Witt vectors. Part 1
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Errata and questions - I+II (only the relevant ones)

Section 4

- "The ring of power series $k((T))$" should be "The ring of power series $k[[T]]$".
- The map σ is never defined. Should it be just a synonym for $\mathbf{f}_p$?
- In the sentence directly following (4.1), I suspect that the term $\sigma^{-1}(x)$ should be $\sigma^{-r}(x)$ instead.
- I think that, at least for the proof of uniqueness, we need to suppose that A is not only complete, but also separated (i. e., Hausdorff) in the $\mathfrak{m}$-adic topology. Otherwise, there is the following counterexample:
  Let $A = \mathbb{Z}_p\left[\frac{1}{p^j}X^{1/p^i} \mid i \text{ and } j \text{ nonnegative integers}\right]$, with $\mathbb{Z}_p$ being the ring of p-adic integers. (Formally speaking, this ring A is defined as

$$A = \mathbb{Z}_p[X_{i,j} \mid i \text{ and } j \text{ nonnegative integers}] \diagup \left(\text{ideal generated by all } pX_{i,j} - X_{i,j-1},\ p^{j(p-1)}X_{i,j}^p - X_{i-1,j}\right),$$

  and we denote $X_{i,j}$ by $\frac{1}{p^j}X^{1/p^i}$.) Let $\mathfrak{m} = pA$. Then, $k = A\diagup\mathfrak{m}$ is simply $\mathbb{F}_p$ (because $\frac{1}{p^j}X^{1\diagup p^i} \in \mathfrak{m}$ for every i and j), and A is complete in the $\mathfrak{m}$-adic topology (because $\frac{1}{p^j}X^{1\diagup p^i} \in \bigcap_{n\in\mathbb{N}} \mathfrak{m}^n$, so when we have a Cauchy sequence of elements in A, we can freely ignore all their components except of the one without the X, which form a converging sequence because $\mathbb{Z}_p$ is p-adically complete). Now, you claim that the Teichmüller system is the unique one which commutes with [forming] p-th powers. This would mean that for every element $f \in k$, there exists only one "p-ancient" representative of f in A (where "p-ancient" means "a p^k-th power for every $k \in \mathbb{N}$"). But in our case, $f = 0$ already has two such representatives: $0 \in A$ and $X \in A$.
  I am not sure whether we also need the separatedness condition for the existence of the Teichmüller representatives.

Section 5

5.4

- In (5.5), there is one closing bracket too little: $(\psi(X^p)^{p^j}$ should be $(\psi(X^p))^{p^j}$.

5.11

- In the first of the two formulas (5.13), the term $s_n(a; b_0)$ should be $s_n(a; b)$. Similarly, $m_n(a; b_0)$ should be $m_n(a; b)$ in the second formula.

5.42

- In the first of the two displayed formulas in the proof of (5.43), the term $w_n(\mathbf{V}_p(a \cdot \mathbf{f}_p b)$ is missing a closing bracket (it should be $w_n(\mathbf{V}_p(a \cdot \mathbf{f}_p b)))$.

Section 6

6.13

- In the first absatz: "This field can be suggestively written $k'\left[T, T^{p^{-1}}, T^{p^{-2}}, ...\right]$." Are you sure you want to say "field", not "ring"?

6.15

- In the last sentence of page 19 (directly before the commutative diagram), you write "If $r_0, r_1, ..., r_n \in \mathfrak{a}$ then $w_n(r_1, r_2, ..., r_n) \in \mathfrak{a}^{n+1}$ ". It seems to me that you want $w_n(r_0, r_1, ..., r_n)$ instead of $w_n(r_1, r_2, ..., r_n)$ here.

- On the left hand side of (6.16), a closing bracket is missing: $q(a_0^{p^n}$ should be $q\left(a_0^{p^n}\right)$.

- On the right hand side of (6.16), I think you want to add "mod $\mathfrak{a}^{n+1}$", just as you did in the next equation.

- *Question:* Shortly before (6.18), what exactly do you mean by "A is of characteristic zero"? That $p \neq 0$, or that p is not a divisor of zero?

- I can't understand the $x_0 + pt_A(x_1)^p + p^2 t_A(x_2)^{p^2} + ...$ part of (6.18). At first, x_0 should be $t_A(x_0)$, or is there some reason to identify these? Then, shouldn't the exponents p, p^2, ... be p^{-1}, p^{-2}, ... instead?

6.19

- *Question:* Do we really not need separatedness of A in the $\mathfrak{m}$-adic topology here?

6.23

- I would replace "A p-ring is (by definition) a unital commutative ring" by "A p-ring is (by definition) a unital commutative ring C " (otherwise, the reader is wondering what the C in pC is supposed to be). Similarly, "a Cohen subring of it is a p-ring" should be "a Cohen subring of it is a p-ring C ".

6.26

- In the centered sentence, I think $C_n(k)$ should be $\mathcal{C}_n(k)$ (a calligraphic C) to match with the notations further below.

Section 7

7.1

- Shortly after (7.7), you write that "(The last equality in (7.7) is immediate from the considerations of the previous chapter.)". I would be more precise here - you just use the middle equation of (6.2), don't you?

Section 8

8.7

- *Question:* In (8.8), what does $\mathfrak{p}$ mean? Is it the operator $\mathfrak{p}$ from (7.6)? And $\mathbf{1}$ means $(1, 0, 0, ..., 0)$?

Section 9

9.1

- In the first line of this subsection, "For each ring" should be "For each ring A" (otherwise, the letter A is never introduced).

- Maybe you should be more explicit about using the multiplication in $1 + tA[[t]]$ as *addition*. People are used to writing an Abelian group both additively and multiplicatively, and when you just say that the power series multiplication makes $\Lambda(A)$ an Abelian group, they will suppose it will be written multiplicatively. When you speak about "The functorial addition" later, they can get confused.

- In (9.6), the μ_f above the second arrow should be a φ_f.

- The very right hand side of (9.7) is missing a closing bracket: $\frac{tf'(t}{f(t)}$ should be $\frac{tf'(t)}{f(t)}$.

- Are you sure about (9.8)? My calculations yield $s_2(a) = a_1^2 - 2a_2$ and $s_3(a) = -a_1^3 + 3a_1a_2 - 3a_3$ instead, which agree with (16.40) (something you could mention as well - the formulas (16.40) and (16.41) hold with Ψ^n replaced by $(-1)^n s_n$ and λ^n replaced by a_n; this "similarity" is not particularly surprising since the crux is the relation between the coefficients of a power series F and those of the power series $t\frac{d}{dt}\log F$).

9.10

- The left hand side of (9.13) should be $\prod_{i=1}^{\infty}\frac{1}{(1-\xi_i t)}$.

9.15

- In (9.16), you write $(f*g)(t) = f(t)$. What do you mean with the $f(t)$?

- In (9.17), you use the notation p_n. It is pretty much clear that this means the n-th power sum, but it would be better explicitly stated (before 9.54).

9.20

- On the very right hand side of (9.24), a factor of t^n is missing (that is, $\sum_{n=1}^{\infty}\sum_{d|n} dx_d^{n/d}$ should be $\sum_{n=1}^{\infty}\sum_{d|n} dx_d^{n/d}t^n$).

- On the left hand side of (9.27), m should be d (or vice versa).

9.30

- In (9.30), you speak of $\mathbf{Symm}_n$, and I suppose you mean the n-th graded component of **Symm**, but with respect to which grading? $\deg h_i = 1$ or $\deg h_i = i$?

9.42

- No mistakes here, but a remark:
  I think it is enough to assume that at least *one* of the sets $\{u_\lambda\}$ and $\{v_\lambda\}$ is a basis for $\mathbf{Symm}_{\mathbb{Q}}$. As an alternative, we could completely omit the basis condition, replacing it by the condition that the power series u_λ and v_λ are homogeneous of degree $\operatorname{wt}\lambda$ for every partition λ, and replacing "bi-orthonormal system" by "bi-orthonormal system of bases". In practice, homogeneity is often easier to prove than the fact that something forms a basis.

9.45

- In (9.50), the term $h_{2,1,1)}$ should be $h_{(2,1,1)}$.

- (9.51) doesn't make much sense to me. An internet source claims that $s_\lambda = \det(h_{\lambda_i+j-i})_{1\leq i,j\leq n}$, which at least has the correct degree. It also claims that $s_\lambda = \det(e_{\mu_i+j-i})_{1\leq i,j\leq n}$ (where μ is the conjugate partition to λ), which could also be worth adding. Maybe you should also remark that h_n is to be read as 0 if $n < 0$.

