

A Solomon Mackey formula for graded bialgebras

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Abstract. Given a graded bialgebra H , we let $\Delta^{[k]} : H \rightarrow H^{\otimes k}$ and $m^{[k]} : H^{\otimes k} \rightarrow H$ be its iterated (co)multiplications for all $k \in \mathbb{N}$. For any k -tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$ of nonnegative integers, and any permutation σ of $\{1, 2, \dots, k\}$, we consider the map $p_{\alpha, \sigma} := m^{[k]} \circ P_{\alpha} \circ \sigma^{-1} \circ \Delta^{[k]} : H \rightarrow H$, where P_{α} denotes the projection of $H^{\otimes k}$ onto its multigraded component $H_{\alpha_1} \otimes H_{\alpha_2} \otimes \dots \otimes H_{\alpha_k}$, and where $\sigma^{-1} : H \rightarrow H$ permutes the tensor factors.

We prove formulas for the composition $p_{\alpha, \sigma} \circ p_{\beta, \tau}$ and the convolution $p_{\alpha, \sigma} \star p_{\beta, \tau}$ of two such maps. When H is cocommutative, these generalize Patras's 1994 results (which, in turn, generalize Solomon's Mackey formula).

We also construct a combinatorial Hopf algebra PNSym ("permuted noncommutative symmetric functions") that governs the maps $p_{\alpha, \sigma}$ for arbitrary connected graded bialgebras H in the same way as the well-known NSym governs them in the cocommutative case. We end by outlining an application to checking identities for connected graded Hopf algebras.

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This is a preliminary report on a project. Its goal is to classify the identities that hold between the natural (\mathbf{k} -linear) operations on the category of graded \mathbf{k} -bialgebras. The following approach is likely improvable (being entirely focussed on unary operations $H \rightarrow H$, despite the multivalued operations $H^{\otimes k} \rightarrow H^{\otimes \ell}$ being probably a more natural object of study) and rather inchoate (the proofs lacking in elegance and readability), but the main result (Theorem 1.18) appears worthy of

dissemination and – to my great surprise – new. Even more surprisingly, a combinatorial Hopf algebra (named PNSym for “permuted noncommutative symmetric functions”) emerges from this study which, too, seems to have hitherto escaped the eyes of the algebraic combinatorics community. Thus I hope that this report will be of some use in the time before the right proofs are found and written up (hopefully without requiring a revision of the respective statements).

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0.1. Introduction

On any bialgebra H , we can define the *Adams operations* $\text{id}^{*k} = m^{[k]} \circ \Delta^{[k]}$ (where $\Delta^{[k]} : H \rightarrow H^{\otimes k}$ is an iterated comultiplication, and $m^{[k]} : H^{\otimes k} \rightarrow H$ is an iterated multiplication¹). These operations are natural in H (that is, equivariant with respect to bialgebra morphisms) and have been studied for years (see, e.g., [Kashin19] and [AguLau14] for some recent work), as have been several similar operators (e.g., [Patras94], [Loday98, §4.5], [PatReu98]) on Hopf algebras and graded bialgebras. More generally, for any bialgebra H , we can define the “twisted Adams operations” $m^{[k]} \circ \sigma^{-1} \circ \Delta^{[k]}$ for any permutation $\sigma \in \mathfrak{S}_k$ (where σ^{-1} acts on $H^{\otimes k}$ by permuting the tensorands)². When H is a graded bialgebra, one can further refine these operations by injecting a projection map P_α between the $m^{[k]}$ and the σ . To be specific: For any k -tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$, we let $P_\alpha : H^{\otimes k} \rightarrow H^{\otimes k}$ be the projection on the α -th multigraded component (i.e., the tensor product of the projections $H \rightarrow H_{\alpha_1}, H \rightarrow H_{\alpha_2}, \dots, H \rightarrow H_{\alpha_k}$).

The resulting “twisted projecting Adams operations” $p_{\alpha, \sigma} := m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]}$ (for $k \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_k$ and $\alpha \in \mathbb{N}^k$) have an interesting history. When H is cocommutative, they have been studied under the name of “opérateurs de descente” by Patras in [Patras94] and used by Reutenauer [Reuten93, §9.1] to prove Solomon’s Mackey formula for the symmetric group. The dual case – when H is commutative – is essentially equivalent. In both of these cases, the permutation σ is immaterial, since it can be swallowed either by the $\Delta^{[k]}$ (when H is cocommutative) or by the $m^{[k]}$ (when H is commutative). Thus, in these cases, the operators depend only on the k -tuple α . Moreover, when the graded bialgebra H is connected, the k -tuple α can be compressed by removing all 0’s from it, thus becoming a composition (a tuple of positive integers). One of Patras’s key results ([Patras94, Théorème II,7], [Reuten93, Theorem 9.2]) is a formula for the composition $p_{\alpha, \text{id}} \circ p_{\beta, \text{id}}$ of two such operators: When H is cocommutative and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$ and

¹These are commonly known as $\Delta^{(k-1)}$ and $m^{(k-1)}$, but we prefer the superscripts to match the number of tensorands.

²The notation \mathfrak{S}_k means the k -th symmetric group. Our use of σ^{-1} instead of σ is meant to simplify the formula for the action.

$\beta = (\beta_1, \beta_2, \dots, \beta_\ell) \in \mathbb{N}^\ell$, it claims that

$$p_{\alpha, \text{id}} \circ p_{\beta, \text{id}} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}), \text{id}} \quad (1)$$

(where $[n] := \{1, 2, \dots, n\}$ for each $n \in \mathbb{N}$, and where $(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})$ denotes the list of all $k\ell$ numbers $\gamma_{i,j}$ listed in lexicographic order of their subscripts)³. When H is furthermore connected (i.e., when $H_0 \cong \mathbf{k}$), we can restrict ourselves to compositions by removing all 0's from our tuples. In that situation, the formula (1) is structurally identical to the expansion of the internal product of two complete homogeneous noncommutative symmetric functions in NSym (see [GKLLRT94, Proposition 5.1]), which reveals that the operators $p_{\alpha, \text{id}}$ are images of the latter functions under an algebra morphism from NSym (under the internal product) to End H . Thus, noncommutative symmetric functions govern many natural operations for cocommutative bialgebras. A dual theory exists for commutative H .

0.2. Plan of this paper

To my knowledge, no analogue of the formula (1) has been proposed for general H . This general case is somewhat complicated by the fact that the operators of the form $p_{\alpha, \text{id}}$ are no longer closed under composition; but the ones of the more general form $p_{\alpha, \sigma}$ still are. The main result of this paper (Theorem 1.18) is a formula that generalizes (1) to this case. I also discuss how the operators $p_{\alpha, \sigma}$ behave under convolution (Proposition 1.15) and tensoring (Proposition 1.37), and show that they can – under certain conditions – be linearly independent (Theorem 1.35). I suspect that the operators $p_{\alpha, \sigma}$ and their infinite \mathbf{k} -linear combinations span all possible natural \mathbf{k} -linear endomorphisms on the category of connected graded bialgebras (i.e., all \mathbf{k} -linear maps $H \rightarrow H$ that are defined for every connected graded bialgebra H and are functorial with respect to graded bialgebra morphisms); while I have been unable to prove this, this conjecture is easily verified for all known natural endomorphisms I have found in the literature.

In Section 2, I define a combinatorial Hopf algebra PNSym (the “permuted noncommutative symmetric functions”) that governs the “twisted projecting descent operations” $p_{\alpha, \sigma}$ on an arbitrary connected graded bialgebra H just like NSym does for the cocommutative case. This Hopf algebra appears to be new, and I would not be surprised if interesting things can be said about it.

In the final Section 3, I discuss an application of the above results: Namely, they can be used to mechanically verify any identity between operators of the form $p_{\alpha, \sigma}$ (and their sums, convolutions and compositions) that hold for arbitrary (connected) graded bialgebras. The subject is still in its early days, and we include 12 open questions (most in the last two sections).

³To be precise, both [Patras94, Théorème II,7] and [Reuten93, Theorem 9.2] are restricting themselves to the case when H is connected.

0.3. Notations

Let $\mathbb{N} = \{0, 1, 2, \dots\}$.

We fix a commutative ring \mathbf{k} , which shall serve as our base ring throughout this paper. In particular, all modules, algebras, coalgebras, bialgebras and Hopf algebras will be over \mathbf{k} . Tensor products and hom spaces are defined over \mathbf{k} as well. Algebras are understood to be unital and associative unless said otherwise.

We will use standard concepts – such as iterated (co)multiplications, tensor products, (co)commutativity, etc. – related to bialgebras (and occasionally Hopf algebras). The reader can find them explained, e.g., in [GriRei20, Chapter 1].

If H is any bialgebra, then the multiplication, the unit, the comultiplication, and the counit of H (regarded as linear maps) will be denoted by

$$\begin{aligned} m_H : H \otimes H &\rightarrow H, & u_H : \mathbf{k} &\rightarrow H, \\ \Delta_H : H &\rightarrow H \otimes H, & \epsilon_H : H &\rightarrow \mathbf{k}, \end{aligned}$$

respectively. If no ambiguity is to be feared, then we will abbreviate them as m , u , Δ and ϵ . We furthermore denote the unity of a ring R (viewed as an element of R) by 1_R .

Graded \mathbf{k} -modules are always understood to be \mathbb{N} -graded, i.e., to have direct sum decompositions $V = \bigoplus_{n \in \mathbb{N}} V_n$. The n -th graded component of a graded \mathbf{k} -module V will be called V_n . If $n < 0$, then this is the zero submodule 0 . Tensor products of graded \mathbf{k} -modules are equipped with the usual grading:

$$\left(A_{(1)} \otimes A_{(2)} \otimes \cdots \otimes A_{(k)} \right)_n = \sum_{\substack{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k; \\ i_1 + i_2 + \cdots + i_k = n}} \left(A_{(1)} \right)_{i_1} \otimes \left(A_{(2)} \right)_{i_2} \otimes \cdots \otimes \left(A_{(k)} \right)_{i_k}.$$

A \mathbf{k} -linear map $f : U \rightarrow V$ between two graded \mathbf{k} -modules U and V is said to be *graded* if every $n \in \mathbb{N}$ satisfies $f(U_n) \subseteq V_n$.

Graded bialgebras are bialgebras whose operations (m , u , Δ and ϵ) are graded \mathbf{k} -linear maps. We do **not** twist our tensor products by the grading (i.e., we do **not** follow the topologists' sign conventions); in particular, the tensor product $A \otimes B$ of two algebras has its multiplication defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, regardless of any possible gradings on A and B .

We recall that a non-graded bialgebra can be viewed as a graded bialgebra concentrated in degree 0 (that is, a graded bialgebra H with $H = H_0$). Thus, all claims about graded bialgebras that we make below can be specialized to non-graded bialgebras.

A graded \mathbf{k} -bialgebra H is said to be *connected* if $H_0 = \mathbf{k} \cdot 1_H$. A known fact (e.g., [GriRei20, Proposition 1.4.16]) says that any connected graded bialgebra is automatically a Hopf algebra, i.e., has an antipode. Hopf algebras are not at the center of our present work, but will occasionally appear in applications.

If H is any \mathbf{k} -module, then $\text{End } H$ shall denote the \mathbf{k} -module of \mathbf{k} -linear maps from H to H . This \mathbf{k} -module $\text{End } H$ becomes an algebra under composition of

maps. As usual, we denote this composition operation by \circ (so that $f \circ g$ means the composition of two maps f and g , sending each $x \in H$ to $f(g(x))$). The neutral element of this composition is the identity map $\text{id}_H \in \text{End } H$.

If H is a bialgebra, then the \mathbf{k} -module $\text{End } H$ has yet another canonical multiplication, known as *convolution* and denoted by \star ; it is defined by⁴

$$f \star g := m_H \circ (f \otimes g) \circ \Delta_H \quad \text{for all } f, g \in \text{End } H.$$

This operation \star is associative and has the neutral element $u_H \circ \epsilon_H$; thus, $\text{End } H$ becomes an algebra under this operation. See [GriRei20, Definition 1.4.1] for more about convolution.

Thus, when H is a bialgebra, $\text{End } H$ becomes a \mathbf{k} -algebra in two natural ways: once using the composition \circ , and once using the convolution \star .

1. The maps $p_{\alpha, \sigma}$ for a graded bialgebra H

1.1. Definitions

Let H be a graded \mathbf{k} -bialgebra. We fix it for the rest of Section 1.

The set $\text{End}_{\text{gr}} H$ of all graded \mathbf{k} -module endomorphisms of H is a \mathbf{k} -submodule of $\text{End } H$, and is preserved under both composition and convolution (and thus is a \mathbf{k} -subalgebra of both the composition algebra $\text{End } H$ and the convolution algebra $\text{End } H$). Furthermore, we can consider the \mathbf{k} -submodule $\mathbf{E}(H)$ of $\text{End}_{\text{gr}} H$ that consists only of those $f \in \text{End}_{\text{gr}} H$ that annihilate all but finitely many graded components of H (that is, that satisfy $f(H_n) = 0$ for all sufficiently high n). This submodule $\mathbf{E}(H)$ is itself graded, with the n -th graded component being canonically isomorphic to $\text{End}(H_n)$. This submodule $\mathbf{E}(H)$, too, is preserved under both composition and convolution (but usually does not contain id_H , whence it is not a subalgebra of the composition algebra $\text{End } H$).

This module $\mathbf{E}(H)$ has been studied, e.g., by Hazewinkel [Hazewi04]. Unlike him, we shall not consider $\mathbf{E}(H)$ for any specific H , but we shall instead focus on the “generic” $\mathbf{E}(H)$. In other words, we will consider the functorial endomorphisms of H defined for all graded bialgebras. Such endomorphisms include

- the projections p_0, p_1, p_2, \dots (where $p_n \in \text{End } H$ is the canonical projection of H on its n -th graded component H_n);
- their convolutions $p_{(i_1, i_2, \dots, i_k)} := p_{i_1} \star p_{i_2} \star \dots \star p_{i_k}$ with $i_1, i_2, \dots, i_k \in \mathbb{N}$;
- the (nonempty) compositions $p_\alpha \circ p_\beta \circ \dots \circ p_\kappa$ of such convolutions.

However, we can define a broader class of such endomorphisms. To do so, we need some notations:

⁴Actually, convolution is defined in the same way for \mathbf{k} -linear maps from any given coalgebra to any given algebra.

Definition 1.1. For each $n \in \mathbb{N}$, we let $p_n : H \rightarrow H$ denote the canonical projection of the graded module H on its n -th graded component H_n .

Definition 1.2. For each $k \in \mathbb{N}$, we set $[k] := \{1, 2, \dots, k\}$.

Definition 1.3. For each $k \in \mathbb{N}$, we let \mathfrak{S}_k denote the k -th symmetric group, i.e., the group of all permutations of $[k]$.

Definition 1.4. For each $k \in \mathbb{N}$, we let the group \mathfrak{S}_k act on $H^{\otimes k}$ from the left by permuting tensorands, according to the rule

$$\sigma \cdot (h_1 \otimes h_2 \otimes \cdots \otimes h_k) = h_{\sigma^{-1}(1)} \otimes h_{\sigma^{-1}(2)} \otimes \cdots \otimes h_{\sigma^{-1}(k)}$$

for all $\sigma \in \mathfrak{S}_k$ and $h_1, h_2, \dots, h_k \in H$.

This is an action by bialgebra endomorphisms.

Definition 1.5. For any $k \in \mathbb{N}$, we let $m^{[k]} : H^{\otimes k} \rightarrow H$ and $\Delta^{[k]} : H \rightarrow H^{\otimes k}$ be the iterated multiplication and iterated comultiplication maps of the bialgebra H (denoted $m^{(k-1)}$ and $\Delta^{(k-1)}$ in [GriRei20, Exercises 1.4.19 and 1.4.20], and denoted $\Pi^{[k]}$ and $\Delta^{[k]}$ in [Patras94]). Note that $m^{[k]}$ sends each pure tensor $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in H^{\otimes k}$ to the product $a_1 a_2 \cdots a_k$, whereas $\Delta^{[k]}$ sends each element $x \in H$ to $\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(k)}$ (using Sweedler notation).

We recall that these iterated multiplications and comultiplications satisfy the rules

$$m^{[u+v]} = m \circ \left(m^{[u]} \otimes m^{[v]} \right) \tag{2}$$

and

$$\Delta^{[u+v]} = \left(\Delta^{[u]} \otimes \Delta^{[v]} \right) \circ \Delta \tag{3}$$

for all $u, v \in \mathbb{N}$. (Indeed, these are the claims of [GriRei20, Exercise 1.4.19(a)] and [GriRei20, Exercise 1.4.20(a)], respectively, applied to $i = u - 1$ and $k = u + v - 1$.) Also, $m^{[1]} = \Delta^{[1]} = \text{id}_H$ and $m^{[0]} = u : \mathbf{k} \rightarrow H$ and $\Delta^{[0]} = \epsilon : H \rightarrow \mathbf{k}$.

Definition 1.6. (a) A *weak composition* means a finite tuple of nonnegative integers.

(b) If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a weak composition, then its *size* $|\alpha|$ is defined to be the number $\alpha_1 + \alpha_2 + \cdots + \alpha_k \in \mathbb{N}$.

Definition 1.7. For any weak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, we define the projection map $P_\alpha : H^{\otimes k} \rightarrow H^{\otimes k}$ to be the tensor product $p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}$ of the \mathbf{k} -linear maps $p_{\alpha_1}, p_{\alpha_2}, \dots, p_{\alpha_k}$. (Thus, if we regard $H^{\otimes k}$ as an \mathbb{N}^k -graded \mathbf{k} -module, then P_α is its projection on its degree- $(\alpha_1, \alpha_2, \dots, \alpha_k)$ component.)

Definition 1.8. For any weak composition $\alpha \in \mathbb{N}^k$ and any permutation $\sigma \in \mathfrak{S}_k$, we define a map $p_{\alpha,\sigma} : H \rightarrow H$ by the formula

$$p_{\alpha,\sigma} := m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]}, \quad (4)$$

where the σ^{-1} really means the action of $\sigma^{-1} \in \mathfrak{S}_k$ on $H^{\otimes k}$ (as in Definition 1.4).

We can rewrite the definition of $p_{\alpha,\sigma}$ in more concrete terms using the Sweedler notation:

Remark 1.9. For any weak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$ and any permutation $\sigma \in \mathfrak{S}_k$, the map $p_{\alpha,\sigma} : H \rightarrow H$ is given by

$$p_{\alpha,\sigma}(x) = \sum_{(x)} p_{\alpha_1}(x_{(\sigma(1))}) p_{\alpha_2}(x_{(\sigma(2))}) \cdots p_{\alpha_k}(x_{(\sigma(k))})$$

for every $x \in H$, where we are using the Sweedler notation $\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(k)}$ for the iterated coproduct $\Delta^{[k]}(x) \in H^{\otimes k}$.

