On the PBW theorem for pre-Lie algebras

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Contents

Introduction

In this note I shall explore the surroundings of Guin's and Oudom's Poincaré-Birkhoff-Witt theorem for Lie algebras obtained from pre-Lie algebras ([GuiOud04, Théorème 3.5], [GuiOud08, Theorem 3.14], [Manchon11, Theorem 1.1], [Schedl10, Corollary 1.3.1]) and, as a consequence, answer a MathOverflow question that I have asked in 2012 [MO102874] about the normal ordered product on differential operators. The results proven here are elementary (and so are their proofs), and few of them are new; yet (in my opinion) there are arguments and remarks worthy of circulation among them (in particular, the question [MO102874] feels natural to me, yet I have to find it in the existing literature).

The paper is long, due to the bulk of straightforward computations and detailed proofs it contains. Since most of the proofs are relatively straightforward (Theorem 4.7 **(h)** is probably the only tricky one), the reader is encouraged to skip them and regard the theorems as exercises. I have also tried to be maximally explicit and technically correct (e.g., I avoid identifying a Lie algebra $\mathfrak{g}$ with its image in its universal enveloping algebra $U(\mathfrak{g})$ because the canonical map $\mathfrak{g} \to U(\mathfrak{g})$ is not always injective over a ring); this, too, is responsible for some of the length of this note.

In Section 1, we state (after fixing notations and reminding the reader of basic terminology) an elementary and simple theorem (Theorem 1.15) about Lie algebra actions. This theorem states that if $\mathfrak{g}$ is a Lie algebra over a commutative ring $\mathbf{k}$, if C is a $\mathbf{k}$-algebra (which, for us, means an associative $\mathbf{k}$-algebra), if $K : \mathfrak{g} \to \operatorname{Der} C$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of derivations of C, and if $f : \mathfrak{g} \to C$ is a $\mathbf{k}$-linear map satisfying

$$f([a,b]) = [f(a), f(b)] + (K(a))(f(b)) - (K(b))(f(a)) \qquad \text{for all } a, b \in \mathfrak{g},$$

then C becomes a $\mathfrak{g}$-module via

$$a \rightharpoonup u = f(a) \cdot u + (K(a))(u) \qquad \text{for all } a \in \mathfrak{g} \text{ and } u \in C$$

(where $a \rightharpoonup u$ is our notation for the action of a on u). We then show some additional properties of this action, the most significant of which is Theorem 1.20. This setting is rather general; we will only end up using a particular case of it in the later sections.

In Section 2, we present the setting of the Poincaré-Birkhoff-Witt theorem(s): a Lie algebra $\mathfrak{g}$, its universal enveloping algebra $U(\mathfrak{g})$, its symmetric algebra $\operatorname{Sym} \mathfrak{g}$, and various homomorphisms between these modules (and their associated graded modules). Everything in this section is well-known, but I found it worthwhile to explicitly state all definitions and basic results, and even prove some of them (Proposition 2.23 and Lemma 2.24 are probably the most important ones), since the available literature leaves too much to the reader and occasionally lacks precision. As a result, this section has become rather long; a reader familiar with the Poincaré-Birkhoff-Witt theorems will probably not be hurt by skipping it entirely.

Section 3 is the heart of this note. Here I first define the notions of left and right pre-Lie algebras, and state their basic properties: viz., that any (associative) algebra is both a left and a right pre-Lie algebra, and that from any left or right pre-Lie algebra A one can construct a Lie algebra A^-. Then, we state the main properties of the Guin-Oudom isomorphism (Theorem 3.10). This is a $\mathbf{k}$-coalgebra isomorphism $U(A^-) \to \operatorname{Sym} A$ defined for every left pre-Lie algebra A. Unlike Guin and Oudom ([GuiOud04] and [GuiOud08]), we construct this isomorphism using Theorem 1.15 and Lemma 2.24 rather than by extending the binary operation of the pre-Lie algebra A to its symmetric algebra $\operatorname{Sym} A$. As a consequence of this isomorphism, the Poincaré-Birkhoff-Witt theorem holds for every Lie algebra of the form A^-, where A is a pre-Lie algebra (Theorem 3.10 **(k)**); this stands in contrast to the usual versions of the Poincaré-Birkhoff-Witt theorem, which make assumptions on the $\mathbf{k}$-module structure of the Lie algebra. We then use this isomorphism to define a commutative multiplication $\boxdot$ on $U(A^-)$ (Corollary 3.11); it is defined by transporting the multiplication from $\operatorname{Sym} A$ to $U(A^-)$ via the Guin-Oudom isomorphism $U(A^-) \to \operatorname{Sym} A$.

In Section 4, we apply the above to the Lie algebra $\mathfrak{gl}_n$ (which has the form A^- for the pre-Lie algebra $A = \mathrm{M}_n(\mathbf{k})$), and relate it to differential operators with polynomial coefficients. Part of the result that we obtain states the following: Assume that $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let G denote the nm-element set $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$. Let $\mathcal{A}$ be the polynomial ring $\mathbf{k}\left[x_{i,j} \mid (i,j) \in G\right]$ in the nm (commuting) indeterminates $x_{i,j}$. Let $\mathcal{D}$ denote the $\mathbf{k}$-subalgebra of $\operatorname{End} \mathcal{A}$ generated by the multiplication operators $x_{i,j}$ (for $(i,j) \in G$) [1] and the differential operators $\dfrac{\partial}{\partial x_{i,j}}$ (for $(i,j) \in G$). (Thus, $\mathcal{D}$ is the $\mathbf{k}$-algebra of differential operators in the $x_{i,j}$ with polynomial coefficients.) Let $\mathcal{A}'$ be the polynomial ring $\mathbf{k}\left[\partial_{i,j} \mid (i,j) \in G\right]$ in the nm (commuting) indeterminates $\partial_{i,j}$. The $\mathbf{k}$-linear map

$$\xi : \mathcal{A} \otimes \mathcal{A}' \to \mathcal{D}, \qquad P \otimes Q \mapsto P \cdot Q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j) \in G}\right)$$

is a $\mathbf{k}$-module isomorphism, but not (in general) a $\mathbf{k}$-algebra homomorphism. However, we can use this $\mathbf{k}$-module isomorphism to transport the (commutative) multiplication on $\mathcal{A} \otimes \mathcal{A}'$ to $\mathcal{D}$. We denote the resulting multiplication by $\boxdot$ (explicitly, it is given by $A \boxdot B = \xi\left(\xi^{-1}(A) \cdot \xi^{-1}(B)\right)$). Next, we define a $\mathbf{k}$-linear map $\omega : \mathfrak{gl}_n \to \mathcal{D}^-$ by

$$\omega\left(E_{i,j}\right) = \sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{j,k}} \qquad \text{for every } (i,j) \in \{1, 2, \ldots, n\}^2 .$$

This ω is actually a Lie algebra homomorphism (Proposition 4.2). By the universal property of the universal enveloping algebra $U(\mathfrak{gl}_n)$, it thus gives rise

[1] When writing $x_{i,j}$ here, we mean the linear map $\mathcal{A} \to \mathcal{A}$, $f \mapsto x_{i,j} \cdot f$.

to a $\mathbf{k}$-algebra homomorphism $\Omega : U(\mathfrak{gl}_n) \to \mathcal{D}$. It turns out that the image $\Omega(U(\mathfrak{gl}_n))$ of this homomorphism is a $\mathbf{k}$-subalgebra of the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$ (Theorem 4.6 **(a)**). Moreover, there exists a commutative multiplication $\boxdot$ on the $\mathbf{k}$-module $U(\mathfrak{gl}_n)$, independent on m, such that Ω is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to (\mathcal{D}, \boxdot)$ (Theorem 4.6 **(b)**). This multiplication $\boxdot$ is obtained by an application of Corollary 3.11 (which explains why we are using the notation $\boxdot$ for two seemingly unrelated multiplications). This answers my MathOverflow question [MO102874] in the affirmative.

In the final Section 5, we shall address an obvious question of symmetry. Namely, the Guin-Oudom isomorphism can be defined for a left pre-Lie algebra, or (similarly) for a right pre-Lie algebra. An (associative) $\mathbf{k}$-algebra can be viewed as a left and a right pre-Lie algebra at the same time; thus it has two Guin-Oudom isomorphisms. How are these two isomorphisms related? We shall show (Theorem 5.22) that they are identical.

0.1. Acknowledgments

This note owes a significant part of its inspiration to Alexander Chervov's comment at [MO102874] and Frédéric Chapoton's answer at [MO102281], the former of which suggested a crucial generalization that made my question [MO102874] a lot more tractable, while the latter referred me (on a related question) to the concept of pre-Lie algebras and Manchon's beautiful exposition [Manchon11] thereof.

1. A simple theorem on Lie algebra actions

We begin with a rather general setting. While I am not aware of any applications of this setting other than the properties of pre-Lie algebras to which it is applied below, I prefer to start with the general and then move on to the particular case, not least because the general case is more "classical" (for example, it does not involve pre-Lie algebras) and has less complexity.

1.1. Notations

Let us first fix some notations:

> **Convention 1.1.** In the following, all rings are associative and with unity.
>
> Fix a commutative ring $\mathbf{k}$ (once and for all). In the following, all $\mathbf{k}$-algebras are associative, unital and central. ("Central" means that $\lambda a = (\lambda \cdot 1_A) a = a (\lambda \cdot 1_A)$ for every $\lambda \in \mathbf{k}$ and every a in the algebra.) All Lie algebras are defined over $\mathbf{k}$. All tensor product signs, all "Hom" signs, and all "End" signs are understood to be defined over $\mathbf{k}$ unless stated otherwise. All "Hom" signs

and all "End" signs refer to homomorphisms (respectively, endomorphisms) of $\mathbf{k}$-modules.

If U and V are two $\mathbf{k}$-submodules of a $\mathbf{k}$-algebra A, then UV denotes the $\mathbf{k}$-submodule of A spanned by $\{uv \mid (u,v) \in U \times V\}$.

If u and v are two elements of a Lie algebra $\mathfrak{g}$, then $[u,v]$ denotes the Lie bracket of $\mathfrak{g}$ evaluated at (u,v).

If A is a $\mathbf{k}$-algebra, then "A-module" means "left A-module" unless explicitly stated otherwise.

The notation $\mathbb{N}$ stands for $\{0,1,2,\ldots\}$ (not $\{1,2,3,\ldots\}$).

If X, Y and Z are three sets, and $f : X \to Y$ and $g : Y \to Z$ are two maps, then $g \circ f$ denotes the map $X \to Z$ which sends every $x \in X$ to $g(f(x))$.

Next, we recall the definition of a $\mathfrak{g}$-module (where $\mathfrak{g}$ is a Lie algebra):

Definition 1.2. Let $\mathfrak{g}$ be a Lie algebra. Let V be a $\mathbf{k}$-module. Let $\mu : \mathfrak{g} \times V \to V$ be a $\mathbf{k}$-bilinear map. We say that (V,μ) is a $\mathfrak{g}$-*module* if and only if

$$(\mu([a,b],v) = \mu(a,\mu(b,v)) - \mu(b,\mu(a,v)) \text{ for every } a \in \mathfrak{g},\ b \in \mathfrak{g} \text{ and } v \in V). \tag{1}$$

If (V,μ) is a $\mathfrak{g}$-module, then the $\mathbf{k}$-bilinear map $\mu : \mathfrak{g} \times V \to V$ is called the *Lie action* of the $\mathfrak{g}$-module V. If (V,μ) is a $\mathfrak{g}$-module, then the $\mathbf{k}$-module V is called the *underlying* $\mathbf{k}$*-module* of (V,μ).

The following conventions will simplify our life somewhat:

Convention 1.3. Let $\mathfrak{g}$ be a Lie algebra. Let (V,μ) be a $\mathfrak{g}$-module.

(a) For any $a \in \mathfrak{g}$ and $v \in V$, we shall abbreviate the term $\mu(a,v)$ by $a \rightharpoonup v$, provided that the map μ is obvious from the context. Using this notation, the relation (1) rewrites as

$$([a,b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v) \text{ for every } a \in \mathfrak{g},\ b \in \mathfrak{g} \text{ and } v \in V). \tag{2}$$

Convention on the precedence of the $\rightharpoonup$ sign: The symbol $\rightharpoonup$ is understood to have the same precedence as the multiplication sign (i.e., it binds as strongly as the multiplication sign). Thus, $a \rightharpoonup v + w$ means $(a \rightharpoonup v) + w$ rather than $a \rightharpoonup (v+w)$, but $a \rightharpoonup v \cdot w$ is undefined (it could mean both $(a \rightharpoonup v) \cdot w$ and $a \rightharpoonup (v \cdot w)$). Application of functions will be supposed to bind more strongly than the $\rightharpoonup$ sign, so that $f(v) \rightharpoonup g(w)$ will mean $(f(v)) \rightharpoonup (g(w))$ (rather than $f(v \rightharpoonup g(w))$ or $(f(v \rightharpoonup g))(w)$ or anything else), but we will often use parentheses in this case to make the correct interpretation of the formula even more obvious.

(b) We shall often refer to the $\mathfrak{g}$-module (V,μ) as "the $\mathbf{k}$-module V, endowed with the $\mathfrak{g}$-action μ" (or by similar formulations). We shall regard (V,μ) as the $\mathbf{k}$-module V equipped with the additional data of the map μ. Thus, when we speak of "elements of (V,μ)" or "maps to (V,μ)" or "$\mathbf{k}$-submodules of

(V,μ)", we shall mean (respectively) "elements of V", "maps to V", or "**k**-submodules of V".

(c) By abuse of notation, we shall write "V is a $\mathfrak{g}$-module" instead of "(V,μ) is a $\mathfrak{g}$-module" when the map μ is clear from the context or has not been introduced yet. (For instance, when we say "Let V be a $\mathfrak{g}$-module", we really mean to say "Let (V,μ) be a $\mathfrak{g}$-module", where the "μ" is some unused symbol. We will mostly be able to avoid referring to this μ, because our notation $a \rightharpoonup v$ for $\mu(a,v)$ makes it possible to talk about values of the Lie action μ without ever mentioning μ.)

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra. Let V be a **k**-module. A $\mathfrak{g}$-*module structure on* V means a map $\mu : \mathfrak{g} \times V \to V$ such that (V,μ) is a $\mathfrak{g}$-module. In other words, a $\mathfrak{g}$-module structure on V means a **k**-bilinear map $\mu : \mathfrak{g} \times V \to V$ such that we have

$$([a,b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v) \text{ for every } a \in \mathfrak{g},\ b \in \mathfrak{g} \text{ and } v \in V),$$

where we denote $\mu(a,m)$ by $a \rightharpoonup m$ for every $a \in \mathfrak{g}$ and $m \in V$. Thus, a $\mathfrak{g}$-module is the same as a **k**-module endowed with a $\mathfrak{g}$-module structure.

Definition 1.5. Let $\mathfrak{g}$ be a Lie algebra. Let V be a $\mathfrak{g}$-module.

(a) A $\mathfrak{g}$-*submodule* of V means a $\mathfrak{g}$-module V' such that the **k**-module V' is a submodule of the **k**-module V, and such that the Lie action of V' is the restriction of the Lie action of V to $\mathfrak{g} \times V'$.

(b) Let W be a further $\mathfrak{g}$-module. A $\mathfrak{g}$-*module homomorphism from* V *to* W means a **k**-linear map $f : V \to W$ satisfying

$$(a \rightharpoonup f(v) = f(a \rightharpoonup v) \qquad \text{for every } a \in \mathfrak{g} \text{ and } v \in V).$$

A $\mathfrak{g}$-*module isomorphism from* V *to* W means an invertible $\mathfrak{g}$-module homomorphism from V to W whose inverse is also a $\mathfrak{g}$-module homomorphism. It is easy to show that any invertible $\mathfrak{g}$-module homomorphism is a $\mathfrak{g}$-module isomorphism.

Thus, we can define a category of $\mathfrak{g}$-modules: Its objects are $\mathfrak{g}$-modules, and its morphisms are $\mathfrak{g}$-module homomorphisms.

Definition 1.6. If A is a **k**-algebra, then the **k**-module A can be endowed with a Lie bracket defined by

$$([a,b] = ab - ba \text{ for every } a \in A \text{ and } b \in A).$$

The **k**-module A thus becomes a Lie algebra. This Lie algebra will be denoted by A^-.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra. Then, $U(\mathfrak{g})$ will denote the universal enveloping algebra of $\mathfrak{g}$. (See Definition 2.22 for the definition of $U(\mathfrak{g})$.) We denote by $\iota_{U,\mathfrak{g}}$ the canonical map $\mathfrak{g} \to U(\mathfrak{g})$. (This map $\iota_{U,\mathfrak{g}}$ is also defined in Definition 2.22 further below.) This map $\iota_{U,\mathfrak{g}}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $(U(\mathfrak{g}))^{-}$. By abuse of notation, some authors write a for the image of an element $a \in \mathfrak{g}$ under the map $\iota_{U,\mathfrak{g}}$; we will not do so, since the map $\iota_{U,\mathfrak{g}}$ is not always injective (although the Poincaré-Birkhoff-Witt theorem shows that it is if $\mathfrak{g}$ is a free $\mathbf{k}$-module or if $\mathbf{k}$ is a $\mathbb{Q}$-algebra).

The classical universal property of the universal enveloping algebra states the following:

Theorem 1.8. Let $\mathfrak{g}$ be a Lie algebra. Let A be a $\mathbf{k}$-algebra. Let $f : \mathfrak{g} \to A^{-}$ be a Lie algebra homomorphism. Then, there exists a unique $\mathbf{k}$-algebra homomorphism $F : U(\mathfrak{g}) \to A$ such that $f = F \circ \iota_{U,\mathfrak{g}}$.

Using this universal property, we can construct a 1-to-1 correspondence between $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules:

Definition 1.9. Let $\mathfrak{g}$ be a Lie algebra.

(a) Every $U(\mathfrak{g})$-module M canonically becomes a $\mathfrak{g}$-module by setting

$$\left(a \rightharpoonup m = \iota_{U,\mathfrak{g}}(a)\, m \qquad \text{for all } a \in \mathfrak{g} \text{ and } m \in M\right).$$

Moreover, any $U(\mathfrak{g})$-module homomorphism between two $U(\mathfrak{g})$-modules becomes a $\mathfrak{g}$-module homomorphism if we regard these $U(\mathfrak{g})$-modules as $\mathfrak{g}$-modules. Thus, we obtain a functor from the category of $U(\mathfrak{g})$-modules to the category of $\mathfrak{g}$-modules.

(b) Every $\mathfrak{g}$-module M canonically becomes a $U(\mathfrak{g})$-module. To define the $U(\mathfrak{g})$-module structure on M, we proceed as follows: Define a map $\varphi : \mathfrak{g} \to \operatorname{End} M$ by

$$\left((\varphi(a))(m) = a \rightharpoonup m \qquad \text{for all } a \in \mathfrak{g} \text{ and } m \in M\right).$$

It is easy to see that this map φ is a Lie algebra homomorphism from $\mathfrak{g}$ to $(\operatorname{End} M)^{-}$. (Indeed, this is a restatement of the axioms of a $\mathfrak{g}$-module; the fact that $\varphi([a,b]) = [\varphi(a), \varphi(b)]$ for all $a, b \in \mathfrak{g}$ is equivalent to the relation (2).) Now, Theorem 1.8 (applied to $A = \operatorname{End} M$ and $f = \varphi$) shows that there exists a unique $\mathbf{k}$-algebra homomorphism $F : U(\mathfrak{g}) \to \operatorname{End} M$ such that $\varphi = F \circ \iota_{U,\mathfrak{g}}$. Consider this F. Now, we define a $U(\mathfrak{g})$-module structure on M by

$$\left(pm = (F(p))(m) \qquad \text{for all } p \in U(\mathfrak{g}) \text{ and } m \in M\right).$$

Thus, every $\mathfrak{g}$-module canonically becomes a $U(\mathfrak{g})$-module. Moreover, any $\mathfrak{g}$-module homomorphism between two $\mathfrak{g}$-modules becomes a $U(\mathfrak{g})$-module homomorphism if we regard these $\mathfrak{g}$-modules as $U(\mathfrak{g})$-modules. Hence, we obtain a functor from the category of $\mathfrak{g}$-modules to the category of $U(\mathfrak{g})$-modules.

(c) In Definition 1.9 **(a)**, we have constructed a functor from the category of $U(\mathfrak{g})$-modules to the category of $\mathfrak{g}$-modules. In Definition 1.9 **(b)**, we have constructed a functor from the category of $\mathfrak{g}$-modules to the category of $U(\mathfrak{g})$-modules. These two functors are mutually inverse. In particular, if M is a $\mathfrak{g}$-module, then the $U(\mathfrak{g})$-module structure on M obtained according to Definition 1.9 **(b)** satisfies

$$\iota_{U,\mathfrak{g}}(a)\, m = a \rightharpoonup m \qquad \text{for every } a \in \mathfrak{g} \text{ and } m \in M.$$

(d) According to Definition 1.9 **(a)**, every $U(\mathfrak{g})$-module canonically becomes a $\mathfrak{g}$-module. In particular, $U(\mathfrak{g})$ itself becomes a $\mathfrak{g}$-module (because $U(\mathfrak{g})$ is a left $U(\mathfrak{g})$-module). This is the $\mathfrak{g}$-module structure on $U(\mathfrak{g})$ "given by left multiplication" (because it satisfies $x \rightharpoonup u = \iota_{U,\mathfrak{g}}(x)\, u$ for every $x \in \mathfrak{g}$ and $u \in U(\mathfrak{g})$). Other canonical $\mathfrak{g}$-module structures on $U(\mathfrak{g})$ exist as well, but we shall not use them for the time being.

Next, let us define the notion of a derivation:

Definition 1.10. Let C be a $\mathbf{k}$-algebra. A $\mathbf{k}$-linear map $d : C \to C$ is said to be a *derivation* of C if and only if it satisfies

$$(d(ab) = ad(b) + d(a)\, b \text{ for every } a \in C \text{ and } b \in C). \tag{3}$$

We let $\operatorname{Der} C$ denote the set of all derivations of C.

We state a few simple properties of derivations:

Proposition 1.11. Let C be a $\mathbf{k}$-algebra.

(a) The set $\operatorname{Der} C$ is a Lie subalgebra of $(\operatorname{End} C)^{-}$.

(b) Let $f \in \operatorname{Der} C$. Let $n \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_n \in C$. Then,

$$f(a_1 a_2 \cdots a_n) = \sum_{i=1}^{n} a_1 a_2 \cdots a_{i-1} f(a_i)\, a_{i+1} a_{i+2} \cdots a_n.$$

(c) Let $f \in \operatorname{Der} C$. Then, $f(1) = 0$.

Proposition 1.11 **(a)** is proven in [Grinbe15, Remark 1.26], and Proposition 1.11 **(b)** is a particular case of [Grinbe15, Theorem 1.14] (for $A = C$ and $M = C$). Proposition 1.11 **(c)** is proven in [Grinbe15, Theorem 1.12].

We shall also use the following three facts:

Lemma 1.12. Let A be a $\mathbf{k}$-algebra. Let $f : A \to A$ be a derivation. Let B be a $\mathbf{k}$-algebra.

(a) The map $f \otimes \operatorname{id}_B : A \otimes B \to A \otimes B$ is a derivation.

(b) The map $\operatorname{id}_B \otimes f : B \otimes A \to B \otimes A$ is a derivation.

Lemma 1.13. Let A be a $\mathbf{k}$-algebra. Let $d : A \to A$ and $e : A \to A$ be two derivations. Let S be a subset of A which generates A as a $\mathbf{k}$-algebra. Assume that $d\mid_S = e\mid_S$. Then, $d = e$.

Lemma 1.14. Let A and B be two $\mathbf{k}$-algebras. Let $f : A \to B$ be a $\mathbf{k}$-algebra homomorphism. Let $d : A \to A$ and $e : B \to B$ be two derivations. Let S be a subset of A which generates A as a $\mathbf{k}$-algebra. Assume that $(f \circ d)\mid_S = (e \circ f)\mid_S$. Then, $f \circ d = e \circ f$.

These three lemmas are elementary and the reader should have no trouble proving them.[2]

1.2. A Lie algebra action

Theorem 1.15. Let $\mathfrak{g}$ be a Lie algebra. Let C be a $\mathbf{k}$-algebra. Let $K : \mathfrak{g} \to \operatorname{Der} C$ be a Lie algebra homomorphism. Let $f : \mathfrak{g} \to C$ be a $\mathbf{k}$-linear map. Assume that

$$f([a,b]) = [f(a), f(b)] + (K(a))(f(b)) - (K(b))(f(a)) \tag{4}$$

for every $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$ (where the Lie bracket $[f(a), f(b)]$ is computed in the Lie algebra C^-).

(a) Then, we can define a $\mathfrak{g}$-module structure on C by setting

$$(a \rightharpoonup u = f(a) \cdot u + (K(a))(u) \text{ for all } a \in \mathfrak{g} \text{ and } u \in C). \tag{5}$$

In the following, we will regard C as a $\mathfrak{g}$-module by means of this $\mathfrak{g}$-module structure.

(b) Being a $\mathfrak{g}$-module, C becomes a $U(\mathfrak{g})$-module. Define a map $\eta : U(\mathfrak{g}) \to C$ by

$$\eta(u) = u1_C \qquad \text{for every } u \in U(\mathfrak{g}).$$

Then, η is a $\mathfrak{g}$-module homomorphism.

(c) For every $a \in \mathfrak{g}$, $b \in C$ and $c \in C$, we have

$$a \rightharpoonup (bc) - b \cdot (a \rightharpoonup c) = (K(a))(b) \cdot c + [f(a), b]\, c.$$

(Here, again, the Lie bracket $[f(a), b]$ is computed in the Lie algebra C^-.)

Proof of Theorem 1.15. **(a)** We define a $\mathbf{k}$-bilinear map $\mu : \mathfrak{g} \times C \to C$ by

$$(\mu(a,u) = f(a) \cdot u + (K(a))(u) \text{ for all } a \in \mathfrak{g} \text{ and } u \in C).$$

We write $a \rightharpoonup u$ for $\mu(a,u)$ whenever $a \in \mathfrak{g}$ and $u \in C$. Thus, (5) holds.

[2]Complete proofs can be found in [Grinbe15]. More precisely: Lemma 1.12 is a particular case of [Grinbe15, Proposition 1.44] (for $M = A$). Lemma 1.13 is a particular case of [Grinbe15, Proposition 1.29] (for $M = A$). Lemma 1.14 is [Grinbe15, Corollary 1.45].

We now have to show that μ is a $\mathfrak{g}$-module structure on C. In other words, we need to show that

$$[a,b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v) \quad \text{for every } a \in \mathfrak{g},\ b \in \mathfrak{g} \text{ and } v \in C. \tag{6}$$

So let us fix $a \in \mathfrak{g}$, $b \in \mathfrak{g}$ and $v \in C$. Recall that $K : \mathfrak{g} \to \operatorname{Der} C$ is a Lie algebra homomorphism; thus,

$$K([a,b]) = [K(a), K(b)] = K(a) \circ K(b) - K(b) \circ K(a).$$

Now, (5) (applied to $[a,b]$ and v instead of a and u) shows that

$$\begin{aligned}
[a,b] \rightharpoonup v &= \underbrace{f([a,b])}_{\substack{=[f(a),f(b)]+(K(a))(f(b))-(K(b))(f(a)) \\ \text{(by (4))}}} \cdot v + \underbrace{(K([a,b]))}_{=K(a)\circ K(b)-K(b)\circ K(a)} (v) \\
&= \underbrace{([f(a), f(b)] + (K(a))(f(b)) - (K(b))(f(a))) \cdot v}_{=[f(a),f(b)]\cdot v+(K(a))(f(b))\cdot v-(K(b))(f(a))\cdot v} \\
&\qquad + \underbrace{(K(a) \circ K(b) - K(b) \circ K(a))(v)}_{=(K(a)\circ K(b))(v)-(K(b)\circ K(a))(v)} \\
&= [f(a), f(b)] \cdot v + (K(a))(f(b)) \cdot v - (K(b))(f(a)) \cdot v \\
&\qquad + (K(a) \circ K(b))(v) - (K(b) \circ K(a))(v).
\end{aligned} \tag{7}$$

On the other hand, (5) (applied to b and v instead of a and u) yields $b \rightharpoonup v = f(b) \cdot v + (K(b))(v)$. But $K\left(\underbrace{a}_{\in \mathfrak{g}}\right) \in K(\mathfrak{g}) \subseteq \operatorname{Der} C$; in other words, $K(a) : C \to C$ is a derivation. Now, (5) (applied to $b \rightharpoonup v$ instead of u) yields

$$\begin{aligned}
&a \rightharpoonup (b \rightharpoonup v) \\
&= f(a) \cdot \underbrace{(b \rightharpoonup v)}_{=f(b)\cdot v+(K(b))(v)} + K(a)\left(\underbrace{b \rightharpoonup v}_{=(f(b)\cdot v+(K(b))(v))}\right) \\
&= \underbrace{f(a) \cdot ((f(b) \cdot v + (K(b))(v)))}_{=f(a)\cdot f(b)\cdot v+f(a)\cdot(K(b))(v)} + \underbrace{K(a)(f(b) \cdot v + (K(b))(v))}_{=K(a)(f(b)\cdot v)+(K(a))((K(b))(v))} \\
&= f(a) \cdot f(b) \cdot v + f(a) \cdot (K(b))(v) \\
&\qquad + \underbrace{(K(a))(f(b) \cdot v)}_{\substack{=f(b)\cdot(K(a))(v)+(K(a))(f(b))\cdot v \\ \text{(since } K(a) \text{ is a derivation)}}} + \underbrace{(K(a))((K(b))(v))}_{=(K(a)\circ K(b))(v)} \\
&= f(a) \cdot f(b) \cdot v + f(a) \cdot (K(b))(v) \\
&\qquad + f(b) \cdot (K(a))(v) + (K(a))(f(b)) \cdot v + (K(a) \circ K(b))(v).
\end{aligned} \tag{8}$$

The same argument (applied to b and a instead of a and b) shows that

$$\begin{aligned}
&b \rightharpoonup (a \rightharpoonup v) \\
&= f(b) \cdot f(a) \cdot v + f(b) \cdot (K(a))(v) \\
&\qquad + f(a) \cdot (K(b))(v) + (K(b))(f(a)) \cdot v + (K(b) \circ K(a))(v).
\end{aligned}$$

Subtracting this equality from (8), we obtain

$$\begin{aligned}
&a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v) \\
&= (f(a) \cdot f(b) \cdot v + f(a) \cdot (K(b))(v) \\
&\qquad\qquad + f(b) \cdot (K(a))(v) + (K(a))(f(b)) \cdot v + (K(a) \circ K(b))(v)) \\
&\qquad - (f(b) \cdot f(a) \cdot v + f(b) \cdot (K(a))(v) \\
&\qquad\qquad + f(a) \cdot (K(b))(v) + (K(b))(f(a)) \cdot v + (K(b) \circ K(a))(v)) \\
&= \underbrace{f(a) \cdot f(b) \cdot v - f(b) \cdot f(a) \cdot v}_{\substack{=(f(a)\cdot f(b) - f(b)\cdot f(a))(v) \\ =[f(a),f(b)]\cdot v \\ \text{(since } f(a)\cdot f(b)-f(b)\cdot f(a)=[f(a),f(b)])}} + (K(a))(f(b)) \cdot v - (K(b))(f(a)) \cdot v \\
&\qquad + (K(a) \circ K(b))(v) - (K(b) \circ K(a))(v) \\
&= [f(a), f(b)] \cdot v + (K(a))(f(b)) \cdot v - (K(b))(f(a)) \cdot v \\
&\qquad + (K(a) \circ K(b))(v) - (K(b) \circ K(a))(v).
\end{aligned}$$

Comparing this with (7), we obtain $[a,b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v)$. Thus, (6) is proven. Hence, μ is a $\mathfrak{g}$-module structure on C. Theorem 1.15 **(a)** is proven.

(b) For every $a \in \mathfrak{g}$ and $u \in U(\mathfrak{g})$, we have

$$\begin{aligned}
&\eta\left(\underbrace{a \rightharpoonup u}_{\substack{=au \\ \text{(by the definition of the} \\ \mathfrak{g}\text{-module structure on } U(\mathfrak{g}))}}\right) \\
&= \eta(au) = au1_C \qquad \text{(by the definition of } \eta) \\
&= \iota_{U,\mathfrak{g}}(a)\, u1_C \qquad \text{(since we use } a \text{ as a shorthand for } \iota_{U,\mathfrak{g}}(a)) \\
&= a \rightharpoonup \underbrace{(u1_C)}_{\substack{=\eta(u) \\ \text{(since } \eta(u)=u1_C)}} \qquad \text{(since } a \rightharpoonup (u1_C) = \iota_{U,\mathfrak{g}}(a)\, u1_C) \\
&= a \rightharpoonup \eta(u).
\end{aligned}$$

In other words, η is a $\mathfrak{g}$-module homomorphism. This proves Theorem 1.15 **(b)**.

(c) Let $a \in \mathfrak{g}$, $b \in C$ and $c \in C$. The map K has target $\operatorname{Der} C$. Hence, $K(a) \in \operatorname{Der} C$. In other words, $K(a) : C \to C$ is a derivation.

The definition of the Lie bracket of C^- shows that $[f(a), b] = f(a) \cdot b - b \cdot f(a)$.

Now, the definition of the $\mathfrak{g}$-module structure on C shows that

$$\begin{aligned} & a \rightharpoonup (bc) \\ & = f(a) \cdot bc + \underbrace{(K(a))(bc)}_{\substack{=b\cdot(K(a))(c)+(K(a))(b)\cdot c \\ \text{(since } K(a) \text{ is a derivation)}}} \\ & = f(a) \cdot bc + b \cdot (K(a))(c) + (K(a))(b) \cdot c. \end{aligned}$$

On the other hand, the definition of the $\mathfrak{g}$-module structure on C shows that $a \rightharpoonup c = f(a) \cdot c + (K(a))(c)$. Hence,

$$b \cdot \underbrace{(a \rightharpoonup c)}_{=f(a)\cdot c+(K(a))(c)} = b \cdot (f(a) \cdot c + (K(a))(c)) = b \cdot f(a) \cdot c + b \cdot (K(a))(c).$$

Thus,

$$\begin{aligned} & \underbrace{a \rightharpoonup (bc)}_{=f(a)\cdot bc+b\cdot(K(a))(c)+(K(a))(b)\cdot c} - \underbrace{b \cdot (a \rightharpoonup c)}_{=b\cdot f(a)\cdot c+b\cdot(K(a))(c)} \\ & = (f(a) \cdot bc + b \cdot (K(a))(c) + (K(a))(b) \cdot c) - (b \cdot f(a) \cdot c + b \cdot (K(a))(c)) \\ & = f(a) \cdot bc + (K(a))(b) \cdot c - b \cdot f(a) \cdot c \\ & = (K(a))(b) \cdot c + \underbrace{f(a) \cdot bc - b \cdot f(a) \cdot c}_{=(f(a)\cdot b-b\cdot f(a))\cdot c} \\ & = (K(a))(b) \cdot c + \underbrace{(f(a) \cdot b - b \cdot f(a))}_{\substack{=[f(a),b] \\ \text{(since } [f(a),b]=f(a)\cdot b-b\cdot f(a))}} \cdot c \\ & = (K(a))(b) \cdot c + [f(a), b]\, c. \end{aligned}$$

This proves Theorem 1.15 **(c)**. □

1.3. The bialgebra case

Convention 1.16. We shall use the notions of $\mathbf{k}$-coalgebras and $\mathbf{k}$-bialgebras. See, for example, [GriRei18, §1] for an introduction to these notions. We always assume $\mathbf{k}$-coalgebras to be counital and coassociative. We will use the notations Δ and ϵ for the comultiplication and the counit of a $\mathbf{k}$-coalgebra.

If $\mathfrak{g}$ is a Lie algebra, then the universal enveloping algebra $U(\mathfrak{g})$ comes equipped with a canonical Hopf algebra structure (obtained by projecting the Hopf algebra structure on the tensor algebra of $\mathfrak{g}$).

Definition 1.17. Let C be a $\mathbf{k}$-coalgebra. Then, a *coderivation* of C means a $\mathbf{k}$-linear map $f : C \to C$ such that $\Delta \circ f = (f \otimes \mathrm{id} + \mathrm{id} \otimes f) \circ \Delta$. We let $\operatorname{Coder} C$ denote the set of all coderivations of a $\mathbf{k}$-coalgebra C. It is known that $\operatorname{Coder} C$ is a Lie subalgebra of $(\operatorname{End} C)^{-}$; however, we shall only be interested in $\operatorname{Coder} C$ as a set.

Proposition 1.18. Let C be a $\mathbf{k}$-coalgebra. Let f be a coderivation of C. Then, $\epsilon \circ f = 0$.

Proposition 1.18, of course, is the "dual" of Proposition 1.11 **(c)** (that is, of the well-known fact that any derivation of a $\mathbf{k}$-algebra sends 1 to 0).

First proof of Proposition 1.18. Let $\kappa_1 : C \to C \otimes \mathbf{k}$ and $\kappa_2 : C \to \mathbf{k} \otimes C$ be the canonical $\mathbf{k}$-module isomorphisms. Let $\kappa : \mathbf{k} \to \mathbf{k} \otimes \mathbf{k}$ be the canonical $\mathbf{k}$-module isomorphism. Then, two of the axioms of a $\mathbf{k}$-coalgebra (applied to the $\mathbf{k}$-coalgebra C) yield $(\mathrm{id} \otimes \epsilon) \circ \Delta = \kappa_1$ and $(\epsilon \otimes \mathrm{id}) \circ \Delta = \kappa_2$. Now,

$$\underbrace{(\epsilon \otimes \epsilon)}_{=(\epsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \epsilon)} \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \underbrace{(\mathrm{id} \otimes \epsilon) \circ \Delta}_{=\kappa_1} = (\epsilon \otimes \mathrm{id}) \circ \kappa_1 = \kappa \circ \epsilon.$$

Therefore,

$$\begin{aligned}
\underbrace{\kappa \circ \epsilon}_{=(\epsilon \otimes \epsilon) \circ \Delta} \circ f &= (\epsilon \otimes \epsilon) \circ \underbrace{\Delta \circ f}_{\substack{=(f \otimes \mathrm{id} + \mathrm{id} \otimes f) \circ \Delta \\ \text{(since } f \text{ is a coderivation)}}} = \underbrace{(\epsilon \otimes \epsilon) \circ (f \otimes \mathrm{id} + \mathrm{id} \otimes f)}_{=(\epsilon \circ f) \otimes (\epsilon \circ \mathrm{id}) + (\epsilon \circ \mathrm{id}) \otimes (\epsilon \circ f)} \circ \Delta \\
&= \left((\epsilon \circ f) \otimes \underbrace{(\epsilon \circ \mathrm{id})}_{=\epsilon} + \underbrace{(\epsilon \circ \mathrm{id})}_{=\epsilon} \otimes (\epsilon \circ f) \right) \circ \Delta = ((\epsilon \circ f) \otimes \epsilon + \epsilon \otimes (\epsilon \circ f)) \circ \Delta \\
&= \underbrace{((\epsilon \circ f) \otimes \epsilon)}_{=((\epsilon \circ f) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \epsilon)} \circ \Delta + \underbrace{(\epsilon \otimes (\epsilon \circ f))}_{=(\mathrm{id} \otimes (\epsilon \circ f)) \circ (\epsilon \otimes \mathrm{id})} \circ \Delta \\
&= ((\epsilon \circ f) \otimes \mathrm{id}) \circ \underbrace{(\mathrm{id} \otimes \epsilon) \circ \Delta}_{=\kappa_1} + (\mathrm{id} \otimes (\epsilon \circ f)) \circ \underbrace{(\epsilon \otimes \mathrm{id}) \circ \Delta}_{=\kappa_2} \\
&= \underbrace{((\epsilon \circ f) \otimes \mathrm{id}) \circ \kappa_1}_{=\kappa \circ (\epsilon \circ f)} + \underbrace{(\mathrm{id} \otimes (\epsilon \circ f)) \circ \kappa_2}_{=\kappa \circ (\epsilon \circ f)} \\
&= \kappa \circ (\epsilon \circ f) + \kappa \circ (\epsilon \circ f) = 2\kappa \circ (\epsilon \circ f) = 2\kappa \circ \epsilon \circ f.
\end{aligned}$$

Subtracting $\kappa \circ \epsilon \circ f$ from this equality, we obtain $0 = \kappa \circ \epsilon \circ f$. Since κ is an isomorphism, we can cancel κ from this equality. As a result, we obtain $0 = \epsilon \circ f$. This proves Proposition 1.18. □

Second proof of Proposition 1.18. Recall that the dual $\mathbf{k}$-module $C^* = \mathrm{Hom}_{\mathbf{k}} (C, \mathbf{k})$ (this is the $\mathbf{k}$-module of all $\mathbf{k}$-linear maps from C to $\mathbf{k}$) canonically becomes a $\mathbf{k}$-algebra. The counit $\epsilon : C \to \mathbf{k}$ of C is the unity of this $\mathbf{k}$-algebra; in other words, $\epsilon = 1_{C^*}$.

Now, the adjoint map $f^* : C^* \to C^*$ of the coderivation $f : C \to C$ is a derivation of the $\mathbf{k}$-algebra C^* [3]. Therefore, f^* sends 1 to 0 (since a well-known result states that any derivation of a $\mathbf{k}$-algebra sends 1 to 0). In other words, $f^* (1_{C^*}) = 0$. Since $1_{C^*} = \epsilon$, this rewrites as $f^* (\epsilon) = 0$. Since $f^* (\epsilon) = \epsilon \circ f$, this rewrites as $\epsilon \circ f = 0$. Proposition 1.18 thus is proven again. □

[3] *Proof.* Let $a \in C^*$ and $b \in C^*$. We must prove that $f^* (ab) = f^* (a) \cdot b + a \cdot f^* (b)$.

Let $\mu_{\mathbf{k}}$ be the canonical $\mathbf{k}$-algebra homomorphism $\mathbf{k} \otimes \mathbf{k} \to \mathbf{k}$, $\lambda \otimes \mu \mapsto \lambda\mu$. (This is,

Definition 1.19. A *primitive* element of a $\mathbf{k}$-bialgebra H means an element $x \in H$ satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$. We let $\operatorname{Prim} H$ denote the set of all primitive elements of a $\mathbf{k}$-bialgebra H. It is well-known that $\operatorname{Prim} H$ is a Lie subalgebra of H^-.

Theorem 1.20. Let $\mathfrak{g}$ be a Lie algebra. Let C be a $\mathbf{k}$-bialgebra. Let $K : \mathfrak{g} \to \operatorname{Der} C$ be a Lie algebra homomorphism such that $K(\mathfrak{g}) \subseteq \operatorname{Coder} C$. Let $f : \mathfrak{g} \to C$ be a $\mathbf{k}$-linear map such that $f(\mathfrak{g}) \subseteq \operatorname{Prim} C$. Assume that (4) holds for every $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$.

Consider the $\mathfrak{g}$-module structure on C defined in Theorem 1.15 **(a)**, and the map $\eta : U(\mathfrak{g}) \to C$ defined in Theorem 1.15 **(b)**.

(a) For every $a \in \mathfrak{g}$, the map $C \to C$, $c \mapsto a \rightharpoonup c$ is a coderivation of C.

(b) The map $\eta : U(\mathfrak{g}) \to C$ is a $\mathbf{k}$-coalgebra homomorphism.

Proof of Theorem 1.20. **(a)** Let $a \in \mathfrak{g}$. Let ζ be the map $C \to C$, $c \mapsto a \rightharpoonup c$. We must prove that this map ζ is a coderivation of C. In other words, we must prove that $\Delta \circ \zeta = (\zeta \otimes \operatorname{id} + \operatorname{id} \otimes \zeta) \circ \Delta$.

We have $K(a) \in K(\mathfrak{g}) \subseteq \operatorname{Coder} C$. In other words, $K(a)$ is a coderivation of C.

Let $u \in C$. Then, by the definition of ζ, we have $\zeta(u) = a \rightharpoonup u = f(a) \cdot u +$

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of course, an isomorphism.) Then, the definition of the multiplication on C^* shows that $ab = \mu_{\mathbf{k}} \circ (a \otimes b) \circ \Delta$. Now, the definition of f^* shows that

$$f^*(ab) = \underbrace{(ab)}_{=\mu_{\mathbf{k}} \circ (a \otimes b) \circ \Delta} \circ f = \mu_{\mathbf{k}} \circ (a \otimes b) \circ \underbrace{\Delta \circ f}_{\substack{=(f \otimes \operatorname{id} + \operatorname{id} \otimes f) \circ \Delta \\ \text{(since } f \text{ is a coderivation)}}} = \mu_{\mathbf{k}} \circ \underbrace{(a \otimes b) \circ (f \otimes \operatorname{id} + \operatorname{id} \otimes f)}_{=(a \otimes b) \circ (f \otimes \operatorname{id}) + (a \otimes b) \circ (\operatorname{id} \otimes f)} \circ \Delta$$

$$= \mu_{\mathbf{k}} \circ \left(\underbrace{(a \otimes b) \circ (f \otimes \operatorname{id})}_{=(a \circ f) \otimes b} + \underbrace{(a \otimes b) \circ (\operatorname{id} \otimes f)}_{=a \otimes (b \circ f)} \right) \circ \Delta$$

$$= \mu_{\mathbf{k}} \circ ((a \circ f) \otimes b + a \otimes (b \circ f)) \circ \Delta = \mu_{\mathbf{k}} \circ ((a \circ f) \otimes b) \circ \Delta + \mu_{\mathbf{k}} \circ (a \otimes (b \circ f)) \circ \Delta.$$

Compared with

$$\underbrace{f^*(a)}_{=a \circ f} \cdot b + a \cdot \underbrace{f^*(b)}_{=b \circ f} = \underbrace{(a \circ f) \cdot b}_{\substack{=\mu_{\mathbf{k}} \circ ((a \circ f) \otimes b) \circ \Delta \\ \text{(by the definition of the} \\ \text{multiplication on } C^*)}} + \underbrace{a \cdot (b \circ f)}_{\substack{=\mu_{\mathbf{k}} \circ (a \otimes (b \circ f)) \circ \Delta \\ \text{(by the definition of the} \\ \text{multiplication on } C^*)}}$$

$$= \mu_{\mathbf{k}} \circ ((a \circ f) \otimes b) \circ \Delta + \mu_{\mathbf{k}} \circ (a \otimes (b \circ f)) \circ \Delta,$$

this shows that $f^*(ab) = f^*(a) \cdot b + a \cdot f^*(b)$.

Thus, we have proven that $f^*(ab) = f^*(a) \cdot b + a \cdot f^*(b)$ for every $a \in C^*$ and $b \in C^*$. In other words, f^* is a derivation of C^*, qed.

$(K(a))(u)$ (by (5)). Hence,

$$\begin{aligned}
(\Delta \circ \zeta)(u) &= \Delta\left(\underbrace{\zeta(u)}_{=f(a)\cdot u+(K(a))(u)}\right) = \Delta\left(f(a)\cdot u + (K(a))(u)\right) \\
&= \underbrace{\Delta(f(a))}_{\substack{=f(a)\otimes 1+1\otimes f(a) \\ (\text{since } f(a)\in f(\mathfrak{g})\subseteq \operatorname{Prim} C)}} \cdot \Delta(u) + \underbrace{\Delta((K(a))(u))}_{=(\Delta\circ K(a))(u)} \\
&\qquad (\text{since } \Delta \text{ is a } \mathbf{k}\text{-algebra homomorphism}) \\
&= (f(a)\otimes 1 + 1\otimes f(a))\cdot \Delta(u) + \underbrace{(\Delta\circ K(a))}_{\substack{=(K(a)\otimes \operatorname{id} + \operatorname{id}\otimes K(a))\circ\Delta \\ (\text{since } K(a) \text{ is a coderivation of } C)}} (u) \\
&= (f(a)\otimes 1 + 1\otimes f(a))\cdot \Delta(u) + \underbrace{((K(a)\otimes \operatorname{id} + \operatorname{id}\otimes K(a))\circ\Delta)(u)}_{=(K(a)\otimes \operatorname{id} + \operatorname{id}\otimes K(a))(\Delta(u))} \\
&= (f(a)\otimes 1 + 1\otimes f(a))\cdot \Delta(u) + (K(a)\otimes \operatorname{id} + \operatorname{id}\otimes K(a))(\Delta(u)).
\end{aligned} \tag{9}$$

But every $p \in C\otimes C$ satisfies

$$(\zeta\otimes \operatorname{id} + \operatorname{id}\otimes\zeta)(p) = (f(a)\otimes 1 + 1\otimes f(a))\cdot p + (K(a)\otimes \operatorname{id} + \operatorname{id}\otimes K(a))(p) \tag{10}$$

[4]. Applying this to $p = \Delta(u)$, we obtain

$$\begin{aligned}
&(\zeta\otimes \operatorname{id} + \operatorname{id}\otimes\zeta)(\Delta(u)) \\
&= (f(a)\otimes 1 + 1\otimes f(a))\cdot \Delta(u) + (K(a)\otimes \operatorname{id} + \operatorname{id}\otimes K(a))(\Delta(u)).
\end{aligned}$$

Compared with (9), this shows that

$$(\Delta\circ\zeta)(u) = (\zeta\otimes \operatorname{id} + \operatorname{id}\otimes\zeta)(\Delta(u)) = ((\zeta\otimes \operatorname{id} + \operatorname{id}\otimes\zeta)\circ\Delta)(u). \tag{11}$$

[4]*Proof.* Let $p \in C\otimes C$. We need to prove the equality (10). Since this equality is $\mathbf{k}$-linear in p, we can WLOG assume that p is a pure tensor (since the pure tensors span the $\mathbf{k}$-module $C\otimes C$). Assume this. Thus, $p = q\otimes r$ for some $q\in C$ and $r \in C$. Consider these q and r. We have

$$\begin{aligned}
&(\zeta\otimes \operatorname{id} + \operatorname{id}\otimes\zeta)\left(\underbrace{p}_{=q\otimes r}\right) \\
&= (\zeta\otimes \operatorname{id} + \operatorname{id}\otimes\zeta)(q\otimes r) \\
&= \underbrace{\zeta(q)}_{\substack{=a\rightharpoonup q \\ (\text{by the definition of } \zeta)}} \otimes \underbrace{\operatorname{id}(r)}_{=r} + \underbrace{\operatorname{id}(q)}_{=q} \otimes \underbrace{\zeta(r)}_{\substack{=a\rightharpoonup r \\ (\text{by the definition of } \zeta)}} \\
&= \underbrace{(a\rightharpoonup q)}_{\substack{=f(a)\cdot q+(K(a))(q) \\ (\text{by } (5))}} \otimes r + q\otimes \underbrace{(a\rightharpoonup r)}_{\substack{=f(a)\cdot r+(K(a))(r) \\ (\text{by } (5))}} \\
&= (f(a)\cdot q + (K(a))(q))\otimes r + q\otimes (f(a)\cdot r + (K(a))(r)) \\
&= f(a)\cdot q\otimes r + (K(a))(q)\otimes r + q\otimes f(a)\cdot r + q\otimes (K(a))(r).
\end{aligned}$$

Now, let us forget that we fixed u. We thus have proven (11) for every $u \in C$. In other words, $\Delta \circ \zeta = (\zeta \otimes \mathrm{id} + \mathrm{id} \otimes \zeta) \circ \Delta$. In other words, ζ is a coderivation of C. This completes our proof of Theorem 1.20 **(a)**.

(b) We need to prove that η is a $\mathbf{k}$-coalgebra homomorphism. In other words, we need to prove that $\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta$ and $\epsilon \circ \eta = \epsilon$. We shall prove $\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta$ first.

We define a map $\Xi : U(\mathfrak{g}) \to \operatorname{End} C$ by setting

$$((\Xi(p))(c) = pc \qquad \text{for every } p \in U(\mathfrak{g}) \text{ and } c \in C).$$

Thus, Ξ is a $\mathbf{k}$-algebra homomorphism. (More precisely, Ξ is the $\mathbf{k}$-algebra homomorphism $U(\mathfrak{g}) \to \operatorname{End} C$ which describes the left $U(\mathfrak{g})$-module C.) Hence, $\Xi \otimes \Xi : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to \operatorname{End} C \otimes \operatorname{End} C$ is a $\mathbf{k}$-algebra homomorphism as well.

We let $\mathbf{z} : \operatorname{End} C \otimes \operatorname{End} C \to \operatorname{End}(C \otimes C)$ be the $\mathbf{k}$-linear map which sends every $f \otimes g \in \operatorname{End} C \otimes \operatorname{End} C$ to the endomorphism $f \otimes g$ of $C \otimes C$. This $\mathbf{z}$ is a $\mathbf{k}$-algebra homomorphism as well.

We define a $\mathbf{k}$-linear map $\Xi' : U(\mathfrak{g}) \to \operatorname{End}(C \otimes C)$ by $\Xi' = \mathbf{z} \circ (\Xi \otimes \Xi) \circ \Delta$. This Ξ' is a $\mathbf{k}$-algebra homomorphism (since it is a composition of three $\mathbf{k}$-algebra homomorphisms).

Now, let $\mathfrak{P}$ be the subset

$$\left\{ p \in U(\mathfrak{g}) \mid \Delta \circ \Xi(p) = \Xi'(p) \circ \Delta \right\}$$

Compared with

$$\begin{aligned}
&(f(a) \otimes 1 + 1 \otimes f(a)) \cdot \underbrace{p}_{=q \otimes r} + (K(a) \otimes \mathrm{id} + \mathrm{id} \otimes K(a)) \left(\underbrace{p}_{=q \otimes r} \right) \\
&= (f(a) \otimes 1 + 1 \otimes f(a)) \cdot (q \otimes r) + (K(a) \otimes \mathrm{id} + \mathrm{id} \otimes K(a))(q \otimes r) \\
&= \underbrace{(f(a) \otimes 1) \cdot (q \otimes r)}_{=f(a) \cdot q \otimes r} + \underbrace{(1 \otimes f(a)) \cdot (q \otimes r)}_{=q \otimes f(a) \cdot r} + \underbrace{(K(a) \otimes \mathrm{id})(q \otimes r)}_{\substack{=(K(a))(q) \otimes \mathrm{id}(r) \\ =(K(a))(q) \otimes r}} + \underbrace{(\mathrm{id} \otimes K(a))(q \otimes r)}_{\substack{=\mathrm{id}(q) \otimes (K(a))(r) \\ =q \otimes (K(a))(r)}} \\
&= f(a) \cdot q \otimes r + q \otimes f(a) \cdot r + (K(a))(q) \otimes r + q \otimes (K(a))(r) \\
&= f(a) \cdot q \otimes r + (K(a))(q) \otimes r + q \otimes f(a) \cdot r + q \otimes (K(a))(r),
\end{aligned}$$

this yields $(\zeta \otimes \mathrm{id} + \mathrm{id} \otimes \zeta)(p) = (f(a) \otimes 1 + 1 \otimes f(a)) \cdot p + (K(a) \otimes \mathrm{id} + \mathrm{id} \otimes K(a))(p)$. This proves (10).

[5] of $U(\mathfrak{g})$. Then, $\mathfrak{P}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$ [6] and satisfies $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{P}$ [7].

But the subset $\iota_{U,\mathfrak{g}}(\mathfrak{g})$ generates the $\mathbf{k}$-algebra $U(\mathfrak{g})$. In other words, every $\mathbf{k}$-subalgebra $\mathfrak{B}$ of $U(\mathfrak{g})$ satisfying $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{B}$ must satisfy $\mathfrak{B} = U(\mathfrak{g})$. Applying

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[5]Keep in mind that $\Delta \circ \Xi(p)$ means $\Delta \circ (\Xi(p))$ rather than $(\Delta \circ \Xi)(p)$.

[6]*Proof.* The set $\mathfrak{P}$ is the set of all $p \in U(\mathfrak{g})$ satisfying the equation $\Delta \circ \Xi(p) = \Xi'(p) \circ \Delta$. Since this equation is $\mathbf{k}$-linear in p, this shows that $\mathfrak{P}$ is a $\mathbf{k}$-submodule of $U(\mathfrak{g})$.

Since Ξ and Ξ' are $\mathbf{k}$-algebra homomorphisms, we have $\Xi(1) = \mathrm{id}$ and $\Xi'(1) = \mathrm{id}$. Hence, $\Delta \circ \underbrace{\Xi(1)}_{=\mathrm{id}} = \Delta$ and $\underbrace{\Xi'(1)}_{=\mathrm{id}} \circ \Delta = \Delta$, so that $\Delta \circ \Xi(1) = \Delta = \Xi'(1) \circ \Delta$. In other words, $1 \in \mathfrak{P}$ (by the definition of $\mathfrak{P}$).

Now, let $a \in \mathfrak{P}$ and $b \in \mathfrak{P}$. We shall prove that $ab \in \mathfrak{P}$.

Indeed, we have $a \in \mathfrak{P}$. In other words, $\Delta \circ \Xi(a) = \Xi'(a) \circ \Delta$ (by the definition of $\mathfrak{P}$). Similarly, from $b \in \mathfrak{P}$, we obtain $\Delta \circ \Xi(b) = \Xi'(b) \circ \Delta$. Now, $\Xi(ab) = \Xi(a) \circ \Xi(b)$ (since Ξ is a $\mathbf{k}$-algebra homomorphism) and $\Xi'(ab) = \Xi'(a) \circ \Xi'(b)$ (since Ξ' is a $\mathbf{k}$-algebra homomorphism). Now,

$$\Delta \circ \underbrace{\Xi(ab)}_{=\Xi(a)\circ\Xi(b)} = \underbrace{\Delta \circ \Xi(a)}_{=\Xi'(a)\circ\Delta} \circ \Xi(b) = \Xi'(a) \circ \underbrace{\Delta \circ \Xi(b)}_{=\Xi'(b)\circ\Delta} = \underbrace{\Xi'(a) \circ \Xi'(b)}_{=\Xi'(ab)} \circ \Delta = \Xi'(ab) \circ \Delta.$$

In other words, $ab \in \mathfrak{P}$ (by the definition of $\mathfrak{P}$).

