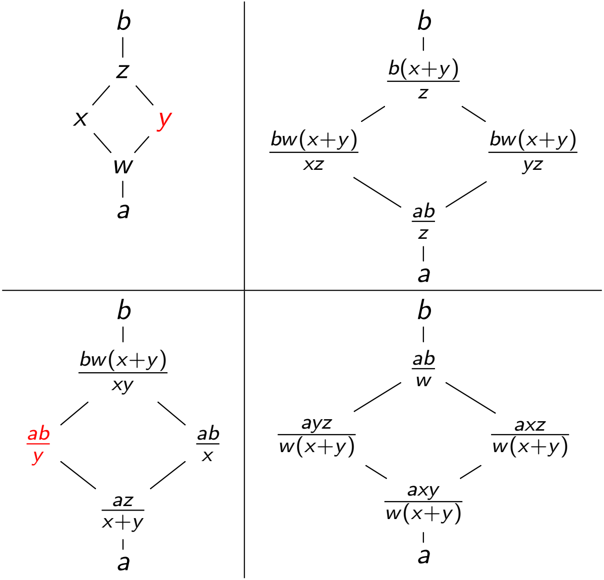

Birational rowmotion: the rectangle case, example

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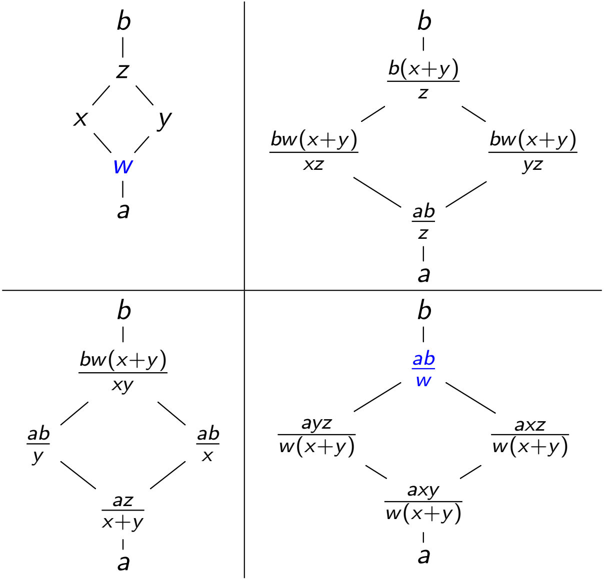

- Inspiration: Alexandre Yu. Volkov, *On Zamolodchikov's Periodicity Conjecture*, `arXiv:hep-th/0606094`.
- We will give only a very vague idea of the proof.
- We WLOG assume that $\mathbb{K}$ is a field. (Everything is polynomial identities.)

- Let $A \in \mathbb{K}^{p \times (p+q)}$ be a matrix with p rows and $p+q$ columns.
- Let A_i be the i-th column of A. Extend to all $i \in \mathbb{Z}$ by setting

$$A_{p+q+i} = (-1)^{p-1} A_i \qquad \text{for all } i.$$

- Let $A[a : b \mid c : d]$ be the matrix whose columns are A_a, A_{a+1}, ..., A_{b-1}, A_c, A_{c+1}, ..., A_{d-1} from left to right.

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- Let $A[a : b \mid c : d]$ be the matrix whose columns are $A_a, A_{a+1}, \ldots, A_{b-1}, A_c, A_{c+1}, \ldots, A_{d-1}$ from left to right.
- For every $j \in \mathbb{Z}$, we define a $\mathbb{K}$-labelling $\operatorname{Grasp}_j A \in \mathbb{K}^{\widehat{P}}$ by

$$\left(\operatorname{Grasp}_j A\right)((i,k)) = \frac{\det\left(A[j+1 : j+i \mid j+i+k-1 : j+p+k]\right)}{\det\left(A[j : j+i \mid j+i+k : j+p+k]\right)}$$

for every $(i,k) \in P$ (this is well-defined for a Zariski-generic A) and $\left(\operatorname{Grasp}_j A\right)(0) = \left(\operatorname{Grasp}_j A\right)(1) = 1$.

- The proof of $\operatorname{ord}(R) = p + q$ now rests on four claims:
    - **Claim 1:** $\operatorname{Grasp}_j A = \operatorname{Grasp}_{p+q+j} A$ for all j and A.
    - **Claim 2:** $R\left(\operatorname{Grasp}_j A\right) = \operatorname{Grasp}_{j-1} A$ for all j and A.
    - **Claim 3:** For almost every $f \in \mathbb{K}^{\widehat{P}}$ satisfying $f(0) = f(1) = 1$, there exists a matrix $A \in \mathbb{K}^{p\times(p+q)}$ such that $\operatorname{Grasp}_0 A = f$.
    - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- Claim 1 is immediate from the definitions.