- *Question:* About the formula (9.52), you write: "This one has the advantage of showing immediately that Schur symmetric functions are positive". But how exactly does it show that? The sum $\sum_\lambda s_\lambda$ is a mix of many different s_λ, and how do you argue why it isn't possible that some negative coefficient in one of the s_λ gets cancelled by a greater positive coefficient in another s_λ ?

9.54

- You write: "This formula can readily be inversted by 'solving' (9.59) using formal exponentials." You could actually add the result of this inversion:

$$h_n = \sum_{\substack{\lambda; \\ \operatorname{wt}\lambda = n}} z_\lambda^{-1} p_\lambda,$$

  with z_λ defined below.

- After (9.62), you write: "showing that suitably normalized p_λ form an orthonormal basis for $\mathbf{Symm}_{\mathbb{Q}}$". I don't think so - we would have to normalize them with the coefficients $\sqrt{z_\lambda}$, but then they would lie in $\mathbf{Symm}_{\mathbb{R}}$ rather than $\mathbf{Symm}_{\mathbb{Q}}$.

9.71

- Replace "$-x_1, x_2, \cdots, x_n, \cdots$" by "$+x_1, -x_2, \cdots, -x_n, \cdots$". Accordingly, change "minus sign" to "plus sign" in footnote [34].

9.73

- In the second formula of (9.75), on the left hand side, $\mu^W_{S,n}(X;Y)$ should be $\mu^W_{P,n}(X;Y)$.

9.78

- I think you use $\mathfrak{p}$ and $\mathfrak{a}$ as synonyms here.
- Shortly after (9.81), "for some unit of A" probably should be "for some unit u of A" (otherwise, it is unclear what u means in $\sigma(c) = uc$).

9.93

- In the second line of this lemma, $\varphi_p(a) \equiv a^p$ should be $\varphi_p(a) \equiv a^p \bmod p$.
- Lemma 9.93 doesn't need the condition that A has characteristic 0, does it? At least, I have seen it stated without this condition in literature, and my proof seems not to require it either.

Section 10

- Are you sure about (10.6)? I think you rather want $\iota_S(h_n) = (-1)^n e_n$, or am I misunderstanding something?

10.14

- Maybe you should say that you denote your graded Hopf algebra by $H = (H_0, H_1, H_2, ...)$. Otherwise, it is never defined what H_0, H_1, H_2, ... mean.

- You write: "They must give the same result because anitpodes (if they exist) are unique". I don't think this is the argument you actually use here. Instead, I believe you mean something along the lines "They must give the same result because the left inverse of id under the convolution product must be equal to the right inverse, provided that both inverses exist".

10.20

- In (10.21), the $\oplus$ sign should be a $\otimes$ sign.

10.23

- In "One axiom that such an object much satisfy", the word "much" should be "must".

- You write: "There is no antipode for the second comultiplication of course (otherwise **Symm** would define a field valued functor)". This argument is literally true (any false assertion would imply another), but actually the second comultiplication wouldn't have an antipode even for a field-valued functor: The zero of a field isn't invertible.

10.25

- There are two subsections numbered 10.25 here: one "10. 25" (with an empty space before the 25), and one "10.25" (without an empty space). The labels (10.20), (10.21), (10.22), (10.23) and (10.24) also occur twice (counting both formulas and subsections).

Section 11

11.1

- In the formula (11.3), the $1 \otimes Z_1$ term should be $1 \otimes Z_n$.

- The **Symm** that appears between (11.6) and (11.7) should be an **NSymm**.

- In the sentence "The kernel of (11.7) is the commutator ideal generated by all elements of the form $Z_i Z_j = Z_j Z_i$, $i, j \in \mathbb{N}$" (quite at the end of subsection 11.1), I think the $=$ sign should be a $-$ sign.

11.19

- The subsection 11.19 begins with "Consider again the ring of polynomials, i.e. power series of bounded degree[44] in a countable infinity of commuting indeterminates, i.e. an element of $\mathbb{Z}[\xi_1, \xi_2, \xi_3, ...]$." I think you don't want to consider the *ring of* polynomials, but you want to consider *one polynomial.*

- In (11.20), ι_1 should be i_1 (under the sum sign).

- In (11.21), ι_1 should be i_1 again (in the index on the left hand side).

11.29

- In (11.30), n should be replaced by m (above the sum sign).

11.33

- On the left hand side of the first equation of (11.35), a bracket is missing (after $Z_\alpha \otimes Z_\beta$).

11.42

- In the second absatz, "comultiplication" is misspelt "comulltiplication".

- One line below this typo, the reference to (11.41) should be a reference to (11.43) instead.

- In the line directly above the diagram (11.43), $ab + bc$ should be $ac + bc$.

11.48

- I am quoting the first sentence of this subsection: "The second multiplication, m_p on **NSymm**". I think it would be more logical to denote it by m_P (with a capital P) or by * (as was done in 11.47).

Section 12

12.4

- The signature of the map $\psi_{n,m}$ should be $T^nM \otimes T^mM \overset{\cong}{\to} T^{n+m}M$ and not $T^nM \otimes T^nM \overset{\cong}{\to} T^{n+m}M$.

- In (12.6), i_M should be replaced by e_M.

12.8

- In the line directly following the equation (12.9), there are two typos: $x_i \in M_{i_1}$ should be $x_k \in M_{i_k}$, and "degree $i_1 + \cdots i_m$" should be "degree $i_1 + \cdots + i_m$" (a plus sign was missing here).

12.11

- Shortly before (12.14), the sentence "It is obvious that the n-th componet of the covering morphism, $\widetilde{\varphi_n} : C \to T^i M$ " has two typos: First, "componet" should be "component"; also, $T^i M$ should be $T^n M$.

12.15

- Directly after the formula (12.16), you speak of a natural morphism $i_C \otimes i_C : C \otimes C \to \operatorname{Free}(C) \otimes \operatorname{Free}(C)$. It is, in my opinion, not that clear what i_C means. You have defined $i_M : M \to TM$ for an Abelian group M, but $\operatorname{Free}(C)$ is not TC but rather $T(\operatorname{Ker}\varepsilon)$. If I am understanding you right, then your i_C means the map

$$C \overset{x \mapsto x - \varepsilon(x) \cdot 1}{\longrightarrow} \operatorname{Ker}\varepsilon \overset{i_{\operatorname{Ker}\varepsilon}}{\longrightarrow} T(\operatorname{Ker}\varepsilon).$$

(This looks even simpler in the case of a connected graded coalgebra C; in this case, the first map $C \overset{x \mapsto x - \varepsilon(x) \cdot 1}{\longrightarrow} \operatorname{Ker}\varepsilon$ simply means factoring the degree-0 part of C away.)

It is more of a mystery to me what you mean by the "fairly obvious (canonical) morphism of Abelian groups $\operatorname{CoFree}(A) \otimes \operatorname{CoFree}(A) \to (A \otimes A) / \mathbb{Z}$". However, if the algebra A is graded and connected (which is the case whenever you actually use this construction), then I can make sense of it: In this case $A/\mathbb{Z} \cong \bigoplus_{i \geq 1} A_i$ canonically, and one can therefore define a morphism of Abelian groups $\operatorname{CoFree}(A) \to A$ by sending $T_0(A/\mathbb{Z})$ to A (by $1 \mapsto 1$) and $T_1(A/\mathbb{Z})$ to A (by $T_1(A/\mathbb{Z}) \cong A/\mathbb{Z} \cong \bigoplus_{i \geq 1} A_i \subseteq A$ canonically) and all $T_i(A/\mathbb{Z})$ for $i > 1$ to 0. This morphism $\operatorname{CoFree}(A) \to A$ induces a morphism $\operatorname{CoFree}(A) \otimes \operatorname{CoFree}(A) \to A \otimes A$, which, composed with the projection $A \otimes A \to (A \otimes A)/\mathbb{Z}$, yields a morphism $\operatorname{CoFree}(A) \otimes \operatorname{CoFree}(A) \to (A \otimes A)/\mathbb{Z}$. But how do you construct the latter morphism if A is *not* graded and connected?

12.20

- In the equation (12.21), the $\otimes$ sign (in $\mathbb{Z} \oplus \mathbb{Z}Z_1 \oplus \mathbb{Z}Z_2 \otimes \cdots$) should be a $\oplus$ sign.

