

Poincaré-Birkhoff-Witt type results for inclusions of Lie algebras

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Poincaré-Birkhoff-Witt type results for inclusions of Lie algebras

Remark on this version:

This is the long (detailed) version of this paper. Its goal is to provide proofs for most of the results involved, even if the results are well-known or their proof is an almost verbatim imitation of another (already given) proof. As a consequence, it is very long (even longer than the supposedly "short" version), so I recommend the short version¹ for reading.

At the moment, Sections 5 and 6 are almost identical in the long and short versions. This will probably be changed when I have some more time.

0.1. Introduction

The goal of this paper is to give an elementary and self-contained proof of the relative Poincaré-Birkhoff-Witt theorem that was formulated and proved by Calaque, Căldăraru and Tu in [2]. While our proof passes the same landmarks as the one given in [2], it will often take a different path in between. In particular, it will completely avoid the use of Koszul algebras and Hopf algebras in the proofs of two crucial lemmata. It will be completely elementary except for applying the (standard, non-relative) Poincaré-Birkhoff-Witt theorem - something I was not able to eschew.

Besides the elementarity, an advantage of our approach is that it applies to a slightly more general setting than the one given in [2]. The proofs of the first two main lemmata still hold true for Lie algebras which are modules over an arbitrary commutative ring k (rather than vector spaces over a field k), as long as a weak splitting condition (which is always satisfied in the case of a field) is satisfied (an inclusion of Lie algebras is supposed to split *as a k -module inclusion*). Unfortunately this generality is lost in the proof of the third main lemma, but it still applies to some rather broad cases encompassing that of k being a field.

Let us sketch the course of action of [2], and meanwhile point out where *our* course of action is going to differ:²

One of the many (albeit not the strongest or most general) avatars of the Poincaré-Birkhoff-Witt theorem states that if k is a field of characteristic 0, and \mathfrak{g} is a k -Lie algebra, then the universal enveloping algebra $U(\mathfrak{g})$ is isomorphic to the symmetric algebra $\text{Sym } \mathfrak{g}$ as a \mathfrak{g} -module.³ Even dropping the characteristic 0 condition, we still know (from another "Poincaré-Birkhoff-Witt theorem") that the canonical filtration of $U(\mathfrak{g})$ (the one obtained from the degree filtration of the tensor algebra $\otimes \mathfrak{g}$) results in

¹available at <http://www.cip.ifi.lmu.de/~grinberg/pbw.pdf>

²Note that the aim of this Introduction is to give an overview of the results some of which we are going to prove in the following, *not* to define and formulate everything in full detail. The reader can safely skip this Introduction: Every notion we define in it will be defined in greater detail (and often in greater generality) in one of the subsequent Sections (unless it will not ever be used outside this Introduction). The situation we consider in this Introduction (a Lie algebra \mathfrak{g} over a field k , and a Lie subalgebra \mathfrak{h} of \mathfrak{g}) will *not* be the situation we consider in the rest of this paper; instead we will consider slightly more general situations in the rest of this paper.

³See Remark 1.67 for the right definition of the \mathfrak{g} -module structure on $U(\mathfrak{g})$.

an associated graded algebra $\text{gr}(U(\mathfrak{g}))$ which is isomorphic to the symmetric algebra $\text{Sym } \mathfrak{g}$ as a \mathfrak{g} -algebra⁴.

The paper [2] is concerned with generalizing these properties to a relative situation, in which we are given a Lie algebra \mathfrak{g} and a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, and we consider the \mathfrak{h} -modules $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ and $\text{Sym}(\mathfrak{g}/\mathfrak{h})$ instead of $U(\mathfrak{g})$ and $\text{Sym } \mathfrak{g}$. (Here, $U(\mathfrak{g}) \cdot \mathfrak{h}$ means the right ideal of $U(\mathfrak{g})$ generated by the image of $\mathfrak{h} \subseteq \mathfrak{g}$ under the canonical map $\mathfrak{g} \rightarrow U(\mathfrak{g})$.) In this relative situation, we do not get much for free anymore, but [2] proves the following results:

- If k is a field of arbitrary characteristic, then we have an isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \cong \text{Sym}(\mathfrak{g}/\mathfrak{h})$ of filtered k -modules (here, the filtration on $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ comes from the canonical filtration on $U(\mathfrak{g})$), even if not necessarily of \mathfrak{h} -modules. This isomorphism needs not be canonical. However, there is a canonical isomorphism of associated graded \mathfrak{h} -modules $\text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \cong \text{Sym}^n(\mathfrak{g}/\mathfrak{h})$ for every $n \in \mathbb{N}$.
- If k is a field of characteristic 0, then we *do* have a canonical isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \cong \text{Sym}(\mathfrak{g}/\mathfrak{h})$ of filtered \mathfrak{h} -modules *if and only if a certain Lie-algebraic condition on \mathfrak{g} and \mathfrak{h} is fulfilled*. This condition takes three equivalent forms (Assertions 2, 3 and 4 in Theorem 0.1), is (comparably) easy to check and is rather often fulfilled in classical cases.

We will now come to the exact statements and strengthenings of these results.

Theorem 1.3 of [2] (the main result of the paper) states:

Theorem 0.1 (Relative Poincaré-Birkhoff-Witt theorem). Let k be a field of characteristic 0, and let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let \mathfrak{n} denote the quotient \mathfrak{h} -module $\mathfrak{g}/\mathfrak{h}$.

Preparations:

1. Consider the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . By using the canonical embedding $\mathfrak{g} \rightarrow U(\mathfrak{g})$ (this is an embedding due to the standard Poincaré-Birkhoff-Witt theorem), we can consider \mathfrak{g} a subset of $U(\mathfrak{g})$, and thus $\mathfrak{h} \subseteq \mathfrak{g} \subseteq U(\mathfrak{g})$.

2. Now, define a new k -Lie algebra $\mathfrak{h}^{(1)}$ as follows (see Proposition 3.69 for a more detailed definition): Let $\text{FreeLie } \mathfrak{g}$ denote the free Lie algebra on the k -module \mathfrak{g} , and let $\iota : \mathfrak{g} \rightarrow \text{FreeLie } \mathfrak{g}$ be the corresponding embedding. Let $\mathfrak{h}^{(1)}$ denote the k -Lie algebra obtained by factoring the free Lie algebra $\text{FreeLie } \mathfrak{g}$ by the Lie ideal generated by its k -submodule $\langle [\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g} \rangle$.

We have a canonical injective k -Lie algebra homomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^{(1)}$ (see Proposition 3.70 (a) for its construction).

3. Let $\tilde{\beta} : \mathfrak{h} \otimes \mathfrak{n} \rightarrow \mathfrak{n}$ be the k -linear map defined by

$$\left(\begin{array}{l} \tilde{\beta}(h \otimes n) = (\text{the action of } h \in \mathfrak{h} \text{ on the element } n \text{ of the } \mathfrak{h}\text{-module } \mathfrak{n}) \\ \text{for every } h \in \mathfrak{h} \text{ and } n \in \mathfrak{n} \end{array} \right).$$

4. Consider the exact sequence $0 \longrightarrow \mathfrak{h} \xrightarrow{\text{inclusion}} \mathfrak{g} \xrightarrow{\text{projection}} \mathfrak{n} \longrightarrow 0$

⁴See Definition 1.70 for the definition of the notion of a \mathfrak{g} -algebra. (It is a very natural notion and probably known in literature under a similar name. Hopf algebraists can translate it as " $U(\mathfrak{g})$ -module algebra".)

of \mathfrak{h} -modules. Tensoring this exact sequence with \mathfrak{n} , we obtain an exact sequence $0 \longrightarrow \mathfrak{h} \otimes \mathfrak{n} \longrightarrow \mathfrak{g} \otimes \mathfrak{n} \longrightarrow \mathfrak{n} \otimes \mathfrak{n} \longrightarrow 0$ of \mathfrak{h} -modules. This exact sequence gives rise to an element of $\text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{h} \otimes \mathfrak{n})$. Applying the map $\text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{h} \otimes \mathfrak{n}) \xrightarrow{\text{Ext}_{\mathfrak{h}}^1(\text{id}, \tilde{\beta})} \text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ (this map is owed to the functoriality of $\text{Ext}_{\mathfrak{h}}^1$) to this element, we obtain an element of $\text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ which we call α .

Statement of the theorem:

The following assertions are equivalent:

Assertion 1: The natural filtration on the \mathfrak{h} -module $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ (the one obtained by quotienting from the natural filtration on $U(\mathfrak{g})$ which, in turn, is obtained by quotienting from the degree filtration on $\otimes \mathfrak{g}$) is \mathfrak{h} -split. (By "mathfrak{h}-split" we mean "split as a filtration of \mathfrak{h} -modules", i. e., the splitting must be \mathfrak{h} -linear.)

Assertion 2: Considering the natural filtration on the \mathfrak{h} -module $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$, there exists an isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \cong \text{Sym } \mathfrak{n}$ of filtered \mathfrak{h} -modules. (Here, an "isomorphism of filtered \mathfrak{h} -modules" means an isomorphism of \mathfrak{h} -modules which respects the filtration, as does its inverse.)

Assertion 3: The class $\alpha \in \text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ is trivial.

Assertion 4: The \mathfrak{h} -module \mathfrak{n} is the restriction of an $\mathfrak{h}^{(1)}$ -module to \mathfrak{h} (via the above-mentioned k -Lie algebra homomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^{(1)}$).

Before we proceed any further, let us note that the equivalence of Assertions 3 and 4 in this theorem is rather easy and was proven in [2] (even in greater generality). More precisely, it is a particular case of the following lemma ([2, Lemma 2.3]):

Lemma 0.2. Let k be a field, and let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let E be an \mathfrak{h} -module. Let \mathfrak{n} denote the quotient \mathfrak{h} -module $\mathfrak{g} / \mathfrak{h}$.

Preparations:

1. Define a Lie algebra $\mathfrak{h}^{(1)}$ as in Theorem 0.1.
2. Let $\tilde{\beta}_E : \mathfrak{h} \otimes E \rightarrow E$ be the k -linear map defined by

$$\left(\begin{array}{l} \tilde{\beta}_E(h \otimes e) = (\text{the action of } h \in \mathfrak{h} \text{ on the element } e \text{ of the } \mathfrak{h}\text{-module } E) \\ \text{for every } h \in \mathfrak{h} \text{ and } e \in E \end{array} \right).$$

3. Consider the exact sequence $0 \longrightarrow \mathfrak{h} \xrightarrow{\text{inclusion}} \mathfrak{g} \xrightarrow{\text{projection}} \mathfrak{n} \longrightarrow 0$ of \mathfrak{h} -modules. Tensoring this exact sequence with E , we obtain an exact sequence $0 \longrightarrow \mathfrak{h} \otimes E \longrightarrow \mathfrak{g} \otimes E \longrightarrow \mathfrak{n} \otimes E \longrightarrow 0$ of \mathfrak{h} -modules. This exact sequence gives rise to an element of $\text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes E, \mathfrak{h} \otimes E)$. Applying the map $\text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes E, \mathfrak{h} \otimes E) \xrightarrow{\text{Ext}_{\mathfrak{h}}^1(\text{id}, \tilde{\beta}_E)} \text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes E, E)$ (this map is owed to the functoriality of $\text{Ext}_{\mathfrak{h}}^1$) to this element, we obtain an element of $\text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes E, E)$ which we call α_E .

Statement of the lemma:

The class $\alpha_E \in \text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes E, E)$ is trivial if and only if the \mathfrak{h} -module E is the restriction of an $\mathfrak{h}^{(1)}$ -module to \mathfrak{h} (via the k -Lie algebra homomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^{(1)}$ mentioned in Theorem 0.1).

This lemma is proven in [2, Lemma 2.3]. The proof generalizes to the case when k is

a commutative ring, as long as we require the inclusion $\mathfrak{h} \rightarrow \mathfrak{g}$ to split as a k -module inclusion.⁵ We are not going to repeat the proof here.

We are actually going to *avoid the use of the Lie algebra* $\mathfrak{h}^{(1)}$ in this paper. While it is a very natural construction, it is rather cumbersome to deal with, and it is nowhere actually used in [2]; the only things used are the notion of an $\mathfrak{h}^{(1)}$ -module and the universal enveloping algebra $U(\mathfrak{h}^{(1)})$. Instead of the notion of an $\mathfrak{h}^{(1)}$ -module, we will use the equivalent notion of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (a notion we define in Definition 3.1, and whose equivalence to that of an $\mathfrak{h}^{(1)}$ -module we prove in Proposition 3.69). Instead of $U(\mathfrak{h}^{(1)})$, we will use a k -algebra $U(\mathfrak{g}, \mathfrak{h})$ that we define in Definition 3.65, and which turns out to be isomorphic to $U(\mathfrak{h}^{(1)})$ (Proposition 3.71). Thus, Assertion 4 of Theorem 0.1 will rewrite as follows:

Assertion 4: The \mathfrak{h} -module \mathfrak{n} is the restriction of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h} .

In a nutshell, a $(\mathfrak{g}, \mathfrak{h})$ -semimodule is the same as a \mathfrak{g} -module, except that we no longer require

$$[a, b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v)$$

to hold for all $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$ (where \rightharpoonup denotes the action of the Lie algebra \mathfrak{g} on the \mathfrak{g} -module/ $(\mathfrak{g}, \mathfrak{h})$ -semimodule), but only require it to hold for all $a \in \mathfrak{h}$ and $b \in \mathfrak{g}$. This is a rather down-to-earth notion, and in my opinion it is much more primordial than that of $\mathfrak{h}^{(1)}$. It actually gives a justification for the interest in $\mathfrak{h}^{(1)}$ - as the Lie algebra whose module category is equivalent to the category of $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

The next step in the proof of Theorem 0.1 is showing the following lemma ([2, Lemma 3.4]):

Lemma 0.3. Let k be a field, and let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let \mathfrak{n} denote the quotient \mathfrak{h} -module $\mathfrak{g}/\mathfrak{h}$.

Let J be the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

of the k -algebra $\otimes \mathfrak{g}$. The degree filtration of the tensor k -algebra $\otimes \mathfrak{g}$ descends to a filtration of the quotient algebra $(\otimes \mathfrak{g})/J$, which we denote by $(F_n)_{n \geq 0}$. This is actually a filtration of the \mathfrak{h} -module $(\otimes \mathfrak{g})/J$.

Then, for every $n \in \mathbb{N}$, the n -th associated graded \mathfrak{h} -module of $(\otimes \mathfrak{g})/J$ with this filtration is isomorphic to $\mathfrak{n}^{\otimes n}$ as \mathfrak{h} -module. In other words, every $n \in \mathbb{N}$ satisfies $F_n/F_{n-1} \cong \mathfrak{n}^{\otimes n}$ as \mathfrak{h} -modules.

This lemma is proven using the theory of Koszul algebras in [2]. We are going to prove it elementarily (by recursive construction of an isomorphism and its inverse) in Section 2. Our elementary approach has the advantage of not depending on homological algebra and thus not requiring k to be a field; we only need the inclusion $\mathfrak{h} \rightarrow \mathfrak{g}$ to split as a k -module inclusion. It would not surprise me if this generality could also be attained by means of the argument from [2] using relative homology, but this would require redoing the theory of Koszul algebras in the relative setting, which was too time consuming a task for me (although probably a rewarding one).

⁵This is a reasonable requirement, as we also need it for Lemma 0.2 to make sense: If we do not require it, it is no longer clear why the sequence $0 \longrightarrow \mathfrak{h} \otimes E \longrightarrow \mathfrak{g} \otimes E \longrightarrow \mathfrak{n} \otimes E \longrightarrow 0$ is exact, but we need this sequence to be exact in order to define the class $\alpha_E \in \text{Ext}_{\mathfrak{h}}^1(\mathfrak{n} \otimes E, \mathfrak{h} \otimes E)$.

Note that the above statement of Lemma 0.3 is not exactly what this lemma wants to state. Just knowing that $F_n/F_{n-1} \cong \mathfrak{n}^{\otimes n}$ as \mathfrak{h} -modules is not enough for us; we need to know that a **very particular** homomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$ is well-defined and an isomorphism. This is what Lemma 0.3 actually should tell, if we would allow it to be twice as long. We refer the reader to Theorem 2.1 (c) below for the "right" statement of this lemma. This "right" statement actually shows that we have a *canonical* isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$. However, we are going to *construct* it by means of a *non-canonical* isomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ (which, however, is non-canonical only by virtue of depending on the choice of a k -vector space complement for \mathfrak{h} in \mathfrak{g}); this will be the isomorphism $\bar{\varphi}$ in Proposition 2.18. The canonicity of the resulting isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$ will come as a surprise.

Lemma 0.3 tells us what the associated graded \mathfrak{h} -modules of the filtered \mathfrak{h} -module $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ are isomorphic to, but it does not directly show how the filtered \mathfrak{h} -module $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ *itself* looks; in fact, passing from a filtered \mathfrak{h} -module to its associated graded \mathfrak{h} -modules entails loss of information (even though a lot of important properties are preserved). However, when a filtration on a filtered \mathfrak{h} -module is \mathfrak{h} -split, then it is determined up to isomorphism by its associated graded \mathfrak{h} -modules. We therefore can ask ourselves when the filtration $(F_n)_{n \geq 0}$ on the filtered \mathfrak{h} -module $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is \mathfrak{h} -split. This is answered by the next lemma, which is [2, Lemma 3.9]:

Lemma 0.4. Let k be a field, and let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let $(F_n)_{n \geq 0}$ be defined as in Lemma 0.3. Let α be defined as in Theorem 0.1.

Then, the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split if and only if the class α is trivial.

Note that one direction of this lemma is more or less straightforward: Namely, if the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split, then abstract nonsense (of the trivial sort) shows that the short exact sequence $0 \longrightarrow F_1/F_0 \xrightarrow{\text{inclusion}} F_2/F_0 \xrightarrow{\text{projection}} F_2/F_1 \longrightarrow 0$ must also be \mathfrak{h} -split, and thus the class α is trivial (because it is, up to isomorphism, the class of this sequence, as [2, Lemma 3.4] shows). We are not going to delve in the details of this argument.

The interesting part is the other direction: to assume that the class α is trivial, and then to show that the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split. In [2], this is proven using a Lie-algebraic analogue of the famous projection formula from representation theory ([2, Lemma 3.8]). The proof uses Hopf algebras (although only as a language - no nontrivial facts are used; as opposed to the proof of Lemma 0.3, this one is completely elementary). Here we are going to give a different proof (somewhat similar to our proof of Lemma 0.3) in Section 4 (more precisely, our Theorem 4.1 (d) yields that the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split even in a more general context than Lemma 0.4 claims it). Both our proof and the proof given in [2] begin by applying the equivalence of Assertions 3 and 4 in Theorem 0.1, so that we know that \mathfrak{n} is the restriction of an $\mathfrak{h}^{(1)}$ -semimodule (i. e., of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule) to \mathfrak{h} , and we want to prove that the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split. Both proofs hold true for k being an arbitrary ring as long as the inclusion $\mathfrak{h} \rightarrow \mathfrak{g}$ splits as a k -module inclusion. Actually, it seems to me that the proofs are kindred (as opposed to the proofs for Lemma 0.3), although written in different lingos.

The next step is the passage from $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ to $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$. This is done in [2, Lemma 4.3]. While the precise assertion of [2, Lemma 4.3] is contained in our Theorem 5.18 (d), its actual significance to the proof lies within the following consequence of [2, Lemma 4.3]:

Lemma 0.5. Let k be a field, and let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let \mathfrak{n} denote the quotient \mathfrak{h} -module $\mathfrak{g}/\mathfrak{h}$.

Let $n \in \mathbb{N}$. Then, there exists a canonical \mathfrak{h} -module isomorphism $\Theta_n : \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ for which the diagram

$$\begin{array}{ccc}
 \text{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\text{gr}_n(\rho \circ \psi)} & \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \\
 \text{gr}_n(\otimes \pi) \downarrow & & \downarrow \Theta_n \\
 \text{gr}_n(\otimes \mathfrak{n}) & & \\
 \text{grad}_{n,n}^{-1} \cong \downarrow & & \\
 \mathfrak{n}^{\otimes n} & \xrightarrow{\text{sym}_{n,n}} & \text{Sym}^n \mathfrak{n}
 \end{array}$$

commutes. Here, π denotes the canonical projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h} = \mathfrak{n}$, while ψ denotes the canonical projection $\otimes \mathfrak{g} \rightarrow U(\mathfrak{g})$, while ρ denotes the canonical projection $U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$, while $\text{grad}_{n,n}^{-1}$ denotes the canonical isomorphism $\mathfrak{n}^{\otimes n} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$, and while $\text{sym}_{n,n}$ denotes the canonical projection $\mathfrak{n}^{\otimes n} \rightarrow \text{Sym}^n \mathfrak{n}$.

This will be proven in parts (c) and (d) of our Corollary 5.19. The proof is identic to that in [2, proof of Lemma 4.3], except that we give more details (as usual) and replace the "k is a field" condition by something more general - albeit not as general as for the results before. In Subsection 6.4, we will somewhat improve this condition.

So what remains is the proof of Theorem 0.1 using all of these lemmata. We already know that Assertions 3 and 4 are equivalent, which allows us to forget Assertion 3. Assertions 1 and 2 are also easily seen to be equivalent (by Proposition 1.118; see the proof of Proposition 5.21 for how this is used). So we only need to show the equivalence between Assertions 1 and 4. We will *not* show that Assertion 1 implies Assertion 4 as this is not difficult and well-explained in [2, proof of Theorem 4.5 (c) \Rightarrow (a)] (and is, apparently, not of too much use: Assertion 1 is much harder to check than Assertion 4). We *will* show that Assertion 4 implies Assertion 1 in Theorem 5.20.

0.2. Remarks on the structure of this paper

The plan of this paper is as follows:

In **Section 1**, we define a number of notions related to Lie algebras and their modules, and prove some basic theorems that will later be used. Every statement in this Section is either well-known or follows easily from well-known facts; most proofs are only given for the sake of completeness and would be more appropriate as solutions to homework exercises in a first course of algebra. Therefore Section 1 can be safely skipped by anyone acquainted to Lie algebra theory, except for Definition 1.70 (this is a well-known notion, but I am not sure whether it is well-known under this exact name) and Remark 1.67 (for the disambiguation of the \mathfrak{g} -module structure on $U(\mathfrak{g})$ that I will be using - as it is one of two different, but equally natural structures).

Section 2 is devoted to the proof of Lemma 0.3 in a more general setting. The proof is based on a k -module homomorphism $\varphi : \otimes \mathfrak{g} \rightarrow \otimes N$ (where N is a k -module complement to \mathfrak{h} in \mathfrak{g}) which is constructed recursively in Definition 2.4. This homomorphism φ was obtained by educated guessing (which, I believe, is the main contribution of this paper) based on experience with similarly constructed maps for Clifford algebras (see Subsection 6.3 for them).

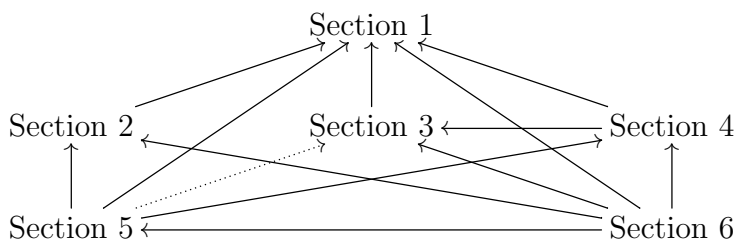
Section 3 defines the notion of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. This is my replacement for the notion of an $\mathfrak{h}^{(1)}$ -module used in [2] (the equivalence to this latter notion is proven in Proposition 3.69) and shares many properties with the familiar notion of \mathfrak{g} -module. We will state some of these properties; many of them are analogous to (and often more general than) similar properties of \mathfrak{g} -modules given in Section 1.

Section 4 proves Lemma 0.4 in the equivalent form given above (instead of assuming that α is trivial, we assume that \mathfrak{n} is the restriction of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule). Again, the proof is given in more generality than Lemma 0.4 itself. The idea of the proof - a recursive construction of a homomorphism $\gamma : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ (this time, as opposed to Section 2, we use $\otimes \mathfrak{n}$ instead of $\otimes N$, albeit these two k -algebras are isomorphic), which is an \mathfrak{h} -module homomorphism this time - is similar to that of Section 2, and so are some further steps of the proof.

Section 5 then states the Poincaré-Birkhoff-Witt theorem in several versions, and completes the proofs of Lemma 0.5 and Theorem 0.1. Again, the situation considered in Section 5 is more general than that of Lemma 0.5 and Theorem 0.1, although not as general as that of Sections 2 and 4.

The final **Section 6** is a kind of odds-and-ends section. It begins with Subsection 6.1, which tries to squeeze out some additional generality from the results of Sections 2 and 4. Subsection 6.3 discusses analogues of the results of [2] in the Clifford algebra of a quadratic space (instead of the universal enveloping algebra of a Lie algebra). Subsection 6.2 is devoted to generalization to Lie superalgebras. Subsection 6.4 extends Theorem 5.18 to a less restrictive case (rather than requiring \mathfrak{h} and N to be free k -modules, we only demand $\mathfrak{g}/\mathfrak{h}$ to be a flat k -module), whose proof is due to Thomas Goodwillie.

Here is a graph depicting the dependencies of the sections of this note on each other:



(arrow means dependency; dotted arrow means very minor dependency).

I have tried to keep this paper as detailed and unambiguous as possible. In particular, I have abdicated many of the common abuses of notation, like silently identifying things which are actually only isomorphic rather than equal⁶, or saying "U and V are

⁶I have not abdicated this completely. I *do* make such identifications in certain places: For example, I identify $V^{\otimes n} \otimes V^{\otimes m}$ with $V^{\otimes(n+m)}$ for any k -module V and $n \in \mathbb{N}$ and $m \in \mathbb{N}$. And I identify $V^{\otimes n}$ with a submodule of $\bigoplus_{n \in \mathbb{N}} V^{\otimes n}$. However, I try to keep these identifications to a minimum; in

isomorphic” when actually meaning the stronger assertion ”a very particular homomorphism $U \rightarrow V$ is an isomorphism”. This noticeably contributes to the length of this paper, but hopefully does so to its readability as well.

Also I have tried to keep theorems self-contained. This means that all notations used in a theorem are defined there, or the places where they are defined are referenced in the theorem. Unsurprisingly, this has stretched the lengths of theorems, but again I hope it was not a vain endeavour.

0.3. Acknowledgements

This paper grew out of a work [2] by Damien Calaque, Andrei Căldăraru, Junwu Tu. I am indebted to Giovanni Felder for acquainting me with this work and to Giovanni Felder and Damien Calaque for inviting me to a research stay at the ETH Zürich.

Andreas Rosenschon advised this thesis and helped out with many valuable discussions. I have learned much of what I am using in this paper from Hans-Jürgen Schneider’s lectures on Hopf algebra and Pavel Etingof’s texts and e-mails.

Further thanks go to my parents for starting off my mathematical education.

0.4. Basic conventions

Before we come to the actual body of this note, let us fix some conventions to prevent misunderstandings from happening:

Convention 0.6. In this note, \mathbb{N} will mean the set $\{0, 1, 2, 3, \dots\}$ (rather than the set $\{1, 2, 3, \dots\}$, which is denoted by \mathbb{N} by various other authors).

Convention 0.7. In this note, a *ring* will always mean a ring with 1. If k is a ring, a *k-algebra* will mean a (not necessarily commutative, but necessarily associative) k -algebra with 1. Sometimes we will use the word ”algebra” as an abbreviation for ” k -algebra”. If L is a k -algebra, then a *left L-module* is always supposed to be a left L -module on which the unity of L acts as the identity. Whenever we use the tensor product sign \otimes without an index, we mean \otimes_k . Similarly, whenever we use the Hom and End signs without index, we mean Hom_k and End_k , respectively.

1. Basics about Lie algebras and their modules

First we are going to recollect the most fundamental definitions and results (and, sometimes, even proofs) from the theory of Lie algebras. While most of these appear in literature, we will recapitulate them already in order to introduce all of the notations that we are going to use.

Almost all results in Section 1 are classical and well-known, so there is no need to prove all of them. However, I will prove *some* of them, in order to counter the

particular I never identify F_n/F_{n-1} with $\mathfrak{n}^{\otimes n}$ in Theorem 2.1 (although $F_n/F_{n-1} \cong \mathfrak{n}^{\otimes n}$ canonically), and I never identify $\text{gr}_n(U(\mathfrak{g}))$ with $\text{Sym}^n \mathfrak{g}$ in Proposition 5.8 (despite the isomorphism $\text{gr}_n(U(\mathfrak{g})) \cong \text{Sym}^n \mathfrak{g}$ when the n -PBW condition is satisfied), and I do not even identify $\text{gr}_p(\otimes V)$ with $V^{\otimes p}$ in Proposition 1.105 (although $\text{gr}_p(\otimes V) \cong V^{\otimes p}$ rather trivially).

unfortunate trend in literature of leaving such proofs to the reader. For example, our Proposition 1.45, which more or less says that the tensor product of several \mathfrak{g} -modules does not depend on the bracketing (in the sense that different bracketings produce different but canonically isomorphic \mathfrak{g} -modules), is known to and considered obvious by everyone working in the field. But I have not seen a detailed proof anywhere in a text. While the proof is straightforward, I did not consider my text complete until the proof was written out.

1.1. Lie algebras

First we recall the basic properties of Lie algebras. Some fundamental definitions:

Definition 1.1. Let k be a commutative ring. A k -Lie algebra will mean a k -module \mathfrak{g} together with a k -bilinear map $\beta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the conditions

$$(\beta(v, v) = 0 \text{ for every } v \in \mathfrak{g}) \quad \text{and} \quad (1)$$

$$(\beta(u, \beta(v, w)) + \beta(v, \beta(w, u)) + \beta(w, \beta(u, v)) = 0 \text{ for every } u \in \mathfrak{g}, v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}). \quad (2)$$

This k -bilinear map $\beta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ will be called the *Lie bracket* of the k -Lie algebra \mathfrak{g} . We will often use the square brackets notation for β , which means that we are going to abbreviate $\beta(v, w)$ by $[v, w]$ for any $v \in \mathfrak{g}$ and $w \in \mathfrak{g}$. Using this notation, the equations (1) and (2) rewrite as

$$([v, v] = 0 \text{ for every } v \in \mathfrak{g}) \quad \text{and} \quad (3)$$

$$([u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \text{ for every } u \in \mathfrak{g}, v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}). \quad (4)$$

The equation (2) (or its equivalent version (4)) is called the *Jacobi identity*.

Also, we will abbreviate the notion " k -Lie algebra" as "Lie algebra", as long as the underlying ring k will be obvious from the context.

Note that Lie algebras are not algebras (in our nomenclature), since we require algebras to be associative.

Convention 1.2. We are going to use the notation $[v, w]$ as a universal notation for the Lie bracket of two elements v and w in a Lie algebra. This means that whenever we have some Lie algebra \mathfrak{g} (it needs not be actually called \mathfrak{g} ; I only refer to it by \mathfrak{g} here in this Convention), and we are given two elements v and w of \mathfrak{g} (they need not be actually called v and w ; I only refer to them by v and w here in this Convention), we will denote by $[v, w]$ the Lie bracket of \mathfrak{g} applied to (v, w) (unless we explicitly stated that the notation $[v, w]$ means something different).

Proposition 1.3. Let k be a commutative ring. Every k -Lie algebra \mathfrak{g} satisfies

$$([v, w] = -[w, v] \text{ for every } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}). \quad (5)$$

(Here, according to Convention 1.2, we denote by $[v, w]$ the Lie bracket of the Lie algebra \mathfrak{g} , applied to (v, w) , and we denote by $[w, v]$ the Lie bracket of the Lie algebra

\mathfrak{g} , applied to (w, v) .) In other words, if β denotes the Lie bracket of \mathfrak{g} , then we have

$$(\beta(v, w) = -\beta(w, v) \text{ for every } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}). \quad (6)$$

Proof of Proposition 1.3. Let $v \in \mathfrak{g}$ and $w \in \mathfrak{g}$ be arbitrary.

Let β denote the Lie bracket of \mathfrak{g} . Applying (1) to $v + w$ instead of v , we obtain $\beta(v + w, v + w) = 0$. Thus,

$$\begin{aligned} 0 &= \beta(v + w, v + w) = \underbrace{\beta(v + w, v)}_{=\beta(v,v)+\beta(w,v)} + \underbrace{\beta(v + w, w)}_{=\beta(v,w)+\beta(w,w)} && \text{(since } \beta \text{ is bilinear)} \\ &= \underbrace{\beta(v, v)}_{=0 \text{ (by (1))}} + \beta(w, v) + \beta(v, w) + \underbrace{\beta(w, w)}_{=0 \text{ (by (1), applied to } w \text{ instead of } v)} = 0 + \beta(w, v) + \beta(v, w) + 0 \\ &= \beta(w, v) + \beta(v, w). \end{aligned}$$

This rewrites as $\beta(v, w) = -\beta(w, v)$. Thus (6) is proven.

According to Convention 1.2, we denote by $[v, w]$ the Lie bracket of \mathfrak{g} , applied to (v, w) . Since the Lie bracket of \mathfrak{g} is β , this becomes $[v, w] = \beta(v, w)$. Similarly, $[w, v] = \beta(w, v)$. Thus, $[v, w] = \beta(v, w) = -\underbrace{\beta(w, v)}_{=[w,v]} = -[w, v]$. This proves (5). We

have thus proven Proposition 1.3.

1.2. Lie subalgebras and Lie algebra homomorphisms

The following definition of the notion of a Lie subalgebra holds little surprise:

Definition 1.4. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a k -submodule of \mathfrak{g} . Then, we say that \mathfrak{h} is a *k -Lie subalgebra* of \mathfrak{g} if every $u \in \mathfrak{h}$ and $v \in \mathfrak{h}$ satisfy $[u, v] \in \mathfrak{h}$.

We will abbreviate " k -Lie subalgebra" as "Lie subalgebra" when k is clear from the context.

This Definition 1.4 is fundamental to this paper, as we are going to study the interplay between the universal enveloping algebra $U(\mathfrak{g})$ (defined in Definition 1.64) with a Lie subalgebra \mathfrak{h} of \mathfrak{g} .

As opposed to this, the following four definitions will only be used marginally (namely, in Subsection 3.11):

Definition 1.5. Let k be a commutative ring. Let \mathfrak{g} and \mathfrak{h} be two k -Lie algebras. Let $f : \mathfrak{g} \rightarrow \mathfrak{h}$ be a map. This map f is said to be a *Lie algebra homomorphism* if and only if it is k -linear and satisfies

$$(f([v, w]) = [f(v), f(w)] \quad \text{for every } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}).$$

(In this equation, according to Convention 1.2, the term $[v, w]$ denotes the Lie bracket of \mathfrak{g} applied to (v, w) , whereas the term $[f(v), f(w)]$ denotes the Lie bracket of \mathfrak{h} applied to $(f(v), f(w))$.)

Definition 1.6. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let S be a subset of \mathfrak{g} .

(1) Consider the subset of \mathfrak{g} which consists of every element which can be obtained by repeated addition, scalar multiplication (i. e., multiplication with elements of k) and forming the Lie bracket from elements of S . This subset is called the *Lie subalgebra of \mathfrak{g} generated by S* . (It is easy to see that this subset indeed is a Lie subalgebra of \mathfrak{g} , so this name is justified. It is also easy to see that this subset is the smallest Lie subalgebra of \mathfrak{g} which contains S as a subset, where "smallest" means "smallest with respect to inclusion".)

(2) The Lie algebra \mathfrak{g} is said to be *generated (as a Lie algebra) by the subset S* (or also *generated (as a Lie algebra) by the elements of S*) if and only if \mathfrak{g} is identical with the Lie subalgebra of \mathfrak{g} generated by S .

(Note that assertions like "The Lie algebra \mathfrak{g} is generated (as a Lie algebra) by the subset S " should never be confused with assertions like "The k -module \mathfrak{g} is generated (as a k -module) by the subset S ", even though every Lie algebra is a k -module. If we have a Lie algebra \mathfrak{g} and we know that the k -module \mathfrak{g} is generated (as a k -module) by some subset S , then we can conclude that the Lie algebra \mathfrak{g} is generated (as a Lie algebra) by S as well; but the converse direction does not hold.)

Definition 1.7. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{i} be a subset of \mathfrak{g} .

(a) We say that \mathfrak{i} is a *Lie ideal* of \mathfrak{g} if and only if \mathfrak{i} is a k -submodule of \mathfrak{g} satisfying

$$([v, x] \in \mathfrak{i} \quad \text{for every } v \in \mathfrak{g} \text{ and } x \in \mathfrak{i}).$$

(b) If \mathfrak{i} is a Lie ideal of \mathfrak{g} , then the k -module $\mathfrak{g}/\mathfrak{i}$ can be made into a k -Lie algebra by setting

$$([\bar{v}, \bar{w}] = \overline{[v, w]} \quad \text{for every } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g})$$

(where for every $t \in \mathfrak{g}$, the residue class of t modulo \mathfrak{i} is denoted by \bar{t}). This k -Lie algebra is indeed well-defined, as can easily be seen.

Definition 1.8. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let S be a subset of \mathfrak{g} .

For any $a \in \mathfrak{g}$, define a map $\text{ad}_a : \mathfrak{g} \rightarrow \mathfrak{g}$ by $(\text{ad}_a(x) = [a, x]$ for every $x \in \mathfrak{g}$).

Consider the subset of \mathfrak{g} which consists of every element which can be obtained by repeated addition, scalar multiplication (i. e., multiplication with elements of k) and application of the maps ad_a (where $a \in \mathfrak{g}$ can be arbitrarily chosen and does not have to be the same each time we apply ad_a) from elements of S . This subset is called the *Lie ideal of \mathfrak{g} generated by S* . (It is easy to see that this subset indeed is a Lie ideal of \mathfrak{g} , so this name is justified. It is also easy to see that this subset is the smallest Lie ideal of \mathfrak{g} which contains S as a subset, where "smallest" means "smallest with respect to inclusion".)

1.3. Modules over Lie algebras

While Lie algebras are interesting for themselves, they are often better understood through their modules. Here is a definition of this notion:

Definition 1.9. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a k -module. Let $\mu : \mathfrak{g} \times V \rightarrow V$ be a k -bilinear map. We say that (V, μ) is a \mathfrak{g} -module if and only if

$$(\mu([a, b], v) = \mu(a, \mu(b, v)) - \mu(b, \mu(a, v)) \text{ for every } a \in \mathfrak{g}, b \in \mathfrak{g} \text{ and } v \in V). \quad (7)$$

If (V, μ) is a \mathfrak{g} -module, then the k -bilinear map $\mu : \mathfrak{g} \times V \rightarrow V$ is called the *Lie action* of the \mathfrak{g} -module V .

Often, when the map μ is obvious from the context, we abbreviate the term $\mu(a, v)$ by $a \rightarrow v$ for any $a \in \mathfrak{g}$ and $v \in V$. Using this notation, the relation (7) rewrites as

$$([a, b] \rightarrow v = a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v) \text{ for every } a \in \mathfrak{g}, b \in \mathfrak{g} \text{ and } v \in V). \quad (8)$$

Also, an abuse of notation allows us to write " V is a \mathfrak{g} -module" instead of " (V, μ) is a \mathfrak{g} -module" if the map μ is clear from the context or has not been introduced yet. Besides, when (V, μ) is a \mathfrak{g} -module, we will say that μ is a *\mathfrak{g} -module structure on V* . In other words, if V is a k -module, then a *\mathfrak{g} -module structure on V* means a map $\mu : \mathfrak{g} \times V \rightarrow V$ such that (V, μ) is a \mathfrak{g} -module. (Thus, in order to make a k -module into a \mathfrak{g} -module, we must define a \mathfrak{g} -module structure on it.)

Convention 1.10. We are going to use the notation $a \rightarrow v$ as a universal notation for the Lie action of a \mathfrak{g} -module. This means that whenever we have some Lie algebra \mathfrak{g} and some \mathfrak{g} -module V (they need not be actually called \mathfrak{g} and V ; I only refer to them as \mathfrak{g} and V here in this Convention), and we are given two elements $a \in \mathfrak{g}$ and $v \in V$ (they need not be actually called a and v ; I only refer to them by a and v here in this Convention), we will denote by $a \rightarrow v$ the Lie action of V applied to (a, v) (unless we explicitly stated that the notation $a \rightarrow v$ means something different).

Convention on the precedence of the \rightarrow sign: When we use the notation $a \rightarrow v$, the \rightarrow sign is supposed to have the same precedence as the multiplication sign (i. e. bind as strongly as the multiplication sign). Thus, $a \rightarrow v + w$ means $(a \rightarrow v) + w$ rather than $a \rightarrow (v + w)$, but $a \rightarrow v \cdot w$ is undefined (it may mean both $(a \rightarrow v) \cdot w$ and $a \rightarrow (v \cdot w)$). Application of functions will be supposed to bind more strongly than the \rightarrow sign, so that $f(v) \rightarrow g(w)$ will mean $(f(v)) \rightarrow (g(w))$ (rather than $f(v \rightarrow g(w))$ or $(f(v \rightarrow g))(w)$ or anything else), but we will often use brackets in this case to make the correct interpretation of the formula even more obvious.

Notational remark. Most authors abbreviate the term $\mu(a, v)$ (where μ is the Lie action of a \mathfrak{g} -module) by $a \cdot v$ or (even shorter) by av , wherever a is an element of a Lie algebra \mathfrak{g} and v is an element of a \mathfrak{g} -module V . However, we cannot afford using this abbreviation, since we will define a \mathfrak{g} -module structure on $\otimes \mathfrak{g}$ which is *not* the left multiplication, so, if we would abbreviate the term $\mu(a, v)$ by $a \cdot v$, we would risk

confusing it with the product of a and v in the tensor algebra $\otimes \mathfrak{g}$. This is why I prefer the abbreviation $a \rightarrow v$.

Remark 1.11. The notion of a " \mathfrak{g} -module" that we defined in Definition 1.9 is often referred to as a "left \mathfrak{g} -module". There is also a similar notion of a "right \mathfrak{g} -module". However, there is not much difference between left \mathfrak{g} -modules and right \mathfrak{g} -modules (in particular, every left \mathfrak{g} -module can be canonically made a right \mathfrak{g} -module and vice versa).⁷ When we speak of \mathfrak{g} -modules, we will always mean left \mathfrak{g} -modules.

Now that we have defined a \mathfrak{g} -module, let us do the next logical step and define a \mathfrak{g} -module homomorphism:

Definition 1.12. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V and W be two \mathfrak{g} -modules. Let $f : V \rightarrow W$ be a k -linear map. Then, f is said to be a \mathfrak{g} -module homomorphism if and only if

$$(f(a \rightarrow v) = a \rightarrow (f(v)) \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V).$$

Often, we will use the words " \mathfrak{g} -module map" or the words "homomorphism of \mathfrak{g} -modules" or the words " \mathfrak{g} -linear map" as synonyms for " \mathfrak{g} -module homomorphism".

It is easy to see that for every commutative ring k and every k -Lie algebra \mathfrak{g} , there is a category whose objects are \mathfrak{g} -modules and whose morphisms are \mathfrak{g} -module homomorphisms. We further define a \mathfrak{g} -module isomorphism as an isomorphism in this category; this is equivalent to the following definition:

Definition 1.13. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V and W be two \mathfrak{g} -modules. Let $f : V \rightarrow W$ be a k -linear map. Then, f is said to be a \mathfrak{g} -module isomorphism if and only if f is an invertible \mathfrak{g} -module homomorphism whose inverse f^{-1} is also a \mathfrak{g} -module homomorphism.

We can easily prove that this definition is somewhat redundant, viz., the condition that f^{-1} be also a \mathfrak{g} -module homomorphism can be omitted:

Proposition 1.14. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V and W be two \mathfrak{g} -modules. Let $f : V \rightarrow W$ be a k -linear map. Then, f is a \mathfrak{g} -module isomorphism if and only if f is an invertible \mathfrak{g} -module homomorphism. In other words, f is a \mathfrak{g} -module isomorphism if and only if f is a \mathfrak{g} -module homomorphism and a k -module isomorphism at the same time.

The proof of this proposition uses the same idea as the standard proof that the inverse of an invertible A -linear map is A -linear (where A is a ring).

It is easy to see that kernels and images of \mathfrak{g} -module isomorphisms are \mathfrak{g} -submodules.

⁷This situation is contradistinctive to the situation for A -modules, where A is an associative algebra. In fact, when A is a non-commutative associative algebra, there is (in general) no way to transform left A -modules into right A -modules.

1.4. Restriction of \mathfrak{g} -modules

If \mathfrak{h} is a Lie subalgebra of a k -Lie algebra \mathfrak{g} , then we can canonically make every \mathfrak{g} -module into an \mathfrak{h} -module according to the following definition:

Definition 1.15. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Then, every \mathfrak{g} -module V canonically becomes an \mathfrak{h} -module (by restricting its Lie action $\mu : \mathfrak{g} \times V \rightarrow V$ to $\mathfrak{h} \times V$). This \mathfrak{h} -module is called the *restriction of V to \mathfrak{h}* , and denoted by $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V$. However, when there is no possibility of confusion, we will denote this \mathfrak{h} -module by V , and we will distinguish it from the original \mathfrak{g} -module V by means of referring to the former one as "the \mathfrak{h} -module V " and referring to the latter one as "the \mathfrak{g} -module V ".

1.5. The \mathfrak{g} -modules \mathfrak{g} and k

Now we notice that the Lie algebra \mathfrak{g} itself is a \mathfrak{g} -module:

Proposition 1.16. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let β be the Lie bracket of \mathfrak{g} (so that $\beta(v, w) = [v, w]$ for all $v \in \mathfrak{g}$ and $w \in \mathfrak{g}$). Then, (\mathfrak{g}, β) is a \mathfrak{g} -module. This \mathfrak{g} -module satisfies

$$v \rightarrow w = [v, w] \quad \text{for all } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}. \quad (9)$$

Definition 1.17. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Whenever we speak of "the \mathfrak{g} -module \mathfrak{g} " without specifying the \mathfrak{g} -module structure, we mean the \mathfrak{g} -module (\mathfrak{g}, β) defined in Proposition 1.16.

Proof of Proposition 1.16. According to Definition 1.9, in order to prove that (\mathfrak{g}, β) is a \mathfrak{g} -module, we need to show that β is a k -bilinear map, and that the equation (7) holds with μ and V replaced by β and \mathfrak{g} .