Now, let us forget that we fixed a and b. We thus have proven that $ab \in \mathfrak{P}$ for every $a \in \mathfrak{P}$ and $b \in \mathfrak{P}$. Combining this with the facts that $\mathfrak{P}$ is a $\mathbf{k}$-submodule of $U(\mathfrak{g})$ and that we have $1 \in \mathfrak{P}$, we conclude that $\mathfrak{P}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$. Qed.

[7]*Proof.* Let $a \in \mathfrak{g}$. We shall show that $\iota_{U,\mathfrak{g}}(a) \in \mathfrak{P}$.

Indeed, let ζ be the map $C \to C,\ c \mapsto a \rightharpoonup c$. Then, this map ζ is a coderivation of C (by Theorem 1.20 **(a)**). In other words, $\Delta \circ \zeta = (\zeta \otimes \mathrm{id} + \mathrm{id} \otimes \zeta) \circ \Delta$.

Now, every $c \in C$ satisfies

$$(\Xi(\iota_{U,\mathfrak{g}}(a)))(c) = \iota_{U,\mathfrak{g}}(a)\, c = a \rightharpoonup c = \zeta(c)$$

(since $\zeta(c)$ is defined to be $a \rightharpoonup c$). In other words, $\Xi(\iota_{U,\mathfrak{g}}(a)) = \zeta$.

On the other hand, Ξ is a $\mathbf{k}$-algebra homomorphism, and thus $\Xi(1) = \mathrm{id}$.

We have $\Delta(\iota_{U,\mathfrak{g}}(a)) = \iota_{U,\mathfrak{g}}(a) \otimes 1 + 1 \otimes \iota_{U,\mathfrak{g}}(a)$ (by the definition of the coalgebra structure on $U(\mathfrak{g})$) and thus

$$\begin{aligned}
\underbrace{\Xi'}_{=\mathbf{z}\circ(\Xi\otimes\Xi)\circ\Delta} (\iota_{U,\mathfrak{g}}(a)) &= (\mathbf{z} \circ (\Xi \otimes \Xi) \circ \Delta)(\iota_{U,\mathfrak{g}}(a)) \\
&= \mathbf{z}\left((\Xi \otimes \Xi) \underbrace{(\Delta(\iota_{U,\mathfrak{g}}(a)))}_{=\iota_{U,\mathfrak{g}}(a)\otimes 1 + 1 \otimes \iota_{U,\mathfrak{g}}(a)} \right) \\
&= \mathbf{z}\left(\underbrace{(\Xi \otimes \Xi)(\iota_{U,\mathfrak{g}}(a) \otimes 1 + 1 \otimes \iota_{U,\mathfrak{g}}(a))}_{=\Xi(\iota_{U,\mathfrak{g}}(a))\otimes\Xi(1)+\Xi(1)\otimes\Xi(\iota_{U,\mathfrak{g}}(a))} \right) \\
&= \mathbf{z}\left(\underbrace{\Xi(\iota_{U,\mathfrak{g}}(a))}_{=\zeta} \otimes \underbrace{\Xi(1)}_{=\mathrm{id}} + \underbrace{\Xi(1)}_{=\mathrm{id}} \otimes \underbrace{\Xi(\iota_{U,\mathfrak{g}}(a))}_{=\zeta} \right) \\
&= \mathbf{z}(\zeta \otimes \mathrm{id} + \mathrm{id} \otimes \zeta) = \zeta \otimes \mathrm{id} + \mathrm{id} \otimes \zeta
\end{aligned}$$

this to $\mathfrak{B} = \mathfrak{P}$, we thus obtain $\mathfrak{P} = U(\mathfrak{g})$ (since $\mathfrak{P}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$ and satisfies $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{P}$). Thus,

$$U(\mathfrak{g}) = \mathfrak{P} = \left\{ p \in U(\mathfrak{g}) \mid \Delta \circ \Xi(p) = \Xi'(p) \circ \Delta \right\}.$$

In other words, every $p \in U(\mathfrak{g})$ satisfies

$$\Delta \circ \Xi(p) = \Xi'(p) \circ \Delta. \tag{12}$$

But every $q \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ satisfies

$$((\mathbf{z} \circ (\Xi \otimes \Xi))(q))(1_C \otimes 1_C) = (\eta \otimes \eta)(q) \tag{13}$$

[8].

Now, let $p \in U(\mathfrak{g})$. We shall prove that $(\Delta \circ \eta)(p) = ((\eta \otimes \eta) \circ \Delta)(p)$.

The definition of Ξ shows that $(\Xi(p))(1_C) = p1_C = \eta(p)$ (since $\eta(p)$ is

(by the definition of the map $\mathbf{z}$). Now,

$$\Delta \circ \underbrace{\Xi(\iota_{U,\mathfrak{g}}(a))}_{=\zeta} = \Delta \circ \zeta = \underbrace{(\zeta \otimes \mathrm{id} + \mathrm{id} \otimes \zeta)}_{=\Xi'(\iota_{U,\mathfrak{g}}(a))} \circ \Delta = \Xi'(\iota_{U,\mathfrak{g}}(a)) \circ \Delta.$$

In other words, $\iota_{U,\mathfrak{g}}(a) \in \mathfrak{P}$ (by the definition of $\mathfrak{P}$).

Now let us forget that we fixed a. We thus have proven that $\iota_{U,\mathfrak{g}}(a) \in \mathfrak{P}$ for every $a \in \mathfrak{g}$. In other words, $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{P}$, qed.

[8] *Proof of (13):* Let $q \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. We must prove the equality (13). This equality is $\mathbf{k}$-linear in q. Hence, we can WLOG assume that q is a pure tensor (since the pure tensors span the $\mathbf{k}$-module $U(\mathfrak{g}) \otimes U(\mathfrak{g})$). Assume this. In other words, $q = u \otimes v$ for some $u \in U(\mathfrak{g})$ and $v \in U(\mathfrak{g})$. Consider these u and v.

The definition of Ξ shows that $(\Xi(u))(1_C) = u1_C = \eta(u)$ (since $\eta(u)$ is defined to be $u1_C$). Similarly, $(\Xi(v))(1_C) = \eta(v)$. Now,

$$\begin{aligned}(\mathbf{z} \circ (\Xi \otimes \Xi))\left(\underbrace{q}_{=u\otimes v}\right) &= (\mathbf{z} \circ (\Xi \otimes \Xi))(u \otimes v) = \mathbf{z}\left(\underbrace{(\Xi \otimes \Xi)(u \otimes v)}_{=\Xi(u)\otimes\Xi(v)}\right) \\ &= \mathbf{z}(\Xi(u) \otimes \Xi(v)) = \Xi(u) \otimes \Xi(v)\end{aligned}$$

(by the definition of $\mathbf{z}$). Hence,

$$\begin{aligned}\left(\underbrace{(\mathbf{z} \circ (\Xi \otimes \Xi))(q)}_{=\Xi(u)\otimes\Xi(v)}\right)(1_C \otimes 1_C) &= (\Xi(u) \otimes \Xi(v))(1_C \otimes 1_C) = \underbrace{(\Xi(u))(1_C)}_{=\eta(u)} \otimes \underbrace{(\Xi(v))(1_C)}_{=\eta(v)} \\ &= \eta(u) \otimes \eta(v) = (\eta \otimes \eta)\left(\underbrace{u \otimes v}_{=q}\right) = (\eta \otimes \eta)(q).\end{aligned}$$

This proves (13).

defined to be $p1_C$). Thus,

$$\begin{aligned}
(\Delta \circ \eta)(p) &= \Delta \left(\underbrace{\eta(p)}_{=(\Xi(p))(1_C)} \right) = \Delta((\Xi(p))(1_C)) = \underbrace{(\Delta \circ \Xi(p))}_{\substack{=\Xi'(p)\circ\Delta \\ \text{(by (12))}}}(1_C) \\
&= (\Xi'(p) \circ \Delta)(1_C) = \left(\underbrace{\Xi'}_{=\mathbf{z}\circ(\Xi\otimes\Xi)\circ\Delta} (p) \right) \left(\underbrace{\Delta(1_C)}_{=1_C\otimes 1_C} \right) \\
&= \underbrace{((\mathbf{z} \circ (\Xi \otimes \Xi) \circ \Delta)(p))}_{=(\mathbf{z}\circ(\Xi\otimes\Xi))(\Delta(p))} (1_C \otimes 1_C) = ((\mathbf{z} \circ (\Xi \otimes \Xi))(\Delta(p)))(1_C \otimes 1_C) \\
&= (\eta \otimes \eta)(\Delta(p)) \qquad \text{(by (13), applied to } q = \Delta(p)) \\
&= ((\eta \otimes \eta) \circ \Delta)(p).
\end{aligned}$$

Now, let us forget that we fixed p. We thus have proven that $(\Delta \circ \eta)(p) = ((\eta \otimes \eta) \circ \Delta)(p)$ for every $p \in U(\mathfrak{g})$. In other words, $\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta$.

It remains to prove that $\epsilon \circ \eta = \epsilon$.

We let $\mathfrak{R}$ be the subset

$$\{p \in U(\mathfrak{g}) \mid \epsilon \circ \Xi(p) = \epsilon(p) \cdot \epsilon\}$$

of $U(\mathfrak{g})$. Then, $\mathfrak{R}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$ [9] and satisfies $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{R}$ [10].

But the subset $\iota_{U,\mathfrak{g}}(\mathfrak{g})$ generates the $\mathbf{k}$-algebra $U(\mathfrak{g})$. In other words, every $\mathbf{k}$-subalgebra $\mathfrak{B}$ of $U(\mathfrak{g})$ satisfying $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{B}$ must satisfy $\mathfrak{B} = U(\mathfrak{g})$. Applying

[9] *Proof.* The set $\mathfrak{R}$ is the set of all $p \in U(\mathfrak{g})$ satisfying the equation $\epsilon \circ \Xi(p) = \epsilon(p) \cdot \epsilon$. Since this equation is $\mathbf{k}$-linear in p, this shows that $\mathfrak{R}$ is a $\mathbf{k}$-submodule of $U(\mathfrak{g})$.

Since Ξ and ϵ are $\mathbf{k}$-algebra homomorphisms, we have $\Xi(1) = \mathrm{id}$ and $\epsilon(1) = 1$. Hence, $\epsilon \circ \underbrace{\Xi(1)}_{=\mathrm{id}} = \epsilon$ and $\underbrace{\epsilon(1)}_{=1} \cdot \epsilon = \epsilon$, so that $\epsilon \circ \Xi(1) = \epsilon = \epsilon(1) \cdot \epsilon$. In other words, $1 \in \mathfrak{R}$ (by the definition of $\mathfrak{R}$).

Now, let $a \in \mathfrak{R}$ and $b \in \mathfrak{R}$. We shall prove that $ab \in \mathfrak{R}$.

Indeed, we have $a \in \mathfrak{R}$. In other words, $\epsilon \circ \Xi(a) = \epsilon(a) \cdot \epsilon$ (by the definition of $\mathfrak{R}$). Similarly, from $b \in \mathfrak{R}$, we obtain $\epsilon \circ \Xi(b) = \epsilon(b) \cdot \epsilon$. Now, $\Xi(ab) = \Xi(a) \circ \Xi(b)$ (since Ξ is a $\mathbf{k}$-algebra homomorphism) and $\epsilon(ab) = \epsilon(a) \cdot \epsilon(b)$ (since ϵ is a $\mathbf{k}$-algebra homomorphism). Now,

$$\epsilon \circ \underbrace{\Xi(ab)}_{=\Xi(a)\circ\Xi(b)} = \underbrace{\epsilon \circ \Xi(a)}_{=\epsilon(a)\cdot\epsilon} \circ \Xi(b) = \epsilon(a) \cdot \underbrace{\epsilon \circ \Xi(b)}_{=\epsilon(b)\cdot\epsilon} = \underbrace{\epsilon(a) \cdot \epsilon(b)}_{=\epsilon(ab)} \cdot \epsilon = \epsilon(ab) \cdot \epsilon.$$

In other words, $ab \in \mathfrak{R}$ (by the definition of $\mathfrak{R}$).

Now, let us forget that we fixed a and b. We thus have proven that $ab \in \mathfrak{R}$ for every $a \in \mathfrak{R}$ and $b \in \mathfrak{R}$. Combining this with the facts that $\mathfrak{R}$ is a $\mathbf{k}$-submodule of $U(\mathfrak{g})$ and that we have $1 \in \mathfrak{R}$, we conclude that $\mathfrak{R}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$. Qed.

[10] *Proof.* Let $a \in \mathfrak{g}$. We shall show that $\iota_{U,\mathfrak{g}}(a) \in \mathfrak{R}$.

Indeed, let ζ be the map $C \to C$, $c \mapsto a \rightharpoonup c$. Then, this map ζ is a coderivation of C (by Theorem 1.20 **(a)**). Hence, $\epsilon \circ \zeta = 0$ (by Proposition 1.18, applied to $f = \zeta$).

this to $\mathfrak{B} = \mathfrak{R}$, we thus obtain $\mathfrak{R} = U(\mathfrak{g})$ (since $\mathfrak{R}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$ and satisfies $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{R}$). Thus,

$$U(\mathfrak{g}) = \mathfrak{R} = \{p \in U(\mathfrak{g}) \mid \epsilon \circ \Xi(p) = \epsilon(p) \cdot \epsilon\}.$$

In other words, every $p \in U(\mathfrak{g})$ satisfies

$$\epsilon \circ \Xi(p) = \epsilon(p) \cdot \epsilon. \tag{14}$$

Now, let $p \in U(\mathfrak{g})$. We shall prove that $(\epsilon \circ \eta)(p) = \epsilon(p)$.

The definition of Ξ shows that $(\Xi(p))(1_C) = p1_C = \eta(p)$ (since $\eta(p)$ is defined to be $p1_C$). Thus,

$$(\epsilon \circ \eta)(p) = \epsilon\left(\underbrace{\eta(p)}_{=(\Xi(p))(1_C)}\right) = \epsilon((\Xi(p))(1_C)) = \underbrace{(\epsilon \circ \Xi(p))}_{\substack{=\epsilon(p)\cdot\epsilon \\ \text{(by (14))}}}(1_C)$$
$$= (\epsilon(p) \cdot \epsilon)(1_C) = \epsilon(p) \cdot \underbrace{\epsilon(1_C)}_{=1} = \epsilon(p).$$

Now, let us forget that we fixed p. We thus have proven that $(\epsilon \circ \eta)(p) = \epsilon(p)$ for every $p \in U(\mathfrak{g})$. In other words, $\epsilon \circ \eta = \epsilon$. As explained above, this concludes the proof of Theorem 1.20 **(b)**. $\square$

2. Filtrations, gradings and the prelude to PBW

In this section, we shall recall some fundamental definitions and facts pertaining to filtered and graded $\mathbf{k}$-modules and $\mathbf{k}$-algebras, and specifically to the (standard) filtration on the universal enveloping algebra $U(\mathfrak{g})$ and its relation to the symmetric algebra $\operatorname{Sym}\mathfrak{g}$. All of what is said and proved in this section is shallow and well-known (and so the section can probably be skimmed at a high

Now, every $c \in C$ satisfies

$$(\Xi(\iota_{U,\mathfrak{g}}(a)))(c) = \iota_{U,\mathfrak{g}}(a)\, c = a \rightharpoonup c = \zeta(c)$$

(since $\zeta(c)$ is defined to be $a \rightharpoonup c$). In other words, $\Xi(\iota_{U,\mathfrak{g}}(a)) = \zeta$.

We have $\epsilon(\iota_{U,\mathfrak{g}}(a)) = 0$ (by the definition of the coalgebra structure on $U(\mathfrak{g})$) and thus $\underbrace{\epsilon(\iota_{U,\mathfrak{g}}(a))}_{=0} \cdot \epsilon = 0$. Now,

$$\epsilon \circ \underbrace{\Xi(\iota_{U,\mathfrak{g}}(a))}_{=\zeta} = \epsilon \circ \zeta = 0 = \epsilon(\iota_{U,\mathfrak{g}}(a)) \cdot \epsilon.$$

In other words, $\iota_{U,\mathfrak{g}}(a) \in \mathfrak{R}$ (by the definition of $\mathfrak{R}$).

Now let us forget that we fixed a. We thus have proven that $\iota_{U,\mathfrak{g}}(a) \in \mathfrak{R}$ for every $a \in \mathfrak{g}$. In other words, $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{R}$, qed.

speed); we nevertheless prefer to be maximally explicit and detailed (although we do not prove the most fundamental results, whose proofs are straightforward and should be in basic abstract algebra texts) in order to fix the terminology (which, unfortunately, is not standard across different texts and authors).

2.1. Definitions on filtered and graded k-modules

We shall now make some definitions concerned with filtered and graded algebras. These definitions are all classical, but the notations are not standardized across the literature (as many authors use the words "filtered" and "graded" in different meanings).

Definition 2.1. **(a)** Let V be a $\mathbf{k}$-module. A $\mathbf{k}$-*module filtration of* V will mean a sequence $(V_n)_{n\geq 0}$ of $\mathbf{k}$-submodules of V such that $\bigcup_{n\geq 0} V_n = V$ and $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$.

(b) A *filtered* $\mathbf{k}$*-module* means a $\mathbf{k}$-module V equipped with a $\mathbf{k}$-module filtration of V. This $\mathbf{k}$-module filtration will be referred to as the *filtration of* V. By abuse of notation, we will often use the same letter V for this filtered $\mathbf{k}$-module and for its underlying $\mathbf{k}$-module V, when the filtration is clear from the context.

(c) Let V and W be two filtered $\mathbf{k}$-modules. Let $(V_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$ be the filtrations of V and W, respectively. Let $f : V \to W$ be a $\mathbf{k}$-linear map. We say that the map f is *filtered* if and only if every $n \in \mathbb{N}$ satisfies $f(V_n) \subseteq W_n$.

(Instead of saying that "the map f is filtered", some authors say that "the map f respects the filtrations $(V_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$" or that "the map f preserves the filtration".)

The filtered $\mathbf{k}$-modules form a category, whose morphisms are the filtered $\mathbf{k}$-linear maps.[11]

Definition 2.2. **(a)** Let V be a $\mathbf{k}$-module. A $\mathbf{k}$-*module grading* on V will mean a sequence $(V_n)_{n\geq 0}$ of $\mathbf{k}$-submodules of V such that $\bigoplus_{n\geq 0} V_n = V$. (Here, the direct sum $\bigoplus_{n\geq 0} V_n$ is internal.)

(b) A *graded* $\mathbf{k}$*-module* means a $\mathbf{k}$-module V equipped with a $\mathbf{k}$-module grading on V. This $\mathbf{k}$-module grading will be referred to as the *grading of* V. By abuse of notation, we will often use the same letter V for this graded $\mathbf{k}$-module and for its underlying $\mathbf{k}$-module V, when the grading is clear from the context.

[11] Nota bene: An isomorphism in this category is an invertible filtered $\mathbf{k}$-linear map whose inverse is again filtered. The "whose inverse is again filtered" requirement cannot be dropped: Not every invertible filtered $\mathbf{k}$-linear map is an isomorphism in this category!

(c) Let V and W be two graded $\mathbf{k}$-modules. Let $(V_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$ be the gradings of V and W, respectively. Let $f : V \to W$ be a $\mathbf{k}$-linear map. We say that the map f is *graded* if and only if every $n \in \mathbb{N}$ satisfies $f(V_n) \subseteq W_n$.

(Instead of saying that "the map f is graded", some authors say that "the map f respects the gradings $(V_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$" or that "the map f preserves the degree".)

The graded $\mathbf{k}$-modules form a category, whose morphisms are the graded $\mathbf{k}$-linear maps.[12]

Definition 2.3. Let V be a graded $\mathbf{k}$-module. Let $(V_n)_{n\geq 0}$ be the grading of V. Thus, $\bigoplus_{n\geq 0} V_n = V$. For every $m \in \mathbb{N}$, define a $\mathbf{k}$-submodule $V_{\leq m}$ of V by

$$V_{\leq m} = \bigoplus_{n=0}^{m} V_n.$$

(This is well-defined, because $\bigoplus_{n=0}^{m} V_n$ is a subsum of the direct sum $\bigoplus_{n\geq 0} V_n$.) Then, $(V_{\leq n})_{n\geq 0}$ is a filtration of the $\mathbf{k}$-module V. Equipped with this filtration, V becomes a filtered $\mathbf{k}$-module. We denote this filtered $\mathbf{k}$-module again by V, since it is defined canonically in terms of V. Thus, every graded $\mathbf{k}$-module V canonically becomes a filtered $\mathbf{k}$-module.

Remark 2.4. Let V and W be two graded $\mathbf{k}$-modules. Thus, both V and W canonically become filtered $\mathbf{k}$-modules (according to Definition 2.3). Every graded $\mathbf{k}$-linear map $f : V \to W$ between the graded $\mathbf{k}$-modules V and W is automatically a filtered $\mathbf{k}$-linear map between the filtered $\mathbf{k}$-modules V and W.

Definition 2.5. Let V be a filtered $\mathbf{k}$-module. Let $(V_n)_{n\geq 0}$ be the filtration of V. We set V_{-1} to be the $\mathbf{k}$-submodule 0 of V; thus, $V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$. For every $m \in \mathbb{N}$, let $\operatorname{gr}_m V$ denote the quotient module V_m / V_{m-1}. Let $\operatorname{gr} V$ denote the graded $\mathbf{k}$-module $\bigoplus_{m\geq 0} \operatorname{gr}_m V$, equipped with the grading $(\operatorname{gr}_m V)_{m\in\mathbb{N}}$. (Here, of course, we identify each $\operatorname{gr}_m V$ with the corresponding submodule of the direct sum $\bigoplus_{m\geq 0} \operatorname{gr}_m V$.) This graded $\mathbf{k}$-module $\operatorname{gr} V$ is known as the *associated graded module* of the filtered $\mathbf{k}$-module V.

For every $m \in \mathbb{N}$ and every $v \in V_m$, we let $[v]_m$ denote the residue class $v + V_{m-1}$ of $v \in V_m$ in $V_m / V_{m-1} = \operatorname{gr}_m V$.

[12]This time, isomorphisms in this category are easy to characterize: They are the invertible graded $\mathbf{k}$-linear maps. Thus, the inverse of any invertible graded $\mathbf{k}$-linear map is again graded. (This is not hard to check.)

Definition 2.6. Let V and W be two filtered $\mathbf{k}$-modules. Then, two graded $\mathbf{k}$-modules $\operatorname{gr} V$ and $\operatorname{gr} W$ are defined according to Definition 2.5.

Let $f : V \to W$ be a filtered $\mathbf{k}$-linear map. Then, for every $m \in \mathbb{N}$, we define a $\mathbf{k}$-linear map $\operatorname{gr}_m f : \operatorname{gr}_m V \to \operatorname{gr}_m W$ as follows:

Let $(V_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$ be the filtrations of V and W, respectively. Then, $f(V_m) \subseteq W_m$ (since the map f is filtered). Hence, the map f restricts to a $\mathbf{k}$-linear map $V_m \to W_m$. This $\mathbf{k}$-linear map sends V_{m-1} to W_{m-1} (since $f(V_{m-1}) \subseteq W_{m-1}$ (again because f is filtered, and since $V_{-1} = 0$)), and thus gives rise to a $\mathbf{k}$-linear map

$$\varphi : V_m / V_{m-1} \to W_m / W_{m-1}, \qquad [v]_m \mapsto [f(v)]_m .$$

We denote this $\mathbf{k}$-linear map φ by $\operatorname{gr}_m f$. Thus, $\operatorname{gr}_m f$ is a $\mathbf{k}$-linear map $V_m/V_{m-1} \to W_m/W_{m-1}$. In other words, $\operatorname{gr}_m f$ is a $\mathbf{k}$-linear map $\operatorname{gr}_m V \to \operatorname{gr}_m W$ (since $\operatorname{gr}_m V = V_m/V_{m-1}$ and $\operatorname{gr}_m W = W_m/W_{m-1}$).

Now, we have defined a $\mathbf{k}$-linear map $\operatorname{gr}_m f : \operatorname{gr}_m V \to \operatorname{gr}_m W$ for every $m \in \mathbb{N}$. This $\operatorname{gr}_m f$ satisfies

$$(\operatorname{gr}_m f)([v]_m) = [f(v)]_m \qquad \text{for every } m \in \mathbb{N} \text{ and } v \in V_m.$$

We finally define a $\mathbf{k}$-linear map $\operatorname{gr} f : \bigoplus_{m\geq 0} \operatorname{gr}_m V \to \bigoplus_{m\geq 0} \operatorname{gr}_m W$ by $\operatorname{gr} f = \bigoplus_{m\geq 0} \operatorname{gr}_m f$. Thus, $\operatorname{gr} f$ is a graded $\mathbf{k}$-linear map from $\operatorname{gr} V$ to $\operatorname{gr} W$ (because $\bigoplus_{m\geq 0} \operatorname{gr}_m V = \operatorname{gr} V$ and $\bigoplus_{m\geq 0} \operatorname{gr}_m W = \operatorname{gr} W$). It satisfies

$$(\operatorname{gr} f)([v]_m) = [f(v)]_m \qquad \text{for every } m \in \mathbb{N} \text{ and } v \in V_m.$$

Definition 2.3 shows a way to interpret every graded $\mathbf{k}$-module as a filtered $\mathbf{k}$-module (though the grading cannot be reconstructed from the filtration, and thus we lose information when we replace the former by the latter). Together with Remark 2.4 (which shows what happens to graded $\mathbf{k}$-linear maps under this interpretation), it therefore defines a functor from the category of graded $\mathbf{k}$-modules (and graded $\mathbf{k}$-linear maps) to the category of filtered $\mathbf{k}$-modules (and filtered $\mathbf{k}$-linear maps). This functor can be regarded as a forgetful functor, since it preserves the underlying set of the graded $\mathbf{k}$-module (although it is not a forgetful functor in the strict meaning of this word, since it replace the grading by a filtration, which is **not** the grading).

Definition 2.5 shows how to construct a graded $\mathbf{k}$-module from any filtered $\mathbf{k}$-module (although not on the same underlying set, and again, at the cost of losing information). Together with Definition 2.6 (which constructs a graded $\mathbf{k}$-linear map from any filtered $\mathbf{k}$-linear map), it thus defines a functor from the category of filtered $\mathbf{k}$-modules (and filtered $\mathbf{k}$-linear maps) to the category of graded $\mathbf{k}$-modules (and graded $\mathbf{k}$-linear maps). This functor does not preserve the underlying set, and it often has the effect of making the underlying set "coarser" in an appropriate informal sense of this word.

So we have found two functors: one from the category of graded $\mathbf{k}$-modules to the category of filtered $\mathbf{k}$-modules, and one in the opposite direction. Neither of these two functors is an isomorphism; however, applying the latter after the former gives a functor naturally isomorphic to the identity:

Definition 2.7. Let V be a graded $\mathbf{k}$-module. Let $(V_n)_{n\geq 0}$ be the grading of V. Consider the $\mathbf{k}$-submodules $V_{\leq m}$ for all $m \in \mathbb{N}$ (defined as in Definition 2.3). Thus, V becomes a filtered $\mathbf{k}$-module (equipped with the filtration $(V_{\leq n})_{n\geq 0}$). Hence, the graded $\mathbf{k}$-module $\operatorname{gr} V$ is well-defined (according to Definition 2.5). For every $m \in \mathbb{N}$, we define a $\mathbf{k}$-linear map

$$\operatorname{grad}_{m,V} : V_m \to V_{\leq m}/V_{\leq m-1}, \qquad v \mapsto [v]_m$$

(making use of the fact that $v \in V_m \subseteq V_{\leq m}$ for every $v \in V_m$). It is easy to see that this map $\operatorname{grad}_{m,V}$ is a $\mathbf{k}$-module isomorphism.

Thus, for every $m \in \mathbb{N}$, the map $\operatorname{grad}_{m,V}$ is a $\mathbf{k}$-module isomorphism from V_m to $V_{\leq m}/V_{\leq m-1} = \operatorname{gr}_m V$. Hence, $\bigoplus\limits_{m\geq 0} \operatorname{grad}_{m,V}$ is a graded $\mathbf{k}$-module isomorphism from V to $\operatorname{gr} V$ (since $V = \bigoplus\limits_{m\geq 0} V_m$ with grading $(V_m)_{m\geq 0}$, and since $\operatorname{gr} V = \bigoplus\limits_{m\geq 0} \operatorname{gr}_m V$ with grading $(\operatorname{gr}_m V)_{m\geq 0}$). We denote this graded $\mathbf{k}$-module isomorphism by grad_V.

Thus, grad_V is a graded $\mathbf{k}$-module isomorphism from V to $\operatorname{gr} V$. It satisfies

$$\operatorname{grad}_V (v) = [v]_m \qquad \text{for every } m \in \mathbb{N} \text{ and } v \in V_m.$$

The inverse $\operatorname{grad}_V^{-1}$ of grad_V is also a graded $\mathbf{k}$-module isomorphism.

Remark 2.8. Let V and W be two graded $\mathbf{k}$-modules. Let $f : V \to W$ be a graded $\mathbf{k}$-linear map. Then, the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\operatorname{grad}_V"'] & W \arrow[d, "\operatorname{grad}_W"] \\ \operatorname{gr} V \arrow[r, "\operatorname{gr} f"'] & \operatorname{gr} W \end{tikzcd}$$

is commutative.

The following (rather well-known) theorem is a basic property of filtered maps; it allows us to use filtrations to prove that certain maps are invertible:

Theorem 2.9. Let V and W be two filtered $\mathbf{k}$-modules. Let $f : V \to W$ be a filtered $\mathbf{k}$-linear map. Assume that the map $\operatorname{gr} f : \operatorname{gr} V \to \operatorname{gr} W$ is invertible. Then, the map f is invertible, and its inverse f^{-1} is again a filtered $\mathbf{k}$-linear map.

Proof of Theorem 2.9. The map $\operatorname{gr} f : \operatorname{gr} V \to \operatorname{gr} W$ is graded, $\mathbf{k}$-linear and invertible. Hence, its inverse $(\operatorname{gr} f)^{-1}$ also is a graded invertible $\mathbf{k}$-linear map.

Let $(V_n)_{n \geq 0}$ be the filtration on V. Let $(W_n)_{n \geq 0}$ be the filtration on W. Thus,

$$f(V_n) \subseteq W_n \qquad \text{for every } n \in \mathbb{N} \tag{15}$$

(since the map f is filtered). As usual, for every negative integer n, we set $V_n = 0$ and $W_n = 0$.

Now,

$$W_n = f(V_n) \qquad \text{for every } n \in \mathbb{N} \cup \{-1\} \tag{16}$$

[13]. Hence,

$$W = \bigcup_{n \in \mathbb{N}} \underbrace{W_n}_{\substack{=f(V_n) \\ \text{(by (16))}}} = \bigcup_{n \in \mathbb{N}} f(V_n) = f\left(\underbrace{\bigcup_{n \in \mathbb{N}} V_n}_{=V}\right) = f(V).$$

[13] *Proof of (16):* We shall prove (16) by induction over n:

Induction base: We have $W_{-1} = 0$. Compared with $f\left(\underbrace{V_{-1}}_{=0}\right) = f(0) = 0$, this yields $W_{-1} = f(V_{-1})$. In other words, (16) holds for $n = -1$. This completes the induction base.

Induction step: Let $N \in \mathbb{N}$. Assume that (16) holds for $n = N - 1$. We must prove that (16) holds for $n = N$.

Let $w \in W_N$. Thus, $[w]_N \in \operatorname{gr}_N W$ is well-defined. Since the map $(\operatorname{gr} f)^{-1}$ is graded, we have $(\operatorname{gr} f)^{-1}([w]_N) \in \operatorname{gr}_N V$ (since $[w]_N \in \operatorname{gr}_N W$). Thus, $(\operatorname{gr} f)^{-1}([w]_N) \in \operatorname{gr}_N V = V_N / V_{N-1}$. In other words, there exists some $v \in V_N$ such that $(\operatorname{gr} f)^{-1}([w]_N) = [v]_N$. Consider this v.

From $(\operatorname{gr} f)^{-1}([w]_N) = [v]_N$, we obtain $[w]_N = (\operatorname{gr} f)([v]_N) = [f(v)]_N$ (by the definition of $\operatorname{gr} f$). In other words, $w \equiv f(v) \bmod W_{N-1}$. In other words, $w - f(v) \in W_{N-1}$.

We assumed that (16) holds for $n = N - 1$. In other words, $W_{N-1} = f(V_{N-1})$. Hence, $w - f(v) \in W_{N-1} = f(V_{N-1})$. Thus,

$$w \in f\left(\underbrace{v}_{\in V_N}\right) + f\left(\underbrace{V_{N-1}}_{\subseteq V_N}\right) \subseteq f(V_N) + f(V_N) \subseteq f(V_N)$$

(since $f(V_N)$ is a $\mathbf{k}$-module).

Let us now forget that we fixed w. We thus have shown that $w \in f(V_N)$ for every $w \in W_N$. In other words, $W_N \subseteq f(V_N)$. Combined with $f(V_N) \subseteq W_N$ (which follows from (15), applied to $n = N$), this yields $W_N = f(V_N)$. In other words, (16) holds for $n = N$. This completes the induction step.

Thus, (16) is proven by induction.

In other words, the map f is surjective.

Furthermore,

$$(\operatorname{Ker} f) \cap V_n = 0 \qquad \text{for every } n \in \mathbb{N} \cup \{-1\} \tag{17}$$

[14]. Now, $\operatorname{Ker} f \subseteq V$, so that $(\operatorname{Ker} f) \cap V = \operatorname{Ker} f$. Hence,

$$\begin{aligned} \operatorname{Ker} f &= (\operatorname{Ker} f) \cap \underbrace{V}_{=\bigcup_{n\in\mathbb{N}} V_n} = (\operatorname{Ker} f) \cap \left(\bigcup_{n\in\mathbb{N}} V_n \right) \\ &= \bigcup_{n\in\mathbb{N}} \underbrace{((\operatorname{Ker} f) \cap V_n)}_{\substack{=0 \\ \text{(by (17))}}} = \bigcup_{n\in\mathbb{N}} 0 = 0. \end{aligned}$$

In other words, the map f is injective. Thus, the map f is both injective and surjective. Hence, the map f is bijective, i.e., invertible. Clearly, its inverse f^{-1} is a $\mathbf{k}$-linear map.

It suffices to prove that this map f^{-1} is filtered. For every $n \in \mathbb{N}$, we have

$$f^{-1} \left(\underbrace{W_n}_{\substack{=f(V_n) \\ \text{(by (16))}}} \right) = f^{-1} \left(f \left(V_n \right) \right) = V_n$$

(since the map f is invertible). Hence, the map f^{-1} is filtered. This completes the proof of Theorem 2.9. $\square$

[14]*Proof of (17):* We shall prove (17) by induction over n:

Induction base: We have $(\operatorname{Ker} f) \cap \underbrace{V_{-1}}_{=0} = (\operatorname{Ker} f) \cap 0 = 0$. In other words, (17) holds for $n = -1$. This completes the induction base.

Induction step: Let $N \in \mathbb{N}$. Assume that (17) holds for $n = N - 1$. We must prove that (17) holds for $n = N$.

We assumed that (17) holds for $n = N - 1$. In other words, $(\operatorname{Ker} f) \cap V_{N-1} = 0$.

Let $v \in (\operatorname{Ker} f) \cap V_N$. Thus, $v \in (\operatorname{Ker} f) \cap V_N \subseteq \operatorname{Ker} f$, so that $f(v) = 0$. Also, $v \in (\operatorname{Ker} f) \cap V_N \subseteq V_N$, and thus $[v]_N \in \operatorname{gr}_N V$ is well-defined. The definition of $\operatorname{gr} f$ shows that $(\operatorname{gr} f)([v]_N) = \left[\underbrace{f(v)}_{=0} \right]_N = [0]_N = 0$. Since the map $\operatorname{gr} f$ is injective (because $\operatorname{gr} f$ is invertible), this shows that $[v]_N = 0$. In other words, $v \in V_{N-1}$. Combined with $v \in \operatorname{Ker} f$, this shows that $v \in (\operatorname{Ker} f) \cap V_{N-1} = 0$. In other words, $v = 0$.

Let us now forget that we fixed v. We thus have proven that $v = 0$ for every $v \in (\operatorname{Ker} f) \cap V_N$. In other words, $(\operatorname{Ker} f) \cap V_N = 0$. In other words, (17) holds for $n = N$. This completes the induction step.

Thus, (17) is proven by induction.

2.2. Definitions on filtered and graded k-algebras

Recall that **k**-algebras are **k**-modules. When a **k**-algebra has a filtration, it may and may not be a filtered **k**-algebra, depending on whether the **k**-algebra structure "plays well" with respect to the filtration. Here is what this means in detail:

Definition 2.10. Let A be a **k**-algebra. Let $(A_n)_{n\geq 0}$ be a filtration on the **k**-module A. Thus, A becomes a filtered **k**-module. We say that the **k**-module A (endowed with the given **k**-algebra structure on A and with the filtration $(A_n)_{n\geq 0}$) is a *filtered* **k***-algebra* if and only if we have

$$1_A \in A_0 \qquad \text{and} \qquad (A_n A_m \subseteq A_{n+m} \qquad \text{for every } n \in \mathbb{N} \text{ and } m \in \mathbb{N}) .$$

Thus, a filtered **k**-algebra automatically is a filtered **k**-module and (at the same time) a **k**-algebra.

We define a similar notation for gradings:

Definition 2.11. Let A be a **k**-algebra. Let $(A_n)_{n\geq 0}$ be a grading on the **k**-module A. Thus, A becomes a graded **k**-module. We say that the **k**-module A (endowed with the given **k**-algebra structure on A and with the grading $(A_n)_{n\geq 0}$) is a *graded* **k***-algebra* if and only if we have

$$1_A \in A_0 \qquad \text{and} \qquad (A_n A_m \subseteq A_{n+m} \qquad \text{for every } n \in \mathbb{N} \text{ and } m \in \mathbb{N}) .$$

Thus, a graded **k**-algebra automatically is a graded **k**-module and (at the same time) a **k**-algebra.

Remark 2.12. Let A be a graded **k**-algebra. Consider the **k**-submodules $A_{\leq m}$ for all $m \in \mathbb{N}$ (defined as in Definition 2.3). Then, A (endowed with the given **k**-algebra structure on A and with the filtration $(A_{\leq n})_{n\geq 0}$) is a filtered **k**-algebra.

We notice that the requirement $1_A \in A_0$ in Definition 2.11 could be dropped (it follows from the other requirements); however, the requirement $1_A \in A_0$ in Definition 2.10 cannot be dropped.

Definition 2.13. Let A be a filtered **k**-algebra. Then, we can define a **k**-algebra structure on the graded **k**-module $\operatorname{gr} A$ as follows:

Let $(A_n)_{n\geq 0}$ be a filtration of the filtered **k**-module A. Then, there exists a unique binary operation $*$ on $\operatorname{gr} A$ which satisfies

$$\left([v]_n * [w]_m = [vw]_{n+m} \qquad \text{for all } n \in \mathbb{N},\, m \in \mathbb{N},\, v \in A_n \text{ and } w \in A_m\right) .$$

The $\mathbf{k}$-module $\operatorname{gr} A$, equipped with this binary operation $*$, becomes a $\mathbf{k}$-algebra with unity $1_{\operatorname{gr} A} = [1_A]_0 \in \operatorname{gr}_0 A \subseteq \operatorname{gr} A$. This $\mathbf{k}$-algebra $\operatorname{gr} A$ furthermore becomes a graded $\mathbf{k}$-algebra (when combined with the grading on $\operatorname{gr} A$). This graded $\mathbf{k}$-algebra $\operatorname{gr} A$ is known as the *associated graded algebra* of the filtered $\mathbf{k}$-algebra A. We shall write the binary operation $*$ as a common multiplication; i.e., we shall write $\alpha\beta$ or $\alpha \cdot \beta$ for $\alpha * \beta$.

Remark 2.14. Let A and B be two filtered $\mathbf{k}$-algebras. Let $f : A \to B$ be a filtered $\mathbf{k}$-algebra homomorphism. Then, $\operatorname{gr} A$ and $\operatorname{gr} B$ are graded $\mathbf{k}$-algebras (according to Definition 2.6). The map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ (defined according to Definition 2.6) is a graded $\mathbf{k}$-algebra homomorphism.

Remark 2.15. Let A be a graded $\mathbf{k}$-algebra. Then, the map $\operatorname{grad}_A : A \to \operatorname{gr} A$ (defined according to Definition 2.7) is a graded $\mathbf{k}$-algebra isomorphism.

Let us record a simple addendum to Theorem 1.15:

Theorem 2.16. Let $\mathfrak{g}$ be a Lie algebra. Let C be a filtered $\mathbf{k}$-algebra. Let $(C_n)_{n\geq 0}$ be the filtration of C. Set $C_{-1} = 0$.

Let $K : \mathfrak{g} \to \operatorname{Der} C$ be a Lie algebra homomorphism. Let $f : \mathfrak{g} \to C$ be a $\mathbf{k}$-linear map. Assume that (4) holds for every $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$ (where the Lie bracket $[f(a), f(b)]$ is computed in the Lie algebra C^-). Define a $\mathfrak{g}$-module structure on C as in Theorem 1.15 **(a)**. Define a map $\eta : U(\mathfrak{g}) \to C$ as in Theorem 1.15 **(b)**.

Assume furthermore that $f(\mathfrak{g}) \subseteq C_1$. Also, assume that the map $K(a) : C \to C$ is filtered for every $a \in \mathfrak{g}$.

Then,

$$\eta\left(\iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)\right) \in f(a_1)\,f(a_2)\cdots f(a_n) + C_{n-1} \qquad (18)$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$.

Proof of Theorem 2.16. We shall prove (18) by induction over n:

Induction base: The equality (18) holds in the case when $n = 0$ [15]. This completes the induction base.

[15] *Proof:* If $a_1, a_2, \ldots, a_0$ are 0 elements of $\mathfrak{g}$, then

$$\begin{aligned}
&\eta\left(\underbrace{\iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_0)}_{=(\text{empty product})=1}\right)\\
&= \eta(1) = 1\cdot 1_C \qquad (\text{by the definition of } \eta)\\
&= 1_C = f(a_1)\,f(a_2)\cdots f(a_0) \qquad (\text{since } f(a_1)\,f(a_2)\cdots f(a_0) = (\text{empty product}) = 1_C)\\
&\in f(a_1)\,f(a_2)\cdots f(a_0) + C_{0-1}
\end{aligned}$$

In other words, (18) holds in the case when $n = 0$. Qed.

Induction step: Let N be a positive integer. Assume that (18) holds in the case when $n = N-1$. We now must show that (18) holds in the case when $n = N$.

We have assumed that (18) holds in the case when $n = N-1$. In other words, we have

$$\eta\left(\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_{N-1}\right)\right) \in f\left(a_1\right)f\left(a_2\right)\cdots f\left(a_{N-1}\right) + C_{(N-1)-1} \quad (19)$$

for every $a_1, a_2, \ldots, a_{N-1} \in \mathfrak{g}$.

Now, let $a_1, a_2, \ldots, a_N \in \mathfrak{g}$.

We have assumed that the map $K(a) : C \to C$ is filtered for every $a \in \mathfrak{g}$. Applying this to $a = a_1$, we conclude that the map $K(a_1) : C \to C$ is filtered.

For every $p \in \{2, 3, \ldots, N\}$, we have $f\left(\underbrace{a_p}_{\in\mathfrak{g}}\right) \in f(\mathfrak{g}) \subseteq C_1$. Multiplying these relations for all $p \in \{2, 3, \ldots, N\}$, we obtain

$$f\left(a_2\right)f\left(a_3\right)\cdots f\left(a_N\right) \in \underbrace{C_1C_1\cdots C_1}_{N-1 \text{ factors}} \subseteq C_{N-1}$$

(since C is a filtered $\mathbf{k}$-algebra). Also, $f\left(\underbrace{a_1}_{\in\mathfrak{g}}\right) \in f(\mathfrak{g}) \subseteq C_1$.

Let g denote the element $\eta\left(\iota_{U,\mathfrak{g}}\left(a_2\right)\iota_{U,\mathfrak{g}}\left(a_3\right)\cdots\iota_{U,\mathfrak{g}}\left(a_N\right)\right)$ of C. Then,

$$g = \eta\left(\iota_{U,\mathfrak{g}}\left(a_2\right)\iota_{U,\mathfrak{g}}\left(a_3\right)\cdots\iota_{U,\mathfrak{g}}\left(a_N\right)\right) \in f\left(a_2\right)f\left(a_3\right)\cdots f\left(a_N\right) + C_{(N-1)-1} \quad (20)$$

(by (19), applied to $a_2, a_3, \ldots, a_{N-1}$ instead of $a_1, a_2, \ldots, a_{N-1}$). Hence,

$$g \in \underbrace{f\left(a_2\right)f\left(a_3\right)\cdots f\left(a_N\right)}_{\in C_{N-1}} + \underbrace{C_{(N-1)-1}}_{\substack{\subseteq C_{N-1} \\ \text{(since } C \text{ is filtered)}}} \subseteq C_{N-1} + C_{N-1} \subseteq C_{N-1}$$

(since C_{N-1} is a $\mathbf{k}$-module). Applying the map $K(a_1)$ to both sides of this relation, we obtain

$$\left(K\left(a_1\right)\right)(g) \in \left(K\left(a_1\right)\right)\left(C_{N-1}\right) \subseteq C_{N-1} \quad (21)$$

(since the map $K(a_1)$ is filtered).

Also, multiplying both sides of the relation (20) with $f(a_1)$ from the left, we obtain

$$\begin{aligned}
f(a_1) \cdot g &\in f(a_1) \cdot \left(f(a_2) f(a_3) \cdots f(a_N) + C_{(N-1)-1} \right) \\
&\subseteq \underbrace{f(a_1) \cdot f(a_2) f(a_3) \cdots f(a_N)}_{= f(a_1) f(a_2) \cdots f(a_N)} + \underbrace{f(a_1)}_{\in C_1} \cdot C_{(N-1)-1} \\
&\subseteq f(a_1) f(a_2) \cdots f(a_N) + \underbrace{C_1 \cdot C_{(N-1)-1}}_{\substack{\subseteq C_{1+((N-1)-1)} \\ \text{(since } C \text{ is a filtered } \mathbf{k}\text{-algebra)}}} \\
&\subseteq f(a_1) f(a_2) \cdots f(a_N) + \underbrace{C_{1+((N-1)-1)}}_{= C_{N-1}} \\
&= f(a_1) f(a_2) \cdots f(a_N) + C_{N-1}. \qquad (22)
\end{aligned}$$

But

$$g = \eta\left(\iota_{U,\mathfrak{g}}(a_2)\, \iota_{U,\mathfrak{g}}(a_3) \cdots \iota_{U,\mathfrak{g}}(a_N)\right) = \left(\iota_{U,\mathfrak{g}}(a_2)\, \iota_{U,\mathfrak{g}}(a_3) \cdots \iota_{U,\mathfrak{g}}(a_N)\right) 1_C \qquad (23)$$

(by the definition of η). Now, the definition of η yields

$$\begin{aligned}
&\eta\left(\iota_{U,\mathfrak{g}}(a_1)\, \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_N)\right) \\
&= \underbrace{\left(\iota_{U,\mathfrak{g}}(a_1)\, \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_N)\right)}_{= \left(\iota_{U,\mathfrak{g}}(a_1)\right) \cdot \left(\iota_{U,\mathfrak{g}}(a_2) \iota_{U,\mathfrak{g}}(a_3) \cdots \iota_{U,\mathfrak{g}}(a_N)\right)} 1_C \\
&= \left(\iota_{U,\mathfrak{g}}(a_1)\right) \cdot \underbrace{\left(\iota_{U,\mathfrak{g}}(a_2)\, \iota_{U,\mathfrak{g}}(a_3) \cdots \iota_{U,\mathfrak{g}}(a_N)\right) 1_C}_{\substack{= g \\ \text{(by (23))}}} = \left(\iota_{U,\mathfrak{g}}(a_1)\right) \cdot g \\
&= a_1 \rightharpoonup g \qquad \text{(by the definition of the } U(\mathfrak{g})\text{-module structure on } C\text{)} \\
&= \underbrace{f(a_1) \cdot g}_{\substack{\in f(a_1) f(a_2) \cdots f(a_N) + C_{N-1} \\ \text{(by (22))}}} + \underbrace{(K(a_1))(g)}_{\substack{\in C_{N-1} \\ \text{(by (21))}}} \\
&\qquad \text{(by the definition of the } \mathfrak{g}\text{-module structure on } C\text{)} \\
&\in f(a_1) f(a_2) \cdots f(a_N) + \underbrace{C_{N-1} + C_{N-1}}_{\substack{\subseteq C_{N-1} \\ \text{(since } C_{N-1} \text{ is a } \mathbf{k}\text{-module)}}} \\
&\subseteq f(a_1) f(a_2) \cdots f(a_N) + C_{N-1}.
\end{aligned}$$

Let us now forget that we fixed $a_1, a_2, \ldots, a_N \in \mathfrak{g}$. We thus have shown that

$$\eta\left(\iota_{U,\mathfrak{g}}(a_1)\, \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_N)\right) \in f(a_1) f(a_2) \cdots f(a_N) + C_{N-1}$$

for every $a_1, a_2, \ldots, a_N \in \mathfrak{g}$. In other words, (18) holds in the case when $n = N$. This completes the induction step. The induction proof of (18) is thus complete. Thus, Theorem 2.16 is proven. □

2.3. $\operatorname{Sym} V$ as a graded $\mathbf{k}$-algebra

Definition 2.17. Let V be a $\mathbf{k}$-module. For every $n \in \mathbb{N}$, we let $V^{\otimes n}$ denote the n-th tensor power of V (that is, the $\mathbf{k}$-module $\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times } V}$). We let $T(V)$ denote the tensor algebra of V. This is the graded $\mathbf{k}$-algebra whose underlying $\mathbf{k}$-module is $\bigoplus_{n \geq 0} V^{\otimes n}$, whose grading is $(V^{\otimes n})_{n \geq 0}$, and whose multiplication is given by

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes b_2 \otimes \cdots \otimes b_m)$$
$$= a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_m$$
$$\text{for every } n \in \mathbb{N},\ m \in \mathbb{N},\ a_1, a_2, \ldots, a_n \in V \text{ and } b_1, b_2, \ldots, b_m \in V.$$

We let $\operatorname{Sym} V$ denote the quotient algebra of the $\mathbf{k}$-algebra $T(V)$ by its ideal generated by the tensors of the form $v \otimes w - w \otimes v$ with $v \in V$ and $w \in V$. We let $\pi_{\operatorname{Sym},V}$ be the canonical projection from $T(V)$ to its quotient $\mathbf{k}$-algebra $\operatorname{Sym} V$. It is well-known that $\operatorname{Sym} V$ (endowed with the grading $\left(\pi_{\operatorname{Sym},V}(V^{\otimes n})\right)_{n \geq 0}$) is a graded $\mathbf{k}$-algebra (since the ideal of $T(V)$ generated by the products of the form $v \otimes w - w \otimes v$ with $v \in V$ and $w \in V$ is a graded $\mathbf{k}$-submodule of $T(V)$). This graded $\mathbf{k}$-algebra $\operatorname{Sym} V$ is known as the *symmetric algebra* of V. For every $n \in \mathbb{N}$, we write $\operatorname{Sym}^n V$ for the $\mathbf{k}$-submodule $\pi_{\operatorname{Sym},V}(V^{\otimes n})$ of $\operatorname{Sym} V$. Thus, $\operatorname{Sym} V = \bigoplus_{n \geq 0} \operatorname{Sym}^n V$, and the grading of $\operatorname{Sym} V$ is $(\operatorname{Sym}^n V)_{n \geq 0}$.

Both $T(V)$ and $\operatorname{Sym} V$ are graded $\mathbf{k}$-algebras, and thus canonically become filtered $\mathbf{k}$-algebras. It is well-known that the $\mathbf{k}$-algebra $\operatorname{Sym} V$ is commutative.

We let $\iota_{T,V}$ be the canonical inclusion map of V into $T(V)$. This map is the composition $V \xrightarrow{\cong} V^{\otimes 1} \xrightarrow{\text{inclusion}} \bigoplus_{n \geq 0} V^{\otimes n} = T(V)$. (Here, the "$T$" in "$\iota_{T,V}$" is not a variable, but stands for the letter "t" in "**t**ensor algebra".) We have $\iota_{T,V}(V) = V^{\otimes 1}$. We will usually (but not always) identify V with the $\mathbf{k}$-submodule $V^{\otimes 1}$ of $T(V)$ via this map $\iota_{T,V}$; this identification is harmless since $\iota_{T,V}$ is injective. Thus, for every $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in V$, we have $a_1 a_2 \cdots a_n = a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in the $\mathbf{k}$-algebra $T(V)$.

We let $\iota_{\operatorname{Sym},V}$ denote the composition $\pi_{\operatorname{Sym},V} \circ \iota_{T,V}$. This composition $\iota_{\operatorname{Sym},V}$ is a $\mathbf{k}$-linear map $V \to \operatorname{Sym} V$. It is well-known that this map $\iota_{\operatorname{Sym},V}$ is injective and satisfies $\iota_{\operatorname{Sym},V}(V) = \operatorname{Sym}^1 V$.

The $\mathbf{k}$-algebra $T(V)$ becomes a $\mathbf{k}$-bialgebra as follows: We define the coproduct Δ of $T(V)$ as the unique $\mathbf{k}$-algebra homomorphism $\Delta : T(V) \to T(V) \otimes T(V)$ satisfying

$$(\Delta(\iota_{T,V}(v)) = \iota_{T,V}(v) \otimes 1 + 1 \otimes \iota_{T,V}(v) \qquad \text{for every } v \in V).$$

We define the counit ϵ of $T(V)$ as the unique $\mathbf{k}$-algebra homomorphism $\epsilon : T(V) \to \mathbf{k}$ satisfying

$$(\epsilon(\iota_{T,V}(v)) = 0 \qquad \text{for every } v \in V).$$

Explicitly, Δ and ϵ are given by the following formulas:

$$\Delta\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) = \sum_{I \subseteq \{1,2,\ldots,n\}} a_I \otimes a_{\{1,2,\ldots,n\} \setminus I};$$
$$\epsilon\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases}$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in V$, where for every subset $J = \{j_1 < j_2 < \cdots < j_k\} \subseteq \{1, 2, \ldots, n\}$, we define a_J to be the tensor $a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_k}$. The $\mathbf{k}$-bialgebra $T(V)$ is actually a Hopf algebra.

It is easy to see that the elements of $\iota_{T,V}(V)$ are primitive elements of $T(V)$. In other words, $\iota_{T,V}(V) \subseteq \operatorname{Prim}(T(V))$.

The $\mathbf{k}$-algebra $\operatorname{Sym} V$ also becomes a $\mathbf{k}$-bialgebra, being a quotient of the $\mathbf{k}$-bialgebra $T(V)$ by a biideal. It is easy to see that $\iota_{\operatorname{Sym},V}(V) \subseteq \operatorname{Prim}(\operatorname{Sym} V)$.

Remark 2.18. Let V be a $\mathbf{k}$-module. Let $n \in \mathbb{N}$.

(a) We have $\pi_{\operatorname{Sym},V}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) = \iota_{\operatorname{Sym},V}(a_1)\, \iota_{\operatorname{Sym},V}(a_2) \cdots \iota_{\operatorname{Sym},V}(a_n)$ for any $a_1, a_2, \ldots, a_n \in V$.

(b) The $\mathbf{k}$-module $\operatorname{Sym}^n V$ is spanned by the elements $\iota_{\operatorname{Sym},V}(a_1)\, \iota_{\operatorname{Sym},V}(a_2) \cdots \iota_{\operatorname{Sym},V}(a_n)$ with $a_1, a_2, \ldots, a_n \in V$.

(c) Assume that n is positive. Then, $\operatorname{Sym}^n V = \left(\operatorname{Sym}^1 V\right)\left(\operatorname{Sym}^{n-1} V\right)$.

Proof of Remark 2.18. **(a)** Let $a_1, a_2, \ldots, a_n \in V$. For any $p \in \{1, 2, \ldots, n\}$, we have

$$\underbrace{\iota_{\operatorname{Sym},V}}_{=\pi_{\operatorname{Sym},V} \circ \iota_{T,V}}\left(a_p\right) = \left(\pi_{\operatorname{Sym},V} \circ \iota_{T,V}\right)\left(a_p\right) = \pi_{\operatorname{Sym},V}\left(\iota_{T,V}\left(a_p\right)\right).$$

Thus,

$$\begin{aligned}
&\iota_{\operatorname{Sym},V}(a_1)\, \iota_{\operatorname{Sym},V}(a_2) \cdots \iota_{\operatorname{Sym},V}(a_n) \\
&= \pi_{\operatorname{Sym},V}\left(\iota_{T,V}(a_1)\right) \pi_{\operatorname{Sym},V}\left(\iota_{T,V}(a_2)\right) \cdots \pi_{\operatorname{Sym},V}\left(\iota_{T,V}(a_n)\right) \\
&= \pi_{\operatorname{Sym},V}\left(\underbrace{\iota_{T,V}(a_1)\, \iota_{T,V}(a_2) \cdots \iota_{T,V}(a_n)}_{\substack{=a_1 \otimes a_2 \otimes \cdots \otimes a_n \\ \text{(by the definition of the product in } T(V))}}\right) \\
&\qquad \left(\text{since } \pi_{\operatorname{Sym},V} \text{ is a } \mathbf{k}\text{-algebra homomorphism}\right) \\
&= \pi_{\operatorname{Sym},V}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right).
\end{aligned}$$

This proves Remark 2.18 **(a)**.

(b) The $\mathbf{k}$-module $V^{\otimes n}$ is spanned by pure tensors. In other words, the $\mathbf{k}$-module $V^{\otimes n}$ is spanned by the elements $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ with $a_1, a_2, \ldots, a_n \in V$.