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  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- Claim 2 is a computation with determinants, which boils down to the three-term Plücker identities:

$$\det\left(A\left[a-1:b \mid c:d+1\right]\right)\cdot\det\left(A\left[a:b+1 \mid c-1:d\right]\right)$$
$$+\det\left(A\left[a:b \mid c-1:d+1\right]\right)\cdot\det\left(A\left[a-1:b+1 \mid c:d\right]\right)$$
$$=\det\left(A\left[a-1:b \mid c-1:d\right]\right)\cdot\det\left(A\left[a:b+1 \mid c:d+1\right]\right).$$

for $A \in \mathbb{K}^{u \times v}$ and $a \leq b$ and $c \leq d$ and $b - a + d - c = u - 2$.

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  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.
- Claim 3 is an annoying (nonlinear) triangularity argument: With the ansatz $A = (I_p \mid B)$ for $B \in \mathbb{K}^{p \times q}$, the equation $\operatorname{Grasp}_0 A = f$ translates into a system of equations in the entries of B which can be solved by elimination.

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  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- Claim 4 follows by recalling $\operatorname{ord}(R) = \operatorname{lcm}\left(n+1, \operatorname{ord}\left(\overline{R}\right)\right)$.

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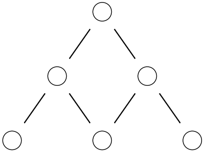
  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- The reciprocity statement can be proven in a similar vein.

- **Theorem (periodicity):** If P is the triangle $\Delta(p) = \{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i+k > p+1\}$ with $p > 2$, then
$$\operatorname{ord}(R) = 2p.$$

Example: The triangle $\Delta(4)$:



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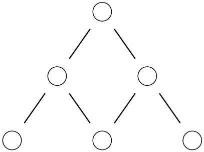

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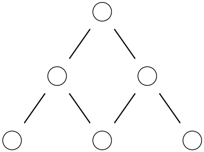

- **Theorem (reciprocity):** R^p reflects any $\mathbb{K}$-labelling across the vertical axis.
- These are precisely the same results as for classical rowmotion.
- The proofs use a "folding"-style argument to reduce this to the rectangle case.

Birational rowmotion: the $\triangleright$-triangle case

- **Theorem (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k\}$, then

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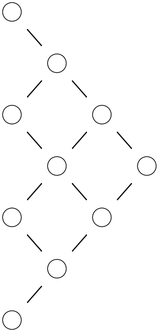
Example: For $p = 4$, this P has the form:



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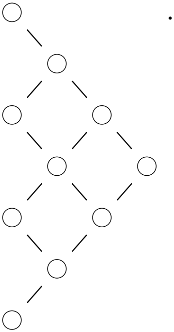

- Again this is reduced to the rectangle case.

- **Conjecture (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ i+k > p+1\}$, then
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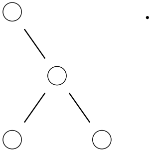

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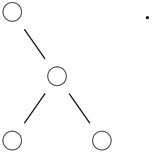


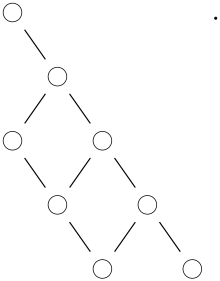
- We proved this for p odd.
- Note that for p even, this is a type-B positive root poset. Armstrong-Stump-Thomas did this for classical rowmotion.

Birational rowmotion: the trapezoid case (Nathan Williams)

- **Conjecture (periodicity):** If P is the trapezoid $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ i+k > p+1;\ k \geq s\}$ for some $0 \leq s \leq p$, then

$$\operatorname{ord}(R) = p.$$

Example: For $p = 6$ and $s = 5$, this P has the form:



- This was observed by Nathan Williams and verified for $p \leq 7$.
- Motivation comes from Williams's "Cataland" philosophy.

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Birational rowmotion: the root system connection (Nathan Williams)

- For what P is $\operatorname{ord}(R) < \infty$? This seems too hard to answer in general.
- **Not true:** for all those P that have nice and small $\operatorname{ord}(\mathbf{r})$'s.
- **However** it seems that $\operatorname{ord}(R) < \infty$ holds if P is **the positive root poset of a coincidental-type root system** (A_n, B_n, H_3), or a **minuscule heap** (see Rush-Shi, section 6).
- But the positive root system of D_4 has $\operatorname{ord}(R) = \infty$.

Application: Promotion on SSYTs

- The following is an application of our result on rectangle-shaped posets.
- It is well known (see Striker-Williams) that **classical** rowmotion ($=$ birational rowmotion over the boolean semiring $\{0, 1\}$) is related to promotion on **two-rowed** semistandard Young tableaux.
- Similarly, **birational** rowmotion over the tropical semiring $\operatorname{Trop} \mathbb{Z}$ relates to **arbitrary** semistandard Young tableaux.
- As an application of the periodicity theorem, we obtain the classical result that promotion done n times on a rectangular semistandard Young tableau with "ceiling" n does nothing.