- In this whole section, when you write Free or CoFree, it is not immediately clear whether you mean the Free rsp. CoFree of an Abelian group, or of an algebra. For instance, when you write that

$$\operatorname{CoFree}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}Z_1 \oplus \mathbb{Z}Z_2 \otimes \cdots$$

(as I said, the $\otimes$ sign should be $\oplus$ here), you mean the cofree comodule over the *Abelian group* $\mathbb{Z}$, not over the *algebra* $\mathbb{Z}$ (since the cofree comodule over the *algebra* $\mathbb{Z}$ would be $\operatorname{CoFree}(\mathbb{Z}/\mathbb{Z})$, which should simply be $\mathbb{Z}$ if I'm not mistaken). On the other hand, when you apply the Free functor to $\operatorname{CoFree}(\mathbb{Z})$, you are thinking of $\operatorname{CoFree}(\mathbb{Z})$ as an algebra (I think).

- In the computation

$$\begin{aligned}\mathbf{QSymm} &= \mathbf{NSymm}^{dual} = \operatorname{Free}(\operatorname{CoFree}(Z))^{dual} \\ &= \operatorname{CoFree}\left(\operatorname{CoFree}(\mathbf{Z})^{dual}\right) = \operatorname{CoFree}\left(\operatorname{Free}\left(\mathbf{Z}^{dual}\right)\right) \\ &= \operatorname{CoFree}(\operatorname{Free}(\mathbf{Z}))\end{aligned}$$

(in the very end of subsection 12.20), the Z in the first line should be a boldface $\mathbf{Z}$.

Section 13

13.1

- In the formula between (13.2) and (13.3), the term $p_3 t$ should be $p_3 t^{2n}$ (in $t\left(nt^{n-1}\left(p_1 + p_2 t^n + p_3 t + ...\right)\right)$).

13.5

- In the first sentence of 13.5, "of the $\Lambda(A)$" should be "of the algebra $\Lambda(A)$".

13.6

- I think that the power series that you call $h(t)$ here (and below) has been denoted by $H(t)$ earlier (e. g., in (9.56)).
- Not a mistake, but something I would add: The last statement that you make in subsection 13.6 is that "The Frobenius Hopf algebra endomorphisms of **Symm** coresponding to the Frobenius operations are (of course) given by $h_r \mapsto Q_{n,r}(h)$." Here I would add that these endomorphisms can be equivalently written in the form $\mathbf{f}_n : P \mapsto P(\xi^n)$ for every $P \in \mathbf{Symm}$ (represented as symmetric function in the ξ_i). In particular, this yields (13.18) and Corollary 13.37 immediately.

13.13

- The proof of (13.14) is rather hard to understand due to the amount of typos: The formula (13.16) should be

$$p^i\left(Q_{p,p^i}\right)^{p^{r-i}} \equiv p^i\left(h_{p^i}\right)^{p^{r-i+1}} \operatorname{mod} p^{r+1}$$

(the $h_{p^{i-1}}$ should be h_{p^i}, the p^{r-i+1} should be an exponent rather than a factor, and you have omitted the $\operatorname{mod} p^{r+1}$ at the end of the formula).
Further, "Cancel mod p^r" should be "cancel mod p^{r+1}" (of course, they cancel mod p^r as well, but this is not enough; for the proof to work, we need cancellation mod p^{r+1}; the "C" in "cancel" should be lowercase, though Micro$oft Word likes to insist otherwise).
The next equation,

$$w_{p^{r+1}}(h) = (h_1)^{p^{r+1}} + p\left(h_p\right)^{p^r} + \cdots p^{r-1}\left(h_{p^{r-1}}\right)^{p^r} + p^r\left(h_{p^r}\right)^p + p^{r+1} h_{p^{r=1}},$$

should be

$$w_{p^{r+1}}(h) = (h_1)^{p^{r+1}} + p(h_p)^{p^r} + \cdots + p^{r-1}(h_{p^{r-1}})^{p^2} + p^r(h_{p^r})^p + p^{r+1}h_{p^{r+1}}$$

(the three typos were: a + sign was missing after the $\cdots$; the exponent p^r after the $\cdots$ should be p^2; the last index $p^{r=1}$ should be p^{r+1}).

- Anyway, I think that there is a simpler proof of (13.14) without any induction. It also proves a generalization of (13.14), namely

$$Q_{p,r}(h) \equiv (h_r)^p \operatorname{mod} p \qquad \text{for any prime } p \text{ and any } r \in \mathbb{N}.$$

In fact,

$$\begin{aligned}
& Q_{p,r}(h) \\
&= \left(\text{coefficient of the power series } \prod_i \left(1-\xi_i^p t\right)^{-1} \text{ before } t^r\right) \\
&= \left(\text{coefficient of the power series } \prod_i \left(1-\xi_i^p t^p\right)^{-1} \text{ before } t^{pr}\right) \\
&\qquad \text{(here, we simply substituted } p^r \text{ for } p \text{ in our power series)} \\
&\equiv \left(\text{coefficient of the power series } \left(\prod_i \left(1-\xi_i t\right)^{-1}\right)^p \text{ before } t^{pr}\right) \\
&\qquad \left(\text{since } \prod_i \left(1-\xi_i^p t^p\right)^{-1} \equiv \prod_i \left(\left(1-\xi_i t\right)^p\right)^{-1} = \left(\prod_i \left(1-\xi_i t\right)^{-1}\right)^p \operatorname{mod} p\right) \\
&\equiv \left(\text{coefficient of the power series } \prod_i \left(1-\xi_i t\right)^{-1} \text{ before } t^r\right)^p \\
&\qquad \left(\text{since for any power series } \alpha = \sum_{i=0}^{\infty} \alpha_i t^i \text{ in the variable } t, \text{ we have } \alpha^p \equiv \sum_{i=0}^{\infty} \alpha_i^p t^{p \cdot i} \operatorname{mod} p\right) \\
&= (h_r)^p \operatorname{mod} p.
\end{aligned}$$

Have I made a mistake here, or is this a correct proof?

- In the formula between (13.20) and (13.21), the last factor $(1-\xi t)^{-1}$ should be $(1-\xi_n t)^{-1}$.

13.25

- In the equation (13.29), the term $p_{r=1}$ should be p_{r-1}.
- There is one closing parenthesis too much in the last displayed congruence of this proof (on page 65, right after "$(-1)^{n+1} h_1^n$").

13.32

- The statement that "it works over any ring" is false. For a counterexample, one can set $R = \mathbb{Q} \times \mathbb{Q}$ (a direct product of two rings both equal to $\mathbb{Q}$) and define $\varphi : \mathbf{Symm}_R \to \mathbf{Symm}_R$ as the unique R-algebra homomorphism sending every p_i (with $i > 0$) to $(1,0)\, p_i + (0,1)\, p_{2i}$. The place where the proof breaks down over this ring R is the claim that "all but one of these coefficients are zero".

  Note that I am **not** saying that this is the only place where the proof of Theorem 13.22 breaks down when $R \neq \mathbb{Z}$. The proof also seems to use the fact that 1 is not divisible by any $r \in \{2, 3, 4, \ldots\}$ in R.

13.42

- In the formula (13.43), the dot between $\Lambda(A) \to \Lambda(A)$ and $a(t) \mapsto a(t)^n$ should actually be a comma.
- It wouldn't harm to mention that $[n]\, p_r = p_{nr}$ for every $r \in \mathbb{N}$. (In fact, seeing how the different endomorphisms of $\Lambda(A)$ act on the power sums p_r is one of the easiest ways to prove the identities in Theorem 13.48.)

13.45

- Again, I would find it useful if you explicitly mention that $\langle u \rangle (p_n) = u^n p_n$ for every $n \in \mathbb{N}$.

13.48

- Talking about the expansions of $r_d(X, Y)$ for $d \geq 2$, you conjecture that "all monomials that possibly can occur do in fact occur with nonzero coefficient". In other words, you conjecture that if d is an integer ≥ 2, then every monomial $X^i Y^{d-i}$ with $0 < i < d$ occurs in the polynomial $r_d(X, Y)$.

  This conjecture is not difficult to prove using the results of Doran from the paper

  [Doran] William F. Doran IV, *A proof of Reutenauer's* $-q_{(n)}$ *Conjecture*, Journal of Combinatorial Theory, Series A, Volume 74, Issue 2, May 1996, Pages 342–344, `http://www.sciencedirect.com/science/article/pii/S0097316596900564` .