Thus, the **k**-module $\pi_{\mathrm{Sym},V}\left(V^{\otimes n}\right)$ is spanned by the elements $\pi_{\mathrm{Sym},V}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right)$ with $a_1, a_2, \ldots, a_n \in V$ (since the map $\pi_{\mathrm{Sym},V}$ is **k**-linear). In other words, the **k**-module $\mathrm{Sym}^n V$ is spanned by the elements $\iota_{\mathrm{Sym},V}\left(a_1\right) \iota_{\mathrm{Sym},V}\left(a_2\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right)$ with $a_1, a_2, \ldots, a_n \in V$ (because $\pi_{\mathrm{Sym},V}\left(V^{\otimes n}\right) = \mathrm{Sym}^n V$ and because every $a_1, a_2, \ldots, a_n \in V$ satisfy $\pi_{\mathrm{Sym},V}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) = \iota_{\mathrm{Sym},V}\left(a_1\right) \iota_{\mathrm{Sym},V}\left(a_2\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right)$). This proves Remark 2.18 **(b)**.

(c) Clearly, $\left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right) \subseteq \mathrm{Sym}^n V$ (since the **k**-algebra $\mathrm{Sym}\, V$ is graded).

Now, let $v \in \mathrm{Sym}^n V$. We shall show that $v \in \left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right)$.

We have $v \in \mathrm{Sym}^n V$. Thus, v is a **k**-linear combination of elements of the form $\iota_{\mathrm{Sym},V}\left(a_1\right) \iota_{\mathrm{Sym},V}\left(a_2\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right)$ with $a_1, a_2, \ldots, a_n \in V$ (due to Remark 2.18 **(b)**).

We need to prove the relation $v \in \left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right)$. This relation is clearly **k**-linear in v. Hence, we can WLOG assume that v has the form $\iota_{\mathrm{Sym},V}\left(a_1\right) \iota_{\mathrm{Sym},V}\left(a_2\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right)$ with $a_1, a_2, \ldots, a_n \in V$ (because v is a **k**-linear combination of elements of the form $\iota_{\mathrm{Sym},V}\left(a_1\right) \iota_{\mathrm{Sym},V}\left(a_2\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right)$ with $a_1, a_2, \ldots, a_n \in V$). Assume this. Thus, $v = \iota_{\mathrm{Sym},V}\left(a_1\right) \iota_{\mathrm{Sym},V}\left(a_2\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right)$ for some $a_1, a_2, \ldots, a_n \in V$. Consider these $a_1, a_2, \ldots, a_n$. We have

$$\begin{aligned} v &= \iota_{\mathrm{Sym},V}\left(a_1\right) \iota_{\mathrm{Sym},V}\left(a_2\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right) \\ &= \underbrace{\iota_{\mathrm{Sym},V}\left(a_1\right)}_{\in \mathrm{Sym}^1 V} \cdot \underbrace{\left(\iota_{\mathrm{Sym},V}\left(a_2\right) \iota_{\mathrm{Sym},V}\left(a_3\right) \cdots \iota_{\mathrm{Sym},V}\left(a_n\right)\right)}_{\substack{\in \left(\mathrm{Sym}^1 V\right)^{n-1} \\ \text{(since } \iota_{\mathrm{Sym},V}\left(a_p\right) \in \mathrm{Sym}^1 V \text{ for every } p \in \{2,3,\ldots,n\})}} \\ &\in \left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right). \end{aligned}$$

Now, let us forget that we have fixed v. We thus have shown $v \in \left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right)$ for every $v \in \mathrm{Sym}^n V$. In other words, $\mathrm{Sym}^n V \subseteq \left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right)$. Combined with $\left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right) \subseteq \mathrm{Sym}^n V$, this shows that $\left(\mathrm{Sym}^1 V\right)\left(\mathrm{Sym}^{n-1} V\right) = \mathrm{Sym}^n V$. Remark 2.18 **(c)** is proven. □

The following fact is well-known as the universal property of the symmetric algebra:

Proposition 2.19. Let V be a **k**-module. Let A be a commutative **k**-algebra. Let $f : V \to A$ be a **k**-linear map. Then, there exists a unique **k**-algebra homomorphism $F : \mathrm{Sym}\, V \to A$ such that $F \circ \iota_{\mathrm{Sym},V} = f$. (See Definition 2.17 for the definition of $\iota_{\mathrm{Sym},V}$.)

Let us also state a "derivational analogue" of this universal property:

Proposition 2.20. Let V be a $\mathbf{k}$-module. Let $f : V \to \operatorname{Sym} V$ be a $\mathbf{k}$-linear map. Then, there exists a unique derivation $F : \operatorname{Sym} V \to \operatorname{Sym} V$ such that $F \circ \iota_{\operatorname{Sym},V} = f$.

Proposition 2.20 follows from [Grinbe15, Proposition 2.7] (applied to $M = \operatorname{Sym} V$). We shall use the following fact, which strengthens Proposition 2.20 in a particular case:

Proposition 2.21. Let V be a $\mathbf{k}$-module. Let $f : V \to V$ be a $\mathbf{k}$-linear map.

(a) There exists a unique derivation $F : \operatorname{Sym} V \to \operatorname{Sym} V$ such that $F \circ \iota_{\operatorname{Sym},V} = \iota_{\operatorname{Sym},V} \circ f$.

We shall denote this derivation F by $\widetilde{f}$.

(b) The map $\widetilde{f} : \operatorname{Sym} V \to \operatorname{Sym} V$ is furthermore a coderivation.

(c) The map $\widetilde{f} : \operatorname{Sym} V \to \operatorname{Sym} V$ is graded.

Proof of Proposition 2.21. **(a)** Proposition 2.20 (applied to $\iota_{\operatorname{Sym},V} \circ f$ instead of f) shows that there exists a unique derivation $F : \operatorname{Sym} V \to \operatorname{Sym} V$ such that $F \circ \iota_{\operatorname{Sym},V} = \iota_{\operatorname{Sym},V} \circ f$. This proves Proposition 2.21 **(a)**.

The map $\widetilde{f}$ was defined as the unique derivation $F : \operatorname{Sym} V \to \operatorname{Sym} V$ such that $F \circ \iota_{\operatorname{Sym},V} = \iota_{\operatorname{Sym},V} \circ f$. Thus, $\widetilde{f}$ is a derivation $\operatorname{Sym} V \to \operatorname{Sym} V$ and satisfies

$$\widetilde{f} \circ \iota_{\operatorname{Sym},V} = \iota_{\operatorname{Sym},V} \circ f.$$

We have $\widetilde{f} \in \operatorname{Der}(\operatorname{Sym} V)$ (since $\widetilde{f}$ is a derivation). For every $v \in V$, we have

$$\widetilde{f}\left(\iota_{\operatorname{Sym},V}(v)\right) = \left(\underbrace{\widetilde{f} \circ \iota_{\operatorname{Sym},V}}_{=\iota_{\operatorname{Sym},V} \circ f}\right)(v) = \left(\iota_{\operatorname{Sym},V} \circ f\right)(v) = \iota_{\operatorname{Sym},V}\left(\underbrace{f(v)}_{\in V}\right) \in \iota_{\operatorname{Sym},V}(V) = \operatorname{Sym}^1 V. \tag{24}$$

(c) Let $n \in \mathbb{N}$. Let $x \in \operatorname{Sym}^n V$. We are going to prove the relation $\widetilde{f}(x) \in \operatorname{Sym}^n V$.

This relation is $\mathbf{k}$-linear in x. Hence, we can WLOG assume that x is of the form $\iota_{\operatorname{Sym},V}(a_1)\, \iota_{\operatorname{Sym},V}(a_2) \cdots \iota_{\operatorname{Sym},V}(a_n)$ for some $a_1, a_2, \ldots, a_n \in V$ (because the $\mathbf{k}$-module $\operatorname{Sym}^n V$ is spanned by the elements $\iota_{\operatorname{Sym},V}(a_1)\, \iota_{\operatorname{Sym},V}(a_2) \cdots \iota_{\operatorname{Sym},V}(a_n)$ with $a_1, a_2, \ldots, a_n \in V$ (by Remark 2.18 **(b)**)). Assume this, and consider these $a_1, a_2, \ldots, a_n$. Thus,

$$x = \iota_{\operatorname{Sym},V}(a_1)\, \iota_{\operatorname{Sym},V}(a_2) \cdots \iota_{\operatorname{Sym},V}(a_n).$$

Applying the map $\widetilde{f}$ to both sides of this equality, we obtain

$$\begin{aligned}
\widetilde{f}(x) &= \widetilde{f}\left(\iota_{\mathrm{Sym},V}(a_1)\,\iota_{\mathrm{Sym},V}(a_2)\cdots\iota_{\mathrm{Sym},V}(a_n)\right) \\
&= \sum_{i=1}^{n} \underbrace{\iota_{\mathrm{Sym},V}(a_1)\,\iota_{\mathrm{Sym},V}(a_2)\cdots\iota_{\mathrm{Sym},V}(a_{i-1})}_{\in \mathrm{Sym}^{i-1} V} \underbrace{\widetilde{f}\left(\iota_{\mathrm{Sym},V}(a_i)\right)}_{\substack{\in \mathrm{Sym}^1 V \\ \text{(by (24), applied} \\ \text{to } v=a_i\text{)}}} \\
&\qquad \underbrace{\iota_{\mathrm{Sym},V}(a_{i+1})\,\iota_{\mathrm{Sym},V}(a_{i+2})\cdots\iota_{\mathrm{Sym},V}(a_n)}_{\in \mathrm{Sym}^{n-i} V} \\
&\qquad \left(\begin{array}{c}\text{by Proposition 1.11 } \textbf{(b)}\text{, applied to } \mathrm{Sym}\, V \text{ and } \iota_{\mathrm{Sym},V}(a_i) \\ \text{instead of } C \text{ and } a_i\end{array}\right) \\
&\in \sum_{i=1}^{n} \underbrace{\left(\mathrm{Sym}^{i-1} V\right)\left(\mathrm{Sym}^{1} V\right)\left(\mathrm{Sym}^{n-i} V\right)}_{\substack{\subseteq \mathrm{Sym}^{(i-1)+1+(n-i)} V \\ \text{(since } \mathrm{Sym}\, V \text{ is a graded } \mathbf{k}\text{-algebra)}}} \subseteq \sum_{i=1}^{n} \underbrace{\mathrm{Sym}^{(i-1)+1+(n-i)} V}_{=\mathrm{Sym}^n V} \\
&= \sum_{i=1}^{n} \mathrm{Sym}^n V \subseteq \mathrm{Sym}^n V
\end{aligned}$$

(since $\mathrm{Sym}^n V$ is a $\mathbf{k}$-module).

Now, let us forget that we fixed x. We thus have proven that $\widetilde{f}(x) \in \mathrm{Sym}^n V$ for every $x \in \mathrm{Sym}^n V$. In other words, $\widetilde{f}(\mathrm{Sym}^n V) \subseteq \mathrm{Sym}^n V$.

Let us now forget that we fixed n. We thus have proven that $\widetilde{f}(\mathrm{Sym}^n V) \subseteq \mathrm{Sym}^n V$ for every $n \in \mathbb{N}$. In other words, the map $\widetilde{f}$ is graded. This proves Proposition 2.21 **(c)**.

(b) Let A be the $\mathbf{k}$-algebra $\mathrm{Sym}\, V$. Then, the comultiplication of the $\mathbf{k}$-coalgebra $\mathrm{Sym}\, V$ is a $\mathbf{k}$-linear map $\Delta : A \to A \otimes A$. This map Δ is a $\mathbf{k}$-algebra homomorphism (since $\mathrm{Sym}\, V$ is a $\mathbf{k}$-bialgebra). The map $\widetilde{f} \otimes \mathrm{id}_A : A \otimes A \to A \otimes A$ is a derivation (by Lemma 1.12 **(a)**, applied to A and $\widetilde{f}$ instead of B and f). In other words, $\widetilde{f} \otimes \mathrm{id}_A \in \mathrm{Der}(A \otimes A)$. The map $\mathrm{id}_A \otimes \widetilde{f} : A \otimes A \to A \otimes A$ is a derivation (by Lemma 1.12 **(b)**, applied to A and $\widetilde{f}$ instead of B and f). In other words, $\mathrm{id}_A \otimes \widetilde{f} \in \mathrm{Der}(A \otimes A)$. But $\mathrm{Der}(A \otimes A)$ is a Lie subalgebra of $(\mathrm{End}(A \otimes A))^-$ (by Proposition 1.11 **(a)**, applied to $C = A \otimes A$), and thus a $\mathbf{k}$-module. Hence, $\widetilde{f} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \widetilde{f} \in \mathrm{Der}(A \otimes A)$ (since both $\widetilde{f} \otimes \mathrm{id}_A$ and $\mathrm{id}_A \otimes \widetilde{f}$ belong to $\mathrm{Der}(A \otimes A)$). In other words, $\widetilde{f} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \widetilde{f} : A \otimes A \to A \otimes A$ is a derivation.

Let S be the subset $\iota_{\mathrm{Sym},V}(V)$ of $\mathrm{Sym}\, V = A$. Thus, $S = \iota_{\mathrm{Sym},V}(V) = \mathrm{Sym}^1 V$. Hence, S generates $\mathrm{Sym}\, V = A$ as a $\mathbf{k}$-algebra.

We have $\widetilde{f} \in \mathrm{Der}(\mathrm{Sym}\, V)$. Thus, Proposition 1.11 **(c)** (applied to $\mathrm{Sym}\, V$ and $\widetilde{f}$ instead of C and f) shows that $\widetilde{f}(1) = 0$.

We have

$$\left(\Delta \circ \widetilde{f}\right) |_S = \left(\left(\widetilde{f} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \widetilde{f}\right) \circ \Delta\right) |_S \tag{25}$$

[16]. Hence, Lemma 1.14 (applied to $A \otimes A$, Δ, $\widetilde{f}$ and $\widetilde{f} \otimes \operatorname{id}_A + \operatorname{id}_A \otimes \widetilde{f}$ instead of B, f, d and e) shows that $\Delta \circ \widetilde{f} = \left(\widetilde{f} \otimes \operatorname{id}_A + \operatorname{id}_A \otimes \widetilde{f}\right) \circ \Delta$. In other words, $\Delta \circ \widetilde{f} = \left(\widetilde{f} \otimes \operatorname{id} + \operatorname{id} \otimes \widetilde{f}\right) \circ \Delta$. In other words, $\widetilde{f}$ is a coderivation (by the definition of a "coderivation"). This proves Proposition 2.21 **(b)**. $\square$

2.4. $U(\mathfrak{g})$ as a filtered k-algebra

[16]*Proof of (25):* Let $s \in S$. We shall show that $\left(\Delta \circ \widetilde{f}\right)(s) = \left(\left(\widetilde{f} \otimes \operatorname{id}_A + \operatorname{id}_A \otimes \widetilde{f}\right) \circ \Delta\right)(s)$.

The definition of the $\mathbf{k}$-bialgebra structure on $\operatorname{Sym} V$ shows that

$$\Delta\left(\iota_{\operatorname{Sym},V}(w)\right) = \iota_{\operatorname{Sym},V}(w) \otimes 1 + 1 \otimes \iota_{\operatorname{Sym},V}(w) \tag{26}$$

for every $w \in V$.

Now, $s \in S = \iota_{\operatorname{Sym},V}(V)$. Hence, $s = \iota_{\operatorname{Sym},V}(v)$ for some $v \in V$. Consider this v. Now,

$$\widetilde{f}\left(\underbrace{s}_{=\iota_{\operatorname{Sym},V}(v)}\right) = \widetilde{f}\left(\iota_{\operatorname{Sym},V}(v)\right) = \underbrace{\left(\widetilde{f} \circ \iota_{\operatorname{Sym},V}\right)}_{=\iota_{\operatorname{Sym},V} \circ f}(v) = \left(\iota_{\operatorname{Sym},V} \circ f\right)(v) = \iota_{\operatorname{Sym},V}(f(v)).$$

Hence,

$$\begin{aligned}\left(\Delta \circ \widetilde{f}\right)(s) &= \Delta\left(\underbrace{\widetilde{f}(s)}_{=\iota_{\operatorname{Sym},V}(f(v))}\right) = \Delta\left(\iota_{\operatorname{Sym},V}(f(v))\right) \\ &= \iota_{\operatorname{Sym},V}(f(v)) \otimes 1 + 1 \otimes \iota_{\operatorname{Sym},V}(f(v))\end{aligned} \tag{27}$$

(by (26), applied to $w = f(v)$).

On the other hand,

$$\begin{aligned}&\left(\left(\widetilde{f} \otimes \operatorname{id}_A + \operatorname{id}_A \otimes \widetilde{f}\right) \circ \Delta\right)(s) \\ &= \left(\widetilde{f} \otimes \operatorname{id}_A + \operatorname{id}_A \otimes \widetilde{f}\right)\left(\Delta\left(\underbrace{s}_{=\iota_{\operatorname{Sym},V}(v)}\right)\right) \\ &= \left(\widetilde{f} \otimes \operatorname{id}_A + \operatorname{id}_A \otimes \widetilde{f}\right)\left(\underbrace{\Delta\left(\iota_{\operatorname{Sym},V}(v)\right)}_{\substack{=\iota_{\operatorname{Sym},V}(v)\otimes 1+1\otimes\iota_{\operatorname{Sym},V}(v) \\ \text{(by (26), applied to } w=v)}}\right) \\ &= \left(\widetilde{f} \otimes \operatorname{id}_A + \operatorname{id}_A \otimes \widetilde{f}\right)\left(\iota_{\operatorname{Sym},V}(v) \otimes 1 + 1 \otimes \iota_{\operatorname{Sym},V}(v)\right) \\ &= \left(\widetilde{f} \otimes \operatorname{id}_A\right)\left(\iota_{\operatorname{Sym},V}(v) \otimes 1 + 1 \otimes \iota_{\operatorname{Sym},V}(v)\right) \\ &\qquad + \left(\operatorname{id}_A \otimes \widetilde{f}\right)\left(\iota_{\operatorname{Sym},V}(v) \otimes 1 + 1 \otimes \iota_{\operatorname{Sym},V}(v)\right).\end{aligned}$$

Definition 2.22. Let $\mathfrak{g}$ be a Lie algebra. We recall that the universal enveloping algebra $U(\mathfrak{g})$ is defined as the quotient algebra of the $\mathbf{k}$-algebra $T(\mathfrak{g})$ by its ideal $J_{\mathfrak{g}}$, where $J_{\mathfrak{g}}$ is the ideal of $T(\mathfrak{g})$ generated by all tensors of the form

$$v \otimes w - w \otimes v - [v, w] \qquad \text{with } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}.$$

We let $\pi_{U,\mathfrak{g}}$ be the canonical projection from $T(\mathfrak{g})$ to its quotient $\mathbf{k}$-algebra

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Since

$$\begin{aligned}
&\left(\widetilde{f} \otimes \mathrm{id}_A\right)\left(\iota_{\mathrm{Sym},V}(v) \otimes 1 + 1 \otimes \iota_{\mathrm{Sym},V}(v)\right) \\
&= \underbrace{\left(\widetilde{f} \otimes \mathrm{id}_A\right)\left(\iota_{\mathrm{Sym},V}(v) \otimes 1\right)}_{=\widetilde{f}\left(\iota_{\mathrm{Sym},V}(v)\right) \otimes \mathrm{id}_A(1)} + \underbrace{\left(\widetilde{f} \otimes \mathrm{id}_A\right)\left(1 \otimes \iota_{\mathrm{Sym},V}(v)\right)}_{=\widetilde{f}(1) \otimes \mathrm{id}_A\left(\iota_{\mathrm{Sym},V}(v)\right)} \\
&= \underbrace{\widetilde{f}\left(\iota_{\mathrm{Sym},V}(v)\right)}_{=\iota_{\mathrm{Sym},V}(f(v))} \otimes \underbrace{\mathrm{id}_A(1)}_{=1} + \underbrace{\widetilde{f}(1)}_{=0} \otimes \mathrm{id}_A\left(\iota_{\mathrm{Sym},V}(v)\right) \\
&= \iota_{\mathrm{Sym},V}(f(v)) \otimes 1 + \underbrace{0 \otimes \mathrm{id}_A\left(\iota_{\mathrm{Sym},V}(v)\right)}_{=0} = \iota_{\mathrm{Sym},V}(f(v)) \otimes 1
\end{aligned}$$

and

$$\begin{aligned}
&\left(\mathrm{id}_A \otimes \widetilde{f}\right)\left(\iota_{\mathrm{Sym},V}(v) \otimes 1 + 1 \otimes \iota_{\mathrm{Sym},V}(v)\right) \\
&= \underbrace{\left(\mathrm{id}_A \otimes \widetilde{f}\right)\left(\iota_{\mathrm{Sym},V}(v) \otimes 1\right)}_{=\mathrm{id}_A\left(\iota_{\mathrm{Sym},V}(v)\right) \otimes \widetilde{f}(1)} + \underbrace{\left(\mathrm{id}_A \otimes \widetilde{f}\right)\left(1 \otimes \iota_{\mathrm{Sym},V}(v)\right)}_{=\mathrm{id}_A(1) \otimes \widetilde{f}\left(\iota_{\mathrm{Sym},V}(v)\right)} \\
&= \mathrm{id}_A\left(\iota_{\mathrm{Sym},V}(v)\right) \otimes \underbrace{\widetilde{f}(1)}_{=0} + \underbrace{\mathrm{id}_A(1)}_{=1} \otimes \underbrace{\widetilde{f}\left(\iota_{\mathrm{Sym},V}(v)\right)}_{=\iota_{\mathrm{Sym},V}(f(v))} \\
&= \underbrace{\mathrm{id}_A\left(\iota_{\mathrm{Sym},V}(v)\right) \otimes 0}_{=0} + 1 \otimes \iota_{\mathrm{Sym},V}(f(v)) = 1 \otimes \iota_{\mathrm{Sym},V}(f(v)),
\end{aligned}$$

this becomes

$$\begin{aligned}
&\left(\left(\widetilde{f} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \widetilde{f}\right) \circ \Delta\right)(s) \\
&= \underbrace{\left(\widetilde{f} \otimes \mathrm{id}_A\right)\left(\iota_{\mathrm{Sym},V}(v) \otimes 1 + 1 \otimes \iota_{\mathrm{Sym},V}(v)\right)}_{=\iota_{\mathrm{Sym},V}(f(v)) \otimes 1} \\
&\qquad + \underbrace{\left(\mathrm{id}_A \otimes \widetilde{f}\right)\left(\iota_{\mathrm{Sym},V}(v) \otimes 1 + 1 \otimes \iota_{\mathrm{Sym},V}(v)\right)}_{=1 \otimes \iota_{\mathrm{Sym},V}(f(v))} \\
&= \iota_{\mathrm{Sym},V}(f(v)) \otimes 1 + 1 \otimes \iota_{\mathrm{Sym},V}(f(v)).
\end{aligned}$$

Compared with (27), this yields $\left(\Delta \circ \widetilde{f}\right)(s) = \left(\left(\widetilde{f} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \widetilde{f}\right) \circ \Delta\right)(s)$.

Now, let us forget that we fixed s. We thus have shown that $\left(\Delta \circ \widetilde{f}\right)(s) = \left(\left(\widetilde{f} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \widetilde{f}\right) \circ \Delta\right)(s)$ for every $s \in S$. In other words, $\left(\Delta \circ \widetilde{f}\right) \mid_S = \left(\left(\widetilde{f} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \widetilde{f}\right) \circ \Delta\right) \mid_S$. This proves (25).

$U(\mathfrak{g})$. (Here, the "U" in "$\pi_{U,\mathfrak{g}}$" is not a variable, but stands for the letter "u" in "**u**niversal enveloping algebra".)

As we know, $T(\mathfrak{g})$ is a graded $\mathbf{k}$-algebra, and thus a filtered $\mathbf{k}$-algebra. Its filtration is $\left(\mathfrak{g}^{\otimes \leq m}\right)_{m\geq 0}$, where we set

$$\mathfrak{g}^{\otimes \leq m} = \bigoplus_{n=0}^{m} \mathfrak{g}^{\otimes n} \qquad \text{for every } m \in \mathbb{N}.$$

This filtration $\left(\mathfrak{g}^{\otimes \leq m}\right)_{m\geq 0}$ on $T(\mathfrak{g})$ gives rise to a filtration $\left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes \leq m}\right)\right)_{m\geq 0}$ on the quotient algebra $U(\mathfrak{g})$. This latter filtration will be denoted by $\left(U_{\leq m}(\mathfrak{g})\right)_{m\geq 0}$ (thus, we set $U_{\leq m}(\mathfrak{g}) = \pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes \leq m}\right)$ for every $m \in \mathbb{N}$), and makes $U(\mathfrak{g})$ into a filtered $\mathbf{k}$-algebra.

If $\mathfrak{g}$ is an abelian Lie algebra (i.e., a Lie algebra satisfying $[v,w] = 0$ for all $v \in \mathfrak{g}$ and $w \in \mathfrak{g}$), then $U(\mathfrak{g}) = \operatorname{Sym}\mathfrak{g}$. However, in general, $U(\mathfrak{g})$ differs noticeably from $\operatorname{Sym}\mathfrak{g}$ (and is usually noncommutative, unlike $\operatorname{Sym}\mathfrak{g}$). Various results comparing $U(\mathfrak{g})$ to $\operatorname{Sym}\mathfrak{g}$ are collectively known as "Poincaré-Birkhoff-Witt theorems"; usually, they state (under certain requirements) either that $U(\mathfrak{g}) \cong \operatorname{Sym}\mathfrak{g}$ as $\mathbf{k}$-modules (or, sometimes, as $\mathfrak{g}$-modules when equipped with appropriate $\mathfrak{g}$-module structures; but almost never as $\mathbf{k}$-algebras), or that $\operatorname{gr}(U(\mathfrak{g})) \cong \operatorname{Sym}\mathfrak{g}$ as $\mathbf{k}$-algebras. See [Grinbe11, Theorem 5.9] for a collection of such "Poincaré-Birkhoff-Witt theorems". In this note, we shall explore one such theorem, which has a condition saying that the Lie algebra structure $\mathfrak{g}$ comes from a pre-Lie algebra structure. (We shall introduce the latter notion in due time.)

Let $\iota_{U,\mathfrak{g}}$ denote the map $\pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$. This map $\iota_{U,\mathfrak{g}}$ is often called the "canonical inclusion of $\mathfrak{g}$ into $U(\mathfrak{g})$", although in some (rather perverse) cases it fails to be injective. For every $a \in \mathfrak{g}$, we have

$$\underbrace{\iota_{U,\mathfrak{g}}}_{=\pi_{U,\mathfrak{g}}\circ\iota_{T,\mathfrak{g}}}(a) = \left(\pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}}\right)(a) = \pi_{U,\mathfrak{g}}\left(\underbrace{\iota_{T,\mathfrak{g}}(a)}_{=a}\right) = \pi_{U,\mathfrak{g}}(a). \qquad (28)$$

It is common to write a for this element $\iota_{U,\mathfrak{g}}(a) = \pi_{U,\mathfrak{g}}(a)$ of $U(\mathfrak{g})$, but this notation is slightly abusive (since, as explained above, $\iota_{U,\mathfrak{g}}$ is not always injective).

Let $\iota_{\operatorname{gr}U,\mathfrak{g}}$ denote the map $\mathfrak{g} \to \operatorname{gr}(U(\mathfrak{g}))$ which sends every $v \in \mathfrak{g}$ to $\left[\iota_{U,\mathfrak{g}}(v)\right]_1 \in \operatorname{gr}_1(U(\mathfrak{g})) \subseteq \operatorname{gr}(U(\mathfrak{g}))$.

Proposition 2.23. Let $\mathfrak{g}$ be a Lie algebra.

(a) The $\mathbf{k}$-algebra $U(\mathfrak{g})$ is generated by its subset $\iota_{U,\mathfrak{g}}(\mathfrak{g})$.

(b) The $\mathbf{k}$-algebra $\operatorname{gr}(U(\mathfrak{g}))$ is generated by its subset $\iota_{\operatorname{gr}U,\mathfrak{g}}(\mathfrak{g})$.

(c) The $\mathbf{k}$-algebra $\operatorname{gr}(U(\mathfrak{g}))$ is commutative.

(d) There exists a unique $\mathbf{k}$-algebra homomorphism $\Phi : \operatorname{Sym}\mathfrak{g} \to \operatorname{gr}(U(\mathfrak{g}))$

such that $\Phi \circ \iota_{\operatorname{Sym},\mathfrak{g}} = \iota_{\operatorname{gr} U,\mathfrak{g}}$. Let us denote this $\mathbf{k}$-algebra homomorphism Φ by $\operatorname{PBW}_{\mathfrak{g}}$, and call it the *PBW homomorphism* of $\mathfrak{g}$. (As usual, "PBW" stands for "Poincaré-Birkhoff-Witt".)

(e) The PBW homomorphism $\operatorname{PBW}_{\mathfrak{g}}$ is graded and surjective.

(f) Let $n \in \mathbb{N}$. The $\mathbf{k}$-module $\operatorname{gr}_n(U(\mathfrak{g}))$ is spanned by the elements $[\iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n$ with $a_1, a_2, \ldots, a_n \in V$.

Proposition 2.23 is well-known, but let us give a proof for the sake of completeness:

Proof of Proposition 2.23. **(a)** The $\mathbf{k}$-algebra $T(\mathfrak{g})$ is generated by its subset $\iota_{T,\mathfrak{g}}(\mathfrak{g})$ (since every tensor is a $\mathbf{k}$-linear combination of pure tensors). Hence, the $\mathbf{k}$-algebra $U(\mathfrak{g}) = \pi_{U,\mathfrak{g}}(T(\mathfrak{g}))$ is generated by its subset $\pi_{U,\mathfrak{g}}(\iota_{T,\mathfrak{g}}(\mathfrak{g})) = \underbrace{(\pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}})}_{=\iota_{U,\mathfrak{g}}}(\mathfrak{g}) = \iota_{U,\mathfrak{g}}(\mathfrak{g})$. This proves Proposition 2.23 **(a)**.

(b) The projection $\pi_{U,\mathfrak{g}} : T(\mathfrak{g}) \to U(\mathfrak{g})$ is filtered (due to the definition of the filtration on $U(\mathfrak{g})$). Hence, it gives rise to a graded $\mathbf{k}$-linear map $\operatorname{gr}(\pi_{U,\mathfrak{g}}) : \operatorname{gr}(T(\mathfrak{g})) \to \operatorname{gr}(U(\mathfrak{g}))$. This $\mathbf{k}$-linear map $\operatorname{gr}(\pi_{U,\mathfrak{g}})$ is a graded $\mathbf{k}$-algebra homomorphism (because of Remark 2.14). It is easy to see that

$$\operatorname{gr}_n(U(\mathfrak{g})) = (\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g}))) \qquad \text{for every } n \in \mathbb{N} \tag{29}$$

[17].

[17]*Proof of (29):* Let $n \in \mathbb{N}$. Let $p \in \operatorname{gr}_n(U(\mathfrak{g}))$. We shall show that $p \in (\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g})))$.

We have $p \in \operatorname{gr}_n(U(\mathfrak{g})) = \left(\pi_{U,\mathfrak{g}}(\mathfrak{g}^{\otimes \leq n})\right) / \left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes \leq (n-1)}\right)\right)$ (since the filtration on $U(\mathfrak{g})$ was defined to be $\left(\pi_{U,\mathfrak{g}}(\mathfrak{g}^{\otimes \leq m})\right)_{m \geq 0}$). Hence, we can write p in the form $p = [q]_n$ for some $q \in \pi_{U,\mathfrak{g}}(\mathfrak{g}^{\otimes \leq n})$. Consider this q. We have $q \in \pi_{U,\mathfrak{g}}(\mathfrak{g}^{\otimes \leq n})$. Thus, $q = \pi_{U,\mathfrak{g}}(Q)$ for some $Q \in \mathfrak{g}^{\otimes \leq n}$. Consider this Q.

We have $Q \in \mathfrak{g}^{\otimes \leq n} = \mathfrak{g}^{\otimes n} + \mathfrak{g}^{\otimes \leq (n-1)}$. Thus, $Q = Q_1 + Q_2$ for some $Q_1 \in \mathfrak{g}^{\otimes n}$ and $Q_2 \in \mathfrak{g}^{\otimes \leq (n-1)}$. Consider these Q_1 and Q_2. We have

$$\begin{aligned} q &= \pi_{U,\mathfrak{g}}\left(\underbrace{Q}_{=Q_1+Q_2}\right) = \pi_{U,\mathfrak{g}}(Q_1 + Q_2) = \pi_{U,\mathfrak{g}}(Q_1) + \pi_{U,\mathfrak{g}}\left(\underbrace{Q_2}_{\in \mathfrak{g}^{\otimes \leq (n-1)}}\right) \\ &\in \pi_{U,\mathfrak{g}}(Q_1) + \pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes \leq (n-1)}\right), \end{aligned}$$

so that $q \equiv \pi_{U,\mathfrak{g}}(Q_1) \bmod \pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes \leq (n-1)}\right)$. In other words, $[q]_n = [\pi_{U,\mathfrak{g}}(Q_1)]_n$.

On the other hand, consider the element $[Q_1]_n$ of $\mathfrak{g}^{\otimes \leq n} / \mathfrak{g}^{\otimes \leq (n-1)} = \operatorname{gr}_n(T(\mathfrak{g}))$. By the definition of $\operatorname{gr}(\pi_{U,\mathfrak{g}})$, we have $(\operatorname{gr}(\pi_{U,\mathfrak{g}}))([Q_1]_n) = [\pi_{U,\mathfrak{g}}(Q_1)]_n$. Comparing this with $[q]_n = [\pi_{U,\mathfrak{g}}(Q_1)]_n$, we obtain $[q]_n = (\operatorname{gr}(\pi_{U,\mathfrak{g}}))([Q_1]_n)$. Hence, $p = [q]_n = (\operatorname{gr}(\pi_{U,\mathfrak{g}}))\left(\underbrace{[Q_1]_n}_{\in \operatorname{gr}_n(T(\mathfrak{g}))}\right) \in (\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g})))$.

Let us now forget that we fixed p. We thus have proven that $p \in (\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g})))$ for every $p \in \operatorname{gr}_n(U(\mathfrak{g}))$. In other words, $\operatorname{gr}_n(U(\mathfrak{g})) \subseteq (\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g})))$. Combined

On the other hand, the $\mathbf{k}$-algebra $T(\mathfrak{g})$ is graded. Thus, $\operatorname{grad}_{T(\mathfrak{g})} : T(\mathfrak{g}) \to \operatorname{gr}(T(\mathfrak{g}))$ is a graded $\mathbf{k}$-algebra isomorphism (by Remark 2.15). Define a $\mathbf{k}$-linear map $\omega : T(\mathfrak{g}) \to \operatorname{gr}(U(\mathfrak{g}))$ by

$$\omega = \operatorname{gr}(\pi_{U,\mathfrak{g}}) \circ \operatorname{grad}_{T(\mathfrak{g})}.$$

Then, ω is a graded $\mathbf{k}$-algebra homomorphism[18]. For every $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathfrak{g}$, we have

$$\omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \left[\iota_{U,\mathfrak{g}}(a_1)\, \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n)\right]_n \qquad (30)$$

[19]. Furthermore, we have

$$\operatorname{gr}_n(U(\mathfrak{g})) = \omega\left(\mathfrak{g}^{\otimes n}\right) \qquad \text{for every } n \in \mathbb{N} \qquad (32)$$

[20].

Let Z be the $\mathbf{k}$-subalgebra of $\operatorname{gr}(U(\mathfrak{g}))$ generated by $\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$. Our goal is to prove that the $\mathbf{k}$-algebra $\operatorname{gr}(U(\mathfrak{g}))$ is generated by its subset $\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$. In other words, our goal is to prove that $\operatorname{gr}(U(\mathfrak{g})) = Z$.

---

with the inclusion $(\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g}))) \subseteq \operatorname{gr}_n(U(\mathfrak{g}))$ (which is obvious, since the map $\operatorname{gr}(\pi_{U,\mathfrak{g}})$ is graded), this results in $\operatorname{gr}_n(U(\mathfrak{g})) = (\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g})))$. This proves (29).

[18] since it is the composition of the graded $\mathbf{k}$-algebra homomorphisms $\operatorname{gr}(\pi_{U,\mathfrak{g}})$ and $\operatorname{grad}_{T(\mathfrak{g})}$

[19] *Proof of (30):* Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathfrak{g}$. Then, $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \mathfrak{g}^{\otimes n}$. Hence, the definition of $\operatorname{grad}_{T(\mathfrak{g})}$ yields $\operatorname{grad}_{T(\mathfrak{g})}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = [a_1 \otimes a_2 \otimes \cdots \otimes a_n]_n$. Now, from $\omega = \operatorname{gr}(\pi_{U,\mathfrak{g}}) \circ \operatorname{grad}_{T(\mathfrak{g})}$, we obtain

$$\begin{aligned}\omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n) &= (\operatorname{gr}(\pi_{U,\mathfrak{g}}))\left(\underbrace{\operatorname{grad}_{T(\mathfrak{g})}(a_1 \otimes a_2 \otimes \cdots \otimes a_n)}_{=[a_1 \otimes a_2 \otimes \cdots \otimes a_n]_n}\right) \\ &= (\operatorname{gr}(\pi_{U,\mathfrak{g}}))\left([a_1 \otimes a_2 \otimes \cdots \otimes a_n]_n\right) = \left[\pi_{U,\mathfrak{g}}(a_1 \otimes a_2 \otimes \cdots \otimes a_n)\right]_n \qquad (31)\end{aligned}$$

(by the definition of $\operatorname{gr}(\pi_{U,\mathfrak{g}})$).

But (28) (applied to $a = a_i$ for each $i \in \{1, 2, \ldots, n\}$, and then multiplied) shows that

$$\begin{aligned}\iota_{U,\mathfrak{g}}(a_1)\, \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n) &= \pi_{U,\mathfrak{g}}(a_1)\, \pi_{U,\mathfrak{g}}(a_2) \cdots \pi_{U,\mathfrak{g}}(a_n) \\ &= \pi_{U,\mathfrak{g}}\underbrace{(a_1 a_2 \cdots a_n)}_{=a_1 \otimes a_2 \otimes \cdots \otimes a_n} \qquad (\text{since } \pi_{U,\mathfrak{g}} \text{ is a } \mathbf{k}\text{-algebra homomorphism}) \\ &= \pi_{U,\mathfrak{g}}(a_1 \otimes a_2 \otimes \cdots \otimes a_n).\end{aligned}$$

Hence, (31) becomes

$$\omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \left[\underbrace{\pi_{U,\mathfrak{g}}(a_1 \otimes a_2 \otimes \cdots \otimes a_n)}_{=\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)}\right]_n = \left[\iota_{U,\mathfrak{g}}(a_1)\, \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n)\right]_n.$$

This proves (30).

[20] *Proof of (32):* Let $n \in \mathbb{N}$. The map $\operatorname{grad}_{T(\mathfrak{g})}$ is a graded $\mathbf{k}$-algebra isomorphism. Thus, it is invertible, and its inverse is again graded. Hence, $\operatorname{grad}_{T(\mathfrak{g})}(\mathfrak{g}^{\otimes n}) = \operatorname{gr}_n(T(\mathfrak{g}))$.

Clearly, $\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g}) \subseteq Z$ (since Z is the $\mathbf{k}$-subalgebra of $\operatorname{gr}(U(\mathfrak{g}))$ generated by $\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$).

Let $n \in \mathbb{N}$. For every $a \in \mathfrak{g}$, we have $\iota_{\operatorname{gr} U,\mathfrak{g}}(a) = [\iota_{U,\mathfrak{g}}(a)]_1$ (by the definition of $\iota_{\operatorname{gr} U,\mathfrak{g}}$). Thus, for every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$, we have

$$\begin{aligned}
&\underbrace{\iota_{\operatorname{gr} U,\mathfrak{g}}(a_1)}_{=[\iota_{U,\mathfrak{g}}(a_1)]_1}\underbrace{\iota_{\operatorname{gr} U,\mathfrak{g}}(a_2)}_{=[\iota_{U,\mathfrak{g}}(a_2)]_1}\cdots\underbrace{\iota_{\operatorname{gr} U,\mathfrak{g}}(a_n)}_{=[\iota_{U,\mathfrak{g}}(a_n)]_1} \\
&= [\iota_{U,\mathfrak{g}}(a_1)]_1 [\iota_{U,\mathfrak{g}}(a_2)]_1 \cdots [\iota_{U,\mathfrak{g}}(a_n)]_1 = [\iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n \\
&\qquad \text{(by the definition of multiplication on } \operatorname{gr}(U(\mathfrak{g}))\text{)} \\
&= \omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \qquad \text{(by (30))}. \qquad (33)
\end{aligned}$$

Hence, for every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$, we have

$$\begin{aligned}
\omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n) &= \iota_{\operatorname{gr} U,\mathfrak{g}}\Bigg(\underbrace{a_1}_{\in\mathfrak{g}}\Bigg)\iota_{\operatorname{gr} U,\mathfrak{g}}\Bigg(\underbrace{a_2}_{\in\mathfrak{g}}\Bigg)\cdots\iota_{\operatorname{gr} U,\mathfrak{g}}\Bigg(\underbrace{a_n}_{\in\mathfrak{g}}\Bigg) \\
&\in \underbrace{\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})\,\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})\cdots\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})}_{n \text{ times}} = \Bigg(\underbrace{\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})}_{\subseteq Z}\Bigg)^n \subseteq Z^n \subseteq Z
\end{aligned} \qquad (34)$$

(since Z is a $\mathbf{k}$-subalgebra).

Now, the $\mathbf{k}$-module $\mathfrak{g}^{\otimes n}$ is spanned by the pure tensors $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$. Hence, its image $\omega(\mathfrak{g}^{\otimes n})$ under the map ω is spanned by the elements $\omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$. Since all of these elements $\omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$ belong to Z (according to (34)), this shows that $\omega(\mathfrak{g}^{\otimes n}) \subseteq Z$.

Now, let us forget that we fixed n. We thus have proven that $\omega(\mathfrak{g}^{\otimes n}) \subseteq Z$ for every $n \in \mathbb{N}$. Now,

$$\operatorname{gr}(U(\mathfrak{g})) = \bigoplus_{n\geq 0}\operatorname{gr}_n(U(\mathfrak{g})) = \sum_{n\geq 0}\underbrace{\operatorname{gr}_n(U(\mathfrak{g}))}_{\substack{=\omega(\mathfrak{g}^{\otimes n}) \\ \text{(by (32))}}} = \sum_{n\geq 0}\underbrace{\omega(\mathfrak{g}^{\otimes n})}_{\subseteq Z} \subseteq \sum_{n\geq 0} Z \subseteq Z.$$

Combined with $Z \subseteq \operatorname{gr}(U(\mathfrak{g}))$ (which is obvious), this shows that $\operatorname{gr}(U(\mathfrak{g})) = Z$. This completes the proof of Proposition 2.23 **(b)**.

Now, $\omega = \operatorname{gr}(\pi_{U,\mathfrak{g}}) \circ \operatorname{grad}_{T(\mathfrak{g})}$, so that

$$\begin{aligned}
\omega(\mathfrak{g}^{\otimes n}) &= \left(\operatorname{gr}(\pi_{U,\mathfrak{g}}) \circ \operatorname{grad}_{T(\mathfrak{g})}\right)(\mathfrak{g}^{\otimes n}) = (\operatorname{gr}(\pi_{U,\mathfrak{g}}))\Bigg(\underbrace{\operatorname{grad}_{T(\mathfrak{g})}(\mathfrak{g}^{\otimes n})}_{=\operatorname{gr}_n(T(\mathfrak{g}))}\Bigg) = (\operatorname{gr}(\pi_{U,\mathfrak{g}}))(\operatorname{gr}_n(T(\mathfrak{g}))) \\
&= \operatorname{gr}_n(U(\mathfrak{g})) \qquad \text{(by (29))}.
\end{aligned}$$

This proves (32).

(c) It is known that a $\mathbf{k}$-algebra is commutative if it has a generating set whose elements commute pairwise.

Now, it is easy to see that the elements of the subset $\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$ of $\operatorname{gr}(U(\mathfrak{g}))$ commute pairwise[21]. Thus the $\mathbf{k}$-algebra $\operatorname{gr}(U(\mathfrak{g}))$ has a generating set whose elements commute pairwise (namely, the set $\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$, because Proposition 2.23 **(b)** shows that this set is a generating set for $\operatorname{gr}(U(\mathfrak{g}))$). Therefore, the $\mathbf{k}$-algebra $\operatorname{gr}(U(\mathfrak{g}))$ is commutative (since a $\mathbf{k}$-algebra is commutative if it has a generating set whose elements pairwise commute). This proves Proposition 2.23 **(c)**.

(d) Proposition 2.23 **(c)** shows that the $\mathbf{k}$-algebra $\operatorname{gr}(U(\mathfrak{g}))$ is commutative. Hence, Proposition 2.19 (applied to $V=\mathfrak{g}$, $A=\operatorname{gr}(U(\mathfrak{g}))$ and $f=\iota_{\operatorname{gr} U,\mathfrak{g}}$) shows that there exists a unique $\mathbf{k}$-algebra homomorphism $F:\operatorname{Sym}\mathfrak{g}\to\operatorname{gr}(U(\mathfrak{g}))$ such that $F\circ\iota_{\operatorname{Sym},\mathfrak{g}}=\iota_{\operatorname{gr} U,\mathfrak{g}}$. Renaming F as Φ in this statement, we obtain precisely

[21] *Proof.* Let $x\in\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$ and $y\in\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$. We must prove that $xy=yx$.

We have $x\in\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$. In other words, there exists $a\in\mathfrak{g}$ such that $x=\iota_{\operatorname{gr} U,\mathfrak{g}}(a)$. Consider this a.

We have $y\in\iota_{\operatorname{gr} U,\mathfrak{g}}(\mathfrak{g})$. In other words, there exists $b\in\mathfrak{g}$ such that $y=\iota_{\operatorname{gr} U,\mathfrak{g}}(b)$. Consider this b.

- The element $a\otimes b-b\otimes a-[a,b]$ of $T(\mathfrak{g})$ belongs to the ideal $J_{\mathfrak{g}}$ defined in Definition 2.22. Since $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ modulo this ideal, this shows that $\pi_{U,\mathfrak{g}}(a\otimes b-b\otimes a-[a,b])=0$. Hence,

$$\begin{aligned}
0&=\pi_{U,\mathfrak{g}}\left(\underbrace{a\otimes b}_{=ab}-\underbrace{b\otimes a}_{=ba}-[a,b]\right)\\
&=\pi_{U,\mathfrak{g}}(ab-ba-[a,b])=\underbrace{\pi_{U,\mathfrak{g}}(a)}_{=\iota_{U,\mathfrak{g}}(a)}\underbrace{\pi_{U,\mathfrak{g}}(b)}_{=\iota_{U,\mathfrak{g}}(b)}-\underbrace{\pi_{U,\mathfrak{g}}(b)}_{=\iota_{U,\mathfrak{g}}(b)}\underbrace{\pi_{U,\mathfrak{g}}(a)}_{=\iota_{U,\mathfrak{g}}(a)}-\pi_{U,\mathfrak{g}}([a,b])\\
&\qquad(\text{since }\pi_{U,\mathfrak{g}}\text{ is a }\mathbf{k}\text{-algebra homomorphism})\\
&=\iota_{U,\mathfrak{g}}(a)\,\iota_{U,\mathfrak{g}}(b)-\iota_{U,\mathfrak{g}}(b)\,\iota_{U,\mathfrak{g}}(a)-\pi_{U,\mathfrak{g}}([a,b]).
\end{aligned}$$

  Thus,

$$\iota_{U,\mathfrak{g}}(a)\,\iota_{U,\mathfrak{g}}(b)-\iota_{U,\mathfrak{g}}(b)\,\iota_{U,\mathfrak{g}}(a)=\pi_{U,\mathfrak{g}}\left(\underbrace{[a,b]}_{\in\mathfrak{g}^{\otimes 1}\subseteq\mathfrak{g}^{\otimes\leq 1}}\right)\in\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes\leq 1}\right). \tag{35}$$

We have $x=\iota_{\operatorname{gr} U,\mathfrak{g}}(a)=[\iota_{U,\mathfrak{g}}(a)]_1$ (by the definition of $\iota_{\operatorname{gr} U,\mathfrak{g}}$) and $y=[\iota_{U,\mathfrak{g}}(b)]_1$ (similarly). Hence,

$$\begin{aligned}
&\underbrace{x}_{=[\iota_{U,\mathfrak{g}}(a)]_1}\underbrace{y}_{=[\iota_{U,\mathfrak{g}}(b)]_1}-\underbrace{y}_{=[\iota_{U,\mathfrak{g}}(b)]_1}\underbrace{x}_{=[\iota_{U,\mathfrak{g}}(a)]_1}\\
&=\underbrace{[\iota_{U,\mathfrak{g}}(a)]_1[\iota_{U,\mathfrak{g}}(b)]_1}_{\substack{=[\iota_{U,\mathfrak{g}}(a)\iota_{U,\mathfrak{g}}(b)]_2\\ (\text{by the definition of the product in }\operatorname{gr}(U(\mathfrak{g})))}}-\underbrace{[\iota_{U,\mathfrak{g}}(b)]_1[\iota_{U,\mathfrak{g}}(a)]_1}_{\substack{=[\iota_{U,\mathfrak{g}}(b)\iota_{U,\mathfrak{g}}(a)]_2\\ (\text{by the definition of the product in }\operatorname{gr}(U(\mathfrak{g})))}}\\
&=[\iota_{U,\mathfrak{g}}(a)\,\iota_{U,\mathfrak{g}}(b)]_2-[\iota_{U,\mathfrak{g}}(b)\,\iota_{U,\mathfrak{g}}(a)]_2=[\iota_{U,\mathfrak{g}}(a)\,\iota_{U,\mathfrak{g}}(b)-\iota_{U,\mathfrak{g}}(b)\,\iota_{U,\mathfrak{g}}(a)]_2=0
\end{aligned}$$

(by (35)). In other words, $xy=yx$, qed.

Proposition 2.23 **(d)**.

(e) The definition of $\mathrm{PBW}_{\mathfrak{g}}$ shows that $\mathrm{PBW}_{\mathfrak{g}} \circ \iota_{\mathrm{Sym},\mathfrak{g}} = \iota_{\mathrm{gr}\,U,\mathfrak{g}}$. Now let $n \in \mathbb{N}$.

Consider the map $\omega : T(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ defined in our above proof of Proposition 2.23 **(c)**. We recall the facts (which we have proven in the proof of Proposition 2.23 **(c)**):

- We have $\omega = \mathrm{gr}(\pi_{U,\mathfrak{g}}) \circ \mathrm{grad}_{T(\mathfrak{g})}$.
- The map ω is a graded $\mathbf{k}$-algebra homomorphism.
- We have $\mathrm{gr}_n(U(\mathfrak{g})) = \omega(\mathfrak{g}^{\otimes n})$ for every $n \in \mathbb{N}$.

We have $\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}} = \omega$ [22]. Now, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathrm{gr}_n\left(U\left(\mathfrak{g}\right)\right) &= \underbrace{\omega}_{=\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}} \left(\mathfrak{g}^{\otimes n}\right) = \left(\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}\right)\left(\mathfrak{g}^{\otimes n}\right) \\ &= \mathrm{PBW}_{\mathfrak{g}} \underbrace{\left(\pi_{\mathrm{Sym},\mathfrak{g}}\left(\mathfrak{g}^{\otimes n}\right)\right)}_{=\mathrm{Sym}^n \mathfrak{g}} = \mathrm{PBW}_{\mathfrak{g}}\left(\mathrm{Sym}^n \mathfrak{g}\right).\end{aligned}$$

Thus, in particular, we have $\mathrm{PBW}_{\mathfrak{g}}\left(\mathrm{Sym}^n \mathfrak{g}\right) \subseteq \mathrm{gr}_n\left(U\left(\mathfrak{g}\right)\right)$ for every $n \in \mathbb{N}$.

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[22] *Proof.* It is well-known that the tensor algebra $T\left(\mathfrak{g}\right)$ is generated by $\mathfrak{g}^{\otimes 1}$ (as a $\mathbf{k}$-algebra).

Now, $\mathrm{PBW}_{\mathfrak{g}}$ and $\pi_{\mathrm{Sym},\mathfrak{g}}$ are $\mathbf{k}$-algebra homomorphisms. Thus, their composition $\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}$ is a $\mathbf{k}$-algebra homomorphism.

Let $w \in \mathfrak{g}^{\otimes 1}$. Then, $w \in \mathfrak{g}^{\otimes 1} = \iota_{T,\mathfrak{g}}\left(\mathfrak{g}\right)$. Hence, $w = \iota_{T,\mathfrak{g}}\left(v\right)$ for some $v \in \mathfrak{g}$. Consider this v. Comparing

$$\begin{aligned}&\left(\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}\right)\left(\underbrace{w}_{=\iota_{T,\mathfrak{g}}(v)}\right) \\ &= \left(\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}\right)\left(\iota_{T,\mathfrak{g}}\left(v\right)\right) = \left(\mathrm{PBW}_{\mathfrak{g}} \circ \underbrace{\pi_{\mathrm{Sym},\mathfrak{g}} \circ \iota_{T,\mathfrak{g}}}_{=\iota_{\mathrm{Sym},\mathfrak{g}}}\right)\left(v\right) \\ &= \left(\underbrace{\mathrm{PBW}_{\mathfrak{g}} \circ \iota_{\mathrm{Sym},\mathfrak{g}}}_{=\iota_{\mathrm{gr}\,U,\mathfrak{g}}}\right)\left(v\right) = \iota_{\mathrm{gr}\,U,\mathfrak{g}}\left(v\right) = \left[\underbrace{\iota_{U,\mathfrak{g}}}_{=\pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}}}\left(v\right)\right]_1 \qquad \left(\text{by the definition of } \iota_{\mathrm{gr}\,U,\mathfrak{g}}\left(v\right)\right) \\ &= \left[\left(\pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}}\right)\left(v\right)\right]_1 = \left[\pi_{U,\mathfrak{g}}\left(\iota_{T,\mathfrak{g}}\left(v\right)\right)\right]_1\end{aligned}$$

with

$$\begin{aligned}&\underbrace{\omega}_{=\mathrm{gr}\left(\pi_{U,\mathfrak{g}}\right) \circ \mathrm{grad}_{T(\mathfrak{g})}}\left(\underbrace{w}_{=\iota_{T,\mathfrak{g}}(v)}\right) = \left(\mathrm{gr}\left(\pi_{U,\mathfrak{g}}\right) \circ \mathrm{grad}_{T(\mathfrak{g})}\right)\left(\iota_{T,\mathfrak{g}}\left(v\right)\right) = \left(\mathrm{gr}\left(\pi_{U,\mathfrak{g}}\right)\right)\left(\underbrace{\mathrm{grad}_{T(\mathfrak{g})}\left(\iota_{T,\mathfrak{g}}\left(v\right)\right)}_{\substack{=\left[\iota_{T,\mathfrak{g}}(v)\right]_1 \\ \text{(by the definition of } \mathrm{grad}_{T(\mathfrak{g})}, \\ \text{since } \iota_{T,\mathfrak{g}}(v) \in \mathfrak{g}^{\otimes 1} \subseteq \mathfrak{g}^{\otimes \leq 1})}}\right) \\ &= \left(\mathrm{gr}\left(\pi_{U,\mathfrak{g}}\right)\right)\left(\left[\iota_{T,\mathfrak{g}}\left(v\right)\right]_1\right) = \left[\pi_{U,\mathfrak{g}}\left(\iota_{T,\mathfrak{g}}\left(v\right)\right)\right]_1,\end{aligned}$$

we obtain $\left(\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}\right)\left(w\right) = \omega\left(w\right)$.

Now let us forget that we fixed w. We thus have shown that $\left(\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}\right)\left(w\right) = \omega\left(w\right)$ for every $w \in \mathfrak{g}^{\otimes 1}$. In other words, the two maps $\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}$ and ω are equal to each other on every element of $\mathfrak{g}^{\otimes 1}$. Since the $\mathbf{k}$-algebra $T\left(\mathfrak{g}\right)$ is generated by $\mathfrak{g}^{\otimes 1}$, this shows that the two maps $\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}}$ and ω must be identical (since these two maps are $\mathbf{k}$-algebra homomorphisms). In other words, $\mathrm{PBW}_{\mathfrak{g}} \circ \pi_{\mathrm{Sym},\mathfrak{g}} = \omega$, qed.

Hence, the map $\operatorname{PBW}_{\mathfrak{g}}$ is graded. Moreover,

$$\begin{aligned} \operatorname{gr}(U(\mathfrak{g})) &= \underbrace{\bigoplus_{n\in\mathbb{N}}}_{=\sum_{n\in\mathbb{N}}} \underbrace{\operatorname{gr}_n(U(\mathfrak{g}))}_{=\operatorname{PBW}_{\mathfrak{g}}(\operatorname{Sym}^n\mathfrak{g})} = \sum_{n\in\mathbb{N}} \operatorname{PBW}_{\mathfrak{g}}(\operatorname{Sym}^n\mathfrak{g}) \\ &= \operatorname{PBW}_{\mathfrak{g}}\left(\underbrace{\sum_{n\in\mathbb{N}} \operatorname{Sym}^n\mathfrak{g}}_{=\operatorname{Sym}\mathfrak{g}}\right) = \operatorname{PBW}_{\mathfrak{g}}(\operatorname{Sym}\mathfrak{g}). \end{aligned}$$

Hence, the map $\operatorname{PBW}_{\mathfrak{g}}$ is surjective. This proves Proposition 2.23 **(e)**.