- This is new and unproven, and inspired by Iyudu/Shkarin, `arXiv:1305.1965v3` (Kontsevich's periodicity conjecture).
- Work in a **skew field**. Write $\overline{m}$ for m^{-1}.
- Define the v-toggle by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

(there are other options as well – so far unexplored).

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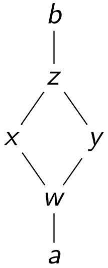
Iteratively apply R to a labelling of the 2×2-rectangle.

$R^0 f =$



Birational rowmotion: noncommutative generalization?

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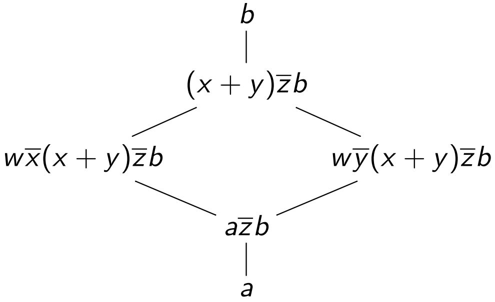

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$R^2 f =$

$$b$$

$$w\left(\overline{x}+\overline{y}\right) b$$

$$a \cdot \overline{x+y} \cdot x\left(\overline{x}+\overline{y}\right) b \qquad\qquad a \cdot \overline{x+y} \cdot y\left(\overline{x}+\overline{y}\right) b$$

$$a\overline{b}z \cdot \overline{x+y} \cdot b$$

$$a$$

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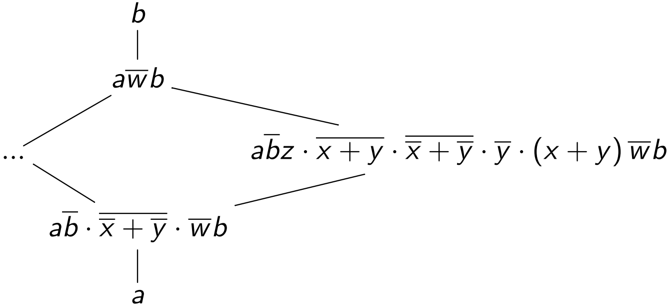
$R^3 f =$



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Iteratively apply R to a labelling of the 2×2-rectangle.
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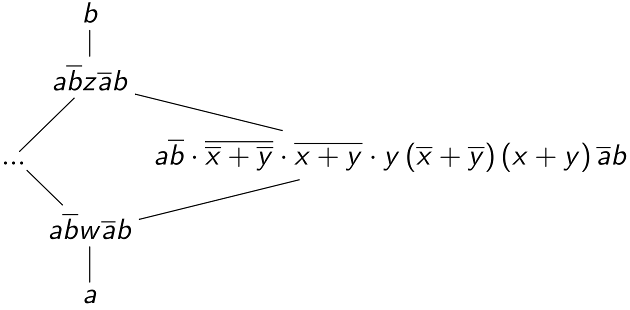

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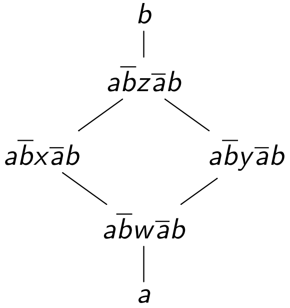

(after nontrivial simplifications).

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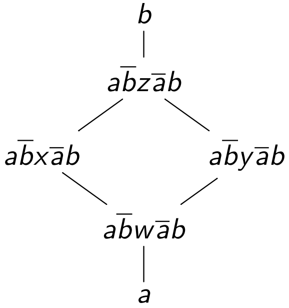

That is, all of our labels got conjugated by $a\overline{b}$. **Is R^{p+q} always conjugation by $f(0) \cdot (f(1))^{-1}$ on a $p \times q$-rectangle?** This is similar to Kontsevich's periodicity. (Noncommutative determinants?)

Announcement

There will be a **workshop on Dynamical Algebraic Combinatorics** at the American Institute of Mathematics the week of **March 23-27, 2015**, organized by Propp, Roby, Striker and Williams.
http://aimath.org/workshops/upcoming/dynalgcomb/
Among the subjects of the workshop:

- everything touched upon in this talk
- yes, that includes Young tableaux and Bender-Knuth, promotion, Lascoux-Schützenberger crystal operators
- homomesies (unsurprisingly)
- alternating sign matrices and gyration
- probably cluster algebras

Acknowledgments

- **Tom Roby**: collaboration
- **Pavlo Pylyavskyy, Gregg Musiker**: suggestions to mimic Volkov's proof of Zamolodchikov conjecture
- **James Propp, David Einstein**: introducing birational rowmotion and conjecturing the rectangle results; helpful advice
- **Nathan Williams**: bringing root systems into play
- **Jessica Striker**: familiarizing the author with rowmotion
- **Alexander Postnikov**: organizing a seminar where the author first met the problem
- **David Einstein, Hugh Thomas**: corrections
- **Sage and Sage-combinat**: computations
- **FPSAC referees**: useful comments

Thank you for listening!

Some references

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