Every $a \in \mathfrak{g}$, $b \in \mathfrak{g}$ and $v \in \mathfrak{g}$ satisfy

$$\begin{aligned} \beta([a, b], v) &= [[a, b], v] && \text{(since } \beta(v, w) = [v, w] \text{ for any } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}) \\ &= -[v, [a, b]] && \text{(due to (5), applied to } [a, b] \text{ and } v \text{ instead of } v \text{ and } w) \\ &= [a, [b, v]] + \left[b, \underbrace{\begin{array}{c} [v, a] \\ = -[a, v] \\ \text{(due to (5), applied} \\ \text{to } a \text{ instead of } w) \end{array}} \right] \\ &= [a, [b, v]] + \left(\text{since (4) (applied to } v, a, b \text{ instead of } u, v, w) \text{ yields } \right. \\ &\quad \left. [v, [a, b]] + [a, [b, v]] + [b, [v, a]] = 0 \right) \\ &= [a, [b, v]] + \underbrace{[b, -[a, v]]}_{= -[b, [a, v]]} = [a, [b, v]] - [b, [a, v]] \\ &\quad \text{(since the Lie bracket is } k\text{-bilinear)} \\ &= \beta(a, \beta(b, v)) - \beta(b, \beta(a, v)) \\ &\quad \text{(since } [v, w] = \beta(v, w) \text{ for any } v \in \mathfrak{g} \text{ and } w \in \mathfrak{g}). \end{aligned}$$

In other words, the equation (7) holds with μ and V replaced by β and \mathfrak{g} . Since β is a k -bilinear map (since β is the Lie bracket of \mathfrak{g} , and the definition of a Lie algebra requires that the Lie bracket is a k -bilinear map), Definition 1.9 therefore yields that (\mathfrak{g}, β) is a \mathfrak{g} -module. This module satisfies $v \rightarrow w = \beta(v, w) = [v, w]$ for all $v \in \mathfrak{g}$ and $w \in \mathfrak{g}$. Thus, Proposition 1.16 is proven.

There is one yet simpler \mathfrak{g} -module:

Proposition 1.18. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Then, $(k, 0)$ is a \mathfrak{g} -module (where 0 denotes the map $\mathfrak{g} \times k \rightarrow k$ which sends everything to zero). This \mathfrak{g} -module satisfies

$$v \rightarrow \lambda = 0 \quad \text{for all } v \in \mathfrak{g} \text{ and } \lambda \in k.$$

Definition 1.19. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Whenever we speak of "the \mathfrak{g} -module k " without specifying the \mathfrak{g} -module structure, we mean the \mathfrak{g} -module $(k, 0)$ defined in Proposition 1.18. This \mathfrak{g} -module is called *the trivial \mathfrak{g} -module*.

1.6. Submodules, factors and direct sums of \mathfrak{g} -modules

There are more interesting \mathfrak{g} -module structures around. One way to obtain them is to factor existing \mathfrak{g} -modules by submodules:

Definition 1.20. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let V be a \mathfrak{g} -module.

(a) A k -submodule W of V is said to be a \mathfrak{g} -submodule of V if and only if

$$(a \rightarrow w \in W \text{ for every } a \in \mathfrak{g} \text{ and } w \in W).$$

In other words, a k -submodule W of V is said to be a \mathfrak{g} -submodule of V if and only if $\mu(\mathfrak{g} \times W) \subseteq W$, where μ denotes the Lie action of V . (We remind ourselves that the Lie action of V means the k -bilinear map $\mu : \mathfrak{g} \times V \rightarrow V$ from Definition 1.9.)

(b) If W is a \mathfrak{g} -submodule of V , then the quotient k -module V/W becomes a \mathfrak{g} -module by setting

$$(a \rightarrow \bar{v} = \overline{a \rightarrow v} \text{ for every } a \in \mathfrak{g} \text{ and } v \in V)$$

(where \bar{u} denotes the residue class of u modulo W for every $u \in V$). (This \mathfrak{g} -module structure is indeed well-defined, as can be easily seen.)

We can also add \mathfrak{g} -modules via the direct sum:

Proposition 1.21. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let V and W be two \mathfrak{g} -modules. Define a map $\mu_{V \oplus W} : \mathfrak{g} \times (V \oplus W) \rightarrow V \oplus W$ by

$$(\mu_{V \oplus W}(a, (v, w))) = (a \rightarrow v, a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W). \quad (10)$$

Then, this map $\mu_{V \oplus W}$ is k -bilinear, and $(V \oplus W, \mu_{V \oplus W})$ is a \mathfrak{g} -module satisfying

$$a \rightarrow (v, w) = (a \rightarrow v, a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W. \quad (11)$$

This proposition is straightforward to prove, so we are not going to elaborate on its proof. Anyway it allows a definition:

Definition 1.22. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let V and W be two \mathfrak{g} -modules.

The \mathfrak{g} -module $(V \oplus W, \mu_{V \oplus W})$ constructed in Proposition 1.21 is called the *direct sum* of the \mathfrak{g} -modules V and W . We are going to denote this \mathfrak{g} -module $(V \oplus W, \mu_{V \oplus W})$ simply by $V \oplus W$.

We can similarly define the direct sum of several (not necessarily just two) \mathfrak{g} -modules:

Proposition 1.23. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let S be a

set. For every $s \in S$, let V_s be a \mathfrak{g} -module. Define a map $\mu_{\oplus} : \mathfrak{g} \times \left(\bigoplus_{s \in S} V_s \right) \rightarrow \bigoplus_{s \in S} V_s$ by

$$\left(\mu_{\oplus} (a, (v_s)_{s \in S}) = (a \rightarrow v_s)_{s \in S} \quad \text{for every } a \in \mathfrak{g} \text{ and every family } (v_s)_{s \in S} \in \bigoplus_{s \in S} V_s \right). \quad (12)$$

Then, this map μ_{\oplus} is k -bilinear, and $\left(\bigoplus_{s \in S} V_s, \mu_{\oplus} \right)$ is a \mathfrak{g} -module satisfying

$$a \rightarrow (v_s)_{s \in S} = (a \rightarrow v_s)_{s \in S} \quad \text{for every } a \in \mathfrak{g} \text{ and every family } (v_s)_{s \in S} \in \bigoplus_{s \in S} V_s. \quad (13)$$

Definition 1.24. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let S be a set. For every $s \in S$, let V_s be a \mathfrak{g} -module.

The \mathfrak{g} -module $\left(\bigoplus_{s \in S} V_s, \mu_{\oplus} \right)$ constructed in Proposition 1.23 is called the *direct sum* of the \mathfrak{g} -modules V_s over all $s \in S$. We are going to denote this \mathfrak{g} -module $\left(\bigoplus_{s \in S} V_s, \mu_{\oplus} \right)$ simply by $\bigoplus_{s \in S} V_s$.

Again, there is nothing substantial to prove here. Notice that if $S = \emptyset$, then $\bigoplus_{s \in S} V_s$ is to be understood as 0.

Working with direct sums is greatly simplified by using the following convention:

Convention 1.25. Let k be a commutative ring. Let S be a set. For every $s \in S$, let V_s be a k -module. For every $t \in S$, we are going to identify the k -module V_t with the image of V_t under the canonical injection $V_t \rightarrow \bigoplus_{s \in S} V_s$. This is an abuse of notation, but a relatively harmless one. It allows us to consider V_t as a k -submodule of the direct sum $\bigoplus_{s \in S} V_s$.

The same applies for \mathfrak{g} -modules:

Proposition 1.26. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let S be a set. For every $s \in S$, let V_s be a \mathfrak{g} -module. In Convention 1.25, we have identified the k -module V_t with the image of V_t under the canonical injection $V_t \rightarrow \bigoplus_{s \in S} V_s$ for every $t \in S$. Thus, by means of this identification, V_t becomes a k -submodule of the direct sum $\bigoplus_{s \in S} V_s$. But actually, something stronger holds: By means of this identification, V_t becomes a \mathfrak{g} -submodule of the direct sum $\bigoplus_{s \in S} V_s$.

We notice an important, even if trivial, fact, which will often be silently used:

Proposition 1.27. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

(a) If V and W are two \mathfrak{g} -modules, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \oplus W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \oplus (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V \oplus W$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \oplus W)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \oplus (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \oplus W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \oplus (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$).

(b) If S is a set, and if V_t is a \mathfrak{g} -module for every $t \in S$, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}\left(\bigoplus_{s \in S} V_s\right) = \bigoplus_{s \in S} (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_s)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $\bigoplus_{s \in S} V_s$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}\left(\bigoplus_{s \in S} V_s\right)$ or $\bigoplus_{s \in S} (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_s)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}\left(\bigoplus_{s \in S} V_s\right) = \bigoplus_{s \in S} (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_s)$).

(c) The \mathfrak{h} -module k is identical with the restriction $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} k$ of the \mathfrak{g} -module k to \mathfrak{h} .

(d) If V is a \mathfrak{g} -module, and if W is a \mathfrak{g} -submodule of V , then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W$ is an \mathfrak{h} -submodule of the \mathfrak{h} -module $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V$ and satisfies $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V/W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) / (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module V/W " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V/W)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) / (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V/W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) / (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$).

1.7. A convention regarding k -spans

Before we proceed any further, let us fix one convention that we are going to use several times in this text:

Convention 1.28. (a) Whenever k is a commutative ring, M is a k -module, and S is a subset of M , we denote by $\langle S \rangle$ the k -submodule of M generated by the elements of S . This k -submodule $\langle S \rangle$ is called the k -linear span (or simply the k -span) of S .

(b) Whenever k is a commutative ring, M is a k -module, Φ is a set, and $P : \Phi \rightarrow M$ is a map (not necessarily a linear map), we denote by $\langle P(v) \mid v \in \Phi \rangle$ the k -submodule $\langle \{P(v) \mid v \in \Phi\} \rangle$ of M . (In other words, $\langle P(v) \mid v \in \Phi \rangle$ is the k -submodule of M generated by the elements $P(v)$ for all $v \in \Phi$.)

Note that some authors use the notation $\langle S \rangle$ for various other things (e. g., the

two-sided ideal generated by S , or the Lie subalgebra generated by S), but *we will only use it for the k -submodule generated by S (as defined in Convention 1.28 (a))*.

Let us record a trivial fact for later use:

Proposition 1.29. Let k be a commutative ring. Let M be a k -module. Let S be a subset of M .

(a) Let Q be a k -submodule of M such that $S \subseteq Q$. Then, $\langle S \rangle \subseteq Q$.

(b) Let R be a k -module, and $f : M \rightarrow R$ be a k -module homomorphism. Then, $f(\langle S \rangle) = \langle f(S) \rangle$.

Proof of Proposition 1.29. (a) Let $\alpha \in \langle S \rangle$. Then, α is a k -linear combination of the elements of S . This means that there exists some $n \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_n$ of k , and some elements s_1, s_2, \dots, s_n of S such that $\alpha = \sum_{i=1}^n \lambda_i s_i$. But $s_i \in Q$ for every $i \in \{1, 2, \dots, n\}$ (since $s_i \in S \subseteq Q$). Thus, $\alpha = \sum_{i=1}^n \lambda_i \underbrace{s_i}_{\in Q} \in \sum_{i=1}^n \lambda_i Q \subseteq Q$ (since Q is a k -module). Thus we have proven that every $\alpha \in \langle S \rangle$ satisfies $\alpha \in Q$. In other words, $\langle S \rangle \subseteq Q$. Thus, Proposition 1.29 (a) is proven.

(b) First let us prove that $f(\langle S \rangle) \subseteq \langle f(S) \rangle$.

In fact, let $\beta \in f(\langle S \rangle)$. Then, there exists some $\alpha \in \langle S \rangle$ such that $\beta = f(\alpha)$. Since $\alpha \in \langle S \rangle$, we know that α is a k -linear combination of the elements of S . This means that there exists some $n \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_n$ of k , and some elements s_1, s_2, \dots, s_n of S such that $\alpha = \sum_{i=1}^n \lambda_i s_i$. Thus, $f(\alpha) = f\left(\sum_{i=1}^n \lambda_i s_i\right) = \sum_{i=1}^n \lambda_i \underbrace{f(s_i)}_{\in f(S) \text{ (since } s_i \in S)}$ (since f is a k -linear map), so that $f(\alpha)$ is a k -linear combination of the elements of $f(S)$. In other words, $f(\alpha) \in \langle f(S) \rangle$. Since $f(\alpha) = \beta$, this becomes $\beta \in \langle f(S) \rangle$. We have thus proven that every $\beta \in f(\langle S \rangle)$ satisfies $\beta \in \langle f(S) \rangle$. In other words, $f(\langle S \rangle) \subseteq \langle f(S) \rangle$.

Let us now show that $\langle f(S) \rangle \subseteq f(\langle S \rangle)$.

In fact, let $\gamma \in \langle f(S) \rangle$. Then, γ is a k -linear combination of the elements of $f(S)$. This means that there exists some $n \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_n$ of k , and some elements t_1, t_2, \dots, t_n of $f(S)$ such that $\gamma = \sum_{i=1}^n \lambda_i t_i$. For every $i \in \{1, 2, \dots, n\}$, there exists some $s_i \in S$ such that $t_i = f(s_i)$ (because $t_i \in f(S)$). Thus, $\gamma = \sum_{i=1}^n \lambda_i \underbrace{t_i}_{=f(s_i)} = \sum_{i=1}^n \lambda_i f(s_i) = f\left(\sum_{i=1}^n \lambda_i s_i\right)$ (since f is a k -linear map). Since $\sum_{i=1}^n \lambda_i s_i \in \langle S \rangle$ (because $s_i \in S$ for every $i \in \{1, 2, \dots, n\}$, and thus $\sum_{i=1}^n \lambda_i s_i$ is a k -linear combination of the elements of S), this yields $\gamma \in f(\langle S \rangle)$. Thus we have proven that every $\gamma \in \langle f(S) \rangle$ satisfies $\gamma \in f(\langle S \rangle)$. In other words, we have proven that $\langle f(S) \rangle \subseteq f(\langle S \rangle)$. Combined with $f(\langle S \rangle) \subseteq \langle f(S) \rangle$, this yields $f(\langle S \rangle) = \langle f(S) \rangle$. This proves Proposition 1.29 (b).

1.8. Tensor products of two \mathfrak{g} -modules

But now let us go back to methods of obtaining new \mathfrak{g} -modules from given \mathfrak{g} -modules. We already know factor modules and direct sums. Another way to construct \mathfrak{g} -modules

is by tensor multiplication. This is based upon the following fact (which will be proven further below):

Proposition 1.30. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let V and W be two \mathfrak{g} -modules. Then, there exists one and only one k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies

$$(m(a, v \otimes w) = (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W). \quad (14)$$

If we denote this map m by $\mu_{V \otimes W}$, then $(V \otimes W, \mu_{V \otimes W})$ is a \mathfrak{g} -module. This \mathfrak{g} -module satisfies

$$a \rightarrow (v \otimes w) = (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W. \quad (15)$$

Definition 1.31. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let V and W be two \mathfrak{g} -modules.

The \mathfrak{g} -module $(V \otimes W, \mu_{V \otimes W})$ constructed in Proposition 1.30 is called the *tensor product* of the \mathfrak{g} -modules V and W . We are going to denote this \mathfrak{g} -module $(V \otimes W, \mu_{V \otimes W})$ simply by $V \otimes W$.

Thus, for any two \mathfrak{g} -modules V and W , the \mathfrak{g} -module $V \otimes W$ satisfies (15).

Proof of Proposition 1.30.

In order to prove Proposition 1.30, we must verify the following assertions:

Assertion α : There exists a k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14).

Assertion β : There exists one and only one k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14).

Assertion γ : If we denote by $\mu_{V \otimes W}$ the k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14), then $(V \otimes W, \mu_{V \otimes W})$ is a \mathfrak{g} -module.

Assertion δ : The \mathfrak{g} -module $(V \otimes W, \mu_{V \otimes W})$ defined in Assertion γ satisfies (15).

Proof of Assertion α : First, we note that for every $a \in \mathfrak{g}$, the map $V \rightarrow V, v \mapsto a \rightarrow v$ is k -linear.⁸ Similarly, for every $a \in \mathfrak{g}$, the map $W \rightarrow W, w \mapsto a \rightarrow w$ is k -linear.

For every $a \in \mathfrak{g}$, let m_a denote the map $V \times W \rightarrow V \otimes W$ given by

$$(m_a(v, w) = (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) \quad \text{for every } v \in V \text{ and } w \in W).$$

This map $m_a : V \times W \rightarrow V \otimes W$ is k -bilinear (since the maps $V \rightarrow V, v \mapsto a \rightarrow v$ and $W \rightarrow W, w \mapsto a \rightarrow w$ are k -linear). Hence, the universal property of the tensor product yields that there exists one and only one k -linear map $\mu_a : V \otimes W \rightarrow V \otimes W$ satisfying $(\mu_a(v \otimes w) = m_a(v, w)$ for every $v \in V$ and $w \in W$). Denoting this map μ_a by \mathfrak{m}_a , we thus have

$$(\mathfrak{m}_a(v \otimes w) = m_a(v, w) \quad \text{for every } v \in V \text{ and } w \in W).$$

Thus, every $v \in V$ and $w \in W$ satisfy

$$\mathfrak{m}_a(v \otimes w) = m_a(v, w) = (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w). \quad (16)$$

⁸This is because $a \rightarrow v$ is the image of (a, v) under the Lie action of \mathfrak{g} , and because the Lie action is k -bilinear.

It is easy to see that

$$(\mathfrak{m}_{\lambda a + \kappa b} = \lambda \mathfrak{m}_a + \kappa \mathfrak{m}_b \quad \text{for every } a \in \mathfrak{g}, b \in \mathfrak{g}, \lambda \in k \text{ and } \kappa \in k). \quad (17)$$

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Now, let $M : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ be the map defined by

$$(M(a, T) = \mathfrak{m}_a(T) \quad \text{for every } a \in \mathfrak{g} \text{ and } T \in V \otimes W).$$

This map M is k -bilinear (because $\mathfrak{m}_a(T)$ depends linearly on a (since \mathfrak{m}_a depends linearly on a , due to (17)) and depends linearly on T (because \mathfrak{m}_a is k -linear)) and satisfies

$$\begin{aligned} M(a, v \otimes w) &= \mathfrak{m}_a(v \otimes w) && \text{(by the definition of } M) \\ &= (a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w) \end{aligned}$$

for every $a \in \mathfrak{g}$, $v \in V$ and $w \in W$. In other words, (14) is satisfied for $m = M$.

Hence, we have shown that there exists a k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14) (namely, the map M). This proves Assertion α .

Proof of Assertion β : Let $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ be a k -bilinear map which satisfies (14). We are going to prove that $m = M$, where M is the map defined in the proof of Assertion α above.

Indeed, for every $a \in \mathfrak{g}$, let $\mathfrak{m}_a : V \otimes W \rightarrow V \otimes W$ be the map \mathfrak{m}_a defined in the proof of Assertion α above, and let $m_a : V \otimes W \rightarrow V \otimes W$ be the map defined by

$$(m_a(T) = m(a, T) \quad \text{for all } a \in \mathfrak{g} \text{ and } T \in V \otimes W).$$

⁹*Proof.* Let $a \in \mathfrak{g}$, $b \in \mathfrak{g}$, $\lambda \in k$ and $\kappa \in k$ be arbitrary. For every $v \in V$ and $w \in W$, we have

$$\mathfrak{m}_a(v \otimes w) = (a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w) \quad \text{(by (16))},$$

$$\mathfrak{m}_b(v \otimes w) = (b \rightharpoonup v) \otimes w + v \otimes (b \rightharpoonup w) \quad \text{(by (16), applied to } b \text{ instead of } a)$$

and

$$\begin{aligned} \mathfrak{m}_{\lambda a + \kappa b}(v \otimes w) &= \left(\begin{array}{c} (\lambda a + \kappa b) \rightharpoonup v \\ \underbrace{= \lambda a \rightharpoonup v + \kappa b \rightharpoonup v}_{\text{(since the Lie action of } V \text{ is } k\text{-bilinear)}} \end{array} \right) \otimes w + v \otimes \left(\begin{array}{c} (\lambda a + \kappa b) \rightharpoonup w \\ \underbrace{= \lambda a \rightharpoonup w + \kappa b \rightharpoonup w}_{\text{(since the Lie action of } W \text{ is } k\text{-bilinear)}} \end{array} \right) \\ &\quad \text{(by (16), applied to } \lambda a + \kappa b \text{ instead of } a) \\ &= (\lambda a \rightharpoonup v + \kappa b \rightharpoonup v) \otimes w + v \otimes (\lambda a \rightharpoonup w + \kappa b \rightharpoonup w) \\ &= \lambda (a \rightharpoonup v) \otimes w + \kappa (b \rightharpoonup v) \otimes w + \lambda v \otimes (a \rightharpoonup w) + \kappa v \otimes (b \rightharpoonup w) \\ &= \underbrace{\lambda (a \rightharpoonup v) \otimes w + \lambda v \otimes (a \rightharpoonup w)}_{= \lambda((a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w))} + \underbrace{\kappa (b \rightharpoonup v) \otimes w + \kappa v \otimes (b \rightharpoonup w)}_{= \kappa((b \rightharpoonup v) \otimes w + v \otimes (b \rightharpoonup w))} \\ &= \lambda \underbrace{((a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w))}_{= \mathfrak{m}_a(v \otimes w)} + \kappa \underbrace{((b \rightharpoonup v) \otimes w + v \otimes (b \rightharpoonup w))}_{= \mathfrak{m}_b(v \otimes w)} \\ &= \lambda \mathfrak{m}_a(v \otimes w) + \kappa \mathfrak{m}_b(v \otimes w) = (\lambda \mathfrak{m}_a + \kappa \mathfrak{m}_b)(v \otimes w). \end{aligned}$$

Thus, the maps $\mathfrak{m}_{\lambda a + \kappa b}$ and $\lambda \mathfrak{m}_a + \kappa \mathfrak{m}_b$ are equal to each other on all pure tensors. But since the pure tensors generate the tensor product $V \otimes W$, two arbitrary k -linear maps defined on $V \otimes W$ must be equal if they are equal to each other on all pure tensors. Thus, since the two k -linear maps $\mathfrak{m}_{\lambda a + \kappa b}$ and $\lambda \mathfrak{m}_a + \kappa \mathfrak{m}_b$ are equal to each other on all pure tensors, we can conclude that these k -linear maps $\mathfrak{m}_{\lambda a + \kappa b}$ and $\lambda \mathfrak{m}_a + \kappa \mathfrak{m}_b$ are equal. That is, $\mathfrak{m}_{\lambda a + \kappa b} = \lambda \mathfrak{m}_a + \kappa \mathfrak{m}_b$, qed.

This map m_a is k -linear (since the map m is k -bilinear). Furthermore, every $v \in V$ and every $w \in W$ satisfy

$$\begin{aligned} m_a(v \otimes w) &= m(a, v \otimes w) = (a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w) \\ &\quad \text{(since we assumed that } m \text{ satisfies (14))} \\ &= \mathbf{m}_a(v \otimes w) \quad \text{(as proved in the proof of Assertion } \alpha \text{).} \end{aligned}$$

In other words, the maps m_a and \mathbf{m}_a are equal to each other on all pure tensors. But since the pure tensors generate the tensor product $V \otimes W$, two arbitrary k -linear maps defined on $V \otimes W$ must be equal if they are equal to each other on all pure tensors. Thus, since the two k -linear maps m_a and \mathbf{m}_a are equal to each other on all pure tensors, we can conclude that these k -linear maps m_a and \mathbf{m}_a are equal. That is, $m_a = \mathbf{m}_a$.

Now, every $a \in \mathfrak{g}$ and $T \in V \otimes W$ satisfy

$$m(a, T) = \underbrace{m_a}_{=\mathbf{m}_a}(T) = \mathbf{m}_a(T) = M(a, T).$$

Thus, $m = M$.

So we have proved that every k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14) must satisfy $m = M$. Hence, there is *at most one* k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14) (because every such map m must satisfy $m = M$, and hence be equal to M). Combined with Assertion α , this yields that there exists one and only one k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14). This proves Assertion β .

Proof of Assertion γ : Let us denote by $\mu_{V \otimes W}$ the k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14). Let $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$. Define a map $P : V \otimes W \rightarrow V \otimes W$ by

$$\left(\begin{array}{l} P(T) = \mu_{V \otimes W}(a, \mu_{V \otimes W}(b, T)) - \mu_{V \otimes W}(b, \mu_{V \otimes W}(a, T)) - \mu_{V \otimes W}([a, b], T) \\ \text{for every } T \in V \otimes W \end{array} \right). \quad (18)$$

This map P is k -linear (due to the k -bilinearity of $\mu_{V \otimes W}$), and thus its kernel $\text{Ker } P$ is a k -submodule of $V \otimes W$.

We have

$$\mu_{V \otimes W}(a, v \otimes w) = (a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W \quad (19)$$

(since $\mu_{V \otimes W}$ is the k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies (14)).

Every $a \in \mathfrak{g}$, $b \in \mathfrak{g}$, $v \in V$ and $w \in W$ satisfy

$$\begin{aligned} \mu_{V \otimes W}(b, v \otimes w) &= (b \rightharpoonup v) \otimes w + v \otimes (b \rightharpoonup w) \\ &\quad \text{(by (19), applied to } b \text{ instead of } a \text{)} \end{aligned}$$

and thus

$$\begin{aligned}
& \mu_{V \otimes W}(a, \mu_{V \otimes W}(b, v \otimes w)) \\
&= \mu_{V \otimes W}(a, (b \rightarrow v) \otimes w + v \otimes (b \rightarrow w)) \\
&= \underbrace{\mu_{V \otimes W}(a, (b \rightarrow v) \otimes w)}_{\substack{=(a \rightarrow (b \rightarrow v)) \otimes w + (b \rightarrow v) \otimes (a \rightarrow w) \\ \text{(by (19), applied to} \\ b \rightarrow v \text{ instead of } v)}} + \underbrace{\mu_{V \otimes W}(v \otimes (b \rightarrow w))}_{\substack{=(a \rightarrow v) \otimes (b \rightarrow w) + v \otimes (a \rightarrow (b \rightarrow w)) \\ \text{(by (19), applied to} \\ b \rightarrow w \text{ instead of } w)}} \\
&\quad \text{(since } \mu_{V \otimes W} \text{ is } k\text{-bilinear)} \\
&= (a \rightarrow (b \rightarrow v)) \otimes w + (b \rightarrow v) \otimes (a \rightarrow w) + (a \rightarrow v) \otimes (b \rightarrow w) + v \otimes (a \rightarrow (b \rightarrow w)). \tag{20}
\end{aligned}$$

The same argument, with a and b transposed, yields

$$\begin{aligned}
& \mu_{V \otimes W}(b, \mu_{V \otimes W}(a, v \otimes w)) \\
&= (b \rightarrow (a \rightarrow v)) \otimes w + (a \rightarrow v) \otimes (b \rightarrow w) + (b \rightarrow v) \otimes (a \rightarrow w) + v \otimes (b \rightarrow (a \rightarrow w)).
\end{aligned}$$

Subtracting this equation from (20), we obtain

$$\begin{aligned}
& \mu_{V \otimes W}(a, \mu_{V \otimes W}(b, v \otimes w)) - \mu_{V \otimes W}(b, \mu_{V \otimes W}(a, v \otimes w)) \\
&= ((a \rightarrow (b \rightarrow v)) \otimes w + (b \rightarrow v) \otimes (a \rightarrow w) + (a \rightarrow v) \otimes (b \rightarrow w) + v \otimes (a \rightarrow (b \rightarrow w))) \\
&\quad - ((b \rightarrow (a \rightarrow v)) \otimes w + (a \rightarrow v) \otimes (b \rightarrow w) + (b \rightarrow v) \otimes (a \rightarrow w) + v \otimes (b \rightarrow (a \rightarrow w))) \\
&= \underbrace{(a \rightarrow (b \rightarrow v)) \otimes w - (b \rightarrow (a \rightarrow v)) \otimes w}_{= (a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v)) \otimes w} + \underbrace{v \otimes (a \rightarrow (b \rightarrow w)) - v \otimes (b \rightarrow (a \rightarrow w))}_{= v \otimes (a \rightarrow (b \rightarrow w) - b \rightarrow (a \rightarrow w))} \\
&\quad \text{(here we just cancelled some equal terms)} \\
&= (a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v)) \otimes w + v \otimes (a \rightarrow (b \rightarrow w) - b \rightarrow (a \rightarrow w)).
\end{aligned}$$

Comparing this to

$$\begin{aligned}
\mu_{V \otimes W}([a, b], v \otimes w) &= \underbrace{([a, b] \rightarrow v)}_{\substack{= a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v) \\ \text{(by (8))}}} \otimes w + v \otimes \underbrace{([a, b] \rightarrow w)}_{\substack{= a \rightarrow (b \rightarrow w) - b \rightarrow (a \rightarrow w) \\ \text{(by (8), applied} \\ \text{to } w \text{ and } W \text{ instead of } v \text{ and } V)}} \\
&\quad \text{(by (19), applied to } [a, b] \text{ instead of } a) \\
&= (a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v)) \otimes w + v \otimes (a \rightarrow (b \rightarrow w) - b \rightarrow (a \rightarrow w)),
\end{aligned}$$

we conclude that

$$\mu_{V \otimes W}(a, \mu_{V \otimes W}(b, v \otimes w)) - \mu_{V \otimes W}(b, \mu_{V \otimes W}(a, v \otimes w)) = \mu_{V \otimes W}([a, b], v \otimes w).$$

Now, (18) (applied to $T = v \otimes w$) yields that

$$P(v \otimes w) = \underbrace{\mu_{V \otimes W}(a, \mu_{V \otimes W}(b, v \otimes w)) - \mu_{V \otimes W}(b, \mu_{V \otimes W}(a, v \otimes w))}_{= \mu_{V \otimes W}([a, b], v \otimes w)} - \mu_{V \otimes W}([a, b], v \otimes w) = 0$$

for every $v \in V$ and $w \in W$. In other words, the kernel $\text{Ker } P$ contains $v \otimes w$ for every $v \in V$ and $w \in W$. Since $\text{Ker } P$ is a k -submodule of $V \otimes W$, we can conclude that

$\text{Ker } P$

$$\begin{aligned}
& \supseteq \text{(the smallest } k\text{-submodule of } V \otimes W \text{ which contains } v \otimes w \text{ for every } v \in V \text{ and } w \in W) \\
&= \langle v \otimes w \mid (v, w) \in V \times W \rangle = V \otimes W
\end{aligned}$$

(because by the definition of the tensor product, $V \otimes W$ equals the span $\langle v \otimes w \mid (v, w) \in V \times W \rangle$). Thus, $P = 0$. Hence, every $T \in V \otimes W$ satisfies $P(T) = 0$. Due to (18), this rewrites as follows: Every $T \in V \otimes W$ satisfies

$$\mu_{V \otimes W}(a, \mu_{V \otimes W}(b, T)) - \mu_{V \otimes W}(b, \mu_{V \otimes W}(a, T)) - \mu_{V \otimes W}([a, b], T) = 0.$$

In other words, every $T \in V \otimes W$ satisfies

$$\mu_{V \otimes W}(a, \mu_{V \otimes W}(b, T)) - \mu_{V \otimes W}(b, \mu_{V \otimes W}(a, T)) = \mu_{V \otimes W}([a, b], T).$$

Since this holds for all $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$, we have thus proven that (7) holds with m and V replaced by $\mu_{V \otimes W}$ and $V \otimes W$. Since $\mu_{V \otimes W}$ is a k -bilinear map, this yields (by Definition 1.9) that $(V \otimes W, \mu_{V \otimes W})$ is a \mathfrak{g} -module. Assertion γ is now proven.

Proof of Assertion δ : Consider the \mathfrak{g} -module $(V \otimes W, \mu_{V \otimes W})$ defined in Assertion γ . This \mathfrak{g} -module satisfies

$$\begin{aligned} a \rightarrow (v \otimes w) &= \mu_{V \otimes W}(a, v \otimes w) && \text{(because the Lie bracket of this } \mathfrak{g}\text{-module is } \mu_{V \otimes W}\text{)} \\ &= (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) && \text{(by (19))} \end{aligned}$$

for every $a \in \mathfrak{g}$, $v \in V$ and $w \in W$. In other words, it satisfies (15). This proves Assertion δ .

Now that we have proven all four assertions α , β , γ and δ , Proposition 1.30 is proven.

Remark 1.32. In most advanced literature, proofs like the one we gave for Proposition 1.30 are not detailed, and not even the necessity of formulating such a proposition is mentioned. As a consequence, the definition of the tensor product of two \mathfrak{g} -modules given in advanced texts about Lie algebras usually does not look like the Definition 1.31 that we gave, but instead roughly looks like follows:

Definition 1.33. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let V and W be two \mathfrak{g} -modules. Then, the k -module $V \otimes W$ becomes a \mathfrak{g} -module by setting

$$a \rightarrow (v \otimes w) = (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W. \quad (21)$$

Such a definition can be confusing to an untrained reader: It is not immediately clear that Definition 1.33 really defines a \mathfrak{g} -module structure on $V \otimes W$, or even just a well-defined k -bilinear map $\mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$. For example, if we would replace the equation (21) by

$$a \rightarrow (v \otimes w) = (a \rightarrow v) \otimes w + v \otimes s \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W,$$

where s is some fixed nonzero element of W , then it would *not* be well-defined, because one and the same pure tensor $T \in V \otimes W$ can be written in the form $v \otimes w$ (with $v \in V$ and $w \in W$) in several different ways, and they lead to different values of $(a \rightarrow v) \otimes w + v \otimes s$. What saves us in the case of (21) is that the right hand side of (21) is linear in a , linear in v and linear in w , and thus (21) can indeed be seen as the definition of a k -bilinear map $\mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$.¹⁰ The details of this

argument we have given in our proof of Proposition 1.30.

A reader well-experienced with bilinear maps and tensor products, of course, will *not* have any trouble with understanding a definition like Definition 1.33. He will (in most cases) be able to check in his mind that it is indeed a legitimate definition, except that sometimes he will need to perform an explicit computation to show that the k -bilinear map indeed satisfies the equation (7) (which is necessary to make sure that we indeed get a \mathfrak{g} -module).

We now move on to showing properties of these tensor products:

Proposition 1.34. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

(a) Let V be a \mathfrak{g} -module. Then, the k -linear map

$$V \rightarrow k \otimes V, \quad v \mapsto 1 \otimes v$$

is a canonical isomorphism of \mathfrak{g} -modules. (Here, as usual, k denotes the \mathfrak{g} -module k defined in Definition 1.19.)

(b) Let V be a \mathfrak{g} -module. Then, the k -linear map

$$V \rightarrow V \otimes k, \quad v \mapsto v \otimes 1$$

is a canonical isomorphism of \mathfrak{g} -modules. (Here, as usual, k denotes the \mathfrak{g} -module k defined in Definition 1.19.)

(c) Let U , V and W be \mathfrak{g} -modules. Then, the k -linear map

$$(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

is a canonical isomorphism of \mathfrak{g} -modules.

Remark 1.35. Part (c) of this proposition suffers from the roughly the same flaw as Definition 1.33: In Proposition 1.34 (c), it is not immediately clear why "the k -linear map

$$(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

" is well-defined. We refer to standard algebra texts for a construction of this map (it is similar to the construction of M in the proof of Assertion α during the proof of Proposition 1.30); for us it is enough to know that it is a k -linear map $(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ which maps $(u \otimes v) \otimes w$ to $u \otimes (v \otimes w)$ for every $u \in U$, $v \in V$ and $w \in W$ (and, in fact, the only such map).

Proof of Proposition 1.34. (a) Let $\delta_{k,V}$ denote the k -module homomorphism

$$V \rightarrow k \otimes V, \quad v \mapsto 1 \otimes v.$$

Then, in order to prove Proposition 1.34 (a), it is enough to show that $\delta_{k,V}$ is a canonical isomorphism of \mathfrak{g} -modules.

¹⁰This still does not show that this k -bilinear map makes $V \otimes W$ into a \mathfrak{g} -module (because to show this we still must prove (7)). But this can be shown separately by a direct computation, which we did in the proof of Assertion γ during the proof of Proposition 1.30.

It is clear that $\delta_{k,V}$ is canonical. Now let us prove that $\delta_{k,V}$ is a homomorphism of \mathfrak{g} -modules:

Every $a \in \mathfrak{g}$ and $v \in V$ satisfy $\delta_{k,V}(v) = 1 \otimes v$ (by the definition of $\delta_{k,V}$) and $\delta_{k,V}(a \rightarrow v) = 1 \otimes (a \rightarrow v)$ (by the definition of $\delta_{k,V}$). Thus, every $a \in \mathfrak{g}$ and $v \in V$ satisfy

$$\begin{aligned} a \rightarrow \underbrace{\delta_{k,V}(v)}_{=1 \otimes v} &= a \rightarrow (1 \otimes v) = \underbrace{(a \rightarrow 1)}_{=0 \text{ (since the Lie action of } k \text{ is 0)}} \otimes v + 1 \otimes (a \rightarrow v) \\ &\quad \text{(by (15), applied to } k, V, 1 \text{ and } v \text{ instead of } V, W, v \text{ and } w) \\ &= \underbrace{0 \otimes v}_{=0} + 1 \otimes (a \rightarrow v) = 1 \otimes (a \rightarrow v) = \delta_{k,V}(a \rightarrow v). \end{aligned}$$

This proves that $\delta_{k,V}$ is a homomorphism of \mathfrak{g} -modules. Since $\delta_{k,V}$ is an isomorphism of k -modules, this yields that $\delta_{k,V}$ is an isomorphism of \mathfrak{g} -modules. Thus, $\delta_{k,V}$ is a canonical isomorphism of \mathfrak{g} -modules. As we said above, this proves Proposition 1.34 (a).

(b) The proof of Proposition 1.34 (b) is completely analogous to the above-given proof of Proposition 1.34 (a), and thus will not be elaborated on.

(c) Let $\alpha_{U,V,W}$ denote the k -linear map

$$(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w).$$

We know from linear algebra that this map $\alpha_{U,V,W}$ is well-defined and a canonical isomorphism of k -modules. Now, we are going to show that this map $\alpha_{U,V,W}$ is an isomorphism of \mathfrak{g} -modules.

The Lie action of a \mathfrak{g} -module is always k -bilinear. Therefore, for every $a \in \mathfrak{g}$, the maps $(U \otimes V) \otimes W \rightarrow (U \otimes V) \otimes W, T \mapsto a \rightarrow T$ and $U \otimes (V \otimes W) \rightarrow U \otimes (V \otimes W), T \mapsto a \rightarrow T$ are k -linear.

For every $a \in \mathfrak{g}$, let $\Phi_a : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ be the k -linear map defined by

$$(\Phi_a(T) = \alpha_{U,V,W}(a \rightarrow T) - a \rightarrow (\alpha_{U,V,W}(T)) \quad \text{for every } T \in (U \otimes V) \otimes W).$$

This map Φ_a is k -linear (because the maps $(U \otimes V) \otimes W \rightarrow (U \otimes V) \otimes W, T \mapsto a \rightarrow T$ and $U \otimes (V \otimes W) \rightarrow U \otimes (V \otimes W), T \mapsto a \rightarrow T$ are k -linear, and so is the map $\alpha_{U,V,W}$). Hence, $\text{Ker } \Phi_a$ is a k -submodule of $(U \otimes V) \otimes W$.

We will now show that $(u \otimes v) \otimes w \in \text{Ker } \Phi_a$ for every $u \in U, v \in V$ and $w \in W$. In

fact, every $u \in U$, $v \in V$ and $w \in W$ satisfy

$$\begin{aligned}
& \Phi_a((u \otimes v) \otimes w) \\
&= \alpha_{U,V,W} \left(\begin{array}{c} \underbrace{a \rightarrow ((u \otimes v) \otimes w)}_{\substack{=(a \rightarrow (u \otimes v)) \otimes w + (u \otimes v) \otimes (a \rightarrow w) \\ \text{(by (15), applied} \\ \text{to } U \otimes V \text{ and } u \otimes v \text{ instead of } V \text{ and } v)}} \end{array} \right) - a \rightarrow \left(\begin{array}{c} \underbrace{\alpha_{U,V,W}((u \otimes v) \otimes w)}_{\substack{=u \otimes (v \otimes w) \\ \text{(by the definition of } \alpha_{U,V,W})}} \end{array} \right) \\
&= \alpha_{U,V,W} \left(\begin{array}{c} \underbrace{(a \rightarrow (u \otimes v))}_{\substack{=(a \rightarrow u) \otimes v + u \otimes (a \rightarrow v) \\ \text{(by (15), applied} \\ \text{to } U, V, u \text{ and } v \text{ instead of } V, W, v \text{ and } w)}} \otimes w + (u \otimes v) \otimes (a \rightarrow w) \end{array} \right) - a \rightarrow (u \otimes (v \otimes w)) \\
&= \alpha_{U,V,W} \left(\begin{array}{c} \underbrace{((a \rightarrow u) \otimes v + u \otimes (a \rightarrow v)) \otimes w + (u \otimes v) \otimes (a \rightarrow w)}_{=((a \rightarrow u) \otimes v) \otimes w + (u \otimes (a \rightarrow v)) \otimes w} \end{array} \right) - a \rightarrow (u \otimes (v \otimes w)) \\
&= \underbrace{\alpha_{U,V,W}(((a \rightarrow u) \otimes v) \otimes w + (u \otimes (a \rightarrow v)) \otimes w + (u \otimes v) \otimes (a \rightarrow w))}_{\substack{=\alpha_{U,V,W}(((a \rightarrow u) \otimes v) \otimes w) + \alpha_{U,V,W}((u \otimes (a \rightarrow v)) \otimes w) + \alpha_{U,V,W}((u \otimes v) \otimes (a \rightarrow w)) \\ \text{(since } \alpha_{U,V,W} \text{ is } k\text{-linear)}}} - a \rightarrow (u \otimes (v \otimes w)) \\
&= \underbrace{\alpha_{U,V,W}(((a \rightarrow u) \otimes v) \otimes w)}_{\substack{=(a \rightarrow u) \otimes (v \otimes w) \\ \text{(by the definition of } \alpha_{U,V,W})}} + \underbrace{\alpha_{U,V,W}((u \otimes (a \rightarrow v)) \otimes w)}_{\substack{=u \otimes ((a \rightarrow v) \otimes w) \\ \text{(by the definition of } \alpha_{U,V,W})}} + \underbrace{\alpha_{U,V,W}((u \otimes v) \otimes (a \rightarrow w))}_{\substack{=u \otimes (v \otimes (a \rightarrow w)) \\ \text{(by the definition of } \alpha_{U,V,W})}} \\
&\quad - \underbrace{a \rightarrow (u \otimes (v \otimes w))}_{\substack{=(a \rightarrow u) \otimes (v \otimes w) + u \otimes (a \rightarrow (v \otimes w)) \\ \text{(by (15), applied} \\ \text{to } U, V \otimes W, u \text{ and } v \otimes w \text{ instead of } V, W, v \text{ and } w)}} \\
&= (a \rightarrow u) \otimes (v \otimes w) + u \otimes ((a \rightarrow v) \otimes w) + u \otimes (v \otimes (a \rightarrow w)) \\
&\quad - ((a \rightarrow u) \otimes (v \otimes w) + u \otimes (a \rightarrow (v \otimes w))) \\
&= u \otimes ((a \rightarrow v) \otimes w) + u \otimes (v \otimes (a \rightarrow w)) - u \otimes \underbrace{(a \rightarrow (v \otimes w))}_{\substack{=(a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) \\ \text{(by (15))}}} \\
&= u \otimes ((a \rightarrow v) \otimes w) + u \otimes (v \otimes (a \rightarrow w)) - \underbrace{u \otimes ((a \rightarrow v) \otimes w + v \otimes (a \rightarrow w))}_{=u \otimes ((a \rightarrow v) \otimes w) + u \otimes (v \otimes (a \rightarrow w))} \\
&= u \otimes ((a \rightarrow v) \otimes w) + u \otimes (v \otimes (a \rightarrow w)) - (u \otimes ((a \rightarrow v) \otimes w) + u \otimes (v \otimes (a \rightarrow w))) = 0.
\end{aligned}$$

Thus, every $u \in U$, $v \in V$ and $w \in W$ satisfy $(u \otimes v) \otimes w \in \text{Ker } \Phi_a$. Hence, $\text{Ker } \Phi_a$ contains $(u \otimes v) \otimes w$ for all $(u, v, w) \in U \times V \times W$. Since $\text{Ker } \Phi_a$ is a k -submodule of $(U \otimes V) \otimes W$, this yields that¹¹

$$\begin{aligned}
\text{Ker } \Phi_a &\supseteq \text{(the smallest } k\text{-submodule of } (U \otimes V) \otimes W \text{ which contains} \\
&\quad (u \otimes v) \otimes w \text{ for all } (u, v, w) \in U \times V \times W) \\
&= \langle (u \otimes v) \otimes w \mid (u, v, w) \in U \times V \times W \rangle.
\end{aligned}$$

¹¹Here, we are using Convention 1.28.

But since $\langle (u \otimes v) \otimes w \mid (u, v, w) \in U \times V \times W \rangle = (U \otimes V) \otimes W$ (this is easy to see¹²), this rewrites as $\text{Ker } \Phi_a \supseteq (U \otimes V) \otimes W$, so that $\Phi_a = 0$. Hence, every $T \in (U \otimes V) \otimes W$ satisfies

$$0 = \Phi_a(T) = \alpha_{U,V,W}(a \rightharpoonup T) - a \rightharpoonup (\alpha_{U,V,W}(T)).$$

In other words, every $T \in (U \otimes V) \otimes W$ satisfies $\alpha_{U,V,W}(a \rightharpoonup T) = a \rightharpoonup (\alpha_{U,V,W}(T))$. This shows that $\alpha_{U,V,W}$ is a \mathfrak{g} -module homomorphism. Since we also know that $\alpha_{U,V,W}$ is a k -module isomorphism, we conclude that $\alpha_{U,V,W}$ is a \mathfrak{g} -module isomorphism.

Thus, $\alpha_{U,V,W}$ is a canonical isomorphism of \mathfrak{g} -modules. Since $\alpha_{U,V,W}$ was defined as the k -linear map

$$(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w),$$

it thus follows that the k -linear map

$$(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

is a canonical isomorphism of \mathfrak{g} -modules. This proves Proposition 1.34 (c).

Corollary 1.36. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Then, the canonical k -module isomorphism $k \otimes V \rightarrow V$ (this is the k -module homomorphism that sends $\lambda \otimes v$ to λv for all $\lambda \in k$ and $v \in V$) is a \mathfrak{g} -module isomorphism.