(f) Consider the map $\omega : T(\mathfrak{g}) \to \operatorname{gr}(U(\mathfrak{g}))$ defined in our above proof of Proposition 2.23 **(c)**. We recall the facts (which we have proven in the proof of Proposition 2.23 **(c)**):

- We have $\omega = \operatorname{gr}(\pi_{U,\mathfrak{g}}) \circ \operatorname{grad}_{T(\mathfrak{g})}$.
- The map ω is a graded $\mathbf{k}$-algebra homomorphism.
- For every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$, we have

$$\iota_{\operatorname{gr}U,\mathfrak{g}}(a_1)\,\iota_{\operatorname{gr}U,\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{gr}U,\mathfrak{g}}(a_n) = \omega(a_1\otimes a_2\otimes\cdots\otimes a_n). \tag{36}$$

- We have $\operatorname{gr}_n(U(\mathfrak{g})) = \omega(\mathfrak{g}^{\otimes n})$.

Now, the $\mathbf{k}$-module $\mathfrak{g}^{\otimes n}$ is spanned by pure tensors. In other words, the $\mathbf{k}$-module $\mathfrak{g}^{\otimes n}$ is spanned by the elements $a_1\otimes a_2\otimes\cdots\otimes a_n$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$. Thus, the $\mathbf{k}$-module $\omega(\mathfrak{g}^{\otimes n})$ is spanned by the elements $\omega(a_1\otimes a_2\otimes\cdots\otimes a_n)$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ (since the map ω is $\mathbf{k}$-linear). In other words, the $\mathbf{k}$-module $\operatorname{gr}_n(U(\mathfrak{g}))$ is spanned by the elements $\iota_{\operatorname{Sym},\mathfrak{g}}(a_1)\,\iota_{\operatorname{Sym},\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}(a_n)$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ (because $\omega(\mathfrak{g}^{\otimes n}) = \operatorname{gr}_n(U(\mathfrak{g}))$ and because every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ satisfy $\omega(a_1\otimes a_2\otimes\cdots\otimes a_n) = \iota_{\operatorname{gr}U,\mathfrak{g}}(a_1)\,\iota_{\operatorname{gr}U,\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{gr}U,\mathfrak{g}}(a_n)$). This proves Proposition 2.23 **(f)**. □

Nothing in Proposition 2.23 states that the map $\operatorname{PBW}_{\mathfrak{g}}$ is bijective. This is, in fact, not true in general; and when it is true, the proof is usually not trivial. One case where it is true is when $\mathfrak{g}$ is a free $\mathbf{k}$-module; another is when $\mathbf{k}$ is a $\mathbb{Q}$-algebra. See [Grinbe11, Theorem 5.9] for a list of such results; they are collectively known as the "Poincaré-Birkhoff-Witt theorems" (although the most commonly encountered one is the one where $\mathfrak{g}$ is assumed to be a free $\mathbf{k}$-algebra). We shall derive another criterion for $\operatorname{PBW}_{\mathfrak{g}}$ to be bijective. First, we state a criterion which (on its own) is not very interesting, but will serve as a lemma:

Lemma 2.24. Let $\mathfrak{g}$ be a Lie algebra. Consider the graded $\mathbf{k}$-algebra homomorphism $\operatorname{PBW}_{\mathfrak{g}}$ constructed in Proposition 2.23.

Let $\phi : U(\mathfrak{g}) \to \operatorname{Sym} \mathfrak{g}$ be a $\mathbf{k}$-linear map. Assume that

$$\begin{aligned} &\phi\left(\iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)\right) \\ &\in \iota_{\operatorname{Sym},\mathfrak{g}}(a_1)\,\iota_{\operatorname{Sym},\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}(a_n) + \sum_{k=0}^{n-1} \operatorname{Sym}^k \mathfrak{g} \end{aligned} \tag{37}$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$. Then:

(a) The map ϕ is a filtered $\mathbf{k}$-module isomorphism.

(b) The inverse ϕ^{-1} of the map ϕ is filtered.

(c) The map $\operatorname{grad}_{\operatorname{Sym}\mathfrak{g}}^{-1} \circ (\operatorname{gr}\phi) : \operatorname{gr}(U(\mathfrak{g})) \to \operatorname{Sym}\mathfrak{g}$ is an inverse to the map $\operatorname{PBW}_{\mathfrak{g}}$. In particular, the map $\operatorname{PBW}_{\mathfrak{g}}$ is invertible.

Proof of Lemma 2.24. We have

$$\phi\left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes n}\right)\right) \subseteq \sum_{k=0}^{n} \operatorname{Sym}^k \mathfrak{g} \qquad \text{for every } n \in \mathbb{N} \tag{38}$$

[23]. Therefore, the map ϕ is filtered[24]. Hence, the graded $\mathbf{k}$-linear map $\operatorname{gr}\phi : \operatorname{gr}(U(\mathfrak{g})) \to \operatorname{gr}(\operatorname{Sym}\mathfrak{g})$ is well-defined.

(c) Let η denote the map $\operatorname{grad}^{-1}_{\operatorname{Sym}\mathfrak{g}} \circ (\operatorname{gr}\phi) : \operatorname{gr}(U(\mathfrak{g})) \to \operatorname{gr}(\operatorname{Sym}\mathfrak{g})$. Clearly, η is a $\mathbf{k}$-linear map.

---

[23]*Proof of (38):* Let $n \in \mathbb{N}$. We must prove that $\phi(\pi_{U,\mathfrak{g}}(v)) \in \sum_{k=0}^{n} \operatorname{Sym}^k \mathfrak{g}$ for every $v \in \mathfrak{g}^{\otimes n}$. So let $v \in \mathfrak{g}^{\otimes n}$. Since this claim is $\mathbf{k}$-linear in v, we can WLOG assume that v is a pure tensor (since the tensor power $\mathfrak{g}^{\otimes n}$ is spanned by pure tensors). Assume this. Thus, $v = a_1 \otimes a_2 \otimes \cdots \otimes a_n$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$. Consider these $a_1, a_2, \ldots, a_n$.

The definition of the product on $T(\mathfrak{g})$ shows that the element $\iota_{T,\mathfrak{g}}(a_1)\,\iota_{T,\mathfrak{g}}(a_2)\cdots\iota_{T,\mathfrak{g}}(a_n)$ equals the tensor $a_1 \otimes a_2 \otimes \cdots \otimes a_n$. Thus, $\iota_{T,\mathfrak{g}}(a_1)\,\iota_{T,\mathfrak{g}}(a_2)\cdots\iota_{T,\mathfrak{g}}(a_n) = a_1 \otimes a_2 \otimes \cdots \otimes a_n = v$. Hence,

$$\begin{aligned}
\pi_{U,\mathfrak{g}}\left(\underbrace{v}_{=\iota_{T,\mathfrak{g}}(a_1)\iota_{T,\mathfrak{g}}(a_2)\cdots\iota_{T,\mathfrak{g}}(a_n)}\right) &= \pi_{U,\mathfrak{g}}\left(\iota_{T,\mathfrak{g}}(a_1)\,\iota_{T,\mathfrak{g}}(a_2)\cdots\iota_{T,\mathfrak{g}}(a_n)\right)\\
&= \pi_{U,\mathfrak{g}}\left(\iota_{T,\mathfrak{g}}(a_1)\right)\pi_{U,\mathfrak{g}}\left(\iota_{T,\mathfrak{g}}(a_2)\right)\cdots\pi_{U,\mathfrak{g}}\left(\iota_{T,\mathfrak{g}}(a_n)\right)\\
&\qquad (\text{since } \pi_{U,\mathfrak{g}} \text{ is a } \mathbf{k}\text{-algebra homomorphism})\\
&= \iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)
\end{aligned}$$

(because every $p \in \{1, 2, \ldots, n\}$ satisfies $\pi_{U,\mathfrak{g}}\left(\iota_{T,\mathfrak{g}}(a_p)\right) = \underbrace{\left(\pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}}\right)}_{=\iota_{U,\mathfrak{g}}}(a_p) = \iota_{U,\mathfrak{g}}(a_p)$).

Applying ϕ to both sides of this equality, we obtain

$$\begin{aligned}
\phi(\pi_{U,\mathfrak{g}}(v)) &= \phi\left(\iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)\right)\\
&\in \underbrace{\iota_{\operatorname{Sym},\mathfrak{g}}(a_1)\,\iota_{\operatorname{Sym},\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}(a_n)}_{\substack{\in\left(\operatorname{Sym}^1\mathfrak{g}\right)^n\\ (\text{since } \iota_{\operatorname{Sym},\mathfrak{g}}(a_p)\in\operatorname{Sym}^1\mathfrak{g} \text{ for every } p\in\{1,2,\ldots,n\})}} + \sum_{k=0}^{n-1}\operatorname{Sym}^k\mathfrak{g} \qquad (\text{by (37)})\\
&\subseteq \underbrace{\left(\operatorname{Sym}^1\mathfrak{g}\right)^n}_{\substack{\subseteq\operatorname{Sym}^n\mathfrak{g}\\ (\text{since } \operatorname{Sym}\mathfrak{g} \text{ is a graded}\\ \mathbf{k}\text{-algebra})}} + \sum_{k=0}^{n-1}\operatorname{Sym}^k\mathfrak{g} = \operatorname{Sym}^n\mathfrak{g} + \sum_{k=0}^{n-1}\operatorname{Sym}^k\mathfrak{g} \subseteq \sum_{k=0}^{n}\operatorname{Sym}^k\mathfrak{g}.
\end{aligned}$$

This is precisely what we wanted to prove. Thus, (38) is proven.

[24]*Proof.* Let $m \in \mathbb{N}$. Then, $\mathfrak{g}^{\otimes \leq m} = \sum_{n=0}^{m} \mathfrak{g}^{\otimes n}$ (by the definition of $\mathfrak{g}^{\otimes \leq m}$), so that

$$\begin{aligned}
\phi\left(\pi_{U,\mathfrak{g}}\left(\underbrace{\mathfrak{g}^{\otimes\leq m}}_{=\sum_{n=0}^{m}\mathfrak{g}^{\otimes n}}\right)\right) &= \phi\left(\pi_{U,\mathfrak{g}}\left(\sum_{n=0}^{m}\mathfrak{g}^{\otimes n}\right)\right) = \sum_{n=0}^{m}\underbrace{\phi\left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes n}\right)\right)}_{\substack{\subseteq\sum_{k=0}^{n}\operatorname{Sym}^k\mathfrak{g}\\ (\text{by (38)})}}\\
&\subseteq \sum_{n=0}^{m}\sum_{k=0}^{n}\operatorname{Sym}^k\mathfrak{g} = \sum_{k=0}^{m}\underbrace{\sum_{n=k}^{m}\operatorname{Sym}^k\mathfrak{g}}_{\subseteq\operatorname{Sym}^k\mathfrak{g}} \subseteq \sum_{k=0}^{m}\operatorname{Sym}^k\mathfrak{g}.
\end{aligned}$$

We have

$$\left(\operatorname{gr}\phi\right)\left(\left[\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\right]_n\right)=\left[\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right)\right]_n \tag{39}$$

for every $n\in\mathbb{N}$ and every $a_1,a_2,\ldots,a_n\in\mathfrak{g}$ [25]. Hence,

$$\eta\left(\left[\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\right]_n\right)=\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right) \tag{40}$$

for every $n\in\mathbb{N}$ and every $a_1,a_2,\ldots,a_n\in\mathfrak{g}$ [26].

We have

$$\operatorname{PBW}_{\mathfrak{g}}\left(\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right)\right)=\left[\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\right]_n \tag{42}$$

Now, let us forget that we fixed m. Thus we have shown that $\phi\left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes\leq m}\right)\right)\subseteq\sum_{k=0}^{m}\operatorname{Sym}^k\mathfrak{g}$ for every $m\in\mathbb{N}$. In other words, the map ϕ is filtered (since $\left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes\leq m}\right)\right)_{m\geq0}$ is the filtration on $U\left(\mathfrak{g}\right)$, and since $\left(\sum_{k=0}^{m}\operatorname{Sym}^k\mathfrak{g}\right)_{m\geq0}$ is the filtration on $\operatorname{Sym}\mathfrak{g}$). Qed.

[25] *Proof of (39):* Let $n\in\mathbb{N}$ and $a_1,a_2,\ldots,a_n\in\mathfrak{g}$. We have $\underbrace{\iota_{U,\mathfrak{g}}}_{=\pi_{U,\mathfrak{g}}\circ\iota_{T,\mathfrak{g}}}\left(a_p\right)=\left(\pi_{U,\mathfrak{g}}\circ\iota_{T,\mathfrak{g}}\right)\left(a_p\right)=\pi_{U,\mathfrak{g}}\left(\underbrace{\iota_{T,\mathfrak{g}}\left(a_p\right)}_{\in\mathfrak{g}^{\otimes1}\subseteq\mathfrak{g}^{\otimes\leq1}}\right)\in\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes\leq1}\right)$ for every $p\in\{1,2,\ldots,n\}$. Hence,

$$\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\in\left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes\leq1}\right)\right)^n\subseteq\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes\leq n}\right)$$

(since $\left(\pi_{U,\mathfrak{g}}\left(\mathfrak{g}^{\otimes\leq m}\right)\right)_{m\geq0}$ is a filtration of the $\mathbf{k}$-algebra $U\left(\mathfrak{g}\right)$). Thus, $\left[\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\right]_n$ is a well-defined element of $\operatorname{gr}_n\left(U\left(\mathfrak{g}\right)\right)$. The definition of $\operatorname{gr}\phi$ therefore yields

$$\begin{aligned}&\left(\operatorname{gr}\phi\right)\left(\left[\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\right]_n\right)\\&=\left[\phi\left(\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\right)\right]_n=\left[\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right)\right]_n\end{aligned}$$

(because of (37), since $\left(\sum_{k=0}^{m}\operatorname{Sym}^k\mathfrak{g}\right)_{m\geq0}$ is the filtration on $\operatorname{Sym}\mathfrak{g}$). This proves (39).

[26] *Proof of (40):* Let $n\in\mathbb{N}$ and $a_1,a_2,\ldots,a_n\in\mathfrak{g}$. We have $\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_p\right)\in\operatorname{Sym}^1\mathfrak{g}$ for every $p\in\{1,2,\ldots,n\}$. Hence,

$$\begin{aligned}\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right)&\in\left(\operatorname{Sym}^1\mathfrak{g}\right)^n\subseteq\operatorname{Sym}^n\mathfrak{g}\qquad\text{(since the }\mathbf{k}\text{-algebra }\operatorname{Sym}\mathfrak{g}\text{ is graded)}\\&\subseteq\sum_{k=0}^{n}\operatorname{Sym}^k\mathfrak{g}.\end{aligned}$$

Thus, $\left[\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right)\right]_n$ is a well-defined element of $\operatorname{gr}_n\left(\operatorname{Sym}\left(\mathfrak{g}\right)\right)$. The definition of $\operatorname{grad}_{\operatorname{Sym}(\mathfrak{g})}$ therefore yields

$$\operatorname{grad}_{\operatorname{Sym}(\mathfrak{g})}\left(\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right)\right)=\left[\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_1\right)\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_2\right)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}\left(a_n\right)\right]_n. \tag{41}$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ [27].

Now,

$$\underbrace{\eta}_{=\operatorname{grad}^{-1}_{\operatorname{Sym}(\mathfrak{g})} \circ (\operatorname{gr}\phi)} \left(\left[\iota_{U,\mathfrak{g}}(a_1) \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n) \right]_n \right)$$
$$= \left(\operatorname{grad}^{-1}_{\operatorname{Sym}(\mathfrak{g})} \circ (\operatorname{gr}\phi) \right) \left(\left[\iota_{U,\mathfrak{g}}(a_1) \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n) \right]_n \right)$$
$$= \operatorname{grad}^{-1}_{\operatorname{Sym}(\mathfrak{g})} \left(\underbrace{(\operatorname{gr}\phi) \left(\left[\iota_{U,\mathfrak{g}}(a_1) \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n) \right]_n \right)}_{\substack{= \left[\iota_{\operatorname{Sym},\mathfrak{g}}(a_1) \iota_{\operatorname{Sym},\mathfrak{g}}(a_2) \cdots \iota_{\operatorname{Sym},\mathfrak{g}}(a_n) \right]_n \\ \text{(by (39))}}} \right)$$
$$= \operatorname{grad}^{-1}_{\operatorname{Sym}(\mathfrak{g})} \left(\left[\iota_{\operatorname{Sym},\mathfrak{g}}(a_1) \iota_{\operatorname{Sym},\mathfrak{g}}(a_2) \cdots \iota_{\operatorname{Sym},\mathfrak{g}}(a_n) \right]_n \right) = \iota_{\operatorname{Sym},\mathfrak{g}}(a_1) \iota_{\operatorname{Sym},\mathfrak{g}}(a_2) \cdots \iota_{\operatorname{Sym},\mathfrak{g}}(a_n)$$

(by (41)), qed.

[27] *Proof of (42):* Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathfrak{g}$. We have $\underbrace{\iota_{U,\mathfrak{g}}}_{= \pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}}} (a_p) = (\pi_{U,\mathfrak{g}} \circ \iota_{T,\mathfrak{g}})(a_p) = \pi_{U,\mathfrak{g}} \left(\underbrace{\iota_{T,\mathfrak{g}}(a_p)}_{\in \mathfrak{g}^{\otimes 1} \subseteq \mathfrak{g}^{\otimes \leq 1}} \right) \in \pi_{U,\mathfrak{g}} \left(\mathfrak{g}^{\otimes \leq 1} \right)$ for every $p \in \{1, 2, \ldots, n\}$. Hence, $\left[\iota_{U,\mathfrak{g}}(a_p) \right]_1$ is well-defined for every $p \in \{1, 2, \ldots, n\}$, and the definition of the product in $\operatorname{gr}(U(\mathfrak{g}))$ yields

$$\left[\iota_{U,\mathfrak{g}}(a_1) \right]_1 \left[\iota_{U,\mathfrak{g}}(a_2) \right]_1 \cdots \left[\iota_{U,\mathfrak{g}}(a_p) \right]_1 = \left[\iota_{U,\mathfrak{g}}(a_1) \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n) \right]_n .$$

The definition of $\operatorname{PBW}_{\mathfrak{g}}$ shows that $\operatorname{PBW}_{\mathfrak{g}} \circ \iota_{\operatorname{Sym},\mathfrak{g}} = \iota_{\operatorname{gr} U,\mathfrak{g}}$.
On the other hand, recall that $\operatorname{PBW}_{\mathfrak{g}}$ is a $\mathbf{k}$-algebra homomorphism. Thus,

$$\operatorname{PBW}_{\mathfrak{g}} \left(\iota_{\operatorname{Sym},\mathfrak{g}}(a_1) \iota_{\operatorname{Sym},\mathfrak{g}}(a_2) \cdots \iota_{\operatorname{Sym},\mathfrak{g}}(a_n) \right)$$
$$= \operatorname{PBW}_{\mathfrak{g}} \left(\iota_{\operatorname{Sym},\mathfrak{g}}(a_1) \right) \operatorname{PBW}_{\mathfrak{g}} \left(\iota_{\operatorname{Sym},\mathfrak{g}}(a_2) \right) \cdots \operatorname{PBW}_{\mathfrak{g}} \left(\iota_{\operatorname{Sym},\mathfrak{g}}(a_n) \right)$$
$$= \left[\iota_{U,\mathfrak{g}}(a_1) \right]_1 \left[\iota_{U,\mathfrak{g}}(a_2) \right]_1 \cdots \left[\iota_{U,\mathfrak{g}}(a_n) \right]_1$$
$$\left(\begin{array}{c} \text{because for every } p \in \{1, 2, \ldots, n\}\text{, we have} \\ \operatorname{PBW}_{\mathfrak{g}} \left(\iota_{\operatorname{Sym},\mathfrak{g}}(a_p) \right) = \underbrace{\left(\operatorname{PBW}_{\mathfrak{g}} \circ \iota_{\operatorname{Sym},\mathfrak{g}} \right)}_{= \iota_{\operatorname{gr} U,\mathfrak{g}}} (a_p) = \iota_{\operatorname{gr} U,\mathfrak{g}}(a_p) = \left[\iota_{U,\mathfrak{g}}(a_p) \right]_1 \\ \text{(by the definition of } \iota_{\operatorname{gr} U,\mathfrak{g}}) \end{array} \right)$$
$$= \left[\iota_{U,\mathfrak{g}}(a_1) \iota_{U,\mathfrak{g}}(a_2) \cdots \iota_{U,\mathfrak{g}}(a_n) \right]_n .$$

This proves (42).

Now, we have $\operatorname{PBW}_{\mathfrak{g}} \circ \eta = \operatorname{id}$ [28] and $\eta \circ \operatorname{PBW}_{\mathfrak{g}} = \operatorname{id}$ [29]. These two equalities show that the maps $\operatorname{PBW}_{\mathfrak{g}}$ and η are mutually inverse. In other words, the map η is an inverse to the map $\operatorname{PBW}_{\mathfrak{g}}$. In other words, the map $\operatorname{grad}^{-1}_{\operatorname{Sym}\mathfrak{g}} \circ (\operatorname{gr}\phi)$ is

[28] *Proof.* Let $v \in \operatorname{gr}(U(\mathfrak{g}))$. We are going to prove the identity $(\operatorname{PBW}_{\mathfrak{g}} \circ \eta)(v) = v$.

Indeed, this identity is **k**-linear in v. Hence, we can WLOG assume that v is a homogeneous element of $\operatorname{gr}(U(\mathfrak{g}))$ (because every element of $\operatorname{gr}(U(\mathfrak{g}))$ is a sum of homogeneous elements). Assume this. Thus, there exists an $n \in \mathbb{N}$ such that $v \in \operatorname{gr}_n(U(\mathfrak{g}))$. Consider this v.

We must prove the identity $(\operatorname{PBW}_{\mathfrak{g}} \circ \eta)(v) = v$. Since this identity is **k**-linear in v, we can WLOG assume that v has the form $[\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ (because the **k**-module $\operatorname{gr}_n(U(\mathfrak{g}))$ is spanned by the elements $[\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ (due to Proposition 2.23 **(f)**)). Assume this. Thus, there exist $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ such that $v = [\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n$. Consider these $a_1, a_2, \ldots, a_n$. Then,

$$\eta\left(\underbrace{v}_{=[\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n}\right) = \eta\left([\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n\right) = \iota_{\operatorname{Sym},\mathfrak{g}}(a_1)\iota_{\operatorname{Sym},\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}(a_n)$$

(by (40)). Applying the map $\operatorname{PBW}_{\mathfrak{g}}$ to both sides of this equality, we obtain

$$\begin{aligned}\operatorname{PBW}_{\mathfrak{g}}(\eta(v)) &= \operatorname{PBW}_{\mathfrak{g}}\left(\iota_{\operatorname{Sym},\mathfrak{g}}(a_1)\iota_{\operatorname{Sym},\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}(a_n)\right)\\ &= [\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n \qquad \text{(by (42))}\\ &= v.\end{aligned}$$

Thus, $(\operatorname{PBW}_{\mathfrak{g}} \circ \eta)(v) = \operatorname{PBW}_{\mathfrak{g}}(\eta(v)) = v$.

Now, let us forget that we have fixed v. Thus, we have shown that $(\operatorname{PBW}_{\mathfrak{g}} \circ \eta)(v) = v$ for every $v \in \operatorname{gr}(U(\mathfrak{g}))$. In other words, $\operatorname{PBW}_{\mathfrak{g}} \circ \eta = \operatorname{id}$, qed.

[29] *Proof.* Let $v \in \operatorname{Sym}\mathfrak{g}$. We are going to prove the identity $(\eta \circ \operatorname{PBW}_{\mathfrak{g}})(v) = v$.

Indeed, this identity is **k**-linear in v. Hence, we can WLOG assume that v is a homogeneous element of $\operatorname{Sym}\mathfrak{g}$ (because every element of $\operatorname{Sym}\mathfrak{g}$ is a sum of homogeneous elements). Assume this. Thus, there exists an $n \in \mathbb{N}$ such that $v \in \operatorname{Sym}^n \mathfrak{g}$. Consider this v.

We must prove the identity $(\eta \circ \operatorname{PBW}_{\mathfrak{g}})(v) = v$. Since this identity is **k**-linear in v, we can WLOG assume that v has the form $\iota_{\operatorname{Sym},V}(a_1)\iota_{\operatorname{Sym},V}(a_2)\cdots\iota_{\operatorname{Sym},V}(a_n)$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ (because the **k**-module $\operatorname{Sym}^n \mathfrak{g}$ is spanned by the elements $\iota_{\operatorname{Sym},V}(a_1)\iota_{\operatorname{Sym},V}(a_2)\cdots\iota_{\operatorname{Sym},V}(a_n)$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ (due to Remark 2.18)). Assume this. Thus, there exist $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ such that $v = \iota_{\operatorname{Sym},V}(a_1)\iota_{\operatorname{Sym},V}(a_2)\cdots\iota_{\operatorname{Sym},V}(a_n)$. Consider these $a_1, a_2, \ldots, a_n$. Then,

$$\begin{aligned}\operatorname{PBW}_{\mathfrak{g}}\left(\underbrace{v}_{=\iota_{\operatorname{Sym},V}(a_1)\iota_{\operatorname{Sym},V}(a_2)\cdots\iota_{\operatorname{Sym},V}(a_n)}\right) &= \operatorname{PBW}_{\mathfrak{g}}\left(\iota_{\operatorname{Sym},V}(a_1)\iota_{\operatorname{Sym},V}(a_2)\cdots\iota_{\operatorname{Sym},V}(a_n)\right)\\ &= [\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n\end{aligned}$$

(by (42)). Applying the map η to both sides of this equality, we obtain

$$\begin{aligned}\eta(\operatorname{PBW}_{\mathfrak{g}}(v)) &= \eta\left([\iota_{U,\mathfrak{g}}(a_1)\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)]_n\right)\\ &= \iota_{\operatorname{Sym},\mathfrak{g}}(a_1)\iota_{\operatorname{Sym},\mathfrak{g}}(a_2)\cdots\iota_{\operatorname{Sym},\mathfrak{g}}(a_n) \qquad \text{(by (40))}\\ &= v.\end{aligned}$$

Thus, $(\eta \circ \operatorname{PBW}_{\mathfrak{g}})(v) = \eta(\operatorname{PBW}_{\mathfrak{g}}(v)) = v$.

an inverse to the map $\mathrm{PBW}_{\mathfrak{g}}$ (since $\eta = \mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}^{-1} \circ (\mathrm{gr}\,\phi)$). Thus, the map $\mathrm{PBW}_{\mathfrak{g}}$ is invertible. Lemma 2.24 **(c)** is proven.

Lemma 2.24 **(c)** also shows that the map $\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}^{-1} \circ (\mathrm{gr}\,\phi)$ is invertible (since it is an inverse to the map $\mathrm{PBW}_{\mathfrak{g}}$). But $\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}$ is also an invertible map (since $\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}$ is a graded **k**-module isomorphism). Hence, the map $\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}} \circ \left(\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}^{-1} \circ (\mathrm{gr}\,\phi)\right)$ is also invertible (being the composition of the two invertible maps $\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}$ and $\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}^{-1} \circ (\mathrm{gr}\,\phi)$). Since $\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}} \circ \left(\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}^{-1} \circ (\mathrm{gr}\,\phi)\right) = \underbrace{\mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}} \circ \mathrm{grad}_{\mathrm{Sym}\,\mathfrak{g}}^{-1}}_{=\mathrm{id}} \circ (\mathrm{gr}\,\phi) = \mathrm{gr}\,\phi$, this rewrites as follows: The map $\mathrm{gr}\,\phi$ is invertible. Therefore, Theorem 2.9 (applied to $V = U(\mathfrak{g})$, $W = \mathrm{Sym}\,\mathfrak{g}$ and $f = \phi$) shows that the map ϕ is invertible, and that its inverse ϕ^{-1} is again a filtered **k**-algebra map. This proves Lemma 2.24 **(b)**.

The map ϕ is a filtered **k**-module isomorphism (since it is filtered, **k**-linear and invertible). This proves Lemma 2.24 **(a)**. □

3. Pre-Lie algebras

3.1. Definitions

We now introduce the main notions of this note:

Definition 3.1. A *left pre-Lie algebra* is a **k**-module A, equipped with a **k**-bilinear map $\mu : A \times A \to A$, satisfying the relation

$$\mu(\mu(a,b),c) - \mu(a,\mu(b,c)) = \mu(\mu(b,a),c) - \mu(b,\mu(a,c)) \qquad \text{for all } a,b,c \in A. \tag{43}$$

When A is a left pre-Lie algebra, and when the map μ is clear from the context, we use the name $\triangleright$ for the map μ and write it in infix notation (i.e., we use the notation $a \triangleright b$ for $\mu(a,b)$ when $a \in A$ and $b \in A$). Thus, the axiom (43) rewrites as

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c) \qquad \text{for all } a,b,c \in A. \tag{44}$$

Definition 3.2. A *right pre-Lie algebra* is a **k**-module A, equipped with a **k**-bilinear map $\mu : A \times A \to A$, satisfying the relation

$$\mu(\mu(a,b),c) - \mu(a,\mu(b,c)) = \mu(\mu(a,c),b) - \mu(a,\mu(c,b)) \qquad \text{for all } a,b,c \in A. \tag{45}$$

Now, let us forget that we have fixed v. Thus, we have shown that $(\eta \circ \mathrm{PBW}_{\mathfrak{g}})(v) = v$ for every $v \in \mathrm{Sym}\,\mathfrak{g}$. In other words, $\eta \circ \mathrm{PBW}_{\mathfrak{g}} = \mathrm{id}$, qed.

When A is a right pre-Lie algebra, and when the map μ is clear from the context, we use the name $\triangleleft$ for the map μ and write it in infix notation (i.e., we use the notation $a \triangleleft b$ for $\mu(a, b)$ when $a \in A$ and $b \in A$). Thus, the axiom (45) rewrites as

$$(a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = (a \triangleleft c) \triangleleft b - a \triangleleft (c \triangleleft b) \qquad \text{for all } a, b, c \in A. \tag{46}$$

Left pre-Lie algebras are also known as left-symmetric algebras, Vinberg algebras, or quasi-associative algebras. Similar epithets exist for right pre-Lie algebras. We shall prove some basic properties of these objects in this section; more advanced (and more interesting) results and examples can be found in the expositions [Manchon11] and [Burde11].

First, let us observe that the notions of left pre-Lie algebras and of right pre-Lie algebras are "symmetric to each other", in the sense that we can translate between the two by reversing the order of the arguments in the binary operation:

Proposition 3.3. Let A be a left pre-Lie algebra. We define a map $\triangleleft : A \times A \to A$ (written in infix notation) by

$$(a \triangleleft b = b \triangleright a \qquad \text{for all } a, b \in A).$$

Then, A (equipped with this map $\triangleleft$) is a right pre-Lie algebra.

Proposition 3.4. Let A be a right pre-Lie algebra. We define a map $\triangleright : A \times A \to A$ (written in infix notation) by

$$(a \triangleright b = b \triangleleft a \qquad \text{for all } a, b \in A).$$

Then, A (equipped with this map $\triangleright$) is a left pre-Lie algebra.

Proof of Proposition 3.3. The map $\triangleright : A \times A \to A$ is $\mathbf{k}$-bilinear (since A is a left pre-Lie algebra). Thus, the element $b \triangleright a \in A$ (for $a, b \in A$) depends $\mathbf{k}$-linearly on each of a and b. In other words, the element $a \triangleleft b$ (for $a, b \in A$) depends $\mathbf{k}$-linearly on each of a and b (since $a \triangleleft b = b \triangleright a$). In other words, the map $\triangleleft$ is $\mathbf{k}$-bilinear.

Also, (44) holds (since A is a left pre-Lie algebra).

Now, we shall prove that $(a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = (a \triangleleft c) \triangleleft b - a \triangleleft (c \triangleleft b)$ for all $a, b, c \in A$.

Indeed, let $a, b, c \in A$. We have

$$\begin{aligned}
&\underbrace{(a \lhd b)}_{\substack{=b \rhd a \\ \text{(by the definition of } \lhd)}} \lhd c - a \lhd \underbrace{(b \lhd c)}_{\substack{=c \rhd b \\ \text{(by the definition of } \lhd)}} \\
&= \underbrace{(b \rhd a) \lhd c}_{\substack{=c \rhd (b \rhd a) \\ \text{(by the definition of } \lhd)}} - \underbrace{a \lhd (c \rhd b)}_{\substack{=(c \rhd b) \rhd a \\ \text{(by the definition of } \lhd)}} \\
&= c \rhd (b \rhd a) - (c \rhd b) \rhd a = -\underbrace{((c \rhd b) \rhd a - c \rhd (b \rhd a))}_{\substack{=(b \rhd c) \rhd a - b \rhd (c \rhd a) \\ \text{(by (44) (applied to} \\ c \text{ and } a \text{ instead of } a \text{ and } c))}} \\
&= -((b \rhd c) \rhd a - b \rhd (c \rhd a)) = b \rhd (c \rhd a) - (b \rhd c) \rhd a.
\end{aligned}$$

Comparing this to

$$\begin{aligned}
&\underbrace{(a \lhd c)}_{\substack{=c \rhd a \\ \text{(by the definition of } \lhd)}} \lhd b - a \lhd \underbrace{(c \lhd b)}_{\substack{=b \rhd c \\ \text{(by the definition of } \lhd)}} \\
&= \underbrace{(c \rhd a) \lhd b}_{\substack{=b \rhd (c \rhd a) \\ \text{(by the definition of } \lhd)}} - \underbrace{a \lhd (b \rhd c)}_{\substack{=(b \rhd c) \rhd a \\ \text{(by the definition of } \lhd)}} \\
&= b \rhd (c \rhd a) - (b \rhd c) \rhd a,
\end{aligned}$$

we obtain $(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b)$.

Let us now forget that we fixed a, b, c. We thus have proven that $(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b)$ for all $a, b, c \in A$. In other words, (46) holds. This (combined with the fact that the map $\lhd$ is $\mathbf{k}$-bilinear) shows that A (equipped with the map $\lhd$) is a right pre-Lie algebra (by the definition of a "right pre-Lie algebra"). This proves Proposition 3.3. $\square$

Proof of Proposition 3.4. This proof is highly similar to the proof of Proposition 3.3 above, and thus is left to the reader. $\square$

The following two propositions show that the notions of (left and right) pre-Lie algebras are a middle ground between (associative) algebras and Lie algebras.

Proposition 3.5. Let A be a $\mathbf{k}$-algebra. (Recall that "$\mathbf{k}$-algebra" means "associative unital $\mathbf{k}$-algebra" for us; that said, unitality is not important in this proposition.)

(a) The $\mathbf{k}$-module A (equipped with the multiplication of A) is a left pre-Lie algebra.

(b) The $\mathbf{k}$-module A (equipped with the multiplication of A) is a right pre-Lie algebra.

Proof of Proposition 3.5. **(a)** The multiplication of A is clearly $\mathbf{k}$-bilinear. It thus remains to prove that (43) holds when μ is taken to be the multiplication of A. In other words, it remains to prove that

$$(ab)\,c - a\,(bc) = (ba)\,c - b\,(ac) \qquad \text{for all } a,b,c \in A.$$

But this obviously follows from comparing $(ab)\,c - a\,(bc) = abc - abc = 0$ with $(ba)\,c - b\,(ac) = bac - bac = 0$. Thus, Proposition 3.5 **(a)** is proven.

(b) The proof of Proposition 3.5 **(b)** is analogous to that of Proposition 3.5 **(a)**, and left to the reader. □

Proposition 3.6. Let A be a left pre-Lie algebra. Define a $\mathbf{k}$-bilinear map $\beta : A \times A \to A$ by

$$\beta\,(a,b) = a \rhd b - b \rhd a \qquad \text{for all } a \in A \text{ and } b \in A.$$

Then, the $\mathbf{k}$-module A, endowed with this map β, is a Lie algebra. This Lie algebra is denoted by A^-. (This clearly generalizes the definition of the Lie algebra A^- when A is a $\mathbf{k}$-algebra.)

Proof of Proposition 3.6. The operation $\rhd$ is $\mathbf{k}$-bilinear (since A is a left pre-Lie algebra). Hence, $a \rhd b - b \rhd a$ (for $a \in A$ and $b \in A$) depends $\mathbf{k}$-linearly on each of a and b. The $\mathbf{k}$-bilinear map β is thus well-defined.

The axiom (44) holds, since A is a left pre-Lie algebra.

We only need to prove that the $\mathbf{k}$-module A, endowed with the map β, is a Lie algebra. To do this, it suffices to verify that

$$(\beta\,(v,v) = 0 \text{ for every } v \in A) \tag{47}$$

and that

$$(\beta\,(u,\beta\,(v,w)) + \beta\,(v,\beta\,(w,u)) + \beta\,(w,\beta\,(u,v)) = 0 \text{ for every } u \in A,\ v \in A \text{ and } w \in A) \tag{48}$$

(because it is clear that the map β is $\mathbf{k}$-bilinear).

Proof of (47): Let $v \in A$. Then, the definition of β shows that $\beta\,(v,v) = v \rhd v - v \rhd v = 0$. Now let us forget that we fixed v. We thus have shown that $\beta\,(v,v) = 0$ for every $v \in A$. In other words, (47) holds.

Proof of (48): Let $u \in A$, $v \in A$ and $w \in A$. Now,

$$\beta\left(u, \underbrace{\beta(v,w)}_{\substack{=v\rhd w-w\rhd v \\ \text{(by the definition of }\beta)}}\right)$$
$$= \beta(u, v\rhd w - w\rhd v) = \underbrace{u\rhd(v\rhd w - w\rhd v)}_{\substack{=u\rhd(v\rhd w)-u\rhd(w\rhd v) \\ \text{(since the operation }\rhd\text{ is }\mathbf{k}\text{-bilinear)}}} - \underbrace{(v\rhd w - w\rhd v)\rhd u}_{\substack{=(v\rhd w)\rhd u-(w\rhd v)\rhd u \\ \text{(since the operation }\rhd\text{ is }\mathbf{k}\text{-bilinear)}}}$$
$$\text{(by the definition of }\beta)$$
$$= (u\rhd(v\rhd w) - u\rhd(w\rhd v)) - ((v\rhd w)\rhd u - (w\rhd v)\rhd u)$$
$$= u\rhd(v\rhd w) - u\rhd(w\rhd v) - (v\rhd w)\rhd u + (w\rhd v)\rhd u. \tag{49}$$

Applying (49) to v, w and u instead of u, v and w, we obtain

$$\beta(v,\beta(w,u)) = v\rhd(w\rhd u) - v\rhd(u\rhd w) - (w\rhd u)\rhd v + (u\rhd w)\rhd v. \tag{50}$$

Applying (49) to w, u and v instead of u, v and w, we obtain

$$\beta(w,\beta(u,v)) = w\rhd(u\rhd v) - w\rhd(v\rhd u) - (u\rhd v)\rhd w + (v\rhd u)\rhd w. \tag{51}$$

Adding together the identities (49), (50) and (51), we obtain

$$\begin{aligned}
&\beta(u,\beta(v,w)) + \beta(v,\beta(w,u)) + \beta(w,\beta(u,v)) \\
&= u\rhd(v\rhd w) - u\rhd(w\rhd v) - (v\rhd w)\rhd u + (w\rhd v)\rhd u \\
&\qquad + v\rhd(w\rhd u) - v\rhd(u\rhd w) - (w\rhd u)\rhd v + (u\rhd w)\rhd v \\
&\qquad + w\rhd(u\rhd v) - w\rhd(v\rhd u) - (u\rhd v)\rhd w + (v\rhd u)\rhd w \\
&= \underbrace{((w\rhd v)\rhd u - w\rhd(v\rhd u))}_{\substack{=(v\rhd w)\rhd u-v\rhd(w\rhd u) \\ \text{(by (44), applied} \\ \text{to }a=w,\ b=v\text{ and }c=u)}} + \underbrace{((u\rhd w)\rhd v - u\rhd(w\rhd v))}_{\substack{=(w\rhd u)\rhd v-w\rhd(u\rhd v) \\ \text{(by (44), applied} \\ \text{to }a=u,\ b=w\text{ and }c=v)}} \\
&\qquad + \underbrace{((v\rhd u)\rhd w - v\rhd(u\rhd w))}_{\substack{=(u\rhd v)\rhd w-u\rhd(v\rhd w) \\ \text{(by (44), applied} \\ \text{to }a=v,\ b=u\text{ and }c=w)}} + u\rhd(v\rhd w) - (v\rhd w)\rhd u \\
&\qquad + v\rhd(w\rhd u) - (w\rhd u)\rhd v + w\rhd(u\rhd v) - (u\rhd v)\rhd w \\
&= (v\rhd w)\rhd u - v\rhd(w\rhd u) + (w\rhd u)\rhd v - w\rhd(u\rhd v) \\
&\qquad + (u\rhd v)\rhd w - u\rhd(v\rhd w) + u\rhd(v\rhd w) - (v\rhd w)\rhd u \\
&\qquad + v\rhd(w\rhd u) - (w\rhd u)\rhd v + w\rhd(u\rhd v) - (u\rhd v)\rhd w \\
&= 0.
\end{aligned}$$

Now, let us forget that we fixed u, v and w. We thus have shown that $\beta(u,\beta(v,w)) + \beta(v,\beta(w,u)) + \beta(w,\beta(u,v)) = 0$ for every $u \in A$, $v \in A$ and $w \in A$. In other words, (48) holds.

Now, we have shown that (47) and (48) hold. Since the map β is $\mathbf{k}$-bilinear, this shows that the $\mathbf{k}$-module A, endowed with this map β, is a Lie algebra (by the definition of a "Lie algebra"). This proves Proposition 3.6. $\square$

Let us state the analogue of Proposition 3.6 for right pre-Lie algebras:

Proposition 3.7. Let A be a right pre-Lie algebra. Define a $\mathbf{k}$-bilinear map $\beta : A \times A \to A$ by

$$\beta(a,b) = a \lhd b - b \lhd a \qquad \text{for all } a \in A \text{ and } b \in A.$$

Then, the $\mathbf{k}$-module A, endowed with this map β, is a Lie algebra. This Lie algebra is denoted by A^-. (This clearly generalizes the definition of the Lie algebra A^- when A is a $\mathbf{k}$-algebra.)

The proof of Proposition 3.7 is highly similar to that of Proposition 3.6; thus we leave it to the reader.

Remark 3.8. It is worth repeating one obvious observation. Namely, let A be a $\mathbf{k}$-algebra. Then, we can construct three Lie algebras, all of which we denote by A^-:

- One Lie algebra denoted by A^- has been defined in Definition 1.6.
- Another Lie algebra denoted by A^- can be obtained from Proposition 3.6, after we first make A into a left pre-Lie algebra (according to Proposition 3.5 **(a)**).
- A third Lie algebra denoted by A^- can be obtained from Proposition 3.7, after we first make A into a right pre-Lie algebra (according to Proposition 3.5 **(b)**).

These three Lie algebras denoted by A^- are **all identical**; thus, the notations do not conflict.

3.2. The Guin-Oudom isomorphism

Definition 3.9. From now on, we shall employ the following abuse of notation: If V is a $\mathbf{k}$-module, then every $v \in V$ will be identified with the element $\iota_{\operatorname{Sym},V}(v)$ of $\operatorname{Sym} V$ and also identified with the element $\iota_{T,V}(v)$ of $T(V)$. This, of course, is dangerous, because a product of the form $v_1 v_2 \cdots v_k$ (with $k \in \mathbb{N}$ and $v_1, v_2, \ldots, v_k \in V$) can mean two different things (namely, the tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ in $T(V)$, and the projection $\pi_{\operatorname{Sym},V}(v_1 \otimes v_2 \otimes \cdots \otimes v_k)$ of this tensor on $\operatorname{Sym} V$). (It becomes even more dangerous when V itself has a $\mathbf{k}$-algebra structure, because then $v_1 v_2 \cdots v_k$ might also mean a product inside V.) However, we shall rely on the context to clear up any ambiguities.

We observe that any k elements $v_1, v_2, \ldots, v_k$ of a $\mathbf{k}$-module V satisfy

$$v_1 v_2 \cdots v_k = \pi_{\operatorname{Sym},V}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) \tag{52}$$

in $\operatorname{Sym} V$.

The following result extends [Manchon11, Theorem 1.1]:

Theorem 3.10. Let A be a left pre-Lie algebra. Recall that a Lie algebra A^- is defined (according to Proposition 3.6).

For any $a \in A$, let left_a be the $\mathbf{k}$-linear map $A \to A,\ b \mapsto a \triangleright b$. For any $a \in A$, we define a derivation $L_a : \operatorname{Sym} A \to \operatorname{Sym} A$ as follows: Proposition 2.20 (applied to $V = A$, $M = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A} \circ \operatorname{left}_a$) shows that there exists a unique derivation $F : \operatorname{Sym} A \to \operatorname{Sym} A$ such that $F \circ \iota_{\operatorname{Sym},A} = \operatorname{left}_a \circ \iota_{\operatorname{Sym},A}$. We let L_a be this derivation.

In the following, we will identify every $a \in A$ with the element $\iota_{\operatorname{Sym},A}(a)$ of $\operatorname{Sym} A$. (This is a particular case of the abuse of notation introduced in Definition 3.9.) Thus, the injection $\iota_{\operatorname{Sym},A} : A \to \operatorname{Sym} A$ becomes an inclusion, and the $\mathbf{k}$-module A becomes a $\mathbf{k}$-submodule of $\operatorname{Sym} A$ (although the left pre-Lie algebra A does **not** become a subalgebra of $\operatorname{Sym} A$). Thus, products such as $b_1 b_2 \cdots b_n$ (where $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in A$) will always mean products inside $\operatorname{Sym} A$. (This notation will not conflict with the binary operation on the left pre-Lie algebra A, since the latter operation is denoted by $\triangleright$.)

(a) If $a \in A$, $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in A$, then

$$L_a(b_1 b_2 \cdots b_n) = \sum_{k=1}^{n} b_1 b_2 \cdots b_{k-1} (a \triangleright b_k) b_{k+1} b_{k+2} \cdots b_n.$$

(b) For every $a \in A$, the map $L_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is graded.

(c) For every $a \in A$, the map $L_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a coderivation.

(d) We define a map $K : A \to \operatorname{Der}(\operatorname{Sym} A)$ by

$$(K(a) = L_a \qquad \text{for every } a \in A).$$

Then, this map K is a Lie algebra homomorphism from A^- to $\operatorname{Der}(\operatorname{Sym} A)$. (See Proposition 1.11 **(a)** for the definition of the Lie algebra $\operatorname{Der}(\operatorname{Sym} A)$.)

(e) We can define an A^--module structure on $\operatorname{Sym} A$ by setting

$$\left(a \rightharpoonup u = au + L_a(u) \text{ for all } a \in A^- \text{ and } u \in \operatorname{Sym} A\right).$$

In the following, we will regard $\operatorname{Sym} A$ as an A^--module by means of this A^--module structure.

(f) Being an A^--module, $\operatorname{Sym} A$ becomes a $U(A^-)$-module. Define a map $\eta : U(A^-) \to \operatorname{Sym} A$ by

$$\eta(u) = u 1_{\operatorname{Sym} A} \qquad \text{for every } u \in U\left(A^-\right).$$

Then, η is an A^--module homomorphism.

(g) We have $\eta\left(\iota_{U,A^-}(a)\right) = a$ for every $a \in A^-$.

(h) The map η is an A^--module isomorphism.

(i) We have

$$\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \in a_1a_2\cdots a_n + \sum_{k=0}^{n-1}\operatorname{Sym}^k A$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A^-$.

(j) The map η and its inverse η^{-1} are filtered (where $U(A^-)$ is endowed with the usual filtration on a universal enveloping algebra).

(k) The map $\operatorname{grad}^{-1}_{\operatorname{Sym} A} \circ (\operatorname{gr}\eta) : \operatorname{gr}(U(A^-)) \to \operatorname{Sym} A$ is an inverse to the map PBW_{A^-}. In particular, the map PBW_{A^-} is invertible.

(l) The map $\eta : U(A^-) \to \operatorname{Sym} A$ is a $\mathbf{k}$-coalgebra isomorphism.

(m) For every $a \in A$, $b \in A$ and $c \in \operatorname{Sym} A$, we have

$$a \rightharpoonup (bc) - b \cdot (a \rightharpoonup c) = (a \triangleright b) \cdot c.$$

We suggest to call the $\mathbf{k}$-coalgebra isomorphism $\eta : U(A^-) \to \operatorname{Sym} A$ in Theorem 3.10 the *Guin-Oudom isomorphism*.

The most striking part of Theorem 3.10 is part **(k)**, which gives a PBW theorem for any Lie algebra of the form A^- for A being a pre-Lie algebra. This result originates in Oudom's and Guin's [GuiOud04, Théorème 3.5] and [GuiOud08, Theorem 3.14][30]. Parts **(a)**, **(b)** and **(d)**–**(k)** of Theorem 3.10 appear (explicitly or implicitly) in [Manchon11, Theorem 1.1 and its proof] and [Manchon15, Theorem 19 and its proof]; parts **(c)** and **(l)** appear in [GuiOud08, Remark 3.3 and Theorem 3.14]. Notice that [Schedl10, Corollary 1.3.1] also follows from Theorem 3.10 **(l)**[31]. We shall mostly derive Theorem 3.10 from what has already been shown.

Proof of Theorem 3.10. For every $a \in A$, we know that $L_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a derivation and satisfies

$$L_a \circ \iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A} \circ \operatorname{left}_a \tag{53}$$

[30] although Oudom and Guin work with right pre-Lie algebras (but this does not change much, as there is a symmetry in the concepts) and approach the theorem in a significantly different way (this actually matters)

[31] although here, again, one has to translate from left to right pre-Lie algebras

[32]. Hence, for every $a \in A$ and $b \in A$, we have

$$L_a\left(\iota_{\mathrm{Sym},A}(b)\right) = \underbrace{\left(L_a \circ \iota_{\mathrm{Sym},A}\right)}_{\substack{=\iota_{\mathrm{Sym},A}\circ \mathrm{left}_a \\ \text{(by (53))}}}(b) = \iota_{\mathrm{Sym},A}\left(\underbrace{\mathrm{left}_a(b)}_{\substack{=a\triangleright b \\ \text{(by the definition} \\ \text{of } \mathrm{left}_a)}}\right) = \iota_{\mathrm{Sym},A}(a \triangleright b). \tag{54}$$

In other words, for every $a \in A$ and $b \in A$, we have

$$L_a(b) = a \triangleright b \tag{55}$$

[33].

(a) Let $a \in A$, $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in A$. The map L_a is a derivation, thus an element of $\mathrm{Der}(\mathrm{Sym}\,A)$. Hence, Proposition 1.11 **(b)** (applied to $C = \mathrm{Sym}\,A$, $f = L_a$ and $a_i = b_i$) yields

$$\begin{aligned} & L_a(b_1 b_2 \cdots b_n) \\ &= \sum_{i=1}^{n} b_1 b_2 \cdots b_{i-1} \underbrace{L_a(b_i)}_{\substack{=a\triangleright b_i \\ \text{(by (55))}}} b_{i+1} b_{i+2} \cdots b_n \\ &= \sum_{i=1}^{n} b_1 b_2 \cdots b_{i-1}(a \triangleright b_i) b_{i+1} b_{i+2} \cdots b_n = \sum_{k=1}^{n} b_1 b_2 \cdots b_{k-1}(a \triangleright b_k) b_{k+1} b_{k+2} \cdots b_n \end{aligned}$$

(here, we have renamed the summation index i as k). This proves Theorem 3.10 **(a)**.

(b) Let $a \in A$.

In Proposition 2.21 (applied to $V = A$ and $f = \mathrm{left}_a$), a map $\widetilde{f} : \mathrm{Sym}\,A \to \mathrm{Sym}\,A$ was defined as the unique derivation $F : \mathrm{Sym}\,A \to \mathrm{Sym}\,A$ such that $F \circ \iota_{\mathrm{Sym},A} = \iota_{\mathrm{Sym},A} \circ \mathrm{left}_a$. This derivation F must clearly be our map L_a (since we know that $L_a : \mathrm{Sym}\,A \to \mathrm{Sym}\,A$ is a derivation and satisfies $L_a \circ \iota_{\mathrm{Sym},A} = \iota_{\mathrm{Sym},A} \circ \mathrm{left}_a$). Hence, the $\widetilde{f}$ defined in Proposition 2.21 (applied to $V = A$ and $f = \mathrm{left}_a$) is our map L_a. Thus, Proposition 2.21 can be applied to $V = A$, $f = \mathrm{left}_a$ and $\widetilde{f} = L_a$.

As a consequence, Proposition 2.21 **(c)** (applied to $V = A$, $f = \mathrm{left}_a$ and $\widetilde{f} = L_a$) shows that the map $L_a : \mathrm{Sym}\,A \to \mathrm{Sym}\,A$ is graded. This proves Theorem 3.10 **(b)**.

[32]*Proof.* Let $a \in A$. Then, L_a is the unique derivation $F : \mathrm{Sym}\,A \to \mathrm{Sym}\,A$ such that $F \circ \iota_{\mathrm{Sym},A} = \iota_{\mathrm{Sym},A} \circ \mathrm{left}_a$. Hence, L_a is a derivation $\mathrm{Sym}\,A \to \mathrm{Sym}\,A$ and satisfies $L_a \circ \iota_{\mathrm{Sym},A} = \iota_{\mathrm{Sym},A} \circ \mathrm{left}_a$. Qed.

[33]since $\iota_{\mathrm{Sym},A}(b) = b$ (because we are identifying $\iota_{\mathrm{Sym},A}(x)$ with x for every $x \in A$) and $\iota_{\mathrm{Sym},A}(a \triangleright b) = a \triangleright b$ (for the same reason)

(c) Let $a \in A$. In our above proof of Theorem 3.10 **(b)**, we have shown that Proposition 2.21 can be applied to $V = A$, $f = \operatorname{left}_a$ and $\widetilde{f} = L_a$. Hence, Proposition 2.21 **(b)** (applied to $V = A$, $f = \operatorname{left}_a$ and $\widetilde{f} = L_a$) shows that the map $L_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a coderivation. Theorem 3.10 **(c)** is thus proven.

(d) The map K is a map from A to $\operatorname{Der}(\operatorname{Sym} A)$, thus a map from A^- to $\operatorname{Der}(\operatorname{Sym} A)$ (since $A^- = A$ as sets).

We shall first prove that

$$K([a,b]) = [K(a), K(b)] \tag{56}$$

for every $a \in A^-$ and $b \in A^-$.

Proof of (56): Recall that we are regarding A as a $\mathbf{k}$-submodule of $\operatorname{Sym} A$ via the injection $\iota_{\operatorname{Sym},A} : A \to \operatorname{Sym} A$. Thus, $A = \iota_{\operatorname{Sym},A}(A) = \operatorname{Sym}^1 A$. Recall that the subset $\operatorname{Sym}^1 A$ of $\operatorname{Sym} A$ generates the $\mathbf{k}$-algebra $\operatorname{Sym} A$. In other words, the subset A of $\operatorname{Sym} A$ generates the $\mathbf{k}$-algebra $\operatorname{Sym} A$ (since $A = \operatorname{Sym}^1 A$).

Let $a \in A^-$ and $b \in A^-$. Then, $K(a)$ and $K(b)$ are elements of $\operatorname{Der}(\operatorname{Sym} A)$. Thus $[K(a), K(b)]$ is an element of $\operatorname{Der}(\operatorname{Sym} A)$ (since $\operatorname{Der}(\operatorname{Sym} A)$ is a Lie algebra), hence a derivation $\operatorname{Sym} A \to \operatorname{Sym} A$. Also, $K([a,b])$ is an element of $\operatorname{Der}(\operatorname{Sym} A)$, hence a derivation $\operatorname{Sym} A \to \operatorname{Sym} A$.

We have $[K(a), K(b)] \mid_A = K([a,b]) \mid_A$ [34]. Hence, Lemma 1.13 (applied to $\operatorname{Sym} A$, $[K(a), K(b)]$, $K([a,b])$ and A instead of A, d, e and S) shows that $[K(a), K(b)] = K([a,b])$. This proves (56).

Next, we notice that

$$K(\lambda a + \mu b) = \lambda K(a) + \mu K(b) \tag{57}$$

for every $a \in A^-$ and $b \in A^-$ and every $\lambda \in \mathbf{k}$ and $\mu \in \mathbf{k}$.

Proof of (57): The proof of (57) is similar to the above proof of (56), but even simpler (because instead of $[a,b] \rhd c = a \rhd (b \rhd c) - b \rhd (a \rhd c)$, we now need

[34] *Proof.* Let $c \in A$. The definition of K yields $K(a) = L_a$, $K(b) = L_b$ and $K([a,b]) = L_{[a,b]}$. Now,

$$\begin{aligned}
&([K(a), K(b)] \mid_A)(c) \\
&= \underbrace{[K(a), K(b)]}_{=K(a)\circ K(b) - K(b)\circ K(a)}(c) = \Big(\underbrace{K(a)}_{=L_a} \circ \underbrace{K(b)}_{=L_b} - \underbrace{K(b)}_{=L_b} \circ \underbrace{K(a)}_{=L_a} \Big)(c) \\
&= (L_a \circ L_b - L_b \circ L_a)(c) = L_a \Bigg(\underbrace{L_b(c)}_{\substack{=b \rhd c \\ \text{(by (55), applied to } b \text{ and } c \\ \text{instead of } a \text{ and } b)}} \Bigg) - L_b \Bigg(\underbrace{L_a(c)}_{\substack{=a \rhd c \\ \text{(by (55), applied to } c \\ \text{instead of } b)}} \Bigg) \\
&= \underbrace{L_a(b \rhd c)}_{\substack{=a \rhd (b \rhd c) \\ \text{(by (55), applied to } b \rhd c \\ \text{instead of } b)}} - \underbrace{L_b(a \rhd c)}_{\substack{=b \rhd (a \rhd c) \\ \text{(by (55), applied to } b \text{ and } a \rhd c \\ \text{instead of } a \text{ and } b)}} \\
&= a \rhd (b \rhd c) - b \rhd (a \rhd c).
\end{aligned}$$

to use the much simpler equality $(\lambda a+\mu b)\rhd c=\lambda(a\rhd c)+\mu(b\rhd c)$). Hence, we leave it to the reader.