  In fact, (9.13) yields $\prod_{i=1}^{\infty} (1 - \xi_i t)^{-1} = 1 + h_1 t + h_2 t^2 + h_3 t^3 + \ldots$ in the ring **Symm** of symmetric functions in the variables $\xi_1, \xi_2, \xi_3, \ldots$. (Notice that the h_n were denoted by a_n in (9.13).) Now, comparing the equality (13.60) with the equality

$$\prod_{i=1}^{\infty} (1 - \xi_i t) = \left(\underbrace{\prod_{i=1}^{\infty} (1 - \xi_i t)^{-1}}_{\substack{= 1 + h_1 t + h_2 t^2 + h_3 t^3 + \ldots \\ = \prod_{d=1}^{\infty} \left(1 - x_d t^d\right)^{-1} \\ \text{(by (9.64))}}} \right)^{-1} = \left(\prod_{d=1}^{\infty} \left(1 - x_d t^d\right)^{-1} \right)^{-1} = \prod_{d=1}^{\infty} \left(1 - x_d t^d\right),$$

we see that for every positive integer d, the polynomial $r_d(X,Y)$ is the evaluation of the symmetric function $x_d \in \mathbf{Symm}$ at (X,Y). Hence, in order to prove your conjecture, it is enough to show that if d is an integer ≥ 2, then the coefficients of the symmetric function x_d in the Schur function basis are nonnegative, and the coefficient in front of the Schur function $s_{(d-1,1)}$ is positive.[1] But this all follows from things proven in [Doran] (in fact, in the induction step, [Doran] shows that

$$-f(n,k) = s_{(n-1,1)} + (\text{things that have nonnegative coefficients in the Schur function basis})$$

with his notations used; but $-f(n,n)$ is x_n).

13.74

- Between (13.75) and (13.76), you speak of the "free algebra" two times. I think that both times, you mean "free commutative algebra".
- In (13.78), replace t by T.
- Three lines below (13.78), "this" is misspelt "his" in "But his last object is easy to describe:".
- Three lines below (13.78), replace t by T (in $\mathbf{CRing}'(\mathbf{Symm}, \mathbf{Z}[t])$).

Section 14

- The last word before (14.2) is "coordinates" in your text. I am not sure, but I tend to believe that you mean "coefficients" here.

14.9

- I think the first equality of (14.10) is wrong.

14.11

- In the third line of (14.12), you speak of a "nontrivial prime number". I would remove the word "nontrivial"; otherwise, it seems like you consider 1 to be a prime number (although trivial), which would mess up the second line of (14.12) (since the value of $\mu(n)$ would depend on whether we count 1 among the r different prime numbers).
- (14.13) should be

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1 \end{cases}.$$

[1] In fact, the Schur function $s_{(d-1,1)}$, evaluated at (X,Y), gives $X^1Y^{d-1} + X^2Y^{d-2} + ... + X^{d-1}Y^1$. Therefore, the nonnegativity of all coefficients of x_d in the Schur function basis, combined with the positivity of the coefficient in front of the Schur function $s_{(d-1,1)}$, yields that the evaluation of x_d at (X,Y) contains all the monomials X^1Y^{d-1}, X^2Y^{d-2}, ..., $X^{d-1}Y^1$.

14.15

- You start with the words "To set the stage here is the abstract situation.", and you then proceed to construct the abstract situation consisting of a surjective morphism $\pi : M \to M_p$ of Abelian groups along with a section $s : M_p \to M$. However, in your concrete application where $M = W(A)$ and $M_p = W_{p^\infty}(A)$, you never actually construct the s in your text. It's not hard (it is the inverse of the isomorphism given in Theorem 14.21), but maybe it should be mentioned.

- Shortly after the formula (14.16), "give prime number" should be "given prime number".

- The $\sum$ sign in formula (14.17) should be $\sum\limits_{n \in \mathbf{N}(p)}$.

- I think the notation $[\frac{\mu(n)}{n}]$ in formula (14.17) requires further explanation. You have defined what $[k]$ means for an *integer* k, but $\frac{\mu(n)}{n}$ is usually not an integer. Actually it would probably be better to leave out the square brackets, so the formula (14.17) would become

$$\Phi_{p-\text{typ}} = \sum_{n \in \mathbf{N}(p)} \frac{\mu(n)}{n} \mathbf{V}_n \mathbf{f}_n.$$

  Still, one fact needs to be proven here: the fact that n is invertible in $\Lambda(A)$ for every $n \in \mathbf{N}(p)$. Or is it really trivial? The simplest proof that comes into my mind is the following: The inverse of n in $\Lambda(A \otimes \mathbb{Q})$ is the series $(1-t)^{-1/n} = \sum\limits_{k=0}^{\infty} \binom{-1/n}{k} (-t)^k \in \Lambda(A \otimes \mathbb{Q})$, and we must prove that the coefficients $\binom{-1/n}{k}$ of this series are well-defined in A. This is equivalent to proving that $\binom{-1/n}{k} \in \mathbb{Z}_{(p)}$ for every $n \in \mathbf{N}(p)$; id est, we must show that $v_p\left(\binom{-1/n}{k}\right) \geq 0$. This is not difficult.

- You could add the words "on $\Lambda(A)$" after (14.17). Not that there really is a danger of confusion, but you have been using the symbols $\mathbf{f}_n$ and $\mathbf{V}_n$ both for the functorial operations on $\Lambda(A)$ and for their defining endomorphisms of **Symm**, and $\mathbf{V}_n \mathbf{f}_n$ in one interpretation means $\mathbf{f}_n \mathbf{V}_n$ in the other (cf. Caveat 13.62).

14.21 (the Proposition)

- There are two things labeled 14.21 (a Proposition and a Theorem).

- In the first line of the proof of Proposition 14.21, $n = \mathbf{N}(p)$ should be $n \in \mathbf{N}(p)$.

- At the end of the proof of Proposition 14.21, $\widehat{\mathrm{A}}$ (in "for some characteristic zero $\mathbb{Z}_{(p)}$-algebra $\widetilde{A}$ covering $\widehat{\mathrm{A}}$") should be A (without hat).

Section 15

15.1

- You write: "It follows with induction using the Newton relations, the fact that $\mathbf{f}_n$ is a ring endomorphism, and duality, that $\langle p_n, e_n \rangle = (-1)^{n+1}$ and hence that $\langle \mathbf{f}_n h_1, e_n \rangle = (-1)^{n+1}$". I think you can replace this by "It follows from (13.30) that $\langle p_n, e_n \rangle = (-1)^{n+1}$ and hence that $\langle \mathbf{f}_n h_1, e_n \rangle = (-1)^{n+1}$".

- In the next sentence ("Using again the Newton relations, ring morphism, and duality and induction one further finds $\langle \mathbf{f}_n h_r, e_{nr} \rangle = 0$") you should require r to be > 1, since it doesn't hold for $r = 1$. However, I am far from sure whether it holds for $r > 1$ either. I don't understand any of your two arguments proving this, and in my opinion, $\langle \mathbf{f}_n h_r, e_{nr} \rangle = \langle h_r, \mathbf{V}_n e_{nr} \rangle$, so if $\langle \mathbf{f}_n h_r, e_{nr} \rangle$ would be 0, then $\langle h_r, \mathbf{V}_n e_{nr} \rangle$ would be 0, which would mean that $\mathbf{V}_n e_{nr}$ has no monomials of the form $m_{(r)} = \xi_1^r + \xi_2^r + ...$, which is equivalent to saying that $\mathbf{V}_n e_{nr} \in \langle p_1, p_2, ..., p_{r-1} \rangle$ over $\mathbb{Q}$, which in turn would yield $e_{nr} \in \langle p_1, p_2, ..., p_{nr-1} \rangle$ over $\mathbb{Q}$ (because $\mathbf{V}_n$ maps p_{nr} to np_r), which is plain wrong.

- At the end of subsection 15.1, the word "tye" should be "type" (in "This is encouraging and suggests that every matrix of the tye specified can arise.").

15.5

- In (15.8), the coefficient b_{21} should be b_2.

15.16

- In footnote [55] (which is somehow stretched over 3 pages), you refer to "the 'caveat' 16.62 below". But 16.62 is not a caveat and doesn't have much to do with this. I think you want to refer to the caveat 13.62.

- In (15.18), the rightmost sum ($\sum\limits_n \mathbf{V}_2 \langle c_{1n} \rangle \mathbf{f}_n$) should be $\sum\limits_n \mathbf{V}_2 \langle c_{2n} \rangle \mathbf{f}_n$.

- One line below (15.18), you write $\langle b_{1n} \rangle + \langle c_{in} \rangle = \sum\limits_{i=1}^{\infty} \mathbf{V}_i \langle r_i(b_{1n}, c_{1n} \rangle \mathbf{f}_i$. There are two mistakes here: The c_{in} should be a c_{1n}, and a closing bracket is missing in $\langle r_i(b_{1n}, c_{1n} \rangle$.