Proof of Corollary 1.36. According to Proposition 1.34 (a), the k -linear map $V \rightarrow k \otimes V$, $v \mapsto 1 \otimes v$ is an isomorphism of \mathfrak{g} -modules. Thus, the inverse of this map $V \rightarrow k \otimes V$, $v \mapsto 1 \otimes v$ is an isomorphism of \mathfrak{g} -modules as well (because the inverse of an isomorphism of \mathfrak{g} -modules is an isomorphism of \mathfrak{g} -modules). But since the inverse of the map $V \rightarrow k \otimes V$, $v \mapsto 1 \otimes v$ is the canonical k -module isomorphism $k \otimes V \rightarrow V$ (this is easy to show), this yields that the canonical k -module isomorphism $k \otimes V \rightarrow V$ is an isomorphism of \mathfrak{g} -modules. In other words, Corollary 1.36 is proven.

¹²*Proof.* Let $T \in (U \otimes V) \otimes W$ be arbitrary. Then, T is a tensor in the tensor product $(U \otimes V) \otimes W$, and thus can be written in the form $T = \sum_{i=1}^p \lambda_i S_i \otimes w_i$ for some $p \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_p$ of k , some elements S_1, S_2, \dots, S_p of $U \otimes V$ and some elements w_1, w_2, \dots, w_p of W (because every tensor is a k -linear combination of pure tensors). For every $i \in \{1, 2, \dots, p\}$, we know that S_i is a tensor in $U \otimes V$, and thus can be written in the form $S_i = \sum_{j=1}^{q_i} \mu_{i,j} u_{i,j} \otimes v_{i,j}$ for some $q_i \in \mathbb{N}$, some elements $\mu_{i,1}, \mu_{i,2}, \dots, \mu_{i,q_i}$ of k , some elements $u_{i,1}, u_{i,2}, \dots, u_{i,q_i}$ of U and some elements $v_{i,1}, v_{i,2}, \dots, v_{i,q_i}$ of V (because every tensor is a k -linear combination of pure tensors). Thus,

$$T = \sum_{i=1}^p \lambda_i \underbrace{S_i}_{=\sum_{j=1}^{q_i} \mu_{i,j} u_{i,j} \otimes v_{i,j}} \otimes w_i = \sum_{i=1}^p \lambda_i \left(\sum_{j=1}^{q_i} \mu_{i,j} u_{i,j} \otimes v_{i,j} \right) \otimes w_i = \sum_{i=1}^p \lambda_i \sum_{j=1}^{q_i} \mu_{i,j} (u_{i,j} \otimes v_{i,j}) \otimes w_i$$

$$\in \langle (u \otimes v) \otimes w \mid (u, v, w) \in U \times V \times W \rangle$$

(because for every $i \in \{1, 2, \dots, p\}$ and every $j \in \{1, 2, \dots, q_i\}$, the tensor $(u_{i,j} \otimes v_{i,j}) \otimes w_i$ has the form $(u \otimes v) \otimes w$ for some $(u, v, w) \in U \times V \times W$ (namely, for $(u, v, w) = (u_{i,j}, v_{i,j}, w_i)$). So we have proven that every $T \in (U \otimes V) \otimes W$ satisfies $T \in \langle (u \otimes v) \otimes w \mid (u, v, w) \in U \times V \times W \rangle$. In other words, $(U \otimes V) \otimes W \subseteq \langle (u \otimes v) \otimes w \mid (u, v, w) \in U \times V \times W \rangle$. Combined with the obvious relation $\langle (u \otimes v) \otimes w \mid (u, v, w) \in U \times V \times W \rangle \subseteq (U \otimes V) \otimes W$, this yields $\langle (u \otimes v) \otimes w \mid (u, v, w) \in U \times V \times W \rangle = (U \otimes V) \otimes W$, qed.

Convention 1.37. Let k be a commutative ring. Let V be a k -module.

We are going to identify the three k -modules $V \otimes k$, $k \otimes V$ and V with each other (due to the canonical isomorphisms $V \rightarrow V \otimes k$ and $V \rightarrow k \otimes V$).

If V is a \mathfrak{g} -module, where \mathfrak{g} is some k -Lie algebra, then this identification will not conflict with the \mathfrak{g} -module structures on $V \otimes k$, $k \otimes V$ and V (because the canonical isomorphisms $V \rightarrow V \otimes k$ and $V \rightarrow k \otimes V$ are \mathfrak{g} -module isomorphisms (as Proposition 1.34 (a) and (b) shows)).

Finally a straightforward fact:

Proposition 1.38. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

Let V , W , V' and W' be four \mathfrak{g} -modules, and let $f : V \rightarrow V'$ and $g : W \rightarrow W'$ be two \mathfrak{g} -module homomorphisms. Then, $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ is a \mathfrak{g} -module homomorphism.

Proof of Proposition 1.38. Let $a \in \mathfrak{g}$ be arbitrary.

Clearly, the maps $V \otimes W \rightarrow V \otimes W, T \mapsto a \rightarrow T$ and $V' \otimes W' \rightarrow V' \otimes W', T \mapsto a \rightarrow T$ are k -linear (because the Lie action of a \mathfrak{g} -module is k -bilinear).

Let $R_a : V \otimes W \rightarrow V' \otimes W'$ be the map defined by

$$(R_a(T) = a \rightarrow ((f \otimes g)(T)) - (f \otimes g)(a \rightarrow T) \quad \text{for every } T \in V \otimes W).$$

This map R_a is k -linear (since the maps $V \otimes W \rightarrow V \otimes W, T \mapsto a \rightarrow T$ and $V' \otimes W' \rightarrow V' \otimes W', T \mapsto a \rightarrow T$ are k -linear, and so is the map $f \otimes g$). Thus, $\text{Ker } R_a$ is a k -submodule of $V \otimes W$.

Now let us prove that

$$\text{every } (v, w) \in V \times W \text{ satisfies } v \otimes w \in \text{Ker } R_a. \quad (22)$$

Proof of (22). Let $(v, w) \in V \times W$ be arbitrary. Then, $v \in V$ and $w \in W$. Now, the

definition of R_a yields

$$\begin{aligned}
& R_a(v \otimes w) \\
&= a \rightarrow \left(\underbrace{(f \otimes g)(v \otimes w)}_{=f(v) \otimes g(w)} \right) - (f \otimes g) \left(\underbrace{a \rightarrow (v \otimes w)}_{\substack{=(a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) \\ \text{(according to (15))}}} \right) \\
&= \underbrace{a \rightarrow (f(v) \otimes g(w))}_{\substack{=(a \rightarrow (f(v))) \otimes g(w) + f(v) \otimes (a \rightarrow (g(w))) \\ \text{(according to (15), \\ applied to } V', W', f(v) \text{ and } g(w) \text{ instead of } V, W, v \text{ and } w)}}} - \underbrace{(f \otimes g)((a \rightarrow v) \otimes w + v \otimes (a \rightarrow w))}_{\substack{=(f \otimes g)((a \rightarrow v) \otimes w) + (f \otimes g)(v \otimes (a \rightarrow w)) \\ \text{(since } f \otimes g \text{ is a } k\text{-linear map)}}} \\
&= ((a \rightarrow (f(v))) \otimes g(w) + f(v) \otimes (a \rightarrow (g(w)))) \\
&\quad - \left(\underbrace{(f \otimes g)((a \rightarrow v) \otimes w)}_{=f(a \rightarrow v) \otimes g(w)} + \underbrace{(f \otimes g)(v \otimes (a \rightarrow w))}_{=f(v) \otimes g(a \rightarrow w)} \right) \\
&= ((a \rightarrow (f(v))) \otimes g(w) + f(v) \otimes (a \rightarrow (g(w)))) \\
&\quad - \left(\underbrace{f(a \rightarrow v)}_{\substack{=a \rightarrow (f(v)) \\ \text{(since } f \text{ is a } \mathfrak{g}\text{-module map)}}} \otimes g(w) + f(v) \otimes \underbrace{g(a \rightarrow w)}_{\substack{=a \rightarrow (g(w)) \\ \text{(since } g \text{ is a } \mathfrak{g}\text{-module map)}}} \right) \\
&= ((a \rightarrow (f(v))) \otimes g(w) + f(v) \otimes (a \rightarrow (g(w)))) \\
&\quad - ((a \rightarrow (f(v))) \otimes g(w) + f(v) \otimes (a \rightarrow (g(w)))) \\
&= 0.
\end{aligned}$$

Thus, $v \otimes w \in \text{Ker } R_a$. This proves (22).

The relation (22) yields $\{v \otimes w \mid (v, w) \in V \times W\} \subseteq \text{Ker } R_a$.

Now, we have $V \otimes W = \langle v \otimes w \mid (v, w) \in V \times W \rangle$ (since the tensor product $V \otimes W$ is generated by its pure tensors). In other words,

$$V \otimes W = \langle v \otimes w \mid (v, w) \in V \times W \rangle = \langle \{v \otimes w \mid (v, w) \in V \times W\} \rangle \subseteq \text{Ker } R_a$$

(due to Proposition 1.29 (a) (applied to $V \otimes W$, $\{v \otimes w \mid (v, w) \in V \times W\}$ and $\text{Ker } R_a$ instead of M , S and Q), which can be applied since $\{v \otimes w \mid (v, w) \in V \times W\} \subseteq \text{Ker } R_a$ and since $\text{Ker } R_a$ is a k -module). Hence, $R_a = 0$. Thus, every $T \in V \otimes W$ satisfies $R_a(T) = 0$. Therefore, every $T \in V \otimes W$ satisfies $(f \otimes g)(a \rightarrow T) = a \rightarrow ((f \otimes g)(T))$ (because $0 = R_a(T) = a \rightarrow ((f \otimes g)(T)) - (f \otimes g)(a \rightarrow T)$ and thus $a \rightarrow ((f \otimes g)(T)) = (f \otimes g)(a \rightarrow T)$). In other words, $f \otimes g$ is a \mathfrak{g} -module homomorphism. This proves Proposition 1.38.

We notice an analogue of Proposition 1.27 for tensor products:

Proposition 1.39. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

If V and W are two \mathfrak{g} -modules, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \otimes W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V \otimes W$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \otimes W)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \otimes W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$).

This follows from the definitions.

1.9. Tensor products of several \mathfrak{g} -modules

We now define multi-factor tensor products of \mathfrak{g} -modules. First we recall one of the possible definitions of the tensor product of several k -modules:

Definition 1.40. Let k be a commutative ring. Let $n \in \mathbb{N}$.

Now, by induction over n , we are going to define a k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary k -modules V_1, V_2, \dots, V_n :

Induction base: For $n = 0$, we define $V_1 \otimes V_2 \otimes \dots \otimes V_n$ as the k -module k .

Induction step: Let $p \in \mathbb{N}$. Assuming that we have defined a k -module $V_1 \otimes V_2 \otimes \dots \otimes V_p$ for any p arbitrary k -modules V_1, V_2, \dots, V_p , we now define a k -module $V_1 \otimes V_2 \otimes \dots \otimes V_{p+1}$ for any $p+1$ arbitrary k -modules V_1, V_2, \dots, V_{p+1} by the equation

$$V_1 \otimes V_2 \otimes \dots \otimes V_{p+1} = V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}). \quad (23)$$

Here, $V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_{p+1})$ is to be understood as the tensor product of the k -module V_1 with the k -module $V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}$ (note that the k -module $V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}$ is already defined because we assumed that we have defined a k -module $V_1 \otimes V_2 \otimes \dots \otimes V_p$ for any p arbitrary k -modules V_1, V_2, \dots, V_p). This completes the inductive definition.

Thus we have defined a k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary k -modules V_1, V_2, \dots, V_n for any $n \in \mathbb{N}$. This k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ is called the *tensor product* of the k -modules V_1, V_2, \dots, V_n .

Remark 1.41. (a) Definition 1.40 is not the only possible definition of the tensor product of several k -modules. One could obtain a different definition by replacing the equation (23) by

$$V_1 \otimes V_2 \otimes \dots \otimes V_{p+1} = (V_1 \otimes V_2 \otimes \dots \otimes V_p) \otimes V_{p+1}.$$

This definition would have given us a *different* k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary k -modules V_1, V_2, \dots, V_n for any $n \in \mathbb{N}$ than the one defined in Definition 1.40. However, this k -module would still be *canonically isomorphic* to the one defined in Definition 1.40, and thus it is commonly considered to be "more or less the same k -module".

There is yet another definition of $V_1 \otimes V_2 \otimes \dots \otimes V_n$, which proceeds by taking the free k -module on the set $V_1 \times V_2 \times \dots \times V_n$ and factoring it modulo a certain submodule. This definition gives yet *another* k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$, but this module is also canonically isomorphic to the k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 1.40, and thus can be considered to be "more or less the same k -module".

(b) Definition 1.40, applied to $n = 1$, defines the tensor product of *one* k -module V_1 as $V_1 \otimes k$. This takes some getting used to, since it seems more natural to define the tensor product of one k -module V_1 simply as V_1 . But this isn't really different because there is a canonical isomorphism of k -modules $V_1 \cong V_1 \otimes k$, so most people consider V_1 to be "more or less the same k -module" as $V_1 \otimes k$.

Now, by analogy, we define the tensor product of several \mathfrak{g} -modules.

Definition 1.42. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$. Now, by induction over n , we are going to define a \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_n :

Induction base: For $n = 0$, we define $V_1 \otimes V_2 \otimes \dots \otimes V_n$ as the \mathfrak{g} -module k defined in Definition 1.19.

Induction step: Let $p \in \mathbb{N}$. Assuming that we have defined a \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_p$ for any p arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_p , we now define a \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_{p+1}$ for any $p+1$ arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_{p+1} by the equation

$$V_1 \otimes V_2 \otimes \dots \otimes V_{p+1} = V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}). \quad (24)$$

Here, $V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_{p+1})$ is to be understood as the tensor product of the \mathfrak{g} -module V_1 with the \mathfrak{g} -module $V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}$ (note that the \mathfrak{g} -module $V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}$ is already defined because we assumed that we have defined a \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_p$ for any p arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_p). This completes the inductive definition.

Thus we have defined a \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_n for any $n \in \mathbb{N}$. This \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ is called the *tensor product* of the \mathfrak{g} -modules V_1, V_2, \dots, V_n .

Remark 1.43. (a) In Definition 1.42, we could have replaced the equation (24) by

$$V_1 \otimes V_2 \otimes \dots \otimes V_{p+1} = (V_1 \otimes V_2 \otimes \dots \otimes V_p) \otimes V_{p+1}.$$

This would have given us a *different* \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_n for any $n \in \mathbb{N}$ than the one defined in Definition 1.42. However, this \mathfrak{g} -module would still be *canonically isomorphic* to the one defined in Definition 1.42 (we will prove this and actually something more general in Proposition 1.45), and thus it is commonly considered to be "more or less the same \mathfrak{g} -module".

(b) Definition 1.42, applied to $n = 1$, defines the tensor product of *one* \mathfrak{g} -module V_1 as $V_1 \otimes k$. This takes some getting used to, since it seems more natural to define the tensor product of one \mathfrak{g} -module V_1 simply as V_1 . But this isn't really different because Proposition 1.34 (b) gives a canonical isomorphism of \mathfrak{g} -modules $V_1 \cong V_1 \otimes k$, so most people consider V_1 to be "more or less the same \mathfrak{g} -module" as $V_1 \otimes k$.

(c) Definition 1.42 does not conflict with Definition 1.40, because the underlying k -module of the \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 1.42 is indeed the k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 1.40. (This is trivial by induction.)

Convention 1.44. A remark about notation is appropriate at this point:

There are two different conflicting notions of a "pure tensor" in a tensor product $V_1 \otimes V_2 \otimes \dots \otimes V_n$ of n arbitrary k -modules V_1, V_2, \dots, V_n , where $n \geq 1$. The one notion defines a "pure tensor" as an element of the form $v \otimes T$ for some $v \in V_1$ and some $T \in V_2 \otimes V_3 \otimes \dots \otimes V_n$ ¹³. The other notion defines a "pure tensor" as an element of the form $v_1 \otimes v_2 \otimes \dots \otimes v_n$ for some $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$. These two notions are not equivalent. In this note, we are going to yield right of way to the second of these notions, i. e. we are going to define a pure tensor in $V_1 \otimes V_2 \otimes \dots \otimes V_n$ as an element of the form $v_1 \otimes v_2 \otimes \dots \otimes v_n$ for some $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$. The first notion, however, will also be used - but we will not call it a "pure tensor" but rather

a "left-induced tensor". Thus we define a *left-induced tensor* in $V_1 \otimes V_2 \otimes \dots \otimes V_n$ as an element of the form $v \otimes T$ for some $v \in V_1$ and some $T \in V_2 \otimes V_3 \otimes \dots \otimes V_n$.

We note that the k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ is generated by its left-induced tensors, but also generated by its pure tensors.

Before we continue, we need a technical result:

Proposition 1.45. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$.

Then, for any n arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_n and every $i \in \{0, 1, \dots, n\}$, the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_n) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_n$ is a \mathfrak{g} -module isomorphism.¹⁴

Proof of Proposition 1.45. Let $\mu_k : k \otimes k \rightarrow k$ denote the multiplication map, i. e. the map which sends $\lambda(1 \otimes 1)$ to λ for every $\lambda \in k$. (This map μ_k is well-defined because every element of $k \otimes k$ can be written uniquely in the form $\lambda(1 \otimes 1)$ with $\lambda \in k$.)

We are going to prove Proposition 1.45 by induction over n :

Induction base: For $n = 0$, Proposition 1.45 states that for any 0 arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_0 and every $i \in \{0\}$, the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_0) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_0$ is a \mathfrak{g} -module isomorphism. But this is true because the k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_0) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_0$ is simply the multiplication map $\mu_k : k \otimes k \rightarrow k$ (because each of the tensor products $V_1 \otimes V_2 \otimes \dots \otimes V_i$, $V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_0$ and $V_1 \otimes V_2 \otimes \dots \otimes V_0$ is an empty tensor product (since $i = 0$ due to $i \in \{0\}$), and therefore equals k), and the multiplication map $\mu_k : k \otimes k \rightarrow k$ is a \mathfrak{g} -module isomorphism (because every $a \in \mathfrak{g}$ and $v \in k \otimes k$ satisfy $\mu_k(a \rightarrow v) = a \rightarrow \mu_k(v)$ ¹⁵). Thus, Proposition 1.45 is true for $n = 0$. This completes the induction base.

¹³In fact, if we look at Definition 1.40, we see that the k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ was defined as $V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_n)$, so it is the k -module $A \otimes B$ where $A = V_1$ and $B = V_2 \otimes V_3 \otimes \dots \otimes V_n$. Since the usual definition of a pure tensor in $A \otimes B$ defines it as an element of the form $v \otimes T$ for some $v \in A$ and $T \in B$, it thus is logical to say that a pure tensor in $V_1 \otimes V_2 \otimes \dots \otimes V_n$ means an element of the form $v \otimes T$ for $v \in V_1$ and $T \in V_2 \otimes V_3 \otimes \dots \otimes V_n$.

¹⁴Here and in the following, whenever I speak of "the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_n) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_n$ ", I mean the k -module homomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_n) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_n$ which sends $(v_1 \otimes v_2 \otimes \dots \otimes v_i) \otimes (v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n)$ to $v_1 \otimes v_2 \otimes \dots \otimes v_n$ for every $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$. This homomorphism is known (from linear algebra) to exist, be unique and be a k -module isomorphism. (This is independent of the \mathfrak{g} -module structures on V_1, V_2, \dots, V_n .)

¹⁵*Proof.* Let $a \in \mathfrak{g}$ and $v \in k \otimes k$. Then, there exists some $\lambda \in k$ such that $v = \lambda(1 \otimes 1)$ (because $(1 \otimes 1)$ is a basis of the k -module $k \otimes k$, since (1) is a basis of the k -module k). Thus,

$$\begin{aligned}
 a \rightarrow v &= a \rightarrow (\lambda(1 \otimes 1)) = \lambda(a \rightarrow (1 \otimes 1)) && \text{(since the Lie action is } k\text{-bilinear)} \\
 &= \lambda \left(\underbrace{(a \rightarrow 1)}_{=0 \text{ (since the Lie action of } k \text{ is 0)}} \otimes 1 + 1 \otimes \underbrace{(a \rightarrow 1)}_{=0 \text{ (since the Lie action of } k \text{ is 0)}} \right) \\
 &\quad \text{(by (15), applied to } V = k, W = k, v = 1 \text{ and } w = 1) \\
 &= \lambda \underbrace{(0 \otimes 1 + 1 \otimes 0)}_{=0} = 0
 \end{aligned}$$

Induction step: Let $N \in \mathbb{N}_+$ be arbitrary. Assume that Proposition 1.45 holds for $n = N - 1$. We now must prove that Proposition 1.45 holds for $n = N$. In other words, we must prove that

$$\left(\begin{array}{l} \text{for any } N \text{ arbitrary } \mathfrak{g}\text{-modules } V_1, V_2, \dots, V_N \text{ and every} \\ i \in \{0, 1, \dots, N\}, \text{ the canonical } k\text{-module isomorphism} \\ (V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_N \\ \text{is a } \mathfrak{g}\text{-module isomorphism} \end{array} \right). \quad (25)$$

We have assume that Proposition 1.45 holds for $n = N$. In other words, we have assumed that

$$\left(\begin{array}{l} \text{for any } N - 1 \text{ arbitrary } \mathfrak{g}\text{-modules } V_1, V_2, \dots, V_{N-1} \text{ and every} \\ i \in \{0, 1, \dots, N - 1\}, \text{ the canonical } k\text{-module isomorphism} \\ (V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_{N-1}) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_{N-1} \\ \text{is a } \mathfrak{g}\text{-module isomorphism} \end{array} \right). \quad (26)$$

Now let us prove (25). Let V_1, V_2, \dots, V_N be N arbitrary \mathfrak{g} -modules. Let $i \in \{0, 1, \dots, N\}$. Let ρ be the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_N$. We are now going to prove that ρ is a \mathfrak{g} -module isomorphism.

We distinguish between two cases:

Case 1: We have $i = 0$.

Case 2: We have $i > 0$.

Let us first consider Case 1: In this case, $i = 0$, so that $V_1 \otimes V_2 \otimes \dots \otimes V_i = k$ and $V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N = V_1 \otimes V_2 \otimes \dots \otimes V_N$. Hence, ρ (being the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_N$) is the canonical k -module isomorphism $k \otimes (V_1 \otimes V_2 \otimes \dots \otimes V_N) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_N$. But this canonical k -module isomorphism is a \mathfrak{g} -module isomorphism (by Corollary 1.36, applied to $V = V_1 \otimes V_2 \otimes \dots \otimes V_N$). Thus, we have proven that ρ is a \mathfrak{g} -module isomorphism in Case 1.

Let us now consider Case 2: In this case, $i > 0$, so we have $V_1 \otimes V_2 \otimes \dots \otimes V_i = V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)$ (as a \mathfrak{g} -module) by the definition of $V_1 \otimes V_2 \otimes \dots \otimes V_i$. Also, $V_1 \otimes V_2 \otimes \dots \otimes V_N = V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_N)$ (as a \mathfrak{g} -module) by the definition of $V_1 \otimes V_2 \otimes \dots \otimes V_N$.

We have denoted by ρ the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_N$. Since $V_1 \otimes V_2 \otimes \dots \otimes V_i = V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)$ and $V_1 \otimes V_2 \otimes \dots \otimes V_N = V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_N)$, this isomorphism ρ thus is the canonical k -module isomorphism $(V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_N)$.

In the following, let id denote the identity $\text{id}_{V_1} : V_1 \rightarrow V_1$.

Let us denote by ρ' the canonical k -module isomorphism

$$(V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_2 \otimes V_3 \otimes \dots \otimes V_N.$$

Then, $\text{id} \otimes \rho'$ is the canonical k -module isomorphism

$$V_1 \otimes ((V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N)) \rightarrow V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_N).$$

and thus $\mu_k(a \rightarrow v) = \mu_k(0) = 0$. Combining this with $a \rightarrow \mu_k(v) = 0$ (since the Lie action of k is 0), we obtain $\mu_k(a \rightarrow v) = a \rightarrow \mu_k(v)$, qed.

We are going to denote the \mathfrak{g} -modules $V_1, V_2 \otimes V_3 \otimes \dots \otimes V_i$ and $V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N$ by U, V and W , respectively. Then,

$$\begin{aligned} U \otimes V &= V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i) = V_1 \otimes V_2 \otimes \dots \otimes V_i && \text{and thus} \\ (U \otimes V) \otimes W &= (V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N). \end{aligned}$$

Now, denote by $\alpha_{U,V,W}$ the k -linear map

$$(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

(this map is well-defined according to linear algebra). From Proposition 1.34 (c), we know that this map $\alpha_{U,V,W}$ is an isomorphism of \mathfrak{g} -modules. Moreover, this map $\alpha_{U,V,W}$ turns out to be a map from $(V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N)$ to $V_1 \otimes ((V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N))$ (because $\alpha_{U,V,W}$ is a map from $(U \otimes V) \otimes W$ to $U \otimes (V \otimes W)$, but we know that $U = V_1, V = V_2 \otimes V_3 \otimes \dots \otimes V_i$ and $W = V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N$), and it is actually the canonical k -module isomorphism

$$\begin{aligned} &(V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \\ &\rightarrow V_1 \otimes ((V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N)) \end{aligned}$$

(because it maps

$$(v_1 \otimes (v_2 \otimes v_3 \otimes \dots \otimes v_i)) \otimes (v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_N)$$

to

$$v_1 \otimes ((v_2 \otimes v_3 \otimes \dots \otimes v_i) \otimes (v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_N))$$

for every $(v_1, v_2, \dots, v_N) \in V_1 \times V_2 \times \dots \times V_N$, according to its definition).

We now recollect our knowledge:

- The map ρ is the canonical k -module isomorphism

$$(V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_N).$$

- The map $\text{id} \otimes \rho'$ is the canonical k -module isomorphism

$$V_1 \otimes ((V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N)) \rightarrow V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_N).$$

- The map $\alpha_{U,V,W}$ is the canonical k -module isomorphism

$$\begin{aligned} &(V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \\ &\rightarrow V_1 \otimes ((V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N)). \end{aligned}$$

Hence, the diagram

$$\begin{array}{ccc} V_1 \otimes ((V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N)) & \xrightarrow{\text{id} \otimes \rho'} & V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_N) \\ \uparrow \alpha_{U,V,W} & \nearrow \rho & \\ (V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_i)) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) & & \end{array}$$

is commutative (because each of its arrows is the canonical k -module isomorphism between the respective objects, and because it is known from linear algebra that canonical isomorphisms between various bracketings of a tensor product form a commutative diagram). In other words, $\rho = (\text{id} \otimes \rho') \circ \alpha_{U,V,W}$.

Now, we know that the canonical k -module isomorphism

$$(V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_2 \otimes V_3 \otimes \dots \otimes V_N$$

is a \mathfrak{g} -module isomorphism (by (26), applied to the \mathfrak{g} -modules V_2, V_3, \dots, V_N and the number $i - 1$ instead of the \mathfrak{g} -modules V_1, V_2, \dots, V_{N-1} and the number i). Since the canonical k -module isomorphism

$$(V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_2 \otimes V_3 \otimes \dots \otimes V_N$$

is ρ' , we thus have shown that ρ' is a \mathfrak{g} -module isomorphism. Combined with the fact that $\text{id} : V_1 \rightarrow V_1$ is a \mathfrak{g} -module isomorphism, this yields that $\text{id} \otimes \rho'$ is a \mathfrak{g} -module homomorphism (by Proposition 1.38, applied to $V_1, V_1, (V_2 \otimes V_3 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N), V_2 \otimes V_3 \otimes \dots \otimes V_N, \text{id}$ and ρ' instead of V, V', W, W', f and g). On the other hand, $\text{id} \otimes \rho'$ is a k -module isomorphism (since both id and ρ' are k -module isomorphisms). Thus, $\text{id} \otimes \rho'$ is a \mathfrak{g} -module homomorphism and a k -module isomorphism at the same time. This yields that $\text{id} \otimes \rho'$ is a \mathfrak{g} -module isomorphism (by an application of Proposition 1.14). Combined with the fact that $\alpha_{U,V,W}$ is a \mathfrak{g} -module isomorphism, this yields that $(\text{id} \otimes \rho') \circ \alpha_{U,V,W}$ is a \mathfrak{g} -module isomorphism. In other words, ρ is a \mathfrak{g} -module isomorphism (since $\rho = (\text{id} \otimes \rho') \circ \alpha_{U,V,W}$).

Hence, in both Cases 1 and 2, we have shown that ρ is a canonical \mathfrak{g} -module isomorphism. So we now know that ρ is a canonical \mathfrak{g} -module isomorphism in every case.

Since ρ is the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_N$, this means that the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_N) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_N$ is a \mathfrak{g} -module isomorphism. Since we have proven this for any N arbitrary \mathfrak{g} -modules V_1, V_2, \dots, V_N and every $i \in \{0, 1, \dots, N\}$, we have thus verified (25). This completes the induction step, and thus the induction proof of Proposition 1.45 is complete.

In Definition 1.42, we defined the \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ by induction over n . It turns out that we can also easily describe the \mathfrak{g} -module structure on $V_1 \otimes V_2 \otimes \dots \otimes V_n$ explicitly. First a convention:

Convention 1.46. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V_1, V_2, \dots, V_n be n arbitrary k -modules. Let $i \in \{1, 2, \dots, n\}$, and let W_i be a k -module. Let $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$ and $w_i \in W_i$ be arbitrary. Then, we are going to denote by

$$v_1 \otimes v_2 \otimes \dots \otimes \begin{array}{|c|} \hline w_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n$$

the result of replacing the i -th tensorand (this tensorand is equal to v_i) of the tensor product $v_1 \otimes v_2 \otimes \dots \otimes v_n$ by the vector w_i . In other words, $v_1 \otimes v_2 \otimes \dots \otimes \begin{array}{|c|} \hline w_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n$ is defined by

$$v_1 \otimes v_2 \otimes \dots \otimes \begin{array}{|c|} \hline w_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n = \underbrace{v_1 \otimes v_2 \otimes \dots \otimes v_{i-1}}_{\text{tensor product of the first } i-1 \text{ vectors } v_\ell} \otimes w_i \otimes \underbrace{v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n}_{\text{tensor product of the last } n-i \text{ vectors } v_\ell}.$$

Proposition 1.47. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$. Let V_1, V_2, \dots, V_n be n arbitrary \mathfrak{g} -modules. Then, the \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 1.42 satisfies

$$a \rightharpoonup (v_1 \otimes v_2 \otimes \dots \otimes v_n) = \sum_{i=1}^n v_1 \otimes v_2 \otimes \dots \otimes \frac{a \rightharpoonup v_i}{v_i} \otimes \dots \otimes v_n$$

for every $a \in \mathfrak{g}$ and every $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$. Here, we are using Convention 1.46.

We are going to prove this later. First an obvious lemma:

Proposition 1.48. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}_+$. Let V_1, V_2, \dots, V_n be n arbitrary \mathfrak{g} -modules. Then, the \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 1.42 satisfies

$$a \rightharpoonup (v \otimes T) = (a \rightharpoonup v) \otimes T + v \otimes (a \rightharpoonup T) \quad (27)$$

for every $a \in \mathfrak{g}$, $v \in V_1$ and $T \in V_2 \otimes V_3 \otimes \dots \otimes V_n$.

Proof of Proposition 1.48. According to Definition 1.42, the \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ is defined as the tensor product $V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_n)$. Hence, applying (15) to $V = V_1$ and $W = V_2 \otimes V_3 \otimes \dots \otimes V_n$, we see that

$$a \rightharpoonup (v \otimes w) = (a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w)$$

for every $a \in \mathfrak{g}$, $v \in V_1$ and $w \in V_2 \otimes V_3 \otimes \dots \otimes V_n$

(because the tensor product $V \otimes W$ of two \mathfrak{g} -modules V and W always satisfies (15)). If we rename w as T , this rewrites as follows:

$$a \rightharpoonup (v \otimes T) = (a \rightharpoonup v) \otimes T + v \otimes (a \rightharpoonup T)$$

for every $a \in \mathfrak{g}$, $v \in V_1$ and $T \in V_2 \otimes V_3 \otimes \dots \otimes V_n$.

This proves Proposition 1.48.

Proof of Proposition 1.47. Let $a \in \mathfrak{g}$ and $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$ be arbitrary. We are now going to show that

$$a \rightharpoonup (v_{n-p+1} \otimes v_{n-p+2} \otimes \dots \otimes v_n) = \sum_{i=n-p+1}^n v_{n-p+1} \otimes v_{n-p+2} \otimes \dots \otimes \frac{a \rightharpoonup v_i}{v_i} \otimes \dots \otimes v_n \quad (28)$$

for every $p \in \{0, 1, \dots, n\}$. (The equation (28) is to be understood as an equation between two elements of $V_{n-p+1} \otimes V_{n-p+2} \otimes \dots \otimes V_n$.)

Proof of (28). In fact, we are going to prove (28) by induction over p :

Induction base: For $p = 0$, the equation (28) holds¹⁶. This completes the induction base.

¹⁶This is because for $p = 0$, we have

$$a \rightharpoonup \underbrace{(v_{n-p+1} \otimes v_{n-p+2} \otimes \dots \otimes v_n)}_{=(\text{empty tensor product})=1} = a \rightharpoonup 1 = 0 \quad (\text{since the Lie action of } k \text{ is } 0)$$

Induction step: Let $q \in \{0, 1, \dots, n-1\}$ be arbitrary. Assume that (28) holds for $p = q$. Now we must show that (28) holds for $p = q + 1$.

We have assumed that (28) holds for $p = q$; in other words,

$$a \rightharpoonup (v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes v_n) = \sum_{i=n-q+1}^n v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes \begin{array}{|c|} \hline a \rightharpoonup v_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n. \quad (29)$$

Applying Proposition 1.48 to the number $q + 1$ and the \mathfrak{g} -modules $V_{n-q}, V_{n-q+1}, \dots, V_n$ instead of the number n and the \mathfrak{g} -modules V_1, V_2, \dots, V_n , we obtain

$$a \rightharpoonup (v \otimes T) = (a \rightharpoonup v) \otimes T + v \otimes (a \rightharpoonup T)$$

for every $a \in \mathfrak{g}$, $v \in V_{n-q}$ and $T \in V_{n-q+1} \otimes V_{n-q+2} \otimes \dots \otimes V_n$. Applying this to $v = v_{n-q}$

and

$$\sum_{i=n-p+1}^n v_{n-p+1} \otimes v_{n-p+2} \otimes \dots \otimes \begin{array}{|c|} \hline a \rightharpoonup v_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n = (\text{empty sum}) = 0.$$

and $T = v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes v_n$, we obtain

$$\begin{aligned}
& a \multimap (v_{n-q} \otimes v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes v_n) \\
= & \underbrace{(a \multimap v_{n-q}) \otimes v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes v_n}_{=v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes} \frac{a \multimap v_{n-q}}{v_{n-q}} \otimes \dots \otimes v_n + v_{n-q} \otimes \underbrace{(a \multimap (v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes v_n))}_{= \sum_{i=n-q+1}^n v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n} \\
& \hspace{15em} \text{(due to (29))} \\
= & v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes \frac{a \multimap v_{n-q}}{v_{n-q}} \otimes \dots \otimes v_n \\
& + v_{n-q} \otimes \underbrace{\sum_{i=n-q+1}^n v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n}_{= \sum_{i=n-q+1}^n v_{n-q} \otimes (v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n)} \\
= & v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes \frac{a \multimap v_{n-q}}{v_{n-q}} \otimes \dots \otimes v_n \\
& + \sum_{i=n-q}^{n-q} v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n \\
& + \sum_{i=n-q+1}^n v_{n-q} \otimes \underbrace{(v_{n-q+1} \otimes v_{n-q+2} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n)}_{=v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n} \\
= & \sum_{i=n-q}^{n-q} v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n \\
& + \sum_{i=n-q+1}^n v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n \\
= & \sum_{i=n-q}^n v_{n-q} \otimes v_{n-q+1} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n \\
= & \sum_{i=n-(q+1)+1}^n v_{n-(q+1)+1} \otimes v_{n-(q+1)+2} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n \\
& \text{(since } n-q = n-(q+1)+1 \text{ and } n-q+1 = n-(q+1)+2 \text{).}
\end{aligned}$$

In other words, (28) holds for $p = q + 1$. This completes the induction step. Thus, the induction proof of (28) is complete.

Now that we have proven (28) for every $p \in \{0, 1, \dots, n\}$, we can apply (28) to $p = n$, and obtain

$$a \multimap (v_{n-n+1} \otimes v_{n-n+2} \otimes \dots \otimes v_n) = \sum_{i=n-n+1}^n v_{n-n+1} \otimes v_{n-n+2} \otimes \dots \otimes \frac{a \multimap v_i}{v_i} \otimes \dots \otimes v_n.$$

Since $n - n + 1 = 1$ and $n - n + 2 = 2$, this simplifies to

$$a \mapsto (v_1 \otimes v_2 \otimes \dots \otimes v_n) = \sum_{i=1}^n v_1 \otimes v_2 \otimes \dots \otimes \begin{array}{|c|} \hline a \mapsto v_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n.$$

This proves Proposition 1.47.

Finally we record the multi-factor version of Proposition 1.38:

Proposition 1.49. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$. Let V_1, V_2, \dots, V_n be n arbitrary \mathfrak{g} -modules. Let V'_1, V'_2, \dots, V'_n be n arbitrary \mathfrak{g} -modules. Let $f_i : V_i \rightarrow V'_i$ be a \mathfrak{g} -module homomorphism for every $i \in \{1, 2, \dots, n\}$. Then, $f_1 \otimes f_2 \otimes \dots \otimes f_n : V_1 \otimes V_2 \otimes \dots \otimes V_n \rightarrow V'_1 \otimes V'_2 \otimes \dots \otimes V'_n$ is a \mathfrak{g} -module homomorphism.

This proposition follows from Proposition 1.38 by induction (the details being left to the reader).

We also notice that Proposition 1.39 has a multi-factor version:

Proposition 1.50. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. If V_1, V_2, \dots, V_n are n arbitrary \mathfrak{g} -modules, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2 \otimes \dots \otimes V_n) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_1) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_2) \otimes \dots \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_n)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2 \otimes \dots \otimes V_n)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_1) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_2) \otimes \dots \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_n)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2 \otimes \dots \otimes V_n) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_1) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_2) \otimes \dots \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_n)$).

1.10. Tensor powers of \mathfrak{g} -modules

Next we define a particular case of tensor products of \mathfrak{g} -modules, namely the tensor powers. First we recall the classical definition of the tensor powers of a k -module:

Definition 1.51. Let k be a commutative ring. Let $n \in \mathbb{N}$. For any k -module V , we define a k -module $V^{\otimes n}$ by $V^{\otimes n} = \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}$. This k -module $V^{\otimes n}$ is called the n -th tensor power of the k -module V .

Remark 1.52. Let k be a commutative ring, and let V be a k -module. Then, $V^{\otimes 0} = k$ (because $V^{\otimes 0} = \underbrace{V \otimes V \otimes \dots \otimes V}_{0 \text{ times}} =$ (tensor product of zero k -modules) $= k$ according to the induction base of Definition 1.40) and $V^{\otimes 1} = V \otimes k$ (because $V^{\otimes 1} = \underbrace{V \otimes V \otimes \dots \otimes V}_{1 \text{ times}} = V \otimes k$ according to the induction step of Definition 1.40).

Since we identify $V \otimes k$ with V , we thus have $V^{\otimes 1} = V$.

Now, by analogy, we define the tensor powers of a \mathfrak{g} -module:

Definition 1.53. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$. For any \mathfrak{g} -module V , we define a \mathfrak{g} -module $V^{\otimes n}$ by $V^{\otimes n} = \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}$. This \mathfrak{g} -module $V^{\otimes n}$ is called the n -th tensor power of the \mathfrak{g} -module V .

Remark 1.54. Let k be a commutative ring, let \mathfrak{g} be a k -Lie algebra, and let V be a \mathfrak{g} -module. Then, $V^{\otimes 0} = k$ (as \mathfrak{g} -modules) and $V^{\otimes 1} = V$ (as \mathfrak{g} -modules), where we identify the \mathfrak{g} -module $V \otimes k$ with V . This is proven the same way as Remark 1.52.

As a consequence of Proposition 1.49, we now have:

Proposition 1.55. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$. Let V and V' be \mathfrak{g} -modules, and let $f : V \rightarrow V'$ be a \mathfrak{g} -module homomorphism. Then, $f^{\otimes n} : V^{\otimes n} \rightarrow V'^{\otimes n}$ is a \mathfrak{g} -module homomorphism.

Here we are using the following convention:

Convention 1.56. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V and V' be k -modules, and let $f : V \rightarrow V'$ be a k -module homomorphism. Then, $f^{\otimes n}$ denotes the k -module homomorphism $\underbrace{f \otimes f \otimes \dots \otimes f}_{n \text{ times}} : \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}} \rightarrow \underbrace{V' \otimes V' \otimes \dots \otimes V'}_{n \text{ times}}$. Since $\underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}} = V^{\otimes n}$ and $\underbrace{V' \otimes V' \otimes \dots \otimes V'}_{n \text{ times}} = V'^{\otimes n}$, this $f^{\otimes n}$ is thus a k -module homomorphism from $V^{\otimes n}$ to $V'^{\otimes n}$.

We notice the following:

Proposition 1.57. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$. Then, for any \mathfrak{g} -module V and every $i \in \{0, 1, \dots, n\}$, the canonical k -module isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$ is a \mathfrak{g} -module isomorphism.¹⁷

This proposition follows directly from applying Proposition 1.45 to V, V, \dots, V instead of V_1, V_2, \dots, V_n .

Convention 1.58. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. For every \mathfrak{g} -module V , every $n \in \mathbb{N}$ and every $i \in \{0, 1, \dots, n\}$, we are going to identify the \mathfrak{g} -module $V^{\otimes i} \otimes V^{\otimes(n-i)}$ with the \mathfrak{g} -module $V^{\otimes n}$ (this is allowed because of Proposition 1.57). In other words, for every \mathfrak{g} -module V , every $a \in \mathbb{N}$ and every $b \in \mathbb{N}$, we are going to identify the \mathfrak{g} -module $V^{\otimes a} \otimes V^{\otimes b}$ with the \mathfrak{g} -module $V^{\otimes(a+b)}$.

Actually, there is no reason to restrict ourselves to \mathfrak{g} -modules in this Convention 1.58; we can do the convention same for k -modules:

¹⁷Here and in the following, whenever I speak of "the canonical k -module isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$ ", I mean the k -module homomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$ which sends $(v_1 \otimes v_2 \otimes \dots \otimes v_i) \otimes (v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n)$ to $v_1 \otimes v_2 \otimes \dots \otimes v_n$ for every $(v_1, v_2, \dots, v_n) \in V^n$. This homomorphism is known (from linear algebra) to exist, be unique and be a k -module isomorphism. It is actually the canonical k -module isomorphism $\underbrace{V \otimes V \otimes \dots \otimes V}_{i \text{ times}} \otimes \underbrace{V \otimes V \otimes \dots \otimes V}_{n-i \text{ times}} \rightarrow \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}$.

Convention 1.59. Let k be a commutative ring. For every k -module V , every $n \in \mathbb{N}$ and every $i \in \{0, 1, \dots, n\}$, we are going to identify the k -module $V^{\otimes i} \otimes V^{\otimes(n-i)}$ with the k -module $V^{\otimes n}$ (this is allowed because of the canonical isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \cong V^{\otimes n}$). In other words, for every k -module V , every $a \in \mathbb{N}$ and every $b \in \mathbb{N}$, we are going to identify the k -module $V^{\otimes a} \otimes V^{\otimes b}$ with the k -module $V^{\otimes(a+b)}$.

Needless to say, Convention 1.59 does not conflict with Convention 1.58, because if V is a \mathfrak{g} -module, then both of these conventions identify $V^{\otimes i} \otimes V^{\otimes(n-i)}$ with $V^{\otimes n}$ by means of the same isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \cong V^{\otimes n}$.

Finally, Proposition 1.50 yields:

Proposition 1.60. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$.

If V is any \mathfrak{g} -module, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V^{\otimes n}) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}V)^{\otimes n}$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V^{\otimes n}$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V^{\otimes n})$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}V)^{\otimes n}$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V^{\otimes n}) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}V)^{\otimes n}$).

1.11. Tensor algebra and universal enveloping algebra

The tensor powers $V^{\otimes n}$ of a k -module V can be combined to a k -module $\otimes V$ which turns out to have an algebra structure: that of the so-called tensor algebra. Let us recall its definition (which can easily be shown to be well-defined):

Definition 1.61. Let k be a commutative ring.

(a) Let V be a k -module. The *tensor algebra* $\otimes V$ of V over k is defined to be the k -algebra formed by the k -module $\bigoplus_{i \in \mathbb{N}} V^{\otimes i} = V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \dots$ equipped with a multiplication which is defined by

$$(a_i)_{i \in \mathbb{N}} \cdot (b_i)_{i \in \mathbb{N}} = \left(\sum_{i=0}^n a_i \otimes b_{n-i} \right)_{n \in \mathbb{N}} \quad \text{for every } (a_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i} \text{ and } (b_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i} \quad (30)$$

(where for every $n \in \mathbb{N}$ and every $i \in \{0, 1, \dots, n\}$, the tensor $a_i \otimes b_{n-i} \in V^{\otimes i} \otimes V^{\otimes(n-i)}$ is considered as an element of $V^{\otimes n}$ due to the canonical identification $V^{\otimes i} \otimes V^{\otimes(n-i)} \cong V^{\otimes n}$ which was defined in Convention 1.59).

The k -module $\otimes V$ itself (without the k -algebra structure) is called the *tensor module* of V .

(b) Let V and W be two k -modules, and let $f : V \rightarrow W$ be a k -module homomorphism. The k -module homomorphisms $f^{\otimes i} : V^{\otimes i} \rightarrow W^{\otimes i}$ for all $i \in \mathbb{N}$ can be combined together to a k -module homomorphism from $V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \dots$ to $W^{\otimes 0} \oplus W^{\otimes 1} \oplus W^{\otimes 2} \oplus \dots$. This homomorphism is called $\otimes f$. Since $V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \dots = \otimes V$ and $W^{\otimes 0} \oplus W^{\otimes 1} \oplus W^{\otimes 2} \oplus \dots = \otimes W$, we see that this homomorphism $\otimes f$ is a k -module homomorphism from $\otimes V$ to $\otimes W$. Moreover, it follows easily from (30) that this $\otimes f$ is actually a k -algebra homomorphism from $\otimes V$ to $\otimes W$.