The following is near-obvious:

Proposition 1.10. Let $\alpha \in \mathbb{N}^k$ be a weak composition, and $\sigma \in \mathfrak{S}_k$ a permutation. Then, the map $p_{\alpha,\sigma}$ is a graded \mathbf{k} -module endomorphism of H that sends $H_{|\alpha|}$ to $H_{|\alpha|}$ and sends all other graded components H_n to 0. Thus, $p_{\alpha,\sigma}$ lies in the $|\alpha|$ -th graded component of $\mathbf{E}(H)$.

Proof. The gradedness of $p_{\alpha,\sigma}$ follows from the fact that all four maps $m^{[k]}$, P_α , σ^{-1} and $\Delta^{[k]}$ in (4) are graded. Thus, the map $p_{\alpha,\sigma}$ sends $H_{|\alpha|}$ to $H_{|\alpha|}$. To see that it sends all other H_n to 0, we only need to observe that for every $n \neq |\alpha|$, we have $(\sigma^{-1} \circ \Delta^{[k]})(H_n) \subseteq (H^{\otimes k})_n$ (since σ^{-1} and $\Delta^{[k]}$ are graded), and that P_α annihilates $(H^{\otimes k})_n$ (since $n \neq |\alpha|$). \square

1.2. Reductions

Let us contrast our maps $p_{\alpha,\sigma}$ to Patras's descent operators p_α (denoted B_α in [Patras94, §II]). We recall how the latter are defined (generalizing slightly from compositions to weak compositions):

Definition 1.11. For any weak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$, we define a map $p_\alpha : H \rightarrow H$ by the formula

$$p_\alpha := p_{\alpha_1} \star p_{\alpha_2} \star \cdots \star p_{\alpha_k}.$$

(Recall that \star denotes convolution.)

Our $p_{\alpha,\sigma}$ recover these for $\sigma = \text{id}$:

Proposition 1.12. Let $\alpha \in \mathbb{N}^k$ be a weak composition. Then, $p_{\alpha,\text{id}} = p_\alpha$ (where id is the identity permutation in \mathfrak{S}_k).

Proof. Write α as $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$. Then, (4) yields

$$p_{\alpha,\text{id}} = m^{[k]} \circ P_\alpha \circ \underbrace{\text{id}^{-1}}_{=\text{id}} \circ \Delta^{[k]} = m^{[k]} \circ P_\alpha \circ \Delta^{[k]}.$$

On the other hand, [GriRei20, Exercise 1.4.23] yields⁵

$$p_{\alpha_1} \star p_{\alpha_2} \star \dots \star p_{\alpha_k} = m^{[k]} \circ \underbrace{(p_{\alpha_1} \otimes p_{\alpha_2} \otimes \dots \otimes p_{\alpha_k})}_{=P_\alpha \text{ (by the definition of } P_\alpha)} \circ \Delta^{[k]} = m^{[k]} \circ P_\alpha \circ \Delta^{[k]}.$$

Comparing these two equalities, we obtain $p_{\alpha,\text{id}} = p_{\alpha_1} \star p_{\alpha_2} \star \dots \star p_{\alpha_k} = p_\alpha$ (since p_α is defined as $p_{\alpha_1} \star p_{\alpha_2} \star \dots \star p_{\alpha_k}$). This proves Proposition 1.12. \square

If the bialgebra H is commutative or cocommutative, then we can bring all our $p_{\alpha,\sigma}$ to the form p_β for some weak composition β :

Proposition 1.13. Let $\alpha \in \mathbb{N}^k$ be a weak composition, and $\sigma \in \mathfrak{S}_k$ a permutation.

(a) If H is commutative, then

$$p_{\alpha,\sigma} = p_{\sigma \cdot \alpha},$$

where we are using the action of \mathfrak{S}_k on \mathbb{N}^k defined by $\sigma \cdot (\alpha_1, \alpha_2, \dots, \alpha_k) := (\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \dots, \alpha_{\sigma^{-1}(k)})$.

(b) If H is cocommutative, then

$$p_{\alpha,\sigma} = p_\alpha.$$

To prove this, we need a simple lemma:

Lemma 1.14. Let f_1, f_2, \dots, f_k be k arbitrary elements of $\text{End } H$. Let $\sigma \in \mathfrak{S}_k$. Then,

$$\sigma \circ (f_1 \otimes f_2 \otimes \dots \otimes f_k) = (f_{\sigma^{-1}(1)} \otimes f_{\sigma^{-1}(2)} \otimes \dots \otimes f_{\sigma^{-1}(k)}) \circ \sigma$$

(where σ and $f_1 \otimes f_2 \otimes \dots \otimes f_k$ and $f_{\sigma^{-1}(1)} \otimes f_{\sigma^{-1}(2)} \otimes \dots \otimes f_{\sigma^{-1}(k)}$ are understood as endomorphisms of $H^{\otimes k}$).

⁵Recall that the maps $m^{[k]}$ and $\Delta^{[k]}$ are called $m^{(k-1)}$ and $\Delta^{(k-1)}$ in [GriRei20].

Proof of Lemma 1.14. Both sides send any given pure tensor $h_1 \otimes h_2 \otimes \cdots \otimes h_k \in H^{\otimes k}$ to

$$f_{\sigma^{-1}(1)}(h_{\sigma^{-1}(1)}) \otimes f_{\sigma^{-1}(2)}(h_{\sigma^{-1}(2)}) \otimes \cdots \otimes f_{\sigma^{-1}(k)}(h_{\sigma^{-1}(k)}).$$

Thus, they agree on all pure tensors, and hence are identical. \square

Proof of Proposition 1.13. (a) Assume that H is commutative. Then, $m^{[k]} \circ \tau = m^{[k]}$ for any $\tau \in \mathfrak{S}_k$ (by [GriRei20, Exercise 1.5.10]). Thus, in particular, $m^{[k]} \circ \sigma^{-1} = m^{[k]}$. However, it is straightforward to see (and actually true for any \mathbf{k} -module in place of H) that

$$\sigma \circ P_\alpha = P_{\sigma \cdot \alpha} \circ \sigma$$

as maps from $H^{\otimes k}$ to $H^{\otimes k}$. (Indeed, this follows from Lemma 1.14 (applied to $f_i = p_{\alpha_i}$), since $P_\alpha = p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}$ and $P_{\sigma \cdot \alpha} = p_{\alpha_{\sigma^{-1}(1)}} \otimes p_{\alpha_{\sigma^{-1}(2)}} \otimes \cdots \otimes p_{\alpha_{\sigma^{-1}(k)}}$.)

Now, the definition of $p_{\alpha, \sigma}$ yields

$$\begin{aligned} p_{\alpha, \sigma} &= m^{[k]} \circ \underbrace{P_\alpha}_{=\sigma^{-1} \circ P_{\sigma \cdot \alpha} \circ \sigma} \circ \sigma^{-1} \circ \Delta^{[k]} \\ &= \underbrace{m^{[k]} \circ \sigma^{-1}}_{=m^{[k]}} \circ P_{\sigma \cdot \alpha} \circ \underbrace{\sigma \circ \sigma^{-1}}_{=\text{id}} \circ \Delta^{[k]} = m^{[k]} \circ P_{\sigma \cdot \alpha} \circ \Delta^{[k]}. \end{aligned}$$

Comparing this with

$$\begin{aligned} p_{\sigma \cdot \alpha} &= p_{\sigma \cdot \alpha, \text{id}} \quad (\text{by Proposition 1.12, applied to } \sigma \cdot \alpha \text{ instead of } \alpha) \\ &= m^{[k]} \circ P_{\sigma \cdot \alpha} \circ \text{id}^{-1} \circ \Delta^{[k]} \quad (\text{by the definition of } p_{\sigma \cdot \alpha, \text{id}}) \\ &= m^{[k]} \circ P_{\sigma \cdot \alpha} \circ \Delta^{[k]}, \end{aligned}$$

we obtain $p_{\alpha, \sigma} = p_{\sigma \cdot \alpha}$. Proposition 1.13 (a) is thus proved.

(b) Assume that H is cocommutative. Then, $\tau \circ \Delta^{[k]} = \Delta^{[k]}$ for any $\tau \in \mathfrak{S}_k$ (by [GriRei20, Exercise 1.5.10]). Thus, in particular, $\sigma^{-1} \circ \Delta^{[k]} = \Delta^{[k]}$. Now, the definition of $p_{\alpha, \sigma}$ yields

$$p_{\alpha, \sigma} = m^{[k]} \circ P_\alpha \circ \underbrace{\sigma^{-1} \circ \Delta^{[k]}}_{=\Delta^{[k]} = \text{id}^{-1} \circ \Delta^{[k]}} = m^{[k]} \circ P_\alpha \circ \text{id}^{-1} \circ \Delta^{[k]}.$$

Comparing this with

$$\begin{aligned} p_\alpha &= p_{\alpha, \text{id}} \quad (\text{by Proposition 1.12}) \\ &= m^{[k]} \circ P_\alpha \circ \text{id}^{-1} \circ \Delta^{[k]} \quad (\text{by the definition of } p_{\alpha, \text{id}}), \end{aligned}$$

we find $p_{\alpha, \sigma} = p_\alpha$. This proves Proposition 1.13 (b). \square

However, in general, when H is neither commutative nor cocommutative, we cannot “simplify” $p_{\alpha,\sigma}$. (See Theorem 1.35 for a concretization of this claim.)

If the graded bialgebra H is connected, then each $p_{\alpha,\sigma}$ for a weak composition α and a permutation σ can be rewritten in the form $p_{\beta,\tau}$ for a composition (not just weak composition) β and a permutation τ . (Indeed, since H is connected, we can remove all $p_0(x_{(i)})$ factors from the product in Definition 1.9.) See Proposition 2.4 below for an explicit statement of this claim.

1.3. The convolution formula

It is easy to see that any convolution of two maps of the form $p_{\alpha,\sigma}$ is again a map of such form:

Proposition 1.15. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation.

Let $\beta = (\beta_1, \beta_2, \dots, \beta_\ell)$ be a weak composition, and let $\tau \in \mathfrak{S}_\ell$ be a permutation.

Let $\alpha\beta$ be the concatenation of α and β ; this is the weak composition $(\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_\ell)$.

Let $\sigma \oplus \tau$ be the image of (σ, τ) under the obvious map $\mathfrak{S}_k \times \mathfrak{S}_\ell \rightarrow \mathfrak{S}_{k+\ell}$. Explicitly, this is the permutation of $[k + \ell]$ that sends each $i \leq k$ to $\sigma(i)$ and sends each $j > k$ to $k + \tau(j - k)$.

Then,

$$p_{\alpha,\sigma} \star p_{\beta,\tau} = p_{\alpha\beta,\sigma \oplus \tau}. \tag{5}$$

Proof. Let us first observe that

$$\underbrace{\sigma^{-1}}_{\substack{\text{this means} \\ \text{the action} \\ \text{of } \sigma^{-1} \text{ on } H^{\otimes k}}} \otimes \underbrace{\tau^{-1}}_{\substack{\text{this means} \\ \text{the action} \\ \text{of } \tau^{-1} \text{ on } H^{\otimes \ell}}} = \underbrace{(\sigma \oplus \tau)^{-1}}_{\substack{\text{this means} \\ \text{the action} \\ \text{of } (\sigma \oplus \tau)^{-1} \text{ on } H^{\otimes (k+\ell)}}}.$$

(This can be shown, e.g., by acting on a pure tensor: Any pure tensor

$$h_1 \otimes h_2 \otimes \dots \otimes h_k \otimes g_1 \otimes g_2 \otimes \dots \otimes g_\ell \in H^{\otimes (k+\ell)}$$

is sent by both $\sigma^{-1} \otimes \tau^{-1}$ and $(\sigma \oplus \tau)^{-1}$ to the same image

$$h_{\sigma(1)} \otimes h_{\sigma(2)} \otimes \dots \otimes h_{\sigma(k)} \otimes g_{\tau(1)} \otimes g_{\tau(2)} \otimes \dots \otimes g_{\tau(\ell)}.$$

Thus, the two \mathbf{k} -linear maps $\sigma^{-1} \otimes \tau^{-1}$ and $(\sigma \oplus \tau)^{-1}$ agree on each pure tensor, and thus are identical.)

The definition of convolution yields $p_{\alpha,\sigma} \star p_{\beta,\tau} = m \circ (p_{\alpha,\sigma} \otimes p_{\beta,\tau}) \circ \Delta$. In view of

$$\begin{aligned}
 & \underbrace{p_{\alpha,\sigma}}_{=m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \text{ (by (4))}} \otimes \underbrace{p_{\beta,\tau}}_{=m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]} \text{ (by (4))}} \\
 &= \left(m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \right) \otimes \left(m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]} \right) \\
 &= \left(m^{[k]} \otimes m^{[\ell]} \right) \circ \underbrace{(P_\alpha \otimes P_\beta)}_{=P_{\alpha\beta}} \circ \underbrace{(\sigma^{-1} \otimes \tau^{-1})}_{=(\sigma \oplus \tau)^{-1}} \circ (\Delta^{[k]} \otimes \Delta^{[\ell]}) \\
 &= \left(m^{[k]} \otimes m^{[\ell]} \right) \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ (\Delta^{[k]} \otimes \Delta^{[\ell]}),
 \end{aligned}$$

we can rewrite this as

$$\begin{aligned}
 p_{\alpha,\sigma} \star p_{\beta,\tau} &= m \circ \underbrace{\left(m^{[k]} \otimes m^{[\ell]} \right)}_{=m^{[k+\ell]} \text{ (by (2))}} \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ \underbrace{\left(\Delta^{[k]} \otimes \Delta^{[\ell]} \right)}_{=\Delta^{[k+\ell]} \text{ (by (3))}} \circ \Delta \\
 &= m^{[k+\ell]} \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ \Delta^{[k+\ell]} = p_{\alpha\beta, \sigma \oplus \tau}
 \end{aligned}$$

(since $p_{\alpha\beta, \sigma \oplus \tau}$ is defined as $m^{[k+\ell]} \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ \Delta^{[k+\ell]}$). This proves Proposition 1.15. \square

1.4. The composition formula: statement

It is more interesting to compute the composition of two maps of the form $p_{\alpha,\sigma}$. It turns out that this is again a \mathbf{k} -linear combination of maps of such form, and the explicit formula is similar to Solomon's Mackey formula for the descent algebra (or, even more closely related, [Reuten93, Theorem 9.2 and §9.5.1]). Before we state the formula, let us introduce one more operation on permutations:

Definition 1.16. Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$ be two permutations. Then, $\tau[\sigma] \in \mathfrak{S}_{k\ell}$ shall denote the permutation of $[k\ell]$ that sends each $\ell(i-1) + j$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\tau(j)-1) + \sigma(i)$.

This is well-defined because each element $p \in [k\ell]$ can be written uniquely in the form $\ell(i-1) + j$ with $i \in [k]$ and $j \in [\ell]$. (To see this, observe that the $k+1$ numbers $0\ell, 1\ell, 2\ell, \dots, k\ell$ subdivide the set $[k\ell]$ into k intervals of length ℓ each.)

Example 1.17. Let $s_1 \in \mathfrak{S}_2$ be the transposition that swaps 1 with 2. Let $\text{id}_3 \in \mathfrak{S}_3$ be the identity permutation. Then, $s_1[\text{id}_3] \in \mathfrak{S}_6$ is the permutation that sends 1, 2, 3, 4, 5, 6 to 2, 4, 6, 1, 3, 5, respectively.

We can now state the explicit formula for composition of $p_{\alpha,\sigma}$'s:

Theorem 1.18. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation.

Let $\beta = (\beta_1, \beta_2, \dots, \beta_\ell)$ be a weak composition, and let $\tau \in \mathfrak{S}_\ell$ be a permutation.

Then,

$$p_{\alpha, \sigma} \circ p_{\beta, \tau} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} p_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}), \tau[\sigma]}.$$

Here, $(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})$ denotes the $k\ell$ -tuple consisting of all $k\ell$ numbers $\gamma_{i,j}$ (for all $i \in [k]$ and $j \in [\ell]$) listed in lexicographically increasing order of subscripts (i.e., the number $\gamma_{i,j}$ comes before $\gamma_{u,v}$ if and only if either $i < u$ or ($i = u$ and $j < v$)).

We note that the sum in Theorem 1.18 can be viewed as a sum over all $k \times \ell$ -matrices $(\gamma_{i,j})_{i \in [k], j \in [\ell]} \in \mathbb{N}^{k \times \ell}$ with row sums α (that is, the sum of all entries in the i -th row of the matrix equals α_i) and column sums β . Such matrices are known as *contingency tables* (with marginal distributions α and β), and have found ample uses in combinatorics.

Example 1.19. Let $s_1 \in \mathfrak{S}_2$ be the transposition that swaps 1 with 2. Then, the permutation $s_1[s_1] \in \mathfrak{S}_4$ sends 1, 2, 3, 4 to 4, 2, 3, 1.

Let us consider two weak compositions (a, b) and (c, d) in \mathbb{N}^2 . Then, Theorem 1.18 says that

$$p_{(a,b), s_1} \circ p_{(c,d), s_1} = \sum_{\substack{\gamma_{1,1}, \gamma_{1,2}, \gamma_{2,1}, \gamma_{2,2} \in \mathbb{N}; \\ \gamma_{1,1} + \gamma_{1,2} = a; \quad \gamma_{2,1} + \gamma_{2,2} = b; \\ \gamma_{1,1} + \gamma_{2,1} = c; \quad \gamma_{1,2} + \gamma_{2,2} = d}} p_{(\gamma_{1,1}, \gamma_{1,2}, \gamma_{2,1}, \gamma_{2,2}), s_1[s_1]}.$$

This is a sum over all 2×2 -matrices $(\gamma_{i,j})_{i \in [2], j \in [2]} \in \mathbb{N}^{2 \times 2}$ with row sums (a, b) and column sums (c, d) . How this sum looks like depends on whether $a + b$ equals $c + d$ or not:

- If $a + b \neq c + d$, then there are no such matrices, and therefore the sum is empty. Thus,

$$p_{(a,b), s_1} \circ p_{(c,d), s_1} = 0 \quad \text{in this case.}$$

This is not surprising, since the image of the map $p_{(c,d), s_1}$ is contained in the graded component H_{c+d} of H , whereas the map $p_{(a,b), s_1}$ is 0 on this component.

- If $a + b = c + d$, then these matrices are precisely the matrices of the form $\begin{pmatrix} i & a-i \\ c-i & i-g \end{pmatrix}$, where $g = c - b = a - d$ and where $i \in \mathbb{Z}$ satisfies

$\max\{0, g\} \leq i \leq \min\{a, c\}$. Thus, our above formula becomes

$$P_{(a,b),s_1} \circ P_{(c,d),s_1} = \sum_{i=\max\{0,g\}}^{\min\{a,c\}} P_{(i, a-i, c-i, i-g), s_1[s_1]} \quad \text{where } g = c - b = a - d.$$

For example, for $a = b = c = d = 1$, this simplifies to

$$P_{(1,1),s_1} \circ P_{(1,1),s_1} = P_{(0,1,1,0),s_1[s_1]} + P_{(1,0,0,1),s_1[s_1]}.$$

This all is not hard to check by hand.

