An exercise on determinant-like sums

Darij Grinberg

January 10, 2019

The following exercise is inspired by Gabriel Dospinescu's [AndDos, Exercise 12.9]:

Exercise 1. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $n \in \mathbb{N}$ and $r \in \mathbb{N}$ with $r > 0$. The *sum* of a matrix shall mean the sum of its entries. If A is any matrix and i and j are two positive integers, then the (i, j)-th entry of A will be denoted by $A_{i,j}$. Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. If $\sigma \in S_n$ is a permutation, then $(-1)^\sigma$ shall denote the sign of σ.

Let m be the minimum sum of an $r \times n$-matrix $M \in \mathbb{N}^{r \times n}$ that has no two equal columns. (Explicitly, m can be computed as follows: Let s be the smallest nonnegative integer satisfying $\sum\limits_{t=0}^{s-1} \binom{r+t-1}{t} \leq n$, and write n in the form $n = \sum\limits_{t=0}^{s-1} \binom{r+t-1}{t} + w$. Then, $m = \sum\limits_{t=0}^{s-1} t \binom{r+t-1}{t} + sw$.)

(a) Let $A \in \mathbb{K}^{n \times n}$ be an $n \times n$-matrix of rank $\leq r$ over a field $\mathbb{K}$. Prove that

$$\sum_{\sigma \in S_n} (-1)^\sigma \left(\sum_{i=1}^{n} A_{i,\sigma(i)} \right)^k = 0$$

for each $k \in \{0, 1, \ldots, m-1\}$.

(b) Let $\mathbb{K}$ be a field of characteristic 0. Prove that m is the smallest $k \in \mathbb{N}$ such that there exists an $n \times n$-matrix $A \in \mathbb{K}^{n \times n}$ of rank $\leq r$ satisfying

$$\sum_{\sigma \in S_n} (-1)^\sigma \left(\sum_{i=1}^{n} A_{i,\sigma(i)} \right)^k \neq 0.$$

Remark 0.1. The "characteristic 0" condition in Exercise 1 **(b)** is important; if $\mathbb{K}$ has positive characteristic, then the smallest k can be larger than m.

Remark 0.2. Exercise 1 can be significantly shortened if we restrict ourselves to the case of fields of characteristic 0. In fact, in this case, part **(a)** can be removed, as its claim is contained in part **(b)**. We can shorten the exercise further if we let the reader figure out the value of m, i.e., if we pose it as follows:

"Let n and r be integers with $n \geq 0$ and $r > 0$. Let $\mathbb{K}$ be a field of characteristic 0. Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. If $\sigma \in S_n$ is a permutation, then $(-1)^\sigma$ shall denote the sign of σ. Find the smallest integer $k \geq 0$ such that there exists an $n \times n$-matrix $\left(A_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n} \in \mathbb{K}^{n \times n}$ of rank $\leq r$ satisfying

$$\sum_{\sigma \in S_n} (-1)^\sigma \left(\sum_{i=1}^{n} A_{i,\sigma(i)} \right)^k \neq 0.$$

"

Remark 0.3. The case $r = n$ of Exercise 1 is contained in [Grinbe15, Exercise 6.54], whereas the case $r = 1$ of Exercise 1 is contained in [Grinbe15, Exercise 6.55] (since an $n \times m$-matrix of rank ≤ 1 can always be written in the form $\left(a_i b_j\right)_{1 \leq i \leq n,\ 1 \leq j \leq m}$ for some $a_1, a_2, \ldots, a_n \in \mathbb{K}$ and $b_1, b_2, \ldots, b_m \in \mathbb{K}$). Both of these cases allow for explicit formulas for $\sum\limits_{\sigma \in S_n} (-1)^\sigma \left(\sum\limits_{i=1}^{n} A_{i,\sigma(i)} \right)^k$ when $k = m$; it is unclear whether such formulas exist for other values of r.

Solution sketch to Exercise 1. If $M \in \mathbb{N}^{r \times n}$ is any matrix, then ΣM shall denote the sum of M; in other words, $\Sigma M = \sum\limits_{i=1}^{n} \sum\limits_{h=1}^{r} M_{h,i} = M_{1,1} + M_{1,2} + \cdots + M_{r,n}$.