(c) Let V be a k -module. Then, according to Convention 1.25, we consider $V^{\otimes n}$ as a k -submodule of the direct sum $\bigoplus_{i \in \mathbb{N}} V^{\otimes i} = \otimes V$ for every $n \in \mathbb{N}$. In particular, every

element of k is considered to be an element of $\otimes V$ by means of the canonical embedding $k = V^{\otimes 0} \subseteq \otimes V$, and every element of V is considered to be an element of $\otimes V$ by means of the canonical embedding $V = V^{\otimes 1} \subseteq \otimes V$. The element $1 \in k \subseteq \otimes V$ is easily seen to be the unity of the tensor algebra $\otimes V$.

Remark 1.62. The formula (30) (which defines the multiplication on the tensor algebra $\otimes V$) is often put in words by saying that "the multiplication in the tensor algebra $\otimes V$ is given by the tensor product". This informal statement tempts many authors (including myself in [35]) to use the sign \otimes for multiplication in the algebra $\otimes V$, that is, to write $u \otimes v$ for the product of any two elements u and v of the tensor algebra $\otimes V$. This notation, however, can collide with the notation $u \otimes v$ for the tensor product of two vectors u and v in a k -module.¹⁸ Due to this possibility of collision, we are *not* going to use the sign \otimes for multiplication in the algebra $\otimes V$ in this paper. Instead we will use the sign \cdot for this multiplication. However, due to (30), we still have

$$(a \cdot b = a \otimes b \quad \text{for any } n \in \mathbb{N}, \text{ any } m \in \mathbb{N}, \text{ any } a \in V^{\otimes n} \text{ and any } b \in V^{\otimes m}), \quad (31)$$

where $a \otimes b$ is considered to be an element of $V^{\otimes(n+m)}$ by means of the identification of $V^{\otimes n} \otimes V^{\otimes m}$ with $V^{\otimes(n+m)}$.

Next we define the universal enveloping algebra of a Lie algebra \mathfrak{g} . First, a convention about the notation we are using:

Convention 1.63. Let k be a commutative ring. Let U be a k -algebra.

(a) If A and B are two k -submodules of U , then we denote by $A \cdot B$ the k -submodule $\langle ab \mid (a, b) \in A \times B \rangle$ of U . We also will sometimes abbreviate $A \cdot B$ by AB .

(b) If A , B and C are three k -submodules of U , then it is easily seen that $(AB)C = A(BC)$, so that we can denote each of the two equal k -submodules $(AB)C$ and $A(BC)$ by ABC .

Definition 1.64. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. We define the *universal enveloping algebra* $U(\mathfrak{g})$ to be the factor algebra $(\otimes \mathfrak{g}) / I_{\mathfrak{g}}$, where $I_{\mathfrak{g}}$ is the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle \cdot (\otimes \mathfrak{g})$$

of the algebra $\otimes \mathfrak{g}$.

¹⁸For example, if z is a vector in the k -module V , then we can define two elements u and v of $\otimes V$ by $u = 1 + z$ and $v = 1 - z$ (where 1 and z are considered to be elements of $\otimes V$ according to Definition 1.61 (c)), and while the product of these elements u and v in $\otimes V$ is the element $(1 + z) \cdot (1 - z) = 1 \cdot 1 - 1 \cdot z + 1 \cdot z - z \otimes z = 1 - z \otimes z \in \otimes V$, the tensor product of these elements u and v is the element $(1 + z) \otimes (1 - z)$ of $(k \oplus V) \otimes (k \oplus V) \cong k \oplus V \oplus V \oplus (V \otimes V)$, which is a different element of a totally different k -module. So if we would use one and the same notation $u \otimes v$ for both the product of u and v in $\otimes V$ and the tensor product of u and v in $(k \oplus V) \otimes (k \oplus V)$, we would have ambiguous notations.

Remark 1.65. In Definition 1.64, the term $\langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle$ is to be understood according to Convention 1.28, and the multiplication sign \cdot in

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle \cdot (\otimes \mathfrak{g})$$

is to be understood according to Convention 1.63. We note that, although the multiplication in $\otimes \mathfrak{g}$ is related to the tensor product by (31), the product $(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle \cdot (\otimes \mathfrak{g})$ has nothing to do with the tensor product $(\otimes \mathfrak{g}) \otimes \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle \otimes (\otimes \mathfrak{g})$!

But we are not going to linger over universal enveloping algebras now. Let us introduce the canonical \mathfrak{g} -module structure on $\otimes V$ for any \mathfrak{g} -module V :

Definition 1.66. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Since $V^{\otimes i}$ is a \mathfrak{g} -module for all $i \in \mathbb{N}$ (by Definition 1.53), the direct sum $\bigoplus_{i \in \mathbb{N}} V^{\otimes i}$ is also a \mathfrak{g} -module (by Definition 1.24). In other words, the tensor algebra $\otimes V$ thus becomes a \mathfrak{g} -module (since $\otimes V = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$). This \mathfrak{g} -module $\otimes V$ is called the *tensor \mathfrak{g} -module* of the \mathfrak{g} -module V .

Whenever we will speak of the \mathfrak{g} -module $\otimes V$, we will be meaning this \mathfrak{g} -module (although there might be many different \mathfrak{g} -module structures on the k -module $\otimes V$).

Remark 1.67. We will often consider the \mathfrak{g} -module $\otimes \mathfrak{g}$. This is the tensor \mathfrak{g} -module of the \mathfrak{g} -module \mathfrak{g} itself. In other words, it is the result of applying Definition 1.66 to $V = \mathfrak{g}$. A look at Proposition 1.47 quickly yields that it satisfies

$$a \rightarrow (v_1 \otimes v_2 \otimes \dots \otimes v_n) = \sum_{i=1}^n v_1 \otimes v_2 \otimes \dots \otimes \frac{[a, v_i]}{v_i} \otimes \dots \otimes v_n$$

for all $n \in \mathbb{N}$, $a \in \mathfrak{g}$ and $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$.

We can show that the ideal $I_{\mathfrak{g}}$ constructed in Definition 1.64 is a \mathfrak{g} -submodule of this \mathfrak{g} -module $\otimes \mathfrak{g}$. (In fact, this follows from Proposition 2.3 (a), applied to $\mathfrak{h} = \mathfrak{g}$.) Thus, $(\otimes \mathfrak{g}) / I_{\mathfrak{g}}$ becomes a \mathfrak{g} -module. Since $(\otimes \mathfrak{g}) / I_{\mathfrak{g}} = U(\mathfrak{g})$, this yields that $U(\mathfrak{g})$ becomes a \mathfrak{g} -module. This \mathfrak{g} -module satisfies

$$a \rightarrow \overline{(v_1 \otimes v_2 \otimes \dots \otimes v_n)} = \sum_{i=1}^n \overline{v_1 \otimes v_2 \otimes \dots \otimes \frac{[a, v_i]}{v_i} \otimes \dots \otimes v_n}$$

for all $n \in \mathbb{N}$, $a \in \mathfrak{g}$ and $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$

(where \overline{T} means the residue class of T modulo $I_{\mathfrak{g}}$ for every $T \in \otimes \mathfrak{g}$). However, this \mathfrak{g} -module structure is not the only interesting \mathfrak{g} -module structure on $U(\mathfrak{g})$. There is a different one, which satisfies

$$a \rightarrow \overline{(v_1 \otimes v_2 \otimes \dots \otimes v_n)} = \overline{a \otimes v_1 \otimes v_2 \otimes \dots \otimes v_n}$$

for all $n \in \mathbb{N}$, $a \in \mathfrak{g}$ and $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$. (32)

It is important not to confuse these two \mathfrak{g} -module structures, as they define two *different* \mathfrak{g} -modules $U(\mathfrak{g})$ (although, as k -modules, they are the same)!

Whenever we will speak of the \mathfrak{g} -module $U(\mathfrak{g})$ in the following, we will mean the \mathfrak{g} -module structure which is obtained by applying Definition 1.66 to $V = \mathfrak{g}$ and setting $U(\mathfrak{g}) = (\otimes \mathfrak{g}) / I_{\mathfrak{g}}$, not the \mathfrak{g} -module structure given by (32)!

It is very easy to see that:

Proposition 1.68. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V and W be two \mathfrak{g} -modules, and let $f : V \rightarrow W$ be a \mathfrak{g} -module homomorphism. Then, $\otimes f : \otimes V \rightarrow \otimes W$ is a \mathfrak{g} -module homomorphism.

Definition 1.69. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Then, according to Proposition 1.26, we consider $V^{\otimes n}$ as a \mathfrak{g} -submodule of the direct sum $\bigoplus_{i \in \mathbb{N}} V^{\otimes i} = \otimes V$ for every $n \in \mathbb{N}$. In particular, $k = V^{\otimes 0}$ and $V = V^{\otimes 1}$ become \mathfrak{g} -submodules of $\otimes V$ this way.

1.12. \mathfrak{g} -algebras

The next notion we introduce is that of a \mathfrak{g} -algebra (also called a \mathfrak{g} -module algebra or \mathfrak{g} -Lie module algebra):

Definition 1.70. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. A \mathfrak{g} -algebra will mean a k -algebra A equipped with a \mathfrak{g} -module structure such that

$$(a \rightarrow uv) = (a \rightarrow u) \cdot v + u \cdot (a \rightarrow v) \text{ for every } a \in \mathfrak{g}, u \in A \text{ and } v \in A. \quad (33)$$

Remark 1.71. In Definition 1.70, when we speak of "a k -algebra A equipped with a \mathfrak{g} -module structure", the words "a \mathfrak{g} -module structure" mean "a \mathfrak{g} -module structure on the underlying k -module of the k -algebra A ". This \mathfrak{g} -module structure must therefore be k -bilinear with respect to the underlying k -module structure of the k -algebra A .

Remark 1.72. Definition 1.70 is often rewritten as follows:

Definition 1.73. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. A \mathfrak{g} -algebra will mean a k -algebra A equipped with a \mathfrak{g} -module structure such that \mathfrak{g} acts on A by means of derivations. Here, we say that " \mathfrak{g} acts on A by means of derivations" if and only if the map $A \rightarrow A, u \mapsto (a \rightarrow u)$ is a derivation for every $a \in \mathfrak{g}$.

This Definition 1.73 is indeed equivalent to Definition 1.70 because the condition that \mathfrak{g} acts on A by means of derivations is equivalent to (33) (as can be easily seen).

It is easy to see that, when \mathfrak{h} is a Lie subalgebra of a k -Lie algebra \mathfrak{g} , every \mathfrak{g} -algebra canonically can be made into an \mathfrak{h} -algebra:

Definition 1.74. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Then, every \mathfrak{g} -algebra A canonically becomes an \mathfrak{h} -algebra. (In fact, the \mathfrak{g} -module A canonically becomes an \mathfrak{h} -module according to Definition 1.15, and thus A is a k -algebra equipped with an \mathfrak{h} -module structure which satisfies (33) with \mathfrak{g} replaced by \mathfrak{h} , so that A thus is an \mathfrak{h} -algebra.). This \mathfrak{h} -algebra is called the *restriction of A to \mathfrak{h}* , and denoted by $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} A$. However, when there is no possibility of confusion, we will denote this \mathfrak{h} -algebra by A , and we will distinguish it from the original \mathfrak{g} -algebra A by means of referring to the former one as "the \mathfrak{h} -algebra A " and referring to the latter one as "the \mathfrak{g} -algebra A ".

A basic property of \mathfrak{g} -algebras:

Proposition 1.75. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let A be a \mathfrak{g} -algebra. Let P and Q be two \mathfrak{g} -submodules of A . Then, $P \cdot Q$ is a \mathfrak{g} -submodule of A .

Remark 1.76. Here, $P \cdot Q$ is to be understood as according to Convention 1.63 (a).

Proof of Proposition 1.75. According to Convention 1.63 (a), the k -submodule $P \cdot Q$ of A is defined by $P \cdot Q = \langle pq \mid (p, q) \in P \times Q \rangle$.

Let $a \in \mathfrak{g}$ be arbitrary.

Now, let $(u, v) \in P \times Q$ be arbitrary. Then, $u \in P$ and $v \in Q$. Since P is a \mathfrak{g} -submodule of A , we have $a \rightarrow u \in P$ (since $u \in P$). Since Q is a \mathfrak{g} -submodule of A , we have $a \rightarrow v \in Q$ (since $v \in Q$). Now, (33) shows us that

$$\begin{aligned} a \rightarrow (uv) &= \underbrace{(a \rightarrow u)}_{\in P} \cdot \underbrace{v}_{\in Q} + \underbrace{u}_{\in P} \cdot \underbrace{(a \rightarrow v)}_{\in Q} \in P \cdot Q + P \cdot Q \\ &\subseteq P \cdot Q \quad (\text{since } P \cdot Q \text{ is a } k\text{-module}). \end{aligned}$$

We have thus proven that

$$a \rightarrow (uv) \in P \cdot Q \text{ for every } (u, v) \in P \times Q. \quad (34)$$

From this it is easily seen that $a \rightarrow t \in P \cdot Q$ for every $t \in P \cdot Q$.¹⁹ Since this holds for every $a \in \mathfrak{g}$, we can conclude that $P \cdot Q$ is a \mathfrak{g} -submodule of A . This proves Proposition 1.75.

¹⁹*Proof.* In fact, let $t \in P \cdot Q$ be arbitrary. Then, $t \in P \cdot Q = \langle pq \mid (p, q) \in P \times Q \rangle$. Hence, t is a k -linear combination of terms of the form pq for various $(p, q) \in P \times Q$. In other words, there exist some $n \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_n$ of k , some elements p_1, p_2, \dots, p_n of P and some elements q_1, q_2, \dots, q_n of Q such that $t = \sum_{i=1}^n \lambda_i p_i q_i$. Hence,

$$\begin{aligned} a \rightarrow t &= a \rightarrow \left(\sum_{i=1}^n \lambda_i p_i q_i \right) = \sum_{i=1}^n \lambda_i \underbrace{a \rightarrow (p_i q_i)}_{\in P \cdot Q} \\ &\quad \text{(this follows from applying (34) to } p_i \text{ and } q_i \text{ instead of } u \text{ and } v) \\ &\quad \text{(since the Lie action of } A \text{ is } k\text{-bilinear)} \\ &\in \sum_{i=1}^n \lambda_i P \cdot Q \subseteq P \cdot Q \quad (\text{since } P \cdot Q \text{ is a } k\text{-module}), \end{aligned}$$

qed.

1.13. $\otimes V$ is a \mathfrak{g} -algebra

Now the prime example of a \mathfrak{g} -algebra is exactly what we would expect:

Proposition 1.77. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

Let V be a \mathfrak{g} -module. If we equip the k -algebra $\otimes V$ (this k -algebra was defined in Definition 1.61 (a)) with the \mathfrak{g} -module structure defined in Definition 1.66, we obtain a \mathfrak{g} -algebra.

Definition 1.78. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

Let V be a \mathfrak{g} -module. The \mathfrak{g} -algebra $\otimes V$ defined in Proposition 1.77 is called the *tensor \mathfrak{g} -algebra* of the \mathfrak{g} -module V . Whenever we will speak of the \mathfrak{g} -algebra $\otimes V$, we will be meaning this tensor \mathfrak{g} -algebra $\otimes V$ (unless we explicitly say that we are talking about a different \mathfrak{g} -algebra structure on $\otimes V$).

Remark 1.79. By far the most important particular case of Proposition 1.77 is the following one: Whenever k is a commutative ring and \mathfrak{g} is a k -Lie algebra, the tensor algebra $\otimes \mathfrak{g}$ of the \mathfrak{g} -module \mathfrak{g} is a \mathfrak{g} -algebra. This is a consequence of Proposition 1.77 (applied to $V = \mathfrak{g}$). We are going to use this several times.

Proof of Proposition 1.77. In the following proof, we are *not* going to identify $V^{\otimes i} \otimes V^{\otimes(n-i)}$ with $V^{\otimes n}$ (for $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, n\}$) as we did in Definition 1.61.²⁰ As a consequence, the equation (30) must be rewritten as

$$\left(\begin{array}{l} (a_i)_{i \in \mathbb{N}} \cdot (b_i)_{i \in \mathbb{N}} = \left(\sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes b_{n-i}) \right)_{n \in \mathbb{N}} \\ \text{for every } (a_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i} \text{ and } (b_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i} \end{array} \right), \quad (35)$$

where $\kappa_{i,n-i}$ denotes the canonical k -module isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$.

Proposition 1.57 states that the canonical k -module isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$ is a \mathfrak{g} -module isomorphism. Since we denoted this isomorphism by $\kappa_{i,n-i}$, we can thus conclude that $\kappa_{i,n-i}$ is a \mathfrak{g} -module isomorphism.

Now, let $u \in \otimes V$ and $v \in \otimes V$ be any two elements. Let $a \in \mathfrak{g}$ be arbitrary. Since $u \in \otimes V = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$, we can write u as a family $(a_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$. Since $v \in \otimes V = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$, we can write v as a family $(b_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$. Thus, $u = (a_i)_{i \in \mathbb{N}}$ and $v = (b_i)_{i \in \mathbb{N}}$, so that

$$uv = (a_i)_{i \in \mathbb{N}} \cdot (b_i)_{i \in \mathbb{N}} = \left(\sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes b_{n-i}) \right)_{n \in \mathbb{N}}$$

(by (35)), and therefore

$$a \rightharpoonup (uv) = a \rightharpoonup \left(\sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes b_{n-i}) \right)_{n \in \mathbb{N}} = \left(a \rightharpoonup \left(\sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes b_{n-i}) \right) \right)_{n \in \mathbb{N}}$$

²⁰The purpose of this is to make the proof more transparent. In fact, identifying $V^{\otimes i} \otimes V^{\otimes(n-i)}$ with $V^{\otimes n}$ would obscure the place where we actually use Proposition 1.57.

(by (13)). Since every $n \in \mathbb{N}$ satisfies

$$\begin{aligned}
a \rightharpoonup \left(\sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes b_{n-i}) \right) &= \sum_{i=0}^n \underbrace{a \rightharpoonup (\kappa_{i,n-i} (a_i \otimes b_{n-i}))}_{\substack{=\kappa_{i,n-i}(a \rightharpoonup (a_i \otimes b_{n-i})) \\ \text{(since } \kappa_{i,n-i} \text{ is a } \mathfrak{g}\text{-module} \\ \text{isomorphism)}}} & \left(\begin{array}{l} \text{since the Lie action of} \\ \otimes V \text{ is } k\text{-bilinear} \end{array} \right) \\
&= \sum_{i=0}^n \kappa_{i,n-i} \left(\begin{array}{c} a \rightharpoonup (a_i \otimes b_{n-i}) \\ = (a \rightharpoonup a_i) \otimes b_{n-i} + a_i \otimes (a \rightharpoonup b_{n-i}) \\ \text{(by (15), applied to } V^{\otimes i}, V^{\otimes(n-i)}, a_i \text{ and } b_{n-i} \\ \text{instead of } V, W, v \text{ and } w) \end{array} \right) \\
&= \sum_{i=0}^n \kappa_{i,n-i} \left(\underbrace{(a \rightharpoonup a_i) \otimes b_{n-i} + a_i \otimes (a \rightharpoonup b_{n-i})}_{\substack{=\kappa_{i,n-i}((a \rightharpoonup a_i) \otimes b_{n-i}) + \kappa_{i,n-i}(a_i \otimes (a \rightharpoonup b_{n-i})) \\ \text{(since } \kappa_{i,n-i} \text{ is linear)}}} \right) \\
&= \sum_{i=0}^n (\kappa_{i,n-i} ((a \rightharpoonup a_i) \otimes b_{n-i}) + \kappa_{i,n-i} (a_i \otimes (a \rightharpoonup b_{n-i}))) \\
&= \sum_{i=0}^n \kappa_{i,n-i} ((a \rightharpoonup a_i) \otimes b_{n-i}) + \sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes (a \rightharpoonup b_{n-i})),
\end{aligned}$$

this rewrites as

$$\begin{aligned}
a \rightharpoonup (uv) &= \left(\begin{array}{c} a \rightharpoonup \left(\sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes b_{n-i}) \right) \\ = \sum_{i=0}^n \kappa_{i,n-i} ((a \rightharpoonup a_i) \otimes b_{n-i}) + \sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes (a \rightharpoonup b_{n-i})) \end{array} \right)_{n \in \mathbb{N}} \\
&= \left(\sum_{i=0}^n \kappa_{i,n-i} ((a \rightharpoonup a_i) \otimes b_{n-i}) + \sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes (a \rightharpoonup b_{n-i})) \right)_{n \in \mathbb{N}} \\
&= \left(\sum_{i=0}^n \kappa_{i,n-i} ((a \rightharpoonup a_i) \otimes b_{n-i}) \right)_{n \in \mathbb{N}} + \left(\sum_{i=0}^n \kappa_{i,n-i} (a_i \otimes (a \rightharpoonup b_{n-i})) \right)_{n \in \mathbb{N}}. \tag{36}
\end{aligned}$$

We have $u = (a_i)_{i \in \mathbb{N}}$ and thus $a \rightharpoonup u = a \rightharpoonup (a_i)_{i \in \mathbb{N}} = (a \rightharpoonup a_i)_{i \in \mathbb{N}}$ (by (13)). Together with $v = (b_i)_{i \in \mathbb{N}}$, this yields

$$(a \rightharpoonup u) \cdot v = (a \rightharpoonup a_i)_{i \in \mathbb{N}} \cdot (b_i)_{i \in \mathbb{N}} = \left(\sum_{i=0}^n \kappa_{i,n-i} ((a \rightharpoonup a_i) \otimes b_{n-i}) \right)_{n \in \mathbb{N}} \tag{37}$$

(by (35), applied to $(a \rightharpoonup a_i)_{i \in \mathbb{N}}$ instead of $(a_i)_{i \in \mathbb{N}}$).

On the other hand, $v = (b_i)_{i \in \mathbb{N}}$ and thus $a \rightharpoonup v = a \rightharpoonup (b_i)_{i \in \mathbb{N}} = (a \rightharpoonup b_i)_{i \in \mathbb{N}}$ (by

(13)). Together with $u = (a_i)_{i \in \mathbb{N}}$, this yields

$$u \cdot (a \rightharpoonup v) = (a_i)_{i \in \mathbb{N}} \cdot (a \rightharpoonup b_i)_{i \in \mathbb{N}} = \left(\sum_{i=0}^n \kappa_{i, n-i} (a_i \otimes (a \rightharpoonup b_{n-i})) \right)_{n \in \mathbb{N}}$$

(by (35), applied to $(a \rightharpoonup b_i)_{i \in \mathbb{N}}$ instead of $(b_i)_{i \in \mathbb{N}}$). Adding this equation to (37), we obtain

$$(a \rightharpoonup u) \cdot v + u \cdot (a \rightharpoonup v) = \left(\sum_{i=0}^n \kappa_{i, n-i} ((a \rightharpoonup a_i) \otimes b_{n-i}) \right)_{n \in \mathbb{N}} + \left(\sum_{i=0}^n \kappa_{i, n-i} (a_i \otimes (a \rightharpoonup b_{n-i})) \right)_{n \in \mathbb{N}}.$$

Compared with (36), this yields $a \rightharpoonup (uv) = (a \rightharpoonup u) \cdot v + u \cdot (a \rightharpoonup v)$. Since this holds for every $a \in \mathfrak{g}$, $u \in \otimes V$ and $v \in \otimes V$, we can conclude that (33) holds with A replaced by $\otimes V$. Hence (by Definition 1.70) we see that $\otimes V$ is a \mathfrak{g} -algebra. This proves Proposition 1.77.

We notice that the \mathfrak{g} -algebra $\otimes V$ behaves under restriction as we would want it to:

Proposition 1.80. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

If V is any \mathfrak{g} -module, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(\otimes V) = \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)$ as \mathfrak{h} -algebras. This allows us to speak of "the \mathfrak{h} -algebra $\otimes V$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(\otimes V)$ or $\otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(\otimes V) = \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)$).

Proving this is a matter of applying the definitions.

A consequence of Proposition 1.77 that we are going to use:

Corollary 1.81. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let V be a \mathfrak{g} -module. Then, $\otimes V$ is a \mathfrak{g} -module (according to Definition 1.66). As we know from Definition 1.15, this $\otimes V$ therefore becomes an \mathfrak{h} -module as well.

(a) Let P and Q be two \mathfrak{h} -submodules of $\otimes V$. Then, $P \cdot Q$ is an \mathfrak{h} -submodule of $\otimes V$ as well.

(b) Let R be an \mathfrak{h} -submodule of $\otimes V$. Then, $(\otimes V) \cdot R \cdot (\otimes V)$ is an \mathfrak{h} -submodule of $\otimes V$ as well.

(c) Let $p \in \mathbb{N}$, and let R be an \mathfrak{h} -submodule of $\otimes V$. Then, $V^{\otimes p} \cdot R$ is an \mathfrak{h} -submodule of $\otimes V$ as well.

Here, we are using Convention 1.63.

Proof of Corollary 1.81. According to Definition 1.74, the \mathfrak{h} -module $\otimes V$, equipped with its k -algebra structure, becomes an \mathfrak{h} -algebra (because it is a \mathfrak{g} -algebra (according to Proposition 1.77)).

(a) Since $\otimes V$ is an \mathfrak{h} -algebra, Proposition 1.75 (applied to $\otimes V$ and \mathfrak{h} instead of A and \mathfrak{g}) yields that $P \cdot Q$ is an \mathfrak{h} -submodule of $\otimes V$. This proves Corollary 1.81 (a).

(b) Corollary 1.81 (a) (applied to $\otimes V$ and R instead of P and Q) yields that $(\otimes V) \cdot R$ is an \mathfrak{h} -submodule of $\otimes V$ (because both $\otimes V$ and R are \mathfrak{h} -submodules of $\otimes V$). Now, Corollary 1.81 (a) (applied to $(\otimes V) \cdot R$ and $\otimes V$ instead of P and Q) yields that $((\otimes V) \cdot R) \cdot (\otimes V)$ is an \mathfrak{h} -submodule of $\otimes V$ (because both $(\otimes V) \cdot R$ and $\otimes V$ are \mathfrak{h} -submodules of $\otimes V$). Since $((\otimes V) \cdot R) \cdot (\otimes V) = (\otimes V) \cdot R \cdot (\otimes V)$, we can thus conclude that $(\otimes V) \cdot R \cdot (\otimes V)$ is an \mathfrak{h} -submodule of $\otimes V$. This proves Corollary 1.81 (b).

(c) We know that $V^{\otimes p}$ is a \mathfrak{g} -submodule of $\otimes V$, and thus an \mathfrak{h} -submodule of $\otimes V$ as well. Now, Corollary 1.81 (a) (applied to $V^{\otimes p}$ and R instead of P and Q) yields that $V^{\otimes p} \cdot R$ is an \mathfrak{h} -submodule of $\otimes V$ (because both $V^{\otimes p}$ and R are \mathfrak{h} -submodules of $\otimes V$). This proves Corollary 1.81 (c).

1.14. \mathfrak{g} -modules are $U(\mathfrak{g})$ -modules

The present Subsection 1.14 is, apart from the (easy) Proposition 1.86, of no importance for this paper, as the results of this subsection will not be used anywhere further in this paper, except in the (equally unimportant) Subsection 3.11. However, it provides some context for the theory of \mathfrak{g} -modules as well as one motivation for considering the universal enveloping algebra $U(\mathfrak{g})$.

Many results about \mathfrak{g} -modules (where \mathfrak{g} is a Lie algebra) are analogous to similar results about A -modules, where A is an associative algebra. For example, Proposition 1.14 is the analogue of the following fact:

Proposition 1.82. Let k be a commutative ring. Let A be a k -algebra. Let V and W be two A -modules. Let $f : V \rightarrow W$ be a k -linear map. Then, f is an A -module isomorphism if and only if f is an invertible A -module homomorphism. In other words, f is an A -module isomorphism if and only if f is an A -module homomorphism and a k -module isomorphism at the same time.

Now, of course, Proposition 1.14 and Proposition 1.82 are both (more or less) trivial, so there is no wonder that they are analogous. However, the analogy between A -modules and \mathfrak{g} -modules goes much further, and there are much deeper theorems about A -modules that have their counterparts for \mathfrak{g} -modules. This has a reason: Namely, the category of \mathfrak{g} -modules is isomorphic to the category of A -modules for a particular (associative) algebra A depending on \mathfrak{g} . Namely, this A is the universal enveloping algebra $U(\mathfrak{g})$ defined in Definition 1.64. More precisely:

Proposition 1.83. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Consider the universal enveloping algebra $U(\mathfrak{g})$ and the ideal $I_{\mathfrak{g}}$ defined in Definition 1.64.

(a) For every \mathfrak{g} -module V , there is one and only one $U(\mathfrak{g})$ -module structure on V satisfying

$$(\bar{a} \cdot v = a \rightarrow v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V)$$

(where \bar{a} denotes the projection of $a \in \mathfrak{g} \subseteq \otimes \mathfrak{g}$ on $(\otimes \mathfrak{g}) / I_{\mathfrak{g}} = U(\mathfrak{g})$). This $U(\mathfrak{g})$ -module structure is canonical. Thus, every \mathfrak{g} -module V canonically becomes a $U(\mathfrak{g})$ -module.

(b) Conversely, for every $U(\mathfrak{g})$ -module V , we can define a \mathfrak{g} -module structure on V by

$$(a \rightarrow v = \bar{a} \cdot v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V)$$

(where \bar{a} denotes the projection of $a \in \mathfrak{g} \subseteq \otimes \mathfrak{g}$ on $(\otimes \mathfrak{g}) / I_{\mathfrak{g}} = U(\mathfrak{g})$). This \mathfrak{g} -module structure is canonical. Thus, every $U(\mathfrak{g})$ -module V canonically becomes a \mathfrak{g} -module.

(c) Let V and W be two \mathfrak{g} -modules. Then, according to Proposition 1.83 (a), each of V and W canonically becomes a $U(\mathfrak{g})$ -module. Let $f : V \rightarrow W$ be a map. Then, f is a homomorphism of \mathfrak{g} -modules if and only if f is a homomorphism of $U(\mathfrak{g})$ -modules.

(d) Let V and W be two $U(\mathfrak{g})$ -modules. Then, according to Proposition 1.83 (b), each of V and W canonically becomes a \mathfrak{g} -module. Let $f : V \rightarrow W$ be a map. Then, f is a homomorphism of \mathfrak{g} -modules if and only if f is a homomorphism of $U(\mathfrak{g})$ -modules.

(e) We can define a functor U_1 from the category of \mathfrak{g} -modules to the category of $U(\mathfrak{g})$ -modules as follows: For every \mathfrak{g} -module V , let $U_1(V)$ be the $U(\mathfrak{g})$ -module V defined in Proposition 1.83 (a). For every homomorphism f between \mathfrak{g} -modules, let $U_1(f)$ be the same homomorphism f , but considered as a homomorphism between $U(\mathfrak{g})$ -modules this time (this is legitimate due to Proposition 1.83 (c)).

(f) We can define a functor U_2 from the category of $U(\mathfrak{g})$ -modules to the category of \mathfrak{g} -modules as follows: For every $U(\mathfrak{g})$ -module V , let $U_2(V)$ be the \mathfrak{g} -module V defined in Proposition 1.83 (b). For every homomorphism f between $U(\mathfrak{g})$ -modules, let $U_2(f)$ be the same homomorphism f , but considered as a homomorphism between \mathfrak{g} -modules this time (this is legitimate due to Proposition 1.83 (d)).

(g) The two functors U_1 and U_2 defined in Proposition 1.83 (e) and (f) are mutually inverse.

(h) Both functors U_1 and U_2 are additive, exact and preserve kernels, cokernels and direct sums.

Note that the two functors U_1 and U_2 are mutually *inverse* (and not just quasi-inverse, as many pairs of functors in algebra tend to be) in Proposition 1.83 (g).

Proposition 1.83 explains why, for a number of results about A -modules (with A an associative algebra), there are analogous results for \mathfrak{g} -modules (with \mathfrak{g} a Lie algebra). However, it does not explain everything; in particular, it does not explain anything about tensor products of \mathfrak{g} -modules, because, in general, there is no reasonable notion of a tensor product between two A -modules, where A is an associative algebra. However, there *is* such a notion when A is a bialgebra. And there is a canonical way to turn $U(\mathfrak{g})$ into a bialgebra, and even into a Hopf algebra (so that, in addition to tensor products of modules, we also get Hom modules). Before I show this way, let us define a Hopf algebra structure on the tensor algebra $\otimes \mathfrak{g}$, and, more generally, on $\otimes V$ for any k -module V :

Proposition 1.84. Let k be a commutative ring. Let V be a k -module.

(a) Let $\varepsilon_{\otimes V} : \otimes V \rightarrow k$ be the projection from the direct sum $\otimes V = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$ onto its addend $V^{\otimes 0} = k$. Then, $\varepsilon_{\otimes V}$ is a k -algebra homomorphism. It is given by

$$\varepsilon_{\otimes V}(v_1 \otimes v_2 \otimes \dots \otimes v_n) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases}$$

for every $n \in \mathbb{N}$ and every $(v_1, v_2, \dots, v_n) \in V^n$.

(b) There exists one and only one k -algebra homomorphism $\mathbf{d} : \otimes V \rightarrow (\otimes V) \otimes (\otimes V)$ satisfying $\mathbf{d}(v) = 1 \otimes v + v \otimes 1$ for every $v \in V$. Denote this homomorphism \mathbf{d} by

$\Delta_{\otimes V}$. Then, this homomorphism $\Delta_{\otimes V}$ is given by

$$\begin{aligned} & \Delta_{\otimes V}(v_1 \otimes v_2 \otimes \dots \otimes v_n) \\ &= \sum_{i=0}^n \sum_{\substack{\pi \in S_n; \\ \pi(1) < \pi(2) < \dots < \pi(i); \\ \pi(i+1) < \pi(i+2) < \dots < \pi(n)}} (v_{\pi(1)} \otimes v_{\pi(2)} \otimes \dots \otimes v_{\pi(i)}) \otimes (v_{\pi(i+1)} \otimes v_{\pi(i+2)} \otimes \dots \otimes v_{\pi(n)}) \\ & \text{for every } n \in \mathbb{N} \text{ and every } (v_1, v_2, \dots, v_n) \in V^n. \end{aligned}$$

(Here, the two tensors $v_{\pi(1)} \otimes v_{\pi(2)} \otimes \dots \otimes v_{\pi(i)}$ and $v_{\pi(i+1)} \otimes v_{\pi(i+2)} \otimes \dots \otimes v_{\pi(n)}$ are considered as elements of the tensor algebra $\otimes V$, and their tensor product is a tensor in $(\otimes V) \otimes (\otimes V)$.)

(c) There exists one and only one k -algebra homomorphism $\mathbf{S} : \otimes V \rightarrow (\otimes V)^{\text{op}}$ satisfying ($\mathbf{S}(v) = -v$ for every $v \in V$). Denote this homomorphism \mathbf{S} by $S_{\otimes V}$. Concretely, this $S_{\otimes V}$ is given by

$$\begin{aligned} S_{\otimes V}(v_1 \otimes v_2 \otimes \dots \otimes v_n) &= (-1)^n v_n \otimes v_{n-1} \otimes \dots \otimes v_1 \\ & \text{for every } n \in \mathbb{N} \text{ and every } (v_1, v_2, \dots, v_n) \in V^n. \end{aligned}$$

(d) The homomorphisms $\Delta_{\otimes V}$, $\varepsilon_{\otimes V}$ and $S_{\otimes V}$ defined above make $\otimes V$ into a Hopf algebra. This Hopf algebra is called the *tensor Hopf algebra* of the k -module V . This Hopf algebra $\otimes V$ is cocommutative.

Proposition 1.85. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. In the following, we will use the notations $U(\mathfrak{g})$ and $I_{\mathfrak{g}}$ which were defined in Definition 1.64. Recall that $U(\mathfrak{g}) = (\otimes \mathfrak{g}) / I_{\mathfrak{g}}$.

(a) The ideal $I_{\mathfrak{g}}$ is a Hopf ideal of the tensor Hopf algebra $\otimes \mathfrak{g}$ of the k -module \mathfrak{g} . (For the definition of the tensor Hopf algebra $\otimes \mathfrak{g}$, see Proposition 1.84.) Thus, $(\otimes \mathfrak{g}) / I_{\mathfrak{g}}$ becomes a Hopf algebra. Since $(\otimes \mathfrak{g}) / I_{\mathfrak{g}} = U(\mathfrak{g})$, this means that $U(\mathfrak{g})$ becomes a Hopf algebra. This Hopf algebra $U(\mathfrak{g})$ is cocommutative.

(b) If the Hopf algebra $U(\mathfrak{g})$ defined in Proposition 1.85 (a) is used to define the tensor product of two $U(\mathfrak{g})$ -modules, then the functors U_1 and U_2 defined in Proposition 1.83 preserve tensor products. These functors as well preserve internal Homs and the ground-field object, if these are also defined using the Hopf algebra structure on $U(\mathfrak{g})$.

A proof of Proposition 1.83 can be found in most standard references about Lie algebras (such as [27, Chapter III, Corollary 3.6]), while a proof of Proposition 1.85 can be found in references for Hopf algebras (such as [26, §1.2.6, Example 6]; more precisely, [26, §1.2.6, Example 6] proves what amounts to an equivalent form of Proposition 1.85 independent of Proposition 1.84). Proposition 1.84 is proven in [26, §1.2.6, Example 8]. All these proofs are rather standard, however. None of these three propositions will be used in the following, except for the easiest part of Proposition 1.85 (a) - namely, the one about the counit of the Hopf algebra $U(\mathfrak{g})$:

Proposition 1.86. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. In the following, we will use the notations $U(\mathfrak{g})$ and $I_{\mathfrak{g}}$ which were defined in Definition 1.64. Recall that $U(\mathfrak{g}) = (\otimes \mathfrak{g}) / I_{\mathfrak{g}}$. Let ψ be the canonical projection from $\otimes \mathfrak{g}$ onto $U(\mathfrak{g})$. Clearly, this projection ψ is a surjective k -algebra homomorphism.

Applying Proposition 1.84 (a) to $V = \mathfrak{g}$, we obtain a k -algebra homomorphism $\varepsilon_{\otimes \mathfrak{g}} : \otimes \mathfrak{g} \rightarrow k$.

(a) Then, $\varepsilon_{\otimes \mathfrak{g}}(I_{\mathfrak{g}}) = 0$. Thus, by the universal property of the factor algebra, there exists one and only one k -algebra homomorphism $\overline{\varepsilon_{\otimes \mathfrak{g}}} : U(\mathfrak{g}) \rightarrow k$ such that $\varepsilon_{\otimes \mathfrak{g}} = \overline{\varepsilon_{\otimes \mathfrak{g}}} \circ \psi$. Denote this homomorphism $\overline{\varepsilon_{\otimes \mathfrak{g}}}$ by $\varepsilon_{U(\mathfrak{g})}$. Then, $\varepsilon_{\otimes \mathfrak{g}} = \varepsilon_{U(\mathfrak{g})} \circ \psi$.

(b) This homomorphism $\varepsilon_{U(\mathfrak{g})}$ satisfies

$$\varepsilon_{U(\mathfrak{g})}(\overline{v_1 \otimes v_2 \otimes \dots \otimes v_n}) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases}$$

for every $n \in \mathbb{N}$ and every $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$.

For the sake of completeness, we record the proof of this proposition:

Proof of Proposition 1.86. (a) In order to prove Proposition 1.86 (a), we only need to show that $\varepsilon_{\otimes \mathfrak{g}}(I_{\mathfrak{g}}) = 0$, because all the other assertions of Proposition 1.86 (a) follow trivially from it.

By the definition of $\varepsilon_{\otimes \mathfrak{g}}$, we know that $\varepsilon_{\otimes \mathfrak{g}}$ is the projection from the direct sum $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$ onto its addend $\mathfrak{g}^{\otimes 0} = k$. Thus, $\varepsilon_{\otimes \mathfrak{g}}$ must map every addend of the direct sum $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$ different from $\mathfrak{g}^{\otimes 0}$ to 0. In other words, every $n \in \mathbb{N}$ such that $n \neq 0$ must satisfy $\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes n}) = 0$ (because whenever $n \in \mathbb{N}$ satisfies $n \neq 0$, the k -submodule $\mathfrak{g}^{\otimes n}$ is an addend of the direct sum $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$ different from $\mathfrak{g}^{\otimes 0}$). Applying this to $n = 1$ yields $\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes 1}) = 0$. On the other hand, applying $\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes n}) = 0$ to $n = 2$ yields $\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes 2}) = 0$. Since $\varepsilon_{\otimes \mathfrak{g}}$ is k -linear, we have $\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes 1} + \mathfrak{g}^{\otimes 2}) = \underbrace{\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes 1})}_{=0} + \underbrace{\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes 2})}_{=0} = 0 + 0 = 0$.

Now, every $(v, w) \in \mathfrak{g} \times \mathfrak{g}$ satisfies

$$v \otimes w - w \otimes v - [v, w] = \underbrace{-[v, w]}_{\in \mathfrak{g}^{\otimes 1}} + \underbrace{v \otimes w - w \otimes v}_{\in \mathfrak{g}^{\otimes 2}} \in \mathfrak{g}^{\otimes 1} + \mathfrak{g}^{\otimes 2}.$$

In other words, $\{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g}\} \subseteq \mathfrak{g}^{\otimes 1} + \mathfrak{g}^{\otimes 2}$. Thus, Proposition 1.29 (a) (applied to $\otimes \mathfrak{g}$, $\{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g}\}$ and $\mathfrak{g}^{\otimes 1} + \mathfrak{g}^{\otimes 2}$ instead of M , S and Q) yields $\langle \{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g}\} \rangle \subseteq \mathfrak{g}^{\otimes 1} + \mathfrak{g}^{\otimes 2}$.

Since $I_{\mathfrak{g}} = (\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle \cdot (\otimes \mathfrak{g})$, we have

$$\begin{aligned} \varepsilon_{\otimes \mathfrak{g}}(I_{\mathfrak{g}}) &= \varepsilon_{\otimes \mathfrak{g}}((\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle \cdot (\otimes \mathfrak{g})) \\ &= \underbrace{\varepsilon_{\otimes \mathfrak{g}}(\otimes \mathfrak{g})}_{\subseteq k} \cdot \varepsilon_{\otimes \mathfrak{g}} \left(\underbrace{\langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle}_{=\langle \{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g}\} \rangle}_{\subseteq \mathfrak{g}^{\otimes 1} + \mathfrak{g}^{\otimes 2}} \right) \cdot \underbrace{\varepsilon_{\otimes \mathfrak{g}}(\otimes \mathfrak{g})}_{\subseteq k} \\ &\quad \text{(because } \varepsilon_{\otimes \mathfrak{g}} \text{ is a } k\text{-algebra homomorphism)} \\ &\subseteq k \cdot \underbrace{\varepsilon_{\otimes \mathfrak{g}}(\mathfrak{g}^{\otimes 1} + \mathfrak{g}^{\otimes 2})}_{=0} \cdot k = k \cdot 0 \cdot k = 0. \end{aligned}$$

In other words, $\varepsilon_{\otimes \mathfrak{g}}(I_{\mathfrak{g}}) = 0$. This completes the proof of Proposition 1.86 (a).

(b) Every $n \in \mathbb{N}$ and every $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$ satisfy $\overline{v_1 \otimes v_2 \otimes \dots \otimes v_n} = \psi(v_1 \otimes v_2 \otimes \dots \otimes v_n)$ (since ψ is the canonical projection from $\otimes \mathfrak{g}$ onto $U(\mathfrak{g})$) and thus

$$\begin{aligned} \varepsilon_{U(\mathfrak{g})}(\overline{v_1 \otimes v_2 \otimes \dots \otimes v_n}) &= \varepsilon_{U(\mathfrak{g})}(\psi(v_1 \otimes v_2 \otimes \dots \otimes v_n)) = \underbrace{(\varepsilon_{U(\mathfrak{g})} \circ \psi)}_{=\varepsilon_{\otimes \mathfrak{g}}}(v_1 \otimes v_2 \otimes \dots \otimes v_n) \\ &= \varepsilon_{\otimes \mathfrak{g}}(v_1 \otimes v_2 \otimes \dots \otimes v_n) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases} \\ &\quad \text{(by Proposition 1.84 (a), applied to } \mathfrak{g} \text{ instead of } V\text{).} \end{aligned}$$

This proves Proposition 1.86 (b).

Also, we note in passing that \mathfrak{g} -algebras (as introduced in Definition 1.70) are left $U(\mathfrak{g})$ -module algebras (where we are using the terminology of Hopf algebra theorists).

1.15. Splitting of exact sequences of \mathfrak{g} -modules

Next we are going to introduce the notion of *splitting* for an exact sequence of \mathfrak{g} -modules. This notion is based on the following known result:

Proposition 1.87. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let A, B and C be three \mathfrak{g} -modules. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two \mathfrak{g} -module homomorphisms such that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence. Then, the following three assertions \mathbb{S}_1 , \mathbb{S}_2 and \mathbb{S}_3 are equivalent:

Assertion \mathbb{S}_1 : There exists a \mathfrak{g} -module homomorphism $f' : B \rightarrow A$ such that $f' \circ f = \text{id}_A$.

Assertion \mathbb{S}_2 : There exists a \mathfrak{g} -module homomorphism $g' : C \rightarrow B$ such that $g \circ g' = \text{id}_C$.

Assertion \mathbb{S}_3 : There exists a \mathfrak{g} -submodule P of B such that $B = f(A) \oplus P$.