Now, the map $K : A^- \to \operatorname{Der}(\operatorname{Sym} A)$ is $\mathbf{k}$-linear (because of (57)) and thus a Lie algebra homomorphism (because of (56)). This proves Theorem 3.10 **(d)**.

(e) The map $K : A^- \to \operatorname{Der}(\operatorname{Sym} A)$ is a Lie algebra homomorphism (according to Theorem 3.10 **(d)**). Moreover, we have

$$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + (K(a))\left(\iota_{\operatorname{Sym},A}(b)\right) - (K(b))\left(\iota_{\operatorname{Sym},A}(a)\right)$$

for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$) [35]. Hence, we can apply Theorem 1.15 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$.

Thus, Theorem 1.15 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$) shows

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Comparing this with

$$\begin{aligned}
&(K([a,b])\mid_A)(c) \\
&= \underbrace{(K([a,b]))}_{=L_{[a,b]}}(c) = L_{[a,b]}(c) = \underbrace{[a,b]}_{\substack{=a\rhd b-b\rhd a \\ \text{(by the definition} \\ \text{of the Lie algebra } A^-)}} \rhd c \qquad \left(\begin{array}{c}\text{by (55), applied to } [a,b] \text{ and } c \\ \text{instead of } a \text{ and } b\end{array}\right) \\
&= (a\rhd b - b\rhd a)\rhd c = (a\rhd b)\rhd c - (b\rhd a)\rhd c = a\rhd(b\rhd c) - b\rhd(a\rhd c) \\
&\qquad \left(\text{since } (a\rhd b)\rhd c - a\rhd(b\rhd c) = (b\rhd a)\rhd c - b\rhd(a\rhd c) \text{ (by (44))}\right),
\end{aligned}$$

we obtain $([K(a),K(b)]\mid_A)(c) = (K([a,b])\mid_A)(c)$.

Now, let us forget that we fixed c. We thus have shown that $([K(a),K(b)]\mid_A)(c) = (K([a,b])\mid_A)(c)$ for every $c \in A$. In other words, $[K(a),K(b)]\mid_A = K([a,b])\mid_A$, qed.

[35] *Proof.* Let $a \in A^-$ and $b \in A^-$. Recall that we are identifying $\iota_{\operatorname{Sym},A}(x)$ with x for every $x \in A$. Thus, $\iota_{\operatorname{Sym},A}([a,b]) = [a,b]$, $\iota_{\operatorname{Sym},A}(a) = a$ and $\iota_{\operatorname{Sym},A}(b) = b$. Moreover, the $\mathbf{k}$-algebra $\operatorname{Sym} A$ is commutative, and thus the Lie bracket of the Lie algebra $(\operatorname{Sym} A)^-$ is identically 0. Hence, $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] = 0$. Now,

$$\begin{aligned}
&\underbrace{\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]}_{=0} + \underbrace{(K(a))}_{\substack{=L_a \\ \text{(by the definition of } K)}}\left(\underbrace{\iota_{\operatorname{Sym},A}(b)}_{=b}\right) - \underbrace{(K(b))}_{\substack{=L_b \\ \text{(by the definition of } K)}}\left(\underbrace{\iota_{\operatorname{Sym},A}(a)}_{=a}\right) \\
&= \underbrace{L_a(b)}_{\substack{=a\rhd b \\ \text{(by (55))}}} - \underbrace{L_b(a)}_{\substack{=b\rhd a \\ \text{(by (55) (applied to} \\ b \text{ and } a \text{ instead of } a \text{ and } b))}} = a\rhd b - b\rhd a.
\end{aligned}$$

Compared with

$$\iota_{\operatorname{Sym},A}([a,b]) = [a,b] = a\rhd b - b\rhd a$$

(by the definition of the Lie bracket on A^-), this shows that

$$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + (K(a))\left(\iota_{\operatorname{Sym},A}(b)\right) - (K(b))\left(\iota_{\operatorname{Sym},A}(a)\right).$$

Qed.

that we can define an A^--module structure on $\operatorname{Sym} A$ by setting

$$\left(a \rightharpoonup u = \iota_{\operatorname{Sym},A}(a) \cdot u + (K(a))(u) \text{ for all } a \in A^- \text{ and } u \in \operatorname{Sym} A\right).$$

In other words, we can define an A^--module structure on $\operatorname{Sym} A$ by setting

$$\left(a \rightharpoonup u = au + L_a(u) \text{ for all } a \in A^- \text{ and } u \in \operatorname{Sym} A\right)$$

(because every $a \in A^-$ and $u \in \operatorname{Sym} A$ satisfy $\underbrace{\iota_{\operatorname{Sym},A}(a)}_{=a} \cdot u + \underbrace{(K(a))}_{\substack{=L_a \\ \text{(by the definition of } K\text{)}}}(u) = au + L_a(u)$). This proves Theorem 3.10 **(e)**.

As a consequence of this proof of Theorem 3.10 **(e)**, we see that the A^--module structure on $\operatorname{Sym} A$ defined in Theorem 3.10 **(e)** is precisely the one that is constructed by Theorem 1.15 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$). We notice further that the map $\eta : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 3.10 **(f)** is precisely the map $\eta : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 1.15 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$) (since these two maps have the same definition).

(f) As we have seen in our proof of Theorem 3.10 **(e)**, we can apply Theorem 1.15 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$. Moreover, the following holds:

- The A^--module structure on $\operatorname{Sym} A$ defined in Theorem 3.10 **(e)** is precisely the one that is constructed by Theorem 1.15 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$).
- The map $\eta : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 3.10 **(f)** is precisely the map $\eta : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 1.15 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$).

Hence, Theorem 1.15 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$) shows that η is an A^--module homomorphism. This proves Theorem 3.10 **(f)**.

(g) Let $a \in A^-$. The map L_a is a derivation. In other words, $L_a \in \operatorname{Der}(\operatorname{Sym} A)$. Thus, $L_a(1) = 0$ (by Proposition 1.11 **(c)**, applied to $\operatorname{Der}(\operatorname{Sym} A)$ and L_a instead of C and f).

Now, the definition of η shows that

$$\begin{aligned}
&\eta\left(\iota_{U,A^-}(a)\right) \\
&= \iota_{U,A^-}(a)\, 1_{\operatorname{Sym} A} = a \rightharpoonup 1_{\operatorname{Sym} A} \\
&\qquad \left(\text{by the definition of the action of } U\left(A^-\right) \text{ on } \operatorname{Sym} A\right) \\
&= \underbrace{a 1_{\operatorname{Sym} A}}_{=a} + \underbrace{L_a\left(1_{\operatorname{Sym} A}\right)}_{=L_a(1)=0} \qquad \left(\text{by the definition of the } A^-\text{-module structure on } \operatorname{Sym} A\right) \\
&= a.
\end{aligned}$$

This proves Theorem 3.10 **(g)**.

(i) We make the following observations:

- The map $K : A^- \to \operatorname{Der}(\operatorname{Sym} A)$ is a Lie algebra homomorphism (according to Theorem 3.10 **(d)**).

- We have

$$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + (K(a))\left(\iota_{\operatorname{Sym},A}(b)\right) - (K(b))\left(\iota_{\operatorname{Sym},A}(a)\right)$$

for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$) [36].

- The A^--module structure on $\operatorname{Sym} A$ defined in Theorem 3.10 **(e)** is precisely the one that is constructed by Theorem 1.15 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$). [37]

- The map $\eta : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 3.10 **(f)** is precisely the map $\eta : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 1.15 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$).

- The filtration of the filtered $\mathbf{k}$-algebra $\operatorname{Sym} A$ is $\left(\bigoplus_{k=0}^{m} \operatorname{Sym}^k A\right)_{m\geq 0}$.

- We have $\iota_{\operatorname{Sym},A}\left(\underbrace{A^-}_{=A}\right) = \iota_{\operatorname{Sym},A}(A) = \operatorname{Sym}^1 A \subseteq \bigoplus_{k=0}^{1} \operatorname{Sym}^k A$.

- The map $K(a) : \operatorname{Sym} A \to \operatorname{Sym} A$ is filtered for every $a \in A^-$ [38].

Hence, we can apply Theorem 2.16 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$, $(C_m)_{m\geq 0} = \left(\bigoplus_{k=0}^{m} \operatorname{Sym}^k A\right)_{m\geq 0}$ and $f = \iota_{\operatorname{Sym},A}$. As a result, we conclude that

$$\begin{aligned}
&\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \\
&\in \iota_{\operatorname{Sym},A}(a_1)\,\iota_{\operatorname{Sym},A}(a_2)\cdots\iota_{\operatorname{Sym},A}(a_n) + \bigoplus_{k=0}^{n-1} \operatorname{Sym}^k A
\end{aligned} \tag{58}$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A^-$. Thus, for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A^-$, we have

$$\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \in a_1 a_2 \cdots a_n + \sum_{k=0}^{n-1} \operatorname{Sym}^k A$$

[36] This was proven in our above proof of Theorem 3.10 **(e)**.

[37] This was proven above.

[38] *Proof.* Let $a \in A^-$. Thus, the definition of K shows that $K(a) = L_a$. Now, the map L_a is graded (by Theorem 3.10 **(b)**), and thus filtered (since any graded $\mathbf{k}$-linear map is filtered). In other words, the map $K(a)$ is filtered (since $K(a) = L_a$). Qed.

[39]. This proves Theorem 3.10 **(i)**.

(h) Recall that η is a $\mathbf{k}$-linear map $U(A^-) \to \operatorname{Sym} A$. In other words, η is a $\mathbf{k}$-linear map $U(A^-) \to \operatorname{Sym}(A^-)$ (since $A^- = A$ as $\mathbf{k}$-modules).

For every $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A^-$, we have

$$\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \in \iota_{\operatorname{Sym},A^-}(a_1)\,\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n) + \sum_{k=0}^{n-1}\operatorname{Sym}^k\left(A^-\right)$$

[40]. Thus, we can apply Lemma 2.24 to $\mathfrak{g} = A^-$ and $\phi = \eta$.

Lemma 2.24 **(a)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta$) shows that the map η is a filtered $\mathbf{k}$-module isomorphism. In particular, the map η is invertible.

But we know that η is an A^--module homomorphism (by Theorem 3.10 **(f)**). Since η is invertible, this shows that η is an A^--module isomorphism. This proves Theorem 3.10 **(h)**.

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[39]*Proof.* Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A^-$.

Recall that we are identifying every $x \in A$ with $\iota_{\operatorname{Sym},A}(x)$. Thus, every $x \in A$ satisfies $x = \iota_{\operatorname{Sym},A}(x) = \iota_{\operatorname{Sym},A^-}(x)$ (since $\iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A^-}$). In particular, every $i \in \{1, 2, \ldots, n\}$ satisfies the identity $a_i = \iota_{\operatorname{Sym},A^-}(a_i)$. Multiplying these identities over all $i \in \{1, 2, \ldots, n\}$, we obtain $a_1 a_2 \cdots a_n = \iota_{\operatorname{Sym},A^-}(a_1)\,\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n)$. Now, (58) becomes

$$\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \in \underbrace{\iota_{\operatorname{Sym},A}(a_1)\,\iota_{\operatorname{Sym},A}(a_2)\cdots\iota_{\operatorname{Sym},A}(a_n)}_{=a_1a_2\cdots a_n} + \underbrace{\bigoplus_{k=0}^{n-1}\operatorname{Sym}^k A}_{\substack{=\sum_{k=0}^{n-1}\operatorname{Sym}^k A \\ \text{(since direct sums} \\ \text{are sums)}}}$$

$$= a_1 a_2 \cdots a_n + \sum_{k=0}^{n-1}\operatorname{Sym}^k A,$$

qed.

[40]*Proof.* Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A^-$. We have $A^- = A$ as $\mathbf{k}$-modules. Hence, $\iota_{\operatorname{Sym},A^-} = \iota_{\operatorname{Sym},A}$.

Recall that we are identifying every $x \in A$ with $\iota_{\operatorname{Sym},A}(x)$. Thus, every $x \in A$ satisfies $x = \iota_{\operatorname{Sym},A}(x) = \iota_{\operatorname{Sym},A^-}(x)$ (since $\iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A^-}$). In particular, every $i \in \{1, 2, \ldots, n\}$ satisfies the identity $a_i = \iota_{\operatorname{Sym},A^-}(a_i)$. Multiplying these identities over all $i \in \{1, 2, \ldots, n\}$, we obtain $a_1 a_2 \cdots a_n = \iota_{\operatorname{Sym},A^-}(a_1)\,\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n)$.

Now, Theorem 3.10 **(i)** shows that

$$\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right)$$
$$\in \underbrace{a_1 a_2 \cdots a_n}_{=\iota_{\operatorname{Sym},A^-}(a_1)\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n)} + \sum_{k=0}^{n-1}\operatorname{Sym}^k \underbrace{A}_{=A^-}$$
$$= \iota_{\operatorname{Sym},A^-}(a_1)\,\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n) + \sum_{k=0}^{n-1}\operatorname{Sym}^k\left(A^-\right),$$

qed.

(j) In our above proof of Theorem 3.10 **(h)**, we have shown that we can apply Lemma 2.24 to $\mathfrak{g} = A^-$ and $\phi = \eta$. Hence, Lemma 2.24 **(a)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta$) shows that the map η is a filtered $\mathbf{k}$-module isomorphism. Also, Lemma 2.24 **(b)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta$) shows that the inverse η^{-1} of the map η is filtered. Thus, both maps η and η^{-1} are filtered. This proves Theorem 3.10 **(j)**.

(k) In our above proof of Theorem 3.10 **(h)**, we have shown that we can apply Lemma 2.24 to $\mathfrak{g} = A^-$ and $\phi = \eta$. Hence, the first claim of Lemma 2.24 **(c)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta$) shows that the map $\operatorname{grad}^{-1}_{\operatorname{Sym}(A^-)} \circ (\operatorname{gr} \eta) : \operatorname{gr}(U(A^-)) \to \operatorname{Sym}(A^-)$ is an inverse to the map PBW_{A^-}. Since $A^- = A$ as $\mathbf{k}$-modules, this rewrites as follows: The map $\operatorname{grad}^{-1}_{\operatorname{Sym} A} \circ (\operatorname{gr} \eta) : \operatorname{gr}(U(A^-)) \to \operatorname{Sym} A$ is an inverse to the map PBW_{A^-}. In particular, the map PBW_{A^-} is invertible. This proves Theorem 3.10 **(k)**.

(l) We first observe the following:

- The map $K : A^- \to \operatorname{Der}(\operatorname{Sym} A)$ is a Lie algebra homomorphism. (This follows from Theorem 3.10 **(d)**.)
- We have $K(A^-) \subseteq \operatorname{Coder}(\operatorname{Sym} A)$ [41].
- We have $\iota_{\operatorname{Sym},A}(A^-) \subseteq \operatorname{Prim}(\operatorname{Sym} A)$ [42].
- The equality
$$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + (K(a))\left(\iota_{\operatorname{Sym},A}(b)\right) - (K(b))\left(\iota_{\operatorname{Sym},A}(a)\right)$$
holds for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$). (This was proven during our above proof of Theorem 3.10 **(e)**.)
- The A^--module structure on $\operatorname{Sym} A$ defined in Theorem 3.10 **(e)** is precisely the one that is constructed by Theorem 1.15 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$). (This follows from our above proof of Theorem 3.10 **(e)**.)
- The map $\eta : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 3.10 **(f)** is precisely the map $\eta : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 1.15 **(b)**

[41] *Proof.* Let $g \in K(A^-)$. Thus, there exists some $a \in A^-$ such that $g = K(a)$. Consider this a. We have $a \in A^- = A$ and $g = K(a) = L_a$ (by the definition of K). But $L_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a coderivation (according to Theorem 3.10 **(c)**). In other words, $L_a \in \operatorname{Coder}(\operatorname{Sym} A)$. Hence, $g = L_a \in \operatorname{Coder}(\operatorname{Sym} A)$.

Now, let us forget that we fixed g. We thus have shown that $g \in \operatorname{Coder}(\operatorname{Sym} A)$ for every $g \in K(A^-)$. In other words, $K(A^-) \subseteq \operatorname{Coder}(\operatorname{Sym} A)$, qed.

[42] *Proof.* Recall that $\iota_{\operatorname{Sym},V}(V) \subseteq \operatorname{Prim}(\operatorname{Sym} V)$ for every $\mathbf{k}$-module V. Applying this to $V = A$, we obtain $\iota_{\operatorname{Sym},A}(A) \subseteq \operatorname{Prim}(\operatorname{Sym} A)$. Since $A^- = A$ as $\mathbf{k}$-modules, this rewrites as $\iota_{\operatorname{Sym},A}(A^-) \subseteq \operatorname{Prim}(\operatorname{Sym} A)$. Qed.

(applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$). (This is clear, because the definitions of these two maps are the same.)

Combining these observations, we see that Theorem 1.20 can be applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$. Consequently, Theorem 1.20 **(b)** shows that the map $\eta : U(A^-) \to \operatorname{Sym} A$ is a $\mathbf{k}$-coalgebra homomorphism. Since η is invertible, this shows that η is a $\mathbf{k}$-coalgebra isomorphism. This finishes the proof of Theorem 3.10 **(l)**.

(m) As we have seen in our proof of Theorem 3.10 **(e)**, we can apply Theorem 1.15 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$. Moreover, the A^--module structure on $\operatorname{Sym} A$ defined in Theorem 3.10 **(e)** is precisely the one that is constructed by Theorem 1.15 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$). Hence, Theorem 1.15 **(c)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$) shows that, for every $a \in A^-$, $b \in \operatorname{Sym} A$ and $c \in \operatorname{Sym} A$, we have

$$a \rightharpoonup (bc) - b \cdot (a \rightharpoonup c) = (K(a))(b) \cdot c + \left[\iota_{\operatorname{Sym},A}(a), b\right] c. \tag{59}$$

Now, fix $a \in A$, $b \in A$ and $c \in \operatorname{Sym} A$. We have $a \in A = A^-$ and $b \in A \subseteq \operatorname{Sym} A$ (since we regard A as a $\mathbf{k}$-submodule of $\operatorname{Sym} A$). Moreover, the $\mathbf{k}$-algebra $\operatorname{Sym} A$ is commutative, and thus the Lie bracket of the Lie algebra $(\operatorname{Sym} A)^-$ is identically 0. Hence, $\left[\iota_{\operatorname{Sym},A}(a), b\right] = 0$. Furthermore, $K(a) = L_a$ (by the definition of K), so that

$$\underbrace{(K(a))}_{=L_a}(b) = L_a(b) = a \triangleright b \qquad \text{(by (55))}.$$

Now, (59) shows that

$$a \rightharpoonup (bc) - b \cdot (a \rightharpoonup c) = \underbrace{(K(a))(b)}_{=a \triangleright b} \cdot c + \underbrace{\left[\iota_{\operatorname{Sym},A}(a), b\right]}_{=0} c = (a \triangleright b) \cdot c + \underbrace{0c}_{=0} = (a \triangleright b) \cdot c.$$

This proves Theorem 3.10 **(m)**. $\square$

Corollary 3.11. Let A be a left pre-Lie algebra. Define the map $\eta : U(A^-) \to \operatorname{Sym} A$ as in Theorem 3.10 **(f)**. Theorem 3.10 **(h)** shows that η is an A^--module isomorphism, thus a $\mathbf{k}$-module isomorphism. Hence, we can define a new, **commutative** multiplication $\boxdot$ on $U(A^-)$ (formally speaking, a $\mathbf{k}$-bilinear map $\boxdot : U(A^-) \times U(A^-) \to U(A^-)$, written in infix notation) by letting

$$x \boxdot y = \eta\left(\eta^{-1}(x) \cdot \eta^{-1}(y)\right) \qquad \text{for all } x \in U\left(A^-\right) \text{ and } y \in U\left(A^-\right).$$

(In other words, this multiplication $\boxdot$ is the result of transporting the multiplication of the $\mathbf{k}$-algebra $\operatorname{Sym} A$ to $U(A^-)$ along the isomorphism η.)

(a) Let us write $(U(A^-), \boxdot)$ for the $\mathbf{k}$-module $U(A^-)$ equipped with the multiplication $\boxdot$. This is a commutative $\mathbf{k}$-algebra with unity $1_{U(A^-)}$.

(b) Let Δ and ϵ denote the comultiplication and the counit of the $\mathbf{k}$-bialgebra $U(A^-)$. The $\mathbf{k}$-algebra $(U(A^-), \boxdot)$, equipped with the comultiplication Δ and the counit ϵ, is a $\mathbf{k}$-bialgebra. We shall denote this $\mathbf{k}$-bialgebra by $(U(A^-), \boxdot)$.

(c) The map η is a $\mathbf{k}$-bialgebra isomorphism $(U(A^-), \boxdot) \to \operatorname{Sym} A$.

(d) We have $1_{(U(A^-),\boxdot)} = 1_{U(A^-)}$.

(e) For every $c \in A^-$, let $\overline{c}$ denote the element $\iota_{U,A^-}(c)$ of $U(A^-)$. For every $n \in \mathbb{N}$, every $a \in A^-$ and every $b_1, b_2, \ldots, b_n \in A^-$, we have

$$\begin{aligned}
&\overline{a} \boxdot \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right) \\
&= \overline{a} \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right) \\
&\qquad - \sum_{k=1}^{n} \overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_{k-1}} \boxdot \left(\overline{a \triangleright b_k}\right) \boxdot \overline{b_{k+1}} \boxdot \overline{b_{k+2}} \boxdot \cdots \boxdot \overline{b_n}.
\end{aligned}$$

Proof of Corollary 3.11. **(a)** The definition of η yields $\eta\left(1_{U(A^-)}\right) = 1_{U(A^-)} 1_{\operatorname{Sym} A} = 1_{\operatorname{Sym} A}$, so that $\eta^{-1}\left(1_{\operatorname{Sym} A}\right) = 1_{U(A^-)}$.

Now, the multiplication of the $\mathbf{k}$-algebra $\operatorname{Sym} A$ is associative and commutative and has neutral element $1_{\operatorname{Sym} A}$. Therefore, the binary operation $\boxdot$ is associative and commutative and has neutral element $\eta^{-1}\left(1_{\operatorname{Sym} A}\right)$ (because the binary operation $\boxdot$ is the result of transporting the multiplication of the $\mathbf{k}$-algebra $\operatorname{Sym} A$ to $U(A^-)$ along the isomorphism η). In other words, the binary operation $\boxdot$ is associative and commutative and has neutral element $1_{U(A^-)}$ (since $\eta^{-1}\left(1_{\operatorname{Sym} A}\right) = 1_{U(A^-)}$). In other words, the $\mathbf{k}$-module $U(A^-)$ equipped with the multiplication $\boxdot$ is a commutative $\mathbf{k}$-algebra with unity $1_{U(A^-)}$. This proves Corollary 3.11 **(a)**.

(b) Let us define a *pre-bialgebra* to mean a $\mathbf{k}$-module V equipped with a $\mathbf{k}$-bilinear map $\mu_V : V \times V \to V$ called its *multiplication*, an element $e_V \in V$ called its *unity*, a $\mathbf{k}$-linear map $\Delta_V : V \to V \otimes V$ called its *comultiplication*, and a $\mathbf{k}$-linear map $\epsilon_V : V \to \mathbf{k}$ called its *counit*. These maps μ_V, Δ_V and ϵ_V and this element e_V are not required to satisfy any axioms (unlike for a $\mathbf{k}$-bialgebra). Thus, a $\mathbf{k}$-bialgebra is a pre-bialgebra satisfying certain axioms. We define a *homomorphism of pre-bialgebras* to be a $\mathbf{k}$-linear map from one pre-bialgebra to another that respects the multiplication, the unity, the comultiplication and the counit. As usual, we define an *isomorphism of pre-bialgebras* to be an invertible homomorphism of pre-bialgebras whose inverse is also a homomorphism of pre-bialgebras. Finally, we say that two pre-bialgebras are *isomorphic* if there exists an isomorphism of pre-bialgebras between them.

Now, let us make one trivial yet crucial observation: If a pre-bialgebra V is isomorphic to a $\mathbf{k}$-bialgebra, then V itself is a $\mathbf{k}$-bialgebra. More precisely, we have the following fact:

Observation 1: If V is a pre-bialgebra, if W is a $\mathbf{k}$-bialgebra, and if $\varphi : V \to W$ is an isomorphism of pre-bialgebras, then V is a $\mathbf{k}$-bialgebra, and φ is a $\mathbf{k}$-bialgebra isomorphism.

Now, the $\mathbf{k}$-module $U(A^-)$, equipped with the multiplication $\boxdot$, the unity $1_{U(A^-)}$, the comultiplication Δ and the counit ϵ is a pre-bialgebra. The map η is an isomorphism of pre-bialgebras from this pre-bialgebra to the $\mathbf{k}$-bialgebra $\operatorname{Sym} A$ [43]. Thus, Observation 1 (applied to $V = U(A^-)$ (equipped with the multiplication $\boxdot$, the unity $1_{U(A^-)}$, the comultiplication Δ and the counit ϵ), $W = \operatorname{Sym} A$ and $\varphi = \eta$) shows that $U(A^-)$ (equipped with the multiplication $\boxdot$, the unity $1_{U(A^-)}$, the comultiplication Δ and the counit ϵ) is a $\mathbf{k}$-bialgebra, and η is a $\mathbf{k}$-bialgebra isomorphism.

In particular, $U(A^-)$ (equipped with the multiplication $\boxdot$, the unity $1_{U(A^-)}$, the comultiplication Δ and the counit ϵ) is a $\mathbf{k}$-bialgebra. In other words, the $\mathbf{k}$-algebra $(U(A^-), \boxdot)$, equipped with the comultiplication Δ and the counit ϵ, is a $\mathbf{k}$-bialgebra. This proves Corollary 3.11 **(b)**.

(c) In our proof of Corollary 3.11 **(b)**, we have shown that η is a $\mathbf{k}$-bialgebra isomorphism. This proves Corollary 3.11 **(c)**.

(d) Corollary 3.11 **(c)** shows that the map η is a $\mathbf{k}$-bialgebra isomorphism $(U(A^-), \boxdot) \to \operatorname{Sym} A$. In particular, the map η is a $\mathbf{k}$-algebra isomorphism $(U(A^-), \boxdot) \to \operatorname{Sym} A$. In particular, the map η is a $\mathbf{k}$-algebra homomorphism $(U(A^-), \boxdot) \to \operatorname{Sym} A$. Hence, $\eta\left(1_{(U(A^-),\boxdot)}\right) = 1_{\operatorname{Sym} A}$.

But the definition of η shows that $\eta\left(1_{U(A^-)}\right) = 1_{U(A^-)} 1_{\operatorname{Sym} A} = 1_{\operatorname{Sym} A}$. Since η is invertible (because η is an isomorphism), this shows that $1_{U(A^-)} = \eta^{-1}\left(1_{\operatorname{Sym} A}\right) = 1_{(U(A^-),\boxdot)}$ (since $\eta\left(1_{(U(A^-),\boxdot)}\right) = 1_{\operatorname{Sym} A}$). This proves Corollary 3.11 **(d)**.

(e) In the following, we will identify every $a \in A$ with the element $\iota_{\operatorname{Sym},A}(a)$ of $\operatorname{Sym} A$ (as it was done in Theorem 3.10).

The map η is a $\mathbf{k}$-algebra isomorphism $(U(A^-), \boxdot) \to \operatorname{Sym} A$ (as we have shown in the proof of Corollary 3.11 **(d)**). Thus, its inverse η^{-1} is a $\mathbf{k}$-algebra isomorphism $\operatorname{Sym} A \to (U(A^-), \boxdot)$.

[43] *Proof.* We must show the following five statements:

1. The map $\eta : U(A^-) \to \operatorname{Sym} A$ is an isomorphism of $\mathbf{k}$-modules.
2. The map η preserves the multiplication (i.e., we have $\eta(x \boxdot y) = \eta(x) \cdot \eta(y)$ for all $x \in U(A^-)$ and $y \in U(A^-)$).
3. The map η preserves the unity (i.e., we have $\eta\left(1_{U(A^-)}\right) = 1_{\operatorname{Sym} A}$).
4. The map η preserves the comultiplication (i.e., we have $(\eta \otimes \eta) \circ \Delta = \Delta_{\operatorname{Sym} A} \circ \eta$).
5. The map η preserves the counit (i.e., we have $\epsilon = \epsilon_{\operatorname{Sym} A} \circ \eta$).

The first and the third of these five statements have already been proven. The second follows from the fact that the binary operation $\boxdot$ is the result of transporting the multiplication of the $\mathbf{k}$-algebra $\operatorname{Sym} A$ to $U(A^-)$ along the isomorphism η. The fourth and fifth statements follow from the fact that η is a coalgebra isomorphism $U(A^-) \to \operatorname{Sym} A$ (by Theorem 3.10 **(l)**). Thus, all five statements are proven, qed.

On the other hand, the map η is an A^--module isomorphism. Hence, its inverse η^{-1} is an A^--module isomorphism as well.

For every $c \in A^-$, we have

$$\eta^{-1}(c) = \overline{c} \tag{60}$$

[44].

Let $n \in \mathbb{N}$. Let $a \in A^-$ and $b_1, b_2, \ldots, b_n \in A^-$. Then, $\overline{a} = \iota_{U,A^-}(a)$ (by the definition of $\overline{a}$).

The definition of the A^--module structure on $\operatorname{Sym} A$ shows that

$$\begin{aligned} a \rightharpoonup (b_1 b_2 \cdots b_n) &= a(b_1 b_2 \cdots b_n) + \underbrace{L_a(b_1 b_2 \cdots b_n)}_{\substack{= \sum_{k=1}^{n} b_1 b_2 \cdots b_{k-1}(a \rhd b_k) b_{k+1} b_{k+2} \cdots b_n \\ \text{(by Theorem 3.10 (a))}}} \\ &= a(b_1 b_2 \cdots b_n) + \sum_{k=1}^{n} b_1 b_2 \cdots b_{k-1}(a \rhd b_k) b_{k+1} b_{k+2} \cdots b_n. \end{aligned}$$

Applying the map η^{-1} to this equality, we obtain

$$\begin{aligned} &\eta^{-1}(a \rightharpoonup (b_1 b_2 \cdots b_n)) \\ &= \eta^{-1}\left(a(b_1 b_2 \cdots b_n) + \sum_{k=1}^{n} b_1 b_2 \cdots b_{k-1}(a \rhd b_k) b_{k+1} b_{k+2} \cdots b_n \right) \\ &= \underbrace{\eta^{-1}(a)}_{\substack{=\overline{a} \\ \text{(by (60))}}} \boxdot \left(\underbrace{\eta^{-1}(b_1)}_{\substack{=\overline{b_1} \\ \text{(by (60))}}} \boxdot \underbrace{\eta^{-1}(b_2)}_{\substack{=\overline{b_2} \\ \text{(by (60))}}} \boxdot \cdots \boxdot \underbrace{\eta^{-1}(b_n)}_{\substack{=\overline{b_n} \\ \text{(by (60))}}} \right) \\ &\qquad + \sum_{k=1}^{n} \underbrace{\eta^{-1}(b_1)}_{\substack{=\overline{b_1} \\ \text{(by (60))}}} \boxdot \underbrace{\eta^{-1}(b_2)}_{\substack{=\overline{b_2} \\ \text{(by (60))}}} \boxdot \cdots \boxdot \underbrace{\eta^{-1}(b_{k-1})}_{\substack{=\overline{b_{k-1}} \\ \text{(by (60))}}} \boxdot \underbrace{\eta^{-1}(a \rhd b_k)}_{\substack{=\overline{a \rhd b_k} \\ \text{(by (60))}}} \\ &\qquad \boxdot \underbrace{\eta^{-1}(b_{k+1})}_{\substack{=\overline{b_{k+1}} \\ \text{(by (60))}}} \boxdot \underbrace{\eta^{-1}(b_{k+2})}_{\substack{=\overline{b_{k+2}} \\ \text{(by (60))}}} \boxdot \cdots \boxdot \underbrace{\eta(b_n)}_{\substack{=\overline{b_n} \\ \text{(by (60))}}} \\ &\qquad \left(\text{since } \eta^{-1} \text{ is a } \mathbf{k}\text{-algebra isomorphism } \operatorname{Sym} A \to \left(U(A^-), \boxdot \right) \right) \\ &= \overline{a} \boxdot \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n} \right) + \sum_{k=1}^{n} \overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_{k-1}} \boxdot \left(\overline{a \rhd b_k} \right) \boxdot \overline{b_{k+1}} \boxdot \overline{b_{k+2}} \boxdot \cdots \boxdot \overline{b_n}. \end{aligned}$$

[44] *Proof of (60):* Let $c \in A^-$. Then, $\overline{c} = \iota_{U,A^-}(c)$ (by the definition of $\overline{c}$). Hence, $\eta\left(\underbrace{\overline{c}}_{=\iota_{U,A^-}(c)}\right) = \eta\left(\iota_{U,A^-}(c)\right) = c$ (by Theorem 3.10 **(g)**, applied to $a = c$). Hence, $\eta^{-1}(c) = \overline{c}$. This proves (60).

Comparing this with

$$\begin{aligned}
&\eta^{-1}\left(a \rightharpoonup \left(b_1 b_2 \cdots b_n\right)\right) \\
&= a \rightharpoonup \underbrace{\eta^{-1}\left(b_1 b_2 \cdots b_n\right)}_{\substack{=\eta^{-1}(b_1)\boxdot\eta^{-1}(b_2)\boxdot\cdots\boxdot\eta^{-1}(b_n) \\ \text{(since } \eta^{-1} \text{ is a } \mathbf{k}\text{-algebra isomorphism } \operatorname{Sym} A \to \left(U\left(A^-\right),\boxdot\right))}} \\
&\qquad \left(\text{since } \eta^{-1} \text{ is an } A^-\text{-module isomorphism}\right) \\
&= a \rightharpoonup \left(\underbrace{\eta^{-1}\left(b_1\right)}_{\substack{=\overline{b_1} \\ \text{(by (60))}}} \boxdot \underbrace{\eta^{-1}\left(b_2\right)}_{\substack{=\overline{b_2} \\ \text{(by (60))}}} \boxdot \cdots \boxdot \underbrace{\eta^{-1}\left(b_n\right)}_{\substack{=\overline{b_n} \\ \text{(by (60))}}}\right) \\
&= a \rightharpoonup \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right) = \underbrace{\iota_{U,A^-}\left(a\right)}_{=\overline{a}} \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right) \\
&\qquad \left(\text{by the definition of the } A^-\text{-module structure on } U\left(A^-\right)\right) \\
&= \overline{a} \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\overline{a} \boxdot \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right) + \sum_{k=1}^{n} \overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_{k-1}} \boxdot \left(\overline{a \triangleright b_k}\right) \boxdot \overline{b_{k+1}} \boxdot \overline{b_{k+2}} \boxdot \cdots \boxdot \overline{b_n} \\
&= \overline{a} \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right).
\end{aligned}$$

In other words,

$$\begin{aligned}
&\overline{a} \boxdot \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right) \\
&= \overline{a} \left(\overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_n}\right) - \sum_{k=1}^{n} \overline{b_1} \boxdot \overline{b_2} \boxdot \cdots \boxdot \overline{b_{k-1}} \boxdot \left(\overline{a \triangleright b_k}\right) \boxdot \overline{b_{k+1}} \boxdot \overline{b_{k+2}} \boxdot \cdots \boxdot \overline{b_n}.
\end{aligned}$$

This proves Corollary 3.11 **(e)**. □

4. The normal ordered product of differential operators

In this section, we shall explore an application of Theorem 3.10 to the calculus of differential operators, answering the question posed in [MO102874].

Definition 4.1. For the rest of Section 4, we shall be working in the following setup:

Assume that $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra (so that $1, 2, 3, \ldots$ are invertible in $\mathbf{k}$). Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let G denote the nm-element set $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$. Let $\mathcal{A}$ be the polynomial ring $\mathbf{k}\left[x_{i,j} \mid (i,j) \in G\right]$ in the nm (commuting) indeterminates $x_{i,j}$. For every $a \in \mathcal{A}$, we shall denote the map $\mathcal{A} \to \mathcal{A}$, $u \mapsto au$ (the "multiplication by a" operator) by a (by abuse of notation). Such an operator will be called a *multiplication operator*.

The abuse of notation that we have just introduced makes certain notations ambiguous. Namely, if $f : \mathcal{A} \to \mathcal{A}$ is a map, and if $a \in \mathcal{A}$ is element, then the expression "fa" (or "$f(a)$") can mean two different things: It can mean the image of a under the map f, or it can mean the composition $f \circ a$ of the map f with the "multiplication by a" operator (which we have just agreed to denote by a). In order to disambiguate such expressions, let us agree to denote the image of a under the map f by $f[a]$ (instead of fa or $f(a)$). Thus, for example, $\dfrac{\partial}{\partial x_{i,j}}\left[x_{i,j}x_{u,v}\right] = x_{u,v}$ (if $(u,v) \neq (i,j)$), whereas $\dfrac{\partial}{\partial x_{i,j}}\left(x_{i,j}x_{u,v}\right)$ is the endomorphism of $\mathcal{A}$ defined as the composition $\dfrac{\partial}{\partial x_{i,j}} \circ \left(x_{i,j}x_{u,v}\right)$.

Let $\mathcal{D}$ denote the $\mathbf{k}$-subalgebra of $\operatorname{End}\mathcal{A}$ generated by the multiplication operators $x_{i,j}$ (for $(i,j) \in G$) and the differential operators $\dfrac{\partial}{\partial x_{i,j}}$ (for $(i,j) \in G$). This $\mathbf{k}$-algebra $\mathcal{D}$ is known as the $\mathbf{k}$-*algebra of differential operators* in the nm variables $x_{i,j}$ *with polynomial coefficients*. It is well-known that

$$\left(M\left(\left(x_{i,j}\right)_{(i,j)\in G}\right) \cdot N\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)\right)_{M \text{ and } N \text{ are two monomials}} \tag{61}$$

is a basis of the $\mathbf{k}$-module $\mathcal{D}$ (where "monomial" means "monomial in the variables $x_{i,j}$ for $(i,j) \in G$").

On the other hand, let $\mathcal{A}'$ be the polynomial ring $\mathbf{k}\left[\partial_{i,j} \mid (i,j) \in G\right]$ in the nm (commuting) indeterminates $\partial_{i,j}$. Let $\xi : \mathcal{A} \otimes \mathcal{A}' \to \mathcal{D}$ be the $\mathbf{k}$-linear map defined by

$$\xi(P \otimes Q) = P \cdot Q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \qquad \text{for all } P \in \mathcal{A} \text{ and } Q \in \mathcal{A}'.$$

Clearly, the family $\left(M\left(\left(x_{i,j}\right)_{(i,j)\in G}\right) \otimes N\left(\left(\partial_{i,j}\right)_{(i,j)\in G}\right)\right)_{M \text{ and } N \text{ are two monomials}}$ is a basis of the $\mathbf{k}$-module $\mathcal{A} \otimes \mathcal{A}'$. The $\mathbf{k}$-linear map ξ takes this basis to the family (61); but this latter family is a basis of the $\mathbf{k}$-module $\mathcal{D}$. Hence, the $\mathbf{k}$-linear map ξ is a $\mathbf{k}$-module isomorphism.

The map ξ is not (in general) a $\mathbf{k}$-algebra homomorphism. Indeed, the $\mathbf{k}$-algebra $\mathcal{A} \otimes \mathcal{A}'$ is commutative, while the $\mathbf{k}$-algebra $\mathcal{D}$ usually is not. However, we can define a new, **commutative** multiplication $\boxdot$ on $\mathcal{D}$ (formally speaking, a $\mathbf{k}$-bilinear map $\boxdot : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$, written in infix notation) by letting

$$A \boxdot B = \xi \left(\xi^{-1} (A) \cdot \xi^{-1} (B) \right) \qquad \text{for all } A \in \mathcal{D} \text{ and } B \in \mathcal{D}.$$

(In other words, this multiplication $\boxdot$ is the result of transporting the multiplication of the $\mathbf{k}$-algebra $\mathcal{A} \otimes \mathcal{A}'$ to $\mathcal{D}$ along the isomorphism ξ.)

The multiplication $\boxdot$ is called the *normal ordered product* (or *normally ordered product*) on $\mathcal{D}$. Let us write $(\mathcal{D}, \boxdot)$ for the $\mathbf{k}$-module $\mathcal{D}$ equipped with the multiplication $\boxdot$. This is a commutative $\mathbf{k}$-algebra.

(The multiplication $\boxdot$ so far has nothing to do with the multiplication $\boxdot$ in Corollary 3.11; but we will relate them soon enough.)

Now, let $\mathfrak{gl}_n$ denote the Lie algebra $(\mathrm{M}_n(\mathbf{k}))^-$ (where $\mathrm{M}_n(\mathbf{k}) = \mathbf{k}^{n \times n}$ is the ring of $n \times n$-matrices over $\mathbf{k}$). The $\mathbf{k}$-module $\mathfrak{gl}_n$ has a basis $(E_{i,j})_{(i,j) \in \{1,2,\ldots,n\}^2}$ consisting of the elementary matrices (i.e., for every $(i,j) \in \{1,2,\ldots,n\}^2$, the matrix $E_{i,j}$ is the $n \times n$-matrix whose (i,j)-th entry is 1 and whose all other entries are 0). We define a $\mathbf{k}$-linear map $\omega : \mathfrak{gl}_n \to \mathcal{D}^-$ by

$$\omega (E_{i,j}) = \sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{j,k}} \qquad \text{for every } (i,j) \in \{1,2,\ldots,n\}^2 .$$

Then, this map $\omega : \mathfrak{gl}_n \to \mathcal{D}^-$ is a Lie algebra homomorphism (this will be proven in Proposition 4.2 below). Thus, Theorem 1.8 (applied to $\mathfrak{gl}_n$, $\mathcal{D}$ and ω instead of $\mathfrak{g}$, A and f) shows that there exists a unique $\mathbf{k}$-algebra homomorphism $F : U(\mathfrak{gl}_n) \to \mathcal{D}$ such that $\omega = F \circ \iota_{U,\mathfrak{gl}_n}$. Denote this F by Ω. Thus, Ω is a $\mathbf{k}$-algebra homomorphism $U(\mathfrak{gl}_n) \to \mathcal{D}$ such that $\omega = \Omega \circ \iota_{U,\mathfrak{gl}_n}$.

Notice that we have required $\mathbf{k}$ to be a $\mathbb{Q}$-algebra in order for ξ to be an isomorphism. We could get rid of this requirement by changing the definition of $\mathcal{D}$ (in such a way that the elements of $\mathcal{D}$ become "formal differential operators" rather than actual endomorphisms of $\mathcal{A}$; this would then require defining the product of the $\mathbf{k}$-algebra $\mathcal{D}$ independently of composition of differential operators). Alternatively, it might be possible to define the multiplication $\boxdot$ without reference to ξ, though this would be significantly more difficult (if possible).

Proposition 4.2. We use the setup of Definition 4.1. The map $\omega : \mathfrak{gl}_n \to \mathcal{D}^-$ is a Lie algebra homomorphism.

Proof of Proposition 4.2. We shall show that

$$\omega ([x,y]) = [\omega (x), \omega (y)] \qquad \text{for every } x \in \mathfrak{gl}_n \text{ and } y \in \mathfrak{gl}_n. \tag{62}$$

Proof of (62): Let $x \in \mathfrak{gl}_n$ and $y \in \mathfrak{gl}_n$. We must prove the equality (62). This equality is $\mathbf{k}$-linear in each of x and y. Hence, we can WLOG assume that both x and y belong to the basis $\left(E_{i,j}\right)_{(i,j)\in\{1,2,\ldots,n\}^2}$ of the $\mathbf{k}$-module $\mathfrak{gl}_n$. Assume this. Thus, $x = E_{i,j}$ and $y = E_{u,v}$ for some $(i,j) \in \{1,2,\ldots,n\}^2$ and $(u,v) \in \{1,2,\ldots,n\}^2$. Consider these (i,j) and (u,v).

We shall use the notation $\delta_{p,q}$ for the integer $\begin{cases} 1, & \text{if } p = q; \\ 0, & \text{if } p \neq q \end{cases}$ whenever p and q are two objects.

It is well-known that the Lie bracket of $\mathfrak{gl}_n$ satisfies

$$\left[E_{i,j}, E_{u,v}\right] = \delta_{j,u} E_{i,v} - \delta_{v,i} E_{u,j}.$$

Hence, $\left[\underbrace{x}_{=E_{i,j}}, \underbrace{y}_{=E_{u,v}}\right] = \left[E_{i,j}, E_{u,v}\right] = \delta_{j,u} E_{i,v} - \delta_{v,i} E_{u,j}$. Applying the map ω to both sides of this equality, we obtain

$$\omega\left(\left[x,y\right]\right) = \omega\left(\delta_{j,u} E_{i,v} - \delta_{v,i} E_{u,j}\right) = \delta_{j,u} \underbrace{\omega\left(E_{i,v}\right)}_{\substack{=\sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{v,k}} \\ \text{(by the definition of } \omega\text{)}}} - \delta_{v,i} \underbrace{\omega\left(E_{u,j}\right)}_{\substack{=\sum_{k=1}^{m} x_{u,k} \frac{\partial}{\partial x_{j,k}} \\ \text{(by the definition of } \omega\text{)}}}$$

$$= \delta_{j,u} \sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{v,k}} - \delta_{v,i} \sum_{k=1}^{m} x_{u,k} \frac{\partial}{\partial x_{j,k}}. \tag{63}$$

On the other hand, it is known that every $(a,b) \in G$ and $(c,d) \in G$ satisfy

$$x_{a,b} x_{c,d} = x_{c,d} x_{a,b}; \tag{64}$$

$$\frac{\partial}{\partial x_{a,b}} \frac{\partial}{\partial x_{c,d}} = \frac{\partial}{\partial x_{c,d}} \frac{\partial}{\partial x_{a,b}}; \tag{65}$$

$$\frac{\partial}{\partial x_{a,b}} \cdot x_{c,d} = x_{c,d} \frac{\partial}{\partial x_{a,b}} + \delta_{a,c} \delta_{b,d}. \tag{66}$$

Now,

$$\omega\left(\underbrace{x}_{=E_{i,j}}\right) = \omega\left(E_{i,j}\right) = \sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{j,k}} \qquad \text{(by the definition of } \omega\text{)}$$

and

$$\omega\left(\underbrace{y}_{=E_{u,v}}\right) = \omega\left(E_{u,v}\right) = \sum_{k=1}^{m} x_{u,k} \frac{\partial}{\partial x_{v,k}} \qquad \text{(by the definition of } \omega\text{)}$$

$$= \sum_{h=1}^{m} x_{u,h} \frac{\partial}{\partial x_{v,h}}.$$

Hence,

$$\begin{aligned}
&\Bigg[\underbrace{\omega(x)}_{=\sum_{k=1}^m x_{i,k}\frac{\partial}{\partial x_{j,k}}}, \underbrace{\omega(y)}_{=\sum_{h=1}^m x_{u,h}\frac{\partial}{\partial x_{v,h}}}\Bigg] \\
&= \left[\sum_{k=1}^m x_{i,k}\frac{\partial}{\partial x_{j,k}}, \sum_{h=1}^m x_{u,h}\frac{\partial}{\partial x_{v,h}}\right] = \sum_{k=1}^m \sum_{h=1}^m \left[x_{i,k}\frac{\partial}{\partial x_{j,k}}, x_{u,h}\frac{\partial}{\partial x_{v,h}}\right].
\end{aligned} \tag{67}$$

But every $k \in \{1, 2, \ldots, m\}$ and $h \in \{1, 2, \ldots, m\}$ satisfy

$$\begin{aligned}
&\left[x_{i,k}\frac{\partial}{\partial x_{j,k}}, x_{u,h}\frac{\partial}{\partial x_{v,h}}\right] \\
&= x_{i,k} \underbrace{\frac{\partial}{\partial x_{j,k}} \cdot x_{u,h}}_{\substack{= x_{u,h}\frac{\partial}{\partial x_{j,k}} + \delta_{j,u}\delta_{k,h} \\ \text{(by (66), applied to} \\ (a,b)=(j,k) \text{ and } (c,d)=(u,h))}} \frac{\partial}{\partial x_{v,h}} - x_{u,h} \underbrace{\frac{\partial}{\partial x_{v,h}} \cdot x_{i,k}}_{\substack{= x_{i,k}\frac{\partial}{\partial x_{v,h}} + \delta_{v,i}\delta_{h,k} \\ \text{(by (66), applied to} \\ (a,b)=(v,h) \text{ and } (c,d)=(i,k))}} \frac{\partial}{\partial x_{j,k}} \\
&= x_{i,k}\left(x_{u,h}\frac{\partial}{\partial x_{j,k}} + \delta_{j,u}\delta_{k,h}\right)\frac{\partial}{\partial x_{v,h}} - x_{u,h}\left(x_{i,k}\frac{\partial}{\partial x_{v,h}} + \delta_{v,i}\delta_{h,k}\right)\frac{\partial}{\partial x_{j,k}} \\
&= \underbrace{x_{i,k}x_{u,h}}_{\substack{= x_{u,h}x_{i,k} \\ \text{(by (64), applied to} \\ (a,b)=(i,k) \text{ and } (c,d)=(u,h))}} \underbrace{\frac{\partial}{\partial x_{j,k}}\frac{\partial}{\partial x_{v,h}}}_{\substack{= \frac{\partial}{\partial x_{v,h}}\frac{\partial}{\partial x_{j,k}} \\ \text{(by (65), applied to} \\ (a,b)=(j,k) \text{ and } (c,d)=(v,h))}} \\
&\qquad + x_{i,k}\delta_{j,u}\delta_{k,h}\frac{\partial}{\partial x_{v,h}} - x_{u,h}x_{i,k}\frac{\partial}{\partial x_{v,h}}\frac{\partial}{\partial x_{j,k}} - x_{u,h}\delta_{v,i}\underbrace{\delta_{h,k}}_{=\delta_{k,h}}\frac{\partial}{\partial x_{j,k}} \\
&= x_{u,h}x_{i,k}\frac{\partial}{\partial x_{v,h}}\frac{\partial}{\partial x_{j,k}} + x_{i,k}\delta_{j,u}\delta_{k,h}\frac{\partial}{\partial x_{v,h}} - x_{u,h}x_{i,k}\frac{\partial}{\partial x_{v,h}}\frac{\partial}{\partial x_{j,k}} - x_{u,h}\delta_{v,i}\delta_{k,h}\frac{\partial}{\partial x_{j,k}} \\
&= x_{i,k}\delta_{j,u}\delta_{k,h}\frac{\partial}{\partial x_{v,h}} - x_{u,h}\delta_{v,i}\delta_{k,h}\frac{\partial}{\partial x_{j,k}} = \delta_{k,h}\left(x_{i,k}\delta_{j,u}\frac{\partial}{\partial x_{v,h}} - x_{u,h}\delta_{v,i}\frac{\partial}{\partial x_{j,k}}\right).
\end{aligned}$$

Thus, (67) becomes

$$\begin{aligned}
&[\omega(x), \omega(y)] \\
&= \sum_{k=1}^{m} \sum_{h=1}^{m} \underbrace{\left[x_{i,k} \frac{\partial}{\partial x_{j,k}}, x_{u,h} \frac{\partial}{\partial x_{v,h}} \right]}_{= \delta_{k,h} \left(x_{i,k} \delta_{j,u} \frac{\partial}{\partial x_{v,h}} - x_{u,h} \delta_{v,i} \frac{\partial}{\partial x_{j,k}} \right)} \\
&= \sum_{k=1}^{m} \underbrace{\sum_{h=1}^{m} \delta_{k,h} \left(x_{i,k} \delta_{j,u} \frac{\partial}{\partial x_{v,h}} - x_{u,h} \delta_{v,i} \frac{\partial}{\partial x_{j,k}} \right)}_{\substack{= x_{i,k} \delta_{j,u} \frac{\partial}{\partial x_{v,k}} - x_{u,k} \delta_{v,i} \frac{\partial}{\partial x_{j,k}} \\ \text{(because the presence of the } \delta_{k,h} \text{ factor in the sum} \\ \text{ensures that only the addend for } h=k \text{ can be nonzero)}}} \\
&= \sum_{k=1}^{m} \left(x_{i,k} \delta_{j,u} \frac{\partial}{\partial x_{v,k}} - x_{u,k} \delta_{v,i} \frac{\partial}{\partial x_{j,k}} \right) \\
&= \delta_{j,u} \sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{v,k}} - \delta_{v,i} \sum_{k=1}^{m} x_{u,k} \frac{\partial}{\partial x_{j,k}} = \omega([x, y]) \qquad \text{(by (63))}.
\end{aligned}$$

This proves (62).

The map $\omega : \mathfrak{gl}_n \to \mathcal{D}^-$ is $\mathbf{k}$-linear. Thus, (62) shows that ω is a Lie algebra homomorphism. This proves Proposition 4.2. □

Corollary 4.3. We use the setup of Definition 4.1.

(a) We can define a $\mathfrak{gl}_n$-module structure on the $\mathbf{k}$-module $\mathcal{D}$ by setting

$$a \rightharpoonup F = \omega(a) \cdot F \qquad \text{for every } a \in \mathfrak{gl}_n \text{ and } F \in \mathcal{D}$$

(where we are using the fact that every $a \in \mathfrak{gl}_n$ satisfies $\omega(a) \in \mathcal{D}^- = \mathcal{D}$). Let us make $\mathcal{D}$ into a $\mathfrak{gl}_n$-module using this $\mathfrak{gl}_n$-module structure.

(b) The map $\Omega : U(\mathfrak{gl}_n) \to \mathcal{D}$ is a $\mathfrak{gl}_n$-module homomorphism.

Proof of Corollary 4.3. **(a)** Define a map $\mu : \mathfrak{gl}_n \times \mathcal{D} \to \mathcal{D}$ by

$$(\mu(a, F) = \omega(a) \cdot F \qquad \text{for every } a \in \mathfrak{gl}_n \text{ and } f \in \mathcal{D}).$$

Clearly, this map μ is $\mathbf{k}$-bilinear (since the map ω is $\mathbf{k}$-linear). Let us denote $\mu(a, v)$ by $a \rightharpoonup v$ for every $a \in \mathfrak{gl}_n$ and $v \in \mathcal{D}$. Thus, we have

$$a \rightharpoonup v = \mu(a, v) = \omega(a) \cdot v \qquad \text{(by the definition of } \mu\text{)} \tag{68}$$

for every $a \in \mathfrak{gl}_n$ and $v \in \mathcal{D}$. Then, we have

$$[a, b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v) \text{ for every } a \in \mathfrak{gl}_n,\ b \in \mathfrak{gl}_n \text{ and } v \in \mathcal{D}$$

[45]. Hence, the map μ is a $\mathfrak{gl}_n$-module structure on the $\mathbf{k}$-module $\mathcal{D}$ (since the map μ is $\mathbf{k}$-bilinear). In other words, we can define a $\mathfrak{gl}_n$-module structure on the $\mathbf{k}$-module $\mathcal{D}$ by setting

$$a \rightharpoonup F = \omega(a) \cdot F \qquad \text{for every } a \in \mathfrak{gl}_n \text{ and } F \in \mathcal{D}$$

(because this definition yields exactly the $\mathfrak{gl}_n$-module structure μ). This proves Corollary 4.3 **(a)**.

(b) Let $a \in \mathfrak{gl}_n$ and $z \in U(\mathfrak{gl}_n)$. We shall show that $\Omega(a \rightharpoonup z) = a \rightharpoonup \Omega(z)$.

Indeed, the definition of the $\mathfrak{gl}_n$-module structure on $U(\mathfrak{gl}_n)$ shows that $a \rightharpoonup z = \iota_{U,\mathfrak{gl}_n}(a) \cdot z$.

On the other hand, the definition of the $\mathfrak{gl}_n$-structure on $\mathcal{D}$ shows that

$$\begin{aligned} a \rightharpoonup \Omega(z) &= \underbrace{\omega}_{=\Omega \circ \iota_{U,\mathfrak{gl}_n}}(a) \cdot \Omega(z) = \underbrace{\left(\Omega \circ \iota_{U,\mathfrak{gl}_n}\right)(a)}_{=\Omega\left(\iota_{U,\mathfrak{gl}_n}(a)\right)} \cdot \Omega(z) \\ &= \Omega\left(\iota_{U,\mathfrak{gl}_n}(a)\right) \cdot \Omega(z) = \Omega\left(\iota_{U,\mathfrak{gl}_n}(a) \cdot z\right) \end{aligned}$$

(since Ω is a $\mathbf{k}$-algebra homomorphism). Comparing this with

$$\Omega\left(\underbrace{a \rightharpoonup z}_{=\iota_{U,\mathfrak{gl}_n}(a) \cdot z}\right) = \Omega\left(\iota_{U,\mathfrak{gl}_n}(a) \cdot z\right),$$

[45]*Proof.* Let $a \in \mathfrak{gl}_n$, $b \in \mathfrak{gl}_n$ and $v \in \mathcal{D}$. Then, ω is a Lie algebra homomorphism from $\mathfrak{gl}_n$ to $\mathcal{D}^-$ (by Proposition 4.2). Thus,

$$\omega([a,b]) = [\omega(a), \omega(b)] = \omega(a) \cdot \omega(b) - \omega(b) \cdot \omega(a)$$

(by the definition of the Lie bracket on $\mathcal{D}^-$). Now,

$$\begin{aligned} &[a,b] \rightharpoonup v \\ &= \underbrace{\omega([a,b])}_{=\omega(a)\cdot\omega(b)-\omega(b)\cdot\omega(a)} \cdot v \qquad (\text{by (68), applied to } [a,b] \text{ instead of } a) \\ &= (\omega(a) \cdot \omega(b) - \omega(b) \cdot \omega(a)) \cdot v \\ &= \omega(a) \cdot \omega(b) \cdot v - \omega(b) \cdot \omega(a) \cdot v. \end{aligned}$$

Comparing this with

$$\begin{aligned} &a \rightharpoonup \underbrace{(b \rightharpoonup v)}_{\substack{=\omega(b)\cdot v \\ \text{(by (68), applied to} \\ b \text{ instead of } a)}} - b \rightharpoonup \underbrace{(a \rightharpoonup v)}_{\substack{=\omega(a)\cdot v \\ \text{(by (68))}}} \\ &= \underbrace{a \rightharpoonup (\omega(b) \cdot v)}_{\substack{=\omega(a)\cdot\omega(b)\cdot v \\ \text{(by (68), applied to} \\ \omega(b)\cdot v \text{ instead of } v)}} - \underbrace{b \rightharpoonup (\omega(a) \cdot v)}_{\substack{=\omega(b)\cdot\omega(a)\cdot v \\ \text{(by (68), applied to} \\ b \text{ and } \omega(a)\cdot v \text{ instead of } a \text{ and } v)}} \\ &= \omega(a) \cdot \omega(b) \cdot v - \omega(b) \cdot \omega(a) \cdot v, \end{aligned}$$

we obtain $[a,b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v)$. Qed.

we obtain $\Omega(a \rightharpoonup z) = a \rightharpoonup \Omega(z)$.