Furthermore, if $M \in \mathbb{N}^{r \times n}$ is any matrix, then $\mathbf{m}(M)$ shall denote the multinomial coefficient

$$\binom{M_{1,1} + M_{1,2} + \cdots + M_{r,n}}{M_{1,1}, M_{1,2}, \ldots, M_{r,n}} = \frac{\left(\sum\limits_{i=1}^{n} \sum\limits_{h=1}^{r} M_{h,i} \right)!}{\prod\limits_{i=1}^{n} \prod\limits_{h=1}^{r} M_{h,i}!} \in \mathbb{N}.$$

The multinomial formula shows that any $r \times n$-matrix $D \in \mathbb{K}^{r \times n}$ satisfies

$$\left(\sum_{i=1}^{n} \sum_{h=1}^{r} D_{h,i} \right)^k = \sum_{\substack{M \in \mathbb{N}^{r \times n}; \\ \Sigma M = k}} \mathbf{m}(M) \prod_{i=1}^{n} \prod_{h=1}^{r} D_{h,i}^{M_{h,i}}. \tag{1}$$

(a) Let $k \in \mathbb{N}$ be arbitrary.

The matrix A has rank $\leq r$. Hence, it can be written in the form $A = BC$ for some $n \times r$-matrix $B \in \mathbb{K}^{n\times r}$ and some $r \times n$-matrix $C \in \mathbb{K}^{r\times n}$ [1]. The equality $A = BC$ leads to

$$A_{i,j} = \sum_{h=1}^{r} B_{i,h} C_{h,j} \tag{2}$$

for each $(i,j) \in \{1,2,\ldots,n\}^2$. Hence,

$$\begin{aligned}
&\sum_{\sigma\in S_n} (-1)^\sigma \left(\sum_{i=1}^{n} \underbrace{A_{i,\sigma(i)}}_{\substack{=\sum_{h=1}^{r} B_{i,h}C_{h,\sigma(i)} \\ \text{(by (2))}}} \right)^k \\
&= \sum_{\sigma\in S_n} (-1)^\sigma \underbrace{\left(\sum_{i=1}^{n} \sum_{h=1}^{r} B_{i,h}C_{h,\sigma(i)} \right)^k}_{\substack{=\sum_{\substack{M\in\mathbb{N}^{r\times n};\\ \Sigma M=k}} \mathbf{m}(M) \prod_{i=1}^{n}\prod_{h=1}^{r} \left(B_{i,h}C_{h,\sigma(i)}\right)^{M_{h,i}} \\ \text{(by (1), applied to the } r\times n\text{-matrix } D\in\mathbb{K}^{r\times n} \\ \text{defined by } D_{h,i}=B_{i,h}C_{h,\sigma(i)})}} \\
&= \sum_{\sigma\in S_n} (-1)^\sigma \sum_{\substack{M\in\mathbb{N}^{r\times n};\\ \Sigma M=k}} \mathbf{m}(M) \underbrace{\prod_{i=1}^{n}\prod_{h=1}^{r} \left(B_{i,h}C_{h,\sigma(i)}\right)^{M_{h,i}}}_{=\left(\prod_{i=1}^{n}\prod_{h=1}^{r} B_{i,h}^{M_{h,i}}\right)\left(\prod_{i=1}^{n}\prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}}\right)} \\
&= \sum_{\sigma\in S_n} (-1)^\sigma \sum_{\substack{M\in\mathbb{N}^{r\times n};\\ \Sigma M=k}} \mathbf{m}(M) \left(\prod_{i=1}^{n}\prod_{h=1}^{r} B_{i,h}^{M_{h,i}}\right) \left(\prod_{i=1}^{n}\prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}}\right) \\
&= \sum_{\substack{M\in\mathbb{N}^{r\times n};\\ \Sigma M=k}} \mathbf{m}(M) \left(\prod_{i=1}^{n}\prod_{h=1}^{r} B_{i,h}^{M_{h,i}}\right) \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^{n}\prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}}.
\end{aligned} \tag{3}$$

Now, for each $M \in \mathbb{N}^{r\times n}$, define an $n \times n$-matrix $D_M \in \mathbb{K}^{n\times n}$ by

$$(D_M)_{i,j} = \prod_{h=1}^{r} C_{h,j}^{M_{h,i}}.$$

[1] This is a well-known fact from linear algebra: see `https://en.wikipedia.org/wiki/Rank_factorization`.