Proof of Proposition 1.87. In order to prove Proposition 1.87, we must show the equivalence of the three assertions \mathbb{S}_1 , \mathbb{S}_2 and \mathbb{S}_3 . In order to achieve this, let us show that $\mathbb{S}_1 \implies \mathbb{S}_3$, $\mathbb{S}_2 \implies \mathbb{S}_3$, $\mathbb{S}_3 \implies \mathbb{S}_1$ and $\mathbb{S}_3 \implies \mathbb{S}_2$:

Proof of the implication $\mathbb{S}_1 \implies \mathbb{S}_3$: Assume that Assertion \mathbb{S}_1 holds. Then, there exists a \mathfrak{g} -module homomorphism $f' : B \rightarrow A$ such that $f' \circ f = \text{id}_A$. Consider this homomorphism f' . Now let $P = \text{Ker } f'$. Then, P is a \mathfrak{g} -submodule of B (since f' is a \mathfrak{g} -module homomorphism, while the kernel of every \mathfrak{g} -module homomorphism is a \mathfrak{g} -submodule). Every $b \in B$ satisfies

$$\begin{aligned} f'(b - f(f'(b))) &= f'(b) - \underbrace{f'(f(f'(b)))}_{=(f' \circ f \circ f')(b)} && \text{(since } f' \text{ is } k\text{-linear)} \\ &= f'(b) - \left(\underbrace{f' \circ f \circ f'}_{=\text{id}_A} \right) (b) = f'(b) - \underbrace{(\text{id}_A \circ f')}_{=f'}(b) = f'(b) - f'(b) = 0, \end{aligned}$$

so that $b - f(f'(b)) \in \text{Ker } f' = P$ and thus

$$b = f \left(\underbrace{f'(b)}_{\in A} \right) + \underbrace{(b - f(f'(b)))}_{\in P} \in f(A) + P.$$

Thus, $B = f(A) + P$. Now let us show that $f(A) \cap P = 0$:

Let $x \in f(A) \cap P$ be arbitrary. Then, $x \in f(A) \cap P \subseteq f(A)$, so there exists some $y \in A$ such that $x = f(y)$. Consider this y . But on the other hand, $x \in f(A) \cap P \subseteq P = \text{Ker } f'$, so that $f'(x) = 0$. Thus, $0 = f' \left(\underbrace{x}_{=f(y)} \right) = f'(f(y)) = \underbrace{(f' \circ f)}_{=\text{id}_A}(y) =$

$\text{id}_A(y) = y$, so that $x = f \left(\underbrace{y}_{=0} \right) = f(0) = 0$ (since f is k -linear). We have thus

proven that every $x \in f(A) \cap P$ satisfies $x = 0$. This shows that $f(A) \cap P = 0$.

Since $B = f(A) + P$ and $f(A) \cap P = 0$, we have $B = f(A) \oplus P$. Thus, we have shown that there exists a \mathfrak{g} -submodule P of B such that $B = f(A) \oplus P$. In other words, Assertion \mathbb{S}_3 holds. This proves the implication $\mathbb{S}_1 \implies \mathbb{S}_3$.

Proof of the implication $\mathbb{S}_2 \implies \mathbb{S}_3$: Assume that Assertion \mathbb{S}_2 holds. Then, there exists a \mathfrak{g} -module homomorphism $g' : C \rightarrow B$ such that $g \circ g' = \text{id}_C$. Consider this homomorphism g' . Now let $P = g'(C)$. Then, P is a \mathfrak{g} -submodule of B (since g' is a \mathfrak{g} -module homomorphism, while the image of every \mathfrak{g} -module homomorphism is a \mathfrak{g} -submodule). Every $b \in B$ satisfies

$$\begin{aligned} g(b - g'(g(b))) &= g(b) - \underbrace{g(g'(g(b)))}_{=(g \circ g' \circ g)(b)} && \text{(since } g \text{ is } k\text{-linear)} \\ &= g(b) - \left(\underbrace{g \circ g' \circ g}_{=\text{id}_C} \right) (b) = g(b) - \underbrace{(\text{id}_C \circ g)}_{=g}(b) = g(b) - g(b) = 0, \end{aligned}$$

so that $b - g'(g(b)) \in \text{Ker } g = f(A)$ (since $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence) and thus

$$b = \underbrace{(b - g'(g(b)))}_{\in f(A)} + g' \left(\underbrace{g(b)}_{\in C} \right) \in f(A) + \underbrace{g'(C)}_{=P} = f(A) + P.$$

Thus, $B = f(A) + P$. Now let us show that $f(A) \cap P = 0$:

Let $x \in f(A) \cap P$ be arbitrary. Then, $x \in f(A) \cap P \subseteq f(A) = \text{Ker } g$. But on the other hand, $x \in f(A) \cap P \subseteq P = g'(C)$, so that there exists some $z \in C$ such that

$x = g'(z)$. Consider this z . Thus, $g \left(\underbrace{g'(z)}_{=x} \right) = g(x) = 0$ (since $x \in \text{Ker } g$). In other

words, $0 = g(g'(z)) = \underbrace{(g \circ g')}_{=\text{id}_C}(z) = \text{id}_C(z) = z$, so that $x = g' \left(\underbrace{z}_{=0} \right) = g'(0) = 0$

(since g' is k -linear). We have thus proven that every $x \in f(A) \cap P$ satisfies $x = 0$. This shows that $f(A) \cap P = 0$.

Since $B = f(A) + P$ and $f(A) \cap P = 0$, we have $B = f(A) \oplus P$. Thus, we have shown that there exists a \mathfrak{g} -submodule P of B such that $B = f(A) \oplus P$. In other words, Assertion \mathbb{S}_3 holds. This proves the implication $\mathbb{S}_2 \implies \mathbb{S}_3$.

Proof of the implication $\mathbb{S}_3 \implies \mathbb{S}_1$: Assume that Assertion \mathbb{S}_3 holds. Then, there exists a \mathfrak{g} -submodule P of B such that $B = f(A) \oplus P$. Consider this \mathfrak{g} -submodule P .

The \mathfrak{g} -module homomorphism $f : A \rightarrow B$ induces a \mathfrak{g} -module homomorphism $f_1 : A \rightarrow f(A)$ defined by $(f_1(a) = f(a))$ for every $a \in A$. Since f is injective (because $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence), this homomorphism f_1 is bijective. That is, f_1 is a k -module isomorphism. Thus f_1 is a \mathfrak{g} -module homomorphism and a k -module isomorphism at the same time. By an application of Proposition 1.14, this yields that f_1 is a \mathfrak{g} -module isomorphism. Thus, its inverse f_1^{-1} also is a \mathfrak{g} -module isomorphism.

On the other hand, let $\pi : f(A) \oplus P \rightarrow f(A)$ be the projection from the direct sum $f(A) \oplus P$ to its addend $f(A)$. Since $f(A)$ and P are \mathfrak{g} -modules, this projection π is a surjective \mathfrak{g} -module homomorphism from $f(A) \oplus P$ onto $f(A)$. Since $f(A) \oplus P = B$, this π is thus a surjective \mathfrak{g} -module homomorphism from B onto $f(A)$.

Let $f' = f_1^{-1} \circ \pi$. Then, f' is a \mathfrak{g} -module homomorphism (since it is the composition of the \mathfrak{g} -module homomorphisms π and f_1^{-1}). Moreover, every $a \in A$ satisfies

$$\pi(f(a)) = f(a) \quad (\text{since } f(a) \in f(A), \text{ while } \pi \text{ is a projection onto } f(A))$$

and thus

$$\left(\underbrace{f'}_{=f_1^{-1} \circ \pi} \circ f \right) (a) = (f_1^{-1} \circ \pi \circ f) (a) = f_1^{-1} \left(\underbrace{\pi(f(a))}_{=f(a)=f_1(a)} \right) = f_1^{-1}(f_1(a)) = a = \text{id}_A(a).$$

Thus, $f' \circ f = \text{id}_A$.

Thus, we have shown that there exists a \mathfrak{g} -module homomorphism $f' : B \rightarrow A$ such that $f' \circ f = \text{id}_A$. In other words, Assertion \mathbb{S}_1 holds. This proves the implication $\mathbb{S}_3 \implies \mathbb{S}_1$.

Proof of the implication $\mathbb{S}_3 \implies \mathbb{S}_2$: Assume that Assertion \mathbb{S}_3 holds. Then, there exists a \mathfrak{g} -submodule P of B such that $B = f(A) \oplus P$. Consider this \mathfrak{g} -submodule P .

Since $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence, we have $g(B) = C$ and $f(A) = \text{Ker } g$.

Define a map $g_1 : P \rightarrow C$ through $g_1 = g|_P$. Then,

$$\text{Ker } g_1 = \text{Ker } (g|_P) = \underbrace{\text{Ker } g}_{=f(A)} \cap P = f(A) \cap P = 0 \quad (\text{since } B = f(A) \oplus P),$$

so that the map g_1 is injective.

On the other hand, $B = f(A) \oplus P = f(A) + P$, so that

$$\begin{aligned} g(B) &= g(f(A) + P) = g \left(\underbrace{f(A)}_{=\text{Ker } g} \right) + \underbrace{g(P)}_{=(g|_P)(P)} \quad (\text{since } g \text{ is } k\text{-linear}) \\ &= \underbrace{g(\text{Ker } g)}_{=0} + \underbrace{(g|_P)(P)}_{=g_1(P)} = 0 + g_1(P) = g_1(P). \end{aligned}$$

Since $g(B) = C$, this becomes $C = g_1(P)$. Thus, g_1 is surjective.

The map g_1 is bijective (since it is injective and surjective), and thus it is a k -module isomorphism. An application of Proposition 1.14 now shows that g_1 is a \mathfrak{g} -module isomorphism (since it is a \mathfrak{g} -module homomorphism and a k -module isomorphism). Thus, its inverse g_1^{-1} also is a \mathfrak{g} -module isomorphism.

Let ι be the canonical inclusion of P in B . Then, ι is a \mathfrak{g} -module homomorphism.

Let $g' = \iota \circ g_1^{-1}$. Then, g' is a \mathfrak{g} -module homomorphism (since it is the composition of the \mathfrak{g} -module homomorphisms g_1^{-1} and ι). Moreover, every $c \in C$ satisfies

$$\begin{aligned} \left(g \circ \underbrace{g'}_{=\iota \circ g_1^{-1}} \right) (c) &= (g \circ \iota \circ g_1^{-1})(c) = \underbrace{g((\iota \circ g_1^{-1})(c))}_{=(g|_P)((\iota \circ g_1^{-1})(c))} = \underbrace{(g|_P)}_{=g_1}((\iota \circ g_1^{-1})(c)) \\ &\quad (\text{since } (\iota \circ g_1^{-1})(c) \in P) \\ &= g_1 \left(\underbrace{(\iota \circ g_1^{-1})(c)}_{=\iota(g_1^{-1}(c))} \right) = g_1 \left(\underbrace{\iota(g_1^{-1}(c))}_{=g_1^{-1}(c)} \right) \\ &\quad (\text{since } \iota \text{ is just an inclusion map}) = g_1(g_1^{-1}(c)) = c = \text{id}_C(c). \end{aligned}$$

Thus, $g \circ g' = \text{id}_C$.

Thus, we have shown that there exists a \mathfrak{g} -module homomorphism $g' : C \rightarrow B$ such that $g \circ g' = \text{id}_C$. In other words, Assertion \mathbb{S}_2 holds. This proves the implication $\mathbb{S}_3 \implies \mathbb{S}_2$.

We have thus shown the four implications $\mathbb{S}_1 \implies \mathbb{S}_3$, $\mathbb{S}_2 \implies \mathbb{S}_3$, $\mathbb{S}_3 \implies \mathbb{S}_1$ and $\mathbb{S}_3 \implies \mathbb{S}_2$. These implications, when combined, yield the equivalence $\mathbb{S}_1 \iff \mathbb{S}_2 \iff \mathbb{S}_3$. This proves Proposition 1.87.

Proposition 1.87 is in no way peculiar to \mathfrak{g} -modules. A completely analogous proposition (with exactly the same proof) would hold if we replace \mathfrak{g} by A and "Lie algebra" by "algebra" throughout Proposition 1.87. More generally, an analogue of Proposition 1.87 holds in every abelian category.

Definition 1.88. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let A , B and C be three \mathfrak{g} -modules. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two \mathfrak{g} -module homomorphisms such that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence. Then, we say that *the exact sequence* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of \mathfrak{g} -modules is \mathfrak{g} -split if and only if the three equivalent assertions \mathbb{S}_1 , \mathbb{S}_2 and \mathbb{S}_3 from Proposition 1.87 are satisfied.

Corollary 1.89. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let A , B and C be three \mathfrak{g} -modules. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two \mathfrak{g} -module homomorphisms such that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence. Then, the exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of \mathfrak{g} -modules is \mathfrak{g} -split if and only if there exists a \mathfrak{g} -submodule P of B such that $B = f(A) \oplus P$.

Proof of Corollary 1.89. We have the following equivalence of assertions:

$$\begin{aligned} & \left(\text{the exact sequence } 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \text{ of } \mathfrak{g}\text{-modules is } \mathfrak{g}\text{-split} \right) \\ \iff & \left(\text{the three equivalent assertions } \mathbb{S}_1, \mathbb{S}_2 \text{ and } \mathbb{S}_3 \text{ from Proposition 1.87 are satisfied} \right) \\ & \left(\text{by Definition 1.88} \right) \\ \iff & \left(\text{the assertion } \mathbb{S}_3 \text{ from Proposition 1.87 is satisfied} \right) \\ & \left(\begin{array}{l} \text{because the assertions } \mathbb{S}_1, \mathbb{S}_2 \text{ and } \mathbb{S}_3 \text{ from Proposition 1.87 are} \\ \text{equivalent, and thus they are all satisfied if any of them is satisfied} \end{array} \right) \\ \iff & \left(\text{there exists a } \mathfrak{g}\text{-submodule } P \text{ of } B \text{ such that } B = f(A) \oplus P \right) \\ & \left(\begin{array}{l} \text{because Assertion } \mathbb{S}_3 \text{ says that there exists a} \\ \mathfrak{g}\text{-submodule } P \text{ of } B \text{ such that } B = f(A) \oplus P \end{array} \right). \end{aligned}$$

This proves Corollary 1.89.

1.16. Filtrations of \mathfrak{g} -modules

Now we are going to define the notion of a *filtration* of a k -module. Note that this is an unknown notion, but it is one of the most overloaded notions in algebra (there are at least four different meanings of a "filtration", and every author defines it to mean the one he wants), so let us settle what *we* are going to call a filtration:

Definition 1.90. Let k be a commutative ring. Let V be a k -module. A *k -module filtration* of V will mean a sequence $(V_n)_{n \geq 0}$ of k -submodules of V such that $\bigcup_{n \geq 0} V_n = V$ and $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$

Similarly we define the notion of a \mathfrak{g} -module filtration:

Definition 1.91. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module.

A \mathfrak{g} -module filtration of V will mean a sequence $(V_n)_{n \geq 0}$ of \mathfrak{g} -submodules of V such that $\bigcup_{n \geq 0} V_n = V$ and $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$

The following is obvious:

Proposition 1.92. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module.

(a) Every \mathfrak{g} -module filtration of V is a k -module filtration of V .

(b) If $(V_n)_{n \geq 0}$ is a k -module filtration of V such that (V_n) is a \mathfrak{g} -submodule of V for every $n \in \mathbb{N}$, then $(V_n)_{n \geq 0}$ is a \mathfrak{g} -module filtration of V .

We also record a definition:

Definition 1.93. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

Let V be a \mathfrak{g} -module. Consider the tensor \mathfrak{g} -module $\otimes V$ of the \mathfrak{g} -module V .

For every $n \in \mathbb{Z}$, let $V^{\otimes \leq n}$ denote the \mathfrak{g} -submodule $\bigoplus_{i=0}^n V^{\otimes i}$ of the \mathfrak{g} -module $\otimes V$. (In

fact, $\bigoplus_{i=0}^n V^{\otimes i}$ is a \mathfrak{g} -submodule of $\otimes V$, because $\otimes V = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$ as a \mathfrak{g} -module.) Note

that this definition yields $V^{\otimes \leq n} = 0$ for every integer $n < 0$.

It is clear that $(V^{\otimes \leq n})_{n \geq 0}$ is a \mathfrak{g} -module filtration of $\otimes V$. This filtration is called the *degree filtration* of $\otimes V$.

We will also use the notation $V^{\otimes \leq n}$ in a slightly more general context:

Definition 1.94. Let k be a commutative ring. Let V be a k -module. Consider the tensor k -module $\otimes V$ of the k -module V .

For every $n \in \mathbb{N}$, let $V^{\otimes \leq n}$ denote the k -submodule $\bigoplus_{i=0}^n V^{\otimes i}$ of the k -module $\otimes V$. It

is clear that $(V^{\otimes \leq n})_{n \geq 0}$ is a k -module filtration of $\otimes V$. This filtration is called the *degree filtration* of $\otimes V$.

We notice a known fact:

Proposition 1.95. Let k be a commutative ring. Let V be a k -module. Then, considering $V^{\otimes n}$ as a k -submodule of $\otimes V$ for every $n \in \mathbb{N}$, we have:

(a) Every $i \in \mathbb{N}$ and $j \in \mathbb{N}$ satisfy $V^{\otimes i} \cdot V^{\otimes j} = V^{\otimes(i+j)}$.

(b) Every $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy $V^{\otimes \leq n} \cdot V^{\otimes \leq m} \subseteq V^{\otimes \leq(n+m)}$.

Proof of Proposition 1.95. (a) Every $i \in \{0, 1, \dots, n\}$ and $j \in \{0, 1, \dots, m\}$ satisfy

$$\begin{aligned}
& V^{\otimes i} \cdot V^{\otimes j} \\
&= \langle ab \mid (a, b) \in V^{\otimes i} \times V^{\otimes j} \rangle = \langle a \otimes b \mid (a, b) \in V^{\otimes i} \times V^{\otimes j} \rangle \\
&\quad \left(\begin{array}{l} \text{since every } (a, b) \in V^{\otimes i} \times V^{\otimes j} \text{ satisfy } ab = a \cdot b = a \otimes b \\ \text{(according to (31) (applied to } i \text{ and } j \text{ instead of } n \text{ and } m\text{)),} \\ \text{because } (a, b) \in V^{\otimes i} \times V^{\otimes j} \text{ yields } a \in V^{\otimes i} \text{ and } b \in V^{\otimes j} \end{array} \right) \\
&= V^{\otimes i} \otimes V^{\otimes j} \quad \left(\begin{array}{l} \text{since the tensor product } V^{\otimes i} \otimes V^{\otimes j} \text{ is spanned by pure tensors,} \\ \text{i. e., by tensors of the form } a \otimes b \text{ for } (a, b) \in V^{\otimes i} \times V^{\otimes j} \end{array} \right) \\
&= V^{\otimes(i+j)}.
\end{aligned}$$

This proves Proposition 1.95 (a).

(b) Definition 1.94 yields

$$V^{\otimes \leq(n+m)} = \bigoplus_{i=0}^{n+m} V^{\otimes i} = \bigoplus_{r=0}^{n+m} V^{\otimes r} \quad (\text{here we renamed } i \text{ as } r \text{ in the sum}).$$

On the other hand, Proposition 1.95 (a) yields that every $i \in \{0, 1, \dots, n\}$ and $j \in \{0, 1, \dots, m\}$ satisfy $V^{\otimes i} \cdot V^{\otimes j} = V^{\otimes(i+j)} \subseteq \bigoplus_{r=0}^{n+m} V^{\otimes r}$ (because $i \in \{0, 1, \dots, n\}$ and $j \in \{0, 1, \dots, m\}$ yield $i + j \in \{0, 1, \dots, n + m\}$, so that $V^{\otimes(i+j)}$ is one of the addends of the direct sum $\bigoplus_{r=0}^{n+m} V^{\otimes r}$). Since $\bigoplus_{r=0}^{n+m} V^{\otimes r} = V^{\otimes \leq(n+m)}$, we have thus proven that every $i \in \{0, 1, \dots, n\}$ and $j \in \{0, 1, \dots, m\}$ satisfy $V^{\otimes i} \cdot V^{\otimes j} \subseteq V^{\otimes \leq(n+m)}$.

Now, Definition 1.94 yields $V^{\otimes \leq n} = \bigoplus_{i=0}^n V^{\otimes i}$, $V^{\otimes \leq m} = \bigoplus_{i=0}^m V^{\otimes i}$ and $V^{\otimes \leq(n+m)} = \bigoplus_{i=0}^{n+m} V^{\otimes i}$.

Since $V^{\otimes \leq n} = \bigoplus_{i=0}^n V^{\otimes i} = \sum_{i=0}^n V^{\otimes i}$ (since direct sums are sums) and

$$\begin{aligned}
V^{\otimes \leq m} &= \bigoplus_{i=0}^m V^{\otimes i} = \bigoplus_{j=0}^m V^{\otimes j} \quad (\text{here we renamed } i \text{ as } j \text{ in the sum}) \\
&= \sum_{j=0}^m V^{\otimes j} \quad (\text{since direct sums are sums}),
\end{aligned}$$

we have

$$\begin{aligned}
\underbrace{V^{\otimes \leq n}}_{= \sum_{i=0}^n V^{\otimes i}} \cdot \underbrace{V^{\otimes \leq m}}_{= \sum_{j=0}^m V^{\otimes j}} &= \left(\sum_{i=0}^n V^{\otimes i} \right) \cdot \left(\sum_{j=0}^m V^{\otimes j} \right) = \sum_{i=0}^n \sum_{j=0}^m \underbrace{V^{\otimes i} \cdot V^{\otimes j}}_{\subseteq V^{\otimes \leq(n+m)}} \subseteq \sum_{i=0}^n \sum_{j=0}^m V^{\otimes \leq(n+m)} \\
&\subseteq V^{\otimes \leq(n+m)} \quad (\text{since } V^{\otimes \leq(n+m)} \text{ is a } k\text{-module}).
\end{aligned}$$

This proves Proposition 1.95 (b).

Our next definition is concerned with k -module homomorphisms:

Definition 1.96. Let k be a commutative ring. Let V and W be two k -modules. Let $f : V \rightarrow W$ be a k -module homomorphism. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V , and let $(W_n)_{n \geq 0}$ be a k -module filtration of W .

We say that *the map f respects the filtrations $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$* if it satisfies ($f(V_n) \subseteq W_n$ for every $n \in \mathbb{N}$). Sometimes we abbreviate "the map f respects the filtrations $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$ " to "*the map f respects the filtration*", as long as the filtrations $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$ can be inferred from the context.

Filtrations of k -modules and homomorphisms respecting them lead to new modules resp. homomorphisms:

Definition 1.97. Let k be a commutative ring.

(a) Let V be a k -module. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V . Then, for every $p \in \mathbb{N}$, we denote the k -module V_p/V_{p-1} (where V_{-1} means 0) by $\text{gr}_p(V, (V_n)_{n \geq 0})$. When the filtration $(V_n)_{n \geq 0}$ is clear from the context, we will abbreviate $\text{gr}_p(V, (V_n)_{n \geq 0})$ by $\text{gr}_p V$.

(b) Let V and W be two k -modules. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V , and let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Let $f : V \rightarrow W$ be a k -module homomorphism respecting the filtration.²¹ Then, for every $p \in \mathbb{N}$, we can define a k -module homomorphism $\text{gr}_p f : \text{gr}_p V \rightarrow \text{gr}_p W$ as follows²²: Since f respects the filtration, we have $f(V_p) \subseteq W_p$. Thus, f induces a k -module homomorphism $f_p : V_p \rightarrow W_p$ defined by ($f_p(v) = f(v)$ for every $v \in V_p$). This homomorphism f_p sends V_{p-1} to W_{p-1} (since $f(V_{p-1}) \subseteq W_{p-1}$, which is because f respects the filtration), and thus gives rise to a k -module homomorphism $f'_p : V_p/V_{p-1} \rightarrow W_p/W_{p-1}$ which satisfies ($f'_p(\bar{v}) = \overline{f_p(v)}$ for every $v \in V_p$) (where \bar{v} denotes the residue class of v modulo V_{p-1} , and $\overline{f_p(v)}$ denotes the residue class of $f_p(v)$ modulo W_{p-1}). Since $V_p/V_{p-1} = \text{gr}_p V$ and $W_p/W_{p-1} = \text{gr}_p W$, this k -module homomorphism $f'_p : V_p/V_{p-1} \rightarrow W_p/W_{p-1}$ is a k -module homomorphism $f'_p : \text{gr}_p V \rightarrow \text{gr}_p W$. We will denote this homomorphism f'_p by $\text{gr}_p f$. (Strictly speaking, the notation $\text{gr}_p f$ is ambiguous, because the homomorphism $\text{gr}_p f$ depends not only on p and f , but also on the filtrations $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$. But we will never run into ambiguities with this notation, because in our cases the filtrations $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$ will always be clear from the context.)

As one could expect, we can get additional structure if we start at \mathfrak{g} -modules and \mathfrak{g} -module homomorphisms:

Proposition 1.98. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

(a) Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . Then, for every $p \in \mathbb{N}$, the k -module $\text{gr}_p V$ canonically becomes a \mathfrak{g} -module, since it is the quotient module V_p/V_{p-1} and since both V_p and V_{p-1} are \mathfrak{g} -modules.

(b) Let V and W be two \mathfrak{g} -modules. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V ,

²¹Of course, "respecting the filtration" means "respecting the filtrations $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$ " here, because the only filtrations of V and W inferrable from the context are $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$.

²²Here, $\text{gr}_p V$ means $\text{gr}_p(V, (V_n)_{n \geq 0})$, and $\text{gr}_p W$ means $\text{gr}_p(W, (W_n)_{n \geq 0})$, because the only filtrations of V and W inferrable from the context are $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$.

and let $(W_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of W . Let $f : V \rightarrow W$ be a \mathfrak{g} -module homomorphism respecting the filtration. Then, for every $p \in \mathbb{N}$, the k -module homomorphism $\text{gr}_p f : \text{gr}_p V \rightarrow \text{gr}_p W$ is a \mathfrak{g} -module homomorphism.

Proof of Proposition 1.98. Part **(a)** of this proposition is obviously true (there is nothing to prove here). Part **(b)** follows from the definitions (the details are left to the reader).

The following fact is also easy to see:

Proposition 1.99. Let k be a commutative ring.

(a) Let V be a k -module. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V . Then, the map $\text{id} : V \rightarrow V$ respects the filtration, and satisfies $\text{gr}_p \text{id} = \text{id}$ for every $p \in \mathbb{N}$.

(b) Let U, V and W be three k -modules. Let $(U_n)_{n \geq 0}$ be a k -module filtration of U . Let $(V_n)_{n \geq 0}$ be a k -module filtration of V . Let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Let $f : U \rightarrow V$ and $g : V \rightarrow W$ be two k -module homomorphisms respecting the filtration. Then, the homomorphism $g \circ f : U \rightarrow W$ also respects the filtration and satisfies $\text{gr}_p g \circ \text{gr}_p f = \text{gr}_p (g \circ f)$ for every $p \in \mathbb{N}$.

(c) Let V and W be two k -modules. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V . Let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Let $f : V \rightarrow W$ and $g : V \rightarrow W$ be two k -module homomorphisms respecting the filtration. Then, $f - g : V \rightarrow W$ also respects the filtration, and satisfies $\text{gr}_p (f - g) = \text{gr}_p f - \text{gr}_p g$ for every $p \in \mathbb{N}$.

(d) Let V and W be two k -modules. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V . Let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Then, the k -module homomorphism $0 : V \rightarrow W$ (which maps everything to 0) respects the filtration and satisfies $\text{gr}_p 0 = 0$ for every $p \in \mathbb{N}$.

In the language of category theory, Proposition 1.99 says that for each $p \in \mathbb{N}$, Definition 1.97 defines an additive functor gr_p from the category of k -modules with filtration (where morphisms are k -module homomorphisms respecting the filtration) to the category of k -modules.

Warning 1.100. Filtrations of k -modules have one somewhat dangerous property: If we have two k -modules V and W with filtrations $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$, respectively, and an isomorphism $f : V \rightarrow W$ of k -modules which respects the filtration, then we *cannot* (in general) be sure that $\text{gr}_p f : \text{gr}_p V \rightarrow \text{gr}_p W$ is an isomorphism for every $p \in \mathbb{N}$. In order to be able to tell that $\text{gr}_p f$ is an isomorphism, we need to require that f^{-1} also respect the filtration. This is enough due to the following fact:

Proposition 1.101. Let k be a commutative ring. Let V and W be two k -modules. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V , and let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Let $f : V \rightarrow W$ be a k -module isomorphism. If each of the maps f and f^{-1} respects the filtration, then $\text{gr}_p f : \text{gr}_p V \rightarrow \text{gr}_p W$ is a k -module isomorphism for every $p \in \mathbb{N}$.

Proof of Proposition 1.101. Assume that each of the maps f and f^{-1} respects the filtration. Then, according to Definition 1.97 **(b)**, both maps $\text{gr}_p f : \text{gr}_p V \rightarrow \text{gr}_p W$ and $\text{gr}_p (f^{-1}) : \text{gr}_p W \rightarrow \text{gr}_p V$ are well-defined. Proposition 1.99 **(b)** (applied to W , $(W_n)_{n \geq 0}$, f^{-1} and f instead of U , $(U_n)_{n \geq 0}$, f and g) now yields that $\text{gr}_p f \circ$

$\text{gr}_p(f^{-1}) = \text{gr}_p(f \circ f^{-1})$ for every $p \in \mathbb{N}$. On the other hand, Proposition 1.99 (b) (applied to V , $(V_n)_{n \geq 0}$, W , $(W_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$ and f^{-1} instead of U , $(U_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$, W , $(W_n)_{n \geq 0}$ and g) now yields that $\text{gr}_p(f^{-1}) \circ \text{gr}_p f = \text{gr}_p(f^{-1} \circ f)$ for every $p \in \mathbb{N}$. Thus, every $p \in \mathbb{N}$ satisfies $\text{gr}_p f \circ \text{gr}_p(f^{-1}) = \text{gr}_p \underbrace{(f \circ f^{-1})}_{= \text{id}} = \text{gr}_p \text{id} = \text{id}$

(according to Proposition 1.99 (a)) and $\text{gr}_p(f^{-1}) \circ \text{gr}_p f = \text{gr}_p \underbrace{(f^{-1} \circ f)}_{= \text{id}} = \text{gr}_p \text{id} = \text{id}$

(according to Proposition 1.99 (a)). This means that for every $p \in \mathbb{N}$, the two k -module homomorphisms $\text{gr}_p f$ and $\text{gr}_p(f^{-1})$ are inverse to each other. Thus, $\text{gr}_p f$ is a k -module isomorphism. This proves Proposition 1.101.

Another easy fact:

Proposition 1.102. Let k be a commutative ring. Let V and W be two k -modules. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V , and let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Let $f : V \rightarrow W$ be a k -module homomorphism which respects the filtration. Let $p \in \mathbb{N}$. If $f(V_p) = W_p$, then $\text{gr}_p f : \text{gr}_p V \rightarrow \text{gr}_p W$ is surjective.

Proof of Proposition 1.102. Assume that $f(V_p) = W_p$.

We are going to use the notations introduced in Definition 1.97 (b). With these notations, $\text{gr}_p f = f'_p$ (because this is how $\text{gr}_p f$ was defined), $\left(f'_p(\bar{v}) = \overline{f_p(v)}\right)$ for every $v \in V_p$ and $(f_p(v) = f(v))$ for every $v \in V_p$.

Now let $w \in \text{gr}_p W$ be arbitrary. Then, $w \in \text{gr}_p W = W_p/W_{p-1}$, so there exists some $u \in W_p$ such that $w = \bar{u}$ (where \bar{u} denotes the residue class of w modulo W_{p-1}). Since $u \in W_p = f(V_p)$, there exists some $v \in V_p$ such that $u = f(v)$. Thus, $\underbrace{(\text{gr}_p f)}_{= f'_p}(\bar{v}) = f'_p(\bar{v}) = \overline{f_p(v)} = \bar{u}$ (since $f_p(v) = f(v) = u$) leads to $(\text{gr}_p f)(\bar{v}) = \bar{u} = w$.

Thus, $w \in (\text{gr}_p f)(\text{gr}_p U)$. Hence, we have shown that every $w \in \text{gr}_p W$ satisfies $w \in (\text{gr}_p f)(\text{gr}_p U)$. In other words, $\text{gr}_p W \subseteq (\text{gr}_p f)(\text{gr}_p U)$. Thus, $\text{gr}_p f : \text{gr}_p V \rightarrow \text{gr}_p W$ is surjective. Proposition 1.102 is now proven.

Proposition 1.103. Let k be a commutative ring. Let V and W be two k -modules. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V , and let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Let $f : V \rightarrow W$ be a k -module homomorphism which respects the filtration. Let $p \in \mathbb{N}$. If $f(V_p) \subseteq W_{p-1}$, then $\text{gr}_p f = 0$.

Proof of Proposition 1.103. Assume that $f(V_p) \subseteq W_{p-1}$.

We are going to use the notations introduced in Definition 1.97 (b). With these notations, $\text{gr}_p f = f'_p$ (because this is how $\text{gr}_p f$ was defined), $\left(f'_p(\bar{v}) = \overline{f_p(v)}\right)$ for every $v \in V_p$ and $(f_p(v) = f(v))$ for every $v \in V_p$. This means that every $v \in V_p$ satisfies $f'_p(\bar{v}) = \overline{f_p(v)} = 0$ (because $v \in V_p$ yields $f(v) \in f(V_p) \subseteq W_{p-1}$ and thus $f_p(v) = f(v) \in W_{p-1}$). Since every element $x \in V_p/V_{p-1}$ can be written in the form $x = \bar{v}$ for some $v \in V_p$, this yields that $f'_p(x) = 0$ for every $x \in V_p/V_{p-1}$ (because writing x in the form $x = \bar{v}$ for some $v \in V_p$, we get $f'_p(x) = f'_p(\bar{v}) = 0$). In other words, $f'_p = 0$. Thus, $\text{gr}_p f = f'_p = 0$. Proposition 1.103 is now proven.

One more triviality:

Proposition 1.104. Let k be a commutative ring. Let V and W be two k -modules. Let $f : V \rightarrow W$ be a k -module homomorphism. Then, the k -module homomorphism $\otimes f : \otimes V \rightarrow \otimes W$ respects the filtration, where the filtrations on $\otimes V$ and $\otimes W$ are the degree filtrations (i. e., the filtration on $\otimes V$ is $(V^{\otimes \leq n})_{n \geq 0}$, and the filtration on $\otimes W$ is $(W^{\otimes \leq n})_{n \geq 0}$).

Another definition related to degree filtrations:

Definition 1.105. Let k be a commutative ring. Let V be a k -module. Let $p \in \mathbb{N}$. Let $\text{grad}_{V,p}$ denote the map $V^{\otimes p} \rightarrow V^{\otimes \leq p} / V^{\otimes \leq (p-1)}$ which sends every $T \in V^{\otimes p}$ to the equivalence class of $T \in V^{\otimes p} \subseteq V^{\otimes \leq p}$ modulo $V^{\otimes \leq (p-1)}$. Since $V^{\otimes \leq p} / V^{\otimes \leq (p-1)} = \text{gr}_p(\otimes V)$ (because the filtration on $\otimes V$ is the degree filtration $(V^{\otimes \leq n})_{n \geq 0}$, and thus $\text{gr}_p(\otimes V)$ is defined as $V^{\otimes \leq p} / V^{\otimes \leq (p-1)}$), we see that this map $\text{grad}_{V,p} : V^{\otimes p} \rightarrow V^{\otimes \leq p} / V^{\otimes \leq (p-1)}$ is a map $V^{\otimes p} \rightarrow \text{gr}_p(\otimes V)$.

Proposition 1.106. Let k be a commutative ring. Let V be a k -module. Let $p \in \mathbb{N}$. Then, the map $\text{grad}_{V,p} : V^{\otimes p} \rightarrow \text{gr}_p(\otimes V)$ defined in Definition 1.105 is a canonical k -module isomorphism.

Proof of Proposition 1.106. We have $V^{\otimes \leq (p-1)} = \bigoplus_{i=0}^{p-1} V^{\otimes i}$ (by the definition of $V^{\otimes \leq (p-1)}$) and

$$\begin{aligned} V^{\otimes \leq p} &= \bigoplus_{i=0}^p V^{\otimes i} && \text{(by the definition of } V^{\otimes \leq p}\text{)} \\ &= \underbrace{\left(\bigoplus_{i=0}^{p-1} V^{\otimes i} \right)}_{=V^{\otimes \leq (p-1)}} \oplus V^{\otimes p} = V^{\otimes \leq (p-1)} \oplus V^{\otimes p}. \end{aligned}$$

Now, we need a known lemma from algebra:

Lemma 1.107. Let A and B be two k -modules. Consider A as a k -submodule of $A \oplus B$. Then, the map $B \rightarrow (A \oplus B) / A$ which sends every $T \in B$ to the equivalence class of $T \in B \subseteq A \oplus B$ modulo A is a k -module isomorphism.

Lemma 1.107 (applied to $A = V^{\otimes \leq (p-1)}$ and $B = V^{\otimes p}$) yields that the map $V^{\otimes p} \rightarrow (V^{\otimes \leq (p-1)} \oplus V^{\otimes p}) / V^{\otimes \leq (p-1)}$ which sends every $T \in V^{\otimes p}$ to the equivalence class of $T \in V^{\otimes p} \subseteq V^{\otimes \leq (p-1)} \oplus V^{\otimes p}$ modulo $V^{\otimes \leq (p-1)}$ is a k -module isomorphism. Since $V^{\otimes \leq (p-1)} \oplus V^{\otimes p} = V^{\otimes \leq p}$, this rewrites as follows: The map $V^{\otimes p} \rightarrow V^{\otimes \leq p} / V^{\otimes \leq (p-1)}$ which sends every $T \in V^{\otimes p}$ to the equivalence class of $T \in V^{\otimes p} \subseteq V^{\otimes \leq (p-1)} \oplus V^{\otimes p}$ modulo $V^{\otimes \leq (p-1)}$ is a k -module isomorphism. Since the map $V^{\otimes p} \rightarrow V^{\otimes \leq p} / V^{\otimes \leq (p-1)}$ which sends every $T \in V^{\otimes p}$ to the equivalence class of $T \in V^{\otimes p} \subseteq V^{\otimes \leq (p-1)} \oplus V^{\otimes p}$ modulo $V^{\otimes \leq (p-1)}$ is simply the map $\text{grad}_{V,p}$ (by the definition of $\text{grad}_{V,p}$), this yields that the map $\text{grad}_{V,p}$ is a k -module isomorphism. Since this map $\text{grad}_{V,p}$ is clearly canonical, we have thus proven Proposition 1.106.

Also, we can easily see that:

Proposition 1.108. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $p \in \mathbb{N}$. Then, the map $\text{grad}_{V,p} : V^{\otimes p} \rightarrow \text{gr}_p(\otimes V)$ defined in Definition 1.105 is a canonical \mathfrak{g} -module isomorphism.

Proof of Proposition 1.108. In order to obtain a proof of Proposition 1.108, it is enough to apply the following changes to the proof of Proposition 1.106:

- Replace "Proposition 1.106" by "Proposition 1.108".
- Replace the words " k -module" by " \mathfrak{g} -module".

This completes the proof of Proposition 1.108.

1.17. Filtrations and isomorphisms

The next proposition will not be used until Section 4, but is still very elementary:

Proposition 1.109. Let k be a commutative ring. Let V be a k -module. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V . Let $f : V \rightarrow V$ be a k -module homomorphism which satisfies

$$(f(V_n) \subseteq V_{n-1} \quad \text{for every } n \in \mathbb{N}),$$

where V_{-1} denotes the k -submodule 0 of V . Then:

- The k -module homomorphism $\text{id} - f$ is an isomorphism.
- Each of the maps $\text{id} - f$ and $(\text{id} - f)^{-1}$ respects the filtration.

Note that this proposition is a generalization of the following well-known fact:

Corollary 1.110. Let k be a commutative ring. Let $m \in \mathbb{N}$. Let $A \in M_m(k)$ be a strictly upper triangular $m \times m$ matrix over k . Then, $I_m - A$ is an invertible matrix, and its inverse is an upper triangular matrix.

The following quick proof of this fact by means of applying Proposition 1.109 is not here because we are going to use it, but rather because it provides a kind of intuition for the meaning of Proposition 1.109 (even if only in a particular case).

Proof of Corollary 1.110 (sketched). Define a filtration $(V_n)_{n \geq 0}$ of the k -module k^m by $\left(V_n = \begin{cases} \langle e_1, e_2, \dots, e_n \rangle, & \text{if } n \leq m; \\ k^m, & \text{if } n > m \end{cases} \text{ for every } n \in \mathbb{N} \right)$, where (e_1, e_2, \dots, e_m) is the standard basis of the k -module k^m . Let f be the k -linear map $k^m \rightarrow k^m$ represented by the matrix A . Then, since A is a strictly upper triangular matrix, it is easy to see that $f(V_n) \subseteq V_{n-1}$ for every $n \in \mathbb{N}$ (where V_{-1} denotes the k -submodule 0 of k^m). Thus, Proposition 1.109 (a) yields that $\text{id} - f$ is an isomorphism, and Proposition 1.109 (b) yields that its inverse $(\text{id} - f)^{-1}$ respects the filtration. Translating this back into the language of matrices, we obtain that $I_m - A$ is an invertible matrix, and its inverse is an upper triangular matrix. This proves Corollary 1.110.

Proof of Proposition 1.109. We notice that every $n \in \mathbb{N}$ satisfies $V_{n-1} \subseteq V_n$ ²³.

²³*Proof.* Let $n \in \mathbb{N}$ be arbitrary. Then, two cases are possible: the case when $n = 0$, and the case when $n > 0$. In the case $n = 0$, we have $V_{n-1} = V_{0-1} = V_{-1} = 0 \subseteq V_n$. In the case $n > 0$, we have $V_{n-1} \subseteq V_n$ (because $(V_n)_{n \geq 0}$ is a k -module filtration). Hence, in both cases $n = 0$ and $n > 0$, we have $V_{n-1} \subseteq V_n$. Thus, $V_{n-1} \subseteq V_n$ always holds.

Now, every $n \in \mathbb{N}$ satisfies $f(V_n) \subseteq V_{n-1} \subseteq V_n$. Hence, the map f respects the filtration. Since the map id also respects the filtration, it follows from Proposition 1.99 (c) (applied to id , f , V and $(V_n)_{n \geq 0}$ instead of f , g , W and $(W_n)_{n \geq 0}$) that the map $\text{id} - f$ also respects the filtration.

Next, let us show that for every $m \in \mathbb{N}$,

$$\text{the map } \sum_{\kappa=0}^m f^\kappa : V \rightarrow V \text{ respects the filtration.} \quad (38)$$

Proof of (38). We are going to verify (38) by induction over m :

Induction base: For $m = 0$, the map $\sum_{\kappa=0}^m f^\kappa : V \rightarrow V$ equals $\sum_{\kappa=0}^0 f^\kappa = f^0 = \text{id}$ and thus respects the filtration (since it is known that id respects the filtration). Thus, (38) holds for $m = 0$. The induction base is thus complete.

Induction step: Let $n \in \mathbb{N}_+$ be arbitrary. Assume that (38) holds for $m = n - 1$. We then must prove that (38) holds for $m = n$.

Since (38) holds for $m = n - 1$, we know that the map $\sum_{\kappa=0}^{n-1} f^\kappa : V \rightarrow V$ respects the filtration. Combined with the fact that the map f respects the filtration, this yields that the map $f \circ \sum_{\kappa=0}^{n-1} f^\kappa$ respects the filtration (according to Proposition 1.99 (b), applied to f , $\sum_{\kappa=0}^{n-1} f^\kappa$, V , $(V_n)_{n \geq 0}$, V and $(V_n)_{n \geq 0}$ instead of g , f , U , $(U_n)_{n \geq 0}$, W and $(W_n)_{n \geq 0}$). Combined with the (trivial) fact that the map $-\text{id}$ respects the filtration, this yields that the map $f \circ \sum_{\kappa=0}^{n-1} f^\kappa - (-\text{id})$ respects the filtration (according to Proposition 1.99 (c), applied to $f \circ \sum_{\kappa=0}^{n-1} f^\kappa$, $-\text{id}$, V and $(V_n)_{n \geq 0}$ instead of f , g , W and $(W_n)_{n \geq 0}$). Since

$$\begin{aligned} f \circ \sum_{\kappa=0}^{n-1} f^\kappa - (-\text{id}) &= \underbrace{f \circ \sum_{\kappa=0}^{n-1} f^\kappa}_{= \sum_{\kappa=0}^{n-1} f \circ f^\kappa} + \underbrace{\text{id}}_{= f^0} = \sum_{\kappa=0}^{n-1} \underbrace{f \circ f^\kappa}_{= f^{\kappa+1}} + f^0 = \sum_{\kappa=0}^{n-1} f^{\kappa+1} + f^0 \\ &= \sum_{\kappa=1}^n f^\kappa + f^0 \quad (\text{here we substituted } \kappa \text{ for } \kappa + 1 \text{ in the sum}) \\ &= \sum_{\kappa=0}^n f^\kappa, \end{aligned}$$

this rewrites as follows: The map $\sum_{\kappa=0}^n f^\kappa$ respects the filtration. In other words, (38) holds for $m = n$. This completes the induction step. Thus (38) is proven for every $m \in \mathbb{N}$.

Next we will prove that every $\ell \in \mathbb{N}$ satisfies

$$f^{\ell+1}(V_\ell) = 0. \quad (39)$$

Proof of (39). We are going to verify (39) by induction over ℓ :

Induction base: For $\ell = 0$, we have

$$\begin{aligned} f^{\ell+1}(V_\ell) &= \underbrace{f^{0+1}}_{=f^1=f}(V_0) = f(V_0) \subseteq V_{0-1} \\ &\quad \text{(due to the condition } (f(V_n) \subseteq V_{n-1} \text{ for every } n \in \mathbb{N}), \text{ applied to } n = 0) \\ &= V_{-1} = 0. \end{aligned}$$

In other words, for $\ell = 0$, we have $f^{\ell+1}(V_\ell) = 0$. Thus, (39) holds for $\ell = 0$. The induction base is thus complete.

Induction step: Let $n \in \mathbb{N}_+$ be arbitrary. Assume that (39) holds for $\ell = n - 1$. We then must prove that (39) holds for $\ell = n$.

Since (39) holds for $\ell = n - 1$, we have $f^{(n-1)+1}(V_{n-1}) = 0$. Now,

$$\underbrace{f^{n+1}}_{=f^n \circ f}(V_n) = (f^n \circ f)(V_n) = \underbrace{f^n}_{=f^{(n-1)+1}}\left(\underbrace{f(V_n)}_{\subseteq V_{n-1}}\right) \subseteq f^{(n-1)+1}(V_{n-1}) = 0,$$

so that $f^{n+1}(V_n) = 0$. In other words, (39) holds for $\ell = n$. This completes the induction step. Thus (39) is proven for every $\ell \in \mathbb{N}$.