Remark 1.20. A bialgebra without a grading can always be interpreted as a graded bialgebra H concentrated in degree 0 (that is, satisfying $H_0 = H$ and $H_k = 0$ for all $k > 0$). Thus, Theorem 1.18 can be applied to bialgebras without a grading. If H is such a bialgebra, then $P_{\alpha,\sigma} = 0$ whenever the weak composition α has any nonzero entry. Thus, the claim of Theorem 1.18 can be greatly simplified in this case. The resulting claim is precisely the equality $\Psi^{(n,\sigma)} \circ \Psi^{(m,\tau)} = \Psi^{(nm, \Phi(\sigma,\tau))}$ in Pirashvili's [Pirash02, §1]. (Note that the $\Phi(\sigma, \tau)$ in [Pirash02, Proposition 5.3] is exactly our $\sigma[\tau]$.)

1.5. The composition formula: lemmas on tensors

As preparation for the proof of Theorem 1.18, we shall first work out how the different kinds of operators composed in (4) (iterated multiplications, comultiplications, projections, permutations) can be “commuted” past each other. We begin with the simplest cases: the ones concerning projections and permutations. These commutation formulas have nothing to do with bialgebras, and would equally make sense for any graded module instead of H .

Recall that \mathbb{N}^k (for any given $k \in \mathbb{N}$) is the set of all weak compositions of length k . The symmetric group \mathfrak{S}_k acts from the right on this set \mathbb{N}^k by permuting the entries: For any $(\gamma_1, \gamma_2, \dots, \gamma_k) \in \mathbb{N}^k$ and $\pi \in \mathfrak{S}_k$, we have

$$(\gamma_1, \gamma_2, \dots, \gamma_k) \cdot \pi = \left(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \dots, \gamma_{\pi(k)} \right).$$

This action has the following property:

Lemma 1.21. For any $\pi \in \mathfrak{S}_k$ and $\gamma \in \mathbb{N}^k$, we have

$$P_\gamma \circ \pi = \pi \circ P_{\gamma \cdot \pi} \tag{6}$$

and

$$\pi \circ P_\gamma = P_{\gamma \cdot \pi^{-1}} \circ \pi. \tag{7}$$

Proof. Let $\pi \in \mathfrak{S}_k$ and $\gamma \in \mathbb{N}^k$. Write γ as $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$. Then, the map $P_\gamma \circ \pi : H^{\otimes k} \rightarrow H^{\otimes k}$ sends each pure tensor $h_1 \otimes h_2 \otimes \dots \otimes h_k$ to

$$\begin{aligned} & P_\gamma (\pi (h_1 \otimes h_2 \otimes \dots \otimes h_k)) \\ &= P_\gamma (h_{\pi^{-1}(1)} \otimes h_{\pi^{-1}(2)} \otimes \dots \otimes h_{\pi^{-1}(k)}) \\ &= p_{\gamma_1} (h_{\pi^{-1}(1)}) \otimes p_{\gamma_2} (h_{\pi^{-1}(2)}) \otimes \dots \otimes p_{\gamma_k} (h_{\pi^{-1}(k)}), \end{aligned}$$

whereas the map $\pi \circ P_{\gamma \cdot \pi} : H^{\otimes k} \rightarrow H^{\otimes k}$ sends it to

$$\begin{aligned} & \pi (P_{\gamma \cdot \pi} (h_1 \otimes h_2 \otimes \dots \otimes h_k)) \\ &= \pi (p_{\gamma_{\pi(1)}} (h_1) \otimes p_{\gamma_{\pi(2)}} (h_2) \otimes \dots \otimes p_{\gamma_{\pi(k)}} (h_k)) \\ & \quad \left(\begin{array}{c} \text{since } \gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \\ \text{entails } \gamma \cdot \pi = (\gamma_{\pi(1)}, \gamma_{\pi(2)}, \dots, \gamma_{\pi(k)}) \end{array} \right) \\ &= p_{\gamma_{\pi(\pi^{-1}(1))}} (h_{\pi^{-1}(1)}) \otimes p_{\gamma_{\pi(\pi^{-1}(2))}} (h_{\pi^{-1}(2)}) \otimes \dots \otimes p_{\gamma_{\pi(\pi^{-1}(k))}} (h_{\pi^{-1}(k)}) \\ &= p_{\gamma_1} (h_{\pi^{-1}(1)}) \otimes p_{\gamma_2} (h_{\pi^{-1}(2)}) \otimes \dots \otimes p_{\gamma_k} (h_{\pi^{-1}(k)}), \end{aligned}$$

which is the same result. Thus, the linear maps $P_\gamma \circ \pi$ and $\pi \circ P_{\gamma \cdot \pi}$ agree on each pure tensor, and therefore are identical. This proves (6).

Applying (6) to $\gamma \cdot \pi^{-1}$ instead of γ , we obtain

$$P_{\gamma \cdot \pi^{-1}} \circ \pi = \pi \circ \underbrace{P_{\gamma \cdot \pi^{-1} \cdot \pi}}_{=P_\gamma} = \pi \circ P_\gamma.$$

This proves (7). □

Moreover, the following holds:

Lemma 1.22. Let $\alpha, \beta \in \mathbb{N}^k$ be two weak compositions. Then,

$$P_\alpha \circ P_\beta = \begin{cases} P_\alpha, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Proof. Obvious. □

Lemma 1.23. Let $k, \ell \in \mathbb{N}$. Let $\zeta \in \mathfrak{S}_{k\ell}$ be the permutation of $[k\ell]$ that sends each $k(j-1) + i$ (with $i \in [k]$ and $j \in [\ell]$) to $\ell(i-1) + j$. This is called the *Zolotarev shuffle* (and appears, e.g., as $\nu^{-1}\mu$ in [Rousse94]).

Let $(h_{i,j})_{i \in [\ell], j \in [k]} \in H^{\ell \times k}$ be any $\ell \times k$ -matrix over H . Then,

$$\begin{aligned}
 & \zeta (h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\
 & \quad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\
 & \quad \otimes \cdots \\
 & \quad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}) \\
 &= h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1} \\
 & \quad \otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2} \\
 & \quad \otimes \cdots \\
 & \quad \otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}.
 \end{aligned} \tag{8}$$

(Here, the $h_{i,j}$ on the left hand side appear in the order of lexicographically increasing pairs (i, j) , whereas the $h_{i,j}$ on the right hand side appear in the order of lexicographically increasing pairs (j, i) .)

Proof. Define two maps λ and ρ from $[k] \times [\ell]$ to $[k\ell]$ by setting

$$\lambda(i, j) := k(j - 1) + i \in [k\ell] \quad \text{and} \quad \rho(i, j) := \ell(i - 1) + j \in [k\ell]$$

for all $(i, j) \in [k] \times [\ell]$. These maps λ and ρ are bijections. In fact, ρ sends each pair (i, j) to the position of (i, j) in the lexicographically ordered Cartesian product $[k] \times [\ell]$, whereas λ sends each pair (i, j) to the position of (j, i) in the lexicographically ordered Cartesian product $[\ell] \times [k]$.

Recall that all $i \in [k]$ and $j \in [\ell]$ satisfy $\zeta(k(j - 1) + i) = \ell(i - 1) + j$ (by the definition of ζ). In other words, all $i \in [k]$ and $j \in [\ell]$ satisfy $\zeta(\lambda(i, j)) = \rho(i, j)$ (since $\lambda(i, j) = k(j - 1) + i$ and $\rho(i, j) = \ell(i - 1) + j$). In other words,

$$\zeta \circ \lambda = \rho. \tag{9}$$

Set $g_{(j,i)} := h_{i,j}$ for all $(j, i) \in [k] \times [\ell]$. Recall that the bijection λ sends each pair (i, j) to the position of (j, i) in the lexicographically ordered Cartesian product $[\ell] \times [k]$. Hence, the list $(\lambda^{-1}(1), \lambda^{-1}(2), \dots, \lambda^{-1}(k\ell))$ consists of all the $k\ell$ pairs $(i, j) \in [k] \times [\ell]$ in the order of (lexicographically) increasing pairs (j, i) . In other words,

$$\begin{aligned}
 (\lambda^{-1}(1), \lambda^{-1}(2), \dots, \lambda^{-1}(k\ell)) &= ((1, 1), (2, 1), \dots, (k, 1), \\
 & \quad (1, 2), (2, 2), \dots, (k, 2), \\
 & \quad \dots, \\
 & \quad (1, \ell), (2, \ell), \dots, (k, \ell)).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)} &= g_{(1,1)} \otimes g_{(2,1)} \otimes \cdots \otimes g_{(k,1)} \\
 &\quad \otimes g_{(1,2)} \otimes g_{(2,2)} \otimes \cdots \otimes g_{(k,2)} \\
 &\quad \otimes \cdots \\
 &\quad \otimes g_{(1,\ell)} \otimes g_{(2,\ell)} \otimes \cdots \otimes g_{(k,\ell)} \\
 &= h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\
 &\quad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\
 &\quad \otimes \cdots \\
 &\quad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}
 \end{aligned}$$

(since $g_{(j,i)} = h_{i,j}$ for all j and i). By a similar argument, we obtain

$$\begin{aligned}
 g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)} &= h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1} \\
 &\quad \otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2} \\
 &\quad \otimes \cdots \\
 &\quad \otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}.
 \end{aligned}$$

In view of these two equalities, we must show that

$$\zeta \left(g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)} \right) = g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)}.$$

Since

$$\zeta \left(g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)} \right) = g_{\lambda^{-1}(\zeta^{-1}(1))} \otimes g_{\lambda^{-1}(\zeta^{-1}(2))} \otimes \cdots \otimes g_{\lambda^{-1}(\zeta^{-1}(k\ell))},$$

this is equivalent to showing that

$$g_{\lambda^{-1}(\zeta^{-1}(1))} \otimes g_{\lambda^{-1}(\zeta^{-1}(2))} \otimes \cdots \otimes g_{\lambda^{-1}(\zeta^{-1}(k\ell))} = g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)}.$$

Thus, we need to check that $\lambda^{-1}(\zeta^{-1}(q)) = \rho^{-1}(q)$ for each $q \in [k\ell]$. In other words, we need to check that $\lambda^{-1} \circ \zeta^{-1} = \rho^{-1}$. But this follows from (9), since $\lambda^{-1} \circ \zeta^{-1} = (\zeta \circ \lambda)^{-1} = \rho^{-1}$ (by (9)). Hence, Lemma 1.23 is proven. \square

Lemma 1.24. Let $k, \ell \in \mathbb{N}$. Let $\sigma \in \mathfrak{S}_k$. Let $\sigma^{\times \ell}$ denote the permutation in $\mathfrak{S}_{k\ell}$ that sends each $\ell(i-1) + j$ (with $i \in [k]$ and $j \in [\ell]$) to $\ell(\sigma(i)-1) + j$.

Let $f : H^{\otimes \ell} \rightarrow H$ be any \mathbf{k} -linear map. Then,

$$\sigma \circ f^{\otimes k} = f^{\otimes k} \circ \sigma^{\times \ell}$$

(as maps from $H^{\otimes k\ell}$ to $H^{\otimes k}$).

Proof. We begin with an even more basic claim:

Claim 1: Let $a_1, a_2, \dots, a_k \in H^{\otimes \ell}$. Then,

$$\sigma^{\times \ell} (a_1 \otimes a_2 \otimes \dots \otimes a_k) = a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \dots \otimes a_{\sigma^{-1}(k)}$$

in $H^{\otimes k\ell}$. (Here, we are identifying $(H^{\otimes \ell})^{\otimes k}$ with $H^{\otimes k\ell}$ in the obvious way.)

Proof of Claim 1. Let $\omega = \sigma^{\times \ell}$.

The equality we need to prove depends linearly on each of a_1, a_2, \dots, a_k . Hence, we can WLOG assume that each of a_1, a_2, \dots, a_k is a pure tensor. Assume this. Thus,

$$a_1 = g_1 \otimes g_2 \otimes \dots \otimes g_\ell, \tag{10}$$

$$a_2 = g_{\ell+1} \otimes g_{\ell+2} \otimes \dots \otimes g_{2\ell}, \tag{11}$$

...

$$a_k = g_{(k-1)\ell+1} \otimes g_{(k-1)\ell+2} \otimes \dots \otimes g_{k\ell} \tag{12}$$

for some $g_1, g_2, \dots, g_{k\ell} \in H$. Consider these $g_1, g_2, \dots, g_{k\ell}$. Thus,

$$a_1 \otimes a_2 \otimes \dots \otimes a_k = g_1 \otimes g_2 \otimes \dots \otimes g_{k\ell}$$

(as we can see by tensoring together the equalities (10), (11), ..., (12)). Applying the permutation $\sigma^{\times \ell} \in \mathfrak{S}_{k\ell}$ to both sides of this equality, we obtain

$$\begin{aligned} & \sigma^{\times \ell} (a_1 \otimes a_2 \otimes \dots \otimes a_k) \\ &= \underbrace{\sigma^{\times \ell}}_{=\omega} (g_1 \otimes g_2 \otimes \dots \otimes g_{k\ell}) \\ &= \omega (g_1 \otimes g_2 \otimes \dots \otimes g_{k\ell}) \\ &= g_{\omega^{-1}(1)} \otimes g_{\omega^{-1}(2)} \otimes \dots \otimes g_{\omega^{-1}(k\ell)}. \end{aligned} \tag{13}$$

We shall now prove that

$$\begin{aligned} & a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \dots \otimes a_{\sigma^{-1}(k)} \\ &= g_{\omega^{-1}(1)} \otimes g_{\omega^{-1}(2)} \otimes \dots \otimes g_{\omega^{-1}(k\ell)}. \end{aligned} \tag{14}$$

Indeed, in order to prove this identity, it clearly suffices to show that

$$a_{\sigma^{-1}(i)} = g_{\omega^{-1}(\ell(i-1)+1)} \otimes g_{\omega^{-1}(\ell(i-1)+2)} \otimes \dots \otimes g_{\omega^{-1}(i\ell)} \tag{15}$$

for each $i \in [k]$ (because then, tensoring the equalities (15) together for all $i \in [k]$ will yield (14)). But this is easy: Let $i \in [k]$. Then, the definition of $a_{\sigma^{-1}(i)}$ yields

$$a_{\sigma^{-1}(i)} = g_{\ell(\sigma^{-1}(i)-1)+1} \otimes g_{\ell(\sigma^{-1}(i)-1)+2} \otimes \dots \otimes g_{\ell(\sigma^{-1}(i))}. \tag{16}$$

But each $j \in [\ell]$ satisfies

$$\ell(\sigma^{-1}(i) - 1) + j = \omega^{-1}(\ell(i-1) + j)$$

(since $\omega = \sigma^{\times \ell}$ was defined to send $\ell(\sigma^{-1}(i) - 1) + j$ to $\ell\left(\underbrace{\sigma(\sigma^{-1}(i))}_{=i} - 1\right) + j = \ell(i - 1) + j$) and thus $g_{\ell(\sigma^{-1}(i)-1)+j} = g_{\omega^{-1}(\ell(i-1)+j)}$. Hence, the right hand side of (16) equals the right hand side of (15). Therefore, (15) follows from (16) (since these two equalities have the same left hand side).

Forget that we fixed i . We thus have proved (15) for each $i \in [k]$. As explained, this proves (14).

Comparing (13) with (14), we find

$$\sigma^{\times \ell}(a_1 \otimes a_2 \otimes \cdots \otimes a_k) = a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}.$$

This proves Claim 1. □

The rest is easy: Let $\mathbf{a} = a_1 \otimes a_2 \otimes \cdots \otimes a_k$ be a pure tensor in $(H^{\otimes \ell})^{\otimes k}$ (with $a_1, a_2, \dots, a_k \in H^{\otimes \ell}$). Then,

$$\begin{aligned} (\sigma \circ f^{\otimes k})(\mathbf{a}) &= (\sigma \circ f^{\otimes k})(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \\ &= \sigma\left(\underbrace{f^{\otimes k}(a_1 \otimes a_2 \otimes \cdots \otimes a_k)}_{=f(a_1) \otimes f(a_2) \otimes \cdots \otimes f(a_k)}\right) \\ &= \sigma(f(a_1) \otimes f(a_2) \otimes \cdots \otimes f(a_k)) \\ &= f(a_{\sigma^{-1}(1)}) \otimes f(a_{\sigma^{-1}(2)}) \otimes \cdots \otimes f(a_{\sigma^{-1}(k)}) \end{aligned}$$

and

$$\begin{aligned} (f^{\otimes k} \circ \sigma^{\times \ell})(\mathbf{a}) &= (f^{\otimes k} \circ \sigma^{\times \ell})(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \\ &= f^{\otimes k}\left(\underbrace{\sigma^{\times \ell}(a_1 \otimes a_2 \otimes \cdots \otimes a_k)}_{\substack{=a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)} \\ \text{(by Claim 1)}}}\right) \\ &= f^{\otimes k}(a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}) \\ &= f(a_{\sigma^{-1}(1)}) \otimes f(a_{\sigma^{-1}(2)}) \otimes \cdots \otimes f(a_{\sigma^{-1}(k)}). \end{aligned}$$

Comparing these two equalities, we find $(\sigma \circ f^{\otimes k})(\mathbf{a}) = (f^{\otimes k} \circ \sigma^{\times \ell})(\mathbf{a})$.

Forget that we fixed \mathbf{a} . We thus have proved that $(\sigma \circ f^{\otimes k})(\mathbf{a}) = (f^{\otimes k} \circ \sigma^{\times \ell})(\mathbf{a})$ for each pure tensor \mathbf{a} in $(H^{\otimes \ell})^{\otimes k}$. In other words, the two maps $\sigma \circ f^{\otimes k}$ and $f^{\otimes k} \circ \sigma^{\times \ell}$ agree on all pure tensors in $(H^{\otimes \ell})^{\otimes k}$. Since these two maps are \mathbf{k} -linear

(and since the pure tensors span $(H^{\otimes \ell})^{\otimes k}$), this entails that they must be identical. In other words, $\sigma \circ f^{\otimes k} = f^{\otimes k} \circ \sigma^{\times \ell}$. This proves Lemma 1.24. \square

Lemma 1.25. Let $k, \ell \in \mathbb{N}$. Let $\tau \in \mathfrak{S}_\ell$. Let $\tau^{k \times}$ denote the permutation in $\mathfrak{S}_{k\ell}$ that sends each $k(j-1) + i$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\tau(j) - 1) + i$.

Let $f : H \rightarrow H^{\otimes k}$ be any \mathbf{k} -linear map. Then,

$$f^{\otimes \ell} \circ \tau = \tau^{k \times} \circ f^{\otimes \ell}$$

(as maps from $H^{\otimes \ell}$ to $H^{\otimes k\ell}$).