Then, this matrix D_M satisfies

$$\det(D_M) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \underbrace{(D_M)_{i,\sigma(i)}}_{\substack{=\prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}} \\ \text{(by the definition of } D_M)}}$$
$$= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}}. \tag{4}$$

If the matrix $M \in \mathbb{N}^{r \times n}$ has two equal columns (say, the p-th and the q-th column of M are equal), then the matrix D_M has two equal rows (viz., its p-th and its q-th row are equal) and thus satisfies

$$\det(D_M) = 0. \tag{5}$$

Now, (3) becomes

$$\sum_{\sigma \in S_n} (-1)^\sigma \left(\sum_{i=1}^{n} A_{i,\sigma(i)} \right)^k$$
$$= \sum_{\substack{M \in \mathbb{N}^{r \times n}; \\ \Sigma M = k}} \mathbf{m}(M) \left(\prod_{i=1}^{n} \prod_{h=1}^{r} B_{i,h}^{M_{h,i}} \right) \underbrace{\sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}}}_{\substack{=\det(D_M) \\ \text{(by (4))}}}$$
$$= \sum_{\substack{M \in \mathbb{N}^{r \times n}; \\ \Sigma M = k}} \mathbf{m}(M) \left(\prod_{i=1}^{n} \prod_{h=1}^{r} B_{i,h}^{M_{h,i}} \right) \det(D_M). \tag{6}$$

Now, assume that $k \in \{0, 1, \ldots, m-1\}$. Then, $k < m$.

But any $r \times n$-matrix $M \in \mathbb{N}^{r \times n}$ having sum $< m$ must have two equal columns (by the definition of m). Hence, any $r \times n$-matrix $M \in \mathbb{N}^{r \times n}$ having sum k must have two equal columns (since it has sum $k < m$). Thus, any $r \times n$-matrix $M \in \mathbb{N}^{r \times n}$ having sum k must satisfy

$$\det(D_M) = 0$$

(by (5)). Now, (6) becomes

$$\sum_{\sigma \in S_n} (-1)^\sigma \left(\sum_{i=1}^{n} A_{i,\sigma(i)} \right)^k = \sum_{\substack{M \in \mathbb{N}^{r \times n}; \\ \Sigma M = k}} \mathbf{m}(M) \left(\prod_{i=1}^{n} \prod_{h=1}^{r} B_{i,h}^{M_{h,i}} \right) \underbrace{\det(D_M)}_{\substack{=0 \\ \text{(by (6))}}} = 0.$$

This solves part **(a)** of the exercise.

(b) Because of part **(a)**, all we need to do is to show that there exists an $n \times n$-matrix $A \in \mathbb{K}^{n\times n}$ of rank $\leq r$ satisfying

$$\sum_{\sigma \in S_n} (-1)^{\sigma} \left(\sum_{i=1}^{n} A_{i,\sigma(i)} \right)^{m} \neq 0.$$

In order to show this, let us assume the contrary. Thus,

$$\sum_{\sigma \in S_n} (-1)^{\sigma} \left(\sum_{i=1}^{n} A_{i,\sigma(i)} \right)^{m} = 0 \qquad (7)$$

for each $n \times n$-matrix $A \in \mathbb{K}^{n\times n}$ of rank $\leq r$. In particular, (7) holds for each $n \times n$-matrix A of the form $A = BC$ with $B \in \mathbb{K}^{n\times r}$ being an $n \times r$-matrix and $C \in \mathbb{K}^{r\times n}$ being an $r \times n$-matrix (because any such matrix A has rank $\leq r$). In other words, each $n \times r$-matrix $B \in \mathbb{K}^{n\times r}$ and each $r \times n$-matrix $C \in \mathbb{K}^{r\times n}$ satisfy

$$\sum_{\substack{M \in \mathbb{N}^{r\times n};\\ \Sigma M = m}} \mathbf{m}(M) \left(\prod_{i=1}^{n} \prod_{h=1}^{r} B_{i,h}^{M_{h,i}} \right) \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} \prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}} = 0 \qquad (8)$$

[2]. Hence, (8) holds as a polynomial identity in the entries $B_{i,j}$ of B and the entries $C_{i,j}$ of C [3]. If we treat the $C_{i,j}$ as constants (temporarily), then the left hand side of (8) is a polynomial in the indeterminates $B_{i,j}$. By comparing coefficients in (8), we thus conclude that each $M \in \mathbb{N}^{r\times n}$ satisfying $\Sigma M = m$ must satisfy

$$\mathbf{m}(M) \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} \prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}} = 0 \qquad (9)$$

(because the monomials $\prod_{i=1}^{n} \prod_{h=1}^{r} B_{i,h}^{M_{h,i}}$ for different $M \in \mathbb{N}^{r\times n}$ are distinct, and thus we can compare coefficients in (8)). Since $\mathbf{m}(M) \neq 0$, we can cancel $\mathbf{m}(M)$ from (9) (since $\mathbb{K}$ has characteristic 0), and thus obtain

$$\sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} \prod_{h=1}^{r} C_{h,\sigma(i)}^{M_{h,i}} = 0. \qquad (10)$$

So (10) is a polynomial identity in the indeterminates $C_{i,j}$ that holds for each $M \in \mathbb{N}^{r\times n}$ satisfying $\Sigma M = m$.