Now, let us define a map $R : V \rightarrow V$ as follows:

For every $v \in V$, let us define the element $R(v)$ as follows:

Since $(V_n)_{n \geq 0}$ is a k -module filtration of V , we have $\bigcup_{n \geq 0} V_n = V$ (by the definition of a filtration). Now, $v \in V = \bigcup_{n \geq 0} V_n$ yields that there exists some $m \in \mathbb{N}$ such that $v \in V_m$. Now, the element $\sum_{\kappa=0}^m f^\kappa(v)$ does *not* depend on m (but only depends on v).²⁴ Therefore, we can define an element $R(v)$ of V (only depending on v) by

²⁴*Proof.* Let m_1 be an element of \mathbb{N} such that $v \in V_{m_1}$. Let m_2 be an element of \mathbb{N} such that $v \in V_{m_2}$.

We are now going to prove that $\sum_{\kappa=0}^{m_1} f^\kappa(v) = \sum_{\kappa=0}^{m_2} f^\kappa(v)$.

In fact, we can WLOG assume that $m_1 \geq m_2$ (otherwise, we switch m_1 and m_2). So let us assume this. We have $v \in V_{m_2}$ and thus $f^{m_2+1}(v) \in f^{m_2+1}(V_{m_2}) = 0$ (by (39), applied to $\ell = m_2$), so that $f^{m_2+1}(v) = 0$. Now, every $\kappa \geq m_2 + 1$ satisfies $f^\kappa = f^{(\kappa-(m_2+1)) \circ (m_2+1)} = f^{\kappa-(m_2+1)} \circ f^{m_2+1}$.

Hence, every $\kappa \geq m_2 + 1$ satisfies $f^\kappa(v) = (f^{\kappa-(m_2+1)} \circ f^{m_2+1})(v) = f^{\kappa-(m_2+1)}\left(\underbrace{f^{m_2+1}(v)}_{=0}\right) =$

$f^{\kappa-(m_2+1)}(0) = 0$ (since $f^{\kappa-(m_2+1)}$ is k -linear). Now,

$$\sum_{\kappa=0}^{m_1} f^\kappa(v) = \sum_{\kappa=0}^{m_2} f^\kappa(v) + \sum_{\kappa=m_2+1}^{m_1} \underbrace{f^\kappa(v)}_{=0 \text{ (since } \kappa \geq m_2+1)} = \sum_{\kappa=0}^{m_2} f^\kappa(v) + \underbrace{\sum_{\kappa=m_2+1}^{m_1} 0}_{=0} = \sum_{\kappa=0}^{m_2} f^\kappa(v).$$

Thus we have proven that, for every element $m_1 \in \mathbb{N}$ such that $v \in V_{m_1}$, and for every element $m_2 \in \mathbb{N}$ such that $v \in V_{m_2}$, we have $\sum_{\kappa=0}^{m_1} f^\kappa(v) = \sum_{\kappa=0}^{m_2} f^\kappa(v)$. In other words, we have proven that

any two values of $m \in \mathbb{N}$ such that $v \in V_m$ lead to the same value for $\sum_{\kappa=0}^m f^\kappa(v)$. In other words,

we have proven that the element $\sum_{\kappa=0}^m f^\kappa(v)$ does *not* depend on m (as long as m satisfies $v \in V_m$).

$$R(v) = \sum_{\kappa=0}^m f^\kappa(v).$$

Thus we have defined an element $R(v) \in V$ for every $v \in V$. In other words, we have thus defined a map $R : V \rightarrow V$.

According to its definition, this map R satisfies

$$R(v) = \sum_{\kappa=0}^m f^\kappa(v) \text{ for every } m \in \mathbb{N} \text{ satisfying } v \in V_m. \quad (40)$$

Now we are going to show that $R \circ (\text{id} - f) = \text{id}$.

Proof of $R \circ (\text{id} - f) = \text{id}$. Let $w \in V$ be arbitrary. Let $v = (\text{id} - f)(w)$.

Since $w \in V = \bigcup_{n \geq 0} V_n$, there exists some $m \in \mathbb{N}$ such that $w \in V_m$. Consider this m .

Then,

$$\begin{aligned} v &= (\text{id} - f)(w) = \underbrace{\text{id}(w)}_{=w \in V_m} - \underbrace{f(w)}_{\in f(V_m) \text{ (since } w \in V_m)} \\ &\subseteq V_m - \underbrace{f(V_m)}_{\subseteq V_m \text{ (since the map } f \text{ respects the filtration)}} \subseteq V_m - V_m \subseteq V_m \quad (\text{since } V_m \text{ is a } k\text{-module}). \end{aligned}$$

Now,

$$\begin{aligned} &(R \circ (\text{id} - f))(w) \\ &= R\left(\underbrace{(\text{id} - f)(w)}_{=v}\right) = R(v) \\ &= \sum_{\kappa=0}^m f^\kappa\left(\underbrace{v}_{=(\text{id} - f)(w) = w - f(w)}\right) \quad (\text{according to (40), since } v \in V_m) \\ &= \sum_{\kappa=0}^m \underbrace{f^\kappa(w - f(w))}_{=f^\kappa(w) - f^\kappa(f(w)) \text{ (since } f^\kappa \text{ is } k\text{-linear)}} = \sum_{\kappa=0}^m \left(f^\kappa(w) - \underbrace{f^\kappa(f(w))}_{=(f^\kappa \circ f)(w)} \right) = \sum_{\kappa=0}^m \left(f^\kappa(w) - \underbrace{(f^\kappa \circ f)(w)}_{=f^{\kappa+1}} \right) \\ &= \sum_{\kappa=0}^m (f^\kappa(w) - f^{\kappa+1}(w)) = \sum_{\kappa=0}^m f^\kappa(w) - \sum_{\kappa=0}^m f^{\kappa+1}(w) = \underbrace{\sum_{\kappa=0}^m f^\kappa(w)}_{=f^0(w) + \sum_{\kappa=1}^m f^\kappa(w)} - \underbrace{\sum_{\kappa=1}^{m+1} f^\kappa(w)}_{=\sum_{\kappa=1}^m f^\kappa(w) + f^{m+1}(w)} \\ &\quad (\text{here, we substituted } \kappa \text{ for } \kappa + 1 \text{ in the second sum}) \\ &= \left(f^0(w) + \sum_{\kappa=1}^m f^\kappa(w) \right) - \left(\sum_{\kappa=1}^m f^\kappa(w) + f^{m+1}(w) \right) = f^0(w) - f^{m+1}(w). \end{aligned}$$

Since $\underbrace{f^0}_{=\text{id}}(w) = \text{id}(w)$ and $f^{m+1}(w) = 0$ (because $w \in V_m$ and thus $f^{m+1}(w) \in f^{m+1}(V_m) = 0$ (due to (39), applied to $\ell = m$), so that $f^{m+1}(w) = 0$), this rewrites

as $(R \circ (\text{id} - f))(w) = \text{id}(w) - 0 = \text{id}(w)$. Since this holds for all $w \in V$, we thus conclude that $R \circ (\text{id} - f) = \text{id}$.

This completes the proof of $R \circ (\text{id} - f) = \text{id}$.

Proof of $(\text{id} - f) \circ R = \text{id}$. Let $v \in V$ be arbitrary.

Since $v \in V = \bigcup_{n \geq 0} V_n$, there exists some $m \in \mathbb{N}$ such that $v \in V_m$. Consider this m .

Then,

$$\begin{aligned}
 ((\text{id} - f) \circ R)(v) &= (\text{id} - f)(R(v)) = (\text{id} - f)\left(\sum_{\kappa=0}^m f^\kappa(v)\right) \\
 &\quad \left(\text{since (40) says that } R(v) = \sum_{\kappa=0}^m f^\kappa(v)\right) \\
 &= \underbrace{\text{id}\left(\sum_{\kappa=0}^m f^\kappa(v)\right)}_{=\sum_{\kappa=0}^m f^\kappa(v)} - \underbrace{f\left(\sum_{\kappa=0}^m f^\kappa(v)\right)}_{=\sum_{\kappa=0}^m f(f^\kappa(v))} \\
 &\quad \text{(since } f \text{ is } k\text{-linear)} \\
 &= \sum_{\kappa=0}^m f^\kappa(v) - \sum_{\kappa=0}^m \underbrace{(f \circ f^\kappa)(v)}_{=f^{\kappa+1}(v)} = \sum_{\kappa=0}^m f^\kappa(v) - \sum_{\kappa=0}^m f^{\kappa+1}(v) \\
 &= \underbrace{\sum_{\kappa=0}^m f^\kappa(v)}_{=f^0(v) + \sum_{\kappa=1}^m f^\kappa(v)} - \underbrace{\sum_{\kappa=1}^{m+1} f^\kappa(v)}_{=\sum_{\kappa=1}^m f^\kappa(v) + f^{m+1}(v)} \\
 &\quad \text{(here, we substituted } \kappa \text{ for } \kappa + 1 \text{ in the second sum)} \\
 &= \left(f^0(v) + \sum_{\kappa=1}^m f^\kappa(v)\right) - \left(\sum_{\kappa=1}^m f^\kappa(v) + f^{m+1}(v)\right) = f^0(v) - f^{m+1}(v).
 \end{aligned}$$

Since $\underbrace{f^0}_{=\text{id}}(v) = \text{id}(v)$ and $f^{m+1}(v) = 0$ (because $v \in V_m$ and thus $f^{m+1}(v) \in f^{m+1}(V_m) = 0$ (due to (39), applied to $\ell = m$), so that $f^{m+1}(v) = 0$), this rewrites as $((\text{id} - f) \circ R)(v) = \text{id}(v) - 0 = \text{id}(v)$. Since this holds for all $v \in V$, we thus conclude that $(\text{id} - f) \circ R = \text{id}$.

The proof of $(\text{id} - f) \circ R = \text{id}$ is thus complete.

We have thus shown that $R \circ (\text{id} - f) = \text{id}$ and $(\text{id} - f) \circ R = \text{id}$. In other words, the two maps $\text{id} - f$ and R are mutually inverse. Hence, the map $\text{id} - f$ is invertible, and its inverse is R .

The k -module homomorphism $\text{id} - f$ is a k -module isomorphism (since it is invertible). This proves Proposition 1.109 (a).

We already know that the map $\text{id} - f$ respects the filtration. Let us now prove that so does R :

Let $n \in \mathbb{N}$ be arbitrary. Every $v \in V_n$ satisfies

$$\begin{aligned} R(v) &= \sum_{\kappa=0}^n f^\kappa(v) && \text{(by (40), applied to } m = n) \\ &= \left(\sum_{\kappa=0}^n f^\kappa \right)(v) \in \left(\sum_{\kappa=0}^n f^\kappa \right)(V_n) && \text{(since } v \in V_n) \\ &\subseteq V_n && \left(\begin{array}{l} \text{since (38) (applied to } m = n) \text{ says that} \\ \text{the map } \sum_{\kappa=0}^n f^\kappa \text{ respects the filtration} \end{array} \right). \end{aligned}$$

In other words, $R(V_n) \subseteq V_n$. Since this holds for every $n \in \mathbb{N}$, we thus have proven that the map R respects the filtration. Since $R = (\text{id} - f)^{-1}$ (because we know that R is the inverse of $\text{id} - f$), this yields that the map $(\text{id} - f)^{-1}$ respects the filtration. Combined with the fact that $\text{id} - f$ respects the filtration, this completes the proof of Proposition 1.109 **(b)**.

1.18. A consequence about isomorphisms

We are not going to use Proposition 1.109 directly; instead we will apply the following corollary:

Corollary 1.111. Let k be a commutative ring. Let V and W be two k -modules. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V . Let $(W_n)_{n \geq 0}$ be a k -module filtration of W . Let $f : V \rightarrow W$ and $g : W \rightarrow V$ be two k -module homomorphisms such that

$$((\text{id} - g \circ f)(V_n) \subseteq V_{n-1} \quad \text{for every } n \in \mathbb{N}),$$

where V_{-1} denotes the k -submodule 0 of V . Assume further that f is a k -module isomorphism, and that f and f^{-1} respect the filtration. Then:

- (a)** The k -module homomorphism g is an isomorphism.
- (b)** Each of the maps g and g^{-1} respects the filtration.

Proof of Corollary 1.111. Since $(\text{id} - g \circ f)(V_n) \subseteq V_{n-1}$ for every $n \in \mathbb{N}$, we can apply Proposition 1.109 to $\text{id} - g \circ f$ instead of f . Hence, Proposition 1.109 **(a)** (applied to $\text{id} - g \circ f$ instead of f) yields that the k -module homomorphism $\text{id} - (\text{id} - g \circ f)$ is an isomorphism. On the other hand, Proposition 1.109 **(b)** (applied to $\text{id} - g \circ f$ instead of f) yields that each of the maps $\text{id} - (\text{id} - g \circ f)$ and $(\text{id} - (\text{id} - g \circ f))^{-1}$ respects the filtration.

We know that $\text{id} - (\text{id} - g \circ f)$ is an isomorphism. Since $\text{id} - (\text{id} - g \circ f) = g \circ f$, we thus conclude that $g \circ f$ is an isomorphism. But since f^{-1} is an isomorphism (this is because f is an isomorphism), the composition $(g \circ f) \circ f^{-1}$ must also be an isomorphism (because $g \circ f$ and f^{-1} are isomorphisms, and because the composition of two isomorphisms is an isomorphism). Since $(g \circ f) \circ f^{-1} = g$, this yields that g is an isomorphism. This proves Corollary 1.111 **(a)**.

Now, we know that each of the maps $\text{id} - (\text{id} - g \circ f)$ and $(\text{id} - (\text{id} - g \circ f))^{-1}$ respects the filtration. Since $\text{id} - (\text{id} - g \circ f) = g \circ f$ and $\underbrace{(\text{id} - (\text{id} - g \circ f))^{-1}}_{=g \circ f} = (g \circ f)^{-1} =$

$f^{-1} \circ g^{-1}$, this result rewrites as follows: Each of the maps $g \circ f$ and $f^{-1} \circ g^{-1}$ respects the filtration.

Since each of the maps $f^{-1} \circ g^{-1}$ and f respects the filtration, Proposition 1.99 (b) (applied to V , $(V_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$, W , $(W_n)_{n \geq 0}$, $f^{-1} \circ g^{-1}$ and f instead of U , $(U_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$, W , $(W_n)_{n \geq 0}$, f and g) yields that the map $f \circ (f^{-1} \circ g^{-1})$ respects the filtration. Since $f \circ (f^{-1} \circ g^{-1}) = g^{-1}$, this rewrites as follows: The map g^{-1} respects the filtration.

Since each of the maps $g \circ f$ and f^{-1} respects the filtration, Proposition 1.99 (b) (applied to W , $(W_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$, f^{-1} and $g \circ f$ instead of U , $(U_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$, W , $(W_n)_{n \geq 0}$, f and g) yields that the map $(g \circ f) \circ f^{-1}$ respects the filtration. Since $(g \circ f) \circ f^{-1} = g$, this rewrites as follows: The map g respects the filtration.

This completes the proof of Corollary 1.111.

1.19. Splitting of filtrations of \mathfrak{g} -modules

Some filtrations are simpler than others. One particular property some \mathfrak{g} -module filtrations have is being *\mathfrak{g} -split*:

Definition 1.112. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . For every integer $n \geq 1$, let ι_n denote the canonical inclusion of V_{n-1} into V_n , and let π_n denote the canonical projection of V_n onto V_n/V_{n-1} . We say that the \mathfrak{g} -module filtration $(V_n)_{n \geq 0}$ is *\mathfrak{g} -split* if and only if

$$\left(\begin{array}{c} \text{the short exact sequence} \\ 0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0 \text{ is } \mathfrak{g}\text{-split for every } n \geq 1 \end{array} \right).$$

The next proposition shows that \mathfrak{g} -split filtrations are those coming from direct sum decompositions of V :

Proposition 1.113. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . Then, the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split if and only if there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V such that $V = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $V_p = \bigoplus_{n=0}^p W_n$.

Note that this Proposition 1.113 depends on the countable Axiom of Choice.

Remark 1.114. A remark about notation: In Proposition 1.113, we could have just as well written $(W_n)_{n \geq 0}$ instead of $(W_n)_{n \in \mathbb{N}}$. In fact, $(W_n)_{n \geq 0}$ and $(W_n)_{n \in \mathbb{N}}$ are two equivalent ways to denote one and the same family. However, we prefer to use the notation $(W_n)_{n \in \mathbb{N}}$ for families of \mathfrak{g} -submodules of V satisfying $V = \bigoplus_{n \in \mathbb{N}} W_n$ (such families are called " \mathfrak{g} -module gradings" of V) as opposed to the notation $(V_n)_{n \geq 0}$ for filtrations, in order to make gradings easier to distinguish from filtrations (there is no deeper reason behind this).

Proof of Proposition 1.113. In order to verify Proposition 1.113, we must show two lemmas:

Lemma 1.115. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . If the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split, then there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V such that $V = \bigoplus_{n \in \mathbb{N}} W_n$

and such that every $p \in \mathbb{N}$ satisfies $V_p = \bigoplus_{n=0}^p W_n$.

Lemma 1.116. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . If there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V such that $V = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$

satisfies $V_p = \bigoplus_{n=0}^p W_n$, then the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split.

Proof of Lemma 1.115. Assume that the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split.

For every $n \geq 1$, let ι_n denote the canonical inclusion of V_{n-1} into V_n , and let π_n denote the canonical projection of V_n onto V_n/V_{n-1} .

The filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split. According to Definition 1.112, this means that the short exact sequence $0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0$ is \mathfrak{g} -split for every $n \geq 1$.

Thus, for every $n \geq 1$, the short exact sequence $0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0$ is \mathfrak{g} -split.

Let $n \in \mathbb{N}_+$ be arbitrary. Then, $n \geq 1$. Applying Corollary 1.89 to V_{n-1} , V_n , V_n/V_{n-1} , ι_n and π_n instead of A , B , C , f and g , we see that the exact sequence $0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0$ is \mathfrak{g} -split if and only if there exists a \mathfrak{g} -submodule P of V_n such that $V_n = \iota_n(V_{n-1}) \oplus P$. Since we know that the exact sequence $0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0$ is \mathfrak{g} -split, we thus conclude that there exists a \mathfrak{g} -submodule P of V_n such that $V_n = \iota_n(V_{n-1}) \oplus P$. Denote this \mathfrak{g} -submodule P by W_n . Then, $V_n = \iota_n(V_{n-1}) \oplus W_n = V_{n-1} \oplus W_n$ (since ι_n is the inclusion map, and thus $\iota_n(V_{n-1}) = V_{n-1}$).

We have thus defined a \mathfrak{g} -submodule W_n of V_n for every $n \in \mathbb{N}_+$, and this \mathfrak{g} -submodule satisfies $V_n = V_{n-1} \oplus W_n$.

Let us additionally define a \mathfrak{g} -submodule W_0 of V_0 by $W_0 = V_0$. Thus, a \mathfrak{g} -submodule W_n of V_n is defined for every $n \in \mathbb{N}$ (because we have defined a \mathfrak{g} -submodule W_n of V_n for every $n \in \mathbb{N}_+$, and we have defined a \mathfrak{g} -submodule W_0 of V_0 , but $\mathbb{N}_+ \cup \{0\} = \mathbb{N}$). Hence, we have defined a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V .

Now let us prove that

$$\text{every } p \in \mathbb{N} \text{ satisfies } V_p = \bigoplus_{n=0}^p W_n. \quad (41)$$

Proof of (41). We are going to verify (41) by induction over p :

Induction base: For $p = 0$, we have $V_p = V_0 = W_0 = \bigoplus_{n=0}^0 W_n = \bigoplus_{n=0}^p W_n$ (since $0 = p$).

In other words, (41) holds for $p = 0$. This completes the induction base.

Induction step: Let $q \in \mathbb{N}_+$ be arbitrary. Assume that (41) is proven for $p = q - 1$. Now we must prove (41) for $p = q$.

Since (41) is proven for $p = q - 1$, we have $V_{q-1} = \bigoplus_{n=0}^{q-1} W_n$. Now, we know that every $n \in \mathbb{N}_+$ satisfies $V_n = V_{n-1} \oplus W_n$. Applying this to $n = q$, we obtain $V_q = V_{q-1} \oplus W_q$, and thus

$$V_q = \underbrace{V_{q-1}}_{= \bigoplus_{n=0}^{q-1} W_n} \oplus W_q = \left(\bigoplus_{n=0}^{q-1} W_n \right) \oplus W_q = \bigoplus_{n=0}^q W_n.$$

Thus, we have proven (41) for $p = q$. This completes the induction step. The proof of (41) is thus complete.

Next we are going to show that $V = \bigoplus_{n \in \mathbb{N}} W_n$.

In fact, we first note that every $p \in \mathbb{N}$ satisfies

$$\begin{aligned} V_p &= \bigoplus_{n=0}^p W_n && \text{(by (41))} \\ &= \sum_{n=0}^p W_n && \text{(since direct sums are sums)} \\ &\subseteq \sum_{n=0}^{\infty} W_n, \end{aligned}$$

and thus

$$\begin{aligned} V &= \bigcup_{p \in \mathbb{N}} \underbrace{V_p}_{\subseteq \sum_{n=0}^{\infty} W_n} && \text{(since } (V_n)_{n \geq 0} \text{ is a } \mathfrak{g}\text{-module filtration of } V) \\ &\subseteq \bigcup_{p \in \mathbb{N}} \sum_{n=0}^{\infty} W_n = \sum_{n=0}^{\infty} W_n. \end{aligned}$$

Combined with the obvious relation $\sum_{n=0}^{\infty} W_n \subseteq V$, this yields $V = \sum_{n=0}^{\infty} W_n$. Now, in

order to prove that $V = \bigoplus_{n \in \mathbb{N}} W_n$, we only need to show that the sum $\sum_{n=0}^{\infty} W_n$ is a direct sum.

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements of V satisfying $(a_n \in W_n \text{ for every } n \in \mathbb{N})$, $(a_n = 0 \text{ for all but finitely many } n \in \mathbb{N})$ and $\sum_{n=0}^{\infty} a_n = 0$.

Since $(a_n = 0 \text{ for all but finitely many } n \in \mathbb{N})$, there exists a finite subset S of \mathbb{N} such that $(a_n = 0 \text{ for all } n \in \mathbb{N} \setminus S)$. Consider this S . Clearly, the set S has an upper bound (because S is finite). Let M be this upper bound. Then, every $n \in S$ satisfies $n \leq M$ (because M is an upper bound of the set S). Hence, every $n \in \mathbb{N}$ satisfying $n > M$ must satisfy $n \notin S$ (because otherwise, it would satisfy $n \in S$, therefore $n \leq M$, which would contradict $n > M$). Hence, every $n \in \mathbb{N}$ satisfying $n > M$ must satisfy $a_n = 0$ (because $n > M$ yields $n \notin S$, so that $n \in \mathbb{N} \setminus S$ and thus $a_n = 0$). On the

other hand, (41) (applied to $p = M$) yields $V_M = \bigoplus_{n=0}^M W_n$. Thus, the sum $\sum_{n=0}^M W_n$ is a direct sum.

Now,

$$0 = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^M a_n + \sum_{n=M+1}^{\infty} \underbrace{a_n}_{\text{(since } n > M)} = \sum_{n=0}^M a_n + \underbrace{\sum_{n=M+1}^{\infty} 0}_{=0} = \sum_{n=0}^M a_n.$$

Now, $(a_n)_{n \in \{0,1,\dots,M\}}$ is an $(M+1)$ -tuple of elements of V satisfying

$(a_n \in W_n \text{ for every } n \in \{0,1,\dots,M\})$ and $\sum_{n=0}^M a_n = 0$. Therefore, $a_n = 0$ for every

$n \in \{0,1,\dots,M\}$ (because the sum $\sum_{n=0}^M W_n$ is a direct sum).

Now, every $n \in \mathbb{N}$ satisfies $a_n = 0$.²⁵

We have thus shown that every sequence $(a_n)_{n \in \mathbb{N}}$ of elements of V satisfying

$(a_n \in W_n \text{ for every } n \in \mathbb{N})$, $(a_n = 0 \text{ for all but finitely many } n \in \mathbb{N})$ and $\sum_{n=0}^{\infty} a_n = 0$

must also satisfy $(a_n = 0 \text{ for every } n \in \mathbb{N})$. In other words, the sum $\sum_{n=0}^{\infty} W_n$ is a di-

rect sum. Hence, $V = \sum_{n=0}^{\infty} W_n$ rewrites as $V = \bigoplus_{n=0}^{\infty} W_n = \bigoplus_{n \in \mathbb{N}} W_n$.

Altogether, we have now proven that there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V such that $V = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $V_p = \bigoplus_{n=0}^p W_n$. This proves Lemma 1.115.

Proof of Lemma 1.116. Assume that there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V such that $V = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $V_p = \bigoplus_{n=0}^p W_n$.

For every $n \geq 1$, let ι_n denote the canonical inclusion of V_{n-1} into V_n , and let π_n denote the canonical projection of V_n onto V_n/V_{n-1} .

We know that every $p \in \mathbb{N}$ satisfies

$$V_p = \bigoplus_{n=0}^p W_n = \bigoplus_{m=0}^p W_m \quad (42)$$

(here, we renamed n as m in the sum).

Now let $n \geq 1$ be arbitrary. Then, $V_n = \bigoplus_{m=0}^n W_m$ (due to (42), applied to $p = n$)

but also $V_{n-1} = \bigoplus_{m=0}^{n-1} W_m$ (due to (42), applied to $p = n-1$). Thus, $V_n = \bigoplus_{m=0}^n W_m =$

²⁵*Proof.* Let $n \in \mathbb{N}$ be arbitrary. Then we must have one of the following two cases: either $n \leq M$ or $n > M$. But if $n \leq M$, then $a_n = 0$ (since $n \leq M$ leads to $n \in \{0,1,\dots,M\}$ and thus $a_n = 0$). On the other hand, if $n > M$, then $a_n = 0$ (as we already know). Thus, in each of the two cases $n \leq M$ and $n > M$, we have $a_n = 0$. Hence, $a_n = 0$ holds in every case.

$\underbrace{\left(\bigoplus_{m=0}^{n-1} W_m\right)}_{=V_{n-1}} \oplus W_n = V_{n-1} \oplus W_n$. Since $V_{n-1} = \iota_n(V_{n-1})$ (because ι_n is the inclusion

map), this rewrites as $V_n = \iota_n(V_{n-1}) \oplus W_n$.

Now, applying Corollary 1.89 to V_{n-1} , V_n , V_n/V_{n-1} , ι_n and π_n instead of A , B , C , f and g , we see that the exact sequence $0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0$ is \mathfrak{g} -split if and only if there exists a \mathfrak{g} -submodule P of V_n such that $V_n = \iota_n(V_{n-1}) \oplus P$. Since we know that there exists a \mathfrak{g} -submodule P of V_n such that $V_n = \iota_n(V_{n-1}) \oplus P$ (namely, $P = W_n$, because $V_n = \iota_n(V_{n-1}) \oplus W_n$), we can thus conclude that the exact sequence $0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0$ is \mathfrak{g} -split.

Hence, we have shown that for every $n \geq 1$, the exact sequence

$0 \longrightarrow V_{n-1} \xrightarrow{\iota_n} V_n \xrightarrow{\pi_n} V_n/V_{n-1} \longrightarrow 0$ is \mathfrak{g} -split. According to Definition 1.112, this means that the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split.

We have hence shown that the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split. Thus, Lemma 1.116 is proven.

Proposition 1.113 immediately follows from Lemma 1.115 and Lemma 1.116.

We will apply Proposition 1.113 in a slightly different version. First a definition:

Definition 1.117. Let k be a commutative ring. Let V and W be two k -modules. Let $f : V \rightarrow W$ be a k -module isomorphism. Let $(V_n)_{n \geq 0}$ be a k -module filtration of V , and let $(W_n)_{n \geq 0}$ be a k -module filtration of W .

The isomorphism f is said to be *bifiltered* if and only if both maps f and f^{-1} respect the filtration.

Proposition 1.118. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . Consider the \mathfrak{g} -module

$\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$, and the filtration $\left(\bigoplus_{p=0}^n \text{gr}_p V\right)_{n \geq 0}$ of this \mathfrak{g} -module $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$.

Then, the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split if and only if there exists a bifiltered \mathfrak{g} -module isomorphism $V \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p V$.

Proof of Proposition 1.118. In order to verify Proposition 1.118, we must show two lemmas:

Lemma 1.119. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . Consider the \mathfrak{g} -module

$\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$, and the filtration $\left(\bigoplus_{p=0}^n \text{gr}_p V\right)_{n \geq 0}$ of this \mathfrak{g} -module $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$.

If the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split, then there exists a bifiltered \mathfrak{g} -module isomorphism $V \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p V$.

Lemma 1.120. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $(V_n)_{n \geq 0}$ be a \mathfrak{g} -module filtration of V . Consider the \mathfrak{g} -module

$\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$, and the filtration $\left(\bigoplus_{p=0}^n \text{gr}_p V\right)_{n \geq 0}$ of this \mathfrak{g} -module $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$.

If there exists a bifiltered \mathfrak{g} -module isomorphism $V \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p V$, then the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split.

Proof of Lemma 1.119. Assume that the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split. Then, Lemma 1.115 yields that there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V such that $V = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $V_p = \bigoplus_{n=0}^p W_n$. Consider this family $(W_n)_{n \in \mathbb{N}}$.

Every $q \in \mathbb{N}$ satisfies $V_{q-1} = \bigoplus_{n=0}^{q-1} W_n$.²⁶ Renaming q as p in this fact, we thus

obtain: Every $p \in \mathbb{N}$ satisfies $V_{p-1} = \bigoplus_{n=0}^{p-1} W_n$. Thus, every $p \in \mathbb{N}$ satisfies

$$V_p = \bigoplus_{n=0}^p W_n = \underbrace{\left(\bigoplus_{n=0}^{p-1} W_n \right)}_{=V_{p-1}} \oplus W_p = V_{p-1} \oplus W_p.$$

Let $p \in \mathbb{N}$.

Let ρ_p be the canonical projection from V_p to V_p/V_{p-1} . Clearly, ρ_p is a surjective \mathfrak{g} -module homomorphism satisfying $\rho_p(V_{p-1}) = 0$.

Let π_p be the projection from the direct sum $V_{p-1} \oplus W_p$ onto its addend W_p . Then, π_p is a surjective \mathfrak{g} -module homomorphism $V_{p-1} \oplus W_p \rightarrow W_p$. Since $V_{p-1} \oplus W_p = V_p$, we thus conclude that π_p is a surjective \mathfrak{g} -module homomorphism $V_p \rightarrow W_p$.

Let ι_p be the injection of the \mathfrak{g} -module W_p in the direct sum $V_{p-1} \oplus W_p$. Then, ι_p is an injective \mathfrak{g} -module homomorphism $W_p \rightarrow V_{p-1} \oplus W_p$. Since $V_{p-1} \oplus W_p = V_p$, we thus conclude that ι_p is an injective \mathfrak{g} -module homomorphism $W_p \rightarrow V_p$.

Since π_p is the projection from the direct sum $V_{p-1} \oplus W_p$ onto its addend W_p , it must send the other addend V_{p-1} to 0. In other words, $\pi_p(V_{p-1}) = 0$. Therefore, by the homomorphism theorem, the homomorphism π_p factors through V_p/V_{p-1} . In other words, there exists a \mathfrak{g} -module homomorphism $\overline{\pi}_p : V_p/V_{p-1} \rightarrow W_p$ such that $\overline{\pi}_p \circ \rho_p = \pi_p$ (since ρ_p is the canonical projection from V_p to V_p/V_{p-1}).

We have thus defined four \mathfrak{g} -module homomorphisms ρ_p , π_p , ι_p and $\overline{\pi}_p$ for every $p \in \mathbb{N}$.

Let $p \in \mathbb{N}$ again.

²⁶*Proof.* Let $q \in \mathbb{N}$. We distinguish between two cases:

Case 1: We have $q = 0$.

Case 2: We have $q \geq 1$.

In Case 1, we have $V_{q-1} = V_{0-1} = V_{-1} = 0$ and $\bigoplus_{n=0}^{q-1} W_n = \bigoplus_{n=0}^{0-1} W_n = (\text{empty sum}) = 0$, so that

$V_{q-1} = \bigoplus_{n=0}^{q-1} W_n$ in Case 1.

In Case 2, we have $V_{q-1} = \bigoplus_{n=0}^{q-1} W_n$ (this follows from $V_p = \bigoplus_{n=0}^p W_n$, applied to $p = q - 1$).

Therefore, we have proven that $V_{q-1} = \bigoplus_{n=0}^{q-1} W_n$ holds in both possible cases 1 and 2. Thus,

$V_{q-1} = \bigoplus_{n=0}^{q-1} W_n$ is proven.

Every $v \in W_p$ satisfies $\iota_p(v) = v$ (since ι_p is the injection of the \mathfrak{g} -module W_p in the direct sum $V_{p-1} \oplus W_p$) and $\pi_p(v) = v$ (since π_p is a projection onto W_p , but $v \in W_p$).

Thus, every $v \in W_p$ satisfies $(\pi_p \circ \iota_p)(v) = \pi_p \left(\underbrace{\iota_p(v)}_{=v} \right) = \pi_p(v) = v = \text{id}(v)$. In other

words, $\pi_p \circ \iota_p = \text{id}$.

Now,

$$\overline{\pi_p} \circ (\rho_p \circ \iota_p) = \underbrace{\overline{\pi_p} \circ \rho_p}_{=\pi_p} \circ \iota_p = \pi_p \circ \iota_p = \text{id}.$$

On the other hand, let us prove that $(\rho_p \circ \iota_p) \circ \overline{\pi_p} = \text{id}$. In fact, let $w \in V_p/V_{p-1}$ be arbitrary. Then, there exists some $v \in V_p$ such that $w = \rho_p(v)$ (since ρ_p is surjective). Consider this v . Since $w = \rho_p(v)$, we have

$$\begin{aligned} ((\rho_p \circ \iota_p) \circ \overline{\pi_p})(w) &= ((\rho_p \circ \iota_p) \circ \overline{\pi_p})(\rho_p(v)) = \left((\rho_p \circ \iota_p) \circ \underbrace{\overline{\pi_p} \circ \rho_p}_{=\pi_p} \right)(v) \\ &= ((\rho_p \circ \iota_p) \circ \pi_p)(v) = \rho_p(\iota_p(\pi_p(v))). \end{aligned}$$

Now, we can write the element $v \in V_p$ in the form $v = v_1 + v_2$ for some $v_1 \in V_{p-1}$ and $v_2 \in W_p$ (because $v \in V_p = V_{p-1} \oplus W_p$). Consider these v_1 and v_2 . Since π_p is the projection from the direct sum $V_{p-1} \oplus W_p$ onto its addend W_p , it satisfies $\pi_p|_{V_{p-1}} = 0$ and $\pi_p|_{W_p} = \text{id}$. Now, $v = v_1 + v_2$ yields

$$\begin{aligned} \pi_p(v) &= \pi_p(v_1 + v_2) = \underbrace{\pi_p(v_1)}_{=0} + \underbrace{\pi_p(v_2)}_{=\text{id}(v_2)} \quad (\text{since } \pi_p \text{ is linear}) \\ &= \underbrace{(\pi_p|_{V_{p-1}})(v_1)}_{=0} + \underbrace{(\pi_p|_{W_p})(v_2)}_{=\text{id}(v_2)} \\ &= \underbrace{(\pi_p|_{V_{p-1}})}_{=0}(v_1) + \underbrace{(\pi_p|_{W_p})}_{=\text{id}}(v_2) = 0(v_1) + \text{id}(v_2) = v_2. \end{aligned}$$

Besides, $\iota_p(v_2) = v_2$ (since ι_p is the injection of the \mathfrak{g} -module W_p in the direct sum $V_{p-1} \oplus W_p$). Thus,

$$((\rho_p \circ \iota_p) \circ \overline{\pi_p})(w) = \rho_p \left(\iota_p \left(\underbrace{\pi_p(v)}_{=v_2} \right) \right) = \rho_p \left(\underbrace{\iota_p(v_2)}_{=v_2} \right) = \rho_p(v_2).$$

Comparing this with

$$\begin{aligned} \text{id}(w) &= w = \rho_p \left(\underbrace{v}_{=v_1+v_2} \right) = \rho_p(v_1 + v_2) = \underbrace{\rho_p(v_1)}_{=0} + \rho_p(v_2) \quad (\text{since } \rho_p \text{ is } k\text{-linear}) \\ &= \rho_p(v_2), \end{aligned}$$

=0 (since $v_1 \in V_{p-1}$
and $\rho_p(V_{p-1})=0$)

we obtain

$$((\rho_p \circ \iota_p) \circ \overline{\pi_p})(w) = \text{id}(w).$$

Since this holds for every $w \in V_p/V_{p-1}$, we have thus shown that $(\rho_p \circ \iota_p) \circ \overline{\pi_p} = \text{id}$.

We have now proven that $\overline{\pi}_p \circ (\rho_p \circ \iota_p) = \text{id}$ and $(\rho_p \circ \iota_p) \circ \overline{\pi}_p = \text{id}$ for every $p \in \mathbb{N}$.

For every $p \in \mathbb{N}$, the \mathfrak{g} -module homomorphism $\overline{\pi}_p : V_p/V_{p-1} \rightarrow W_p$ is a \mathfrak{g} -module homomorphism $\text{gr}_p V \rightarrow W_p$ (since $\text{gr}_p V = V_p/V_{p-1}$ by the definition of $\text{gr}_p V$). Therefore, the direct sum $\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p$ is a \mathfrak{g} -module homomorphism $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V \rightarrow \bigoplus_{p \in \mathbb{N}} W_p$. Since $V = \bigoplus_{n \in \mathbb{N}} W_n = \bigoplus_{p \in \mathbb{N}} W_p$ (here, we renamed n as p), we can thus conclude that $\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p$ is a \mathfrak{g} -module homomorphism $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V \rightarrow V$.

For every $p \in \mathbb{N}$, the \mathfrak{g} -module homomorphism $\rho_p \circ \iota_p : W_p \rightarrow V_p/V_{p-1}$ is a \mathfrak{g} -module homomorphism $W_p \rightarrow \text{gr}_p V$ (since $\text{gr}_p V = V_p/V_{p-1}$ by the definition of $\text{gr}_p V$). Therefore, the direct sum $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ is a \mathfrak{g} -module homomorphism $\bigoplus_{p \in \mathbb{N}} W_p \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p V$.

Since $V = \bigoplus_{n \in \mathbb{N}} W_n = \bigoplus_{p \in \mathbb{N}} W_p$ (here, we renamed n as p), we can thus conclude that

$$\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p \text{ is a } \mathfrak{g}\text{-module homomorphism } V \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p V.$$

Since

$$\left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p \right) \circ \left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p \right) = \bigoplus_{p \in \mathbb{N}} \underbrace{\overline{\pi}_p \circ (\rho_p \circ \iota_p)}_{=\text{id}} = \bigoplus_{p \in \mathbb{N}} \text{id} = \text{id}$$

and

$$\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p \right) \circ \left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p \right) = \bigoplus_{p \in \mathbb{N}} \underbrace{(\rho_p \circ \iota_p) \circ \overline{\pi}_p}_{=\text{id}} = \bigoplus_{p \in \mathbb{N}} \text{id} = \text{id},$$

the two \mathfrak{g} -module homomorphisms $\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p$ and $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ are mutually inverse. Thus, these two \mathfrak{g} -module homomorphisms $\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p$ and $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ are \mathfrak{g} -module isomorphisms

$$\text{and satisfy } \left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p \right)^{-1} = \bigoplus_{p \in \mathbb{N}} \overline{\pi}_p.$$

Every $m \in \mathbb{N}$ satisfies $\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p \right) (V_m) \subseteq \bigoplus_{p=0}^m \text{gr}_p V$.²⁷ If we rename m as n in

this fact, we obtain the following result: Every $n \in \mathbb{N}$ satisfies $\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p \right) (V_n) \subseteq$

$\bigoplus_{p=0}^n \text{gr}_p V$. This yields that the map $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ respects the filtration (since the filtration

on V is $(V_n)_{n \geq 0}$, while the filtration on $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$ is $\left(\bigoplus_{p=0}^n \text{gr}_p V \right)_{n \geq 0}$).

²⁷*Proof.* Let $m \in \mathbb{N}$ be arbitrary. Let $v \in V_m$ be arbitrary. Then, $v \in V_m = \bigoplus_{n=0}^m W_n$ (due to

$V_p = \bigoplus_{n=0}^p W_n$, applied to $p = m$). Thus, there exist $m + 1$ elements w_0, w_1, \dots, w_m of V such

that $(w_i \in W_i \text{ for every } i \in \{0, 1, \dots, m\})$ and $v = \sum_{i=0}^m w_i$. Consider these elements w_0, w_1, \dots, w_m .

On the other hand, every $m \in \mathbb{N}$ satisfies $\left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p\right) \left(\bigoplus_{p=0}^m \text{gr}_p V\right) \subseteq V_m$.²⁸ If we rename m as n in this fact, we obtain the following result: Every $n \in \mathbb{N}$ satisfies $\left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p\right) \left(\bigoplus_{p=0}^n \text{gr}_p V\right) \subseteq V_n$. This yields that the map $\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p$ respects the filtration (since the filtration on V is $(V_n)_{n \geq 0}$, while the filtration on $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$ is $\left(\bigoplus_{p=0}^n \text{gr}_p V\right)_{n \geq 0}$).

Then,

$$\begin{aligned}
\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right)(v) &= \left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right) \left(\sum_{i=0}^m w_i\right) && \left(\text{since } v = \sum_{i=0}^m w_i\right) \\
&= \sum_{i=0}^m \underbrace{\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right)}_{\substack{= (\rho_i \circ \iota_i)(w_i) \\ \text{(since } w_i \in W_i)}}(w_i) && \left(\text{since } \bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p \text{ is } k\text{-linear}\right) \\
&= \sum_{i=0}^m \underbrace{(\rho_i \circ \iota_i)(w_i)}_{\substack{\in \text{gr}_i V \\ \text{(since } \rho_i \circ \iota_i: W_i \rightarrow \text{gr}_i V)}} \in \sum_{i=0}^m \text{gr}_i V = \sum_{p=0}^m \text{gr}_p V && \text{(here, we renamed } i \text{ as } p) \\
&= \bigoplus_{p=0}^m \text{gr}_p V && \left(\text{since the sum } \sum_{p=0}^m \text{gr}_p V \text{ is a direct sum}\right).
\end{aligned}$$

Since this holds for every $v \in V_m$, we obtain $\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right)(V_m) \subseteq \bigoplus_{p=0}^m \text{gr}_p V$, qed.

²⁸*Proof.* Let $m \in \mathbb{N}$ be arbitrary. Let $v \in \bigoplus_{p=0}^m \text{gr}_p V$ be arbitrary. Then, there exist $m+1$ elements v_0, v_1, \dots, v_m of $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$ such that $(v_i \in \text{gr}_i V \text{ for every } i \in \{0, 1, \dots, m\})$ and $v = \sum_{i=0}^m v_i$. Consider these elements v_0, v_1, \dots, v_m . Then,

$$\begin{aligned}
\left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p\right)(v) &= \left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p\right) \left(\sum_{i=0}^m v_i\right) && \left(\text{since } v = \sum_{i=0}^m v_i\right) \\
&= \sum_{i=0}^m \underbrace{\left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p\right)}_{\substack{= \overline{\pi}_i(v_i) \\ \text{(since } v_i \in \text{gr}_i V)}}(v_i) && \left(\text{since } \bigoplus_{p \in \mathbb{N}} \overline{\pi}_p \text{ is } k\text{-linear}\right) \\
&= \sum_{i=0}^m \underbrace{\overline{\pi}_i(v_i)}_{\substack{\in W_i \\ \text{(since } \overline{\pi}_i: \text{gr}_i V \rightarrow W_i)}} \in \sum_{i=0}^m W_i = \sum_{n=0}^m W_n && \text{(here, we renamed } i \text{ as } n)
\end{aligned}$$

and

$$\begin{aligned}
V_m &= \bigoplus_{n=0}^m W_n && \left(\text{due to } V_p = \bigoplus_{n=0}^p W_n, \text{ applied to } p = m\right) \\
&= \sum_{n=0}^m W_n && \text{(since direct sums are sums),}
\end{aligned}$$

Since $\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right)^{-1} = \bigoplus_{p \in \mathbb{N}} \overline{\pi}_p$, this rewrites as follows: The map $\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right)^{-1}$ respects the filtration.

According to Definition 1.117 (applied to $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$, $\left(\bigoplus_{p=0}^n \text{gr}_p V\right)_{n \geq 0}$ and $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ instead of W , $(W_n)_{n \geq 0}$ and f), the isomorphism $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ is bifiltered if and only if

both maps $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ and $\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right)^{-1}$ respect the filtration. Since we know that

both maps $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ and $\left(\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p\right)^{-1}$ respect the filtration, we thus conclude that the isomorphism $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$ is bifiltered.

Thus we know that there exists a bifiltered \mathfrak{g} -module isomorphism $V \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p V$ (namely, the isomorphism $\bigoplus_{p \in \mathbb{N}} \rho_p \circ \iota_p$). This proves Lemma 1.119.

Proof of Lemma 1.120. Assume that there exists a bifiltered \mathfrak{g} -module isomorphism $V \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p V$. Let f be such an isomorphism.

For each $n \in \mathbb{N}$, let W_n be the \mathfrak{g} -submodule $f^{-1}(\text{gr}_n V)$ of V (here, of course, $\text{gr}_n V$ is considered as a \mathfrak{g} -submodule of $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$).

Then,

$$\begin{aligned} V &= f^{-1}\left(\bigoplus_{p \in \mathbb{N}} \text{gr}_p V\right) && \text{(since } f \text{ is a } k\text{-module isomorphism)} \\ &= f^{-1}\left(\bigoplus_{n \in \mathbb{N}} \text{gr}_n V\right) && \text{(here, we renamed } p \text{ as } n) \\ &= \bigoplus_{n \in \mathbb{N}} \underbrace{f^{-1}(\text{gr}_n V)}_{=W_n} && \text{(since } f \text{ is a } k\text{-module isomorphism)} \\ &= \bigoplus_{n \in \mathbb{N}} W_n. \end{aligned}$$

Also, Definition 1.117 (applied to $\bigoplus_{p \in \mathbb{N}} \text{gr}_p V$ and $\left(\bigoplus_{p=0}^n \text{gr}_p V\right)_{n \geq 0}$ instead of W and $(W_n)_{n \geq 0}$) yields that the map f is bifiltered if and only if both maps f and f^{-1} respect the filtration. Since we know that f is bifiltered, we thus conclude that both maps f and f^{-1} respect the filtration.

so that

$$\left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p\right)(v) \in \sum_{n=0}^m W_n = V_m.$$

Since this holds for every $v \in \bigoplus_{p=0}^m \text{gr}_p V$, we obtain $\left(\bigoplus_{p \in \mathbb{N}} \overline{\pi}_p\right)\left(\bigoplus_{p=0}^m \text{gr}_p V\right) \subseteq V_m$, qed.