Proof. We begin with an even more basic claim:

Claim 1: Let $a_1, a_2, \dots, a_\ell \in H^{\otimes k}$. Then,

$$\tau^{k \times} (a_1 \otimes a_2 \otimes \dots \otimes a_\ell) = a_{\tau^{-1}(1)} \otimes a_{\tau^{-1}(2)} \otimes \dots \otimes a_{\tau^{-1}(\ell)}$$

in $H^{\otimes k\ell}$. (Here, we are identifying $(H^{\otimes k})^{\otimes \ell}$ with $H^{\otimes k\ell}$ in the obvious way.)

Proof of Claim 1. Analogous to the Claim 1 in our above proof of Lemma 1.24. \square

Now, let $\mathbf{a} = a_1 \otimes a_2 \otimes \dots \otimes a_\ell$ be a pure tensor in $H^{\otimes \ell}$ (with $a_1, a_2, \dots, a_\ell \in H$). Then,

$$\begin{aligned} (f^{\otimes \ell} \circ \tau)(\mathbf{a}) &= (f^{\otimes \ell} \circ \tau)(a_1 \otimes a_2 \otimes \dots \otimes a_\ell) \\ &= f^{\otimes \ell} \left(\underbrace{\tau(a_1 \otimes a_2 \otimes \dots \otimes a_\ell)}_{=a_{\tau^{-1}(1)} \otimes a_{\tau^{-1}(2)} \otimes \dots \otimes a_{\tau^{-1}(\ell)}} \right) \\ &= f^{\otimes \ell} (a_{\tau^{-1}(1)} \otimes a_{\tau^{-1}(2)} \otimes \dots \otimes a_{\tau^{-1}(\ell)}) \\ &= f(a_{\tau^{-1}(1)}) \otimes f(a_{\tau^{-1}(2)}) \otimes \dots \otimes f(a_{\tau^{-1}(\ell)}) \end{aligned}$$

and

$$\begin{aligned} (\tau^{k \times} \circ f^{\otimes \ell})(\mathbf{a}) &= (\tau^{k \times} \circ f^{\otimes \ell})(a_1 \otimes a_2 \otimes \dots \otimes a_\ell) \\ &= \tau^{k \times} \left(\underbrace{f^{\otimes \ell}(a_1 \otimes a_2 \otimes \dots \otimes a_\ell)}_{=f(a_1) \otimes f(a_2) \otimes \dots \otimes f(a_\ell)} \right) \\ &= \tau^{k \times} (f(a_1) \otimes f(a_2) \otimes \dots \otimes f(a_\ell)) \\ &= f(a_{\tau^{-1}(1)}) \otimes f(a_{\tau^{-1}(2)}) \otimes \dots \otimes f(a_{\tau^{-1}(\ell)}) \end{aligned}$$

(by Claim 1, applied to $f(a_i)$ instead of a_i). Comparing these two equalities, we find $(f^{\otimes \ell} \circ \tau)(\mathbf{a}) = (\tau^{k \times} \circ f^{\otimes \ell})(\mathbf{a})$.

Forget that we fixed \mathbf{a} . We thus have proved that $(f^{\otimes \ell} \circ \tau)(\mathbf{a}) = (\tau^{k \times} \circ f^{\otimes \ell})(\mathbf{a})$ for each pure tensor \mathbf{a} in $H^{\otimes \ell}$. In other words, the two maps $f^{\otimes \ell} \circ \tau$ and $\tau^{k \times} \circ f^{\otimes \ell}$ agree on all pure tensors in $H^{\otimes \ell}$. Since these two maps are \mathbf{k} -linear (and since the pure tensors span $H^{\otimes \ell}$), this entails that they must be identical. In other words, $f^{\otimes \ell} \circ \tau = \tau^{k \times} \circ f^{\otimes \ell}$. This proves Lemma 1.25. \square

Lemma 1.26. Let $k, \ell \in \mathbb{N}$. Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$. Define a permutation $\sigma^{\times \ell} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.24, and define a permutation $\tau^{k \times} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.25. Define a permutation $\zeta \in \mathfrak{S}_{k\ell}$ as in Lemma 1.23. Recall also the permutation $\tau[\sigma]$ defined in Definition 1.16. Then,

$$\tau^{k \times} \circ \zeta^{-1} \circ \sigma^{\times \ell} = \tau[\sigma] \quad (17)$$

and

$$(\sigma^{\times \ell})^{-1} \circ \zeta \circ (\tau^{k \times})^{-1} = (\tau[\sigma])^{-1}. \quad (18)$$

Proof. Let us first prove (17). Let $n \in [k\ell]$. Write n in the form $n = \ell(i-1) + j$ for some $i \in [k]$ and $j \in [\ell]$. (This is clearly possible.) Thus,

$$\sigma^{\times \ell}(n) = \sigma^{\times \ell}(\ell(i-1) + j) = \ell(\sigma(i) - 1) + j$$

(by the definition of $\sigma^{\times \ell}$). Hence,

$$\zeta^{-1}(\sigma^{\times \ell}(n)) = \zeta^{-1}(\ell(\sigma(i) - 1) + j) = k(j-1) + \sigma(i)$$

(since the definition of ζ yields $\zeta(k(j-1) + \sigma(i)) = \ell(\sigma(i) - 1) + j$). Hence,

$$\tau^{k \times}(\zeta^{-1}(\sigma^{\times \ell}(n))) = \tau^{k \times}(k(j-1) + \sigma(i)) = k(\tau(j) - 1) + \sigma(i)$$

(by the definition of $\tau^{k \times}$). Comparing this with

$$\begin{aligned} (\tau[\sigma])(n) &= (\tau[\sigma])(\ell(i-1) + j) && \text{(since } n = \ell(i-1) + j) \\ &= k(\tau(j) - 1) + \sigma(i) && \text{(by the definition of } \tau[\sigma]), \end{aligned}$$

we obtain $\tau^{k \times}(\zeta^{-1}(\sigma^{\times \ell}(n))) = (\tau[\sigma])(n)$.

Forget that we fixed n . We thus have shown that $\tau^{k \times}(\zeta^{-1}(\sigma^{\times \ell}(n))) = (\tau[\sigma])(n)$ for each $n \in [k\ell]$. In other words, $\tau^{k \times} \circ \zeta^{-1} \circ \sigma^{\times \ell} = \tau[\sigma]$. This proves (17).

Now, taking inverses on both sides of (17), we obtain $(\tau^{k \times} \circ \zeta^{-1} \circ \sigma^{\times \ell})^{-1} = (\tau[\sigma])^{-1}$. In other words, $(\sigma^{\times \ell})^{-1} \circ \zeta \circ (\tau^{k \times})^{-1} = (\tau[\sigma])^{-1}$ (since $(\tau^{k \times} \circ \zeta^{-1} \circ \sigma^{\times \ell})^{-1} = (\sigma^{\times \ell})^{-1} \circ \zeta \circ (\tau^{k \times})^{-1}$). This proves (18). \square

Recall again that each symmetric group \mathfrak{S}_k acts on the corresponding set \mathbb{N}^k of k -tuples from the right. For this action, we have two further elementary combinatorial properties:

Lemma 1.27. Let $k, \ell \in \mathbb{N}$. Define a permutation $\zeta \in \mathfrak{S}_{k\ell}$ as in Lemma 1.23.

Let $\delta_{i,j} \in \mathbb{N}$ be a nonnegative integer for each $i \in [k]$ and $j \in [\ell]$. Then,

$$(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell}) \cdot \zeta = (\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell}).$$

(Here, $(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})$ denotes the list of all $k\ell$ numbers $\delta_{i,j}$ in the order of lexicographically increasing pairs (i, j) , whereas $(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})$ denotes the list of all $k\ell$ numbers $\delta_{i,j}$ in the order of lexicographically increasing pairs (j, i) .)

Proof. Let us write $\delta_{(i,j)}$ for $\delta_{i,j}$. Thus, δ_p is defined for any pair $p \in [k] \times [\ell]$.

Define the two bijections λ and ρ as in the proof of Lemma 1.23. Then,

$$\begin{aligned} (\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell}) &= (\delta_{\rho^{-1}(1)}, \delta_{\rho^{-1}(2)}, \dots, \delta_{\rho^{-1}(k\ell)}) \quad \text{and} \\ (\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell}) &= (\delta_{\lambda^{-1}(1)}, \delta_{\lambda^{-1}(2)}, \dots, \delta_{\lambda^{-1}(k\ell)}). \end{aligned}$$

Thus,

$$\begin{aligned} &\underbrace{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})}_{=(\delta_{\rho^{-1}(1)}, \delta_{\rho^{-1}(2)}, \dots, \delta_{\rho^{-1}(k\ell)})} \cdot \zeta = (\delta_{\rho^{-1}(1)}, \delta_{\rho^{-1}(2)}, \dots, \delta_{\rho^{-1}(k\ell)}) \cdot \zeta \\ &= (\delta_{\rho^{-1}(\zeta(1))}, \delta_{\rho^{-1}(\zeta(2))}, \dots, \delta_{\rho^{-1}(\zeta(k\ell))}) \\ &= (\delta_{\lambda^{-1}(1)}, \delta_{\lambda^{-1}(2)}, \dots, \delta_{\lambda^{-1}(k\ell)}) \\ &\quad \left(\begin{array}{l} \text{since each } i \in [k\ell] \\ \text{satisfies } \rho^{-1}(\zeta(i)) = \lambda^{-1}(i) \\ \text{(since (9) yields } \rho^{-1} \circ \zeta = \lambda^{-1}) \\ \text{and thus } \delta_{\rho^{-1}(\zeta(i))} = \delta_{\lambda^{-1}(i)} \end{array} \right) \\ &= (\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell}), \end{aligned}$$

and this proves Lemma 1.27. □

Lemma 1.28. Let $k, \ell \in \mathbb{N}$. Let $\sigma \in \mathfrak{S}_k$ be any permutation. Define a permutation $\sigma^{\times \ell} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.24.

Let $\delta_{i,j} \in \mathbb{N}$ be a nonnegative integer for each $i \in [k]$ and $j \in [\ell]$. Then,

$$(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell}) \cdot \sigma^{\times \ell} = (\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell}).$$

(Here, $(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})$ denotes the list of all $k\ell$ numbers $\delta_{i,j}$ in the order of lexicographically increasing pairs (i, j) , whereas $(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})$ denotes the list of all $k\ell$ numbers $\delta_{\sigma(i),j}$ in the same order.)

Proof. Let us denote the $k\ell$ -tuples $(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})$ and $(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})$ as $(\alpha_1, \alpha_2, \dots, \alpha_{k\ell})$ and $(\beta_1, \beta_2, \dots, \beta_{k\ell})$, respectively. Then, each $i \in [k]$ and $j \in [\ell]$ satisfy

$$\alpha_{\ell(i-1)+j} = \delta_{i,j} \tag{19}$$

(since $(\alpha_1, \alpha_2, \dots, \alpha_{k\ell}) = (\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})$) and

$$\beta_{\ell(i-1)+j} = \delta_{\sigma(i),j} \tag{20}$$

(since $(\beta_1, \beta_2, \dots, \beta_{k\ell}) = (\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})$).

Let ω be the permutation $\sigma^{\times \ell}$. Then, for each $i \in [k]$ and $j \in [\ell]$, we have

$$\begin{aligned} \omega(\ell(i-1) + j) &= \sigma^{\times \ell}(\ell(i-1) + j) \\ &= \ell(\sigma(i) - 1) + j \end{aligned} \tag{21}$$

(by the definition of $\sigma^{\times \ell}$).

Now, let $n \in [k\ell]$. Write n in the form $n = \ell(i-1) + j$ for some $i \in [k]$ and $j \in [\ell]$. (This is clearly possible.) Thus, $\omega(n) = \omega(\ell(i-1) + j) = \ell(\sigma(i) - 1) + j$ (by (21)). This yields

$$\begin{aligned} \alpha_{\omega(n)} &= \alpha_{\ell(\sigma(i)-1)+j} \\ &= \delta_{\sigma(i),j} \quad (\text{by (19), applied to } \sigma(i) \text{ instead of } i) \\ &= \beta_n \quad (\text{by (20)}). \end{aligned}$$

Forget that we fixed n . We thus have shown that $\alpha_{\omega(n)} = \beta_n$ for each $n \in [k\ell]$.

Now,

$$\begin{aligned} &\underbrace{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})}_{=(\alpha_1, \alpha_2, \dots, \alpha_{k\ell})} \cdot \underbrace{\sigma^{\times \ell}}_{=\omega} \\ &= (\alpha_1, \alpha_2, \dots, \alpha_{k\ell}) \cdot \omega \\ &= (\alpha_{\omega(1)}, \alpha_{\omega(2)}, \dots, \alpha_{\omega(k\ell)}) \\ &= (\beta_1, \beta_2, \dots, \beta_{k\ell}) \quad (\text{since } \alpha_{\omega(n)} = \beta_n \text{ for each } n \in [k\ell]) \\ &= (\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell}). \end{aligned}$$

This proves Lemma 1.28. □

1.6. The composition formula: lemmas on bialgebras

Now, we step to some lemmas that rely on the bialgebra structure on H .

Lemma 1.29. Let $k, \ell \in \mathbb{N}$. Then, $m^{[k]} \circ (m^{[\ell]})^{\otimes k} = m^{[k\ell]}$.

Proof. We induct on k . The *base case* ($k = 0$) is trivial (since $(m^{[\ell]})^{\otimes 0} = \text{id}_{\mathbf{k}}$ and $0 = 0\ell$). For the *induction step*, we fix a $k \in \mathbb{N}$, and we assume (as the induction hypothesis) that $m^{[k]} \circ (m^{[\ell]})^{\otimes k} = m^{[k\ell]}$. We must now prove that $m^{[k+1]} \circ (m^{[\ell]})^{\otimes(k+1)} = m^{[(k+1)\ell]}$. But $(k+1)\ell = k\ell + \ell$, and thus

$$\begin{aligned}
 m^{[(k+1)\ell]} &= m^{[k\ell+\ell]} = m \circ \left(\underbrace{m^{[k\ell]}}_{=m^{[k]} \circ (m^{[\ell]})^{\otimes k}} \otimes \underbrace{m^{[\ell]}}_{=\text{id} \circ m^{[\ell]}} \right) && \text{(by (2))} \\
 &= m \circ \left(\underbrace{\left((m^{[k]} \circ (m^{[\ell]})^{\otimes k}) \otimes (\text{id} \circ m^{[\ell]}) \right)}_{=(m^{[k]} \otimes \text{id}) \circ ((m^{[\ell]})^{\otimes k} \otimes m^{[\ell]})} \right) \\
 &= m \circ \left(\underbrace{m^{[k]} \otimes \underbrace{\text{id}}_{=m^{[1]}}}_{=m^{[k+1]}} \right) \circ \underbrace{\left((m^{[\ell]})^{\otimes k} \otimes m^{[\ell]} \right)}_{=(m^{[\ell]})^{\otimes(k+1)}} \\
 &= \underbrace{m \circ (m^{[k]} \otimes m^{[1]})}_{=m^{[k+1]} \text{ (by (2))}} \circ (m^{[\ell]})^{\otimes(k+1)} = m^{[k+1]} \circ (m^{[\ell]})^{\otimes(k+1)}.
 \end{aligned}$$

Thus, $m^{[k+1]} \circ (m^{[\ell]})^{\otimes(k+1)} = m^{[(k+1)\ell]}$ is proved, so that the induction is complete. This proves Lemma 1.29. \square

Lemma 1.30. Let $k, \ell \in \mathbb{N}$. Then, $(\Delta^{[k]})^{\otimes \ell} \circ \Delta^{[\ell]} = \Delta^{[k\ell]}$.

Proof. This is dual to Lemma 1.29, so the same proof can be used (with all arrows reversed). \square

Lemma 1.31. Let $k, \ell \in \mathbb{N}$. Define a permutation $\zeta \in \mathfrak{S}_{k\ell}$ as in Lemma 1.23. Then,

$$(m^{[\ell]})^{\otimes k} \circ \zeta \circ (\Delta^{[k]})^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}.$$

Proof. From [GriRei20, Exercise 1.4.22(c)] (applied to $k-1$ and $\ell-1$ instead of k and ℓ), we obtain

$$m_{H^{\otimes k}}^{[\ell]} \circ (\Delta^{[k]})^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}, \tag{22}$$

where $m_{H^{\otimes k}}^{[\ell]}$ is the map defined just as $m^{[\ell]}$ but for the algebra $H^{\otimes k}$ instead of H . However, it is easy to see (and should be known) that

$$m_{H^{\otimes k}}^{[\ell]} = \left(m^{[\ell]}\right)^{\otimes k} \circ \zeta. \quad (23)$$

Indeed, this can be checked on pure tensors, using Lemma 1.23.⁶

Using (23), we can rewrite (22) as

$$\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}.$$

⁶Proof. If $(h_{i,j})_{i \in [\ell], j \in [k]} \in H^{\ell \times k}$ is any $\ell \times k$ -matrix over H , then

$$\begin{aligned} & \left(\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta \right) (h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\ & \quad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\ & \quad \otimes \cdots \\ & \quad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}) \\ &= \left(m^{[\ell]}\right)^{\otimes k} (h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1} \\ & \quad \otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2} \\ & \quad \otimes \cdots \\ & \quad \otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}) \\ & \quad \left(\begin{array}{c} \text{here, we have applied the map } \left(m^{[\ell]}\right)^{\otimes k} \\ \text{to both sides of the equality (8)} \end{array} \right) \\ &= \underbrace{m^{[\ell]}(h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1})}_{=h_{1,1}h_{2,1}\cdots h_{\ell,1}} \\ & \quad \otimes \underbrace{m^{[\ell]}(h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2})}_{=h_{1,2}h_{2,2}\cdots h_{\ell,2}} \\ & \quad \otimes \cdots \\ & \quad \otimes \underbrace{m^{[\ell]}(h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k})}_{=h_{1,k}h_{2,k}\cdots h_{\ell,k}} \\ &= h_{1,1}h_{2,1} \cdots h_{\ell,1} \otimes h_{1,2}h_{2,2} \cdots h_{\ell,2} \otimes \cdots \otimes h_{1,k}h_{2,k} \cdots h_{\ell,k} \\ &= (h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k}) \cdot (h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k}) \cdots \cdots (h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}) \\ &= m_{H^{\otimes k}}^{[\ell]} (h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\ & \quad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\ & \quad \otimes \cdots \\ & \quad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}). \end{aligned}$$

In other words, the two maps $\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta$ and $m_{H^{\otimes k}}^{[\ell]}$ agree on each pure tensor. Since these two maps are \mathbf{k} -linear, they must therefore be identical (since the pure tensors span $H^{\otimes k\ell}$). In other words, $\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta = m_{H^{\otimes k}}^{[\ell]}$. This proves (23).