- Between (15.18) and (15.19), you write: "What is left is a sum

$$\mathbf{V}_2 \langle r_2 (b_{1n}, c_{1n}) \rangle \mathbf{f}_2 + \sum_n \mathbf{V}_2 \langle b_{2n} \rangle \mathbf{f}_n + \sum_n \mathbf{V}_2 \langle c_{2n} \rangle \mathbf{f}_n$$

". You have forgotten a $\sum\limits_n$ sign at the beginning of this expression.

15.20

- In the first sentence of subsection 15.20, "unital-commutatve-ringvalued" should be "unital-commutative-ringvalued". But you can actually leave out this word completely, because by saying "functor on **CRing** to itself" you already make clear that this functor is unital-commutative-ringvalued.

- There is an (in my opinion) simpler proof that the operations of the form $\sum\limits_{n=1}^{\infty} \mathbf{V}_n \langle x_n \rangle \mathbf{f}_n$ (with $x_n \in R$) form a subring of $\operatorname{End} W_R$ isomorphic to the ring $W(R)$ of Witt vectors. This proof goes as follows:
  Instead of considering the operations $\sum\limits_{n=1}^{\infty} \mathbf{V}_n \langle x_n \rangle \mathbf{f}_n$ on Λ_R, we consider their determining endomorphisms $\sum\limits_{n=1}^{\infty} \mathbf{f}_n \langle x_n \rangle \mathbf{V}_n$ of **Symm** (as we remember, operations on $\Lambda(A)$ and endomorphisms of **Symm** are contravariantly equivalent). In order to prove that these "behave" like the Witt vectors $x \in W(R)$, we must show that

$$\sum_{n=1}^{\infty} \mathbf{f}_n \langle x_n \rangle \mathbf{V}_n + \sum_{n=1}^{\infty} \mathbf{f}_n \langle y_n \rangle \mathbf{V}_n = \sum_{n=1}^{\infty} \mathbf{f}_n \langle \mu_{S,n}(x,y) \rangle \mathbf{V}_n \qquad \text{and}$$
$$\sum_{n=1}^{\infty} \mathbf{f}_n \langle x_n \rangle \mathbf{V}_n \cdot \sum_{n=1}^{\infty} \mathbf{f}_n \langle y_n \rangle \mathbf{V}_n = \sum_{n=1}^{\infty} \mathbf{f}_n \langle \mu_{P,n}(x,y) \rangle \mathbf{V}_n$$

  (in **Symm**) for any two Witt vectors $x \in W(R)$ and $y \in W(R)$ (where $\mu_{S,n}$ and $\mu_{P,n}$ are as defined in (14.3)) (that the 0 and the 1 behave as the 0 and the 1 is pretty obvious). But this can be seen by letting these endomorphisms operate on the power sums p_r, because any Witt vector $x \in W(R)$ satisfies

$$\left(\sum_{n=1}^{\infty} \mathbf{f}_n \langle x_n \rangle \mathbf{V}_n\right)(p_r) = \sum_{n=1}^{\infty} \mathbf{f}_n \left(\langle x_n \rangle \left(\underbrace{\mathbf{V}_n(p_r)}_{= \begin{cases} np_{r/n}, & \text{if } n \mid r; \\ 0, & \text{if } n \nmid r \end{cases}} \right)\right) = \sum_{n \mid r} \mathbf{f}_n \left(\langle x_n \rangle (np_{r/n}) \right)$$
$$= \sum_{n \mid r} n\mathbf{f}_n \left(\underbrace{\langle x_n \rangle (p_{r/n})}_{= x_n^{r/n} p_{r/n}} \right) = \sum_{n \mid r} n x_n^{r/n} \underbrace{\mathbf{f}_n(p_{r/n})}_{= p_{n \cdot r/n} = p_r} = \sum_{n \mid r} n x_n^{r/n} p_r = w_r(x)\, p_r,$$

  and the Witt vector addition and multiplication formulas do the rest of the work.

15.24

- The word "denotes" is misspelt "de notes" in the context "where $x^{\mathbf{f}_p}$ de notes the Witt vector arising from applying the p-th Frobenius operation to the Witt vector x".

- In the same sentence, you define the notation $x^{\mathbf{f}_p}$ but you never use it. Instead, you use $x^{(p)}$, which means something different from $x^{\mathbf{f}_p}$. Maybe you should replace the definition of $x^{\mathbf{f}_p}$ by that of $x^{(p)}$. (However, you should then explain what $x^{\mathbf{V}_p}$ means further below.)

15.25

- Footnote [57] is only readable if one copy-pastes it from the PDF viewer. Besides, it is placed two pages after it is referenced, so it is very hard to find. (Here is the content of this footnote: "The 'WH' here stands for 'Witt-Hopf' (as a sort of mnemonic).")

15.27

- Replace "$U_0, U_1, U_2, \cdots$" by "$U_1, U_2, U_3, \cdots$" in (15.28). Similarly, replace "$(G_0(U), G_1(U), G_2(U), \cdots)$" by "$(G_1(U), G_2(U), G_3(U), \cdots)$" between (15.29) and (15.30). (Or, alternatively, replace "$A^{\mathbf{N}}$" by "$A^{\{0,1,2,\cdots\}}$" in (15.28), and start the indexing in (15.30) with 0. This is probably the better solution, given that in §15.32 your indexes start at 0.)

- Between (15.29) and (15.30), you write: "A Hopf algebra endomorphism of this Hopf algebra". Maybe you should make it clearer that you mean the Hopf algebra $\left(A^{\mathbb{N}}, +, \cdot, \mu_S, \varepsilon_S\right)$ (and not $\left(A^{\mathbb{N}}, +, \cdot, \mu_P, \varepsilon_P\right)$, which is only a bialgebra). Actually, you don't have to mention μ_P and ε_P at all in (15.29) - they have nothing to do with the infinite dimensional *additive* group.

15.32

- In the second line of subsection 15.32, the series of polynomials $(\varphi_0, \varphi_1, \varphi_0, \cdots)$ should be $(\varphi_0, \varphi_1, \varphi_2, \cdots)$.

- One line above (15.33), the series of polynomials $(\varphi_0, \varphi_1, \varphi_0, \cdots)$ should again be $(\varphi_0, \varphi_1, \varphi_2, \cdots)$.

15.42

- In the sentence "Also recall that (see 13.13) the sequence of polynomials that define the Frobenius endomorphism on $WK_k^{(p)}$ is", I guess $WK_k^{(p)}$ should be $WH_k^{(p)}$.

15.46

- In (15.49), there is one + sign too much (namely, the + sign directly before the $\mapsto$ arrow).

Section 16

16.1

- The commutative diagram

$$\begin{array}{ccc} & & \Lambda(A) \\ & \overset{\exists^1\ \tilde{\varphi}}{\nearrow} & \downarrow \pi \\ S & \xrightarrow[\varphi]{} & A \end{array}$$

  bleeds into the footnotes. I don't know much about such issues in office programs, but inserting empty lines to get the diagram into the next page could help.

16.4

- In footnote [63], add a closing bracket in "[239, 324".

16.14

- In formula (16.15), the exponent $n-1$ (in σ^{n-1}) should be $n-i$ instead.
- Formula (16.15) needs the condition $n > 0$ (otherwise, the right hand side should be 1).
- On the right hand side of the equation (16.19), the product $\xi_{i_1}\xi_{i_1}\cdots\xi_{i_1}$ should be $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}$.

16.20

- In footnote [65], you write "Then the composite is the additive map (in general nothing more) $[n]$, the n-fold sum of the identity (under convolution)." It is clear what you mean here, but maybe you should say "the n-fold power" rather than "the n-fold sum". One is used to treating the convolution as a multiplication, not as an addition.

16.35

- Lemma 16.35 is wrong as stated. You should suppose that the rings A and B are of characteristic zero. Or maybe one of them is already enough - but if none of them is, then there may be counterexamples[2].

16.45

- Replace "takes the quotient $W(k) \longrightarrow W_{p^\infty}$" by "takes the quotient $W(k) \longrightarrow W_{p^\infty}(k)$".