Let us now forget that we fixed a and z. We thus have shown that $\Omega(a \rightharpoonup z) = a \rightharpoonup \Omega(z)$ for every $a \in \mathfrak{gl}_n$ and $z \in U(\mathfrak{gl}_n)$. In other words, the map $\Omega : U(\mathfrak{gl}_n) \to \mathcal{D}$ is a $\mathfrak{gl}_n$-module homomorphism. This proves Corollary 4.3 **(b)**. $\square$

Proposition 4.4. We use the setup of Definition 4.1. Then, $\omega(\mathfrak{gl}_n) \subseteq \operatorname{Der}\mathcal{A}$.

Proof of Proposition 4.4. We want to prove that $\omega(\mathfrak{gl}_n) \subseteq \operatorname{Der}\mathcal{A}$. In other words, we want to prove that $\omega(b) \in \operatorname{Der}\mathcal{A}$ for every $b \in \mathfrak{gl}_n$. So let us fix $b \in \mathfrak{gl}_n$.

We must prove the relation $\omega(b) \in \operatorname{Der}\mathcal{A}$. This relation is $\mathbf{k}$-linear in b. Hence, we can WLOG assume that b belongs to the basis $(E_{i,j})_{(i,j)\in\{1,2,\ldots,n\}^2}$ of the $\mathbf{k}$-module $\mathfrak{gl}_n$. Assume this. Thus, $b = E_{i,j}$ for some $(i,j) \in \{1,2,\ldots,n\}^2$. Consider this (i,j). Hence,

$$\omega\left(\underbrace{b}_{=E_{i,j}}\right) = \omega(E_{i,j}) = \sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}} \qquad \text{(by the definition of } \omega\text{)}. \tag{69}$$

Now, let $p \in \mathcal{A}$ and $q \in \mathcal{A}$.

Let $k \in \{1,2,\ldots,m\}$. Then, $\dfrac{\partial}{\partial x_{j,k}} \in \operatorname{End}\mathcal{A}$ is a derivation of $\mathcal{A}$ (since $\dfrac{\partial}{\partial x_{j,k}}$ is a partial derivative operator). Hence,

$$\frac{\partial}{\partial x_{j,k}}[pq] = p\frac{\partial}{\partial x_{j,k}}[q] + \frac{\partial}{\partial x_{j,k}}[p]\cdot q. \tag{70}$$

(Recall that we are using the notation $f[a]$ for the image of an element $a \in \mathcal{A}$ under a map $f : \mathcal{A} \to \mathcal{A}$.)

Now, let us forget that we fixed k. We thus have proven (70) for every $k \in$

$\{1, 2, \ldots, m\}$. Now,

$$\begin{aligned}
&\underbrace{(\omega(b))}_{\substack{=\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}} \\ \text{(by (69))}}} [pq] \\
&= \left(\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}\right)[pq] = \sum_{k=1}^{m} x_{i,k} \underbrace{\frac{\partial}{\partial x_{j,k}}[pq]}_{\substack{=p\frac{\partial}{\partial x_{j,k}}[q]+\frac{\partial}{\partial x_{j,k}}[p]\cdot q \\ \text{(by (70))}}} \\
&= \sum_{k=1}^{m} x_{i,k}\left(p\frac{\partial}{\partial x_{j,k}}[q] + \frac{\partial}{\partial x_{j,k}}[p]\cdot q\right) \\
&= \sum_{k=1}^{m} \underbrace{x_{i,k}p}_{=px_{i,k}} \frac{\partial}{\partial x_{j,k}}[q] + \underbrace{\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}[p]\cdot q}_{=\left(\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}[p]\right)\cdot q} \\
&= \underbrace{\sum_{k=1}^{m} px_{i,k}\frac{\partial}{\partial x_{j,k}}[q]}_{\substack{=p\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}[q] \\ =p\left(\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}\right)[q]}} + \underbrace{\left(\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}[p]\right)}_{=\left(\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}\right)[p]}\cdot q \\
&= p\underbrace{\left(\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}\right)}_{\substack{=\omega(b) \\ \text{(by (69))}}}[q] + \underbrace{\left(\sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}\right)}_{\substack{=\omega(b) \\ \text{(by (69))}}}[p]\cdot q \\
&= p(\omega(b))[q] + (\omega(b))[p]\cdot q.
\end{aligned}$$

Now let us forget that we fixed p and q. We thus have shown that $(\omega(b))[pq] = p(\omega(b))[q] + (\omega(b))[p]\cdot q$ for every $p \in \mathcal{A}$ and $q \in \mathcal{A}$. In other words, the map $\omega(b)$ is a derivation. In other words, $\omega(b) \in \operatorname{Der}\mathcal{A}$. This proves Proposition 4.4. □

Proposition 4.5. We use the setup of Definition 4.1.

(a) The map ξ is a $\mathbf{k}$-algebra isomorphism from $\mathcal{A}\otimes\mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$.

(b) We have $1_{(\mathcal{D},\boxdot)} = 1_{\mathcal{D}}$.

Proof of Proposition 4.5. **(a)** The map ξ is a $\mathbf{k}$-module isomorphism $\mathcal{A} \otimes \mathcal{A}' \to \mathcal{D}$, and thus is invertible. Moreover, it sends products in $\mathcal{A} \otimes \mathcal{A}'$ to products inside the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$ (i.e., it satisfies $\xi(ab) = \xi(a) \boxdot \xi(b)$ for every $a \in \mathcal{A} \otimes \mathcal{A}'$ and $b \in \mathcal{A} \otimes \mathcal{A}'$), because the multiplication $\boxdot$ on $\mathcal{D}$ is the result of transporting the multiplication of the $\mathbf{k}$-algebra $\mathcal{A} \otimes \mathcal{A}'$ to $\mathcal{D}$ along the isomorphism ξ. Hence, this map ξ is a $\mathbf{k}$-algebra isomorphism from $\mathcal{A} \otimes \mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$ (because any invertible $\mathbf{k}$-linear map between two $\mathbf{k}$-algebras which sends products in its domain to products inside its target must be a $\mathbf{k}$-algebra isomorphism). This proves Proposition 4.5 **(a)**.

(b) Proposition 4.5 **(a)** shows that the map ξ is a $\mathbf{k}$-algebra isomorphism from $\mathcal{A} \otimes \mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$. Hence, ξ sends the unity of the $\mathbf{k}$-algebra $\mathcal{A} \otimes \mathcal{A}'$ to the unity of the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$. In other words, $\xi(1_{\mathcal{A}\otimes\mathcal{A}'}) = 1_{(\mathcal{D},\boxdot)}$. Thus,

$$\begin{aligned}
1_{(\mathcal{D},\boxdot)} &= \xi\left(\underbrace{1_{\mathcal{A}\otimes\mathcal{A}'}}_{=1_{\mathcal{A}}\otimes 1_{\mathcal{A}'}}\right) = \xi\left(1_{\mathcal{A}} \otimes 1_{\mathcal{A}'}\right) \\
&= \underbrace{1_{\mathcal{A}}}_{=1} \cdot \underbrace{1_{\mathcal{A}'}\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=1} \qquad \text{(by the definition of } \xi\text{)} \\
&= 1 \cdot 1 = 1 = 1_{\mathcal{D}}.
\end{aligned}$$

This proves Proposition 4.5 **(b)**. $\square$

The following theorem states a positive answer to MathOverflow question [MO102874]:

Theorem 4.6. We use the setup of Definition 4.1.

(a) The set $\Omega(U(\mathfrak{gl}_n))$ is a $\mathbf{k}$-subalgebra of the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$.

(b) There exists a commutative multiplication $\boxdot$ on the $\mathbf{k}$-module $U(\mathfrak{gl}_n)$ such that Ω is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to (\mathcal{D}, \boxdot)$. Moreover, this multiplication $\boxdot$ can be chosen to be independent on m.

We shall prove Theorem 4.6 by constructing the multiplication $\boxdot$ whose existence is alleged in Theorem 4.6 **(b)** using Theorem 3.10:

Theorem 4.7. We use the setup of Definition 4.1.

The $\mathbf{k}$-module $\mathrm{M}_n(\mathbf{k})$ (equipped with the multiplication of $\mathrm{M}_n(\mathbf{k})$) is a left pre-Lie algebra (by Proposition 3.5 **(a)**). Hence, we can construct a map $\eta : U\left((\mathrm{M}_n(\mathbf{k}))^-\right) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ according to Theorem 3.10 **(f)** (applied to $A = \mathrm{M}_n(\mathbf{k})$). The map η is a map from $U\left((\mathrm{M}_n(\mathbf{k}))^-\right)$ to $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$, therefore a map from $U(\mathfrak{gl}_n)$ to $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ (since $(\mathrm{M}_n(\mathbf{k}))^- = \mathfrak{gl}_n$).

(a) The map η is a $\mathbf{k}$-module isomorphism. Hence, we can define a new, **commutative** multiplication $\boxdot$ on $U(\mathfrak{gl}_n)$ (formally speaking, a $\mathbf{k}$-bilinear map $\boxdot : U(\mathfrak{gl}_n) \times U(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$, written in infix notation) by letting

$$x \boxdot y = \eta\left(\eta^{-1}(x) \cdot \eta^{-1}(y)\right) \qquad \text{for all } x \in U(\mathfrak{gl}_n) \text{ and } y \in U(\mathfrak{gl}_n).$$

(In other words, this multiplication $\boxdot$ is the result of transporting the multiplication of the $\mathbf{k}$-algebra $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ to $U(\mathfrak{gl}_n)$ along the isomorphism η.)

(b) Let us write $(U(\mathfrak{gl}_n), \boxdot)$ for the $\mathbf{k}$-module $U(\mathfrak{gl}_n)$ equipped with the multiplication $\boxdot$. This is a commutative $\mathbf{k}$-algebra with unity $1_{U(\mathfrak{gl}_n)}$.

(c) Let Δ and ϵ denote the comultiplication and the counit of the $\mathbf{k}$-bialgebra $U(\mathfrak{gl}_n)$. The $\mathbf{k}$-algebra $(U(\mathfrak{gl}_n), \boxdot)$, equipped with the comultiplication Δ and the counit ϵ, is a $\mathbf{k}$-bialgebra. We shall denote this $\mathbf{k}$-bialgebra by $(U(\mathfrak{gl}_n), \boxdot)$.

(d) The map η is a $\mathbf{k}$-bialgebra isomorphism $(U(\mathfrak{gl}_n), \boxdot) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$.

(e) For every $p \in \mathcal{A}$, $q \in \mathcal{A}'$ and $h \in \mathcal{D}$, we have

$$h \boxdot \xi(p \otimes q) = p \cdot h \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right).$$

(f) For every $p \in \mathcal{A}$, $q \in \mathcal{A}'$ and $a \in \mathfrak{gl}_n$, we have

$$\omega(a) \cdot \xi(p \otimes q) = \omega(a) \boxdot \xi(p \otimes q) + \xi((\omega(a))[p] \otimes q).$$

(Recall that we are using the notation $f[a]$ for the image of an element $a \in \mathcal{A}$ under a map $f : \mathcal{A} \to \mathcal{A}$.)

(g) For every $a \in \mathfrak{gl}_n$, $b \in \mathfrak{gl}_n$ and $h \in \mathcal{D}$, we have

$$\omega(a) \cdot (\omega(b) \boxdot h) = \omega(b) \boxdot (\omega(a) \cdot h) + \omega(a \triangleright b) \boxdot h.$$

(h) The map Ω is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to (\mathcal{D}, \boxdot)$.

Proof of Theorem 4.7. Define a $(\mathrm{M}_n(\mathbf{k}))^-$-module structure on $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ as in Theorem 3.10 **(e)** (applied to $A = \mathrm{M}_n(\mathbf{k})$). Thus, $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ is an $(\mathrm{M}_n(\mathbf{k}))^-$-module. In other words, $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ is a $\mathfrak{gl}_n$-module (since $\mathfrak{gl}_n = \mathrm{M}_n(\mathbf{k})$).

(a) Theorem 3.10 **(h)** (applied to $A = \mathrm{M}_n(\mathbf{k})$) shows that the map η is an $(\mathrm{M}_n(\mathbf{k}))^-$-module isomorphism. Thus, in particular, the map η is a $\mathbf{k}$-module isomorphism. This proves Theorem 4.7 **(a)** (since the remaining claims of Theorem 4.7 **(a)** are obvious).

(b) Theorem 4.7 **(b)** follows from Corollary 3.11 **(a)** (applied to $A = \mathrm{M}_n(\mathbf{k})$), upon realizing that $(\mathrm{M}_n(\mathbf{k}))^- = \mathfrak{gl}_n$.

(c) Theorem 4.7 **(c)** follows from Corollary 3.11 **(b)** (applied to $A = \mathrm{M}_n(\mathbf{k})$), upon realizing that $(\mathrm{M}_n(\mathbf{k}))^- = \mathfrak{gl}_n$.

(d) Theorem 4.7 **(d)** follows from Corollary 3.11 **(c)** (applied to $A = \mathrm{M}_n(\mathbf{k})$), upon realizing that $(\mathrm{M}_n(\mathbf{k}))^- = \mathfrak{gl}_n$.

(e) The map ξ is a $\mathbf{k}$-algebra isomorphism from $\mathcal{A} \otimes \mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$ (by Proposition 4.5 **(a)**), therefore also a $\mathbf{k}$-algebra homomorphism from $\mathcal{A} \otimes \mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$.

Let $p \in \mathcal{A}$ and $q \in \mathcal{A}'$. We are first going to prove that

$$\xi(u) \boxdot \xi(p \otimes q) = p \cdot \xi(u) \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \tag{71}$$

for every $u \in \mathcal{A} \otimes \mathcal{A}'$.

Proof of (71): Let $u \in \mathcal{A} \otimes \mathcal{A}'$. We need to prove the equality (71). This equality is $\mathbf{k}$-linear in u. Hence, we can WLOG assume that u is a pure tensor (since the $\mathbf{k}$-module $\mathcal{A} \otimes \mathcal{A}'$ is spanned by the pure tensors). Assume this. Thus, $u = p' \otimes q'$ for some $p' \in \mathcal{A}$ and $q' \in \mathcal{A}'$. Consider these p' and q'. (The notations p' and q' have nothing to do with derivatives.) We have

$$\xi\left(\underbrace{u}_{=p'\otimes q'}\right) = \xi(p' \otimes q') = p' \cdot q'\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \tag{72}$$

(by the definition of ξ). On the other hand, recall that the map ξ is a $\mathbf{k}$-algebra homomorphism from $\mathcal{A} \otimes \mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$. Hence,

$$\begin{aligned}
\xi(u) \boxdot \xi(p \otimes q) &= \xi\left(\underbrace{u}_{=p'\otimes q'} \cdot (p \otimes q)\right) = \xi\left(\underbrace{(p' \otimes q') \cdot (p \otimes q)}_{=p'p\otimes q'q}\right) \\
&= \xi\left(\underbrace{p'p}_{\substack{=pp' \\ \text{(since the } \mathbf{k}\text{-algebra } \mathcal{A} \\ \text{is commutative)}}} \otimes q'q\right) = \xi(pp' \otimes q'q) \\
&= \underbrace{(pp')}_{=p\cdot p'} \cdot \underbrace{(q'q)\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=q'\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)\cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)} \qquad \text{(by the definition of } \xi\text{)} \\
&= \underbrace{p \cdot p' \cdot q'\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{\substack{=\xi(u) \\ \text{(by (72))}}} \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \\
&= p \cdot \xi(u) \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right).
\end{aligned}$$

This proves (72).

Now, let $h \in \mathcal{D}$. Let $u = \xi^{-1}(h)$. This u is a well-defined element of $\mathcal{A} \otimes \mathcal{A}'$ (since ξ is a $\mathbf{k}$-module isomorphism). Now, $\xi(u) = h$ (since $u = \xi^{-1}(h)$), so that $h = \xi(u)$. Thus,

$$\begin{aligned}
\underbrace{h}_{=\xi(u)} \boxdot \xi(p \otimes q) = \xi(u) \boxdot \xi(p \otimes q) &= p \cdot \underbrace{\xi(u)}_{=h} \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \\
&\qquad \text{(by (72))} \\
&= p \cdot h \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right).
\end{aligned}$$

This proves Theorem 4.7 **(e)**.

(f) Let $p \in \mathcal{A}$, $q \in \mathcal{A}'$ and $a \in \mathfrak{gl}_n$.

Let h denote the element $\omega(a)$ of $\mathcal{D}$. Then, $h = \omega\left(\underbrace{a}_{\in\mathfrak{gl}_n}\right) \in \omega(\mathfrak{gl}_n) \subseteq \operatorname{Der}\mathcal{A}$ (by Proposition 4.4). In other words, h is a derivation of $\mathcal{A}$. Thus, we have

$$h[s_1 s_2] = s_1 \cdot h[s_2] + h[s_1] \cdot s_2 \tag{73}$$

for every $s_1 \in \mathcal{A}$ and $s_2 \in \mathcal{A}$.

Let q_∂ denote the element $q\left(\left(\dfrac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \in \mathcal{D}$. Thus, $q\left(\left(\dfrac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) = q_\partial$.

Let $r \in \mathcal{A}$.

The definition of ξ shows that

$$\xi(p \otimes q) = p \cdot \underbrace{q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=q_\partial} = p \cdot q_\partial$$

and

$$\xi(h[p] \otimes q) = h[p] \cdot \underbrace{q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=q_\partial} = h[p] \cdot q_\partial.$$

But Theorem 4.7 **(e)** shows that

$$h \boxdot \xi(p \otimes q) = p \cdot h \cdot \underbrace{q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=q_\partial} = p \cdot h \cdot q_\partial,$$

and thus

$$\underbrace{(h \boxdot \xi(p \otimes q))}_{=p\cdot h\cdot q_\partial}[r] = (p \cdot h \cdot q_\partial)[r] = p \cdot \underbrace{(h \cdot q_\partial)[r]}_{=h[q_\partial[r]]} = p \cdot h[q_\partial[r]]. \tag{74}$$

Moreover,

$$\underbrace{(\xi(h[p] \otimes q))}_{=h[p]\cdot q_\partial}[r] = (h[p] \cdot q_\partial)[r] = h[p] \cdot q_\partial[r]. \tag{75}$$

Also,

$$\begin{aligned}
\Big(h\cdot\underbrace{\xi\left(p\otimes q\right)}_{=p\cdot q_\partial}\Big)\left[r\right] &= \left(h\cdot p\cdot q_\partial\right)\left[r\right] = h\Big[\underbrace{\left(p\cdot q_\partial\right)\left[r\right]}_{=p\cdot q_\partial[r]}\Big] = h\left[p\cdot q_\partial\left[r\right]\right]\\
&= \underbrace{p\cdot h\left[q_\partial\left[r\right]\right]}_{\substack{=(h\boxdot\xi(p\otimes q))[r]\\ \text{(by (74))}}} + \underbrace{h\left[p\right]\cdot q_\partial\left[r\right]}_{\substack{=(\xi(h[p]\otimes q))[r]\\ \text{(by (75))}}}\\
&\qquad\text{(by (73), applied to } s_1 = p \text{ and } s_2 = q_\partial\left[r\right])\\
&= \left(h\boxdot\xi\left(p\otimes q\right)\right)\left[r\right] + \left(\xi\left(h\left[p\right]\otimes q\right)\right)\left[r\right]\\
&= \left(h\boxdot\xi\left(p\otimes q\right) + \xi\left(h\left[p\right]\otimes q\right)\right)\left[r\right].
\end{aligned}$$

Now, let us forget that we fixed r. We thus have shown that

$$\left(h\cdot\xi\left(p\otimes q\right)\right)\left[r\right] = \left(h\boxdot\xi\left(p\otimes q\right) + \xi\left(h\left[p\right]\otimes q\right)\right)\left[r\right]$$

for every $r\in\mathcal{A}$. In other words,

$$h\cdot\xi\left(p\otimes q\right) = h\boxdot\xi\left(p\otimes q\right) + \xi\left(h\left[p\right]\otimes q\right).$$

Since $h=\omega\left(a\right)$, this rewrites as

$$\omega\left(a\right)\cdot\xi\left(p\otimes q\right) = \omega\left(a\right)\boxdot\xi\left(p\otimes q\right) + \xi\left(\left(\omega\left(a\right)\right)\left[p\right]\otimes q\right).$$

This proves Theorem 4.7 **(f)**.

(g) Recall that the $\mathbf{k}$-module $\mathrm{M}_n\left(\mathbf{k}\right)$ (equipped with the multiplication of $\mathrm{M}_n\left(\mathbf{k}\right)$) is a left pre-Lie algebra. Hence, if $a\in\mathrm{M}_n\left(\mathbf{k}\right)$ and $b\in\mathrm{M}_n\left(\mathbf{k}\right)$ are two matrices, then $a\rhd b$ is the product of the matrices a and b inside $\mathrm{M}_n\left(\mathbf{k}\right)$. Hence, for every $\left(u,v\right)\in\left\{1,2,\ldots,n\right\}^2$ and $\left(y,z\right)\in\left\{1,2,\ldots,n\right\}^2$, we have

$$\begin{aligned}
E_{u,v}\rhd E_{y,z} &= \left(\text{the product of the matrices } E_{u,v} \text{ and } E_{y,z} \text{ inside } \mathrm{M}_n\left(\mathbf{k}\right)\right)\\
&= \delta_{v,y}E_{u,z} \qquad (76)
\end{aligned}$$

(by the standard multiplication rule for elementary matrices).

Next, let us observe that

$$\omega\left(E_{u,v}\right)\boxdot\xi\left(p\otimes q\right) = \sum_{k=1}^{m}\xi\left(x_{u,k}p\otimes\partial_{v,k}q\right) \qquad (77)$$

for every $\left(u,v\right)\in\left\{1,2,\ldots,n\right\}^2$, $p\in\mathcal{A}$ and $q\in\mathcal{A}'$.

Proof of (77): Let $(u,v) \in \{1,2,\ldots,n\}^2$, $p \in \mathcal{A}$ and $q \in \mathcal{A}'$. We have

$$\begin{aligned}
&\omega\left(E_{u,v}\right) \boxdot \xi\left(p \otimes q\right) \\
&= p \cdot \underbrace{\omega\left(E_{u,v}\right)}_{\substack{=\sum\limits_{k=1}^{m} x_{u,k}\frac{\partial}{\partial x_{v,k}} \\ \text{(by the definition of } \omega\text{)}}} \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \\
&\qquad \left(\text{by Theorem 4.7 } \textbf{(e)} \text{ (applied to } h = \omega\left(E_{u,v}\right)\text{)}\right) \\
&= p \cdot \left(\sum_{k=1}^{m} x_{u,k}\frac{\partial}{\partial x_{v,k}}\right) \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \\
&= \sum_{k=1}^{m} p x_{u,k}\frac{\partial}{\partial x_{v,k}} \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right).
\end{aligned}$$

Comparing this with

$$\begin{aligned}
&\sum_{k=1}^{m} \underbrace{\xi\left(x_{u,k}p \otimes \partial_{v,k}q\right)}_{\substack{=\left(x_{u,k}p\right)\cdot\left(\partial_{v,k}q\right)\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \\ \text{(by the definition of } \xi\text{)}}} \\
&= \sum_{k=1}^{m} \underbrace{\left(x_{u,k}p\right)}_{=px_{u,k}} \cdot \underbrace{\left(\partial_{v,k}q\right)\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=\partial_{v,k}\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)\cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)} \\
&= \sum_{k=1}^{m} px_{u,k} \cdot \underbrace{\partial_{v,k}\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=\frac{\partial}{\partial x_{v,k}}} \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right) \\
&= \sum_{k=1}^{m} px_{u,k}\frac{\partial}{\partial x_{v,k}} \cdot q\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right),
\end{aligned}$$

this yields $\omega\left(E_{u,v}\right) \boxdot \xi\left(p \otimes q\right) = \sum\limits_{k=1}^{m} \xi\left(x_{u,k}p \otimes \partial_{v,k}q\right)$. This proves (77).

We are now going to show that

$$\omega\left(a\right) \cdot \left(\omega\left(b\right) \boxdot \xi\left(s\right)\right) = \omega\left(b\right) \boxdot \left(\omega\left(a\right) \cdot \xi\left(s\right)\right) + \omega\left(a \triangleright b\right) \boxdot \xi\left(s\right) \qquad (78)$$

for every $a \in \mathfrak{gl}_n$, $b \in \mathfrak{gl}_n$ and $s \in \mathcal{A} \otimes \mathcal{A}'$.

Proof of (71): Let $a \in \mathfrak{gl}_n$, $b \in \mathfrak{gl}_n$ and $s \in \mathcal{A} \otimes \mathcal{A}'$.

We want to prove the equality (71). This equality is $\mathbf{k}$-linear in s. Hence, we can WLOG assume that s is a pure tensor (since the $\mathbf{k}$-module $\mathcal{A} \otimes \mathcal{A}'$ is spanned by the pure tensors). Assume this. Thus, $s = p \otimes q$ for some $p \in \mathcal{A}$ and $q \in \mathcal{A}$. Consider these p and q.

We want to prove the equality (71). This equality is $\mathbf{k}$-linear in each of a and b. Hence, we can WLOG assume that both a and b belong to the basis $\left(E_{i,j}\right)_{(i,j)\in\{1,2,\ldots,n\}^2}$ of the $\mathbf{k}$-module $\mathfrak{gl}_n$. Assume this. Thus, $a = E_{u,v}$ and $b = E_{y,z}$ for some $(u,v) \in \{1,2,\ldots,n\}^2$ and some $(y,z) \in \{1,2,\ldots,n\}^2$. Consider this (u,v) and this (y,z).

Let h denote the element $\omega(a)$ of $\mathcal{D}$. Then, $h = \omega\left(\underbrace{a}_{\in\mathfrak{gl}_n}\right) \in \omega(\mathfrak{gl}_n) \subseteq \operatorname{Der}\mathcal{A}$ (by Proposition 4.4). In other words, h is a derivation of $\mathcal{A}$. Thus, we have

$$h[s_1 s_2] = s_1 \cdot h[s_2] + h[s_1] \cdot s_2 \tag{79}$$

for every $s_1 \in \mathcal{A}$ and $s_2 \in \mathcal{A}$.

We have

$$\underbrace{a}_{=E_{u,v}} \rhd \underbrace{b}_{=E_{y,z}} = E_{u,v} \rhd E_{y,z} = \delta_{v,y} E_{u,z} \tag{80}$$

(by (76)). Hence,

$$h\left[x_{y,\ell}\right] = \delta_{v,y} x_{u,\ell} \tag{81}$$

for every $\ell \in \{1,2,\ldots,m\}$ [46].

We have

$$\omega\left(\underbrace{b}_{=E_{y,z}}\right) = \omega\left(E_{y,z}\right) = \sum_{k=1}^{m} x_{y,k} \frac{\partial}{\partial x_{z,k}} \qquad \text{(by the definition of } \omega\text{)}. \tag{82}$$

On the other hand,

$$\begin{aligned} \omega(a) \cdot \xi\left(\underbrace{s}_{=p\otimes q}\right) &= \omega(a) \cdot \xi(p \otimes q) = \omega(a) \boxdot \xi\left(\underbrace{p \otimes q}_{=s}\right) + \xi\left(\underbrace{(\omega(a))}_{=h}[p] \otimes q\right) \\ &\qquad \text{(by Theorem 4.7 } \textbf{(f)}\text{)} \\ &= \omega(a) \boxdot \xi(s) + \xi(h[p] \otimes q). \end{aligned}$$

[46] *Proof of (81):* Let $\ell \in \{1,2,\ldots,m\}$. Then, $h = \omega\left(\underbrace{a}_{=E_{u,v}}\right) = \omega(E_{u,v}) = \sum_{k=1}^{m} x_{u,k} \frac{\partial}{\partial x_{v,k}}$ (by the

Hence,

$$\begin{aligned}
&\omega\left(b\right)\boxdot\underbrace{\left(\omega\left(a\right)\cdot\xi\left(s\right)\right)}_{=\omega(a)\boxdot\xi(s)+\xi(h[p]\otimes q)}\\
&=\omega\left(b\right)\boxdot\left(\omega\left(a\right)\boxdot\xi\left(s\right)+\xi\left(h\left[p\right]\otimes q\right)\right)\\
&=\underbrace{\omega\left(b\right)\boxdot\omega\left(a\right)}_{\substack{=\omega(a)\boxdot\omega(b)\\ \text{(since the }\mathbf{k}\text{-algebra }(\mathcal{D},\boxdot)\\ \text{is commutative)}}}\boxdot\xi\left(s\right)+\omega\left(\underbrace{b}_{=E_{y,z}}\right)\boxdot\xi\left(h\left[p\right]\otimes q\right)\\
&=\omega\left(a\right)\boxdot\omega\left(b\right)\boxdot\xi\left(s\right)+\underbrace{\omega\left(E_{u,z}\right)\boxdot\xi\left(h\left[p\right]\otimes q\right)}_{\substack{=\sum_{k=1}^{m}\xi\left(x_{y,k}\cdot h[p]\otimes\partial_{z,k}q\right)\\ \text{(by (77), applied to}\\ h[p]\text{ and }(y,z)\text{ instead of }p\text{ and }(u,v))}}\\
&=\omega\left(a\right)\boxdot\omega\left(b\right)\boxdot\xi\left(s\right)+\sum_{k=1}^{m}\xi\left(x_{y,k}\cdot h\left[p\right]\otimes\partial_{z,k}q\right). \qquad (83)
\end{aligned}$$

On the other hand,

$$\omega\left(\underbrace{b}_{=E_{y,z}}\right)\boxdot\xi\left(\underbrace{s}_{=p\otimes q}\right)=\omega\left(E_{y,z}\right)\boxdot\xi\left(p\otimes q\right)=\sum_{k=1}^{m}\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right) \qquad (84)$$

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definition of ω). Hence,

$$\begin{aligned}
&\underbrace{h}_{=\sum_{k=1}^{m}x_{u,k}\frac{\partial}{\partial x_{v,k}}}\left[x_{y,\ell}\right]\\
&=\left(\sum_{k=1}^{m}x_{u,k}\frac{\partial}{\partial x_{v,k}}\right)\left[x_{y,\ell}\right]=\underbrace{\sum_{k=1}^{m}}_{=\sum_{k\in\{1,2,\ldots,m\}}}x_{u,k}\underbrace{\frac{\partial}{\partial x_{v,k}}\left[x_{y,\ell}\right]}_{=\delta_{(v,k),(y,\ell)}=\delta_{v,y}\delta_{k,\ell}}=\sum_{k\in\{1,2,\ldots,m\}}x_{u,k}\delta_{v,y}\delta_{k,\ell}\\
&=x_{u,\ell}\delta_{v,y}\underbrace{\delta_{\ell,\ell}}_{=1}+\sum_{\substack{k\in\{1,2,\ldots,m\};\\ k\neq\ell}}x_{u,k}\delta_{v,y}\underbrace{\delta_{k,\ell}}_{\substack{=0\\ \text{(since }k\neq\ell)}}\\
&\qquad\text{(here, we have split off the addend for }k=\ell\text{ from the sum)}\\
&=x_{u,\ell}\delta_{v,y}+\underbrace{\sum_{\substack{k\in\{1,2,\ldots,m\};\\ k\neq\ell}}x_{u,k}\delta_{v,y}0}_{=0}=x_{u,\ell}\delta_{v,y}=\delta_{v,y}x_{u,\ell}.
\end{aligned}$$

This proves (81).

(by (77) (applied to (y,z) instead of (u,v))). Hence,

$$\begin{aligned}
&\omega(a)\cdot \underbrace{(\omega(b)\boxdot\xi(s))}_{=\sum_{k=1}^{m}\xi(x_{y,k}p\otimes\partial_{z,k}q)} \\
&=\omega(a)\cdot\sum_{k=1}^{m}\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right)=\sum_{k=1}^{m}\underbrace{\omega(a)\cdot\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right)}_{\substack{=\omega(a)\boxdot\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right)+\xi\left((\omega(a))\left[x_{y,k}p\right]\otimes\partial_{z,k}q\right)\\ \text{(by Theorem 4.7 \textbf{(f)} (applied to } x_{y,k}p \text{ and } \partial_{z,k}q\\ \text{instead of } p \text{ and } q))}} \\
&=\sum_{k=1}^{m}\left(\omega(a)\boxdot\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right)+\xi\left(\underbrace{(\omega(a))}_{=h}\left[x_{y,k}p\right]\otimes\partial_{z,k}q\right)\right) \\
&=\underbrace{\sum_{k=1}^{m}\omega(a)\boxdot\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right)}_{=\omega(a)\boxdot\sum_{k=1}^{m}\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right)}+\sum_{k=1}^{m}\xi\left(h\left[x_{y,k}p\right]\otimes\partial_{z,k}q\right) \\
&=\omega(a)\boxdot\underbrace{\sum_{k=1}^{m}\xi\left(x_{y,k}p\otimes\partial_{z,k}q\right)}_{\substack{=\omega(b)\boxdot\xi(s)\\ \text{(by (84))}}}+\sum_{k=1}^{m}\xi\left(h\left[x_{y,k}p\right]\otimes\partial_{z,k}q\right) \\
&=\omega(a)\boxdot\omega(b)\boxdot\xi(s)+\sum_{k=1}^{m}\xi\left(h\left[x_{y,k}p\right]\otimes\partial_{z,k}q\right).
\end{aligned}\tag{85}$$

But for every $k\in\{1,2,\ldots,m\}$, we have

$$\begin{aligned}
&\xi\left(\underbrace{h\left[x_{y,k}p\right]}_{\substack{=x_{y,k}\cdot h[p]+h\left[x_{y,k}\right]\cdot p\\ \text{(by (79), applied to}\\ s_1=x_{y,k}\text{ and } s_2=p)}}\otimes\partial_{z,k}q\right) \\
&=\xi\left(\underbrace{\left(x_{y,k}\cdot h[p]+h\left[x_{y,k}\right]\cdot p\right)\otimes\partial_{z,k}q}_{=x_{y,k}\cdot h[p]\otimes\partial_{z,k}q+h\left[x_{y,k}\right]\cdot p\otimes\partial_{z,k}q}\right) \\
&=\xi\left(x_{y,k}\cdot h[p]\otimes\partial_{z,k}q+h\left[x_{y,k}\right]\cdot p\otimes\partial_{z,k}q\right) \\
&=\xi\left(x_{y,k}\cdot h[p]\otimes\partial_{z,k}q\right)+\xi\left(h\left[x_{y,k}\right]\cdot p\otimes\partial_{z,k}q\right).
\end{aligned}$$

Thus, (85) becomes

$$\begin{aligned}
&\omega\left(a\right)\cdot\left(\omega\left(b\right)\boxdot\xi\left(s\right)\right)\\
&=\omega\left(a\right)\boxdot\omega\left(b\right)\boxdot\xi\left(s\right)+\sum_{k=1}^{m}\underbrace{\xi\left(h\left[x_{y,k}p\right]\otimes\partial_{z,k}q\right)}_{=\xi\left(x_{y,k}\cdot h\left[p\right]\otimes\partial_{z,k}q\right)+\xi\left(h\left[x_{y,k}\right]\cdot p\otimes\partial_{z,k}q\right)}\\
&=\omega\left(a\right)\boxdot\omega\left(b\right)\boxdot\xi\left(s\right)+\underbrace{\sum_{k=1}^{m}\left(\xi\left(x_{y,k}\cdot h\left[p\right]\otimes\partial_{z,k}q\right)+\xi\left(h\left[x_{y,k}\right]\cdot p\otimes\partial_{z,k}q\right)\right)}_{=\sum_{k=1}^{m}\xi\left(x_{y,k}\cdot h\left[p\right]\otimes\partial_{z,k}q\right)+\sum_{k=1}^{m}\xi\left(h\left[x_{y,k}\right]\cdot p\otimes\partial_{z,k}q\right)}\\
&=\underbrace{\omega\left(a\right)\boxdot\omega\left(b\right)\boxdot\xi\left(s\right)+\sum_{k=1}^{m}\xi\left(x_{y,k}\cdot h\left[p\right]\otimes\partial_{z,k}q\right)}_{\substack{=\omega\left(b\right)\boxdot\left(\omega\left(a\right)\cdot\xi\left(s\right)\right)\\ \text{(by (83))}}}+\sum_{k=1}^{m}\xi\left(\underbrace{h\left[x_{y,k}\right]}_{\substack{=\delta_{v,y}x_{u,k}\\ \text{(by (81)}\\ \text{(applied to }\ell=k\text{))}}}\cdot p\otimes\partial_{z,k}q\right)\\
&=\omega\left(b\right)\boxdot\left(\omega\left(a\right)\cdot\xi\left(s\right)\right)+\sum_{k=1}^{m}\underbrace{\xi\left(\delta_{v,y}x_{u,k}\cdot p\otimes\partial_{z,k}q\right)}_{\substack{=\delta_{v,y}\xi\left(x_{u,k}\cdot p\otimes\partial_{z,k}q\right)\\ =\delta_{v,y}\xi\left(x_{u,k}p\otimes\partial_{z,k}q\right)}}\\
&=\omega\left(b\right)\boxdot\left(\omega\left(a\right)\cdot\xi\left(s\right)\right)+\sum_{k=1}^{m}\delta_{v,y}\xi\left(x_{u,k}p\otimes\partial_{z,k}q\right).
\end{aligned}\tag{86}$$

On the other hand, applying the map ω to the equality (80), we obtain

$$\omega\left(a\rhd b\right)=\omega\left(\delta_{v,y}E_{u,z}\right)=\delta_{v,y}\omega\left(E_{u,z}\right).$$

Hence,

$$\begin{aligned}
\underbrace{\omega\left(a\rhd b\right)}_{=\delta_{v,y}\omega\left(E_{u,z}\right)}\boxdot\xi\left(\underbrace{s}_{=p\otimes q}\right)&=\delta_{v,y}\underbrace{\omega\left(E_{u,z}\right)\boxdot\xi\left(p\otimes q\right)}_{\substack{=\sum_{k=1}^{m}\xi\left(x_{u,k}p\otimes\partial_{z,k}q\right)\\ \text{(by (77) (applied to }(u,z)\\ \text{instead of }(u,v)\text{))}}}\\
&=\delta_{v,y}\sum_{k=1}^{m}\xi\left(x_{u,k}p\otimes\partial_{z,k}q\right)=\sum_{k=1}^{m}\delta_{v,y}\xi\left(x_{u,k}p\otimes\partial_{z,k}q\right).
\end{aligned}\tag{87}$$

Hence, (86) becomes

$$\begin{aligned}
&\omega(a)\cdot(\omega(b)\boxdot\xi(s)) \\
&=\omega(b)\boxdot(\omega(a)\cdot\xi(s))+\underbrace{\sum_{k=1}^{m}\delta_{v,y}\xi\left(x_{u,k}p\otimes\partial_{z,k}q\right)}_{\substack{=\omega(a\rhd b)\boxdot\xi(s)\\ \text{(by (87))}}} \\
&=\omega(b)\boxdot(\omega(a)\cdot\xi(s))+\omega(a\rhd b)\boxdot\xi(s).
\end{aligned}$$

Thus, (78) is proven.

Now, fix $a\in\mathfrak{gl}_n$, $b\in\mathfrak{gl}_n$ and $h\in\mathcal{D}$. Let $s=\xi^{-1}(h)$. This s is a well-defined element of $\mathcal{A}\otimes\mathcal{A}'$ (since ξ is a $\mathbf{k}$-module isomorphism). Now, $\xi(s)=h$ (since $s=\xi^{-1}(h)$), so that $h=\xi(s)$. Thus,

$$\begin{aligned}
&\omega(a)\cdot\left(\omega(b)\boxdot\underbrace{h}_{=\xi(s)}\right) \\
&=\omega(a)\cdot(\omega(b)\boxdot\xi(s))=\omega(b)\boxdot\left(\omega(a)\cdot\underbrace{\xi(s)}_{=h}\right)+\omega(a\rhd b)\boxdot\underbrace{\xi(s)}_{=h} \qquad \text{(by (78))} \\
&=\omega(b)\boxdot(\omega(a)\cdot h)+\omega(a\rhd b)\boxdot h.
\end{aligned}$$

This proves Theorem 4.7 **(g)**.

(h) This proof is going to be somewhat complicated, so we subdivide it into several steps for the reader's convenience.

Step 1: Recall that $\left(E_{i,j}\right)_{(i,j)\in\{1,2,\ldots,n\}^2}$ is a basis of the $\mathbf{k}$-module $\mathfrak{gl}_n=\mathrm{M}_n(\mathbf{k})$. Hence, we can define a $\mathbf{k}$-linear map $\phi:\mathrm{M}_n(\mathbf{k})\to\mathcal{A}\otimes\mathcal{A}'$ by

$$\phi\left(E_{i,j}\right)=\sum_{k=1}^{m}x_{i,k}\otimes\partial_{j,k} \qquad \text{for every } (i,j)\in\{1,2,\ldots,n\}^2.$$

Consider this map ϕ. Proposition 2.19 (applied to $V=\mathrm{M}_n(\mathbf{k})$, $A=\mathcal{A}\otimes\mathcal{A}'$ and $f=\phi$) shows that there exists a unique $\mathbf{k}$-algebra homomorphism $F:\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))\to\mathcal{A}\otimes\mathcal{A}'$ such that $F\circ\iota_{\operatorname{Sym},\mathrm{M}_n(\mathbf{k})}=\phi$. Let us denote this F by Φ. Thus, Φ is a $\mathbf{k}$-algebra homomorphism $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))\to\mathcal{A}\otimes\mathcal{A}'$ such that $\Phi\circ\iota_{\operatorname{Sym},\mathrm{M}_n(\mathbf{k})}=\phi$.

Step 2: Let us make a convention. Namely, we shall employ the abuse of notation introduced in Definition 3.9 (applied to $V=\mathrm{M}_n(\mathbf{k})$). Thus, we will identify every $a\in\mathrm{M}_n(\mathbf{k})$ with the element $\iota_{\operatorname{Sym},\mathrm{M}_n(\mathbf{k})}(a)$ of $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$. As a consequence, products such as $b_1b_2\cdots b_p$ (where $p\in\mathbb{N}$ and $b_1,b_2,\ldots,b_p\in\mathrm{M}_n(\mathbf{k})$) will always mean products inside $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ (and not products inside $\mathrm{M}_n(\mathbf{k})$). In particular, we are **not** going to denote the product of two elements a and b of the matrix ring $\mathrm{M}_n(\mathbf{k})$ by ab, because the notation ab will mean the product of the elements a and b of $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$.

Fortunately, we have a different notation for products inside $\mathrm{M}_n(\mathbf{k})$. Namely, recall that the $\mathbf{k}$-module $\mathrm{M}_n(\mathbf{k})$ (equipped with the multiplication of $\mathrm{M}_n(\mathbf{k})$) is a left pre-Lie algebra. Hence, if $a \in \mathrm{M}_n(\mathbf{k})$ and $b \in \mathrm{M}_n(\mathbf{k})$ are two matrices, then $a \triangleright b$ is the product of the matrices a and b inside $\mathrm{M}_n(\mathbf{k})$. Hence, for every $(u,v) \in \{1,2,\ldots,n\}^2$ and $(i,j) \in \{1,2,\ldots,n\}^2$, we have

$$\begin{aligned} E_{u,v} \triangleright E_{i,j} &= \left(\text{the product of the matrices } E_{u,v} \text{ and } E_{i,j} \text{ inside } \mathrm{M}_n(\mathbf{k})\right) \\ &= \delta_{v,i} E_{u,j} \end{aligned} \tag{88}$$

(by the standard multiplication rule for elementary matrices).

We identify every $a \in \mathrm{M}_n(\mathbf{k})$ with the element $\iota_{\mathrm{Sym},\mathrm{M}_n(\mathbf{k})}(a)$ of $\mathrm{Sym}(\mathrm{M}_n(\mathbf{k}))$. Thus, $\mathrm{M}_n(\mathbf{k}) = \iota_{\mathrm{Sym},\mathrm{M}_n(\mathbf{k})}(\mathrm{M}_n(\mathbf{k})) = \mathrm{Sym}^1(\mathrm{M}_n(\mathbf{k}))$.

Step 3: The map ξ is a $\mathbf{k}$-algebra isomorphism from $\mathcal{A} \otimes \mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$ (by Proposition 4.5 **(a)**), therefore also a $\mathbf{k}$-algebra homomorphism from $\mathcal{A} \otimes \mathcal{A}'$ to the $\mathbf{k}$-algebra $(\mathcal{D}, \boxdot)$.

Step 4: Now, define a map $\psi : \mathrm{Sym}(\mathrm{M}_n(\mathbf{k})) \to \mathcal{D}$ by

$$\psi = \xi \circ \Phi.$$

Then, ψ is the composition of the two $\mathbf{k}$-algebra homomorphisms $\xi : \mathcal{A} \otimes \mathcal{A}' \to (\mathcal{D}, \boxdot)$ and $\Phi : \mathrm{Sym}(\mathrm{M}_n(\mathbf{k})) \to \mathcal{A} \otimes \mathcal{A}'$. Hence, ψ is a $\mathbf{k}$-algebra homomorphism from $\mathrm{Sym}(\mathrm{M}_n(\mathbf{k}))$ to $(\mathcal{D}, \boxdot)$.

The following diagram shows the most important maps that we have so far introduced:

$$\begin{tikzcd} (U(\mathfrak{gl}_n), \boxdot) \arrow[r, "\eta"] \arrow[d, "\Omega"'] & \mathrm{Sym}(\mathrm{M}_n(\mathbf{k})) \arrow[d, "\Phi"] \arrow[dl, "\psi"] \\ (\mathcal{D}, \boxdot) & \mathcal{A} \otimes \mathcal{A}' \arrow[l, "\xi"] \end{tikzcd}.$$

We do not yet know whether the left triangle of this diagram is commutative (although we will later prove this, in Step 10). The right triangle is commutative by definition of ψ. All arrows in this diagram except for Ω are known to be $\mathbf{k}$-algebra homomorphisms; proving this for Ω is our goal.

Step 5: For every $(i,j) \in \{1,2,\ldots,n\}^2$, we have

$$\Phi\left(E_{i,j}\right) = \sum_{k=1}^{m} x_{i,k} \otimes \partial_{j,k} \tag{89}$$

[47] and

$$\psi\left(E_{i,j}\right) = \sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{j,k}} \tag{90}$$

[47] *Proof of (89):* Let $(i,j) \in \{1,2,\ldots,n\}^2$. We have $\Phi \circ \iota_{\mathrm{Sym},\mathrm{M}_n(\mathbf{k})} = \phi$. Hence,

[48]. Hence,

$$\psi(a) = \omega(a) \qquad \text{for every } a \in \mathfrak{gl}_n \tag{92}$$

[49].

Step 6: Recall that we have defined a $\mathfrak{gl}_n$-module structure on the $\mathbf{k}$-module

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$\left(\Phi \circ \iota_{\mathrm{Sym},\mathrm{M}_n(\mathbf{k})}\right)(E_{i,j}) = \phi(E_{i,j}) = \sum_{k=1}^{m} x_{i,k} \otimes \partial_{j,k}$ (by the definition of ϕ). Compared with

$$\left(\Phi \circ \iota_{\mathrm{Sym},\mathrm{M}_n(\mathbf{k})}\right)(E_{i,j}) = \Phi\left(\underbrace{\iota_{\mathrm{Sym},\mathrm{M}_n(\mathbf{k})}(E_{i,j})}_{\substack{=E_{i,j}\\ \text{(by our abuse of notation)}}}\right) = \Phi(E_{i,j}),$$

this shows that $\Phi(E_{i,j}) = \sum_{k=1}^{m} x_{i,k} \otimes \partial_{j,k}$. This proves (89).

[48]*Proof of (90):* Every $u \in \{1, 2, \ldots, n\}$, $v \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$ satisfy

$$\begin{aligned}\xi(x_{u,k} \otimes \partial_{v,k}) &= x_{u,k} \cdot \underbrace{\partial_{v,k}\left(\left(\frac{\partial}{\partial x_{i,j}}\right)_{(i,j)\in G}\right)}_{=\frac{\partial}{\partial x_{v,k}}} \qquad \text{(by the definition of } \xi\text{)} \\ &= x_{u,k}\frac{\partial}{\partial x_{v,k}}.\end{aligned} \tag{91}$$

Now, let $(i, j) \in \{1, 2, \ldots, n\}^2$. Then,

$$\begin{aligned}\underbrace{\psi}_{=\xi\circ\Phi}(E_{i,j}) &= (\xi \circ \Phi)(E_{i,j}) = \xi\left(\underbrace{\Phi(E_{i,j})}_{\substack{=\sum_{k=1}^{m} x_{i,k}\otimes\partial_{j,k}\\ \text{(by (89))}}}\right) = \xi\left(\sum_{k=1}^{m} x_{i,k} \otimes \partial_{j,k}\right) \\ &= \sum_{k=1}^{m} \underbrace{\xi\left(x_{i,k} \otimes \partial_{j,k}\right)}_{\substack{=x_{i,k}\frac{\partial}{\partial x_{j,k}}\\ \text{(by (91), applied to } u=i \text{ and } v=j)}} = \sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}}.\end{aligned}$$

This proves (90).

[49]*Proof of (92):* Let $a \in \mathfrak{gl}_n$. We must prove the equality $\psi(a) = \omega(a)$. This equality is $\mathbf{k}$-linear in a. Hence, we can WLOG assume that a belongs to the basis $(E_{i,j})_{(i,j)\in\{1,2,\ldots,n\}^2}$ of the $\mathbf{k}$-module $\mathfrak{gl}_n$. Assume this. Thus, $a = E_{i,j}$ for some $(i, j) \in \{1, 2, \ldots, n\}^2$. Consider this (i, j). Comparing

$$\psi\left(\underbrace{a}_{=E_{i,j}}\right) = \psi(E_{i,j}) = \sum_{k=1}^{m} x_{i,k}\frac{\partial}{\partial x_{j,k}} \qquad \text{(by (90))}$$

$\mathcal{D}$ in Corollary 4.3. Let us make $\mathcal{D}$ into a $\mathfrak{gl}_n$-module using this $\mathfrak{gl}_n$-module structure.

On the other hand, recall that $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ is a $\mathfrak{gl}_n$-module. This $\mathfrak{gl}_n$-module satisfies

$$a \rightharpoonup 1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))} = a \qquad \text{for every } a \in \mathfrak{gl}_n \tag{93}$$

[50].

Step 7: We shall now show that

$$\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z) \qquad \text{for every } N \in \mathbb{N},\ a \in \mathfrak{gl}_n \text{ and } z \in \operatorname{Sym}^N(\mathrm{M}_n(\mathbf{k})). \tag{94}$$

Proof of (94): We shall prove (94) by induction over N:

Induction base: We have $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$ for every $a \in \mathfrak{gl}_n$ and $z \in \operatorname{Sym}^0(\mathrm{M}_n(\mathbf{k}))$ [51]. In other words, (94) holds for $N = 0$. This completes the induction base.

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with

$$\omega\left(\underbrace{a}_{=E_{i,j}}\right) = \omega(E_{i,j}) = \sum_{k=1}^{m} x_{i,k} \frac{\partial}{\partial x_{j,k}} \qquad \text{(by the definition of } \omega\text{)},$$

we obtain $\psi(a) = \omega(a)$. This proves (92).

[50]*Proof of (93):* For every $a \in \mathrm{M}_n(\mathbf{k})$, we define a derivation $L_a : \operatorname{Sym}(\mathrm{M}_n(\mathbf{k})) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ as in Theorem 3.10 (applied to $A = \mathrm{M}_n(\mathbf{k})$).

Let $a \in \mathfrak{gl}_n$. Thus, $a \in \mathfrak{gl}_n = \mathrm{M}_n(\mathbf{k})$. Thus, L_a is a derivation $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k})) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$. In other words, $L_a \in \operatorname{Der}(\operatorname{Sym}(\mathrm{M}_n(\mathbf{k})))$. Hence, Proposition 1.11 **(c)** (applied to $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ and L_a instead of C and f) shows that $L_a(1) = 0$.

Now, the definition of the $(\mathrm{M}_n(\mathbf{k}))^-$-module structure on $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ shows that

$$a \rightharpoonup 1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))} = \underbrace{a \cdot 1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}}_{=a} + L_a\left(\underbrace{1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}}_{=1}\right) = a + \underbrace{L_a(1)}_{=0} = a.$$

This proves (93).

[51]*Proof.* Let $a \in \mathfrak{gl}_n$ and $z \in \operatorname{Sym}^0(\mathrm{M}_n(\mathbf{k}))$. Recall that $\operatorname{Sym}^0(\mathrm{M}_n(\mathbf{k}))$ is a free $\mathbf{k}$-module with basis $\left(1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}\right)$.

We need to prove the equality $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$. This equality is $\mathbf{k}$-linear in z. Hence, we can WLOG assume that z belongs to the basis $\left(1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}\right)$ of the $\mathbf{k}$-module $\operatorname{Sym}^0(\mathrm{M}_n(\mathbf{k}))$. Assume this. Thus, $z = 1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}$. Now, $\psi\left(\underbrace{z}_{=1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}}\right) = \psi\left(1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}\right) = 1_{(\mathcal{D},\boxdot)}$ (since ψ is a $\mathbf{k}$-algebra homomorphism from $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ to $(\mathcal{D},\boxdot)$), so that $\psi\left(1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}\right) = 1_{(\mathcal{D},\boxdot)} = 1_{\mathcal{D}}$ (by Proposition 4.5 **(b)**). Now,

$$\begin{aligned} a \rightharpoonup \psi(z) &= \omega(a) \cdot \underbrace{\psi(z)}_{=1_{\mathcal{D}}} \qquad \text{(by the definition of the } \mathfrak{gl}_n\text{-structure on } \mathcal{D}\text{)} \\ &= \omega(a) \cdot 1_{\mathcal{D}} = \omega(a). \end{aligned}$$

Induction step: Let M be a positive integer. Assume that (94) holds for $N = M-1$. We must prove that (94) holds for $N = M$.

We have assumed that (94) holds for $n = N-1$. In other words, we have

$$\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z) \qquad \text{for every } a \in \mathfrak{gl}_n \text{ and } z \in \operatorname{Sym}^{M-1}(\mathrm{M}_n(\mathbf{k})). \tag{95}$$

Now, let $a \in \mathfrak{gl}_n$ and $z \in \operatorname{Sym}^M(\mathrm{M}_n(\mathbf{k}))$. We are going to show that $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$.

We have $z \in \operatorname{Sym}^M(\mathrm{M}_n(\mathbf{k})) = \left(\operatorname{Sym}^1(\mathrm{M}_n(\mathbf{k}))\right)\left(\operatorname{Sym}^{M-1}(\mathrm{M}_n(\mathbf{k}))\right)$ (by Remark 2.18 **(c)** (applied to $\mathrm{M}_n(\mathbf{k})$ and M instead of V and n)). Thus, z is a $\mathbf{k}$-linear combination of elements of the form bu with $b \in \operatorname{Sym}^1(\mathrm{M}_n(\mathbf{k}))$ and $u \in \operatorname{Sym}^{M-1}(\mathrm{M}_n(\mathbf{k}))$.

Now, we must prove the equality $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$. This equality is $\mathbf{k}$-linear in z. Hence, we can WLOG assume that z has the form bu with $b \in \operatorname{Sym}^1(\mathrm{M}_n(\mathbf{k}))$ and $u \in \operatorname{Sym}^{M-1}(\mathrm{M}_n(\mathbf{k}))$ (because z is a $\mathbf{k}$-linear combination of elements of the form bu with $b \in \operatorname{Sym}^1(\mathrm{M}_n(\mathbf{k}))$ and $u \in \operatorname{Sym}^{M-1}(\mathrm{M}_n(\mathbf{k}))$). Assume this. Hence, $z = bu$ for some $b \in \operatorname{Sym}^1(\mathrm{M}_n(\mathbf{k}))$ and $u \in \operatorname{Sym}^{M-1}(\mathrm{M}_n(\mathbf{k}))$. Consider this b and u.

We have $b \in \operatorname{Sym}^1(\mathrm{M}_n(\mathbf{k})) = \mathrm{M}_n(\mathbf{k}) = \mathfrak{gl}_n$. Hence, (92) (applied to b instead of a) shows that $\psi(b) = \omega(b)$.

Now, $a \in \mathfrak{gl}_n = \mathrm{M}_n(\mathbf{k})$ and $b \in \mathrm{M}_n(\mathbf{k})$. Hence, $a \triangleright b \in \mathrm{M}_n(\mathbf{k})$. Hence, (92) (applied to $a \triangleright b$ instead of a) shows that $\psi(a \triangleright b) = \omega(a \triangleright b)$.

Applying (95) to u instead of z, we see that

$$\psi(a \rightharpoonup u) = a \rightharpoonup \psi(u) = \omega(a) \cdot \psi(u) \tag{96}$$

(by the definition of the $\mathfrak{gl}_n$-module structure on $\mathcal{D}$). On the other hand,

$$a \rightharpoonup \psi\left(\underbrace{z}_{=bu}\right) = a \rightharpoonup \psi(bu) = \omega(a) \cdot \psi(bu) \tag{97}$$

(by the definition of the $\mathfrak{gl}_n$-module structure on $\mathcal{D}$).