This proves Lemma 1.31. □

The next two lemmas combine the algebra and coalgebra structures with the grading:

Lemma 1.32. Let $k, \ell \in \mathbb{N}$. Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \in \mathbb{N}^k$. Then,

$$P_\gamma \circ (m^{[\ell]})^{\otimes k} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \gamma_i \text{ for all } i \in [k]}} (m^{[\ell]})^{\otimes k} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}.$$

Here, $(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})$ denotes the $k\ell$ -tuple consisting of all $k\ell$ numbers $\gamma_{i,j}$ (for all $i \in [k]$ and $j \in [\ell]$) listed in lexicographically increasing order of subscripts.

Proof. Let $i \in \mathbb{N}$. Recall that $p_i : H \rightarrow H$ denotes the projection of the graded \mathbf{k} -module H onto its i -th graded component. Likewise, let $p'_i : H^{\otimes \ell} \rightarrow H^{\otimes \ell}$ denote the projection of the graded \mathbf{k} -module $H^{\otimes \ell}$ onto its i -th graded component. The map $m^{[\ell]} : H^{\otimes \ell} \rightarrow H$ is graded⁷. Thus, it commutes with the projection onto the i -th graded component. In other words,

$$p_i \circ m^{[\ell]} = m^{[\ell]} \circ p'_i. \tag{24}$$

However, the definition of the grading on $H^{\otimes \ell}$ yields that the i -th graded component of $H^{\otimes \ell}$ is $\bigoplus_{\substack{(i_1, i_2, \dots, i_\ell) \in \mathbb{N}^\ell; \\ i_1 + i_2 + \dots + i_\ell = i}} H_{i_1} \otimes H_{i_2} \otimes \dots \otimes H_{i_\ell}$. Hence, it is easy to see that the

projection p'_i onto this component is

$$\sum_{\substack{(i_1, i_2, \dots, i_\ell) \in \mathbb{N}^\ell; \\ i_1 + i_2 + \dots + i_\ell = i}} p_{i_1} \otimes p_{i_2} \otimes \dots \otimes p_{i_\ell}.$$

In view of this, we can rewrite (24) as

$$\begin{aligned} p_i \circ m^{[\ell]} &= m^{[\ell]} \circ \left(\sum_{\substack{(i_1, i_2, \dots, i_\ell) \in \mathbb{N}^\ell; \\ i_1 + i_2 + \dots + i_\ell = i}} p_{i_1} \otimes p_{i_2} \otimes \dots \otimes p_{i_\ell} \right) \\ &= \sum_{\substack{(i_1, i_2, \dots, i_\ell) \in \mathbb{N}^\ell; \\ i_1 + i_2 + \dots + i_\ell = i}} m^{[\ell]} \circ (p_{i_1} \otimes p_{i_2} \otimes \dots \otimes p_{i_\ell}) \end{aligned} \tag{25}$$

(here, we have distributed the summation sign to the beginning of the product, since all the maps involved are linear).

⁷This follows by a straightforward induction on ℓ from the gradedness of the map $m : H \otimes H \rightarrow H$.

Forget that we fixed i . We have thus proved the equality (25) for each $i \in \mathbb{N}$.

Now, from $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$, we obtain $P_\gamma = p_{\gamma_1} \otimes p_{\gamma_2} \otimes \dots \otimes p_{\gamma_k}$ (by the definition of P_γ). Hence,

$$\begin{aligned}
 & \underbrace{P_\gamma}_{=p_{\gamma_1} \otimes p_{\gamma_2} \otimes \dots \otimes p_{\gamma_k}} \circ \underbrace{(m^{[\ell]})^{\otimes k}}_{=m^{[\ell]} \otimes m^{[\ell]} \otimes \dots \otimes m^{[\ell]}} \\
 &= (p_{\gamma_1} \otimes p_{\gamma_2} \otimes \dots \otimes p_{\gamma_k}) \circ (m^{[\ell]} \otimes m^{[\ell]} \otimes \dots \otimes m^{[\ell]}) \\
 &= (p_{\gamma_1} \circ m^{[\ell]}) \otimes (p_{\gamma_2} \circ m^{[\ell]}) \otimes \dots \otimes (p_{\gamma_k} \circ m^{[\ell]}) \\
 &= \bigotimes_{s=1}^k (p_{\gamma_s} \circ m^{[\ell]}) \quad \left(\text{here, we use the symbol } \bigotimes_{s=1}^k f_s \text{ for } f_1 \otimes f_2 \otimes \dots \otimes f_k \right) \\
 &= \bigotimes_{s=1}^k \left(\sum_{\substack{(i_1, i_2, \dots, i_\ell) \in \mathbb{N}^\ell; \\ i_1 + i_2 + \dots + i_\ell = \gamma_s}} m^{[\ell]} \circ (p_{i_1} \otimes p_{i_2} \otimes \dots \otimes p_{i_\ell}) \right) \\
 & \quad \left(\text{here, we have applied (25) to } i = \gamma_s \text{ for each } s \in [k] \right) \\
 &= \sum_{\substack{\gamma_{s,j} \in \mathbb{N} \text{ for all } s \in [k] \text{ and } j \in [\ell]; \\ \gamma_{s,1} + \gamma_{s,2} + \dots + \gamma_{s,\ell} = \gamma_s \text{ for all } s \in [k]}} \underbrace{\bigotimes_{s=1}^k (m^{[\ell]} \circ (p_{\gamma_{s,1}} \otimes p_{\gamma_{s,2}} \otimes \dots \otimes p_{\gamma_{s,\ell}}))}_{= \left(\bigotimes_{s=1}^k m^{[\ell]} \right) \circ \left(\bigotimes_{s=1}^k (p_{\gamma_{s,1}} \otimes p_{\gamma_{s,2}} \otimes \dots \otimes p_{\gamma_{s,\ell}}) \right)} \\
 & \quad \left(\begin{array}{l} \text{by the product rule for tensor products, which} \\ \text{says } \bigotimes_{s=1}^k \sum_{a \in A_s} f_{s,a} = \sum_{(a_1, a_2, \dots, a_k) \in A_1 \times A_2 \times \dots \times A_k} \bigotimes_{s=1}^k f_{s,a_s}, \\ \text{and which we here have applied} \\ \text{to } A_s = \{(i_1, i_2, \dots, i_\ell) \in \mathbb{N}^\ell \mid i_1 + i_2 + \dots + i_\ell = \gamma_s\} \\ \text{and } f_{s,(i_1, i_2, \dots, i_\ell)} = m^{[\ell]} \circ (p_{i_1} \otimes p_{i_2} \otimes \dots \otimes p_{i_\ell}) \end{array} \right) \\
 &= \sum_{\substack{\gamma_{s,j} \in \mathbb{N} \text{ for all } s \in [k] \text{ and } j \in [\ell]; \\ \gamma_{s,1} + \gamma_{s,2} + \dots + \gamma_{s,\ell} = \gamma_s \text{ for all } s \in [k]}} \underbrace{\left(\bigotimes_{s=1}^k m^{[\ell]} \right)}_{=(m^{[\ell]})^{\otimes k}} \circ \underbrace{\left(\bigotimes_{s=1}^k (p_{\gamma_{s,1}} \otimes p_{\gamma_{s,2}} \otimes \dots \otimes p_{\gamma_{s,\ell}}) \right)}_{=P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \text{ (by the definition of } P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})})} \\
 &= \sum_{\substack{\gamma_{s,j} \in \mathbb{N} \text{ for all } s \in [k] \text{ and } j \in [\ell]; \\ \gamma_{s,1} + \gamma_{s,2} + \dots + \gamma_{s,\ell} = \gamma_s \text{ for all } s \in [k]}} (m^{[\ell]})^{\otimes k} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \gamma_i \text{ for all } i \in [k]}} (m^{[\ell]})^{\otimes k} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}.
 \end{aligned}$$

This proves Lemma 1.32. □

Lemma 1.33. Let $k, \ell \in \mathbb{N}$. Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \in \mathbb{N}^k$. Then,

$$\left(\Delta^{[\ell]}\right)^{\otimes k} \circ P_\gamma = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \gamma_i \text{ for all } i \in [k]}} P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \circ \left(\Delta^{[\ell]}\right)^{\otimes k}.$$

Here, $(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})$ denotes the $k\ell$ -tuple consisting of all $k\ell$ numbers $\gamma_{i,j}$ (for all $i \in [k]$ and $j \in [\ell]$) listed in lexicographically increasing order of subscripts.

Proof. This is the dual statement to Lemma 1.32, and is proved by the same argument “with all arrows reversed” (meaning that any composition $f_1 \circ f_2 \circ \dots \circ f_\ell$ of several maps should be reversed to become $f_\ell \circ f_{\ell-1} \circ \dots \circ f_1$) and with all m ’s replaced by Δ ’s. □

1.7. The composition formula: proof

We are finally able to prove Theorem 1.18:

Proof of Theorem 1.18. The formula (4) yields

$$p_{\alpha,\sigma} = m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \quad \text{and} \quad (26)$$

$$p_{\beta,\tau} = m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]}. \quad (27)$$

Multiplying these two equalities (using the operation \circ), we obtain

$$p_{\alpha,\sigma} \circ p_{\beta,\tau} = m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \circ m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]}.$$

Our main task now is to “commute” the operators in this equality past each other, moving both m factors to the left, both Δ factors to the right, and ideally obtaining some expression of the form $m^{[k\ell]} \circ P_\gamma \circ \rho^{-1} \circ \Delta^{[k\ell]}$ (more precisely, it will be a sum of such expressions). For this purpose, we shall derive some “commutation-like” formulas.

Lemma 1.32 (applied to α and α_i instead of γ and γ_i) yields

$$\begin{aligned} & P_\alpha \circ \left(m^{[\ell]}\right)^{\otimes k} \\ &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \left(m^{[\ell]}\right)^{\otimes k} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}. \end{aligned} \quad (28)$$

Lemma 1.33 (applied to k, ℓ, β and β_i instead of ℓ, k, γ and γ_i) yields

$$\begin{aligned}
 & \left(\Delta^{[k]}\right)^{\otimes \ell} \circ P_{\beta} \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [\ell] \text{ and } j \in [k]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,k} = \beta_i \text{ for all } i \in [\ell]}} P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{\ell,k})} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
 &= \sum_{\substack{\delta_{j,i} \in \mathbb{N} \text{ for all } i \in [\ell] \text{ and } j \in [k]; \\ \delta_{1,i} + \delta_{2,i} + \dots + \delta_{k,i} = \beta_i \text{ for all } i \in [\ell]}} P_{(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
 &\quad \text{(here, we have renamed the } \gamma_{i,j} \text{ as } \delta_{j,i}\text{)} \\
 &= \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } j \in [\ell] \text{ and } i \in [k]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} P_{(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
 &\quad \text{(here, we have renamed the indices } i \text{ and } j \text{ as } j \text{ and } i\text{)} \\
 &= \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} P_{(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \tag{29}
 \end{aligned}$$

(here, we have just rewritten the “ $\delta_{i,j} \in \mathbb{N}$ for all $j \in [\ell]$ and $i \in [k]$ ” under the summation sign as “ $\delta_{i,j} \in \mathbb{N}$ for all $i \in [k]$ and $j \in [\ell]$ ”).

Next, we shall use a nearly trivial fact: If f, g, u, v are four maps such that the compositions $f \circ u$ and $g \circ v$ are well-defined (i.e., the target of u is the domain of f , and the target of v is the domain of g) and satisfy $f \circ u = g \circ v$, and if the maps f and v are invertible, then

$$u \circ v^{-1} = f^{-1} \circ g \tag{30}$$

(since $\underbrace{f \circ u}_{=g \circ v} \circ v^{-1} = g \circ \underbrace{v \circ v^{-1}}_{=id} = g$ and thus $g = f \circ u \circ v^{-1}$, so that $f^{-1} \circ g = \underbrace{f^{-1} \circ f}_{=id} \circ u \circ v^{-1} = u \circ v^{-1}$).

Define a permutation $\sigma^{\times \ell} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.24. Define a permutation $\tau^{k \times} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.25. Define a permutation $\zeta \in \mathfrak{S}_{k\ell}$ as in Lemma 1.23.

Lemma 1.24 (applied to $f = m^{[\ell]}$) yields $\sigma \circ \left(m^{[\ell]}\right)^{\otimes k} = \left(m^{[\ell]}\right)^{\otimes k} \circ \sigma^{\times \ell}$. Hence, (30) (applied to $f = \sigma$ and $u = \left(m^{[\ell]}\right)^{\otimes k}$ and $g = \left(m^{[\ell]}\right)^{\otimes k}$ and $v = \sigma^{\times \ell}$) yields

$$\left(m^{[\ell]}\right)^{\otimes k} \circ \left(\sigma^{\times \ell}\right)^{-1} = \sigma^{-1} \circ \left(m^{[\ell]}\right)^{\otimes k}. \tag{31}$$

Lemma 1.25 (applied to $f = \Delta^{[k]}$) yields $\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \tau = \tau^{k \times} \circ \left(\Delta^{[k]}\right)^{\otimes \ell}$. In other words, $\tau^{k \times} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \left(\Delta^{[k]}\right)^{\otimes \ell} \circ \tau$. Hence, (30) (applied to $f = \tau^{k \times}$ and $u =$

$(\Delta^{[k]})^{\otimes \ell}$ and $g = (\Delta^{[k]})^{\otimes \ell}$ and $v = \tau$ yields

$$(\Delta^{[k]})^{\otimes \ell} \circ \tau^{-1} = (\tau^{k \times})^{-1} \circ (\Delta^{[k]})^{\otimes \ell}. \quad (32)$$

Furthermore, if $\delta_{i,j} \in \mathbb{N}$ is a nonnegative integer for each $i \in [k]$ and $j \in [\ell]$, then we have

$$\begin{aligned} & P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})} \circ \zeta \\ &= \zeta \circ P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})} \cdot \zeta \\ & \quad \text{(by (6), applied to } \pi = \zeta \text{ and } \gamma = (\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})) \\ &= \zeta \circ P_{(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})} \end{aligned} \quad (33)$$

(since Lemma 1.27 yields $(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell}) \cdot \zeta = (\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})$) and

$$\begin{aligned} & P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})} \circ \sigma^{\times \ell} \\ &= \sigma^{\times \ell} \circ P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})} \cdot \sigma^{\times \ell} \\ & \quad \text{(by (6), applied to } \pi = \sigma^{\times \ell} \text{ and } \gamma = (\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})) \\ &= \sigma^{\times \ell} \circ P_{(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})} \end{aligned}$$

(since Lemma 1.28 yields $(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell}) \cdot \sigma^{\times \ell} = (\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})$) and therefore $\sigma^{\times \ell} \circ P_{(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})} = P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})} \circ \sigma^{\times \ell}$, so that

$$P_{(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})} \circ (\sigma^{\times \ell})^{-1} = (\sigma^{\times \ell})^{-1} \circ P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})} \quad (34)$$

(by (30), applied to $f = \sigma^{\times \ell}$ and $u = P_{(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})}$ and $g = P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})}$ and $v = \sigma^{\times \ell}$).

Now, (26) and (27) yield

$$\begin{aligned}
 & p_{\alpha, \sigma} \circ p_{\beta, \tau} \\
 &= m^{[k]} \circ P_{\alpha} \circ \sigma^{-1} \circ \underbrace{\Delta^{[k]} \circ m^{[\ell]}}_{=(m^{[\ell]})^{\otimes k} \circ \zeta \circ (\Delta^{[k]})^{\otimes \ell} \text{ (by Lemma 1.31)}} \circ P_{\beta} \circ \tau^{-1} \circ \Delta^{[\ell]} \\
 &= m^{[k]} \circ P_{\alpha} \circ \underbrace{\sigma^{-1} \circ (m^{[\ell]})^{\otimes k}}_{=(m^{[\ell]})^{\otimes k} \circ (\sigma^{\times \ell})^{-1} \text{ (by (31))}} \circ \zeta \\
 &\quad \circ \underbrace{(\Delta^{[k]})^{\otimes \ell} \circ P_{\beta}}_{\substack{= \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} P_{(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})} \circ (\Delta^{[k]})^{\otimes \ell} \text{ (by (29))}}} \circ \tau^{-1} \circ \Delta^{[\ell]} \\
 &= \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k]} \circ \underbrace{P_{\alpha} \circ (m^{[\ell]})^{\otimes k}}_{=(m^{[\ell]})^{\otimes k} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \text{ (by (28))}}} \\
 &\quad \circ (\sigma^{\times \ell})^{-1} \circ \zeta \circ P_{(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})} \circ \underbrace{(\Delta^{[k]})^{\otimes \ell} \circ \tau^{-1} \circ \Delta^{[\ell]}}_{=(\tau^{k \times})^{-1} \circ (\Delta^{[k]})^{\otimes \ell} \text{ (by (32))}} \\
 &\quad \left(\text{here, we have distributed the summation sign to the} \right. \\
 &\quad \left. \text{beginning of the product, since all our maps are } \mathbf{k}\text{-linear} \right) \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} \\
 &\quad \underbrace{m^{[k]} \circ (m^{[\ell]})^{\otimes k}}_{=m^{[k\ell]} \text{ (by Lemma 1.29)}} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \circ (\sigma^{\times \ell})^{-1} \\
 &\quad \circ \zeta \circ P_{(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{k,\ell})} \circ (\tau^{k \times})^{-1} \circ \underbrace{(\Delta^{[k]})^{\otimes \ell} \circ \Delta^{[\ell]}}_{= \Delta^{[k\ell]} \text{ (by Lemma 1.30)}} \\
 &\quad \left(\text{here, again, we have distributed the summation sign to the} \right. \\
 &\quad \left. \text{beginning of the product, since all our maps are } \mathbf{k}\text{-linear} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k\ell]} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \circ \underbrace{\left(\sigma^{\times \ell} \right)^{-1} \circ P_{(\delta_{1,1}, \delta_{1,2}, \dots, \delta_{k,\ell})}}_{= P_{(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})} \circ (\sigma^{\times \ell})^{-1} \text{ (by (34))}} \circ \zeta \circ \left(\tau^{k \times} \right)^{-1} \circ \Delta^{[k\ell]} \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k\ell]} \circ \underbrace{P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \circ P_{(\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})}}_{= \begin{cases} P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}, & \text{if } (\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}) = (\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell}); \\ 0, & \text{otherwise} \end{cases}} \\
 &\quad \underbrace{\left(\sigma^{\times \ell} \right)^{-1} \circ \zeta \circ \left(\tau^{k \times} \right)^{-1} \circ \Delta^{[k\ell]}}_{= (\tau[\sigma])^{-1} \text{ (by (18))}}; \quad \text{(by Lemma 1.22)} \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k\ell]} \circ \begin{cases} P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}, & \text{if } (\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}) = (\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell}); \\ 0, & \text{otherwise} \end{cases} \\
 &\quad \circ (\tau[\sigma])^{-1} \circ \Delta^{[k\ell]} \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k\ell]} \circ \begin{cases} P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}, & \text{if } \gamma_{i,j} = \delta_{\sigma(i),j} \text{ for all } i, j; \\ 0, & \text{otherwise} \end{cases} \\
 &\quad \circ (\tau[\sigma])^{-1} \circ \Delta^{[k\ell]} \\
 &\quad \left(\text{since the condition } "(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}) = (\delta_{\sigma(1),1}, \delta_{\sigma(1),2}, \dots, \delta_{\sigma(k),\ell})" \right. \\
 &\quad \left. \text{is equivalent to } "\gamma_{i,j} = \delta_{\sigma(i),j} \text{ for all } i, j" \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \dots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]; \\ \gamma_{i,j} = \delta_{\sigma(i),j} \text{ for all } i,j}} m^{[k\ell]} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \circ (\tau[\sigma])^{-1} \circ \Delta^{[k\ell]} \\
 &\quad \left(\begin{array}{l} \text{here, we have discarded all the addends that do not} \\ \text{satisfy } (\gamma_{i,j} = \delta_{\sigma(i),j} \text{ for all } i,j), \text{ because these addends are 0} \end{array} \right) \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{\sigma^{-1}(1),j} + \gamma_{\sigma^{-1}(2),j} + \dots + \gamma_{\sigma^{-1}(k),j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k\ell]} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \circ (\tau[\sigma])^{-1} \circ \Delta^{[k\ell]} \\
 &\quad \left(\begin{array}{l} \text{here, we have combined the two summation signs} \\ \text{into a single one, by rewriting } \delta_{i,j} \text{ as } \gamma_{\sigma^{-1}(i),j} \\ \text{(since the condition " } \gamma_{i,j} = \delta_{\sigma(i),j} \text{ for all } i,j \text{"} \\ \text{ensures that } \delta_{i,j} = \gamma_{\sigma^{-1}(i),j} \text{ for all } i,j, \text{ and therefore} \\ \text{the } \delta_{i,j} \text{ are uniquely determined by the } \gamma_{i,j} \end{array} \right) \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} \underbrace{m^{[k\ell]} \circ P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})} \circ (\tau[\sigma])^{-1} \circ \Delta^{[k\ell]}}_{=P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}), \tau[\sigma]} \text{ (by (4))}} \\
 &\quad \left(\begin{array}{l} \text{here, we have rewritten the " } \gamma_{\sigma^{-1}(1),j} + \gamma_{\sigma^{-1}(2),j} + \dots + \gamma_{\sigma^{-1}(k),j} \text{"} \\ \text{under the summation sign as " } \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} \text{"} \\ \text{because } \sigma \in \mathfrak{S}_k \text{ is a bijection from } [k] \text{ to } [k] \end{array} \right) \\
 &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} P_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}), \tau[\sigma]}
 \end{aligned}$$

This proves Theorem 1.18. □

Question 1.34. In [Pang21, Proposition 12], Pang generalized (1) to Hopf algebras with involutions (as long as they are commutative or cocommutative). Is a similar generalization possible for Theorem 1.18?