[2] see `http://mathoverflow.net/questions/13486/is-every-adams-ring-morphism-a-lambda-ring-morphism`

- This is not a mistake, but I would find it useful if you would interpolate (between 16.45 and 16.46 maybe?) the following remark:

  16.45a. *Remark.* Let A be a commutative ring. Let $A^{\mathbf{N}}$ be the $\mathbf{N}$-fold direct power of the ring A (with componentwise addition and multiplication). Define a map $\mathfrak{s}_A : \Lambda(A) \to A^{\mathbf{N}}$ by

  $$\left(\mathfrak{s}_A(a) = (s_n(a))_{n\in\mathbf{N}} \qquad \text{for every } a \in \Lambda(A)\right).$$

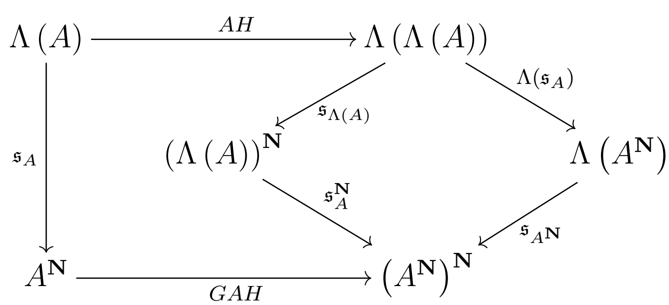
  Note that $\mathfrak{s}_A$ is a ring homomorphism (since all s_n are ring homomorphisms). Thus, we have defined a natural homomorphism $\mathfrak{s} := (\mathfrak{s}_A)_{A\in\mathbf{CRing}}$ from Λ to $(\cdot)^{\mathbf{N}}$.

  Also, for any ring A, define a map $GAH : A^{\mathbf{N}} \to \left(A^{\mathbf{N}}\right)^{\mathbf{N}}$ by

  $$\left(GAH\left((a_n)_{n\in\mathbf{N}}\right) = \left((a_{mn})_{n\in\mathbf{N}}\right)_{m\in\mathbf{N}} \qquad \text{for every } (a_n)_{n\in\mathbf{N}} \in A^{\mathbf{N}}\right).$$

  Clearly, GAH is a ring homomorphism.

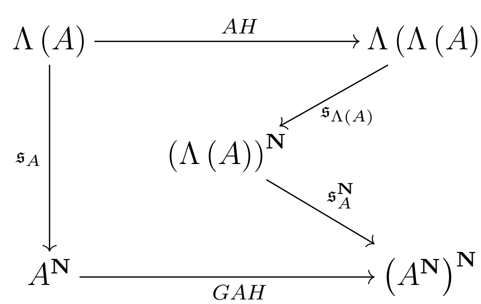
  Now, for every commutative ring A, the diagram

  (16.45b)

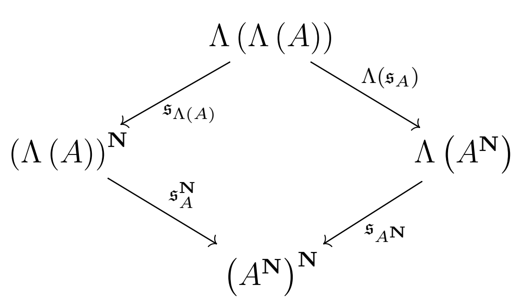
  commutes.

  Proof. Clearly, we only need to show that the diagram (16.45b) commutes (since all the other assertions are trivial). To show this, it needs to be proven that the diagrams

  (16.45c)

  and

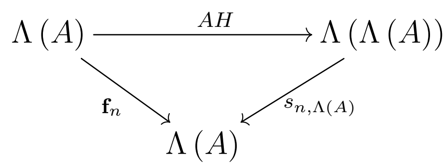
  (16.45d)

commute. Since the commutativity of (16.45d) is trivial (in fact, it follows from the fact that $\mathfrak{s}$ is a natural transformation while $\mathfrak{s}_A$ is a ring homomorphism), we only need to prove the commutativity of (16.45c).

Now, define a map $\mathbf{f} : \Lambda(A) \to (\Lambda(A))^{\mathbf{N}}$ by

$$\left(\mathbf{f}(a) = (\mathbf{f}_n(a))_{n \in \mathbf{N}} \qquad \text{for every } a \in \Lambda(A)\right).$$

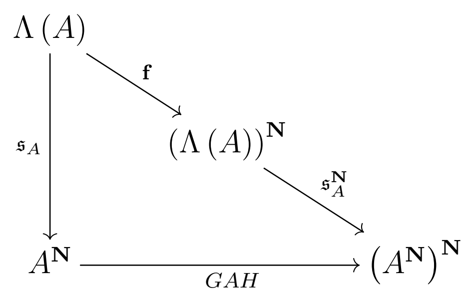
Since the diagram



commutes for every $n \geq 1$, we have

$$\mathbf{f}_n(a) = s_{n,\Lambda(A)}(AH(a)) \qquad \text{for every } n \geq 1 \text{ and every } a \in \Lambda(A).$$

Now, the diagram



commutes (because every $a \in \Lambda(t)$ satisfies

$$\mathfrak{s}_A^{\mathbf{N}} \left(\underbrace{\mathbf{f}(a)}_{\substack{=(\mathbf{f}_n(a))_{n\in\mathbf{N}} \\ \text{(by the definition of } \mathbf{f}\text{)}}} \right)$$

$$= \mathfrak{s}_A^{\mathbf{N}} \left((\mathbf{f}_n(a))_{n\in\mathbf{N}} \right) = (\mathfrak{s}_A(\mathbf{f}_n(a)))_{n\in\mathbf{N}} = \left(\underbrace{\mathfrak{s}_A(\mathbf{f}_m(a))}_{\substack{=(s_n(\mathbf{f}_m(a)))_{n\in\mathbf{N}} \\ \text{(by the definition of } \mathfrak{s}_A\text{)}}} \right)_{m\in\mathbf{N}}$$

(here, we renamed the index n as m)

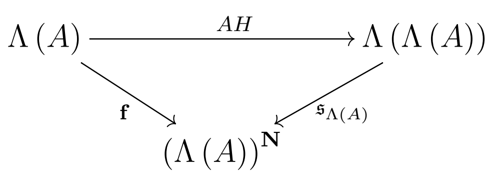
$$= \left(\left(\underbrace{s_n(\mathbf{f}_m(a))}_{\substack{=s_{mn}(a) \\ \text{(by (13.68), applied to} \\ n,\, m \text{ and } a \text{ instead of } r,\, n \text{ and } a(t))}} \right)_{n\in\mathbf{N}} \right)_{m\in\mathbf{N}} = \left((s_{mn}(a))_{n\in\mathbf{N}} \right)_{m\in\mathbf{N}}$$

$$= GAH \left(\underbrace{(s_n(a))_{n\in\mathbf{N}}}_{=\mathfrak{s}_A(a)} \right)$$

$$\left(\text{since } GAH\left((s_n(a))_{n\in\mathbf{N}}\right) = \left((s_{mn}(a))_{n\in\mathbf{N}}\right)_{m\in\mathbf{N}} \text{ by the definition of } GAH \right)$$

$$= GAH(\mathfrak{s}_A(a))$$

). Combined with the commutative diagram

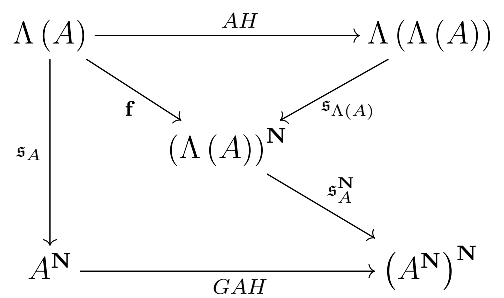


(in fact this diagram commutes because every $a \in \Lambda(A)$ satisfies

$$\mathbf{f}(a) = \left(\underbrace{\mathbf{f}_n(a)}_{=s_{n,\Lambda(A)}(AH(a))} \right)_{n\in\mathbf{N}} = \left(s_{n,\Lambda(A)}(AH(a)) \right)_{n\in\mathbf{N}} = \mathfrak{s}_{\Lambda(A)}(AH(a))$$

$$\left(\text{since } \mathfrak{s}_{\Lambda(A)}(AH(a)) = \left(s_{n,\Lambda(A)}(AH(a)) \right)_{n\in\mathbf{N}} \text{ by the definition of } \mathfrak{s}_{\Lambda(A)} \right)$$

), this yields the commutativity of the diagram



As a consequence, the diagram (16.45c) commutes. This completes the proof of 16.45a.

Remark 16.45a shows how the Artin-Hasse exponential AH is written in terms of ghost components, and explains how Auer's functorial morphism $W_{MN}(A) \to W_M(W_N(A))$ defined in [28] comes from the Artin-Hasse homomorphism $AH : \Lambda(A) \to \Lambda(\Lambda(A))$. (In fact, the definition of Auer's functorial morphism in [28] was more or less by requiring it to have a diagram similar to (16.45b) commute.)