Every $n \in \mathbb{N}$ satisfies $f(V_n) \subseteq \bigoplus_{p=0}^n \text{gr}_p V$ (since f respects the filtration) and $f^{-1}\left(\bigoplus_{p=0}^n \text{gr}_p V\right) \subseteq V_n$ (since f^{-1} respects the filtration). Thus, every $n \in \mathbb{N}$ satisfies $V_n \subseteq f^{-1}\left(\bigoplus_{p=0}^n \text{gr}_p V\right)$ (since $f(V_n) \subseteq \bigoplus_{p=0}^n \text{gr}_p V$). Combined with $f^{-1}\left(\bigoplus_{p=0}^n \text{gr}_p V\right) \subseteq V_n$, this yields that $V_n = f^{-1}\left(\bigoplus_{p=0}^n \text{gr}_p V\right)$ for every $n \in \mathbb{N}$. Renaming n and p as p and n in this fact, we obtain the following result: $V_p = f^{-1}\left(\bigoplus_{n=0}^p \text{gr}_n V\right)$ for every $p \in \mathbb{N}$.

Thus, every $p \in \mathbb{N}$ satisfies

$$\begin{aligned} V_p &= f^{-1}\left(\bigoplus_{n=0}^p \text{gr}_n V\right) = \bigoplus_{n=0}^p \underbrace{f^{-1}(\text{gr}_n V)}_{=W_n} \\ &\quad \text{(since } f \text{ is a } k\text{-module isomorphism)} \\ &= \bigoplus_{n=0}^p W_n. \end{aligned}$$

Thus we have found a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{g} -submodules of V such that $V = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $V_p = \bigoplus_{n=0}^p W_n$. Therefore, we can apply Lemma 1.116, and as a result we obtain that the filtration $(V_n)_{n \geq 0}$ is \mathfrak{g} -split. This proves Lemma 1.120.

Proposition 1.118 immediately follows from Lemma 1.119 and Lemma 1.120.

1.20. A trivial lemma

Here is one lemma, hardly worth the name, that we are going to use later in the proof:

Lemma 1.121. Let k be a commutative ring. Let \mathfrak{h} be a k -Lie algebra. Let A , B and C be three \mathfrak{h} -modules. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two k -module homomorphisms such that f is a surjective \mathfrak{h} -module homomorphism. Assume that $g \circ f$ is an \mathfrak{h} -module homomorphism. Then, g is an \mathfrak{h} -module homomorphism.

Proof of Lemma 1.121. Let $a \in \mathfrak{h}$ and $v \in B$ be arbitrary.

Since f is surjective, there is a $u \in A$ such that $v = f(u)$. Thus,

$$a \rightharpoonup v = a \rightharpoonup (f(u)) = f(a \rightharpoonup u) \quad \text{(since } f \text{ is an } \mathfrak{h}\text{-module homomorphism),}$$

and thus

$$\begin{aligned} g(a \rightharpoonup v) &= g(f(a \rightharpoonup u)) = (g \circ f)(a \rightharpoonup u) \\ &= a \rightharpoonup \underbrace{(g \circ f)(u)}_{=g(f(u))} \quad \text{(since } g \circ f \text{ is an } \mathfrak{h}\text{-module homomorphism)} \\ &= a \rightharpoonup \left(g \left(\underbrace{f(u)}_{=v} \right) \right) = a \rightharpoonup (g(v)). \end{aligned}$$

Thus, we have shown that every $a \in \mathfrak{h}$ and $v \in B$ satisfy $g(a \rightarrow v) = a \rightarrow (g(v))$. In other words, g is an \mathfrak{h} -module homomorphism. This proves Lemma 1.121.

1.21. A variation on the nine lemma

The following fact is completely unrelated to Lie algebras. It is one of several algebraic statements related to the nine lemma, but having both weaker assertions and weaker conditions. We record it here to use it in Section 5:

Proposition 1.122. Let k be a commutative ring. Let A, B, C and D be k -modules, and let $x : A \rightarrow B$, $y : A \rightarrow C$, $z : B \rightarrow D$ and $w : C \rightarrow D$ be k -linear maps such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ y \downarrow & & \downarrow z \\ C & \xrightarrow{w} & D \end{array} \quad (43)$$

commutes. Assume that $\text{Ker } z \subseteq x(\text{Ker } y)$. Further assume that y is surjective. Then, $\text{Ker } w = y(\text{Ker } x)$.

Proof of Proposition 1.122. We know that the diagram

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ y \downarrow & & \downarrow z \\ C & \xrightarrow{w} & D \end{array}$$

commutes. In other words, $w \circ y = z \circ x$.

We have

$$w(y(\text{Ker } x)) = \underbrace{(w \circ y)}_{=z \circ x}(\text{Ker } x) = (z \circ x)(\text{Ker } x) = z\left(\underbrace{x(\text{Ker } x)}_{=0}\right) = z(0) = 0$$

(since z is k -linear),

and thus $y(\text{Ker } x) \subseteq \text{Ker } w$. We will now prove that $\text{Ker } w \subseteq y(\text{Ker } x)$:

Let $c \in \text{Ker } w$ be arbitrary. Then, $w(c) = 0$. Now, since y is surjective, there exists some $a \in A$ such that $c = y(a)$. Consider this a . Then,

$$0 = w\left(\underbrace{c}_{=y(a)}\right) = w(y(a)) = \underbrace{(w \circ y)}_{=z \circ x}(a) = (z \circ x)(a) = z(x(a)),$$

so that $x(a) \in \text{Ker } z \subseteq x(\text{Ker } y)$. Thus, there exists some $a' \in \text{Ker } y$ such that $x(a) = x(a')$. Consider this a' . Since x is k -linear, we have $x(a - a') = \underbrace{x(a) - x(a')}_{=x(a')}$

$x(a') - x(a') = 0$, so that $a - a' \in \text{Ker } x$. Thus, $y(a - a') \in y(\text{Ker } x)$. But since

$$\begin{aligned} y(a - a') &= \underbrace{y(a)}_{=c} - \underbrace{y(a')}_{=0 \text{ (since } a' \in \text{Ker } y)} && \text{(since } y \text{ is } k\text{-linear)} \\ &= c - 0 = c, \end{aligned}$$

this rewrites as $c \in y(\text{Ker } x)$.

We have thus shown that every $c \in \text{Ker } w$ satisfies $c \in y(\text{Ker } x)$. Thus, $\text{Ker } w \subseteq y(\text{Ker } x)$. Combined with $y(\text{Ker } x) \subseteq \text{Ker } w$, this yields $\text{Ker } w = y(\text{Ker } x)$. This proves Proposition 1.122.

Note that we would not lose any generality if we would replace k by \mathbb{Z} in the statement of Proposition 1.122, because every k -module is an abelian group, i. e., a \mathbb{Z} -module (with additional structure). We could actually generalize Proposition 1.122 by replacing " k -modules " by " groups " (not necessarily abelian), but we will not have any use for Proposition 1.122 in this generality here.

2. The isomorphism $\text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \cong \mathfrak{n}^{\otimes n}$

2.1. Statement of the theorem

In this chapter, we are going to show the following fact (which generalizes Lemma 3.4 in [2]):

Theorem 2.1. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Assume that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion (but not necessarily as an \mathfrak{h} -module inclusion).

Let J be the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

of the k -algebra $\otimes \mathfrak{g}$.

As we know, \mathfrak{g} is a \mathfrak{g} -module, and thus also an \mathfrak{h} -module (by Definition 1.15). Let $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$. This \mathfrak{n} is an \mathfrak{h} -module (because both \mathfrak{g} and \mathfrak{h} are \mathfrak{h} -modules).

Let $\pi : \mathfrak{g} \rightarrow \mathfrak{n}$ be the canonical projection with kernel \mathfrak{h} . Obviously, π is an \mathfrak{h} -module homomorphism. Thus, $\otimes \pi : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ is also an \mathfrak{h} -module homomorphism (according to Proposition 1.68).

We consider \mathfrak{h} as an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ by means of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow \otimes \mathfrak{g}$.

(a) Both J and $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$. Thus, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is an \mathfrak{h} -module. Let $\zeta : \otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ be the canonical projection. Then, ζ is an \mathfrak{h} -module homomorphism.

(b) For every $n \in \mathbb{N}$, let F_n be the \mathfrak{h} -submodule $\zeta(\mathfrak{g}^{\otimes \leq n})$ of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$.²⁹ Also define an \mathfrak{h} -submodule F_{-1} of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ by $F_{-1} = 0$. Then, $(F_n)_{n \geq -1}$ is an \mathfrak{h} -module filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ and satisfies $F_n / F_{n-1} \cong \mathfrak{n}^{\otimes n}$ as \mathfrak{h} -modules for every $n \in \mathbb{N}$.

(c) Let $n \in \mathbb{N}$. There exists one and only one k -module homomorphism $\Omega_n : F_n / F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ for which the diagram

$$\begin{array}{ccc} \text{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\text{gr}_n(\otimes \pi)} & \text{gr}_n(\otimes \mathfrak{n}) \\ \text{gr}_n \zeta \downarrow & & \uparrow \\ \text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) = F_n / F_{n-1} & \xrightarrow{\Omega_n} & \text{gr}_n(\otimes \mathfrak{n}) \end{array} \quad (44)$$

commutes. Denote this homomorphism Ω_n by ω_n . Then, ω_n is an \mathfrak{h} -module isomorphism, and the diagram

$$\begin{array}{ccc} \text{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\text{gr}_n(\otimes \pi)} & \text{gr}_n(\otimes \mathfrak{n}) \\ \text{gr}_n \zeta \downarrow & & \uparrow \\ \text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) = F_n / F_{n-1} & \xrightarrow{\omega_n} & \text{gr}_n(\otimes \mathfrak{n}) \end{array} \quad (45)$$

commutes.

Applying Definition 1.105 to \mathfrak{n} and n instead of V and p , we obtain a map $\text{grad}_{\mathfrak{n},n} : \mathfrak{n}^{\otimes n} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$. According to Proposition 1.108 (applied to \mathfrak{h} , n and \mathfrak{n} instead of \mathfrak{g} , p and V), this map $\text{grad}_{\mathfrak{n},n}$ is a canonical \mathfrak{h} -module isomorphism. Thus, its inverse $\text{grad}_{\mathfrak{n},n}^{-1}$ is an \mathfrak{h} -module isomorphism as well. The composition $\text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n : F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$ is an \mathfrak{h} -module isomorphism (because ω_n and $\text{grad}_{\mathfrak{n},n}^{-1}$ are \mathfrak{h} -module isomorphisms).

We are going to prove this theorem at the end of Section 2 (by means of explicitly constructing an isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$), after paving our way with some auxiliary results. But let us first remark what this theorem actually says:

Theorem 2.1 (a) is a result of computational nature and does not require the condition that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion. We are going to prove it in Proposition 2.3 (a). This is *not* the difficult part of Theorem 2.1.

Theorem 2.1 (b) (or, more precisely, its " $F_n/F_{n-1} \cong \mathfrak{n}^{\otimes n}$ as \mathfrak{h} -modules" part) is what Lemma 3.4 in [2] states (except that [2] only considers the case when k is a field, and of course the condition that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module becomes void in this case).

Theorem 2.1 (c), however, is what Lemma 3.4 in [2] actually **means to state**. In [2], as in many papers on algebra, there is a slight discrepancy between the statements of some results and the actual meaning that the authors give to these results. This discrepancy manifests itself in the fact that Lemma 3.4 in [2] formally only states that $F_n/F_{n-1} \cong \mathfrak{n}^{\otimes n}$ as \mathfrak{h} -modules, i. e., that there exists some \mathfrak{h} -module isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$, but the authors actually **want to say that a very particular map** $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$ is an \mathfrak{h} -module isomorphism (namely, the map $\text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n$ of our Theorem 2.1 (c)). This assertion (that $\text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n$ is an \mathfrak{h} -module isomorphism) is a stronger assertion than just the existence of an \mathfrak{h} -module isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$, and **this stronger assertion is the one actually used later on** (for example, in the proof of Theorem 4.4 in [2]). Therefore, if we would only prove Theorem 2.1 (b) (but not Theorem 2.1 (c)), we would miss some of the tools we would need later on.

Also note that Theorem 2.1 (c) shows that not only does there exist an \mathfrak{h} -module isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$, but also that there exists an \mathfrak{h} -module isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$ independent of choice of the splitting of the k -module inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. See Proposition 2.21 for the details of this.

Remark 2.2. The condition that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ split *as a k -module inclusion cannot be removed from Theorem 2.1*, as the counterexample $k = \mathbb{Z}$, $\mathfrak{g} = \mathfrak{sl}_2\mathbb{Q}$, $\mathfrak{h} = (\mathfrak{sl}_2\mathbb{Q}) \cap \mathbb{Z}^{2 \times 2}$ shows. (In this counterexample, $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ contains $\mathfrak{g}^{\otimes i}$ for all $i \geq 2$, and thus we can easily see that $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$ contains $\mathfrak{g}^{\otimes i}$ for all $i \geq 1$, so that $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \cong \mathbb{Z}$ and $F_1 = F_0$, whereas $\mathfrak{n}^{\otimes 1} = (\mathfrak{sl}_2\mathbb{Q}) / ((\mathfrak{sl}_2\mathbb{Q}) \cap \mathbb{Z}^{2 \times 2}) \neq 0$.)

I do not know whether a flatness hypothesis of some kind (like requiring $\mathfrak{g}/\mathfrak{h}$ to be a flat k -module) could be used instead of this splitting condition.

²⁹In fact, $\zeta(\mathfrak{g}^{\otimes \leq n})$ is indeed an \mathfrak{h} -submodule (because ζ is an \mathfrak{h} -module homomorphism and because $\mathfrak{g}^{\otimes \leq n}$ is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$).

2.2. J and $(\otimes \mathfrak{g}) \mathfrak{h}$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$

Let us start by proving Theorem 2.1 (a) without the condition that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion. More precisely let us prove the following:

Proposition 2.3. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .
Let J be the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

of the k -algebra $\otimes \mathfrak{g}$.

As we know, \mathfrak{g} is a \mathfrak{g} -module, and thus also an \mathfrak{h} -module (by Definition 1.15). We consider \mathfrak{h} as an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ by means of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow \otimes \mathfrak{g}$.

(a) Both J and $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$. Thus, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is an \mathfrak{h} -module. Let $\zeta : \otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ be the canonical projection. Then, ζ is an \mathfrak{h} -module homomorphism.

(b) Let J_0 denote the k -submodule $\langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle$ of $\otimes \mathfrak{g}$. Then, $J = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g})$. Both J_0 and $J_0 \cdot (\otimes \mathfrak{g})$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$.

Proof of Proposition 2.3. (b) Clearly,

$$J = (\otimes \mathfrak{g}) \cdot \underbrace{\langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle}_{=J_0} \cdot (\otimes \mathfrak{g}) = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g}).$$

Next, we are going to prove that J_0 is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$.

Clearly, $J_0 = \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle$ yields that

$$v \otimes w - w \otimes v - [v, w] \in J_0 \quad \text{for every } v \in \mathfrak{g} \text{ and } w \in \mathfrak{h}. \quad (46)$$

Let $a \in \mathfrak{h}$ be arbitrary. Let us denote by L_a the map $\otimes \mathfrak{g} \rightarrow \otimes \mathfrak{g}$, $x \mapsto a \rightharpoonup x$. This map L_a is k -linear (since the Lie action of $\otimes \mathfrak{g}$ is k -bilinear).

We are now going to prove that

$$a \rightharpoonup (p \otimes q - q \otimes p - [p, q]) \in J_0 \quad \text{for every } p \in \mathfrak{g} \text{ and } q \in \mathfrak{h}. \quad (47)$$

Proof of (47). Let $p \in \mathfrak{g}$ and $q \in \mathfrak{h}$ be arbitrary. Then, $[q, a] = -[a, q]$ (by (5), applied to q and a instead of v and w) and $[q, [a, p]] = -[[a, p], q]$ (by (5), applied to q and $[a, p]$ instead of v and w). Also, note that $[a, p] \in \mathfrak{h}$ (because $a \in \mathfrak{h}$ and $p \in \mathfrak{h}$, and because \mathfrak{h} is a Lie subalgebra of \mathfrak{g}). Furthermore, (9) (applied to a and p instead of v and w) yields $a \rightharpoonup p = [a, p]$. On the other hand, (9) (applied to a and q instead of v and w) yields $a \rightharpoonup q = [a, q]$. Besides, (9) (applied to a and $[p, q]$ instead of v and w) yields

$$\begin{aligned} a \rightharpoonup [p, q] &= [a, [p, q]] = - \left[p, \underbrace{[q, a]}_{=-[a, q]} \right] - \underbrace{[q, [a, p]]}_{=-[[a, p], q]} \\ &\quad \left(\text{since (4) (applied to } a, p, q \text{ instead of } u, v, w \text{) yields} \right. \\ &\quad \left. [a, [p, q]] + [p, [q, a]] + [q, [a, p]] = 0 \right) \\ &= \underbrace{-[p, -[a, q]]}_{=[p, [a, q]]} - (-[[a, p], q]) = [p, [a, q]] + [[a, p], q]. \end{aligned}$$

Now,

$$\begin{aligned}
& a \rightharpoonup (p \otimes q - q \otimes p - [p, q]) \\
&= \underbrace{a \rightharpoonup (p \otimes q)}_{\substack{=(a \rightharpoonup p) \otimes q + p \otimes (a \rightharpoonup q) \\ \text{(by (15), applied to} \\ p \text{ and } q \text{ instead of } v \text{ and } w)}} - \underbrace{a \rightharpoonup (q \otimes p)}_{\substack{=(a \rightharpoonup q) \otimes p + q \otimes (a \rightharpoonup p) \\ \text{(by (15), applied to} \\ q \text{ and } p \text{ instead of } v \text{ and } w)}} - a \rightharpoonup [p, q] \\
&\quad \text{(since the Lie action of } \otimes \mathfrak{g} \text{ is } k\text{-bilinear)} \\
&= \underbrace{(a \rightharpoonup p)}_{=[a,p]} \otimes q + p \otimes \underbrace{(a \rightharpoonup q)}_{=[a,q]} - \underbrace{(a \rightharpoonup q)}_{=[a,q]} \otimes p - q \otimes \underbrace{(a \rightharpoonup p)}_{=[a,p]} - \underbrace{a \rightharpoonup [p, q]}_{=[p,[a,q]]+[a,p],q} \\
&= [a, p] \otimes q + p \otimes [a, q] - [a, q] \otimes p - q \otimes [a, p] - [p, [a, q]] - [[a, p], q] \\
&= \underbrace{([a, p] \otimes q - q \otimes [a, p] - [[a, p], q])}_{\substack{\in J_0 \text{ (by (46) (applied to } v=[a,p] \text{ and } w=q), \text{ since } [a,p] \in \mathfrak{h})}} + \underbrace{(p \otimes [a, q] - [a, q] \otimes p - [p, [a, q]])}_{\substack{\in J_0 \text{ (by (46) (applied to } v=p \text{ and } w=[a,q], \text{ since } p \in \mathfrak{h})}} \\
&\in J_0 + J_0 \subseteq J_0 \quad \text{(since } J_0 \text{ is a } k\text{-module)}.
\end{aligned}$$

This proves (47).

Now we have:

$$\{a \rightharpoonup (v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\} \subseteq J_0. \quad (48)$$

Proof of (48). Let $\gamma \in \{a \rightharpoonup (v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}$ be arbitrary. Then, γ has the form $\gamma = a \rightharpoonup (p \otimes q - q \otimes p - [p, q])$ for some $(p, q) \in \mathfrak{g} \times \mathfrak{h}$. Hence, $\gamma = a \rightharpoonup (p \otimes q - q \otimes p - [p, q]) \in J_0$ (by (47)). Thus, we have shown that every $\gamma \in \{a \rightharpoonup (v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}$ satisfies $\gamma \in J_0$. In other words, $\{a \rightharpoonup (v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\} \subseteq J_0$. This proves (48).

Now,

$$\begin{aligned}
J_0 &= \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \\
&= \langle \{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\} \rangle \quad \text{(by Convention 1.28 (b))}
\end{aligned}$$

and thus

$$\begin{aligned}
L_a(J_0) &= L_a(\langle \{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\} \rangle) \\
&= \left\langle \underbrace{L_a(\{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\})}_{=\{L_a(v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}} \right\rangle \\
&\quad \left(\begin{array}{l} \text{according to Proposition 1.29 (b) (applied to } M = \otimes \mathfrak{g}, R = \otimes \mathfrak{g}, \\ f = L_a \text{ and } S = \{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}) \end{array} \right) \\
&= \langle \{L_a(v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\} \rangle \\
&= \langle \{a \rightharpoonup (v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\} \rangle \\
&\quad \left(\begin{array}{l} \text{since every } (v, w) \in \mathfrak{g} \times \mathfrak{h} \text{ satisfies} \\ L_a(v \otimes w - w \otimes v - [v, w]) = a \rightharpoonup (v \otimes w - w \otimes v - [v, w]) \\ \text{by the definition of } L_a \end{array} \right) \\
&\subseteq J_0
\end{aligned}$$

(according to Proposition 1.29 (a) (applied to $M = \otimes \mathfrak{g}$, $Q = J_0$ and $S = \{a \rightharpoonup (v \otimes w - w \otimes v - [v, w]) \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}$), because of (48)).

Thus, every $a \in \mathfrak{h}$ and every $x \in J_0$ satisfy $a \rightarrow x \in J_0$ (since $L_a(x) = a \rightarrow x$ by the definition of L_a , and thus $a \rightarrow x = L_a \left(\underbrace{x}_{\in J_0} \right) \in L_a(J_0) \subseteq J_0$). In other words, J_0 is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$. Now, Corollary 1.81 (a) (applied to $V = \mathfrak{g}$, $P = J_0$ and $Q = \otimes \mathfrak{g}$) yields that $J_0 \cdot (\otimes \mathfrak{g})$ is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$. This completes the proof of Proposition 2.3 (b).

(a) Since both $\otimes \mathfrak{g}$ and \mathfrak{h} are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$, it follows from Corollary 1.81 (a) (applied to $V = \mathfrak{g}$, $P = \otimes \mathfrak{g}$ and $Q = \mathfrak{h}$) that $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$.

Since J_0 is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$, Corollary 1.81 (b) (applied to $V = \mathfrak{g}$ and $R = J_0$) yields that $(\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g})$ is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$. Since $(\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g}) = J$ by Proposition 2.3 (b), we conclude that J is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$.

Since we now know that both J and $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$, the sum $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$ must also be an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$. Thus, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is an \mathfrak{h} -module. Therefore, the canonical projection $\zeta : \otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is an \mathfrak{h} -module homomorphism. This completes the proof of Proposition 2.3 (a).

Thus we have proved Theorem 2.1 (a) (because it trivially follows from Proposition 2.3 (a)).

2.3. Planning the proof of Theorem 2.1 (b) and (c)

Now that we have proven part (a) of Theorem 2.1 without even using the whole assumptions, it is time to sketch our further procedure to prove the hard one - part (b) and (c).

In the situation of Theorem 2.1, there exists a k -submodule N of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus N$ (because the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion). The projection $\pi : \mathfrak{g} \rightarrow \mathfrak{n}$ has the property that $\pi|_N : N \rightarrow \mathfrak{n}$ is an isomorphism (this is easily seen). We are going to define a projection $\varphi : \otimes \mathfrak{g} \rightarrow \otimes N$ which, composed with the isomorphism $\otimes(\pi|_N) : \otimes N \rightarrow \otimes \mathfrak{n}$, will entail a homomorphism $\otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ (of k -modules). This homomorphism respects the filtration and (as we will prove) satisfies $\varphi(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$, so we will conclude that φ induces a homomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$. This homomorphism is a k -module isomorphism (as we will, again, prove), but is not an \mathfrak{h} -module isomorphism in general. But as it respects the filtration, it yields a homomorphism from F_n / F_{n-1} to $\mathfrak{n}^{\otimes n}$, which will prove to be an \mathfrak{h} -module isomorphism. Additionally it turns out that this isomorphism does not depend on the (non-canonical) choice of N and can be described by a simple formula.

We now proceed to the execution of this plan.

2.4. Definitions and basic properties of N and φ

Definition 2.4. Consider the situation of Theorem 2.1.

(a) Since the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion, there exists a k -submodule N of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus N$ (as k -modules).

Let $s : \mathfrak{g} \rightarrow \mathfrak{h}$ be the projection from \mathfrak{g} on \mathfrak{h} with kernel N (this projection exists since $\mathfrak{g} = \mathfrak{h} \oplus N$).

Let $t : \mathfrak{g} \rightarrow N$ be the projection from \mathfrak{g} on N with kernel \mathfrak{h} (this projection exists since $\mathfrak{g} = \mathfrak{h} \oplus N$).

Note that both s and t are k -module maps (but not necessarily \mathfrak{h} -module maps; actually this would not even make sense for t).

(b) Let us now define a k -linear map $\varphi_p : \mathfrak{g}^{\otimes p} \rightarrow \otimes N$ for every $p \in \mathbb{N}$. We are going to define this map φ_p by induction over p :

Induction base: For $p = 0$, define the map $\varphi_0 : \mathfrak{g}^{\otimes 0} \rightarrow \otimes N$ by

$$(\varphi_0(\lambda) = \lambda \quad \text{for every } \lambda \in \mathfrak{g}^{\otimes 0}) \quad (49)$$

(this definition makes sense since $\mathfrak{g}^{\otimes 0} = k \subseteq \otimes N$).

Induction step: For any $p > 0$, we assume that the map $\varphi_{p-1} : \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes N$ is already defined, and now we define a map $\varphi_p : \mathfrak{g}^{\otimes p} \rightarrow \otimes N$ as follows: The map

$$\mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes N, \quad (u, U) \mapsto t(u) \cdot \varphi_{p-1}(U) + \varphi_{p-1}(s(u) \rightarrow U)$$

is k -bilinear (because the maps φ_{p-1} , t and s are k -linear and the Lie action of $\otimes \mathfrak{g}$ is k -bilinear). Thus, by the universal property of the tensor product, this map gives rise to a k -linear map $\mathfrak{g} \otimes \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes N$ which sends $u \otimes U$ to $t(u) \cdot \varphi_{p-1}(U) + \varphi_{p-1}(s(u) \rightarrow U)$ for every $(u, U) \in \mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)}$. This k -linear map is going to be denoted by φ_p . It is a map from $\mathfrak{g}^{\otimes p}$ to $\otimes N$ because $\mathfrak{g} \otimes \mathfrak{g}^{\otimes(p-1)} = \mathfrak{g}^{\otimes p}$.

This completes the inductive definition of φ_p for every $p \in \mathbb{N}$.

(c) Now, we define a k -linear map $\varphi : \otimes \mathfrak{g} \rightarrow \otimes N$ as follows: The sum $\sum_{i \in \mathbb{N}} \varphi_i$ of the maps $\varphi_i : \mathfrak{g}^{\otimes i} \rightarrow \otimes N$ is a map from $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$ to $\otimes N$. Since $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i} = \otimes \mathfrak{g}$, the sum $\sum_{i \in \mathbb{N}} \varphi_i$ of the maps $\varphi_i : \mathfrak{g}^{\otimes i} \rightarrow \otimes N$ is thus a map from $\otimes \mathfrak{g}$ to $\otimes N$. Denote this map by φ .

Convention 2.5. Throughout the rest of Section 2, we are going to work in the situation of Definition 2.4. So, for example, when we refer to \mathfrak{h} , we mean the Lie subalgebra \mathfrak{h} of Theorem 2.1, and when we refer to t , we mean the map t of Definition 2.4.

Remark 2.6. **(a)** As a consequence of the inductive step in the definition of φ_p (in Definition 2.4 **(b)**), we know that for every $p > 0$, the map φ_p is the k -linear map $\mathfrak{g} \otimes \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes N$ which sends $u \otimes U$ to $t(u) \cdot \varphi_{p-1}(U) + \varphi_{p-1}(s(u) \rightarrow U)$ for every $(u, U) \in \mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)}$. In other words,

$$\varphi_p(u \otimes U) = t(u) \cdot \varphi_{p-1}(U) + \varphi_{p-1}(s(u) \rightarrow U) \quad \text{for every } (u, U) \in \mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)}. \quad (50)$$

(b) According to Definition 2.4 **(c)**, the map $\varphi : \otimes \mathfrak{g} \rightarrow \otimes N$ is the sum $\sum_{i \in \mathbb{N}} \varphi_i$ of the maps $\varphi_i : \mathfrak{g}^{\otimes i} \rightarrow \otimes N$. Hence,

$$\varphi(T) = \left(\sum_{i \in \mathbb{N}} \varphi_i \right) (T) = \varphi_p(T) \quad \text{for every } p \in \mathbb{N} \text{ and every } T \in \mathfrak{g}^{\otimes p}. \quad (51)$$

(c) Every $\lambda \in k$ satisfies

$$\begin{aligned}\varphi(\lambda) &= \varphi_0(\lambda) && \text{(by (51) (applied to } p=0 \text{ and } T=\lambda), \text{ because } \lambda \in k = \mathfrak{g}^{\otimes 0}) \\ &= \lambda && \text{(by (49)).}\end{aligned}\tag{52}$$

This yields, in particular, that $\varphi(k) = k$.

(d) Every $u \in \mathfrak{g}$ satisfies

$$\begin{aligned}\varphi(u) &= \varphi_1(u) && \text{(by (51) (applied to } p=1 \text{ and } T=u), \text{ because } u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}) \\ &= \varphi_1(u \otimes 1) && \text{(since } u = u \otimes 1 \text{ under the identification } \mathfrak{g} \cong \mathfrak{g} \otimes k) \\ &= t(u) \cdot \underbrace{\varphi_{1-1}(1)}_{=\varphi_0} + \underbrace{\varphi_{1-1}}_{=\varphi_0} \left(\underbrace{s(u) \rightarrow 1}_{=0 \text{ (since the Lie action of } k \text{ is 0)}} \right) \\ &\quad \text{(by (50), applied to } p=1 \text{ and } U=1) \\ &= t(u) \cdot \underbrace{\varphi_0(1)}_{=1 \text{ (by (49), applied to } \lambda=1)} + \underbrace{\varphi_0(0)}_{=0 \text{ (since } \varphi_0 \text{ is linear)}} = t(u) \cdot 1 + 0 = t(u).\end{aligned}\tag{53}$$

This yields $\varphi(\mathfrak{g}) = t(\mathfrak{g}) = N$ (since t is a projection of \mathfrak{g} on N).

(e) We now know that $\varphi(k) = k$ and $\varphi(\mathfrak{g}) \subseteq N$. But in general, we cannot generalize this to $\varphi(\mathfrak{g}^{\otimes p}) \subseteq N^{\otimes p}$ for all $p \in \mathbb{N}$. However, Proposition 2.12 will give us a weaker result that is actually true.

First let us show an extension of (50) to all of $\otimes \mathfrak{g}$:

Proposition 2.7. Every $u \in \mathfrak{g}$ and $U \in \otimes \mathfrak{g}$ satisfy

$$\varphi(u \cdot U) = t(u) \cdot \varphi(U) + \varphi(s(u) \rightarrow U).\tag{54}$$

Proof of Proposition 2.7. Let $u \in \mathfrak{g}$ and $U \in \otimes \mathfrak{g}$ be arbitrary. Since $U \in \otimes \mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$, we can write U as a family $U = (U_i)_{i \in \mathbb{N}}$, where $(U_i \in \mathfrak{g}^{\otimes i} \text{ for every } i \in \mathbb{N})$. Taking into account that we consider $\mathfrak{g}^{\otimes p}$ as a k -submodule of $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$ for every $p \in \mathbb{N}$, we have $(U_i)_{i \in \mathbb{N}} = \sum_{i \in \mathbb{N}} U_i$, and thus $U = (U_i)_{i \in \mathbb{N}} = \sum_{i \in \mathbb{N}} U_i$. Thus,

$$\begin{aligned}s(u) \rightarrow U &= s(u) \rightarrow \left(\sum_{i \in \mathbb{N}} U_i \right) = \sum_{i \in \mathbb{N}} (s(u) \rightarrow U_i) \\ &\quad \text{(since the Lie action of } \otimes \mathfrak{g} \text{ is } k\text{-bilinear).}\end{aligned}$$

On the other hand, every $i \in \mathbb{N}$ satisfies $s(u) \rightarrow U_i \in \mathfrak{g}^{\otimes i}$ (because $U_i \in \mathfrak{g}^{\otimes i}$ and because $\mathfrak{g}^{\otimes i}$ is a \mathfrak{g} -module) and $\underbrace{u}_{\in \mathfrak{g}} \otimes \underbrace{U_i}_{\in \mathfrak{g}^{\otimes i}} \in \mathfrak{g} \otimes \mathfrak{g}^{\otimes i} = \mathfrak{g}^{\otimes(i+1)}$. Also, since $\varphi = \sum_{i \in \mathbb{N}} \varphi_i = \sum_{p \in \mathbb{N}} \varphi_p$

is the sum of the maps $\varphi_p : \mathfrak{g}^{\otimes p} \rightarrow \otimes N$ for all $p \in \mathbb{N}$, we have

$$\varphi(v) = \varphi_i(v) \quad \text{for every } v \in \mathfrak{g}^{\otimes i}.\tag{55}$$

Similarly,

$$\varphi(v) = \varphi_{i+1}(v) \quad \text{for every } v \in \mathfrak{g}^{\otimes(i+1)}. \quad (56)$$

Every $i \in \mathbb{N}$ satisfies $\varphi(u \otimes U_i) = \varphi_{i+1}(u \otimes U_i)$ according to (56) (applied to $u \otimes U_i$ instead of v), because $u \otimes U_i \in \mathfrak{g}^{\otimes(i+1)}$.

Every $i \in \mathbb{N}$ satisfies $\varphi(U_i) = \varphi_i(U_i)$ according to (55) (applied to U_i instead of v), since $U_i \in \mathfrak{g}^{\otimes i}$.

Every $i \in \mathbb{N}$ satisfies $\varphi(s(u) \rightarrow U_i) = \varphi_i(s(u) \rightarrow U_i)$ according to (55) (applied to $s(u) \rightarrow U_i$ instead of v), since $s(u) \rightarrow U_i \in \mathfrak{g}^{\otimes i}$.

Now, $U = \sum_{i \in \mathbb{N}} U_i$ leads to

$u \cdot U$

$$\begin{aligned} &= u \cdot \sum_{i \in \mathbb{N}} U_i = \sum_{i \in \mathbb{N}} u \cdot U_i = \sum_{i \in \mathbb{N}} u \otimes U_i \\ &\quad \left(\begin{array}{c} \text{since every } i \in \mathbb{N} \text{ satisfies } u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1} \text{ and } U_i \in \mathfrak{g}^{\otimes i} \text{ and therefore} \\ u \cdot U_i = u \otimes U_i \text{ (due to (31), applied to } u, U_i, 1 \text{ and } i \text{ instead of } a, b, n \text{ and } m) \end{array} \right), \end{aligned}$$

and therefore

$$\begin{aligned} \varphi(u \cdot U) &= \varphi \left(\sum_{i \in \mathbb{N}} u \otimes U_i \right) = \sum_{i \in \mathbb{N}} \underbrace{\varphi(u \otimes U_i)}_{=\varphi_{i+1}(u \otimes U_i)} \quad (\text{since } \varphi \text{ is } k\text{-linear}) \\ &= \sum_{i \in \mathbb{N}} \varphi_{i+1}(u \otimes U_i) = \sum_{i \in \mathbb{N}} \left(t(u) \cdot \underbrace{\varphi_{(i+1)-1}(U_i)}_{=\varphi_i} + \underbrace{\varphi_{(i+1)-1}(s(u) \rightarrow U_i)}_{=\varphi_i} \right) \\ &\quad \left(\begin{array}{c} \text{since every } i \in \mathbb{N} \text{ satisfies} \\ \varphi_{i+1}(u \otimes U_i) = t(u) \cdot \varphi_{(i+1)-1}(U_i) + \varphi_{(i+1)-1}(s(u) \rightarrow U_i) \\ \text{according to (50) (applied to } i+1 \text{ and } U_i \text{ instead of } p \text{ and } U) \end{array} \right) \\ &= \sum_{i \in \mathbb{N}} (t(u) \cdot \varphi_i(U_i) + \varphi_i(s(u) \rightarrow U_i)) = \sum_{i \in \mathbb{N}} t(u) \cdot \underbrace{\varphi_i(U_i)}_{=\varphi(U_i)} + \sum_{i \in \mathbb{N}} \underbrace{\varphi_i(s(u) \rightarrow U_i)}_{=\varphi(s(u) \rightarrow U_i)} \\ &= \sum_{i \in \mathbb{N}} t(u) \cdot \varphi(U_i) + \sum_{i \in \mathbb{N}} \varphi(s(u) \rightarrow U_i). \end{aligned}$$

Since

$$\begin{aligned} \sum_{i \in \mathbb{N}} t(u) \cdot \varphi(U_i) &= t(u) \cdot \underbrace{\sum_{i \in \mathbb{N}} \varphi(U_i)}_{=\varphi\left(\sum_{i \in \mathbb{N}} U_i\right)} \\ &\quad (\text{since } \varphi \text{ is } k\text{-linear}) \\ &= t(u) \cdot \varphi \left(\underbrace{\sum_{i \in \mathbb{N}} U_i}_{=U} \right) = t(u) \cdot \varphi(U) \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \varphi(s(u) \rightarrow U_i) &= \varphi \left(\underbrace{\sum_{i \in \mathbb{N}} (s(u) \rightarrow U_i)}_{=s(u) \rightarrow U} \right) && \text{(since } \varphi \text{ is } k\text{-linear)} \\ &= \varphi(s(u) \rightarrow U), \end{aligned}$$

this becomes

$$\varphi(u \cdot U) = \underbrace{\sum_{i \in \mathbb{N}} t(u) \cdot \varphi(U_i)}_{=t(u) \cdot \varphi(U)} + \underbrace{\sum_{i \in \mathbb{N}} \varphi(s(u) \rightarrow U_i)}_{=\varphi(s(u) \rightarrow U)} = t(u) \cdot \varphi(U) + \varphi(s(u) \rightarrow U).$$

This proves Proposition 2.7.

Two obvious corollaries of Proposition 2.7:

Corollary 2.8. Every $u \in \mathfrak{h}$ and $U \in \otimes \mathfrak{g}$ satisfy

$$\varphi(u \cdot U) = \varphi(u \rightarrow U). \quad (57)$$

Corollary 2.9. Every $u \in N$ and $U \in \otimes \mathfrak{g}$ satisfy

$$\varphi(u \cdot U) = u \cdot \varphi(U). \quad (58)$$

Proof of Corollary 2.8. Since $u \in \mathfrak{h}$, we have $t(u) = 0$ (since t is a projection with kernel \mathfrak{h}) and $s(u) = u$ (since s is a projection on \mathfrak{h}). Thus, (54) yields

$$\varphi(u \cdot U) = \underbrace{t(u)}_{=0} \cdot \varphi(U) + \varphi \left(\underbrace{s(u)}_{=u} \rightarrow U \right) = \underbrace{0 \cdot \varphi(U)}_{=0} + \varphi(u \rightarrow U) = \varphi(u \rightarrow U).$$

This proves Corollary 2.8.

Proof of Corollary 2.9. Since $u \in N$, we have $s(u) = 0$ (since s is a projection with kernel N) and $t(u) = u$ (since t is a projection on N). Thus, (54) yields

$$\varphi(u \cdot U) = \underbrace{t(u)}_{=u} \cdot \varphi(U) + \varphi \left(\underbrace{s(u)}_{=0} \rightarrow U \right) = u \cdot \varphi(U) + \varphi \left(\underbrace{0 \rightarrow U}_{=0} \right) = u \cdot \varphi(U) + \underbrace{\varphi(0)}_{=0} = u \cdot \varphi(U).$$

This proves Corollary 2.9.

Before we go on, let us make another convention:

Convention 2.10. Let V be any k -module. Let $p \in \mathbb{N}_+$.

According to Convention 1.44 (applied to $V_1 = V, V_2 = V, \dots, V_n = V$ and $n = p$), a *left-induced tensor* in $\underbrace{V \otimes V \otimes \dots \otimes V}_{p \text{ times}}$ means an element of the form $v \otimes T$ for some

$v \in V$ and some $T \in \underbrace{V \otimes V \otimes \dots \otimes V}_{p-1 \text{ times}}$.

Since $\underbrace{V \otimes V \otimes \dots \otimes V}_{p \text{ times}} = V^{\otimes p}$ and $\underbrace{V \otimes V \otimes \dots \otimes V}_{p-1 \text{ times}} \in V^{\otimes(p-1)}$, we can rewrite this

fact as follows: A *left-induced tensor* in $V^{\otimes p}$ means an element of the form $v \otimes T$ for some $v \in V$ and some $T \in V^{\otimes(p-1)}$.

We denote by $V_{\text{lind}}^{\otimes p}$ the subset of $V^{\otimes p}$ consisting of all left-induced tensors in $V^{\otimes p}$.

Proposition 2.11. Let V be any k -module. Let $p \in \mathbb{N}_+$. Then, $V^{\otimes p} = \langle V_{\text{lind}}^{\otimes p} \rangle$.

Proof of Proposition 2.11. According to what we said in Convention 1.44, we know the following fact: For every n arbitrary k -modules V_1, V_2, \dots, V_n , where $n \geq 1$, the k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ is generated by its left-induced tensors. Applying this to $V_1 = V, V_2 = V, \dots, V_n = V$ and $n = p$, we conclude that the k -module $\underbrace{V \otimes V \otimes \dots \otimes V}_{p \text{ times}}$

is generated by its left-induced tensors. Since $\underbrace{V \otimes V \otimes \dots \otimes V}_{p \text{ times}} = V^{\otimes p}$, this rewrites as

follows: The k -module $V^{\otimes p}$ is generated by its left-induced tensors. In other words: The k -module $V^{\otimes p}$ is generated by $V_{\text{lind}}^{\otimes p}$ (because the set of all left-induced tensors in $V^{\otimes p}$ is $V_{\text{lind}}^{\otimes p}$). In other words, $V^{\otimes p} = \langle V_{\text{lind}}^{\otimes p} \rangle$. This proves Proposition 2.11.

The next (rather trivial) property will again be proven by induction:

Proposition 2.12. The map $\varphi : \otimes \mathfrak{g} \rightarrow \otimes N$ respects the filtration. Here, the filtration on $\otimes \mathfrak{g}$ is the degree filtration $(\mathfrak{g}^{\otimes \leq n})_{n \geq 0}$, and the filtration on $\otimes N$ is the degree filtration $(N^{\otimes \leq n})_{n \geq 0}$.

Proof of Proposition 2.12. We are going to show that

$$\varphi(\mathfrak{g}^{\otimes p}) \subseteq N^{\otimes \leq p} \text{ for every } p \in \mathbb{N}. \quad (59)$$

Proof of (59). We will prove (59) by induction over p :

Induction base: We know that $\mathfrak{g}^{\otimes 0} = k$ and $N^{\otimes \leq 0} = \bigoplus_{i=0}^0 N^{\otimes i} = N^{\otimes 0} = k$, and we know (from Remark 2.6 (c)) that $\varphi(k) = k$. Thus, $\mathfrak{g}^{\otimes 0} = k$ yields $\varphi(\mathfrak{g}^{\otimes 0}) = \varphi(k) = k = N^{\otimes \leq 0}$. In other words, (59) holds for $p = 0$. This completes the induction base.

Induction step: Let $q \in \mathbb{N}_+$. Assume that (59) holds for $p = q - 1$. Now we must prove that (59) holds for $p = q$.

We have assumed that (59) holds for $p = q - 1$. In other words, $\varphi(\mathfrak{g}^{\otimes(q-1)}) \subseteq N^{\otimes \leq(q-1)}$.

Now we are going to prove that every $U \in \varphi(\mathfrak{g}_{\text{lind}}^{\otimes q})$ satisfies $U \in N^{\otimes \leq q}$.

In fact, let $U \in \varphi(\mathfrak{g}_{\text{lind}}^{\otimes q})$ be arbitrary. Then, there exists a $U' \in \mathfrak{g}_{\text{lind}}^{\otimes q}$ such that $U = \varphi(U')$ (since $U \in \varphi(\mathfrak{g}_{\text{lind}}^{\otimes q})$). Consider this U' . This U' is a left-induced tensor in $\mathfrak{g}^{\otimes q}$ (since $U' \in \mathfrak{g}_{\text{lind}}^{\otimes q}$); this means that there exists some $v \in \mathfrak{g}$ and $T \in \mathfrak{g}^{\otimes(q-1)}$ such that $U' = v \otimes T$. Consider these v and T . Since $v \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $T \in \mathfrak{g}^{\otimes(q-1)}$, we have $v \cdot T = v \otimes T$ (due to (31), applied to $v, T, 1$ and $q - 1$ instead of a, b, n and m).