1.8. Linear independence of $p_{\alpha, \sigma}$'s

Both Theorem 1.18 and Proposition 1.15 hold for arbitrary graded (not necessarily connected) bialgebras. Let us now focus our view on the connected graded bialgebras (which, as we recall, are automatically Hopf algebras).

Using Theorem 1.18 and (5), we can expand any nested convolution-and-composition of $p_{\alpha, \sigma}$'s as a \mathbf{k} -linear combination of single $p_{\alpha, \sigma}$'s. This allows us to mechanically

prove equalities for $p_{\alpha,\sigma}$'s that involve only convolution, composition and \mathbf{k} -linear combination and that are supposed to be valid for any connected graded Hopf algebra H . The reason why this works is the following "generic linear independence" theorem:

Theorem 1.35. (a) There is a connected graded Hopf algebra H such that the family

$$(p_{\alpha,\sigma})_{\substack{k \in \mathbb{N}; \\ \alpha \text{ is a composition of length } k; \\ \sigma \in \mathfrak{S}_k}}$$

(of endomorphisms of H) is \mathbf{k} -linearly independent, and such that each H_n is a free \mathbf{k} -module.

(b) Let $n \in \mathbb{N}$. Then, there is a connected graded Hopf algebra H such that the family

$$(p_{\alpha,\sigma})_{\substack{k \in \mathbb{N}; \\ \alpha \text{ is a composition of length } k \text{ and size } < n; \\ \sigma \in \mathfrak{S}_k}}$$

(of endomorphisms of H) is \mathbf{k} -linearly independent, and such that each H_n is a free \mathbf{k} -module with a finite basis.

Proof of Theorem 1.35 (sketched). **(b)** Let H be the free \mathbf{k} -algebra with generators

$$x_{i,j} \quad \text{with } i, j \in \mathbb{Z} \text{ satisfying } 1 \leq i < j \leq n.$$

We also set

$$x_{k,k} := 1_H \quad \text{for each } k \in \{1, 2, \dots, n\}.$$

We make the \mathbf{k} -algebra H graded by declaring each $x_{i,j}$ to be homogeneous of degree $j - i$. We define a comultiplication $\Delta : H \rightarrow H \otimes H$ on H to be the \mathbf{k} -algebra homomorphism that satisfies

$$\Delta(x_{i,j}) = \sum_{k=i}^j x_{i,k} \otimes x_{k,j} \quad \text{for each } i, j \in \mathbb{Z} \text{ satisfying } 1 \leq i < j \leq n.$$

We define a counit $\epsilon : H \rightarrow \mathbf{k}$ in the obvious way to preserve the grading (so that $\epsilon(x_{i,j}) = 0$ whenever $i < j$). It is easy to see that H is a connected graded \mathbf{k} -bialgebra, thus a connected graded Hopf algebra.⁸

It is easy to see that

$$\Delta^{[k]}(x_{i,j}) = \sum_{i=u_0 \leq u_1 \leq \dots \leq u_k=j} x_{u_0,u_1} \otimes x_{u_1,u_2} \otimes \dots \otimes x_{u_{k-1},u_k}$$

⁸This H resembles certain bialgebras that appear in the literature. In particular, it can be regarded either as a noncommutative version of a reduced incidence algebra of the chain poset with n elements, or as a noncommutative variant of the subalgebra of the Schur algebra corresponding to the upper-triangular matrices with equal numbers on the diagonal. But these connections are tangential for our purpose.

for any $i \leq j$ and any $k \in \mathbb{N}$. Thus, for any $i \leq j$ and any $k \in \mathbb{N}$ and any permutation $\sigma \in \mathfrak{S}_k$, we have

$$\begin{aligned} (\sigma^{-1} \circ \Delta^{[k]})(x_{i,j}) &= \sigma^{-1} \cdot \sum_{i=u_0 \leq u_1 \leq \dots \leq u_k=j} x_{u_0, u_1} \otimes x_{u_1, u_2} \otimes \dots \otimes x_{u_{k-1}, u_k} \\ &= \sum_{i=u_0 \leq u_1 \leq \dots \leq u_k=j} x_{u_{\sigma(1)-1}, u_{\sigma(1)}} \otimes x_{u_{\sigma(2)-1}, u_{\sigma(2)}} \otimes \dots \otimes x_{u_{\sigma(k)-1}, u_{\sigma(k)}}. \end{aligned}$$

Hence, for any weak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$ and any $\sigma \in \mathfrak{S}_k$ and any $1 \leq i < j \leq n$ satisfying $j - i = |\alpha|$, we have

$$p_{\alpha, \sigma}(x_{i,j}) = x_{u_{\sigma(1)-1}, u_{\sigma(1)}} x_{u_{\sigma(2)-1}, u_{\sigma(2)}} \dots x_{u_{\sigma(k)-1}, u_{\sigma(k)'}}$$

where $(u_0 \leq u_1 \leq \dots \leq u_k)$ is the unique weakly increasing sequence of integers satisfying $u_0 = i$ and $u_k = j$ and $u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i$ for all $i \in [k]$. Hence, for any choice of $1 \leq i < j \leq n$, the images $p_{\alpha, \sigma}(x_{i,j})$ as α runs over all compositions of $j - i$ and σ runs over all permutations of $[\ell(\alpha)]$ are distinct monomials and therefore are \mathbf{k} -linearly independent. This yields the \mathbf{k} -linear independence of the family

$$(p_{\alpha, \sigma})_{\substack{k \in \mathbb{N}; \\ \alpha \text{ is a composition of length } k \text{ and size } s; \\ \sigma \in \mathfrak{S}_k}}$$

for any given $s \in \{0, 1, \dots, n - 1\}$. Since each $p_{\alpha, \sigma}$ lies in $\text{End}_{\mathbf{k}}(H_{|\alpha|})$, we thus obtain the \mathbf{k} -linear independence of the entire family

$$(p_{\alpha, \sigma})_{\substack{k \in \mathbb{N}; \\ \alpha \text{ is a composition of length } k \text{ and size } < n; \\ \sigma \in \mathfrak{S}_k}}$$

(since the sum $\sum_{s=0}^{n-1} \text{End}_{\mathbf{k}}(H_s)$ is a direct sum). This proves Theorem 1.35 (b).

(a) As for part (b), but remove “ $\leq n$ ” everywhere. □

Question 1.36. Is there a graded Hopf algebra H such that the family

$$(p_{\alpha, \sigma})_{k \in \mathbb{N}; \alpha \in \mathbb{N}^k; \sigma \in \mathfrak{S}_k}$$

(of endomorphisms of H) is \mathbf{k} -linearly independent?

1.9. The tensor product formula

We shall now connect the $p_{\alpha, \sigma}$ operators on different graded bialgebras:

Proposition 1.37. Let H and G be two graded bialgebras. Let α be a weak composition of length k , and let $\sigma \in \mathfrak{S}_k$ be a permutation. Then,

$$(p_{\alpha,\sigma} \text{ for } H \otimes G) = \sum_{\substack{\beta,\gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} (p_{\beta,\sigma} \text{ for } H) \otimes (p_{\gamma,\sigma} \text{ for } G)$$

as endomorphisms of $H \otimes G$.

Proof sketch. This is fairly straightforward using Sweedler notation:

The definition of the grading on $H \otimes G$ shows that $(H \otimes G)_a = \bigoplus_{\substack{b,c \in \mathbb{N}; \\ b+c=a}} H_b \otimes G_c$ for

any $a \in \mathbb{N}$. Hence,

$$p_a(u \otimes v) = \sum_{\substack{b,c \in \mathbb{N}; \\ b+c=a}} p_b(u) \otimes p_c(v) \tag{35}$$

for any $u \in H$, any $v \in G$ and any $a \in \mathbb{N}$.

Write the weak composition $\alpha \in \mathbb{N}^k$ as $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$. Consider a pure

tensor $h \otimes g \in H \otimes G$. Then, Remark 1.9 yields

$$\begin{aligned}
 & p_{\alpha, \sigma}(h \otimes g) \\
 &= \sum_{(h \otimes g)} p_{\alpha_1} \left((h \otimes g)_{(\sigma(1))} \right) p_{\alpha_2} \left((h \otimes g)_{(\sigma(2))} \right) \cdots p_{\alpha_k} \left((h \otimes g)_{(\sigma(k))} \right) \\
 &= \sum_{(h \otimes g)} \prod_{i=1}^k p_{\alpha_i} \left(\underbrace{(h \otimes g)_{(\sigma(i))}}_{=h_{(\sigma(i))} \otimes g_{(\sigma(i))}} \right) \\
 &= \sum_{(h), (g)} \prod_{i=1}^k \underbrace{p_{\alpha_i} \left(h_{(\sigma(i))} \otimes g_{(\sigma(i))} \right)}_{= \sum_{\substack{b, c \in \mathbb{N}; \\ b+c=\alpha_i}} p_b(h_{(\sigma(i))}) \otimes p_c(g_{(\sigma(i))})} \\
 &\hspace{10em} \text{(by (35))} \\
 &= \sum_{(h), (g)} \prod_{i=1}^k \sum_{\substack{b, c \in \mathbb{N}; \\ b+c=\alpha_i}} p_b \left(h_{(\sigma(i))} \right) \otimes p_c \left(g_{(\sigma(i))} \right) \\
 &= \sum_{(h), (g)} \sum_{\substack{\beta_i, \gamma_i \in \mathbb{N} \text{ for all } i \in [k]; \\ \beta_i + \gamma_i = \alpha_i \text{ for all } i \in [k]}} \prod_{i=1}^k \left(p_{\beta_i} \left(h_{(\sigma(i))} \right) \otimes p_{\gamma_i} \left(g_{(\sigma(i))} \right) \right) \hspace{10em} \text{(by the product rule)} \\
 &= \sum_{\substack{\beta_i, \gamma_i \in \mathbb{N} \text{ for all } i \in [k]; \\ \beta_i + \gamma_i = \alpha_i \text{ for all } i \in [k]}} \sum_{(h), (g)} \underbrace{\prod_{i=1}^k \left(p_{\beta_i} \left(h_{(\sigma(i))} \right) \otimes p_{\gamma_i} \left(g_{(\sigma(i))} \right) \right)}_{= \left(\prod_{i=1}^k p_{\beta_i} \left(h_{(\sigma(i))} \right) \right) \otimes \left(\prod_{i=1}^k p_{\gamma_i} \left(g_{(\sigma(i))} \right) \right)} \\
 &= \sum_{\substack{\beta = (\beta_1, \beta_2, \dots, \beta_k) \text{ and} \\ \gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \\ \text{weak compositions;} \\ \text{entrywise sum } \beta + \gamma = \alpha}} \sum_{(h), (g)} \left(\prod_{i=1}^k p_{\beta_i} \left(h_{(\sigma(i))} \right) \right) \otimes \left(\prod_{i=1}^k p_{\gamma_i} \left(g_{(\sigma(i))} \right) \right) \\
 &= \sum_{\substack{\beta = (\beta_1, \beta_2, \dots, \beta_k) \text{ and} \\ \gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \\ \text{weak compositions;} \\ \text{entrywise sum } \beta + \gamma = \alpha}} \underbrace{\left(\sum_{(h)} \prod_{i=1}^k p_{\beta_i} \left(h_{(\sigma(i))} \right) \right)}_{= p_{\beta, \sigma}(h) \text{ (by Remark 1.9)}} \otimes \underbrace{\left(\sum_{(g)} \prod_{i=1}^k p_{\gamma_i} \left(g_{(\sigma(i))} \right) \right)}_{= p_{\gamma, \sigma}(g) \text{ (by Remark 1.9)}}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{\beta=(\beta_1,\beta_2,\dots,\beta_k) \text{ and} \\ \gamma=(\gamma_1,\gamma_2,\dots,\gamma_k) \\ \text{weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} p_{\beta,\sigma}(h) \otimes p_{\gamma,\sigma}(g) = \sum_{\substack{\beta,\gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} p_{\beta,\sigma}(h) \otimes p_{\gamma,\sigma}(g) \\
 &= \left(\sum_{\substack{\beta,\gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} (p_{\beta,\sigma} \text{ for } H) \otimes (p_{\gamma,\sigma} \text{ for } G) \right) (h \otimes g).
 \end{aligned}$$

Hence, the two \mathbf{k} -linear maps

$$(p_{\alpha,\sigma} \text{ for } H \otimes G) \text{ and } \sum_{\substack{\beta,\gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} (p_{\beta,\sigma} \text{ for } H) \otimes (p_{\gamma,\sigma} \text{ for } G)$$

agree on each pure tensor. Therefore, they are identical. This proves Proposition 1.37. \square

1.10. Have we found them all?

Now let H be a connected graded bialgebra. As we already mentioned, each $p_{\alpha,\sigma}$ belongs to the \mathbf{k} -module $\mathbf{E}(H)$ of all graded \mathbf{k} -module endomorphisms of H that annihilate all but finitely many degrees of H . Thus, the same holds for any \mathbf{k} -linear combination of the $p_{\alpha,\sigma}$. The fact that each $p_{\alpha,\sigma}$ annihilates all but the $|\alpha|$ -th graded component of H allows us to form infinite \mathbf{k} -linear combinations $\sum_{k \in \mathbb{N}} \sum_{\alpha \in \{1,2,3,\dots\}^k} \sum_{\sigma \in \mathfrak{S}_k} \lambda_{\alpha,\sigma} p_{\alpha,\sigma}$ of these $p_{\alpha,\sigma}$ as well. Each such combination belongs to $\text{End}_{\text{gr}} H$ (although not to $\mathbf{E}(H)$ any more) and (since it is natural in H) is therefore a natural graded \mathbf{k} -module endomorphism of H defined for any connected graded bialgebra H .

Question 1.38. Are these combinations the only natural graded \mathbf{k} -module endomorphisms of H defined for any connected graded bialgebra H ?

In other words: Let g be a natural graded \mathbf{k} -module endomorphism on the category of connected graded \mathbf{k} -Hopf algebras. (That is, for each connected graded Hopf algebra H , we have a graded \mathbf{k} -module endomorphism g_H , and each graded Hopf algebra morphism $\varphi : H \rightarrow H'$ gives a commutative diagram.) Is it true that g is an infinite \mathbf{k} -linear combination of $p_{\alpha,\sigma}$'s?

I have so far been unable to answer this question, for lack of convenient free objects in the relevant category. Nevertheless, I suspect that the answer is positive (i.e., every natural endomorphism is an infinite \mathbf{k} -linear combination of $p_{\alpha,\sigma}$'s), at least when \mathbf{k} is a field of characteristic 0.

Question 1.38 can also be asked without the first "graded". That is, we can ask about natural \mathbf{k} -module endomorphisms of H that are not necessarily graded.

A similar question can be asked for general (as opposed to connected) graded bialgebras. However, it requires some more care in its formulation, as not every infinite \mathbf{k} -linear combination $\sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^k} \sum_{\sigma \in \mathfrak{S}_k} \lambda_{\alpha, \sigma} p_{\alpha, \sigma}$ is well-defined (and we cannot restrict the second sum to the $\alpha \in \{1, 2, 3, \dots\}^k$ only).

2. The combinatorial Hopf algebra PNSym

Let us now recall the connected graded Hopf algebra NSym of noncommutative symmetric functions (introduced in [GKLLRT94], recently exposed in [GriRei20, §5.4] and [Meliot17, Definition 6.1]⁹). As a \mathbf{k} -algebra, it is free with countably many generators $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \dots$, and its comultiplication is given by $\Delta(\mathbf{H}_m) = \sum_{i=0}^m \mathbf{H}_i \otimes \mathbf{H}_{m-i}$, where $\mathbf{H}_0 := 1$. (We are using the notation \mathbf{H}_k for the k -th complete homogeneous noncommutative symmetric function in NSym, which is denoted H_k in [GriRei20, Theorem 5.4.2], and denoted S_k in [GKLLRT94, (22)] and [Meliot17].)

For any weak composition $(\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{N}^n$, we set $\mathbf{H}_{(\delta_1, \delta_2, \dots, \delta_n)} := \mathbf{H}_{\delta_1} \mathbf{H}_{\delta_2} \cdots \mathbf{H}_{\delta_n} \in \text{NSym}$. Thus, the family $(\mathbf{H}_\alpha)_\alpha$ is a basis of the \mathbf{k} -module NSym. (Note that \mathbf{H}_α is called S^α in [GKLLRT94].)