16.48

- The equality sign in (16.50) should be a $\equiv$ sign.

16.54

- Between (16.55) and (16.56), replace $m_C : T(C) \longrightarrow TT(C) = T(T(C))$ by $m_C : TT(C) = T(T(C)) \longrightarrow T(C)$.

- One line below (16.57), you write: "And a coalgebra for the comonad (T, μ, ε) is an object in the category $\mathcal{C}$ [...]". This should be "And a coalgebra for the comonad (T, μ, ε) is an object C in the category $\mathcal{C}$ [...]" (otherwise, the notation C is never defined).

- In (16.58), you have one bracket too much (in the expression $(T(\sigma)\sigma)$.

16.59

- In the second line of subsection 16.59, you write: "with the comonad morphism given by the Artin-Hasse exponential". Here, "morphism" should be replaced by "morphisms" (since there are two of them).

- In the commutative diagram (16.61), the id arrow must be directed from the lower right A to the upper left A. (Otherwise, it yields a diagram which does not commute, because $\sigma_t \circ \mathrm{id} \circ s_1$ is generally $\neq \mathrm{id}$.)

16.62

- The right hand side of the formula (16.64) should be $(-1)^{n+1}(a_n + P(a_1, \cdots, a_{n-1}))$.

16.65

- On page 99, replace "for which the morphisms (16.66)" by "for which the morphisms (16.67)".

16.70

- Replace "A word is Lyndon" by "A nonempty word is Lyndon".

16.71

- Replace "α primitive word" by "α primitive Lyndon word".

16.72

- You write: "(b) The corresponding Adams operations as described by (16.66)". But the Adams operations are described by (16.67), not (16.66).

16.73

- In the proof of 16.73, you refer to (16.66). I think you mean (16.67) here.

16.74

- The proof of the theorem begins with: "As is easily verified from e.g. (16.58) the λ-ring structure on **Symm** satisfies $\lambda^n(e_1) = e_n$." I don't see how this is easily verified from (16.58). Here are two proofs that the λ-ring structure on **Symm** satisfies $\lambda^n(e_1) = e_n$:

  First proof of $\lambda^n(e_1) = e_n$ for all nonnegative integers n:

  For every positive integer m, we have $\Psi^m(e_1) = p_m$ in **Symm** (because

$$\Psi^m\left(\underbrace{e_1}_{=\xi_1+\xi_2+\xi_3+\cdots=\sum_{i=1}^{\infty}\xi_i}\right) = \Psi^m\left(\sum_{i=1}^{\infty}\xi_i\right) = \sum_{i=1}^{\infty}\underbrace{\Psi^m(\xi_i)}_{\substack{=\xi_i^m \\ \text{(by (16.67), applied to} \\ m \text{ instead of } n)}} \qquad \text{(since } \Psi^m \text{ is a ring homomorphism)}$$

$$= \sum_{i=1}^{\infty}\xi_i^m = \xi_1^m + \xi_2^m + \xi_3^m + \cdots = p_m$$

). Now, for every integer $n \geq 0$, we have

$$n!\lambda^n = \det \begin{pmatrix} \Psi^1 & 1 & 0 & \cdots & 0 \\ \Psi^2 & \Psi^1 & 2 & \ddots & \vdots \\ \Psi^3 & \Psi^2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \Psi^1 & n-1 \\ \Psi^n & \Psi^{n-1} & \cdots & \Psi^2 & \Psi^1 \end{pmatrix} \qquad \text{(by (16.68))},$$

so that

$$\begin{aligned} n!\lambda^n (e_1) &= \left(\det \begin{pmatrix} \Psi^1 & 1 & 0 & \cdots & 0 \\ \Psi^2 & \Psi^1 & 2 & \ddots & \vdots \\ \Psi^3 & \Psi^2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \Psi^1 & n-1 \\ \Psi^n & \Psi^{n-1} & \cdots & \Psi^2 & \Psi^1 \end{pmatrix} \right) (e_1) \\ &= \det \begin{pmatrix} \Psi^1 (e_1) & 1 & 0 & \cdots & 0 \\ \Psi^2 (e_1) & \Psi^1 (e_1) & 2 & \ddots & \vdots \\ \Psi^3 (e_1) & \Psi^2 (e_1) & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \Psi^1 (e_1) & n-1 \\ \Psi^n (e_1) & \Psi^{n-1} (e_1) & \cdots & \Psi^2 (e_1) & \Psi^1 (e_1) \end{pmatrix} \\ &= \det \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \ddots & \vdots \\ p_3 & p_2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & p_1 & n-1 \\ p_n & p_{n-1} & \cdots & p_2 & p_1 \end{pmatrix} \qquad \text{(since } \Psi^m (e_1) = p_m \text{ for every positive integer } m\text{)} \\ &= e_n \qquad \text{(by a formula for symmetric functions)}, \end{aligned}$$

qed.

Second proof of $\lambda^n (e_1) = e_n$ *for all nonnegative integers* n*:*

Since the map

$$\mathbf{Z} [\xi] \to \Lambda (\mathbf{Z} [\xi]), \qquad x \mapsto \lambda_{-t} (x)^{-1}$$

is a λ-ring homomorphism (according to 16.4), we have $\left(\lambda_{-t} \left(\sum_{i=1}^{\infty} \xi_i \right) \right)^{-1} =$

$\prod_{i=1}^{\infty} (\lambda_{-t}(\xi_i))^{-1}$. Now, since $e_1 = \xi_1 + \xi_2 + \xi_3 + \cdots = \sum_{i=1}^{\infty} \xi_i$, we have

$$\lambda_{-t}(e_1) = \lambda_{-t}\left(\sum_{i=1}^{\infty} \xi_i\right) = \left(\underbrace{\left(\lambda_{-t}\left(\sum_{i=1}^{\infty} \xi_i\right)\right)^{-1}}_{=\prod_{i=1}^{\infty}(\lambda_{-t}(\xi_i))^{-1}}\right)^{-1} = \left(\prod_{i=1}^{\infty} (\lambda_{-t}(\xi_i))^{-1}\right)^{-1}$$

$$= \prod_{i=1}^{\infty} \underbrace{\lambda_{-t}(\xi_i)}_{\substack{=\sum_{j\geq 0} \lambda^j(\xi_i)(-t)^j \\ (\text{since } \lambda_t(\xi_i) = \sum_{j\geq 0} \lambda^j(\xi_i) t^j)}} = \prod_{i=1}^{\infty} \left(\underbrace{\sum_{j\geq 0} \lambda^j(\xi_i)(-t)^j}_{=\lambda^0(\xi_i)+\lambda^1(\xi_i)(-t)+\sum_{j\geq 2}\lambda^j(\xi_i)(-t)^j}\right)$$

$$= \prod_{i=1}^{\infty} \left(\underbrace{\lambda^0(\xi_i)}_{=1} + \underbrace{\lambda^1(\xi_i)}_{=\xi_i}(-t) + \sum_{j\geq 2} \underbrace{\lambda^j(\xi_i)}_{\substack{=0 \\ (\text{by (16.66), since } j\geq 2)}} (-t)^j\right)$$

$$= \prod_{i=1}^{\infty} \left(\underbrace{1 + \xi_i(-t)}_{=1-\xi_i t} + \underbrace{\sum_{j\geq 2} 0(-t)^j}_{=0}\right) = \prod_{i=1}^{\infty} (1 - \xi_i t) = \sum_{d\geq 0} (-1)^d e_d t^d$$

$$(\text{by a known formula from symmetric function theory}).$$

Compared with

$$\lambda_{-t}(e_1) = \sum_{d\geq 0} \lambda^d(e_1) \underbrace{(-t)^d}_{=(-1)^d t^d} \qquad \left(\text{since } \lambda_t(e_1) = \sum_{d\geq 0} \lambda^d(e_1) t^d\right)$$
$$= \sum_{d\geq 0} (-1)^d \lambda^d(e_1) t^d,$$

this yields $\sum_{d\geq 0} (-1)^d \lambda^d(e_1) t^d = \sum_{d\geq 0} (-1)^d e_d t^d$. Comparing coefficients in front of t^n on both sides of this equation, we obtain $(-1)^n \lambda^n(e_1) = (-1)^n e_n$ for every nonnegative integer n. In other words, $\lambda^n(e_1) = e_n$ for every nonnegative integer n, qed.

Is one of the two proofs above the proof you mean? If so, what does it have to do with (16.58)?