But recall that ψ is a $\mathbf{k}$-algebra homomorphism from $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ to $(\mathcal{D}, \boxdot)$. Hence, $\psi(bu) = \underbrace{\psi(b)}_{=\omega(b)} \boxdot \psi(u) = \omega(b) \boxdot \psi(u)$. Hence, (97) becomes

$$\begin{aligned} a \rightharpoonup \psi(z) &= \omega(a) \cdot \underbrace{\psi(bu)}_{=\omega(b) \boxdot \psi(u)} = \omega(a) \cdot (\omega(b) \boxdot \psi(u)) \\ &= \omega(b) \boxdot (\omega(a) \cdot \psi(u)) + \omega(a \triangleright b) \boxdot \psi(u) \\ &\qquad \text{(by Theorem 4.7 } \textbf{(g)} \text{ (applied to } h = \psi(u)\text{))}. \end{aligned} \tag{98}$$

But

$$a \rightharpoonup \underbrace{z}_{=1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}} = a \rightharpoonup 1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))} = a \qquad \text{(by (93))}.$$

Applying the map ψ to this equality, we obtain $\psi(a \rightharpoonup z) = \psi(a) = \omega(a)$ (by (92)). Comparing this with $a \rightharpoonup \psi(z) = \omega(a)$, we obtain $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$. Qed.

But $a \in \mathrm{M}_n(\mathbf{k})$ and $b \in \mathrm{M}_n(\mathbf{k})$. Hence, Theorem 3.10 **(m)** (applied to $A = \mathrm{M}_n(\mathbf{k})$ and $c = u$) shows that

$$a \rightharpoonup (bu) - b \cdot (a \rightharpoonup u) = (a \triangleright b) \cdot u.$$

Hence, $a \rightharpoonup (bu) = b \cdot (a \rightharpoonup u) + (a \triangleright b) \cdot u$. Now,

$$a \rightharpoonup \underbrace{z}_{=bu} = a \rightharpoonup (bu) = b \cdot (a \rightharpoonup u) + (a \triangleright b) \cdot u.$$

Applying the map ψ to this equality, we obtain

$$\begin{aligned} \psi(a \rightharpoonup z) &= \psi(b \cdot (a \rightharpoonup u) + (a \triangleright b) \cdot u) = \underbrace{\psi(b)}_{=\omega(b)} \boxdot \underbrace{\psi(a \rightharpoonup u)}_{\substack{=\omega(a)\cdot\psi(u) \\ \text{(by (96))}}} + \underbrace{\psi(a \triangleright b)}_{=\omega(a \triangleright b)} \boxdot \psi(u) \\ &\qquad \left(\begin{array}{c} \text{since } \psi \text{ is a } \mathbf{k}\text{-algebra homomorphism} \\ \text{from } \operatorname{Sym}(\mathrm{M}_n(\mathbf{k})) \text{ to } (\mathcal{D}, \boxdot) \end{array} \right) \\ &= \omega(b) \boxdot (\omega(a) \cdot \psi(u)) + \omega(a \triangleright b) \boxdot \psi(u). \end{aligned}$$

Comparing this with (98), we obtain $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$.

Let us now forget that we fixed a and z. We thus have shown that

$$\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z) \qquad \text{for every } a \in \mathfrak{gl}_n \text{ and } z \in \operatorname{Sym}^M(\mathrm{M}_n(\mathbf{k})).$$

In other words, (94) holds for $N = M$. This completes the induction step. Hence, the induction proof of (94) is complete.

Step 8: We will now show that

$$\psi \text{ is a } \mathfrak{gl}_n\text{-module homomorphism from } \operatorname{Sym}(\mathrm{M}_n(\mathbf{k})) \text{ to } \mathcal{D}. \tag{99}$$

Proof of (99): We want to prove that ψ is a $\mathfrak{gl}_n$-module homomorphism from $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ to $\mathcal{D}$. In other words, we want to prove that $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$ for every $a \in \mathfrak{gl}_n$ and $z \in \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$. So let us fix $a \in \mathfrak{gl}_n$ and $z \in \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$.

We want to prove the equality $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$. This equality is $\mathbf{k}$-linear in z. Hence, we can WLOG assume that z is a homogeneous element of $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ (since every element of $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ is a $\mathbf{k}$-linear combination of homogeneous elements). Assume this. Thus, $z \in \operatorname{Sym}^N(\mathrm{M}_n(\mathbf{k}))$ for some $N \in \mathbb{N}$. Consider this N. Now, (94) shows that $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$.

Now, let us forget that we fixed a and z. We thus have shown that $\psi(a \rightharpoonup z) = a \rightharpoonup \psi(z)$ for every $a \in \mathfrak{gl}_n$ and $z \in \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$. In other words, ψ is a $\mathfrak{gl}_n$-module homomorphism from $\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ to $\mathcal{D}$. This proves (99).

Step 9: The definition of the map η shows that $\eta\left(1_{U(\mathfrak{gl}_n)}\right) = 1_{U(\mathfrak{gl}_n)} \cdot 1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))} =$

$1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}$. Hence,

$$\begin{aligned}
(\psi \circ \eta)\left(1_{U(\mathfrak{gl}_n)}\right) &= \psi\left(\underbrace{\eta\left(1_{U(\mathfrak{gl}_n)}\right)}_{=1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}}\right) = \psi\left(1_{\operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))}\right) = 1_{(\mathcal{D},\boxdot)} \\
&\qquad \left(\begin{array}{c}\text{since } \psi \text{ is a } \mathbf{k}\text{-algebra homomorphism} \\ \text{from } \operatorname{Sym}(\mathrm{M}_n(\mathbf{k})) \text{ to } (\mathcal{D},\boxdot)\end{array}\right) \\
&= 1_{\mathcal{D}} \qquad (\text{by Proposition 4.5 } \textbf{(b)}).
\end{aligned}$$

On the other hand, $\Omega\left(1_{U(\mathfrak{gl}_n)}\right) = 1_{\mathcal{D}}$ (since Ω is a $\mathbf{k}$-algebra homomorphism from $U(\mathfrak{gl}_n)$ to $\mathcal{D}$).

Step 10: Theorem 3.10 **(h)** (applied to $A = \mathrm{M}_n(\mathbf{k})$) shows that the map η is an $(\mathrm{M}_n(\mathbf{k}))^-$-module isomorphism. In other words, the map η is an $\mathfrak{gl}_n$-module isomorphism (since $\mathfrak{gl}_n = (\mathrm{M}_n(\mathbf{k}))^-$).

But (99) shows that the map ψ is a $\mathfrak{gl}_n$-module homomorphism.

The map $\psi \circ \eta : U(\mathfrak{gl}_n) \to \mathcal{D}$ is a $\mathfrak{gl}_n$-module homomorphism (since it is the composition of the $\mathfrak{gl}_n$-module homomorphisms $\eta : U(\mathfrak{gl}_n) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ and $\psi : \operatorname{Sym}(\mathrm{M}_n(\mathbf{k})) \to \mathcal{D}$). The map Ω is also a $\mathfrak{gl}_n$-module homomorphism (by Corollary 4.3 **(b)**).

Now, we are going to prove that $\psi \circ \eta = \Omega$.

Indeed, let $z \in U(\mathfrak{gl}_n)$. We shall show that $(\psi \circ \eta)(z) = \Omega(z)$.

The $\mathbf{k}$-modules $U(\mathfrak{gl}_n)$ and $\mathcal{D}$ are $\mathfrak{gl}_n$-modules, and thus left $U(\mathfrak{gl}_n)$-modules (since every $\mathfrak{gl}_n$-module is a left $U(\mathfrak{gl}_n)$-module). The maps $\psi \circ \eta$ and Ω are $\mathfrak{gl}_n$-module homomorphisms, and thus left $U(\mathfrak{gl}_n)$-module homomorphisms (since every $\mathfrak{gl}_n$-module homomorphism is a left $U(\mathfrak{gl}_n)$-module homomorphism).

The left $U(\mathfrak{gl}_n)$-module structure on $U(\mathfrak{gl}_n)$ is given by left multiplication. Hence, $z \cdot 1_{U(\mathfrak{gl}_n)} = z$. Now, $\psi \circ \eta$ is a left $U(\mathfrak{gl}_n)$-module homomorphism. Hence,

$$(\psi \circ \eta)\left(z \cdot 1_{U(\mathfrak{gl}_n)}\right) = z \rightharpoonup \underbrace{(\psi \circ \eta)\left(1_{U(\mathfrak{gl}_n)}\right)}_{=1_{\mathcal{D}}} = z \rightharpoonup 1_{\mathcal{D}}.$$

Since $z \cdot 1_{U(\mathfrak{gl}_n)} = z$, this rewrites as $(\psi \circ \eta)(z) = z \rightharpoonup 1_{\mathcal{D}}$.

On the other hand, Ω is a left $U(\mathfrak{gl}_n)$-module homomorphism. Hence,

$$\Omega\left(z \cdot 1_{U(\mathfrak{gl}_n)}\right) = z \rightharpoonup \underbrace{\Omega\left(1_{U(\mathfrak{gl}_n)}\right)}_{=1_{\mathcal{D}}} = z \rightharpoonup 1_{\mathcal{D}}.$$

Since $z \cdot 1_{U(\mathfrak{gl}_n)} = z$, this rewrites as $\Omega(z) = z \rightharpoonup 1_{\mathcal{D}}$. Comparing this with $(\psi \circ \eta)(z) = z \rightharpoonup 1_{\mathcal{D}}$, we obtain $(\psi \circ \eta)(z) = \Omega(z)$.

Let us now forget that we fixed z. We thus have shown that $(\psi \circ \eta)(z) = \Omega(z)$ for every $z \in U(\mathfrak{gl}_n)$. In other words, $\psi \circ \eta = \Omega$.

Step 11: Theorem 4.7 **(b)** shows that the map η is a $\mathbf{k}$-bialgebra isomorphism $(U(\mathfrak{gl}_n), \boxdot) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$. In particular, the map η is a $\mathbf{k}$-algebra isomorphism $(U(\mathfrak{gl}_n), \boxdot) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$. Hence, η is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$.

The map $\psi \circ \eta$ is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to (\mathcal{D}, \boxdot)$ (since it is the composition of the $\mathbf{k}$-algebra homomorphisms $\eta : (U(\mathfrak{gl}_n), \boxdot) \to \operatorname{Sym}(\mathrm{M}_n(\mathbf{k}))$ and $\psi : \operatorname{Sym}(\mathrm{M}_n(\mathbf{k})) \to (\mathcal{D}, \boxdot)$). Since $\psi \circ \eta = \Omega$, we can rewrite this as follows: The map Ω is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to (\mathcal{D}, \boxdot)$. This proves Theorem 4.7 **(h)**. □

Now that Theorem 4.7 is proven, we can finally prove Theorem 4.6:

Proof of Theorem 4.6. **(b)** Consider the multiplication $\boxdot$ on the $\mathbf{k}$-module $U(\mathfrak{gl}_n)$ defined in Theorem 4.7 **(a)**. This multiplication is commutative (according to Theorem 4.7 **(a)**) and has the property that Ω is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to (\mathcal{D}, \boxdot)$ (according to Theorem 4.7 **(h)**). Moreover, it is independent on m (due to its construction). Thus, Theorem 4.6 **(b)** is proven.

(a) Consider the multiplication $\boxdot$ on the $\mathbf{k}$-module $U(\mathfrak{gl}_n)$ defined in Theorem 4.7 **(a)**. Theorem 4.7 **(h)** shows that Ω is a $\mathbf{k}$-algebra homomorphism $(U(\mathfrak{gl}_n), \boxdot) \to (\mathcal{D}, \boxdot)$. Hence, $\Omega(U(\mathfrak{gl}_n))$ is the image of the $\mathbf{k}$-algebra $(U(\mathfrak{gl}_n), \boxdot)$ under a $\mathbf{k}$-algebra homomorphism, and thus itself a $\mathbf{k}$-subalgebra of $(\mathcal{D}, \boxdot)$. This proves Theorem 4.6 **(a)**. □

5. Commutation of left and right actions

Many of the results we have stated above were not "left-right symmetric": For instance, Definition 1.2 involves a map $\mu : \mathfrak{g} \times V \to V$, whereas we could just as well consider $\mathbf{k}$-modules V equipped with a map $\mu : V \times \mathfrak{g} \to \mathfrak{g}$. Theorem 1.15 constructed a left $U(\mathfrak{g})$-module structure on C, but one could also ask for a right $U(\mathfrak{g})$-module structure. Theorem 3.10 concerns left pre-Lie algebras, but we could just as well state an analogous result for right pre-Lie algebras[52]. And so on.

Stating and proving these analogues is a rather unexciting and straightforward task; one essentially needs to "turn all products around" (i.e., replacing ab by ba, replacing $\mu(a, v)$ by $\mu(v, a)$, replacing $a \triangleright b$ by $b \triangleleft a$, etc.). What is more interesting is to combine these analogues with the original results and to see how they interact. As an example of such an interaction, we can recall that an (associative) $\mathbf{k}$-algebra A can be regarded both as a left pre-Lie algebra (according to Proposition 3.5 **(a)**) and as a right pre-Lie algebra (according to Proposition 3.5 **(b)**). The left pre-Lie algebra structure on A allows us to apply Theorem 3.10 to this A, thus obtaining a map $\eta : U(A^-) \to \operatorname{Sym} A$. The right pre-Lie algebra structure on A allows us to apply Theorem 5.21 (the straightforward analogue

[52]Right pre-Lie algebras are a "symmetric" notion to that of left pre-Lie algebras; see Proposition 3.3 and Proposition 3.4.

of Theorem 3.10 for right pre-Lie algebras) to this A, thus obtaining a second map $\eta' : U(A^-) \to \operatorname{Sym} A$ [53]. How are these two maps η and η' related? We will see (in Theorem 5.22 below) that they are identical.

5.1. Right $\mathfrak{g}$-modules

Let us, however, return to a more general setting. In Definition 1.2, we have defined the notion of "$\mathfrak{g}$-modules"; we shall sometimes refer to them as "left $\mathfrak{g}$-modules" from now on. Let us now define a notion of "right $\mathfrak{g}$-modules", which is a "right" analogue to the notion of $\mathfrak{g}$-modules introduced in Definition 1.2:

Definition 5.1. Let $\mathfrak{g}$ be a Lie algebra. Let V be a $\mathbf{k}$-module. Let $\mu : V \times \mathfrak{g} \to V$ be a $\mathbf{k}$-bilinear map. We say that (V, μ) is a *right* $\mathfrak{g}$*-module* if and only if

$$(\mu(v,[a,b]) = \mu(\mu(v,a),b) - \mu(\mu(v,b),a) \text{ for every } a \in \mathfrak{g},\ b \in \mathfrak{g} \text{ and } v \in V). \tag{100}$$

If (V, μ) is a right $\mathfrak{g}$-module, then the $\mathbf{k}$-bilinear map $\mu : V \times \mathfrak{g} \to V$ is called the *Lie action* of the right $\mathfrak{g}$-module V. If (V, μ) is a right $\mathfrak{g}$-module, then the $\mathbf{k}$-module V is called the *underlying* $\mathbf{k}$*-module* of (V, μ).

Similarly to Convention 1.3, we now make the following convention:

Convention 5.2. Let $\mathfrak{g}$ be a Lie algebra. Let (V, μ) be a right $\mathfrak{g}$-module.

(a) For any $a \in \mathfrak{g}$ and $v \in V$, we shall abbreviate the term $\mu(v,a)$ by $v \leftharpoonup a$, provided that the map μ is obvious from the context. Using this notation, the relation (100) rewrites as

$$(v \leftharpoonup [a,b] = (v \leftharpoonup a) \leftharpoonup b - (v \leftharpoonup b) \leftharpoonup a \text{ for every } a \in \mathfrak{g},\ b \in \mathfrak{g} \text{ and } v \in V). \tag{101}$$

Convention on the precedence of the $\leftharpoonup$ sign: The symbol $\leftharpoonup$ is understood to have the same precedence as the multiplication sign (i.e., it binds as strongly as the multiplication sign). Thus, $v + w \leftharpoonup a$ means $v + (w \leftharpoonup a)$ rather than $(v+w) \leftharpoonup a$, but $v \cdot w \leftharpoonup a$ is undefined (it could mean both $v \cdot (w \leftharpoonup a)$ and $(v \cdot w) \leftharpoonup a$). Application of functions will be supposed to bind more strongly than the $\rightharpoonup$ sign, so that $f(v) \leftharpoonup g(w)$ will mean $(f(v)) \leftharpoonup (g(w))$ (rather than $f(v \leftharpoonup g(w))$ or $(f(v \leftharpoonup g))(w)$ or anything else), but we will often use parentheses in this case to make the correct interpretation of the formula even more obvious.

(b) We shall often refer to the right $\mathfrak{g}$-module (V, μ) as "the $\mathbf{k}$-module V, endowed with the right $\mathfrak{g}$-action μ" (or by similar formulations). We shall regard (V, μ) as the $\mathbf{k}$-module V equipped with the additional data of the map μ. Thus, when we speak of "elements of (V, μ)" or "maps to (V, μ)"

[53] The A^- appearing in this map η' is the same as the A^- appearing in the map $\eta : U(A^-) \to \operatorname{Sym} A$, because of Remark 3.8.

or "$\mathbf{k}$-submodules of (V, μ)", we shall mean (respectively) "elements of V", "maps to V", or "$\mathbf{k}$-submodules of V".

(c) By abuse of notation, we shall write "V is a right $\mathfrak{g}$-module" instead of "(V, μ) is a right $\mathfrak{g}$-module" when the map μ is clear from the context or has not been introduced yet.

Definition 5.3. Let $\mathfrak{g}$ be a Lie algebra. Let V be a $\mathbf{k}$-module. A *right $\mathfrak{g}$-module structure on V* means a map $\mu : V \times \mathfrak{g} \to V$ such that (V, μ) is a right $\mathfrak{g}$-module. In other words, a right $\mathfrak{g}$-module structure on V means a $\mathbf{k}$-bilinear map $\mu : V \times \mathfrak{g} \to V$ such that we have

$$(v \leftharpoonup [a,b] = (v \leftharpoonup a) \leftharpoonup b - (v \leftharpoonup b) \leftharpoonup a \text{ for every } a \in \mathfrak{g}, b \in \mathfrak{g} \text{ and } v \in V),$$

where we denote $\mu(m, a)$ by $m \leftharpoonup a$ for every $a \in \mathfrak{g}$ and $m \in V$. Thus, a right $\mathfrak{g}$-module is the same as a $\mathbf{k}$-module endowed with a right $\mathfrak{g}$-module structure.

Definition 5.4. Let $\mathfrak{g}$ be a Lie algebra. Let V be a right $\mathfrak{g}$-module.

(a) A *right $\mathfrak{g}$-submodule* of V means a right $\mathfrak{g}$-module V' such that the $\mathbf{k}$-module V' is a submodule of the $\mathbf{k}$-module V, and such that the Lie action of V' is the restriction of the Lie action of V to $V' \times \mathfrak{g}$.

(b) Let W be a further right $\mathfrak{g}$-module. A *right $\mathfrak{g}$-module homomorphism from V to W* means a $\mathbf{k}$-linear map $f : V \to W$ satisfying

$$(f(v) \leftharpoonup a = f(v \leftharpoonup a) \qquad \text{for every } a \in \mathfrak{g} \text{ and } v \in V).$$

A *right $\mathfrak{g}$-module isomorphism from V to W* means an invertible right $\mathfrak{g}$-module homomorphism from V to W whose inverse is also a right $\mathfrak{g}$-module homomorphism. It is easy to show that any invertible right $\mathfrak{g}$-module homomorphism is a $\mathfrak{g}$-module isomorphism.

Thus, we can define a category of right $\mathfrak{g}$-modules: Its objects are right $\mathfrak{g}$-modules, and its morphisms are right $\mathfrak{g}$-module homomorphisms.

You will not encounter right $\mathfrak{g}$-modules in the literature too often (unlike right A-modules over a $\mathbf{k}$-algebra A). This has a rather simple reason: They are in 1-to-1 correspondence with (left) $\mathfrak{g}$-modules. Namely, the following holds:

Remark 5.5. Let $\mathfrak{g}$ be a Lie algebra.

(a) If V is a left $\mathfrak{g}$-module, then we can define a right $\mathfrak{g}$-module structure on V by

$$(v \leftharpoonup a = -a \rightharpoonup v \qquad \text{for every } a \in \mathfrak{g} \text{ and } v \in V).$$

Thus, every left $\mathfrak{g}$-module becomes a right $\mathfrak{g}$-module.

(b) If V is a right $\mathfrak{g}$-module, then we can define a right $\mathfrak{g}$-module structure on V by

$$(a \rightharpoonup v = -v \leftharpoonup a \qquad \text{for every } a \in \mathfrak{g} \text{ and } v \in V).$$

Thus, every right $\mathfrak{g}$-module becomes a left $\mathfrak{g}$-module.

(c) If V and W are two left $\mathfrak{g}$-modules, and if $f : V \to W$ is a left $\mathfrak{g}$-module homomorphism, then $f : V \to W$ is also a right $\mathfrak{g}$-module homomorphism (where V and W become right $\mathfrak{g}$-modules according to Remark 5.5 **(a)**). Thus, we obtain a functor from the category of left $\mathfrak{g}$-modules to the category of right $\mathfrak{g}$-modules.

(d) Similarly, we obtain a functor from the category of right $\mathfrak{g}$-modules to the category of left $\mathfrak{g}$-modules.

(e) These two functors are mutually inverse.

We leave the proof of Remark 5.5 to the reader; it is an easy consequence of the fact that any two elements a and b of a Lie algebra $\mathfrak{g}$ satisfy $[a,b] = -[b,a]$.

Remark 5.5 shows that (roughly speaking) everything that can be said using the notion of right $\mathfrak{g}$-modules can also be said without it. Nevertheless, we find this notion useful, because when we will state analogues of our results above (e.g., of Theorem 3.10), it will help to make the analogy obvious.

5.2. Opposite algebras

Next, we shall define some concepts which boil down to "turning around the product" in a $\mathbf{k}$-algebra, in a Lie algebra and in a pre-Lie algebra:

Definition 5.6. Let A be a $\mathbf{k}$-algebra. Then, we define A^{op} to be the $\mathbf{k}$-module A, equipped with the multiplication $* : A \times A \to A$ (written in infix notation) which is defined by

$$(a * b = ba \qquad \text{for every } a \in A \text{ and } b \in A).$$

It is easy to see that A^{op} is again a $\mathbf{k}$-algebra, with the same unity as A. (Roughly speaking, A^{op} is the same $\mathbf{k}$-algebra as A, except that the order of the factors in its multiplication has been turned around.) We call A^{op} the *opposite algebra* of A.

Definition 5.7. Let $\mathfrak{g}$ be a Lie algebra. Then, we define $\mathfrak{g}^{\operatorname{op}}$ to be the $\mathbf{k}$-module $\mathfrak{g}$, equipped with the Lie bracket $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is defined by

$$(\lambda(a,b) = [b,a] \qquad \text{for every } a \in \mathfrak{g} \text{ and } b \in \mathfrak{g}).$$

It is easy to see that $\mathfrak{g}^{\operatorname{op}}$ is again a Lie algebra. (Roughly speaking, $\mathfrak{g}^{\operatorname{op}}$ is the same Lie algebra as $\mathfrak{g}$, except that the order of the arguments in its Lie bracket has been turned around.) We call $\mathfrak{g}^{\operatorname{op}}$ the *opposite Lie algebra* of $\mathfrak{g}$.

Definition 5.8. Let A be a left pre-Lie algebra. Then, we define A^{op} to be the $\mathbf{k}$-module A, equipped with the map $\triangleleft : A \times A \to A$ (written in infix notation) which is defined by

$$(a \triangleleft b = b \triangleright a \qquad \text{for all } a, b \in A).$$

Proposition 3.3 shows that A^{op} is a right pre-Lie algebra. We call A^{op} the *opposite pre-Lie algebra* of A.

Definition 5.9. Let A be a right pre-Lie algebra. Then, we define A^{op} to be the $\mathbf{k}$-module A, equipped with the map $\triangleright$: $A \times A \to A$ (written in infix notation) which is defined by

$$(a \triangleright b = b \triangleleft a \qquad \text{for all } a, b \in A).$$

Proposition 3.4 shows that A^{op} is a left pre-Lie algebra. We call A^{op} the *opposite pre-Lie algebra* of A.

The four definitions that we just made are similar to each other: Each of them transforms an algebraic structure by switching the order of arguments in its binary operation. (Similar operations exist for groups, monoids, etc..) However, their properties differ. For example, every Lie algebra $\mathfrak{g}$ is isomorphic to its opposite Lie algebra $\mathfrak{g}^{\mathrm{op}}$:

Remark 5.10. Let $\mathfrak{g}$ be a Lie algebra. Then, the map

$$\mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}, \qquad x \mapsto -x$$

is a Lie algebra isomorphism.

However, in general, a $\mathbf{k}$-algebra A is not isomorphic to A^{op}, and for pre-Lie algebras, such a statement would not even make sense. Also, the opposite Lie algebra $\mathfrak{g}^{\mathrm{op}}$ of a Lie algebra $\mathfrak{g}$ can be simply obtained by multiplying the Lie bracket of $\mathfrak{g}$ by -1.

Remark 5.11. Let A be a $\mathbf{k}$-algebra. Then, $(A^{\mathrm{op}})^{-} = (A^{-})^{\mathrm{op}}$.

Definition 5.12. Let V be a $\mathbf{k}$-module. We let $\mathrm{End}^{\mathrm{op}} V$ be the $\mathbf{k}$-module $\mathrm{End}\, V$, equipped with the multiplication $* : (\mathrm{End}\, V) \times (\mathrm{End}\, V) \to \mathrm{End}\, V$ (written in infix notation) which is defined by

$$(a * b = b \circ a \qquad \text{for every } a \in \mathrm{End}\, V \text{ and } b \in \mathrm{End}\, V).$$

This $\mathrm{End}^{\mathrm{op}} V$ is a $\mathbf{k}$-algebra. Actually, $\mathrm{End}^{\mathrm{op}} V = (\mathrm{End}\, V)^{\mathrm{op}}$ as $\mathbf{k}$-algebras.

We have introduced the special notation $\mathrm{End}^{\mathrm{op}} V$ (instead of using $(\mathrm{End}\, V)^{\mathrm{op}}$) because we want to stress that $\mathrm{End}^{\mathrm{op}} V$ is an analogue of $\mathrm{End}\, V$ (rather than just the opposite algebra of $\mathrm{End}\, V$). More precisely, $\mathrm{End}^{\mathrm{op}} V$ is defined in the same way as $\operatorname{End} V$, with the only difference that the multiplication now corresponds to the composition of maps "in the opposite order". The **k**-algebra $\operatorname{End}^{\operatorname{op}} V$ plays the same role in regard to right modules as the **k**-algebra $\operatorname{End} V$ plays in regard to left modules; more precisely, we have the following two analogous propositions:

Proposition 5.13. Let V be a **k**-module. Let A be a **k**-algebra. Left A-module structures on V are in a 1-to-1 correspondence with **k**-algebra homomorphisms $A \to \operatorname{End} V$; this correspondence is defined as follows: Given a left A-module structure on V, we can define a **k**-algebra homomorphism $\Phi : A \to \operatorname{End} V$ by setting

$$((\Phi(a))(v) = av \qquad \text{for every } a \in A \text{ and } v \in V).$$

Conversely, given a **k**-algebra homomorphism $\Phi : A \to \operatorname{End} V$, we can define a left A-module structure on V by setting

$$(av = (\Phi(a))(v) \qquad \text{for every } a \in A \text{ and } v \in V).$$

Proposition 5.14. Let V be a **k**-module. Let A be a **k**-algebra. Right A-module structures on V are in a 1-to-1 correspondence with **k**-algebra homomorphisms $A \to \operatorname{End}^{\operatorname{op}} V$; this correspondence is defined as follows: Given a right A-module structure on V, we can define a **k**-algebra homomorphism $\Phi : A \to \operatorname{End}^{\operatorname{op}} V$ by setting

$$((\Phi(a))(v) = va \qquad \text{for every } a \in A \text{ and } v \in V).$$

Conversely, given a **k**-algebra homomorphism $\Phi : A \to \operatorname{End}^{\operatorname{op}} V$, we can define a right A-module structure on V by setting

$$(va = (\Phi(a))(v) \qquad \text{for every } a \in A \text{ and } v \in V).$$

The analogy between Proposition 5.13 and Proposition 5.14 would become even more obvious if we would write the action of $\Phi(a)$ on the right of the v in Proposition 5.14 (that is, if we would write $v(\Phi(a))$ instead of $(\Phi(a))(v)$); but we prefer to keep to the more standard notations.

We can now state the analogue of Definition 1.9 for right $\mathfrak{g}$-modules:

Definition 5.15. Let $\mathfrak{g}$ be a Lie algebra.

(a) Every right $U(\mathfrak{g})$-module M canonically becomes a right $\mathfrak{g}$-module by setting

$$(m \leftharpoonup a = m\iota_{U,\mathfrak{g}}(a) \qquad \text{for all } a \in \mathfrak{g} \text{ and } m \in M).$$

Moreover, any right $U(\mathfrak{g})$-module homomorphism between two right $U(\mathfrak{g})$-modules becomes a right $\mathfrak{g}$-module homomorphism if we regard these right

$U(\mathfrak{g})$-modules as right $\mathfrak{g}$-modules. Thus, we obtain a functor from the category of right $U(\mathfrak{g})$-modules to the category of right $\mathfrak{g}$-modules.

(b) Every right $\mathfrak{g}$-module M canonically becomes a right $U(\mathfrak{g})$-module. To define the right $U(\mathfrak{g})$-module structure on M, we proceed as follows: Define a map $\varphi : \mathfrak{g} \to \operatorname{End}^{\operatorname{op}} M$ by

$$((\varphi(a))(m) = m \leftharpoonup a \qquad \text{for all } a \in \mathfrak{g} \text{ and } m \in M).$$

It is easy to see that this map φ is a Lie algebra homomorphism from $\mathfrak{g}$ to $(\operatorname{End}^{\operatorname{op}} M)^{-}$. (Indeed, this is a restatement of the axioms of a right $\mathfrak{g}$-module; the fact that $\varphi([a,b]) = [\varphi(a), \varphi(b)]$ for all $a, b \in \mathfrak{g}$ is equivalent to the relation (101).) Now, Theorem 1.8 (applied to $A = \operatorname{End}^{\operatorname{op}} M$ and $f = \varphi$) shows that there exists a unique $\mathbf{k}$-algebra homomorphism $F : U(\mathfrak{g}) \to \operatorname{End}^{\operatorname{op}} M$ such that $\varphi = F \circ \iota_{U,\mathfrak{g}}$. Consider this F. Now, we define a right $U(\mathfrak{g})$-module structure on M by

$$(mp = (F(p))(m) \qquad \text{for all } p \in U(\mathfrak{g}) \text{ and } m \in M).$$

Thus, every right $\mathfrak{g}$-module canonically becomes a right $U(\mathfrak{g})$-module. Moreover, any right $\mathfrak{g}$-module homomorphism between two right $\mathfrak{g}$-modules becomes a right $U(\mathfrak{g})$-module homomorphism if we regard these right $\mathfrak{g}$-modules as right $U(\mathfrak{g})$-modules. Hence, we obtain a functor from the category of right $\mathfrak{g}$-modules to the category of right $U(\mathfrak{g})$-modules.

(c) In Definition 5.15 **(a)**, we have constructed a functor from the category of right $U(\mathfrak{g})$-modules to the category of right $\mathfrak{g}$-modules. In Definition 5.15 **(b)**, we have constructed a functor from the category of right $\mathfrak{g}$-modules to the category of right $U(\mathfrak{g})$-modules. These two functors are mutually inverse. In particular, if M is a right $\mathfrak{g}$-module, then the right $U(\mathfrak{g})$-module structure on M obtained according to Definition 5.15 **(b)** satisfies

$$m\iota_{U,\mathfrak{g}}(a) = m \leftharpoonup a \qquad \text{for every } a \in \mathfrak{g} \text{ and } m \in M.$$

(d) According to Definition 5.15 **(a)**, every right $U(\mathfrak{g})$-module canonically becomes a right $\mathfrak{g}$-module. In particular, $U(\mathfrak{g})$ itself becomes a right $\mathfrak{g}$-module (because $U(\mathfrak{g})$ is a right $U(\mathfrak{g})$-module). This is the right $\mathfrak{g}$-module structure on $U(\mathfrak{g})$ "given by right multiplication" (because it satisfies $u \leftharpoonup x = u\iota_{U,\mathfrak{g}}(x)$ for every $x \in \mathfrak{g}$ and $u \in U(\mathfrak{g})$). Other canonical right $\mathfrak{g}$-module structures on $U(\mathfrak{g})$ exist as well, but we shall not use them for the time being.

We furthermore define a "right analogue" of $\operatorname{Der} C$:

Definition 5.16. Let C be a $\mathbf{k}$-algebra. We can regard $\operatorname{Der} C$ not only as a subset of $\operatorname{End} C$, but also as a subset of $\operatorname{End}^{\operatorname{op}} C$ (since $\operatorname{End}^{\operatorname{op}} C = \operatorname{End} C$ as sets). Then, $\operatorname{Der} C$ is a Lie subalgebra of $(\operatorname{End}^{\operatorname{op}} C)^{-}$. (This is an analogue of Proposition 1.11 **(a)**.) We define $\operatorname{Der}^{\operatorname{op}} C$ to be the Lie subalgebra $\operatorname{Der} C$ of

$\left(\operatorname{End}^{\operatorname{op}} C\right)^{-}$. (Thus, $\operatorname{Der}^{\operatorname{op}} C = \operatorname{Der} C$ as sets and as $\mathbf{k}$-modules, but $\operatorname{Der}^{\operatorname{op}} C = (\operatorname{Der} C)^{\operatorname{op}}$ as Lie algebras.)

5.3. Analogues of results on Lie algebra actions

We can now state the analogue of Theorem 1.15 for right $\mathfrak{g}$-modules:

Theorem 5.17. Let $\mathfrak{g}$ be a Lie algebra. Let C be a $\mathbf{k}$-algebra. Let $K' : \mathfrak{g} \to \operatorname{Der}^{\operatorname{op}} C$ be a Lie algebra homomorphism. Let $f' : \mathfrak{g} \to C$ be a $\mathbf{k}$-linear map. Assume that

$$f'([a,b]) = \left[f'(a), f'(b)\right] + \left(K'(b)\right)\left(f'(a)\right) - \left(K'(a)\right)\left(f'(b)\right) \tag{102}$$

for every $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$ (where the Lie bracket $[f'(a), f'(b)]$ is computed in the Lie algebra C^{-}).

(a) Then, we can define a right $\mathfrak{g}$-module structure on C by setting

$$\left(u \leftharpoonup a = u \cdot f'(a) + \left(K'(a)\right)(u) \text{ for all } a \in \mathfrak{g} \text{ and } u \in C\right). \tag{103}$$

In the following, we will regard C as a right $\mathfrak{g}$-module by means of this right $\mathfrak{g}$-module structure.

(b) Being a right $\mathfrak{g}$-module, C becomes a right $U(\mathfrak{g})$-module. Define a map $\eta' : U(\mathfrak{g}) \to C$ by

$$\eta'(u) = 1_C u \qquad \text{for every } u \in U(\mathfrak{g}).$$

Then, η' is a right $\mathfrak{g}$-module homomorphism.

(c) For every $a \in \mathfrak{g}$, $b \in C$ and $c \in C$, we have

$$(cb) \leftharpoonup a - (c \leftharpoonup a) \cdot b = c \cdot \left(K'(a)\right)(b) - c\left[f'(a), b\right].$$

(Here, again, the Lie bracket $[f'(a), b]$ is computed in the Lie algebra C^{-}.)

Proof of Theorem 5.17. In order to prove Theorem 5.17, it suffices to follow the proof of Theorem 1.15, making only some minor changes. For example, "f", "K", "$\operatorname{Der} C$" and "η" have to be replaced by "f'", "K'", "$\operatorname{Der}^{\operatorname{op}} C$" and "$\eta'$"; some plus signs have to be changed to minus signs and vice versa; and in various products the factors have to be reordered. Thus, for example:

- The map $\mu : \mathfrak{g} \times C \to C$ in the proof of Theorem 1.15 has to be replaced by a map $\mu : C \times \mathfrak{g} \to C$.

- The equality

$$K([a,b]) = [K(a), K(b)] = K(a) \circ K(b) - K(b) \circ K(a)$$

has to be replaced by

$$K'([a,b]) = \left[K'(a), K'(b)\right] = K'(b) \circ K'(a) - K'(a) \circ K'(b)$$

(since $[K'(a), K'(b)]$ now has to be understood as a Lie bracket inside $\operatorname{Der}^{\mathrm{op}} C$ and not inside $\operatorname{Der} C$).

□

Next, let us state the analogue of Theorem 1.20 for right $\mathfrak{g}$-modules:

Theorem 5.18. Let $\mathfrak{g}$ be a Lie algebra. Let C be a $\mathbf{k}$-bialgebra. Let $K' : \mathfrak{g} \to \operatorname{Der}^{\mathrm{op}} C$ be a Lie algebra homomorphism such that $K'(\mathfrak{g}) \subseteq \operatorname{Coder} C$. (We are regarding $\operatorname{Coder} C$ as a set here, whence we do not have to replace it by its "right analogue".) Let $f' : \mathfrak{g} \to C$ be a $\mathbf{k}$-linear map such that $f'(\mathfrak{g}) \subseteq \operatorname{Prim} C$. Assume that (102) holds for every $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$.

Consider the right $\mathfrak{g}$-module structure on C defined in Theorem 5.17 **(a)**, and the map $\eta' : U(\mathfrak{g}) \to C$ defined in Theorem 5.17 **(b)**.

(a) For every $a \in \mathfrak{g}$, the map $C \to C,\ c \mapsto c \leftharpoonup a$ is a coderivation of C.

(b) The map $\eta' : U(\mathfrak{g}) \to C$ is a $\mathbf{k}$-coalgebra homomorphism.

Proof of Theorem 5.18. In order to prove Theorem 5.18, it suffices to follow the proof of Theorem 1.20, making only some minor changes. For example, "f", "K", "$\operatorname{Der} C$" and "η" have to be replaced by "f'", "K'", "$\operatorname{Der}^{\mathrm{op}} C$" and "$\eta'$"; some plus signs have to be changed to minus signs and vice versa; and in various products the factors have to be reordered. Here are some concrete examples of what these changes mean:

- The map ζ, which was defined as the map $C \to C,\ c \mapsto a \rightharpoonup c$, must now be defined as the map $C \to C,\ c \mapsto c \leftharpoonup a$ instead.
- The equality (9) has to be replaced by

$$(\Delta \circ \zeta)(u) = \Delta(u) \cdot \left(f'(a) \otimes 1 + 1 \otimes f'(a)\right) + \left(K'(a) \otimes \operatorname{id} + \operatorname{id} \otimes K'(a)\right)(\Delta(u)).$$

- The equality (10) has to be replaced by

$$(\zeta \otimes \operatorname{id} + \operatorname{id} \otimes \zeta)(p) = p \cdot \left(f'(a) \otimes 1 + 1 \otimes f'(a)\right) + \left(K'(a) \otimes \operatorname{id} + \operatorname{id} \otimes K'(a)\right)(p).$$

- The $\mathbf{k}$-algebra homomorphism $\Xi : U(\mathfrak{g}) \to \operatorname{End} C$ has to be replaced by a $\mathbf{k}$-algebra homomorphism $\Xi : U(\mathfrak{g}) \to \operatorname{End}^{\mathrm{op}} C$ defined by setting

$$((\Xi(p))(c) = cp \qquad \text{for every } p \in U(\mathfrak{g}) \text{ and } c \in C). \tag{104}$$

- The map $\mathbf{z}$ needs not be changed, but it now has to be considered as a $\mathbf{k}$-algebra homomorphism from $\operatorname{End}^{\mathrm{op}} C \otimes \operatorname{End}^{\mathrm{op}} C$ to $\operatorname{End}^{\mathrm{op}}(C \otimes C)$.

- The map Ξ' is defined in the same manner as in the proof of Theorem 1.20, but now using the new map Ξ (defined by (104)); thus, this map Ξ' will be a $\mathbf{k}$-algebra homomorphism $U(\mathfrak{g}) \to \operatorname{End}^{\operatorname{op}}(C \otimes C)$.

- The proof of the fact that $\mathfrak{P}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$ must be modified: For example, "$\Xi(ab) = \Xi(a) \circ \Xi(b)$" has to be replaced by "$\Xi(ab) = \Xi(b) \circ \Xi(a)$" (since Ξ is no longer a $\mathbf{k}$-algebra homomorphism from $U(\mathfrak{g})$ to $\operatorname{End} C$, but now is a $\mathbf{k}$-algebra homomorphism from $U(\mathfrak{g})$ to $\operatorname{End}^{\operatorname{op}} C$); similarly for Ξ'.

- The proof of $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{P}$ has to be modified as well: For example, "$(\Xi(\iota_{U,\mathfrak{g}}(a)))(c) = \iota_{U,\mathfrak{g}}(a)\, c = a \rightharpoonup c = \zeta(c)$" now must be replaced by "$(\Xi(\iota_{U,\mathfrak{g}}(a)))(c) = c\iota_{U,\mathfrak{g}}(a) = c \leftharpoonup a = \zeta(c)$".

- The proof of the fact that $\mathfrak{R}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$ must be modified (similarly to the proof of the fact that $\mathfrak{P}$ is a $\mathbf{k}$-subalgebra of $U(\mathfrak{g})$).

- The proof of $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{R}$ must be modified (similarly to the proof of $\iota_{U,\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{P}$).

The reader should be able to do all these changes without trouble. $\square$

Furthermore, let us state the analogue of Theorem 2.16:

Theorem 5.19. Let $\mathfrak{g}$ be a Lie algebra. Let C be a filtered $\mathbf{k}$-algebra. Let $(C_n)_{n\geq 0}$ be the filtration of C. Set $C_{-1} = 0$.

Let $K' : \mathfrak{g} \to \operatorname{Der}^{\operatorname{op}} C$ be a Lie algebra homomorphism. Let $f' : \mathfrak{g} \to C$ be a $\mathbf{k}$-linear map. Assume that (102) holds for every $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$ (where the Lie bracket $[f'(a), f'(b)]$ is computed in the Lie algebra C^-). Define a right $\mathfrak{g}$-module structure on C as in Theorem 5.17 **(a)**. Define a map $\eta' : U(\mathfrak{g}) \to C$ as in Theorem 5.17 **(b)**.

Assume furthermore that $f'(\mathfrak{g}) \subseteq C_1$. Also, assume that the map $K'(a) : C \to C$ is filtered for every $a \in \mathfrak{g}$.

Then,

$$\eta'(\iota_{U,\mathfrak{g}}(a_1)\,\iota_{U,\mathfrak{g}}(a_2)\cdots\iota_{U,\mathfrak{g}}(a_n)) \in f'(a_1)\, f'(a_2)\cdots f'(a_n) + C_{n-1} \qquad (105)$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$.

Proof of Theorem 5.19. We shall first show a slightly restated version of the claim of Theorem 5.19:

Claim 1: We have

$$\eta'(\iota_{U,\mathfrak{g}}(a_n)\,\iota_{U,\mathfrak{g}}(a_{n-1})\cdots\iota_{U,\mathfrak{g}}(a_1)) \in f'(a_n)\, f'(a_{n-1})\cdots f'(a_1) + C_{n-1}$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in \mathfrak{g}$.

Proof of Claim 1. In order to prove Claim 1, it suffices to follow the proof of Theorem 2.16, making only some minor changes. For example, "f", "K", "$\operatorname{Der} C$" and "η" have to be replaced by "f'", "K'", "$\operatorname{Der}^{\mathrm{op}} C$" and "$\eta'$"; some plus signs have to be changed to minus signs and vice versa; and in various products the factors have to be reordered. Here are some concrete examples of what these changes mean:

- The equality (19) has to be replaced by
$$\eta'\left(\iota_{U,\mathfrak{g}}\left(a_{N-1}\right)\iota_{U,\mathfrak{g}}\left(a_{N-2}\right)\cdots\iota_{U,\mathfrak{g}}\left(a_1\right)\right)\in f'\left(a_{N-1}\right)f\left(a_{N-2}\right)\cdots f\left(a_1\right)+C_{(N-1)-1}. \tag{106}$$
- The element g has to be replaced by the element $\eta'\left(\iota_{U,\mathfrak{g}}\left(a_N\right)\iota_{U,\mathfrak{g}}\left(a_{N-1}\right)\cdots\iota_{U,\mathfrak{g}}\left(a_2\right)\right)$.
- The product $f\left(a_2\right)f\left(a_3\right)\cdots f\left(a_N\right)$ has to be replaced by $f'\left(a_N\right)f'\left(a_{N-1}\right)\cdots f'\left(a_2\right)$.
- Instead of multiplying both sides of the relation (20) with $f\left(a_1\right)$ from the left, we need to multiply both sides of the relation (106) with $f\left(a_1\right)$ from the right.
- The equality (23) has to be replaced by
$$g=\eta'\left(\iota_{U,\mathfrak{g}}\left(a_N\right)\iota_{U,\mathfrak{g}}\left(a_{N-1}\right)\cdots\iota_{U,\mathfrak{g}}\left(a_2\right)\right)=1_C\left(\iota_{U,\mathfrak{g}}\left(a_N\right)\iota_{U,\mathfrak{g}}\left(a_{N-1}\right)\cdots\iota_{U,\mathfrak{g}}\left(a_2\right)\right).$$

Thus, Claim 1 is proven.

Now, let $n\in\mathbb{N}$ and $a_1,a_2,\ldots,a_n\in\mathfrak{g}$. Then, Claim 1 (applied to $a_n,a_{n-1},\ldots,a_1$ instead of $a_1,a_2,\ldots,a_n$) shows that
$$\eta'\left(\iota_{U,\mathfrak{g}}\left(a_1\right)\iota_{U,\mathfrak{g}}\left(a_2\right)\cdots\iota_{U,\mathfrak{g}}\left(a_n\right)\right)\in f'\left(a_1\right)f'\left(a_2\right)\cdots f'\left(a_n\right)+C_{n-1}.$$

This proves Theorem 5.19. □

5.4. A theorem connecting left and right actions

As we have said, what we have done so far in Section 5 was rather unexciting: We have just stated analogues of previous results; the proofs of these analogues were mostly analogous to the proofs of those latter results. Now, however, we shall connect the analogues with the original results, to obtain something new:

Theorem 5.20. Let $\mathfrak{g}$ be a Lie algebra. Let C be a $\mathbf{k}$-algebra.

Let $K:\mathfrak{g}\to\operatorname{Der} C$ be a Lie algebra homomorphism. Let $f:\mathfrak{g}\to C$ be a $\mathbf{k}$-linear map. Assume that (4) holds for every $a\in\mathfrak{g}$ and $b\in\mathfrak{g}$ (where the Lie bracket $\left[f\left(a\right),f\left(b\right)\right]$ is computed in the Lie algebra C^{-}). Theorem 1.15 **(a)** shows that we can define a $\mathfrak{g}$-module structure on C by setting
$$\left(a\rightharpoonup u=f\left(a\right)\cdot u+\left(K\left(a\right)\right)\left(u\right)\text{ for all }a\in\mathfrak{g}\text{ and }u\in C\right).$$

In the following, we will regard C as a $\mathfrak{g}$-module by means of this $\mathfrak{g}$-module structure.

Let $K' : \mathfrak{g} \to \operatorname{Der}^{\operatorname{op}} C$ be a Lie algebra homomorphism. Let $f' : \mathfrak{g} \to C$ be a $\mathbf{k}$-linear map. Assume that (102) holds for every $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$ (where the Lie bracket $[f'(a), f'(b)]$ is computed in the Lie algebra C^-). Theorem 5.17 **(a)** shows that we can define a right $\mathfrak{g}$-module structure on C by setting

$$\left(u \leftharpoonup a = u \cdot f'(a) + \left(K'(a)\right)(u) \text{ for all } a \in \mathfrak{g} \text{ and } u \in C\right).$$

In the following, we will regard C as a right $\mathfrak{g}$-module by means of this right $\mathfrak{g}$-module structure.

Furthermore, assume that

$$\left(K(a) \circ K'(b) - K'(b) \circ K(a)\right)(u) = \left(K'(b)\right)(f(a)) \cdot u - u \cdot (K(a))\left(f'(b)\right) \tag{107}$$

for all $a \in \mathfrak{g}$, $b \in \mathfrak{g}$ and $u \in C$. (Here, "$K(a) \circ K'(b)$" has to be read as $(K(a)) \circ (K'(b))$, and similarly for the expression "$K'(b) \circ K(a)$".)

(a) The left $\mathfrak{g}$-module structure and the right $\mathfrak{g}$-module structure defined on C satisfy

$$a \rightharpoonup (u \leftharpoonup b) = (a \rightharpoonup u) \leftharpoonup b$$

for all $a \in \mathfrak{g}$, $b \in \mathfrak{g}$ and $u \in C$.

(b) The $\mathbf{k}$-module C is a left $\mathfrak{g}$-module, and thus becomes a left $U(\mathfrak{g})$-module. Also, the $\mathbf{k}$-module C is a right $\mathfrak{g}$-module, and thus becomes a right $U(\mathfrak{g})$-module.

The left $U(\mathfrak{g})$-module structure on C and the right $U(\mathfrak{g})$-module structure on C can be combined to form a $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodule structure on C.

(c) Assume further that $f = f'$.

Consider the map $\eta : U(\mathfrak{g}) \to C$ defined in Theorem 1.15 **(b)**. Consider the map $\eta' : U(\mathfrak{g}) \to C$ defined in Theorem 5.17 **(b)**. Then, $\eta = \eta'$.

Proof of Theorem 5.20. **(a)** Let $a \in \mathfrak{g}$, $b \in \mathfrak{g}$ and $u \in C$. We have $K\left(\underbrace{a}_{\in \mathfrak{g}}\right) \in K(\mathfrak{g}) \subseteq \operatorname{Der} C$. In other words, $K(a) : C \to C$ is a derivation. Also, $K'\left(\underbrace{b}_{\in \mathfrak{g}}\right) \in K'(\mathfrak{g}) \subseteq \operatorname{Der}^{\operatorname{op}} C = \operatorname{Der} C$. In other words, $K'(b) : C \to C$ is a derivation. Now, subtracting the equality

$$
\begin{aligned}
a \rightharpoonup & \underbrace{(u \leftharpoonup b)}_{\substack{=u\cdot f'(b)+(K'(b))(u) \\ \text{(by the definition of the right} \\ \mathfrak{g}\text{-module structure on } C)}} \\
&= a \rightharpoonup \left(u \cdot f'(b) + \left(K'(b)\right)(u)\right) \\
&= \underbrace{a \rightharpoonup \left(u \cdot f'(b)\right)}_{\substack{=f(a)\cdot u\cdot f'(b)+(K(a))(u\cdot f'(b)) \\ \text{(by the definition of the left} \\ \mathfrak{g}\text{-module structure on } C)}} + \underbrace{a \rightharpoonup \left(K'(b)\right)(u)}_{\substack{=f(a)\cdot(K'(b))(u)+(K(a))((K'(b))(u)) \\ \text{(by the definition of the left} \\ \mathfrak{g}\text{-module structure on } C)}} \\
&= f(a) \cdot u \cdot f'(b) + \underbrace{(K(a))\left(u \cdot f'(b)\right)}_{\substack{=u\cdot(K(a))(f'(b))+(K(a))(u)\cdot f'(b) \\ \text{(since } K(a) \text{ is a derivation)}}} \\
&\qquad + f(a) \cdot \left(K'(b)\right)(u) + \underbrace{(K(a))\left(\left(K'(b)\right)(u)\right)}_{=(K(a)\circ K'(b))(u)} \\
&= f(a) \cdot u \cdot f'(b) + u \cdot (K(a))\left(f'(b)\right) + (K(a))(u) \cdot f'(b) \\
&\qquad + f(a) \cdot \left(K'(b)\right)(u) + \left(K(a) \circ K'(b)\right)(u)
\end{aligned}
$$

from the equality

$$
\begin{aligned}
& \underbrace{(a \rightharpoonup u)}_{\substack{=f(a)\cdot u+(K(a))(u) \\ \text{(by the definition of the left} \\ \mathfrak{g}\text{-module structure on } C)}} \leftharpoonup b \\
&= (f(a) \cdot u + (K(a))(u)) \leftharpoonup b \\
&= \underbrace{(f(a) \cdot u) \leftharpoonup b}_{\substack{=f(a)\cdot u\cdot f'(b)+(K'(b))(f(a)\cdot u) \\ \text{(by the definition of the right} \\ \mathfrak{g}\text{-module structure on } C)}} + \underbrace{(K(a))(u) \leftharpoonup b}_{\substack{=(K(a))(u)\cdot f'(b)+(K'(b))((K(a))(u)) \\ \text{(by the definition of the right} \\ \mathfrak{g}\text{-module structure on } C)}} \\
&= f(a) \cdot u \cdot f'(b) + \underbrace{\left(K'(b)\right)(f(a) \cdot u)}_{\substack{=f(a)\cdot(K'(b))(u)+(K'(b))(f(a))\cdot u \\ \text{(since } K'(b) \text{ is a derivation)}}} \\
&\qquad + (K(a))(u) \cdot f'(b) + \underbrace{\left(K'(b)\right)((K(a))(u))}_{=(K'(b)\circ K(a))(u)} \\
&= f(a) \cdot u \cdot f'(b) + f(a) \cdot \left(K'(b)\right)(u) + \left(K'(b)\right)(f(a)) \cdot u \\
&\qquad + (K(a))(u) \cdot f'(b) + \left(K'(b) \circ K(a)\right)(u),
\end{aligned}
$$

we obtain

$$\begin{aligned}
&(a \rightharpoonup u) \leftharpoonup b - a \rightharpoonup (u \leftharpoonup b) \\
&= \big(f(a) \cdot u \cdot f'(b) + f(a) \cdot \big(K'(b)\big)(u) + \big(K'(b)\big)(f(a)) \cdot u \\
&\qquad\qquad + (K(a))(u) \cdot f'(b) + \big(K'(b) \circ K(a)\big)(u)\big) \\
&\qquad - \big(f(a) \cdot u \cdot f'(b) + u \cdot (K(a))\big(f'(b)\big) + (K(a))(u) \cdot f'(b) \\
&\qquad\qquad + f(a) \cdot \big(K'(b)\big)(u) + \big(K(a) \circ K'(b)\big)(u)\big) \\
&= \big(K'(b)\big)(f(a)) \cdot u + \big(K'(b) \circ K(a)\big)(u) - u \cdot (K(a))\big(f'(b)\big) - \big(K(a) \circ K'(b)\big)(u) \\
&= \underbrace{\big(K'(b)\big)(f(a)) \cdot u - u \cdot (K(a))\big(f'(b)\big)}_{\substack{=(K(a)\circ K'(b) - K'(b)\circ K(a))(u) \\ \text{(by (107))}}} + \big(K'(b) \circ K(a)\big)(u) - \big(K(a) \circ K'(b)\big)(u) \\
&= \underbrace{\big(K(a) \circ K'(b) - K'(b) \circ K(a)\big)(u)}_{=(K(a)\circ K'(b))(u) - (K'(b)\circ K(a))(u)} + \big(K'(b) \circ K(a)\big)(u) - \big(K(a) \circ K'(b)\big)(u) \\
&= \big(K(a) \circ K'(b)\big)(u) - \big(K'(b) \circ K(a)\big)(u) + \big(K'(b) \circ K(a)\big)(u) - \big(K(a) \circ K'(b)\big)(u) \\
&= 0.
\end{aligned}$$

In other words, $a \rightharpoonup (u \leftharpoonup b) = (a \rightharpoonup u) \leftharpoonup b$. This proves Theorem 5.20 **(a)**.

(b) Fix $a \in \mathfrak{g}$. Let λ_a be the map $C \to C$, $u \mapsto a \rightharpoonup u$. Then, the map λ_a is a right $\mathfrak{g}$-module homomorphism[54], and thus is a right $U(\mathfrak{g})$-module homomorphism (since every right $\mathfrak{g}$-module homomorphism is a right $U(\mathfrak{g})$-module homomorphism). Thus, we have

$$a \rightharpoonup (uq) = (a \rightharpoonup u) q \qquad \text{for every } u \in C \text{ and } q \in U(\mathfrak{g}) \tag{108}$$

[55].

Let us now forget that we fixed a. We thus have proven (108) for every $a \in \mathfrak{g}$.

Let us now fix $q \in U(\mathfrak{g})$. Let ρ_q be the map $C \to C$, $u \mapsto uq$. Then, the map ρ_q is a left $\mathfrak{g}$-module homomorphism[56], and thus is a left $U(\mathfrak{g})$-module homomorphism (since every left $\mathfrak{g}$-module homomorphism is a left $U(\mathfrak{g})$-module homomorphism). Thus, we have

$$p(uq) = (pu) q \qquad \text{for every } u \in C \text{ and } p \in U(\mathfrak{g}) \tag{109}$$

[54] *Proof.* Let $b \in \mathfrak{g}$ and $u \in C$. Then, the definition of λ_a shows that $\lambda_a(u \leftharpoonup b) = a \rightharpoonup (u \leftharpoonup b) = (a \rightharpoonup u) \leftharpoonup b$ (by Theorem 5.20 **(a)**). Also, $\lambda_a(u) = a \rightharpoonup u$ (by the definition of λ_a), so that $\underbrace{\lambda_a(u)}_{=a\rightharpoonup u} \leftharpoonup b = (a \rightharpoonup u) \leftharpoonup b$. Comparing this with $\lambda_a(u \leftharpoonup b) = (a \rightharpoonup u) \leftharpoonup b$, we obtain $\lambda_a(u \leftharpoonup b) = \lambda_a(u) \leftharpoonup b$.