An *internal product* $*$ is defined on NSym in [GKLLRT94, §5.1]. It is explicitly given by the formula

$$\mathbf{H}_\alpha * \mathbf{H}_\beta = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} \mathbf{H}_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}$$

for all compositions α and β (see [GKLLRT94, Proposition 5.1]). As mentioned in the introduction, Patras's composition formula (1) is structurally identical to this expression. This shows that any cocommutative connected graded bialgebra H is a module over the non-unital algebra $\text{NSym}^{(2)}$ of noncommutative symmetric functions equipped with its internal product. (Each complete noncommutative symmetric function \mathbf{H}_α acts as the operator $p_{\alpha, \text{id}}$ on H .)

It is natural to ask whether a similar construction can be made for any connected graded bialgebra H using Theorem 1.18. Thus, we are looking for a non-unital algebra that contains elements $F_{\alpha, \sigma}$ for all compositions $\alpha \in \mathbb{N}^k$ and all permutations $\sigma \in \mathfrak{S}_k$, and which acts on any connected graded bialgebra H by having each $F_{\alpha, \sigma}$ act as the operator $p_{\alpha, \sigma}$.

In this section, we shall construct such an algebra – which I name $\text{PNSym}^{(2)}$ for “permuted noncommutative symmetric functions”. Besides having an “internal

⁹These three references give slightly different definitions of this Hopf algebra NSym, but all these definitions are easily seen to be equivalent (e.g., using [GKLLRT94, Note 3.5, Proposition 3.8, Proposition 3.9] and [Meliot17, Proposition 6.2, Proposition 6.3]). They also denote it by different symbols: It is called **Sym** in [GKLLRT94], called NSym in [GriRei20, §5.4], and called NCSym in [Meliot17, Definition 6.1] (a name that means a different Hopf algebra in most of the literature).

product” $*$, it has an “external product” \cdot (corresponding to the convolution of operators on H) and a coproduct Δ (corresponding to acting on a tensor product of bialgebras).

2.1. Mopiscotions

To construct this algebra, we need to make some implicit things explicit and introduce some more notation:

Definition 2.1. A *mopiscotion* (short for “permuted composition”) is a pair (α, σ) , where α is a composition of length k (for some $k \in \mathbb{N}$) and σ is a permutation in \mathfrak{S}_k .

Definition 2.2. A *weak mopiscotion* is a pair (α, σ) , where α is a weak composition of length k (for some $k \in \mathbb{N}$) and σ is a permutation in \mathfrak{S}_k .

Definition 2.3. Let (α, σ) be a weak mopiscotion, with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\sigma \in \mathfrak{S}_k$. Let $(j_1 < j_2 < \dots < j_h)$ be the list of all elements i of $[k]$ satisfying $\alpha_i \neq 0$, in increasing order. Let $\tau \in \mathfrak{S}_h$ be the standardization of the list $(\sigma(j_1), \sigma(j_2), \dots, \sigma(j_h))$. (See [GriRei20, Definition 5.3.3] for the meaning of “standardization”.) Let α^{red} denote the composition $(\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_h})$ (which consists of all nonzero entries of α). Then, we define $(\alpha, \sigma)^{\text{red}}$ to be the mopiscotion $(\alpha^{\text{red}}, \tau)$.

For example,

$$((3, 0, 1, 2, 0), [4, 5, 1, 3, 2])^{\text{red}} = ((3, 1, 2), [3, 1, 2]),$$

where the square brackets indicate a permutation written in one-line notation. For another example,

$$((3, 0, 1, 2, 0), [4, 1, 3, 2, 5])^{\text{red}} = ((3, 1, 2), [3, 2, 1]).$$

Clearly, if (α, σ) is a mopiscotion, then $(\alpha, \sigma)^{\text{red}} = (\alpha, \sigma)$.

The following is easy to see:

Proposition 2.4. Let H be a connected graded bialgebra. Let (α, σ) be a weak mopiscotion, and let $(\beta, \tau) = (\alpha, \sigma)^{\text{red}}$. Then,

$$p_{\alpha, \sigma} = p_{\beta, \tau}.$$

Proof idea. The connectedness of H yields $p_0(h) = \varepsilon(h) \cdot 1_H$ for each $h \in H$. Hence, in the formula

$$p_{\alpha, \sigma}(x) = \sum_{(x)} p_{\alpha_1}(x_{(\sigma(1))}) p_{\alpha_2}(x_{(\sigma(2))}) \cdots p_{\alpha_k}(x_{(\sigma(k))})$$

(from Remark 1.9), all the factors $p_{\alpha_i}(x_{(\sigma(i))})$ with $\alpha_i = 0$ can be rewritten as $\varepsilon(x_{(\sigma(i))})$ (the 1_H gets swallowed by the product) and thus can be removed completely (using the $\sum_{(h)} \varepsilon(h_{(1)}) h_{(2)} = \sum_{(h)} h_{(1)} \varepsilon(h_{(2)}) = h$ axiom of a coalgebra), as long as we remember to adjust the permutation σ accordingly (removing its value $\sigma(i)$ and decreasing all values larger than $\sigma(i)$ by 1). At the end of this process, we end up with $p_{\beta,\tau}(x)$. This shows that $p_{\alpha,\sigma}(x) = p_{\beta,\tau}(x)$ for each $x \in H$. Proposition 2.4 follows. \square

2.2. PNSym

Definition 2.5. Let PNSym be the free \mathbf{k} -module with basis $(F_{\alpha,\sigma})_{(\alpha,\sigma)}$ is a mopiscotion.

For any weak mopiscotion (α, σ) , we set

$$F_{\alpha,\sigma} := F_{\beta,\tau},$$

where $(\beta, \tau) = (\alpha, \sigma)^{\text{red}}$.

Define two multiplications on PNSym : one “external multiplication” (which mirrors convolution of $p_{\alpha,\sigma}$ ’s as expressed in Proposition 1.15) given by

$$F_{\alpha,\sigma} \cdot F_{\beta,\tau} = F_{\alpha\beta,\sigma\oplus\tau};$$

and another “internal multiplication” (which mirrors composition of $p_{\alpha,\sigma}$ ’s as expressed in Theorem 1.18) given by

$$F_{\alpha,\sigma} * F_{\beta,\tau} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} F_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell}), \tau[\sigma]}$$

(where $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^\ell$). Also, we define a comultiplication $\Delta : \text{PNSym} \rightarrow \text{PNSym} \otimes \text{PNSym}$ on PNSym by

$$\Delta(F_{\alpha,\sigma}) = \sum_{\substack{\beta, \gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta + \gamma = \alpha}} F_{\beta,\sigma} \otimes F_{\gamma,\sigma}$$

(mirroring the formula from Proposition 1.37).

We also equip the \mathbf{k} -module PNSym with a grading by letting each $F_{\alpha,\sigma}$ be homogeneous of degree $|\alpha|$.

These operations on PNSym behave as nicely as the analogous operations on NSym :

Theorem 2.6. The \mathbf{k} -module PNSym becomes a connected graded cocommutative Hopf algebra when equipped with the external multiplication \cdot , and a (non-graded) non-unital bialgebra when equipped with the internal multiplication $*$. In particular, both multiplications are associative.

There are two ways to prove this. I shall very briefly outline both:

First proof idea for Theorem 2.6. Most claims can be derived from properties of the operators $p_{\alpha,\sigma}$, using the H from Theorem 1.35 (a) as a faithful representation.

For an example, let us prove that the internal multiplication $*$ on PNSym is associative.

Let H be any connected graded \mathbf{k} -bialgebra. Let $\text{ev}_H : \text{PNSym} \rightarrow \text{End } H$ be the \mathbf{k} -linear map that sends any $F_{\alpha,\sigma}$ to the operator $p_{\alpha,\sigma} \in \text{End } H$ for any mopiscotion (α, σ) . Note that

$$\text{ev}_H(F_{\alpha,\sigma}) = p_{\alpha,\sigma} \tag{36}$$

is true not only for all mopiscotions (α, σ) , but also for all weak mopiscotions (α, σ) (because if (α, σ) is any weak mopiscotion, and if $(\beta, \tau) = (\alpha, \sigma)^{\text{red}}$, then $p_{\alpha,\sigma} = p_{\beta,\tau}$ and $F_{\alpha,\sigma} = F_{\beta,\tau}$).

Now, let H be the connected graded Hopf algebra H from Theorem 1.35 (a). Then, Theorem 1.35 (a) says that the family $(p_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a mopiscotion}}$ is \mathbf{k} -linearly independent. Hence, the linear map ev_H is injective.

The formula for $F_{\alpha,\sigma} * F_{\beta,\tau}$ that we used to define the internal multiplication $*$ is very similar to the formula for $p_{\alpha,\sigma} \circ p_{\beta,\tau}$ in Theorem 1.18. In view of (36), this entails that

$$\text{ev}_H(F_{\alpha,\sigma} * F_{\beta,\tau}) = p_{\alpha,\sigma} \circ p_{\beta,\tau} = \text{ev}_H(F_{\alpha,\sigma}) \circ \text{ev}_H(F_{\beta,\tau})$$

for any two mopiscotions (α, σ) and (β, τ) . By bilinearity, this entails that

$$\text{ev}_H(f * g) = (\text{ev}_H f) \circ (\text{ev}_H g)$$

for any $f, g \in \text{PNSym}$. Thus, the injective \mathbf{k} -linear map $\text{ev}_H : \text{PNSym} \rightarrow \text{End } H$ embeds the \mathbf{k} -module PNSym with its binary operation $*$ into the algebra $\text{End } H$ with its binary operation \circ . Since the latter operation \circ is associative, it thus follows that the former operation $*$ is associative as well.

Similarly, we can show that the operation \cdot on PNSym is associative and unital with the unity $1 = F_{\emptyset,\emptyset}$ (although this is pretty obvious).

It is very easy to see that the cooperation Δ is coassociative, counital and cocommutative. It is also clear that both the multiplication \cdot and the comultiplication Δ on PNSym are graded.

The next difficulty is to prove that Δ is a \mathbf{k} -algebra homomorphism, i.e., that $\Delta(fg) = \Delta(f) \cdot \Delta(g)$ for all $f, g \in \text{PNSym}$ (where the “ \cdot ” on the right hand side is the extension of the external multiplication \cdot to $\text{PNSym} \otimes \text{PNSym}$). Here, we can argue as above, using the fact (a consequence of Theorem 1.37) that

$$\begin{aligned} \text{ev}_{H \otimes H}(f) &= (\text{ev}_H \otimes \text{ev}_H)(\Delta(f)) \in \text{End}(H \otimes H) \\ &\text{for every } f \in \text{PNSym} \end{aligned}$$

to make sense of Δ), and using the fact that the map

$$\text{ev}_H \otimes \text{ev}_H : \text{PNSym} \otimes \text{PNSym} \rightarrow \text{End } H \otimes \text{End } H \rightarrow \text{End } (H \otimes H)$$

is injective (this is not hard to show using the argument used in the proof of Theorem 1.35).

What we have shown so far yields that PNSym (equipped with \cdot and Δ) is a connected graded \mathbf{k} -bialgebra. Thus, PNSym is a Hopf algebra (since any connected graded \mathbf{k} -bialgebra is a Hopf algebra).

It remains to show that PNSym (equipped with $*$ and Δ) is a non-unital \mathbf{k} -bialgebra. Having already verified that $*$ is associative, we only need to show that $\Delta(f * g) = \Delta(f) * \Delta(g)$ for all $f, g \in \text{PNSym}$. But this is similar to the proof of $\Delta(fg) = \Delta(f) \cdot \Delta(g)$ above. Thus, the proof of Theorem 2.6 is complete. \square

Second proof idea for Theorem 2.6. There is also a more direct combinatorial approach to this theorem. First, we shall define two smaller bialgebras NNSym and Perm , and then present PNSym as a quotient of their tensor product $\text{NNSym} \otimes \text{Perm}$.

Here are some details:

We define NNSym to be the free \mathbf{k} -module with basis $(C_\alpha)_\alpha$ is a weak composition. We equip this \mathbf{k} -module NNSym with an “external multiplication” defined by

$$C_\alpha \cdot C_\beta = C_{\alpha\beta}$$

(where $\alpha\beta$ is the concatenation of α and β), and an “internal multiplication” defined by

$$C_\alpha * C_\beta = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \dots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \dots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} C_{(\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{k,\ell})}$$

(where $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^\ell$), and a comultiplication $\Delta : \text{NNSym} \rightarrow \text{NNSym} \otimes \text{NNSym}$ defined by

$$\Delta(C_\alpha) = \sum_{\substack{\beta, \gamma \in \mathbb{N}^k; \\ \text{entrywise sum } \beta + \gamma = \alpha}} C_\beta \otimes C_\gamma \quad \text{for any } \alpha \in \mathbb{N}^k.$$

It is not too hard to show that NNSym thus becomes a graded (but not connected!) cocommutative \mathbf{k} -bialgebra when equipped with the external multiplication \cdot , and a (non-graded) non-unital bialgebra when equipped with the internal multiplication $*$. (Indeed, this NNSym is a mild variation on the Hopf algebra NSym of noncommutative symmetric functions, which is studied (e.g.) in [GKLLRT94] or [GriRei20, §5.4]; the only difference is that compositions have been replaced by weak compositions.)

We define \mathfrak{S} to be the disjoint union $\bigsqcup_{k \in \mathbb{N}} \mathfrak{S}_k$ of all symmetric groups \mathfrak{S}_k for all $k \in \mathbb{N}$. We define Perm to be the free \mathbf{k} -module with basis $(P_\sigma)_{\sigma \in \mathfrak{S}}$. We equip this \mathbf{k} -module Perm with an “external multiplication” \cdot defined by

$$P_\sigma \cdot P_\tau = P_{\sigma \oplus \tau},$$

and an “internal multiplication” $*$ defined by

$$P_\sigma * P_\tau = P_{\tau[\sigma]},$$

and a comultiplication $\Delta : \text{Perm} \rightarrow \text{Perm} \otimes \text{Perm}$ defined by

$$\Delta(P_\sigma) = P_\sigma \otimes P_\sigma.$$

It is not too hard to show that Perm becomes a (non-graded) cocommutative \mathbf{k} -bialgebra when equipped with either of the two multiplications. Indeed, in both cases, it becomes the monoid algebra of an appropriate monoid on the set Perm . (Checking the associativity of the internal multiplication is a neat exercise in combinatorics.¹⁰)

Now, let PNNSym be the tensor product $\text{NNSym} \otimes \text{Perm}$. We equip this tensor product PNNSym with an “external multiplication” (obtained by tensoring the external multiplications of NNSym and of Perm), an “internal multiplication” (similarly) and a comultiplication (similarly). We furthermore set

$$\widehat{F}_{\alpha,\sigma} := C_\alpha \otimes P_\sigma \in \text{PNNSym} \quad \text{for any weak mopiscotion } (\alpha, \sigma).$$

Then, $(\widehat{F}_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a weak mopiscotion}}$ is a basis of the \mathbf{k} -module PNNSym , and our operations \cdot , $*$ and Δ on PNNSym satisfy the same relations for this basis as the

¹⁰Let me sketch a way to make the argument elegant and at least somewhat transparent.

We must show that $\tau[\sigma[\rho]] = (\tau[\sigma])[\rho]$ for any three permutations $\tau, \sigma, \rho \in \mathfrak{S}$.

We can regard the set \mathfrak{S} as a skeletal groupoid with objects $0, 1, 2, \dots$ and morphism sets $\mathfrak{S}(k, k) = \mathfrak{S}_k$ and $\mathfrak{S}(k, \ell) = \emptyset$ for all $k \neq \ell$. However, the definition of $\tau[\sigma]$ becomes cleaner if we “de-skeletize” \mathfrak{S} to a larger category. Namely, we define a *tormutation* to be a bijection (not necessarily order-preserving) between two finite totally ordered sets. Clearly, each permutation $\sigma \in \mathfrak{S}_k$ is a tormutation $[k] \rightarrow [k]$. Conversely, any tormutation $\phi : A \rightarrow B$ induces a canonical permutation $\bar{\phi} \in \mathfrak{S}_{|A|}$ by the rule

$$\bar{\phi} := \text{inc}_{B \rightarrow [|B|]} \circ \phi \circ \text{inc}_{[|A|] \rightarrow A} : [|A|] \rightarrow [|B|],$$

where $\text{inc}_{X \rightarrow Y}$ denotes the unique order isomorphism between two given finite totally ordered sets X and Y . Thus, the skeletal groupoid \mathfrak{S} is a skeleton of the groupoid $\widetilde{\mathfrak{S}}$ whose objects are the finite totally ordered sets and whose morphisms are the tormutations.

Given two tormutations $\phi : A \rightarrow B$ and $\phi' : A' \rightarrow B'$, we now define a tormutation $\phi' \langle \phi \rangle : A \times A' \rightarrow B' \times B$ by

$$(\phi' \langle \phi \rangle)(a, a') = (\phi'(a'), \phi(a)) \quad \text{for all } (a, a') \in A \times A'.$$

In other words, $\phi' \langle \phi \rangle$ applies ϕ and ϕ' to the respective entries of the input, then swaps the outputs.

It is now easy to see that $\overline{\phi' \langle \phi \rangle} = \overline{\phi'} [\overline{\phi}]$ for any two tormutations ϕ and ϕ' . Thus, in order to prove that $\tau[\sigma[\rho]] = (\tau[\sigma])[\rho]$ for any three permutations $\tau, \sigma, \rho \in \mathfrak{S}$, it suffices to show that $\overline{\phi'' \langle \phi' \langle \phi \rangle \rangle} = \overline{\phi'' \langle \phi' \rangle} \langle \overline{\phi} \rangle$ for any three tormutations ϕ, ϕ', ϕ'' . But the latter is easy: The map $\overline{\phi'' \langle \phi' \langle \phi \rangle \rangle}$ sends each $((a, a'), a'')$ to $(\phi''(a''), (\phi'(a'), \phi(a)))$, whereas the map $\overline{\phi'' \langle \phi' \rangle} \langle \overline{\phi} \rangle$ sends each $(a, (a', a''))$ to $((\phi''(a''), \phi'(a')), \phi(a))$. Because of the canonical order isomorphism $A \times (B \times C) \cong (A \times B) \times C$ for any three totally ordered sets A, B, C , these two maps are therefore equivalent, i.e., have the same canonical permutation.

analogous operations on PNSym do for the basis $(F_{\alpha,\sigma})_{(\alpha,\sigma)}$ is a mopiscotion (we just have to replace each “ F ” by “ \widehat{F} ”). It is thus easy to show that PNNSym is a graded (but not connected) cocommutative \mathbf{k} -bialgebra when equipped with \cdot , and a (non-graded) non-unital \mathbf{k} -bialgebra when equipped with $*$. (Here we use the facts that the tensor product of two cocommutative bialgebras is a cocommutative bialgebra, and that the tensor product of two non-unital bialgebras is a non-unital bialgebra.)