But there exists an $r \times n$-matrix $M \in \mathbb{N}^{r\times n}$ that has no two equal columns and satisfies $\Sigma M = m$ (by the definition of m). Consider this matrix M.

[2] In fact, applying (3) to $k = m$, we conclude that the left hand side of (7) equals the left hand side of (8). Thus, the equality (8) follows from (7).

[3] This is because the field $\mathbb{K}$ is infinite (since it has characteristic 0).

There exists a $(p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$ such that the integers $\sum\limits_{h=1}^{r} p_h M_{h,i}$ for $i \in \{1, 2, \ldots, n\}$ are distinct[4]. Pick such a $(p_1, p_2, \ldots, p_r)$. For each $i \in \{1, 2, \ldots, n\}$, define a $q_i \in \mathbb{N}$ by $q_i = \sum\limits_{h=1}^{r} p_h M_{h,i}$. Then, the integers $q_1, q_2, \ldots, q_n$ are distinct (since the integers $\sum\limits_{h=1}^{r} p_h M_{h,i}$ for $i \in \{1, 2, \ldots, n\}$ are distinct).

Introduce n new indeterminates $z_1, z_2, \ldots, z_n$. Substitute $z_j^{p_i}$ for each $C_{i,j}$ in the identity (10). The result is the polynomial identity

$$\sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \prod_{h=1}^{r} \left(z_{\sigma(i)}^{p_h} \right)^{M_{h,i}} = 0.$$

Thus,

$$0 = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \underbrace{\prod_{h=1}^{r} \left(z_{\sigma(i)}^{p_h} \right)^{M_{h,i}}}_{\substack{= z_{\sigma(i)}^{\sum_{h=1}^{r} p_h M_{h,i}} = z_{\sigma(i)}^{q_i} \\ \text{(since } \sum_{h=1}^{r} p_h M_{h,i} = q_i \text{)}}} = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} z_{\sigma(i)}^{q_i}. \tag{12}$$

However, it is well-known that $\sum\limits_{\sigma \in S_n} (-1)^\sigma \prod\limits_{i=1}^{n} z_{\sigma(i)}^{q_i} \neq 0$ in $\mathbb{Q}[z_1, z_2, \ldots, z_n]$ [5]. This contradicts (12). This contradiction completes the proof. □

[4]*Proof.* Assume the contrary. Thus, for each $(p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$, there are at least two equal integers among the integers $\sum\limits_{h=1}^{r} p_h M_{h,i}$ for $i \in \{1, 2, \ldots, n\}$. In other words, for each $(p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$, we have

$$\prod_{1 \leq i < j \leq n} \left(\sum_{h=1}^{r} p_h M_{h,i} - \sum_{h=1}^{r} p_h M_{h,j} \right) = 0.$$

Since $\mathbb{N}^r$ is Zariski-dense in the $\mathbb{Q}$-vector space $\mathbb{Q}^r$, this shows that we have

$$\prod_{1 \leq i < j \leq n} \left(\sum_{h=1}^{r} x_h M_{h,i} - \sum_{h=1}^{r} x_h M_{h,j} \right) = 0$$

in the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_r]$. Since $\mathbb{Q}[x_1, x_2, \ldots, x_r]$ is an integral domain, we conclude that there exists a pair (i, j) of integers satisfying $1 \leq i < j \leq n$ and

$$\sum_{h=1}^{r} x_h M_{h,i} - \sum_{h=1}^{r} x_h M_{h,j} = 0. \tag{11}$$

Consider this (i, j). From (11), we easily conclude that $M_{h,i} = M_{h,j}$ for all $h \in \{1, 2, \ldots, r\}$. In other words, the i-th column of M equals the j-th column of M. This contradicts the fact that the matrix M has no two equal columns. This contradiction completes the proof.

[5]*Proof.* Recall that the integers $q_1, q_2, \ldots, q_n$ are distinct. Thus, there are no cancellations in the

References

[AndDos] Titu Andreescu, Gabriel Dospinescu, *Problems from the Book*, XYZ Press 2008.

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10 .

sum $\sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} z_{\sigma(i)}^{q_i}$; that is, any two $\sigma \in S_n$ give rise to distinct monomials. Hence, this sum is $\neq 0$. Qed.