16.75

- In (b), you misspell "comonad" as "comad".

- In (b), replace "the coalgebras af" by "the coalgebras for".
- Replace "expliit" by "explicit" on the last line of 16.75.

16.76

- In (16.78), "the unique morphism of λ-rings" should be "the unique morphism ϕ_a of λ-rings" (otherwise, ϕ_a is never introduced).

16.80

- On page 102, replace "Write $g \in \mathbf{Symm} \subset \mathbf{Z}$" by "Write $g \in \mathbf{Symm} \subset \mathbf{Z}[\xi]$".
- On page 103, replace "(16.82)-(16.83)" by "(16.81)-(16.82)".
- Shortly after (16.82), in "a plethysm that was used in 9.93 above", I think you are referring to 9.63 rather than to 9.93.

16.84

- I would rather see the equalities (16.86) replaced by

$$\beta_f(a+b) = \sum_i \left(f'_{S,i} \circ a\right)\left(f''_{S,i} \circ b\right), \qquad \beta_f(ab) = \sum_i \left(f'_{P,i} \circ a\right)\left(f''_{P,i} \circ b\right),$$

  unless you have somewhere defined $f(a)$ to be a different notation for $f \circ a$.

- On page 103, you write: "As morphisms of rings $\mathbf{Symm} \overset{\varphi_a,\varphi_b}{\longrightarrow} A$." I would rather replace this by something like the following: "Let φ_a and φ_b the Witt vectors $\sigma_t(a)$ and $\sigma_t(b)$ considered as morphisms of rings $\mathbf{Symm} \to A$. Then, φ_a and φ_b are actually λ-ring morphisms (because the power series$\sigma_t(a)$ and $\sigma_t(b)$ have the property that their n-th coordinates are σ^n of their first coordinates for every n), and thus coincide with the maps α_a and α_b defined according to (16.78)."
- On page 103, replace "$\mu_{\mathbf{S}}$" by "μ_S" in the composition describing the sum Witt vector. (The S should not be boldfaced.)
- On page 103, replace "μ_p" by "μ_P".

Section 17

- On the left hand side of (17.1), there are two a_2t^2 terms. Obviously, one of them should be an a_3t^3 term.

17.3

- It might be good to point out that the notion of "primitive" defined here is not the same as the notion of "primitive" defined in subsection 16.70.
- Directly after (17.6), you write: "Then β_n is the rank of the homogeneous component of degree n of the free associative algebra over $\mathbf{Z}$ in α symbols." Of course, this is just a complicated way to say that $\beta_n = \alpha^n$, which agrees with (17.6).

17.10

- Between (17.11) and (17.12), you write: "All three identities (17.8), (17.8), (17.11)". One of the two (17.8)'s should be (17.9) here.

17.15

- The first sentence of subsection 17.15 reads: "As an Abelian group the necklace ring over a ring A is the infinite product $A^{\mathbb{N}}$ of all sequences $(a_1, a_2, a_3, \cdots)$, $a_i \in A$ with component wise addition." It is clear what you mean here, but it would make more sense if you add the word "consisting" between "infinite product $A^{\mathbb{N}}$" and "of all sequences" (since it is an infinite product of sets, not an infinite product of sequences).

17.29

- How do you define $(1 - t_n)^{-c_n}$ (in (17.30)) unless A is a binomial ring (and not just any torsion free ring)?

17.38

- Nothing wrong here, but you could be more explicit about the construction of $Bin^U(A)$. You tell it that it is "much like $IVal[X]$ compared to $\mathbb{Z}[X]$", which sounds as if it were so complicated that you would like to avoid it. Actually, $Bin^U(A)$ is the subring of $\mathbb{N}^{-1}A$ (this is the localization of the ring A at the multiplicative subset $\{1, 2, 3, ...\}$) formed by all elements of the form $\frac{a(a-1)...(a-n+1)}{n!}$ for $a \in A$ and $n \in \mathbb{N}$. This is what Theorem 7.1 (1)-(2) in [134] actually claims.

Section 18

18.24

- Between (18.26) and (18.27), you say that "H is coassociative and cocommutative". Maybe you actually mean "commutative and cocommutative" here? Maybe you want to assume cocommutativity (additionally to commutativity) throughout the whole 18.24? I am far from sure about this, as I haven't read Zelevinsky's monograph yet and I don't have your [199], but otherwise I don't

understand why $\beta_n(x)$ is symmetric in the variables ξ_1, ..., ξ_n.
(Note that in your "Niceness theorems" article, you claim that the Bernstein morphism is "defined for any commutative associative graded connected Hopf algebra H". Here I think you omitted the cocommutativity assumption.)

- It also seems to me that you want H to be connected, or how do you prove otherwise that β_n stabilizes in n?

- In footnote [90], I have a feeling that **Symm** means $(\mathbf{Symm}, \mu_P, \varepsilon_P)$ rather than $(\mathbf{Symm}, \mu_S, \varepsilon_S)$ as usual. Maybe worth a mention.

18.28

- In the very first absatz of 18.28, you write $\overline{a}_1 \geq \overline{a}_2 \geq \cdots \geq a_n$. Here, a_n should be $\overline{a}_n$.

Section 19

19.4

- In the second line of this subsection, "H-ser" should be "H-set".

19.16

- Directly after (19.17), you write: "This induces ring morphisms $\widehat{B}(\mathbb{Z}) \to \mathbb{Z}$". Maybe it would be better to say "This induces ring morphisms $\varphi_{n\mathbb{Z}} : \widehat{B}(\mathbb{Z}) \to \mathbb{Z}$".

19.21

- In the formula (19.24), $\varphi_{r/n}$ should be $\varphi_{(r/n)\mathbb{Z}}$.

- The left hand side of (19.29) should be $\varphi_{n\mathbb{Z}}(T(x_1, x_2, x_3, \cdots))$.

19.31

- The formula (19.34) should be

$$X \mapsto 1 + \sum_{n=1}^{\infty} (S^n X) t^n$$

or, equivalently,

$$X \mapsto \sum_{n=0}^{\infty} (S^n X) t^n$$

(since $S^0 X = \{\varnothing\}$ is the trivial G-set, which is the 1 in the Burnside ring).

- On the right hand side of (19.38), a $\cdots$ sign is missing (the equation ends with a $+$ sign).

19.41

- "quotients group" (in the second line of this subsection) should be "quotient groups".

19.43

- Replace "$\alpha \mapsto \alpha \circ h$" by "$\alpha \mapsto h \circ \alpha$".
- Directly before (19.44), "such that for all open subgroups" should be "such that for all open subgroups U " (in order to introduce U).
- (19.45) is not really clear: it should be an equation, not just a term. I guess you mean $\psi_U^A(\alpha) = ...$
- Directly after (19.45), you refer twice to "the sum (9.45)". I think you mean "the sum (19.45)" both of the times here.
- Two lines below (19.45), you write "such that U is subconjugate to U". Obviously you mean "such that U is subconjugate to V".

Appendix

- **A.7:** Footnote [107] on page 129 is wrong: The given definition of e^σ does not ensure that $e^{\sigma\tau} = (e^\sigma)^\tau$.
- **A.7:** On page 129, you write: "a power series with all but finitely many of the coefficients a_e unequal to zero". The "unequal" should be "equal".

References

- **[88]:** This reference should be: "Christol, Gilles, *Opération de Cartier et vecteurs de Witt*, Séminaire Delange-Pisot-Poitou, Théorie des nombres, tome 12 (1970-1971), exp. n° 13, pp. 1-7". The reference you gave seems to have been mixed together with [152].
- **[192]:** A more precise weblink for this reference is

  `https://repository.cwi.nl/noauth/search/fullrecord.php?publnr=10033`

  (though the scan is buggy at times).
- **[195]:** The arXiv ID for this is math/0410366, not math/0410365.
- **[197]:** A more precise weblink for this reference is

  `http://www.sciencedirect.com/science/article/pii/0021869383900807`

  for the published version and

  `https://repository.cwi.nl/noauth/search/fullrecord.php?publnr=10215`

  for a preprint.

- **[200]:** A more precise weblink for this reference is

  `https://repository.cwi.nl/noauth/search/fullrecord.php?publnr=10344`

  for a preprint and

  `https://repository.cwi.nl/noauth/search/fullrecord.php?publnr=10341`

  for the published version.

- **[201]:** This weblink no longer gives a downloadable file.

- [**215**] and [**216**] seem to be the same reference ([215] has a slight typo). It looks like [215] is never cited in the text, though (unless you are citing by interval like [210-219]).