As we said, PNNSym is almost the PNSym that we care about. But PNNSym does not satisfy the rule

$$F_{\alpha,\sigma} = F_{\beta,\tau} \quad \text{for } (\beta,\tau) = (\alpha,\sigma)^{\text{red}}$$

that is fundamental to the definition of PNSym . Hence, PNSym is not quite PNNSym but rather a quotient of PNNSym . To be specific, we define a \mathbf{k} -submodule I_{red} of PNNSym by

$$\begin{aligned} I_{\text{red}} &:= \text{span}_{\mathbf{k}} \left(\widehat{F}_{\alpha,\sigma} - \widehat{F}_{\beta,\tau} \mid (\beta,\tau) = (\alpha,\sigma)^{\text{red}} \right) \\ &= \text{span}_{\mathbf{k}} \left(\widehat{F}_{\alpha,\sigma} - \widehat{F}_{\beta,\tau} \mid (\beta,\tau)^{\text{red}} = (\alpha,\sigma)^{\text{red}} \right). \end{aligned}$$

It is not too hard to show that this I_{red} is an ideal of PNNSym with respect to both \cdot and $*$ and a coideal with respect to Δ . Hence, the quotient $\text{PNNSym} / I_{\text{red}}$ inherits all operations of PNNSym , thus becoming a graded bialgebra under \cdot and Δ and a non-unital bialgebra under $*$ and Δ . Moreover, the graded bialgebra $\text{PNNSym} / I_{\text{red}}$ is connected (since $(\alpha,\sigma)^{\text{red}} = (\emptyset,\emptyset)$ whenever $|\alpha| = 0$), and thus is a Hopf algebra. As we recall, this means that PNSym is a well-defined connected graded Hopf algebra (since $\text{PNSym} \cong \text{PNNSym} / I_{\text{red}}$). This completes the proof of Theorem 2.6 again. \square

Proposition 2.7. Let $n \in \mathbb{N}$. Then, the n -th graded component PNSym_n of PNSym is a free \mathbf{k} -module of rank

$$\sum_{k=0}^n \binom{n-1}{n-k} k!.$$

Proof idea. Clearly, PNSym_n is a free \mathbf{k} -module with a basis consisting of all $F_{\alpha,\sigma}$ where (α,σ) ranges over all mopiscotions satisfying $|\alpha| = n$. It remains to show that the number of such mopiscotions is $\sum_{k=0}^n \binom{n-1}{n-k} k!$. But this is easy: For any k , the number of such mopiscotions in which α has length k is $\binom{n-1}{n-k} k!$ (since there are $\binom{n-1}{n-k}$ compositions of n into k parts, and $k!$ permutations $\sigma \in \mathfrak{S}_k$). \square

We note that the addend $\binom{n-1}{n-k}k!$ in Proposition 2.7 can also be rewritten as $k \cdot (n-1)(n-2) \cdots (n-k+1)$ when n is positive (but not when $n=0$).

Here are the ranks of the free \mathbf{k} -modules PNSym_n for the first few values of n :

n	0	1	2	3	4	5	6	7
$\text{rank}(\text{PNSym}_n)$	1	1	3	11	49	261	1631	11743

Starting at $n=1$, this sequence of ranks is known to the OEIS as Sequence A001339, and has appeared in the theory of combinatorial Hopf algebras before ([HiNoTh06, §3.8.3]), although the Hopf algebra considered there appears to be different (perhaps dual to ours?).¹¹

Theorem 2.8. Let $\text{PNSym}^{(2)}$ be the non-unital algebra PNSym with multiplication $*$. Then, every connected graded bialgebra H becomes a $\text{PNSym}^{(2)}$ -module, with $F_{\alpha,\sigma}$ acting as $p_{\alpha,\sigma}$. Moreover, the action of an external product uv of two elements $u, v \in \text{PNSym}$ on H is the convolution of the action of u with the action of v .

Proof idea. The first claim follows from Theorem 1.18, the second from Proposition 1.15. □

There is also an analogue of the “splitting formula” ([GKLLRT94, Proposition 5.2]) for PNSym , connecting the two products (internal and external) with the comultiplication:

Theorem 2.9. Let $f, g, h \in \text{PNSym}$. Write $\Delta(h)$ as $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ (using Sweedler notation). Then,

$$(fg) * h = \sum_{(h)} (f * h_{(1)}) (g * h_{(2)}).$$

¹¹Some words about the connection to [HiNoTh06, §3.8.3] are in order. One of the many concepts studied in [HiNoTh06] is the *stalactic equivalence*, an equivalence relation on the words over a given alphabet A . It is the equivalence relation on A^* defined by the single axiom $uawav \equiv uaaav$ for any words $u, v, w \in A^*$ and any letter $a \in A$. It is easy to see that two words $u, v \in A^*$ are equivalent if and only if they contain each letter the same number of times, and the order in which the letters **first** appear in each word is the same for both words. Thus, each stalactic equivalence class is uniquely determined by the multiplicities of its letters and by the order in which they first appear. For *initial* words (i.e., for words over the alphabet $\{1, 2, 3, \dots\}$ whose set of letters is $[k]$ for some $k \in \mathbb{N}$), this data is equivalent to a mopiscotion (α, σ) (where α determines the multiplicities of letters, and σ determines their order of first appearance). The number of stalactic equivalence classes of initial words of length n thus equals the number of mopiscotions (α, σ) with $|\alpha| = n$.

(To be more precise, the analogue of [GKLLRT94, Proposition 5.2] would be the generalization of this formula to iterated coproducts $\Delta^{[r]}(h)$, but this generalization follows from Theorem 2.9 by a straightforward induction on r .)

Proof idea for Theorem 2.9. Fairly easy using Definition 2.5. (An important first step is to show that the formulas for $F_{\alpha,\sigma} \cdot F_{\beta,\tau}$ and $F_{\alpha,\sigma} * F_{\beta,\tau}$ hold not only for mopiscotions (α, σ) and (β, τ) but also for weak mopiscotions (α, σ) and (β, τ) .) \square

Remark 2.10. The \mathbf{k} -linear map

$$\begin{aligned} \mathfrak{p} : \text{PNSym} &\rightarrow \text{NSym}, \\ F_{\alpha,\sigma} &\mapsto \mathbf{H}_\alpha \quad \text{for any mopiscotion } (\alpha, \sigma) \end{aligned}$$

is a surjection that respects all structures (external and internal multiplication, comultiplication and grading).

This surjection is furthermore split: The \mathbf{k} -linear map

$$\begin{aligned} \mathfrak{i} : \text{NSym} &\rightarrow \text{PNSym}, \\ \mathbf{H}_\alpha &\mapsto F_{\alpha,\text{id}} \quad \text{for any composition } \alpha \end{aligned}$$

(where id denotes the identity permutation in \mathfrak{S}_k where α has length k) is an injection that is right-inverse to \mathfrak{p} and respects external multiplication, comultiplication and grading (but does not respect internal multiplication).

Proposition 2.11. The \mathbf{k} -algebra PNSym (equipped with the external multiplication) is free.

Proof idea. This algebra is just the monoid algebra of the monoid of mopiscotions under the operation $(\alpha, \sigma) \cdot (\beta, \tau) = (\alpha\beta, \sigma \oplus \tau)$. But this monoid is free, with the generators being the mopiscotions (α, σ) for which the permutation σ is connected¹² (as can be easily verified). \square

2.3. Questions

Much remains to be understood about PNSym . Many combinatorial Hopf algebras can be embedded into algebras of noncommutative formal power series (i.e., completions of free algebras). For instance, NSym has such an embedding¹³.

Question 2.12. Does PNSym have such an embedding as well?

We can ask about some other features that certain combinatorial Hopf algebras have:

¹²See [GriRei20, Exercise 8.1.10] for the definition of a connected permutation.

¹³This is explained, e.g., in [GriRei20, §8.1]: Namely, [GriRei20, Corollary 8.1.14(b)] embeds NSym in the Hopf algebra FQSym , whereas [GriRei20, (8.1.3)] embeds FQSym in the algebra $\mathbf{k}\langle\langle X_1, X_2, X_3, \dots \rangle\rangle$ of noncommutative formal power series.

Question 2.13. Does PNSym have a categorification (e.g., a presentation as K_0 of some category)?

Question 2.14. Is there a cancellation-free formula for the antipode of PNSym ?

Question 2.15. Is PNSym isomorphic to the dual of the commutative combinatorial Hopf algebra arising from [HiNoTh06, §3.8.3]?

Question 2.16. What are the primitive elements of PNSym ? We recall that the primitive elements of NSym correspond to the *Lie quasi-idempotents* in the descent algebra ([GKLLRT94, Corollary 5.17]), and include several renowned families, such as the noncommutative power sums of the first two kinds ([GKLLRT94, §5.2], [GriRei20, Exercises 5.4.5 and 5.4.12]) and the third kind ([KrLeTh97, Definition 5.26]) and the Klyachko elements ([KrLeTh97, §6.2]). What is the meaning of primitive elements of PNSym ? Do some of them act as quasi-idempotents on every connected graded bialgebra H ?

Question 2.17. Is the dual of PNSym a polynomial ring? (For comparison, the dual of NSym is QSym , which is a polynomial ring by a result of Hazewinkel [GriRei20, Theorem 6.4.3].)

Recall the nonunital algebra $\text{PNSym}^{(2)}$ (that is, PNSym equipped with the internal product $*$). It is a lift of the nonunital algebra $\text{NSym}^{(2)}$ (that is, NSym equipped with the internal product $*$), whose n -th graded component $\text{NSym}_n^{(2)}$ is known to be isomorphic to the descent algebra of the symmetric group \mathfrak{S}_n (see [GKLLRT94, §5.1]). Thus, $\text{PNSym}^{(2)}$ can be viewed as a “twisted” version of the descent algebra (though apparently not in the same sense as in [PatSch06]).

Question 2.18. Is the nonunital algebra $\text{PNSym}^{(2)}$ isomorphic to its own opposite algebra?

Question 2.19. If \mathbf{k} is a field of characteristic 0, what are the Jacobson radical and the semisimple quotient of the non-unital algebra $\text{PNSym}^{(2)}$? (All signs point to the semisimple quotient of the n -th graded component $\text{PNSym}_n^{(2)}$ being the center of the symmetric group algebra $\mathbf{k}[\mathfrak{S}_n]$, just as for $\text{NSym}_n^{(2)}$.)

2.4. Aguiar and Mahajan

We finish with some cryptic remarks connecting PNSym with Aguiar’s and Mahajan’s theory of Hopf monoids.

In [AguMah20, §1.9.3], Aguiar and Mahajan define the *Janus algebra* $J[\mathcal{A}]$ of a hyperplane arrangement \mathcal{A} . When \mathcal{A} is the braid arrangement in \mathbb{R}^n , this Janus

algebra $J[\mathcal{A}]$ has a basis indexed by pairs (U, V) , where U and V are two set compositions (i.e., ordered set partitions) of $[n]$ that define the same (unordered) set partition (i.e., the blocks of U are the blocks of V , perhaps in a different order)¹⁴. Such pairs (U, V) can equivalently be viewed as pairs (U, σ) , where U is a set composition of $[n]$ into k blocks (for some $k \in \mathbb{N}$), and where $\sigma \in \mathfrak{S}_k$ is a permutation (the one that permutes the blocks of U to obtain V). However, viewing them as pairs (U, V) exposes a symmetry that would be hidden in the (U, σ) formulation. Note that [AguMah20, Proposition 11.6] is an analogue of our Theorem 2.8 for \mathcal{A} -bimonoids instead of connected graded bialgebras.

Let us continue with the case when \mathcal{A} is the braid arrangement in \mathbb{R}^n . The symmetric group \mathfrak{S}_n acts on the Janus algebra $J[\mathcal{A}]$ by the rule $\tau \cdot (U, V) = (\tau(U), \tau(V))$ (for any $\tau \in \mathfrak{S}_n$), or (in the (U, σ) formulation) by the rule $\tau \cdot (U, \sigma) = (\tau(U), \sigma)$. The invariant space (more precisely, \mathbf{k} -module) $J[\mathcal{A}]^{\mathfrak{S}_n}$ under this action is a subalgebra of the Janus algebra $J[\mathcal{A}]$, called the *invariant Janus algebra*. It has a basis formed by the orbit sums, which are in bijection with the mopiscotions (α, σ) satisfying $|\alpha| = n$ (since an orbit of a set composition of $[n]$ under the symmetric group \mathfrak{S}_n is essentially an integer composition of n , recording the sizes of the blocks).

Question 2.20. Is $J[\mathcal{A}]^{\mathfrak{S}_n}$ isomorphic (as an algebra) to the n -th graded component of PNSym equipped with the internal multiplication $*$?

If so, then it stands to reason that results by Aguiar and Mahajan should be translatable into properties of PNSym , at least as far as the internal multiplication is concerned¹⁵. The invariant Janus algebra $J[\mathcal{A}]^{\mathfrak{S}_n}$ (and even a q -deformation thereof) is studied by Aguiar and Mahajan in [AguMah22, §1.12.3], although establishing a dictionary between the geometric language of [AguMah22] and our combinatorial one is a rather daunting task.

3. Application to identity checking

In this last section, we shall outline a way in which the above results can be used. For an example, let us prove that every connected graded bialgebra H satisfies

$$(p_1 \star p_2 - p_2 \star p_1)^5 = 0 \tag{37}$$

(where the 5-th power is taken with respect to the composition \circ). To prove this equality, we can rewrite the left hand side as $(p_{(1,2),\text{id}} - p_{(2,1),\text{id}})^5$ (where id is the identity permutation in \mathfrak{S}_2), and expand this (using Theorem 1.18 and Proposition

¹⁴In the language of hyperplane arrangements, set compositions are called *faces*, and our pairs (U, V) of set compositions are called *bifaces*. Thus the name “Janus algebra”.

¹⁵The external multiplication and the coproduct are less likely to arise in this way, since they relate different graded components and thus should correspond not to a single braid arrangement but rather a whole family thereof.

1.15) as a \mathbb{Z} -linear combination of $p_{\alpha,\sigma}$'s. According to Theorem 1.35, for the above equality (37) to hold, this \mathbb{Z} -linear combination has to be trivial (i.e., all coefficients must be zeroes), which we can directly check. Alternatively, using Theorem 2.8, we can reduce the equality (37) to the equality

$$\left(F_{(1,2),\text{id}} - F_{(2,1),\text{id}} \right)^{*5} = 0 \quad \text{in PNSym}$$

(where the “*5” exponent means a 5-th power with respect to the internal product $*$). This equality is straightforward to check using the definition of $*$.

Likewise, any identity such as (37) (that is, any identity whose both sides are formed from the maps $p_{\alpha,\sigma}$ by addition, convolution and composition) can be proved mechanically using computations inside PNSym. This approach can be useful even if we don't end up making the computations. For example, it shows that when proving an identity such as (37) (with integer coefficients), it suffices to prove it for $\mathbf{k} = \mathbb{Q}$ (since the Hopf algebra PNSym with all its structures is defined over \mathbb{Z} and is free as a \mathbf{k} -module, so that its \mathbb{Z} -version embeds naturally into its \mathbb{Q} -version). Moreover, it suffices to prove it in the case when all graded components H_n of H are finite-dimensional \mathbf{k} -vector spaces (thanks to Theorem 1.35 (b)). This makes a number of methods available (graded duals, coradical filtration¹⁶) that could not be used in the a-priori generality of a connected graded bialgebra over an arbitrary commutative ring. While this is perhaps not very unexpected, it is reassuring and helpful.

This all can be applied to a slightly wider class of identities. Indeed, while in (37) we don't allow the use of the identity map $\text{id}_H : H \rightarrow H$, we can actually express id_H as the infinite sum $p_0 + p_1 + p_2 + \dots$. Infinite sums are not allowed in (37), but we can replace them by finite partial sums if we restrict ourselves to the submodule $H_0 + H_1 + \dots + H_k$ of H . For example, if we want to prove the equality

$$(p_1 \star \text{id}_H - 2 \text{id}_H) \circ (p_1 \star \text{id}_H)^2 = 0 \quad \text{on } H_2$$

for any connected graded bialgebra H (this identity is part of [Grinbe22, Theorem 1.4 (b)]), then we can replace each id_H by $p_0 + p_1 + p_2$, which renders the equality amenable to our PNSym-method. The antipode S of a connected graded Hopf algebra H is also not directly of a form supported by our method, but can be written as an infinite sum

$$\sum_{k \in \mathbb{N}} (-1)^k (p_1 + p_2 + p_3 + \dots)^{\star k} = \sum_{k \in \mathbb{N}} (-1)^k \sum_{\alpha \in \mathbb{N}^k \text{ is a composition}} p_{\alpha, \text{id}},$$

which – when restricted to a submodule $H_0 + H_1 + \dots + H_k$ – can be replaced by a finite partial sum. Thus, claims such as Aguiar's and Lauve's “ $(S^2 - \text{id})^k = 0$ on H_k for any connected graded Hopf algebra H ” can (for a fixed $k \in \mathbb{N}$) be checked

¹⁶Both of these methods have been used in proving such identities (see, e.g., [AguLau14], [Pang18, §3]).

using the PNSym-method (although this particular claim has been extended and proved in a much more elementary way in [Grinbe21]). I believe that the field of Hopf algebra identities is vast and mostly unexplored so far.

Question 3.1. Is there a similar method for checking identities for general (not just connected) graded bialgebras? It appears natural to use the bialgebra PNNSym (constructed in our second proof of Theorem 2.6) instead of PNSym here, but I don't know whether every identity can be proved in this way, since I don't know if there is an analogue of Theorem 1.35 for weak mopiscotions and arbitrary graded bialgebras (see Question 1.36).

Question 3.2. Is there a similar method for checking identities between maps $H^{\otimes k} \rightarrow H^{\otimes \ell}$ as opposed to only maps $H \rightarrow H$?

The same problem without the grading has been solved by categorical methods in Pirashvili's 2002 paper [Pirash02].

We end this section with an open question that generalizes (37):

Question 3.3. Let $i, j \in \mathbb{N}$. What is the smallest integer $k(i, j) \in \mathbb{N}$ for which every connected graded bialgebra H satisfies

$$(p_i \star p_j - p_j \star p_i)^{k(i,j)} = 0$$

(where the power is with respect to composition)? In other words, what is the smallest integer $k(i, j) \in \mathbb{N}$ for which

$$\left(F_{(i,j),\text{id}} - F_{(j,i),\text{id}} \right)^{*k(i,j)} = 0 \quad \text{in PNSym}^{(2)} ?$$

Some known values (computed using SageMath) are

$$\begin{aligned} k(0, j) &= 1, & k(i, i) &= 1, & k(1, 2) &= 5, & k(1, 3) &= 7, \\ k(1, 4) &= 9, & k(1, 5) &= 11, & k(1, 6) &= 13, & k(2, 3) &= 9, \\ k(2, 4) &= 9. \end{aligned}$$

(Of course, $k(i, j) = k(j, i)$ for all i and j .) It is striking that all these known values are odd.

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