Now, let us forget that we fixed b and u. We thus have proven that $\lambda_a(u \leftharpoonup b) = \lambda_a(u) \leftharpoonup b$ for every $b \in \mathfrak{g}$ and $u \in C$. In other words, λ_a is a right $\mathfrak{g}$-module homomorphism. Qed.

[55] *Proof of (108):* Let $u \in C$ and $q \in U(\mathfrak{g})$. The definition of λ_a shows that $\lambda_a(uq) = a \rightharpoonup (uq)$. But λ_a is a right $U(\mathfrak{g})$-module homomorphism. Hence, $\lambda_a(uq) = \underbrace{\lambda_a(u)}_{\substack{=a\rightharpoonup u \\ \text{(by the definition} \\ \text{of } \lambda_a)}} q = (a \rightharpoonup u) q$.

Comparing this with $\lambda_a(uq) = a \rightharpoonup (uq)$, we obtain $a \rightharpoonup (uq) = (a \rightharpoonup u) q$. This proves (108).

[56] *Proof.* Let $a \in \mathfrak{g}$ and $u \in C$. The definition of ρ_q shows that $\rho_q(u) = uq$, so that

[57].

Let us now forget that we fixed q. We thus have proven (108) for every $q \in U(\mathfrak{g})$. In other words, we have $p(uq) = (pu)q$ for every $u \in C$, $p \in U(\mathfrak{g})$ and $q \in U(\mathfrak{g})$. In other words, the left $U(\mathfrak{g})$-module structure on C and the right $U(\mathfrak{g})$-module structure on C can be combined to form a $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodule structure on C. This proves Theorem 5.20 **(b)**.

$a \rightharpoonup \underbrace{\rho_q(u)}_{=uq} = a \rightharpoonup (uq) = (a \rightharpoonup u)q$ (by (108)). On the other hand, the definition of ρ_q shows that $\rho_q(a \rightharpoonup u) = (a \rightharpoonup u)q$. Comparing this with $a \rightharpoonup \rho_q(u) = (a \rightharpoonup u)q$, we obtain $\rho_q(a \rightharpoonup u) = a \rightharpoonup \rho_q(u)$.

Now, let us forget that we fixed a and u. We thus have proven that $\rho_q(a \rightharpoonup u) = a \rightharpoonup \rho_q(u)$ for every $a \in \mathfrak{g}$ and $u \in C$. In other words, the map ρ_q is a left $\mathfrak{g}$-module homomorphism. Qed.

[57]*Proof of (109):* Let $u \in C$ and $p \in U(\mathfrak{g})$. The definition of ρ_q shows that $\rho_q(pu) = (pu)q$. But ρ_q is a left $U(\mathfrak{g})$-module homomorphism. Hence, $\rho_q(pu) = p \underbrace{\rho_q(u)}_{\substack{=uq \\ \text{(by the definition} \\ \text{of } \rho_q)}} = p(uq)$.

Comparing this with $\rho_q(pu) = (pu)q$, we obtain $p(uq) = (pu)q$. This proves (109).

(c) The map $\eta' : U(\mathfrak{g}) \to C$ is a left $\mathfrak{g}$-module homomorphism[58], and therefore is a left $U(\mathfrak{g})$-module homomorphism.

---

[58] *Proof.* Let $a \in \mathfrak{g}$ and $u \in U(\mathfrak{g})$.

We have $K'\left(\underbrace{a}_{\in\mathfrak{g}}\right) \in K'(\mathfrak{g}) \subseteq \operatorname{Der}^{\mathrm{op}} C = \operatorname{Der} C$. Hence, Proposition 1.11 **(c)** (applied to $K'(a)$ instead of f) shows that $(K'(a))(1) = 0$. Thus, $(K'(a))\left(\underbrace{1_C}_{=1}\right) = (K'(a))(1) = 0$.

We have $K\left(\underbrace{a}_{\in\mathfrak{g}}\right) \in K(\mathfrak{g}) \subseteq \operatorname{Der} C$. Hence, Proposition 1.11 **(c)** (applied to $K(a)$ instead of f) shows that $(K(a))(1) = 0$. Thus, $(K(a))\left(\underbrace{1_C}_{=1}\right) = (K(a))(1) = 0$.

The definition of the right $U(\mathfrak{g})$-module structure on C shows that

$$\begin{aligned}
1_C \iota_{U,\mathfrak{g}}(a) &= 1_C \leftharpoondown a = 1_C \cdot f'(a) + \underbrace{(K'(a))(1_C)}_{=0} \\
&\qquad \text{(by the definition of the right } \mathfrak{g}\text{-module structure on } C) \\
&= 1_C \cdot f'(a) = f'(a).
\end{aligned}$$

Comparing this with

$$\begin{aligned}
a \rightharpoonup 1_C &= f(a) \cdot 1_C + \underbrace{(K(a))(1_C)}_{=0} \\
&\qquad \text{(by the definition of the left } \mathfrak{g}\text{-module structure on } C) \\
&= f(a) \cdot 1_C = \underbrace{f}_{=f'}(a) = f'(a),
\end{aligned}$$

we obtain $1_C \iota_{U,\mathfrak{g}}(a) = a \rightharpoonup 1_C$.

The definition of η' shows that $\eta'(u) = 1_C u$. Hence,

$$a \rightharpoonup \underbrace{\eta'(u)}_{=1_C u} = a \rightharpoonup (1_C u) = (a \rightharpoonup 1_C) u$$

(by (108) (applied to 1_C and u instead of u and q)). But the definition of η' also shows that

$$\begin{aligned}
\eta'(a \rightharpoonup u) &= 1_C \underbrace{(a \rightharpoonup u)}_{\substack{=\iota_{U,\mathfrak{g}}(a)u \\ \text{(by the definition of the left} \\ \mathfrak{g}\text{-module structure on } U(\mathfrak{g}))}} = 1_C (\iota_{U,\mathfrak{g}}(a) u) \\
&= \underbrace{(1_C \iota_{U,\mathfrak{g}}(a))}_{=a \rightharpoonup 1_C} u = (a \rightharpoonup 1_C) u.
\end{aligned}$$

Comparing this with $a \rightharpoonup \eta'(u) = (a \rightharpoonup 1_C) u$, we obtain $\eta'(a \rightharpoonup u) = a \rightharpoonup \eta'(u)$.

Now, let us forget that we fixed a and u. We thus have shown that $\eta'(a \rightharpoonup u) = a \rightharpoonup \eta'(u)$ for every $a \in \mathfrak{g}$ and $u \in U(\mathfrak{g})$. In other words, $\eta' : U(\mathfrak{g}) \to C$ is a left $\mathfrak{g}$-module homomorphism, qed.

Now, every $u \in U(\mathfrak{g})$ satisfies

$$\eta'\left(\underbrace{u}_{=u\cdot 1}\right) = \eta'(u\cdot 1) = u \cdot \underbrace{\eta'(1)}_{\substack{=1\cdot 1_C \\ \text{(by the definition of } \eta')}} \qquad \left(\text{since } \eta' \text{ is a left } U(\mathfrak{g})\text{-module homomorphism}\right)$$
$$= u\cdot 1\cdot 1_C = u1_C$$
$$= \eta(u) \qquad \left(\text{since } \eta(u) = u1_C \text{ (by the definition of } \eta)\right).$$

In other words, $\eta' = \eta$. This proves Theorem 5.20 **(c)**. □

5.5. The right Guin-Oudom isomorphism

We can now prove the analogue of Theorem 3.10 for right pre-Lie algebras:

Theorem 5.21. Let A be a right pre-Lie algebra. Recall that a Lie algebra A^- is defined (according to Proposition 3.7).

For any $a \in A$, let right_a be the $\mathbf{k}$-linear map $A \to A$, $b \mapsto b \lhd a$. For any $a \in A$, we define a derivation $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ as follows: Proposition 2.20 (applied to $V = A$, $M = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A} \circ \operatorname{right}_a$) shows that there exists a unique derivation $F : \operatorname{Sym} A \to \operatorname{Sym} A$ such that $F \circ \iota_{\operatorname{Sym},A} = \operatorname{right}_a \circ \iota_{\operatorname{Sym},A}$. We let R_a be this derivation.

In the following, we will identify every $a \in A$ with the element $\iota_{\operatorname{Sym},A}(a)$ of $\operatorname{Sym} A$. (This is a particular case of the abuse of notation introduced in Definition 3.9.) Thus, the injection $\iota_{\operatorname{Sym},A} : A \to \operatorname{Sym} A$ becomes an inclusion, and the $\mathbf{k}$-module A becomes a $\mathbf{k}$-submodule of $\operatorname{Sym} A$ (although the right pre-Lie algebra A does **not** become a subalgebra of $\operatorname{Sym} A$). Thus, products such as $b_1 b_2 \cdots b_n$ (where $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in A$) will always mean products inside $\operatorname{Sym} A$. (This notation will not conflict with the binary operation on the right pre-Lie algebra A, since the latter operation is denoted by $\lhd$.)

(a) If $a \in A$, $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in A$, then

$$R_a(b_1 b_2 \cdots b_n) = \sum_{k=1}^{n} b_1 b_2 \cdots b_{k-1} (b_k \lhd a) b_{k+1} b_{k+2} \cdots b_n.$$

(b) For every $a \in A$, the map $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is graded.

(c) For every $a \in A$, the map $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a coderivation.

(d) We define a map $K' : A \to \operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$ by

$$\left(K'(a) = R_a \qquad \text{for every } a \in A\right).$$

Then, this map K' is a Lie algebra homomorphism from A^- to $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$. (See Definition 5.16 for the definition of the Lie algebra $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$.)

(e) We can define a right A^{-}-module structure on $\operatorname{Sym} A$ by setting

$$\left(u \leftharpoonup a = ua + R_a(u) \text{ for all } a \in A^{-} \text{ and } u \in \operatorname{Sym} A\right).$$

In the following, we will regard $\operatorname{Sym} A$ as a right A^{-}-module by means of this right A^{-}-module structure.

(f) Being a right A^{-}-module, $\operatorname{Sym} A$ becomes a right $U(A^{-})$-module. Define a map $\eta' : U(A^{-}) \to \operatorname{Sym} A$ by

$$\eta'(u) = 1_{\operatorname{Sym} A} u \qquad \text{for every } u \in U\left(A^{-}\right).$$

Then, η' is a right A^{-}-module homomorphism.

(g) We have $\eta'\left(\iota_{U,A^{-}}(a)\right) = a$ for every $a \in A^{-}$.

(h) The map η' is a right A^{-}-module isomorphism.

(i) We have

$$\eta'\left(\iota_{U,A^{-}}(a_1)\,\iota_{U,A^{-}}(a_2)\cdots\iota_{U,A^{-}}(a_n)\right) \in a_1 a_2 \cdots a_n + \sum_{k=0}^{n-1} \operatorname{Sym}^k A$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A^{-}$.

(j) The map η' and its inverse $(\eta')^{-1}$ are filtered (where $U(A^{-})$ is endowed with the usual filtration on a universal enveloping algebra).

(k) The map $\operatorname{grad}^{-1}_{\operatorname{Sym} A} \circ (\operatorname{gr}(\eta')) : \operatorname{gr}(U(A^{-})) \to \operatorname{Sym} A$ is an inverse to the map $\operatorname{PBW}_{A^{-}}$. In particular, the map $\operatorname{PBW}_{A^{-}}$ is invertible.

(l) The map $\eta' : U(A^{-}) \to \operatorname{Sym} A$ is a $\mathbf{k}$-coalgebra isomorphism.

(m) For every $a \in A$, $b \in A$ and $c \in \operatorname{Sym} A$, we have

$$(cb) \leftharpoonup a - (c \leftharpoonup a) \cdot b = c \cdot (b \lhd a).$$

The proof of Theorem 5.21 is analogous to that of Theorem 3.10. I shall, however, spell it out in full instead of just pointing out the necessary changes, since this is the only way I can make sure that I don't make mistakes.

Proof of Theorem 5.21. For every $a \in A$, we know that $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a derivation and satisfies

$$R_a \circ \iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A} \circ \operatorname{left}_a \tag{110}$$

[59]. Hence, for every $a \in A$ and $b \in A$, we have

$$R_a\left(\iota_{\operatorname{Sym},A}(b)\right) = \underbrace{\left(R_a \circ \iota_{\operatorname{Sym},A}\right)}_{\substack{=\iota_{\operatorname{Sym},A}\circ \operatorname{right}_a \\ \text{(by (110))}}}(b) = \iota_{\operatorname{Sym},A}\left(\underbrace{\operatorname{right}_a(b)}_{\substack{=b\triangleleft a \\ \text{(by the definition} \\ \text{of } \operatorname{right}_a\text{)}}}\right)$$

$$= \iota_{\operatorname{Sym},A}(b \triangleleft a). \qquad (111)$$

In other words, for every $a \in A$ and $b \in A$, we have

$$R_a(b) = b \triangleleft a \qquad (112)$$

[60].

(a) Let $a \in A$, $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in A$. The map R_a is a derivation, thus an element of $\operatorname{Der}(\operatorname{Sym} A)$. Hence, Proposition 1.11 **(b)** (applied to $C = \operatorname{Sym} A$, $f = R_a$ and $a_i = b_i$) yields

$$R_a(b_1 b_2 \cdots b_n)$$
$$= \sum_{i=1}^{n} b_1 b_2 \cdots b_{i-1} \underbrace{R_a(b_i)}_{\substack{=b_i\triangleleft a \\ \text{(by (112))}}} b_{i+1} b_{i+2} \cdots b_n$$
$$= \sum_{i=1}^{n} b_1 b_2 \cdots b_{i-1} (b_i \triangleleft a) b_{i+1} b_{i+2} \cdots b_n = \sum_{k=1}^{n} b_1 b_2 \cdots b_{k-1} (b_k \triangleleft a) b_{k+1} b_{k+2} \cdots b_n$$

(here, we have renamed the summation index i as k). This proves Theorem 5.21 **(a)**.

(b) Let $a \in A$.

In Proposition 2.21 (applied to $V = A$ and $f = \operatorname{right}_a$), a map $\widetilde{f} : \operatorname{Sym} A \to \operatorname{Sym} A$ was defined as the unique derivation $F : \operatorname{Sym} A \to \operatorname{Sym} A$ such that $F \circ \iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A} \circ \operatorname{right}_a$. This derivation F must clearly be our map R_a (since we know that $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a derivation and satisfies $R_a \circ \iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A} \circ \operatorname{right}_a$). Hence, the $\widetilde{f}$ defined in Proposition 2.21 (applied to $V = A$ and $f = \operatorname{right}_a$) is our map R_a. Thus, Proposition 2.21 can be applied to $V = A$, $f = \operatorname{right}_a$ and $\widetilde{f} = R_a$.

As a consequence, Proposition 2.21 **(c)** (applied to $V = A$, $f = \operatorname{right}_a$ and $\widetilde{f} = R_a$) shows that the map $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is graded. This proves Theorem 5.21 **(b)**.

[59]*Proof.* Let $a \in A$. Then, R_a is the unique derivation $F : \operatorname{Sym} A \to \operatorname{Sym} A$ such that $F \circ \iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A} \circ \operatorname{right}_a$. Hence, R_a is a derivation $\operatorname{Sym} A \to \operatorname{Sym} A$ and satisfies $L_a \circ \iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A} \circ \operatorname{right}_a$. Qed.

[60]since $\iota_{\operatorname{Sym},A}(b) = b$ (because we are identifying $\iota_{\operatorname{Sym},A}(x)$ with x for every $x \in A$) and $\iota_{\operatorname{Sym},A}(b \triangleleft a) = b \triangleleft a$ (for the same reason)

(c) Let $a \in A$. In our above proof of Theorem 5.21 **(b)**, we have shown that Proposition 2.21 can be applied to $V = A$, $f = \operatorname{right}_a$ and $\widetilde{f} = R_a$. Hence, Proposition 2.21 **(b)** (applied to $V = A$, $f = \operatorname{right}_a$ and $\widetilde{f} = R_a$) shows that the map $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a coderivation. Theorem 5.21 **(c)** is thus proven.

(d) The map K' is a map from A to $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$, thus a map from A^- to $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$ (since $A^- = A$ as sets).

We shall first prove that

$$K'([a,b]) = [K'(a), K'(b)] \tag{113}$$

for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $[K'(a), K'(b)]$ is computed in $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$).

Proof of (113): Recall that we are regarding A as a $\mathbf{k}$-submodule of $\operatorname{Sym} A$ via the injection $\iota_{\operatorname{Sym},A} : A \to \operatorname{Sym} A$. Thus, $A = \iota_{\operatorname{Sym},A}(A) = \operatorname{Sym}^1 A$. Recall that the subset $\operatorname{Sym}^1 A$ of $\operatorname{Sym} A$ generates the $\mathbf{k}$-algebra $\operatorname{Sym} A$. In other words, the subset A of $\operatorname{Sym} A$ generates the $\mathbf{k}$-algebra $\operatorname{Sym} A$ (since $A = \operatorname{Sym}^1 A$).

Let $a \in A^-$ and $b \in A^-$. Then, $K'(a)$ and $K'(b)$ are elements of $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$. Thus $[K'(a), K'(b)]$ is an element of $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$ (since $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$ is a Lie algebra), hence an element of $\operatorname{Der}(\operatorname{Sym} A)$ (since $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A) = \operatorname{Der}(\operatorname{Sym} A)$ as sets), hence a derivation $\operatorname{Sym} A \to \operatorname{Sym} A$. Also, $K'([a,b])$ is an element of $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A) = \operatorname{Der}(\operatorname{Sym} A)$, hence a derivation $\operatorname{Sym} A \to \operatorname{Sym} A$.

We have $[K'(a), K'(b)] \mid_A = K'([a,b]) \mid_A$ [61]. Hence, Lemma 1.13 (applied to $\operatorname{Sym} A$, $[K'(a), K'(b)]$, $K'([a,b])$ and A instead of A, d, e and S) shows that $[K'(a), K'(b)] = K'([a,b])$. This proves (113).

Next, we notice that

$$K'(\lambda a + \mu b) = \lambda K'(a) + \mu K'(b) \tag{114}$$

[61] *Proof.* Let $c \in A$. The definition of K' yields $K'(a) = R_a$, $K'(b) = R_b$ and $K'([a,b]) = R_{[a,b]}$. Now,

$$\begin{aligned}
&\left([K'(a), K'(b)] \mid_A\right)(c) \\
&= \underbrace{[K'(a), K'(b)]}_{\substack{=K'(b)\circ K'(a)-K'(a)\circ K'(b) \\ \text{(since the Lie bracket used here} \\ \text{is that of } \operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A))}}(c) = \left(\underbrace{K'(b)}_{=R_b} \circ \underbrace{K'(a)}_{=R_a} - \underbrace{K'(a)}_{=R_a} \circ \underbrace{K'(b)}_{=R_b}\right)(c) \\
&= (R_b \circ R_a - R_a \circ R_b)(c) = R_b\left(\underbrace{R_a(c)}_{\substack{=c\lhd a \\ \text{(by (112), applied to } c \\ \text{instead of } b)}}\right) - R_a\left(\underbrace{R_b(c)}_{\substack{=c\lhd b \\ \text{(by (112), applied to } b \text{ and } c \\ \text{instead of } a \text{ and } b)}}\right) \\
&= \underbrace{R_b(c \lhd a)}_{\substack{=(c\lhd a)\lhd b \\ \text{(by (112), applied to } b \text{ and } c\lhd a \\ \text{instead of } a \text{ and } b)}} - \underbrace{R_a(c \lhd b)}_{\substack{=(c\lhd b)\lhd a \\ \text{(by (112), applied to } c\lhd b \\ \text{instead of } b)}} \\
&= (c \lhd a) \lhd b - (c \lhd b) \lhd a.
\end{aligned}$$

for every $a \in A^-$ and $b \in A^-$ and every $\lambda \in \mathbf{k}$ and $\mu \in \mathbf{k}$.

Proof of (114): The proof of (114) is similar to the above proof of (113), but even simpler (because instead of $c \lhd [a,b] = (c \lhd a) \lhd b - (c \lhd b) \lhd a$, we now need to use the much simpler equality $c \lhd (\lambda a + \mu b) = \lambda (c \lhd a) + \mu (c \lhd b)$). Hence, we leave it to the reader.

Now, the map $K' : A^- \to \operatorname{Der}^{\mathrm{op}} (\operatorname{Sym} A)$ is $\mathbf{k}$-linear (because of (114)) and thus a Lie algebra homomorphism (because of (113)). This proves Theorem 5.21 **(d)**.

(e) The map $K' : A^- \to \operatorname{Der}^{\mathrm{op}} (\operatorname{Sym} A)$ is a Lie algebra homomorphism (according to Theorem 5.21 **(d)**). Moreover, we have

$$\iota_{\operatorname{Sym},A} ([a,b]) = \left[\iota_{\operatorname{Sym},A} (a), \iota_{\operatorname{Sym},A} (b)\right] + \left(K' (b)\right) \left(\iota_{\operatorname{Sym},A} (a)\right) - \left(K' (a)\right) \left(\iota_{\operatorname{Sym},A} (b)\right)$$

for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A} (a), \iota_{\operatorname{Sym},A} (b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$) [62]. Hence, we can apply Theorem 5.17

Comparing this with

$$\begin{aligned} &\left(K' ([a,b]) \mid_A\right) (c) \\ &= \underbrace{\left(K' ([a,b])\right)}_{=R_{[a,b]}} (c) = R_{[a,b]} (c) = c \lhd \underbrace{[a,b]}_{\substack{=a \lhd b - b \lhd a \\ \text{(by the definition} \\ \text{of the Lie algebra } A^-)}} \qquad \left(\begin{array}{c}\text{by (112), applied to } [a,b] \text{ and } c \\ \text{instead of } a \text{ and } b\end{array}\right) \\ &= c \lhd (a \lhd b - b \lhd a) = c \lhd (a \lhd b) - c \lhd (b \lhd a) = (c \lhd a) \lhd b - (c \lhd b) \lhd a \\ &\qquad \left(\begin{array}{c}\text{since } (c \lhd b) \lhd a - c \lhd (b \lhd a) = (c \lhd a) \lhd b - c \lhd (a \lhd b) \\ \text{(by (46), applied to } c \text{ and } a \text{ instead of } a \text{ and } c)\end{array}\right), \end{aligned}$$

we obtain $\left(\left[K' (a), K' (b)\right] \mid_A\right) (c) = \left(K' ([a,b]) \mid_A\right) (c)$.

Now, let us forget that we fixed c. We thus have shown that $\left(\left[K' (a), K' (b)\right] \mid_A\right) (c) = \left(K' ([a,b]) \mid_A\right) (c)$ for every $c \in A$. In other words, $\left[K' (a), K' (b)\right] \mid_A = K' ([a,b]) \mid_A$, qed.

[62] *Proof.* Let $a \in A^-$ and $b \in A^-$. Recall that we are identifying $\iota_{\operatorname{Sym},A} (x)$ with x for every $x \in A$. Thus, $\iota_{\operatorname{Sym},A} ([a,b]) = [a,b]$, $\iota_{\operatorname{Sym},A} (a) = a$ and $\iota_{\operatorname{Sym},A} (b) = b$. Moreover, the $\mathbf{k}$-algebra $\operatorname{Sym} A$ is commutative, and thus the Lie bracket of the Lie algebra $(\operatorname{Sym} A)^-$ is identically 0. Hence, $\left[\iota_{\operatorname{Sym},A} (a), \iota_{\operatorname{Sym},A} (b)\right] = 0$. Now,

$$\begin{aligned} &\underbrace{\left[\iota_{\operatorname{Sym},A} (a), \iota_{\operatorname{Sym},A} (b)\right]}_{=0} + \underbrace{\left(K' (b)\right)}_{\substack{=R_b \\ \text{(by the definition of } K')}} \left(\underbrace{\iota_{\operatorname{Sym},A} (a)}_{=a}\right) - \underbrace{\left(K' (a)\right)}_{\substack{=R_a \\ \text{(by the definition of } K')}} \left(\underbrace{\iota_{\operatorname{Sym},A} (b)}_{=b}\right) \\ &= \underbrace{R_b (a)}_{\substack{=a \lhd b \\ \text{(by (112) (applied to} \\ b \text{ and } a \text{ instead of } a \text{ and } b))}} - \underbrace{R_a (b)}_{\substack{=b \lhd a \\ \text{(by (112))}}} = a \lhd b - b \lhd a. \end{aligned}$$

Compared with

$$\iota_{\operatorname{Sym},A} ([a,b]) = [a,b] = a \lhd b - b \lhd a$$

(by the definition of the Lie bracket on A^-), this shows that

$$\iota_{\operatorname{Sym},A} ([a,b]) = \left[\iota_{\operatorname{Sym},A} (a), \iota_{\operatorname{Sym},A} (b)\right] + \left(K' (b)\right) \left(\iota_{\operatorname{Sym},A} (a)\right) - \left(K' (a)\right) \left(\iota_{\operatorname{Sym},A} (b)\right).$$

to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$.

Thus, Theorem 5.17 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$) shows that we can define a right A^--module structure on $\operatorname{Sym} A$ by setting

$$\left(u \leftharpoonup a = u \cdot \iota_{\operatorname{Sym},A}(a) + \left(K'(a)\right)(u) \text{ for all } a \in A^- \text{ and } u \in \operatorname{Sym} A\right).$$

In other words, we can define a right A^--module structure on $\operatorname{Sym} A$ by setting

$$\left(u \leftharpoonup a = ua + R_a(u) \text{ for all } a \in A^- \text{ and } u \in \operatorname{Sym} A\right)$$

(because every $a \in A^-$ and $u \in \operatorname{Sym} A$ satisfy $u \cdot \underbrace{\iota_{\operatorname{Sym},A}(a)}_{=a} + \underbrace{\left(K'(a)\right)}_{\substack{=R_a \\ \text{(by the definition of } K')}}(u) = ua + R_a(u)$). This proves Theorem 5.21 **(e)**.

As a consequence of this proof of Theorem 5.21 **(e)**, we see that the A^--module structure on $\operatorname{Sym} A$ defined in Theorem 5.21 **(e)** is precisely the one that is constructed by Theorem 5.17 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$). We notice further that the map $\eta : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 5.21 **(f)** is precisely the map $\eta : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 5.17 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$) (since these two maps have the same definition).

(f) As we have seen in our proof of Theorem 5.21 **(e)**, we can apply Theorem 5.17 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$. Moreover, the following holds:

- The right A^--module structure on $\operatorname{Sym} A$ defined in Theorem 5.21 **(e)** is precisely the one that is constructed by Theorem 5.17 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$).
- The map $\eta' : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 5.21 **(f)** is precisely the map $\eta' : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 5.17 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$).

Hence, Theorem 5.17 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$) shows that η' is a right A^--module homomorphism. This proves Theorem 5.21 **(f)**.

(g) Let $a \in A^-$. The map R_a is a derivation. In other words, $R_a \in \operatorname{Der}(\operatorname{Sym} A)$. Thus, $R_a(1) = 0$ (by Proposition 1.11 **(c)**, applied to $\operatorname{Der}(\operatorname{Sym} A)$ and R_a instead of C and f).

Qed.

Now, the definition of η' shows that

$$\begin{aligned}
&\eta'\left(\iota_{U,A^-}(a)\right) \\
&= 1_{\operatorname{Sym} A}\iota_{U,A^-}(a) = 1_{\operatorname{Sym} A} \leftharpoonup a \\
&\qquad \left(\text{by the definition of the right action of } U\left(A^-\right) \text{ on } \operatorname{Sym} A\right) \\
&= \underbrace{1_{\operatorname{Sym} A} a}_{=a} + \underbrace{R_a\left(1_{\operatorname{Sym} A}\right)}_{=R_a(1)=0} \qquad \left(\text{by the definition of the right } A^-\text{-module structure on } \operatorname{Sym} A\right) \\
&= a.
\end{aligned}$$

This proves Theorem 5.21 **(g)**.

(i) We make the following observations:

- The map $K' : A^- \to \operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$ is a Lie algebra homomorphism (according to Theorem 5.21 **(d)**).

- We have

  $$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + \left(K'(b)\right)\left(\iota_{\operatorname{Sym},A}(a)\right) - \left(K'(a)\right)\left(\iota_{\operatorname{Sym},A}(b)\right)$$

  for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$) [63].

- The right A^--module structure on $\operatorname{Sym} A$ defined in Theorem 5.21 **(e)** is precisely the one that is constructed by Theorem 5.17 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$). [64]

- The map $\eta' : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 5.21 **(f)** is precisely the map $\eta' : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 5.17 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$).

- The filtration of the filtered $\mathbf{k}$-algebra $\operatorname{Sym} A$ is $\left(\bigoplus_{k=0}^{m} \operatorname{Sym}^k A\right)_{m\geq 0}$.

- We have $\iota_{\operatorname{Sym},A}\left(\underbrace{A^-}_{=A}\right) = \iota_{\operatorname{Sym},A}(A) = \operatorname{Sym}^1 A \subseteq \bigoplus_{k=0}^{1} \operatorname{Sym}^k A$.

- The map $K'(a) : \operatorname{Sym} A \to \operatorname{Sym} A$ is filtered for every $a \in A^-$ [65].

[63] This was proven in our above proof of Theorem 5.21 **(e)**.

[64] This was proven above.

[65] *Proof.* Let $a \in A^-$. Thus, the definition of K' shows that $K'(a) = R_a$. Now, the map R_a is graded (by Theorem 5.21 **(b)**), and thus filtered (since any graded $\mathbf{k}$-linear map is filtered). In other words, the map $K'(a)$ is filtered (since $K'(a) = R_a$). Qed.

Hence, we can apply Theorem 5.19 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$, $(C_m)_{m\geq 0} = \left(\bigoplus_{k=0}^m \operatorname{Sym}^k A\right)_{m\geq 0}$ and $f' = \iota_{\operatorname{Sym},A}$. As a result, we conclude that

$$\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \\ \in \iota_{\operatorname{Sym},A}(a_1)\,\iota_{\operatorname{Sym},A}(a_2)\cdots\iota_{\operatorname{Sym},A}(a_n) + \bigoplus_{k=0}^{n-1}\operatorname{Sym}^k A \qquad (115)$$

for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A^-$. Thus, for every $n \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A^-$, we have

$$\eta\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \in a_1a_2\cdots a_n + \sum_{k=0}^{n-1}\operatorname{Sym}^k A$$

[66]. This proves Theorem 5.21 **(i)**.

(h) Recall that η' is a $\mathbf{k}$-linear map $U(A^-) \to \operatorname{Sym} A$. In other words, η' is a $\mathbf{k}$-linear map $U(A^-) \to \operatorname{Sym}(A^-)$ (since $A^- = A$ as $\mathbf{k}$-modules).

For every $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A^-$, we have

$$\eta'\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \\ \in \iota_{\operatorname{Sym},A^-}(a_1)\,\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n) + \sum_{k=0}^{n-1}\operatorname{Sym}^k\left(A^-\right)$$

[67]. Thus, we can apply Lemma 2.24 to $\mathfrak{g} = A^-$ and $\phi = \eta'$.

[66] *Proof.* Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A^-$.

Recall that we are identifying every $x \in A$ with $\iota_{\operatorname{Sym},A}(x)$. Thus, every $x \in A$ satisfies $x = \iota_{\operatorname{Sym},A}(x) = \iota_{\operatorname{Sym},A^-}(x)$ (since $\iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A^-}$). In particular, every $i \in \{1, 2, \ldots, n\}$ satisfies the identity $a_i = \iota_{\operatorname{Sym},A^-}(a_i)$. Multiplying these identities over all $i \in \{1, 2, \ldots, n\}$, we obtain $a_1a_2\cdots a_n = \iota_{\operatorname{Sym},A^-}(a_1)\,\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n)$. Now, (115) becomes

$$\eta'\left(\iota_{U,A^-}(a_1)\,\iota_{U,A^-}(a_2)\cdots\iota_{U,A^-}(a_n)\right) \in \underbrace{\iota_{\operatorname{Sym},A}(a_1)\,\iota_{\operatorname{Sym},A}(a_2)\cdots\iota_{\operatorname{Sym},A}(a_n)}_{=a_1a_2\cdots a_n} + \underbrace{\bigoplus_{k=0}^{n-1}\operatorname{Sym}^k A}_{\substack{=\sum_{k=0}^{n-1}\operatorname{Sym}^k A \\ \text{(since direct sums} \\ \text{are sums)}}}$$

$$= a_1a_2\cdots a_n + \sum_{k=0}^{n-1}\operatorname{Sym}^k A,$$

qed.

[67] *Proof.* Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A^-$. We have $A^- = A$ as $\mathbf{k}$-modules. Hence, $\iota_{\operatorname{Sym},A^-} = \iota_{\operatorname{Sym},A}$.

Recall that we are identifying every $x \in A$ with $\iota_{\operatorname{Sym},A}(x)$. Thus, every $x \in A$ satisfies $x = \iota_{\operatorname{Sym},A}(x) = \iota_{\operatorname{Sym},A^-}(x)$ (since $\iota_{\operatorname{Sym},A} = \iota_{\operatorname{Sym},A^-}$). In particular, every $i \in \{1, 2, \ldots, n\}$ satisfies the identity $a_i = \iota_{\operatorname{Sym},A^-}(a_i)$. Multiplying these identities over all $i \in \{1, 2, \ldots, n\}$, Lemma 2.24 **(a)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta'$) shows that the map η' is a filtered $\mathbf{k}$-module isomorphism. In particular, the map η' is invertible.

But we know that η' is a right A^--module homomorphism (by Theorem 5.21 **(f)**). Since η' is invertible, this shows that η' is a right A^--module isomorphism. This proves Theorem 5.21 **(h)**.

(j) In our above proof of Theorem 5.21 **(h)**, we have shown that we can apply Lemma 2.24 to $\mathfrak{g} = A^-$ and $\phi = \eta'$. Hence, Lemma 2.24 **(a)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta'$) shows that the map η' is a filtered $\mathbf{k}$-module isomorphism. Also, Lemma 2.24 **(b)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta'$) shows that the inverse $(\eta')^{-1}$ of the map η' is filtered. Thus, both maps η' and $(\eta')^{-1}$ are filtered. This proves Theorem 5.21 **(j)**.

(k) In our above proof of Theorem 5.21 **(h)**, we have shown that we can apply Lemma 2.24 to $\mathfrak{g} = A^-$ and $\phi = \eta'$. Hence, the first claim of Lemma 2.24 **(c)** (applied to $\mathfrak{g} = A^-$ and $\phi = \eta'$) shows that the map $\operatorname{grad}^{-1}_{\operatorname{Sym}(A^-)} \circ (\operatorname{gr}(\eta')) : \operatorname{gr}(U(A^-)) \to \operatorname{Sym}(A^-)$ is an inverse to the map PBW_{A^-}. Since $A^- = A$ as $\mathbf{k}$-modules, this rewrites as follows: The map $\operatorname{grad}^{-1}_{\operatorname{Sym} A} \circ (\operatorname{gr}(\eta')) : \operatorname{gr}(U(A^-)) \to \operatorname{Sym} A$ is an inverse to the map PBW_{A^-}. In particular, the map PBW_{A^-} is invertible. This proves Theorem 5.21 **(k)**.

(l) We first observe the following:

- The map $K' : A^- \to \operatorname{Der}(\operatorname{Sym} A)$ is a Lie algebra homomorphism. (This follows from Theorem 5.21 **(d)**.)
- We have $K'(A^-) \subseteq \operatorname{Coder}(\operatorname{Sym} A)$ [68].
- We have $\iota_{\operatorname{Sym},A}(A^-) \subseteq \operatorname{Prim}(\operatorname{Sym} A)$ [69].

we obtain $a_1 a_2 \cdots a_n = \iota_{\operatorname{Sym},A^-}(a_1)\, \iota_{\operatorname{Sym},A^-}(a_2) \cdots \iota_{\operatorname{Sym},A^-}(a_n)$.

Now, Theorem 5.21 **(i)** shows that

$$\begin{aligned}
&\eta'\left(\iota_{U,A^-}(a_1)\, \iota_{U,A^-}(a_2) \cdots \iota_{U,A^-}(a_n)\right) \\
&\in \underbrace{a_1 a_2 \cdots a_n}_{=\iota_{\operatorname{Sym},A^-}(a_1)\iota_{\operatorname{Sym},A^-}(a_2)\cdots\iota_{\operatorname{Sym},A^-}(a_n)} + \sum_{k=0}^{n-1} \operatorname{Sym}^k \underbrace{A}_{=A^-} \\
&= \iota_{\operatorname{Sym},A^-}(a_1)\, \iota_{\operatorname{Sym},A^-}(a_2) \cdots \iota_{\operatorname{Sym},A^-}(a_n) + \sum_{k=0}^{n-1} \operatorname{Sym}^k (A^-),
\end{aligned}$$

qed.

[68] *Proof.* Let $g \in K'(A^-)$. Thus, there exists some $a \in A^-$ such that $g = K'(a)$. Consider this a. We have $a \in A^- = A$ and $g = K'(a) = R_a$ (by the definition of K'). But $R_a : \operatorname{Sym} A \to \operatorname{Sym} A$ is a coderivation (according to Theorem 5.21 **(c)**). In other words, $R_a \in \operatorname{Coder}(\operatorname{Sym} A)$. Hence, $g = R_a \in \operatorname{Coder}(\operatorname{Sym} A)$.

Now, let us forget that we fixed g. We thus have shown that $g \in \operatorname{Coder}(\operatorname{Sym} A)$ for every $g \in K'(A^-)$. In other words, $K'(A^-) \subseteq \operatorname{Coder}(\operatorname{Sym} A)$, qed.

[69] *Proof.* Recall that $\iota_{\operatorname{Sym},V}(V) \subseteq \operatorname{Prim}(\operatorname{Sym} V)$ for every $\mathbf{k}$-module V. Applying this to $V = A$, we obtain $\iota_{\operatorname{Sym},A}(A) \subseteq \operatorname{Prim}(\operatorname{Sym} A)$. Since $A^- = A$ as $\mathbf{k}$-modules, this rewrites as $\iota_{\operatorname{Sym},A}(A^-) \subseteq \operatorname{Prim}(\operatorname{Sym} A)$. Qed.

- The equality

  $$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + \left(K'(b)\right)\left(\iota_{\operatorname{Sym},A}(a)\right) - \left(K'(a)\right)\left(\iota_{\operatorname{Sym},A}(b)\right)$$

  holds for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$). (This was proven during our above proof of Theorem 5.21 **(e)**.)

- The right A^--module structure on $\operatorname{Sym} A$ defined in Theorem 5.21 **(e)** is precisely the one that is constructed by Theorem 5.17 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$). (This follows from our above proof of Theorem 5.21 **(e)**.)

- The map $\eta' : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 5.21 **(f)** is precisely the map $\eta' : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 5.17 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$). (This is clear, because the definitions of these two maps are the same.)

Combining these observations, we see that Theorem 5.18 can be applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$. Consequently, Theorem 5.18 **(b)** shows that the map $\eta' : U(A^-) \to \operatorname{Sym} A$ is a $\mathbf{k}$-coalgebra homomorphism. Since η' is invertible, this shows that η' is a $\mathbf{k}$-coalgebra isomorphism. This finishes the proof of Theorem 5.21 **(l)**.

(m) As we have seen in our proof of Theorem 5.21 **(e)**, we can apply Theorem 5.17 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$. Moreover, the right A^--module structure on $\operatorname{Sym} A$ defined in Theorem 5.21 **(e)** is precisely the one that is constructed by Theorem 5.17 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$). Hence, Theorem 5.17 **(c)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$) shows that, for every $a \in A^-$, $b \in \operatorname{Sym} A$ and $c \in \operatorname{Sym} A$, we have

$$(cb) \leftharpoonup a - (c \leftharpoonup a) \cdot b = c \cdot \left(K'(a)\right)(b) - c\left[\iota_{\operatorname{Sym},A}(a), b\right]. \qquad (116)$$

Now, fix $a \in A$, $b \in A$ and $c \in \operatorname{Sym} A$. We have $a \in A = A^-$ and $b \in A \subseteq \operatorname{Sym} A$ (since we regard A as a $\mathbf{k}$-submodule of $\operatorname{Sym} A$). Moreover, the $\mathbf{k}$-algebra $\operatorname{Sym} A$ is commutative, and thus the Lie bracket of the Lie algebra $(\operatorname{Sym} A)^-$ is identically 0. Hence, $\left[\iota_{\operatorname{Sym},A}(a), b\right] = 0$. Furthermore, $K'(a) = R_a$ (by the definition of K'), so that

$$\underbrace{\left(K'(a)\right)}_{=R_a}(b) = R_a(b) = b \triangleleft a \qquad \text{(by (112))}.$$

Now, (116) shows that

$$(cb) \leftharpoonup a - (c \leftharpoonup a) \cdot b = c \cdot \underbrace{\left(K'(a)\right)(b)}_{=b \triangleleft a} - c\underbrace{\left[\iota_{\operatorname{Sym},A}(a), b\right]}_{=0} = c \cdot (b \triangleleft a) - \underbrace{c0}_{=0} = c \cdot (b \triangleleft a).$$

This proves Theorem 5.21 **(m)**. $\square$

5.6. The case of an associative algebra

Now we can see Theorem 3.10 with Theorem 5.21 interact:

Theorem 5.22. Let A be a $\mathbf{k}$-algebra. Thus, A becomes a left pre-Lie algebra (according to Proposition 3.5 **(a)**) and a right pre-Lie algebra (according to Proposition 3.5 **(b)**). Consider the map $\eta : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 3.10 **(f)** (using the left pre-Lie algebra structure on A). Also, consider the map $\eta' : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 5.21 **(f)** (using the right pre-Lie algebra structure on A).

Then, $\eta = \eta'$.

Before we prove this theorem, let us state a trivial lemma:

Lemma 5.23. Let A be a $\mathbf{k}$-algebra. Thus, A becomes a left pre-Lie algebra (according to Proposition 3.5 **(a)**) and a right pre-Lie algebra (according to Proposition 3.5 **(b)**).

(a) We have $a \triangleright b = a \triangleleft b$ for any $a \in A$ and $b \in A$.

(b) We have $a \triangleright (c \triangleleft b) = (a \triangleright c) \triangleleft b$ for any $a \in A$, $b \in A$ and $c \in A$.

Proof of Lemma 5.23. **(a)** Let $a \in A$ and $b \in A$. We have $a \triangleright b = ab$ (by the definition of the left pre-Lie algebra structure on A). We also have $a \triangleleft b = ab$ (by the definition of the right pre-Lie algebra structure on A). Comparing this with $a \triangleright b = ab$, we obtain $a \triangleright b = a \triangleleft b$. This proves Lemma 5.23 **(a)**.

(b) Let $a \in A$, $b \in A$ and $c \in A$. We have $c \triangleleft b = cb$ (by the definition of the right pre-Lie algebra structure on A). We also have $a \triangleright c = ac$ (by the definition of the left pre-Lie algebra structure on A). Now, the definition of the left pre-Lie algebra structure on A shows that

$$a \triangleright (c \triangleleft b) = a \underbrace{(c \triangleleft b)}_{=cb} = acb.$$

On the other hand, the definition of the right pre-Lie algebra structure on A shows that

$$(a \triangleright c) \triangleleft b = \underbrace{(a \triangleright c)}_{=ac} b = acb.$$

Comparing this with $a \triangleright (c \triangleleft b) = acb$, we obtain $a \triangleright (c \triangleleft b) = (a \triangleright c) \triangleleft b$. This proves Lemma 5.23 **(b)**. $\square$

Proof of Theorem 5.22. We shall use all notations introduced in Theorem 3.10 (in particular, the map $K : A \to \operatorname{Der}(\operatorname{Sym} A)$ and the left A^--module structure on $\operatorname{Sym} A$). We shall also use all notations introduced in Theorem 5.21 (in particular, the map $K' : A \to \operatorname{Der}^{\mathrm{op}}(\operatorname{Sym} A)$ and the right A^--module structure on $\operatorname{Sym} A$).

Our setup causes a notational ambiguity: When a and b are two elements of A, then the expression "ab" might mean the product of a and b in the $\mathbf{k}$-algebra

A, but can also mean the product ab of the elements a and b of the symmetric algebra $\operatorname{Sym} A$. We resolve this ambiguity by agreeing **not** to use the notation ab with the former meaning. Thus, in this proof, a product such as ab (with $a \in A$ and $b \in A$) shall always mean the product of a and b regarded as elements of the symmetric algebra $\operatorname{Sym} A$, rather than the product of a and b regarded as elements of A.

For every $a \in A^-$ and $b \in A^-$, we have

$$K(a) \circ K'(b) = K'(b) \circ K(a) \tag{117}$$

[70].

[70]*Proof of (117):* Let $a \in A^-$ and $b \in A^-$.

Recall that we are regarding A as a $\mathbf{k}$-submodule of $\operatorname{Sym} A$ via the injection $\iota_{\operatorname{Sym},A} : A \to \operatorname{Sym} A$. Thus, $A = \iota_{\operatorname{Sym},A}(A) = \operatorname{Sym}^1 A$. Recall that the subset $\operatorname{Sym}^1 A$ of $\operatorname{Sym} A$ generates the $\mathbf{k}$-algebra $\operatorname{Sym} A$. In other words, the subset A of $\operatorname{Sym} A$ generates the $\mathbf{k}$-algebra $\operatorname{Sym} A$ (since $A = \operatorname{Sym}^1 A$).

The target of the map K' is $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A) = \operatorname{Der}(\operatorname{Sym} A)$. Thus, $K'(b) \in \operatorname{Der}(\operatorname{Sym} A)$. Also, $K(a) \in \operatorname{Der}(\operatorname{Sym} A)$ (since the target of the map K is $\operatorname{Der}(\operatorname{Sym} A)$).

Now, $K(a)$ and $K'(b)$ both are elements of the Lie algebra $\operatorname{Der}(\operatorname{Sym} A)$. Hence, $[K(a), K'(b)] \in \operatorname{Der}(\operatorname{Sym} A)$, where we are using the Lie bracket of the Lie algebra $\operatorname{Der}(\operatorname{Sym} A)$ (not of the Lie algebra $\operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$). In other words, $[K(a), K'(b)]$ is a derivation $\operatorname{Sym} A \to \operatorname{Sym} A$.

The map $0 : \operatorname{Sym} A \to \operatorname{Sym} A$ is also a derivation.

Now, we shall show that $[K(a), K'(b)] \mid_A = 0 \mid_A$.

Indeed, let $c \in A$. Lemma 5.23 **(b)** shows that $a \rhd (c \lhd b) = (a \rhd c) \lhd b$.

The definition of K yields $K(a) = L_a$. The definition of K' yields $K'(b) = R_b$. Now,

$$\begin{aligned}
&\left(\left[K(a), K'(b)\right] \mid_A\right)(c) \\
&= \underbrace{\left[K(a), K'(b)\right]}_{\substack{=K(a)\circ K'(b)-K'(b)\circ K(a) \\ \text{(since the Lie bracket used here} \\ \text{is that of } \operatorname{Der}(\operatorname{Sym} A))}}(c) = \left(\underbrace{K(a)}_{=L_a} \circ \underbrace{K'(b)}_{=R_b} - \underbrace{K'(b)}_{=R_b} \circ \underbrace{K(a)}_{=L_a}\right)(c) \\
&= (L_a \circ R_b - R_b \circ L_a)(c) = L_a\left(\underbrace{R_b(c)}_{\substack{=c\lhd b \\ \text{(by (112), applied to } b \text{ and } c \\ \text{instead of } a \text{ and } b)}}\right) - R_b\left(\underbrace{L_a(c)}_{\substack{=a\rhd c \\ \text{(by (55), applied to } c \\ \text{instead of } b)}}\right) \\
&= \underbrace{L_a(c \lhd b)}_{\substack{=a\rhd(c\lhd b) \\ \text{(by (55), applied to } c\lhd b \\ \text{instead of } b)}} - \underbrace{R_b(a \rhd c)}_{\substack{=(a\rhd c)\lhd b \\ \text{(by (112), applied to } b \text{ and } a\rhd c \\ \text{instead of } a \text{ and } b)}} \\
&= \underbrace{a \rhd (c \lhd b)}_{=(a\rhd c)\lhd b} - (a \rhd c) \lhd b = (a \rhd c) \lhd b - (a \rhd c) \lhd b = 0.
\end{aligned}$$

Comparing this with $(0 \mid_A)(c) = 0(c) = 0$, we obtain $([K(a), K'(b)] \mid_A)(c) = (0 \mid_A)(c)$.

Now, let us forget that we fixed c. We thus have shown that $([K(a), K'(b)] \mid_A)(c) = (0 \mid_A)(c)$ for every $c \in A$. In other words, $[K(a), K'(b)] \mid_A = 0 \mid_A$. Hence, Lemma 1.13

In the proof of Theorem 3.10 **(e)**, we have shown that the following holds:

- The map $K : A^- \to \operatorname{Der}(\operatorname{Sym} A)$ is a Lie algebra homomorphism.
- We have

  $$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + (K(a))\left(\iota_{\operatorname{Sym},A}(b)\right) - (K(b))\left(\iota_{\operatorname{Sym},A}(a)\right)$$

  for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$).
- We can apply Theorem 1.15 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$.
- The A^--module structure on $\operatorname{Sym} A$ defined in Theorem 3.10 **(e)** is precisely the one that is constructed by Theorem 1.15 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$).
- The map $\eta : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 3.10 **(f)** is precisely the map $\eta : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 1.15 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f = \iota_{\operatorname{Sym},A}$).

Similarly, in the proof of Theorem 5.21 **(e)**, we have shown that the following holds:

- The map $K' : A^- \to \operatorname{Der}^{\operatorname{op}}(\operatorname{Sym} A)$ is a Lie algebra homomorphism.
- We have

  $$\iota_{\operatorname{Sym},A}([a,b]) = \left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right] + \left(K'(b)\right)\left(\iota_{\operatorname{Sym},A}(a)\right) - \left(K'(a)\right)\left(\iota_{\operatorname{Sym},A}(b)\right)$$

  for every $a \in A^-$ and $b \in A^-$ (where the Lie bracket $\left[\iota_{\operatorname{Sym},A}(a), \iota_{\operatorname{Sym},A}(b)\right]$ is computed in the Lie algebra $(\operatorname{Sym} A)^-$).
- We can apply Theorem 5.17 to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$.
- The right A^--module structure on $\operatorname{Sym} A$ defined in Theorem 5.21 **(e)** is precisely the one that is constructed by Theorem 5.17 **(a)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$).
- The map $\eta' : U(A^-) \to \operatorname{Sym} A$ defined in Theorem 5.21 **(f)** is precisely the map $\eta' : U(A^-) \to \operatorname{Sym} A$ that is constructed by Theorem 5.17 **(b)** (applied to $\mathfrak{g} = A^-$, $C = \operatorname{Sym} A$ and $f' = \iota_{\operatorname{Sym},A}$).

Furthermore, we observe the following:

(applied to $\operatorname{Sym} A$, $[K(a), K'(b)]$, 0 and A instead of A, d, e and S) shows that $[K(a), K'(b)] = 0$. Since $[K(a), K'(b)] = K(a) \circ K'(b) - K'(b) \circ K(a)$, this rewrites as $K(a) \circ K'(b) - K'(b) \circ K(a) = 0$. In other words, $K(a) \circ K'(b) = K'(b) \circ K(a)$. This proves (117).

- We have
$$\left(K\left(a\right)\circ K'\left(b\right)-K'\left(b\right)\circ K\left(a\right)\right)\left(u\right)=\left(K'\left(b\right)\right)\left(\iota_{\operatorname{Sym},A}\left(a\right)\right)\cdot u-u\cdot\left(K\left(a\right)\right)\left(\iota_{\operatorname{Sym},A}\left(b\right)\right)$$
for all $a\in A^{-}$, $b\in A^{-}$ and $u\in\operatorname{Sym}A$ [71].

- We have $\iota_{\operatorname{Sym},A}=\iota_{\operatorname{Sym},A}$.

Combining all the observations in the bullet points made above, we conclude that we can apply Theorem 5.20 **(c)** to $\mathfrak{g}=A^{-}$, $C=\operatorname{Sym}A$, $f=\iota_{\operatorname{Sym},A}$ and $f'=\iota_{\operatorname{Sym},A}$. As a result, we obtain $\eta=\eta'$. This proves Theorem 5.22. $\square$

[71] *Proof.* Let $a\in A^{-}$, $b\in A^{-}$ and $u\in\operatorname{Sym}A$.

Recall that we are identifying $\iota_{\operatorname{Sym},A}\left(x\right)$ with x for every $x\in A$. Thus, $\iota_{\operatorname{Sym},A}\left(a\right)=a$ and $\iota_{\operatorname{Sym},A}\left(b\right)=b$.

The definition of K yields $K\left(a\right)=L_{a}$. Hence,
$$\begin{aligned}\underbrace{\left(K\left(a\right)\right)}_{=L_{a}}\left(\underbrace{\iota_{\operatorname{Sym},A}\left(b\right)}_{=b}\right)&=L_{a}\left(b\right)=a\rhd b\qquad\left(\text{by (55)}\right)\\&=a\lhd b\qquad\left(\text{by Lemma 5.23 }\textbf{(a)}\right).\end{aligned}$$

On the other hand, the definition of K' yields $K'\left(b\right)=R_{b}$. Now,
$$\underbrace{\left(K'\left(b\right)\right)}_{=R_{b}}\left(\underbrace{\iota_{\operatorname{Sym},A}\left(a\right)}_{=a}\right)=R_{b}\left(a\right)=a\lhd b$$
(by (112), applied to b and a instead of a and b).

Now,
$$\begin{aligned}&\left(\underbrace{K\left(a\right)\circ K'\left(b\right)}_{\substack{=K'\left(b\right)\circ K\left(a\right)\\\text{(by (117))}}}-K'\left(b\right)\circ K\left(a\right)\right)\left(u\right)\\&=\underbrace{\left(K'\left(b\right)\circ K\left(a\right)-K'\left(b\right)\circ K\left(a\right)\right)}_{=0}\left(u\right)=0\left(u\right)=0.\end{aligned}$$

Comparing this with
$$\begin{aligned}\underbrace{\left(K'\left(b\right)\right)\left(\iota_{\operatorname{Sym},A}\left(a\right)\right)}_{=a\lhd b}\cdot u-u\cdot\underbrace{\left(K\left(a\right)\right)\left(\iota_{\operatorname{Sym},A}\left(b\right)\right)}_{=a\lhd b}&=\left(a\lhd b\right)\cdot u-\underbrace{u\cdot\left(a\lhd b\right)}_{\substack{=\left(a\lhd b\right)\cdot u\\\text{(since the }\mathbf{k}\text{-algebra }\operatorname{Sym}A\\\text{is commutative)}}}\\&=\left(a\lhd b\right)\cdot u-\left(a\lhd b\right)\cdot u=0,\end{aligned}$$
we obtain
$$\left(K\left(a\right)\circ K'\left(b\right)-K'\left(b\right)\circ K\left(a\right)\right)\left(u\right)=\left(K'\left(b\right)\right)\left(\iota_{\operatorname{Sym},A}\left(a\right)\right)\cdot u-u\cdot\left(K\left(a\right)\right)\left(\iota_{\operatorname{Sym},A}\left(b\right)\right),$$
qed.

References

[Manchon11] Dominique Manchon, *A short survey on pre-Lie algebras*, E. Schrödinger Institut Lectures in Math. Phys., Eur. Math. Soc, A. Carey Ed. (2011),
available at `http://math.univ-bpclermont.fr/~manchon/#Listedesarticles` .

[Manchon15] Dominique Manchon, *Algebraic background for numerical methods, control theory and renormalization*, Proceedings of the COmbinatorics and COntrol conference, Benasque, Spring 2010 (to appear),
available at `arXiv:1501.07205v1`.

[GuiOud04] Jean-Michel Oudom, Daniel Guin, *Sur l'algèbre enveloppante d'une algèbre pré-Lie*, C. R. Acad. Sci. Paris, Ser. I 340 (2005), pp. 331–336.
`http://www.sciencedirect.com/science/article/pii/S1631073X05000269`

[GuiOud08] Jean-Michel Oudom, Daniel Guin, *On the Lie enveloping algebra of a pre-Lie algebra*, J. K-Theory 2 (2008), pp. 147–167, doi:10.1017/is008001011jkt037.
A preprint version of this paper is available at `arXiv:math/0404457v1`.

[Schedl10] Travis Schedler, *Connes-Kreimer quantizations and PBW theorems for pre-Lie algebras*, arXiv:0907.1717v2.

[Burde11] Dietrich Burde, *Left-symmetric algebras, or pre-Lie algebras in geometry and physics*, Central European Journal of Mathematics 4, Nr. 3, 323–357 (2006),
available at
`https://homepage.univie.ac.at/Dietrich.Burde/papers/burde_24_pre_lie.pdf` .

A preprint version of this paper is available at `arXiv:math-ph/0509016v2`.

[Grinbe11] Darij Grinberg, *Poincaré-Birkhoff-Witt type results for inclusions of Lie algebras*, updated version of a diploma thesis at the LMU Munich, Version 1.5 (October 22, 2013).
`http://www.cip.ifi.lmu.de/~grinberg/algebra/algebra.html#pbw`

[GriRei18] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, version of 11 May 2018, `arXiv:1409.8356v5`.
See also `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf` for a version that gets updated.

[MO102281] Darij Grinberg (question) and Frédéric Chapoton (answer), *A mysterious Heisenberg algebra identity from Sylvester, 1867*, MathOverflow question #102281.

[MO102874] Darij Grinberg (question) and Alexander Chervov (comments), *Does the normal ordered product on differential operators lift to $U(\mathfrak{gl}_n)$?*, MathOverflow question #102874.

[Grinbe15] Darij Grinberg, *Collected trivialities on algebra derivations*, December 4, 2016.
`http://www.cip.ifi.lmu.de/~grinberg/algebra/derivat.pdf`