Now,

$$\begin{aligned}
U &= \varphi \left(\underbrace{U'}_{=v \otimes T = v \cdot T} \right) = \varphi(v \cdot T) = \underbrace{t(v)}_{\in N = N^{\otimes 1} \subseteq N^{\otimes \leq 1}} \cdot \varphi \left(\underbrace{T}_{\in \mathfrak{g}^{\otimes (q-1)}} \right) + \varphi \left(\underbrace{s(v) \rightarrow T}_{\substack{\in \mathfrak{g}^{\otimes (q-1)} \text{ (since } T \in \mathfrak{g}^{\otimes (q-1)} \\ \text{and since } \mathfrak{g}^{\otimes (q-1)} \text{ is a } \mathfrak{g}\text{-module)}}} \right) \\
&\quad \text{(by (54), applied to } v \text{ and } T \text{ instead of } u \text{ and } U) \\
&\in N^{\otimes \leq 1} \cdot \underbrace{\varphi(\mathfrak{g}^{\otimes (q-1)})}_{\subseteq N^{\otimes \leq (q-1)}} + \underbrace{\varphi(\mathfrak{g}^{\otimes (q-1)})}_{\substack{\subseteq N^{\otimes \leq (q-1)} \subseteq N^{\otimes \leq q} \\ \text{(since } (N^{\otimes \leq n})_{n \geq 0} \text{ is a filtration)}}} \subseteq N^{\otimes \leq 1} \cdot N^{\otimes \leq (q-1)} + N^{\otimes \leq q}.
\end{aligned}$$

Since Proposition 1.95 **(b)** (applied to N , 1 and $q - 1$ instead of V , n and m) yields $N^{\otimes \leq 1} \cdot N^{\otimes \leq (q-1)} \subseteq N^{\otimes \leq (1+(q-1))} = N^{\otimes \leq q}$, this becomes

$$U \in \underbrace{N^{\otimes \leq 1} \cdot N^{\otimes \leq (q-1)}}_{\subseteq N^{\otimes \leq q}} + N^{\otimes \leq q} \subseteq N^{\otimes \leq q} + N^{\otimes \leq q} \subseteq N^{\otimes \leq q} \quad (\text{since } N^{\otimes \leq q} \text{ is a } k\text{-module}).$$

We have thus proven that every $U \in \varphi(\mathfrak{g}_{\text{lind}}^{\otimes q})$ satisfies $U \in N^{\otimes \leq q}$. In other words, $\varphi(\mathfrak{g}_{\text{lind}}^{\otimes q}) \subseteq N^{\otimes \leq q}$.

Now, Proposition 2.11 (applied to q and \mathfrak{g} instead of p and V) says $\mathfrak{g}^{\otimes q} = \langle \mathfrak{g}_{\text{lind}}^{\otimes q} \rangle$, and thus

$$\begin{aligned}
\varphi(\mathfrak{g}^{\otimes q}) &= \varphi(\langle \mathfrak{g}_{\text{lind}}^{\otimes q} \rangle) = \langle \varphi(\mathfrak{g}_{\text{lind}}^{\otimes q}) \rangle \\
&\quad \left(\begin{array}{c} \text{by Proposition 1.29 (b) (applied to} \\ \mathfrak{g}^{\otimes q}, \mathfrak{g}_{\text{lind}}^{\otimes q}, \otimes N \text{ and } \varphi \text{ instead of } M, S, R \text{ and } f) \end{array} \right) \\
&\subseteq N^{\otimes \leq q} \quad \left(\begin{array}{c} \text{by Proposition 1.29 (a) (applied to } \otimes N, \varphi(\mathfrak{g}_{\text{lind}}^{\otimes q}) \text{ and } N^{\otimes \leq q} \\ \text{instead of } M, S \text{ and } Q), \text{ since } \varphi(\mathfrak{g}_{\text{lind}}^{\otimes q}) \subseteq N^{\otimes \leq q} \end{array} \right).
\end{aligned}$$

In other words, (59) holds for $p = q$. This completes the induction step, and thus the induction proof of (59) is done.

Now, let $m \in \mathbb{N}$. The definition of $\mathfrak{g}^{\otimes \leq m}$ says $\mathfrak{g}^{\otimes \leq m} = \bigoplus_{i=0}^m \mathfrak{g}^{\otimes i}$, so that $\mathfrak{g}^{\otimes \leq m} = \bigoplus_{i=0}^m \mathfrak{g}^{\otimes i} = \sum_{i=0}^m \mathfrak{g}^{\otimes i}$ (since direct sums are sums) and thus

$$\begin{aligned}
\varphi(\mathfrak{g}^{\otimes \leq m}) &= \varphi \left(\sum_{i=0}^m \mathfrak{g}^{\otimes i} \right) = \sum_{i=0}^m \underbrace{\varphi(\mathfrak{g}^{\otimes i})}_{\substack{\subseteq N^{\otimes \leq i} \\ \text{(by (59), applied} \\ \text{to } i \text{ instead of } p)}} \quad (\text{since } \varphi \text{ is } k\text{-linear}) \\
&\subseteq \sum_{i=0}^m \underbrace{N^{\otimes \leq i}}_{\substack{\subseteq N^{\otimes \leq m} \text{ (since } i \leq m \\ \text{and since } (N^{\otimes \leq n})_{n \geq 0} \text{ is} \\ \text{a filtration)}}} \subseteq \sum_{i=0}^m N^{\otimes \leq m} \subseteq N^{\otimes \leq m} \quad (\text{since } N^{\otimes \leq m} \text{ is a } k\text{-module}).
\end{aligned}$$

Since this holds for every $m \in \mathbb{N}$, this yields that φ respects the filtration. This proves Proposition 2.12.

The next, still very trivial, result shows how φ acts on $\otimes N$ (when $\otimes N$ is considered as a k -subalgebra of $\otimes \mathfrak{g}$):

Proposition 2.13. Let us consider $\otimes N$ as a k -subalgebra of $\otimes \mathfrak{g}$ (since N is a k -submodule of \mathfrak{g}). Then, $\varphi|_{\otimes N} = \text{id}_{\otimes N}$.

Proof of Proposition 2.13. We are going to prove that

$$N^{\otimes p} \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N}) \quad \text{for every } p \in \mathbb{N}. \quad (60)$$

Proof of (60). We will prove (60) by induction over p :

Induction base: Every $\lambda \in N^{\otimes 0}$ satisfies

$$(\varphi|_{\otimes N} - \text{id}_{\otimes N})(\lambda) = \underbrace{(\varphi|_{\otimes N})(\lambda)}_{\substack{=\varphi(\lambda)=\lambda \text{ (by (52))} \\ \text{since } \lambda \in N^{\otimes 0}=k}} - \underbrace{\text{id}_{\otimes N}(\lambda)}_{=\lambda} = \lambda - \lambda = 0.$$

Thus, every $\lambda \in N^{\otimes 0}$ satisfies $\lambda \in \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$. In other words, $N^{\otimes 0} \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$. Hence, (60) holds for $p = 0$. This completes the induction base.

Induction step: Let $q \in \mathbb{N}_+$. Assume that (60) holds for $p = q - 1$. Now we must prove that (60) holds for $p = q$.

We have assumed that (60) holds for $p = q - 1$. In other words, $N^{\otimes(q-1)} \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$.

Now we are going to prove that every $T \in N_{\text{lind}}^{\otimes q}$ satisfies $T \in \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$.

In fact, let $T \in N_{\text{lind}}^{\otimes q}$ be arbitrary. Then, T is a left-induced tensor in $N^{\otimes q}$ (since $T \in N_{\text{lind}}^{\otimes q}$); this means that there exist some $u \in N$ and $U \in N^{\otimes(q-1)}$ such that $T = u \otimes U$. Consider these u and U . Since $u \in N \subseteq \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $U \in N^{\otimes(q-1)} \subseteq \mathfrak{g}^{\otimes(q-1)}$, we have $u \cdot U = u \otimes U$ (due to (31), applied to $u, U, 1$ and $q - 1$ instead of a, b, n and m).

On the other hand, $U \in N^{\otimes(q-1)} \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$ yields $(\varphi|_{\otimes N} - \text{id}_{\otimes N})(U) = 0$, so that

$$0 = (\varphi|_{\otimes N} - \text{id}_{\otimes N})(U) = \underbrace{(\varphi|_{\otimes N})(U)}_{=\varphi(U)} - \underbrace{\text{id}_{\otimes N}(U)}_{=U} = \varphi(U) - U$$

and thus $\varphi(U) = U$.

Now, $T = u \otimes U = u \cdot U$ yields

$$\begin{aligned} \varphi(T) &= \varphi(u \cdot U) = u \cdot \underbrace{\varphi(U)}_{=U} && \text{(due to (58))} \\ &= u \cdot U = T. \end{aligned}$$

Hence,

$$(\varphi|_{\otimes N} - \text{id}_{\otimes N})(T) = \underbrace{(\varphi|_{\otimes N})(T)}_{=\varphi(T)=T} - \underbrace{\text{id}_{\otimes N}(T)}_{=T} = T - T = 0,$$

and thus $T \in \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$.

We have thus proven that every $T \in N_{\text{lind}}^{\otimes q}$ satisfies $T \in \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$. In other words, $N_{\text{lind}}^{\otimes q} \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$.

Thus, Proposition 1.29 (a) (applied to $\otimes N$, $N_{\text{lind}}^{\otimes q}$ and $\text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$ instead of M, S and Q) yields $\langle N_{\text{lind}}^{\otimes q} \rangle \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$. Since Proposition 2.11 (applied to q and N instead of p and V) says $N^{\otimes q} = \langle N_{\text{lind}}^{\otimes q} \rangle$, we thus conclude that $N^{\otimes q} =$

$\langle N_{\text{lind}}^{\otimes q} \rangle \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N})$. In other words, (60) holds for $p = q$. This completes the induction step, and thus the induction proof of (60) is done.

Now,

$$\begin{aligned} \otimes N &= \bigoplus_{p \in \mathbb{N}} N^{\otimes p} = \sum_{p \in \mathbb{N}} \underbrace{N^{\otimes p}}_{\substack{\subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N}) \\ \text{(by (60))}}} \quad (\text{since direct sums are sums}) \\ &\subseteq \sum_{p \in \mathbb{N}} \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N}) \subseteq \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N}) \\ &\quad (\text{since } \text{Ker}(\varphi|_{\otimes N} - \text{id}_{\otimes N}) \text{ is a } k\text{-module}), \end{aligned}$$

so that $\varphi|_{\otimes N} - \text{id}_{\otimes N} = 0$ and thus $\varphi|_{\otimes N} = \text{id}_{\otimes N}$. This proves Proposition 2.13.

As a consequence of Proposition 2.13, we can easily see:

Corollary 2.14. The map φ satisfies $\varphi^2 = \varphi$.

But we are not going to use this.

2.5. A lemma on φ and \mathfrak{h} -submodules of $\otimes \mathfrak{g}$

Our next plan is showing that $\varphi(J) = 0$ and $\varphi((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$. We will simplify this task by the following lemma:

Lemma 2.15. Let C be an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ satisfying $\varphi(C) = 0$. Then, $\varphi((\otimes \mathfrak{g}) \cdot C) = 0$.

Proof of Lemma 2.15. We are going to prove that

$$\varphi(\mathfrak{g}^{\otimes p} \cdot C) = 0 \quad \text{for every } p \in \mathbb{N}. \quad (61)$$

Proof of (61). We will prove (61) by induction over p :

Induction base: We have $\mathfrak{g}^{\otimes 0} = k$ and thus $\mathfrak{g}^{\otimes 0} \cdot C = k \cdot C = C$ (since C is a k -submodule), so that $\varphi(\mathfrak{g}^{\otimes 0} \cdot C) = \varphi(C) = 0$. Thus, (61) holds for $p = 0$. This completes the induction base.

Induction step: Let $q \in \mathbb{N}_+$. Assume that (61) holds for $p = q - 1$. Now we must prove that (61) holds for $p = q$.

We have assumed that (61) holds for $p = q - 1$. In other words, $\varphi(\mathfrak{g}^{\otimes(q-1)} \cdot C) = 0$.

Let $D = \mathfrak{g}^{\otimes(q-1)} \cdot C$. Corollary 1.81 (c) (applied to \mathfrak{g} , $q - 1$ and C instead of V , p and R) yields that $\mathfrak{g}^{\otimes(q-1)} \cdot C$ is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ (since C is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$). Thus, D is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ (since $D = \mathfrak{g}^{\otimes(q-1)} \cdot C$). Also, $D = \mathfrak{g}^{\otimes(q-1)} \cdot C$ yields $\varphi(D) = \varphi(\mathfrak{g}^{\otimes(q-1)} \cdot C) = 0$.

On the other hand, Proposition 1.95 (a) (applied to $V = \mathfrak{g}$, $i = 1$ and $j = q - 1$) yields $\mathfrak{g}^{\otimes 1} \cdot \mathfrak{g}^{\otimes(q-1)} = \mathfrak{g}^{\otimes(1+(q-1))} = \mathfrak{g}^{\otimes q}$. Thus,

$$\underbrace{\mathfrak{g}^{\otimes q}}_{=\mathfrak{g}^{\otimes 1} \cdot \mathfrak{g}^{\otimes(q-1)}} \cdot C = \underbrace{\mathfrak{g}^{\otimes 1}}_{=\mathfrak{g}} \cdot \underbrace{\mathfrak{g}^{\otimes(q-1)} \cdot C}_{=D} = \mathfrak{g} \cdot D.$$

We are now in the following situation: We know that D is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ satisfying $\varphi(D) = 0$. We want to prove that $\varphi(\mathfrak{g} \cdot D) = 0$.

Let E be the subset $\{u \cdot U \mid (u, U) \in \mathfrak{g} \times D\}$ of $\otimes \mathfrak{g}$. Then,

$$\mathfrak{g} \cdot D = \langle u \cdot U \mid (u, U) \in \mathfrak{g} \times D \rangle = \left\langle \underbrace{\{u \cdot U \mid (u, U) \in \mathfrak{g} \times D\}}_{=E} \right\rangle = \langle E \rangle.$$

Now we are going to show that $E \subseteq \text{Ker } \varphi$.

In fact, let $T \in E$ be arbitrary. Then, $T \in E = \{u \cdot U \mid (u, U) \in \mathfrak{g} \times D\}$, so that there exists some $(u, U) \in \mathfrak{g} \times D$ such that $T = u \cdot U$. Consider this (u, U) . Clearly, $u \in \mathfrak{g}$ and $U \in D$. Now, (54) yields

$$\begin{aligned} \varphi(u \cdot U) &= t(u) \cdot \varphi \left(\underbrace{U}_{\in D} \right) + \varphi \left(\underbrace{s(u) \rightarrow U}_{\substack{\in D \text{ (since } s(u) \in \mathfrak{h}, \text{ since } U \in D \\ \text{and since } D \text{ is an } \mathfrak{h}\text{-module)}}} \right) \\ &\in t(u) \cdot \underbrace{\varphi(D)}_{=0} + \underbrace{\varphi(D)}_{=0} = t(u) \cdot 0 + 0 = 0, \end{aligned}$$

so that $\varphi(u \cdot U) = 0$. Since $u \cdot U = T$, this rewrites as $\varphi(T) = 0$. Thus, $T \in \text{Ker } \varphi$.

We have therefore shown that every $T \in E$ satisfies $T \in \text{Ker } \varphi$. Thus, $E \subseteq \text{Ker } \varphi$.

Consequently, Proposition 1.29 (a) (applied to $\otimes \mathfrak{g}$, E and $\text{Ker } \varphi$ instead of M , S and Q) yields $\langle E \rangle \subseteq \text{Ker } \varphi$ (since $\text{Ker } \varphi$ is a k -submodule of $\otimes \mathfrak{g}$, since φ is k -linear). Since $\langle E \rangle = \mathfrak{g} \cdot D$, this rewrites as $\mathfrak{g} \cdot D \subseteq \text{Ker } \varphi$, and thus $\varphi(\mathfrak{g} \cdot D) = 0$. Since $\mathfrak{g}^{\otimes q} \cdot C = \mathfrak{g} \cdot D$, this becomes $\varphi(\mathfrak{g}^{\otimes q} \cdot C) = 0$. In other words, (61) holds for $p = q$. This completes the induction step, and thus the induction proof of (61) is done.

Now, $\otimes \mathfrak{g} = \bigoplus_{p \in \mathbb{N}} \mathfrak{g}^{\otimes p} = \sum_{p \in \mathbb{N}} \mathfrak{g}^{\otimes p}$ (since direct sums are sums) yields $(\otimes \mathfrak{g}) \cdot C =$

$$\left(\sum_{p \in \mathbb{N}} \mathfrak{g}^{\otimes p} \right) \cdot C = \sum_{p \in \mathbb{N}} (\mathfrak{g}^{\otimes p} \cdot C) \text{ and thus}$$

$$\begin{aligned} \varphi((\otimes \mathfrak{g}) \cdot C) &= \varphi \left(\sum_{p \in \mathbb{N}} (\mathfrak{g}^{\otimes p} \cdot C) \right) = \sum_{p \in \mathbb{N}} \underbrace{\varphi(\mathfrak{g}^{\otimes p} \cdot C)}_{=0 \text{ (due to (61))}} \quad (\text{since } \varphi \text{ is } k\text{-linear}) \\ &= \sum_{p \in \mathbb{N}} 0 = 0. \end{aligned}$$

This proves Lemma 2.15.

2.6. $\varphi(J) = 0$ and $\varphi((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$

We are now ready to prove the following facts:

■ **Proposition 2.16.** We have $\varphi(J) = 0$.

■ **Proposition 2.17.** We have $\varphi((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$.

Proof of Proposition 2.16. Consider the k -submodule J_0 defined in Proposition 2.3 (b).

We are going to prove that $\varphi(J_0 \cdot (\otimes \mathfrak{g})) = 0$ now (this will quickly yield $\varphi(J) = 0$ then, due to $J = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g})$ and Lemma 2.15).

Let S_0 denote the subset $\{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}$ of $\otimes \mathfrak{g}$. Then, we know from the definition of J_0 that

$$\begin{aligned} J_0 &= \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \\ &= \left\langle \underbrace{\{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}}_{=S_0} \right\rangle = \langle S_0 \rangle. \end{aligned}$$

Now, let $T \in \otimes \mathfrak{g}$ be arbitrary. Let $\varrho_T : \otimes \mathfrak{g} \rightarrow \otimes N$ be the map defined by

$$(\varrho_T(U) = \varphi(U \cdot T) \quad \text{for every } U \in \otimes \mathfrak{g}).$$

This map ϱ_T is k -linear (since it is the composition of the two k -linear maps φ and $\otimes \mathfrak{g} \rightarrow \otimes \mathfrak{g}, U \mapsto U \cdot T$). Thus, $\text{Ker } \varrho_T$ is a k -submodule of $\otimes \mathfrak{g}$.

We are now going to prove that every $s \in S_0$ satisfy $s \in \text{Ker } \varrho_T$.

In fact, let $s \in S_0$ be arbitrary. Since $s \in S_0 = \{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}$, there exists some $(v, w) \in \mathfrak{g} \times \mathfrak{h}$ such that $s = v \otimes w - w \otimes v - [v, w]$. So we have $v \in \mathfrak{g}$ and $w \in \mathfrak{h}$. Note that $v \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $w \in \mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ yield $v \cdot w = v \otimes w$ (by (31), applied to 1 and 1 instead of n and m), and similarly $w \cdot v = w \otimes v$. Using these observations, we see that

$$s = \underbrace{v \otimes w}_{=v \cdot w} - \underbrace{w \otimes v}_{=w \cdot v} - [v, w] = v \cdot w - w \cdot v - [v, w].$$

Now, we are going to prove that $\varrho_T(s) = 0$. In order to do this, we will compute $\varrho_T(v \cdot w)$, $\varrho_T(w \cdot v)$ and $\varrho_T([v, w])$. First let us compute $\varphi(v \cdot T)$ and $\varphi(w \cdot T)$.

We have $\varphi(v \cdot T) = t(v) \cdot \varphi(T) + \varphi(s(v) \rightarrow T)$ (according to (54), applied to $u = v$ and $U = T$) and $\varphi(w \cdot T) = \varphi(w \rightarrow T)$ (according to (57), applied to $u = w$ and $U = T$).

The definition of ϱ_T yields

$$\begin{aligned}
\varrho_T(v \cdot w) &= \varphi(v \cdot w \cdot T) = t(v) \cdot \underbrace{\varphi(w \cdot T)}_{=\varphi(w \rightarrow T)} + \varphi \left(\underbrace{s(v) \rightarrow (w \cdot T)}_{=\underbrace{(s(v) \rightarrow w) \cdot T + w \cdot (s(v) \rightarrow T)}_{\substack{\text{(by (33) (applied to } \otimes \mathfrak{g}, s(v), w \text{ and } T \\ \text{instead of } A, a, u \text{ and } v), \text{ because } \otimes \mathfrak{g} \text{ is a } \mathfrak{g}\text{-algebra)}}}} \right) \\
&\quad \text{(according to (54), applied to } u = v \text{ and } U = w \cdot T) \\
&= t(v) \cdot \varphi(w \rightarrow T) + \underbrace{\varphi((s(v) \rightarrow w) \cdot T + w \cdot (s(v) \rightarrow T))}_{=\varphi((s(v) \rightarrow w) \cdot T) + \varphi(w \cdot (s(v) \rightarrow T))} \\
&\quad \quad \quad \text{(since } \varphi \text{ is } k\text{-linear)} \\
&= t(v) \cdot \varphi(w \rightarrow T) + \varphi \left(\underbrace{(s(v) \rightarrow w)}_{=\underbrace{[s(v), w]}_{\substack{\text{(according to (9), applied} \\ \text{to } s(v) \text{ instead of } v)}}}} \cdot T \right) + \underbrace{\varphi(w \cdot (s(v) \rightarrow T))}_{=\varphi(w \rightarrow (s(v) \rightarrow T))} \\
&\quad \quad \quad \text{(by (57), applied} \\
&\quad \quad \quad \text{to } u = w \text{ and } U = s(v) \rightarrow T) \\
&= t(v) \cdot \varphi(w \rightarrow T) + \underbrace{\varphi([s(v), w] \cdot T)}_{=\varphi([s(v), w] \rightarrow T)} + \varphi(w \rightarrow (s(v) \rightarrow T)) \\
&\quad \quad \quad \text{(by (57) (applied} \\
&\quad \quad \quad \text{to } u = [s(v), w] \text{ and } U = T, \text{ because} \\
&\quad \quad \quad \text{[} s(v), w \text{]} \in \mathfrak{h} \text{ (since } s(v) \in \mathfrak{h} \text{ and } w \in \mathfrak{h} \text{ and} \\
&\quad \quad \quad \text{since } \mathfrak{h} \text{ is a Lie subalgebra of } \mathfrak{g})) \\
&= t(v) \cdot \varphi(w \rightarrow T) + \varphi([s(v), w] \rightarrow T) + \varphi(w \rightarrow (s(v) \rightarrow T)). \tag{62}
\end{aligned}$$

The definition of ϱ_T yields

$$\begin{aligned}
\varrho_T(w \cdot v) &= \varphi(w \cdot v \cdot T) = \varphi \left(\underbrace{w \rightarrow (v \cdot T)}_{=\underbrace{(w \rightarrow v) \cdot T + v \cdot (w \rightarrow T)}_{\substack{\text{(by (33) (applied to } \otimes \mathfrak{g}, w, v \text{ and } T \\ \text{instead of } A, a, u \text{ and } v), \text{ because } \otimes \mathfrak{g} \text{ is a } \mathfrak{g}\text{-algebra)}}}} \right) \\
&\quad \text{(by (57), applied to } u = w \text{ and } U = v \cdot T) \\
&= \varphi((w \rightarrow v) \cdot T + v \cdot (w \rightarrow T)) \\
&= \varphi \left(\underbrace{(w \rightarrow v)}_{=\underbrace{[w, v]}_{\substack{\text{(according to (9), applied} \\ \text{to } w \text{ and } v \text{ instead of } v \text{ and } w)}}}} \cdot T \right) + \underbrace{\varphi(v \cdot (w \rightarrow T))}_{=\underbrace{t(v) \cdot \varphi(w \rightarrow T) + \varphi(s(v) \rightarrow (w \rightarrow T))}_{\substack{\text{(according to (54),} \\ \text{applied to } u = v \text{ and } U = w \rightarrow T)}}} \\
&\quad \text{(since } \varphi \text{ is } k\text{-linear)} \\
&= \varphi([w, v] \cdot T) + t(v) \cdot \varphi(w \rightarrow T) + \varphi(s(v) \rightarrow (w \rightarrow T)). \tag{63}
\end{aligned}$$

Finally, the definition of ϱ_T yields

$$\varrho_T([v, w]) = \varphi \left(\underbrace{[v, w]}_{\substack{=-[w, v] \\ \text{(due to (5))}}} \cdot T \right) = \varphi(-[w, v] \cdot T) = -\varphi([w, v] \cdot T) \quad (64)$$

(since φ is k -linear).

Now, $s = v \cdot w - w \cdot v - [v, w]$ yields

$$\begin{aligned} & \varrho_T(s) \\ &= \varrho_T(v \cdot w - w \cdot v - [v, w]) \\ &= \underbrace{\varrho_T(v \cdot w)}_{\substack{=t(v) \cdot \varphi(w \rightarrow T) + \varphi([s(v), w] \rightarrow T) + \varphi(w \rightarrow (s(v) \rightarrow T)) \\ \text{(due to (62))}}} - \underbrace{\varrho_T(w \cdot v)}_{\substack{=\varphi([w, v] \cdot T) + t(v) \cdot \varphi(w \rightarrow T) + \varphi(s(v) \rightarrow (w \rightarrow T)) \\ \text{(due to (63))}}} - \underbrace{\varrho_T([v, w])}_{\substack{=-\varphi([w, v] \cdot T) \\ \text{(due to (64))}}} \\ & \quad \text{(since } \varrho_T \text{ is } k\text{-linear)} \\ &= (t(v) \cdot \varphi(w \rightarrow T) + \varphi([s(v), w] \rightarrow T) + \varphi(w \rightarrow (s(v) \rightarrow T))) \\ & \quad - (\varphi([w, v] \cdot T) + t(v) \cdot \varphi(w \rightarrow T) + \varphi(s(v) \rightarrow (w \rightarrow T))) - (-\varphi([w, v] \cdot T)) \\ &= \varphi([s(v), w] \rightarrow T) + \varphi(w \rightarrow (s(v) \rightarrow T)) - \varphi(s(v) \rightarrow (w \rightarrow T)) \\ & \quad \text{(after some obvious cancellations)} \\ &= \varphi \left(\underbrace{[s(v), w] \rightarrow T}_{\substack{=s(v) \rightarrow (w \rightarrow T) - w \rightarrow (s(v) \rightarrow T) \\ \text{(according to (8) (applied to} \\ \otimes \mathfrak{g}, s(v), w \text{ and } T \text{ instead of } V, a, b \text{ and } v), \\ \text{since } \otimes \mathfrak{g} \text{ is a } \mathfrak{g}\text{-module)}}} + w \rightarrow (s(v) \rightarrow T) - s(v) \rightarrow (w \rightarrow T) \right) \\ & \quad \text{(since } \varphi \text{ is } k\text{-linear)} \\ &= \varphi \left(\underbrace{s(v) \rightarrow (w \rightarrow T) - w \rightarrow (s(v) \rightarrow T) + w \rightarrow (s(v) \rightarrow T) - s(v) \rightarrow (w \rightarrow T)}_{=0} \right) \\ &= \varphi(0) = 0, \end{aligned}$$

so that $s \in \text{Ker } \varrho_T$.

We have thus proven that every $s \in S_0$ satisfies $s \in \text{Ker } \varrho_T$. Thus, $S_0 \subseteq \text{Ker } \varrho_T$. Since $\text{Ker } \varrho_T$ is a k -module, Proposition 1.29 (a) (applied to $\otimes \mathfrak{g}$, S_0 and $\text{Ker } \varrho_T$ instead of M , S and Q) now yields $\langle S_0 \rangle \subseteq \text{Ker } \varrho_T$. Since $\langle S_0 \rangle = J_0$, this rewrites as $J_0 \subseteq \text{Ker } \varrho_T$. Thus, $\varrho_T(J_0) = 0$. In other words, every $j \in J_0$ satisfies $\varrho_T(j) = 0$. Since $\varrho_T(j) = \varphi(j \cdot T)$ (by the definition of ϱ_T), this rewrites as $\varphi(j \cdot T) = 0$, and thus $j \cdot T \in \text{Ker } \varphi$ for every $j \in J_0$.

We have thus shown that every $j \in J_0$ and $T \in \otimes \mathfrak{g}$ satisfy $j \cdot T \in \text{Ker } \varphi$. In other words, every $(j, T) \in J_0 \times (\otimes \mathfrak{g})$ satisfies $j \cdot T \in \text{Ker } \varphi$. In other words, $\{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \subseteq \text{Ker } \varphi$.

Now, applying Proposition 1.29 (a) to $\otimes \mathfrak{g}$, $\{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\}$ and $\text{Ker } \varphi$ instead of M , S and Q , we see that $\langle \{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \rangle \subseteq \text{Ker } \varphi$ (since

$\{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \subseteq \text{Ker } \varphi$ and since $\text{Ker } \varphi$ is a k -module). Now

$$J_0 \cdot (\otimes \mathfrak{g}) = \langle j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g}) \rangle = \langle \{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \rangle \subseteq \text{Ker } \varphi$$

and thus $\varphi(J_0 \cdot (\otimes \mathfrak{g})) = 0$.

Now, $J_0 \cdot (\otimes \mathfrak{g})$ is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ (according to Proposition 2.3 (b)). Thus, Lemma 2.15 (applied to $C = J_0 \cdot (\otimes \mathfrak{g})$) yields that $\varphi((\otimes \mathfrak{g}) \cdot (J_0 \cdot (\otimes \mathfrak{g}))) = 0$ (because $\varphi(J_0 \cdot (\otimes \mathfrak{g})) = 0$). Since $(\otimes \mathfrak{g}) \cdot (J_0 \cdot (\otimes \mathfrak{g})) = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g}) = J$ (according to Proposition 2.3 (b) again), this rewrites as $\varphi(J) = 0$. Thus, Proposition 2.16 is proven.

Proof of Proposition 2.17. Every $u \in \mathfrak{h}$ satisfies

$$\begin{aligned} \varphi(u) &= t(u) && \text{(according to (53), since } u \in \mathfrak{h} \subseteq \mathfrak{g}) \\ &= 0 && \text{(since } u \in \mathfrak{h}, \text{ while } t \text{ is a projection with kernel } \mathfrak{h}). \end{aligned}$$

Thus, $\varphi(\mathfrak{h}) = 0$. Since we know that \mathfrak{h} is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$, we can thus follow from Lemma 2.15 (applied to $C = \mathfrak{h}$) that $\varphi((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$ (because $\varphi(\mathfrak{h}) = 0$). Thus, Proposition 2.17 is proven.

2.7. φ induces a filtered k -module isomorphism

We recall that, in order to prove Theorem 2.1 (b), we want to construct an \mathfrak{h} -module isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$. We will do this by constructing a k -module isomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ which respects the filtration. The associated graded morphism of this isomorphism will then turn out to be an \mathfrak{h} -module isomorphism.

Here is how we construct our k -module isomorphism:

- Proposition 2.18.** (a) The homomorphism φ is surjective and satisfies $\text{Ker } \varphi = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$. Thus, φ induces a k -module isomorphism $\bar{\varphi} : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes N$ which satisfies $\varphi = \bar{\varphi} \circ \zeta$ (where ζ denotes the canonical projection $\otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ as in Theorem 2.1).
- (b) The isomorphism $\bar{\varphi}$ respects the filtration. Here, the filtration on $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is given by $(F_n)_{n \geq 0}$, and the filtration on $\otimes N$ is given by $(N^{\otimes \leq n})_{n \geq 0}$.
- (c) The k -module homomorphism $\pi|_N : N \rightarrow \mathfrak{n}$ is an isomorphism. Thus, the k -algebra homomorphism $\otimes(\pi|_N) : \otimes N \rightarrow \otimes \mathfrak{n}$ is an isomorphism as well. We are going to denote this isomorphism $\otimes(\pi|_N)$ by η . The homomorphism η respects the filtration (where the filtrations on $\otimes N$ and $\otimes \mathfrak{n}$ are the degree filtrations, as usual).
- (d) Let $\iota : \otimes N \rightarrow \otimes \mathfrak{g}$ be the canonical inclusion of $\otimes N$ in $\otimes \mathfrak{g}$. Then, the inverse $\bar{\varphi}^{-1}$ of the isomorphism $\bar{\varphi}$ equals $\zeta \circ \iota$. This inverse $\bar{\varphi}^{-1}$ also respects the filtration.
- (e) The composition $\eta \circ \bar{\varphi} : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ respects the filtration. For every $p \in \mathbb{N}$, the homomorphism $\text{gr}_p(\eta \circ \bar{\varphi}) : \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_p(\otimes \mathfrak{n})$ is an \mathfrak{h} -module isomorphism.

In order to prove this, a very technical result:

Lemma 2.19. We have $\otimes \mathfrak{g} = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$ (where we, as before, identify $\otimes N$ with a k -subalgebra of $\otimes \mathfrak{g}$).

There are two ways to prove this lemma. Let us sketch the first one (which is hard to formalize, but rather straightforward) and detail the second one (which is easier to formalize, but is not the simplest proof).

First proof of Lemma 2.19 (sketched). It is enough to prove that every pure tensor in $\otimes \mathfrak{g}$ lies in $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$ (because the pure tensors generate $\otimes \mathfrak{g}$ as a k -module). So we must show that $g_1 \otimes g_2 \otimes \dots \otimes g_n \in J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$ for every $n \in \mathbb{N}$ and any elements g_1, g_2, \dots, g_n of \mathfrak{g} .

Since $g_i = t(g_i) + s(g_i)$ for every $i \in \{1, 2, \dots, n\}$, this rewrites as $(t(g_1) + s(g_1)) \otimes (t(g_2) + s(g_2)) \otimes \dots \otimes (t(g_n) + s(g_n)) \in J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$. Expanding the left hand side, it becomes a sum of 2^n addends. One of these addends is $t(g_1) \otimes t(g_2) \otimes \dots \otimes t(g_n)$ and lies in $\otimes N$. Every of the other $2^n - 1$ addends is a pure tensor with at least one tensorand lying in \mathfrak{h} . Some of these tensors have their rightmost tensorand lying in \mathfrak{h} , which means that these tensors lie in $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$. The other ones still have a tensorand lying in \mathfrak{h} , but it is not their rightmost tensorand. Transform each of these latter tensors according to the following rewriting rule:

$$\begin{aligned} & v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes v_\ell \otimes v_{\ell+1} \otimes v_{\ell+2} \otimes \dots \otimes v_n \\ & \rightarrow v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes v_{\ell+1} \otimes v_\ell \otimes v_{\ell+2} \otimes \dots \otimes v_n \\ & \quad - v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes [v_{\ell+1}, v_\ell] \otimes v_{\ell+2} \otimes \dots \otimes v_n \\ & \quad - v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes (v_{\ell+1} \otimes v_\ell - v_\ell \otimes v_{\ell+1} - [v_{\ell+1}, v_\ell]) \otimes v_{\ell+2} \otimes \dots \otimes v_n \\ & \quad (\text{whenever } v_1, v_2, \dots, v_n \in \mathfrak{g} \text{ and } v_\ell \in \mathfrak{h}). \end{aligned}$$

This rewriting rule splits the tensor $v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes v_\ell \otimes v_{\ell+1} \otimes v_{\ell+2} \otimes \dots \otimes v_n$ into three tensors:

- a tensor $v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes v_{\ell+1} \otimes v_\ell \otimes v_{\ell+2} \otimes \dots \otimes v_n$ (we call this tensor the "primary product") which still has a tensorand lying in \mathfrak{h} (namely, v_ℓ), but now this tensorand has moved one step to the right;
- a tensor $-v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes [v_{\ell+1}, v_\ell] \otimes v_{\ell+2} \otimes \dots \otimes v_n$ (we call this tensor a "fission product") which is of degree smaller than n ;
- a tensor $-v_1 \otimes v_2 \otimes \dots \otimes v_{\ell-1} \otimes (v_{\ell+1} \otimes v_\ell - v_\ell \otimes v_{\ell+1} - [v_{\ell+1}, v_\ell]) \otimes v_{\ell+2} \otimes \dots \otimes v_n$ (we call this tensor a "J-product") which lies in J .

After applying this rewriting rule to all the tensors to which it can be applied, let us apply it again to the primary products, then again to the resulting primary products, etc. - until each of the primary products has its rightmost tensorand lying in \mathfrak{h} and thus cannot be rewritten anymore. As a result, we obtain a sum of tensors whose rightmost tensorand lies in \mathfrak{h} (these tensors clearly lie in $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$), of "fission products" (these are tensors of degree smaller than n) and of "J-products" (which lie in J). And do not forget the tensor $t(g_1) \otimes t(g_2) \otimes \dots \otimes t(g_n)$ which lies in $\otimes N$. As a result we

know that

$$\begin{aligned}
g_1 \otimes g_2 \otimes \dots \otimes g_n &= (\text{sum of some tensors in } (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \\
&\quad + (\text{sum of some tensors of degree smaller than } n) \\
&\quad + (\text{sum of some tensors in } J) + (\text{a tensor in } \otimes N) \\
&\in (\otimes \mathfrak{g}) \cdot \mathfrak{h} + \mathfrak{g}^{\otimes \leq (n-1)} + J + (\otimes N).
\end{aligned}$$

Since this holds for every pure tensor $g_1 \otimes g_2 \otimes \dots \otimes g_n \in \mathfrak{g}^{\otimes n}$, this yields $\mathfrak{g}^{\otimes n} \subseteq (\otimes \mathfrak{g}) \cdot \mathfrak{h} + \mathfrak{g}^{\otimes \leq (n-1)} + J + (\otimes N)$. But if we do an induction over n , we can assume that we already know that $\mathfrak{g}^{\otimes \leq (n-1)} \subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$, so that this becomes

$$\mathfrak{g}^{\otimes n} \subseteq (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)) + J + (\otimes N) = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N),$$

completing the induction.

So much for the first proof of Lemma 2.19.

Second proof of Lemma 2.19. Let us first show that

$$J \cdot (\otimes \mathfrak{g}) = J \quad \text{and} \quad (\otimes \mathfrak{g}) \cdot J = J \quad (65)$$

Proof of (65). In fact, Proposition 2.3 **(b)** yields $J = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g})$ (where J_0 was defined in that proposition). This leads to $J \cdot (\otimes \mathfrak{g}) = (\otimes \mathfrak{g}) \cdot J_0 \cdot \underbrace{(\otimes \mathfrak{g}) \cdot (\otimes \mathfrak{g})}_{=\otimes \mathfrak{g}} = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g}) = J$, but also leads to $(\otimes \mathfrak{g}) \cdot J = \underbrace{(\otimes \mathfrak{g}) \cdot (\otimes \mathfrak{g})}_{=\otimes \mathfrak{g}} \cdot J_0 \cdot (\otimes \mathfrak{g}) \subseteq (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g}) = J$.

This proves (65).

Next, we consider the k -submodule J_0 of $\otimes \mathfrak{g}$ defined in Proposition 2.3 **(b)**. This k -submodule J_0 was defined as $\langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle$. Hence, obviously,

$$v \otimes w - w \otimes v - [v, w] \in J_0 \quad \text{for every } (v, w) \in \mathfrak{g} \times \mathfrak{h}. \quad (66)$$

Now we are going to show that

$$vw - wv - [v, w] \in J \quad \text{for any } v \in \mathfrak{g} \text{ and } w \in \mathfrak{h} \quad (67)$$

(where vw means the product of v and w in $\otimes \mathfrak{g}$, and similarly wv means the product of w and v in $\otimes \mathfrak{g}$).

Proof of (67). Clearly, $v \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $w \in \mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ yield $v \cdot w = v \otimes w$ (by (31), applied to 1 and 1 instead of n and m), and similarly $w \cdot v = w \otimes v$. On the other hand, $v \in \mathfrak{g}$ and $w \in \mathfrak{h}$ lead to $(v, w) \in \mathfrak{g} \times \mathfrak{h}$. Now,

$$\begin{aligned}
\underbrace{vw}_{=v \cdot w = v \otimes w} - \underbrace{wv}_{=w \cdot v = w \otimes v} - [v, w] &= v \otimes w - w \otimes v - [v, w] = \underbrace{1}_{\in \otimes \mathfrak{g}} \cdot \underbrace{(v \otimes w - w \otimes v - [v, w])}_{\in J_0 \text{ (according to (66))}} \cdot \underbrace{1}_{\in \otimes \mathfrak{g}} \\
&\in (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g}) = J \quad \text{(by Proposition 2.3 (b)).}
\end{aligned}$$

This proves (67).

Next, let us show that

$$\mathfrak{h}\mathfrak{g} \subseteq J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}. \quad (68)$$

Proof of (68). Let G be the subset $\{hg \mid (h, g) \in \mathfrak{h} \times \mathfrak{g}\}$ of $\otimes \mathfrak{g}$. We are going to prove that $G \subseteq J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}$ now.

Indeed, let $U \in G$ be arbitrary. Then, $U \in G = \{hg \mid (h, g) \in \mathfrak{h} \times \mathfrak{g}\}$, so there exists some $(w, v) \in \mathfrak{h} \times \mathfrak{g}$ such that $U = wv$. Thus, $w \in \mathfrak{h}$ and $v \in \mathfrak{g}$, and $U = wv$. Now,

$$\begin{aligned} U = wv &= - \underbrace{(vw - wv - [v, w])}_{\in J \text{ (according to (67))}} + \underbrace{v}_{\in \mathfrak{g}} \underbrace{w}_{\in \mathfrak{h}} + \underbrace{(-[v, w])}_{\in \mathfrak{g}} \in \underbrace{-J}_{=J \text{ (since } J \text{ is a } k\text{-module)}} + \mathfrak{g}\mathfrak{h} + \mathfrak{g} \\ &= J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}. \end{aligned}$$

We have thus shown that every $U \in G$ satisfies $U \in J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}$. In other words, $G \subseteq J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}$.

Therefore, Proposition 1.29 (a) (applied to $\otimes \mathfrak{g}$, G and $J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}$ instead of M , S and Q) yields $\langle G \rangle \subseteq J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}$. Since

$$\begin{aligned} \langle G \rangle &= \langle \{hg \mid (h, g) \in \mathfrak{h} \times \mathfrak{g}\} \rangle && \text{(since } G = \{hg \mid (h, g) \in \mathfrak{h} \times \mathfrak{g}\}) \\ &= \langle hg \mid (h, g) \in \mathfrak{h} \times \mathfrak{g} \rangle = \mathfrak{h}\mathfrak{g}, \end{aligned}$$

this rewrites as $\mathfrak{h}\mathfrak{g} \subseteq J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}$. Thus, (68) is proven.

Let us next conclude that

$$\mathfrak{g}^{\otimes(p-1)}\mathfrak{h}\mathfrak{g} \subseteq J + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p} \quad \text{for every } p \in \mathbb{N}_+. \quad (69)$$

Proof of (69). Proposition 1.95 (a) (applied to \mathfrak{g} , $p-1$ and 1 instead of V , i and j) yields $\mathfrak{g}^{\otimes(p-1)} \cdot \mathfrak{g}^{\otimes 1} = \mathfrak{g}^{\otimes((p-1)+1)} = \mathfrak{g}^{\otimes p}$. Since $\mathfrak{g}^{\otimes 1} = \mathfrak{g}$, this becomes $\mathfrak{g}^{\otimes(p-1)} \cdot \mathfrak{g} = \mathfrak{g}^{\otimes p}$. In other words, $\mathfrak{g}^{\otimes(p-1)}\mathfrak{g} = \mathfrak{g}^{\otimes p}$.

From (68), we have

$$\begin{aligned} \mathfrak{g}^{\otimes(p-1)} \underbrace{\mathfrak{h}\mathfrak{g}}_{\subseteq J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}} &\subseteq \mathfrak{g}^{\otimes(p-1)} (J + \mathfrak{g}\mathfrak{h} + \mathfrak{g}) = \underbrace{\mathfrak{g}^{\otimes(p-1)} J}_{\subseteq \otimes \mathfrak{g}} + \underbrace{\mathfrak{g}^{\otimes(p-1)} \mathfrak{g}\mathfrak{h}}_{=\mathfrak{g}^{\otimes p}} + \underbrace{\mathfrak{g}^{\otimes(p-1)} \mathfrak{g}}_{=\mathfrak{g}^{\otimes p} \subseteq \mathfrak{g}^{\otimes \leq p}} \\ &\subseteq \underbrace{(\otimes \mathfrak{g}) \cdot J}_{=J \text{ (by (65))}} + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p} \subseteq J + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p}. \end{aligned}$$

This proves (69).

Our next step is to prove that

$$\mathfrak{g}^{\otimes q} \subseteq \mathfrak{g}^{\otimes \leq(q-1)} + J + \mathfrak{g}^{\otimes(q-1)}\mathfrak{h} + N^{\otimes q} \quad \text{for every } q \in \mathbb{N}_+. \quad (70)$$

Proof of (70). We are going to prove (70) by induction over q :

Induction base: In the case $q = 1$, we have $\mathfrak{g}^{\otimes q} = \mathfrak{g}^{\otimes 1} = \mathfrak{g} = \mathfrak{h} \oplus N = \mathfrak{h} + N$ and

$$\mathfrak{g}^{\otimes \leq(q-1)} + J + \mathfrak{g}^{\otimes(q-1)}\mathfrak{h} + N^{\otimes q} \supseteq \underbrace{\mathfrak{g}^{\otimes(q-1)}}_{=\mathfrak{g}^{\otimes(1-1)}=\mathfrak{g}^{\otimes 0}=k} \mathfrak{h} + \underbrace{N^{\otimes q}}_{=N^{\otimes 1}=N} = \underbrace{k\mathfrak{h}}_{=\mathfrak{h} \text{ (since } \mathfrak{h} \text{ is a } k\text{-module)}} + N = \mathfrak{h} + N.$$

Thus, in the case $q = 1$, we have $\mathfrak{g}^{\otimes q} = \mathfrak{h} + N \subseteq \mathfrak{g}^{\otimes \leq(q-1)} + J + \mathfrak{g}^{\otimes(q-1)}\mathfrak{h} + N^{\otimes q}$. In other words, (70) holds for $q = 1$. This completes the induction base.

Induction step: Let $p \in \mathbb{N}_+$ be arbitrary. Assume that (70) holds for $q = p$. Now our task is to prove that (70) holds for $q = p + 1$ as well.

Since (70) holds for $q = p$, we have $\mathfrak{g}^{\otimes p} \subseteq \mathfrak{g}^{\otimes \leq(p-1)} + J + \mathfrak{g}^{\otimes(p-1)}\mathfrak{h} + N^{\otimes p}$.

Now, Proposition 1.95 (a) (applied to \mathfrak{g} , p and 1 instead of V , i and j) yields $\mathfrak{g}^{\otimes p} \cdot \mathfrak{g}^{\otimes 1} = \mathfrak{g}^{\otimes(p+1)}$. Since $\mathfrak{g}^{\otimes 1} = \mathfrak{g}$, this becomes $\mathfrak{g}^{\otimes p} \cdot \mathfrak{g} = \mathfrak{g}^{\otimes(p+1)}$. Thus,

$$\begin{aligned}
\mathfrak{g}^{\otimes(p+1)} &= \underbrace{\mathfrak{g}^{\otimes p}}_{\subseteq \mathfrak{g}^{\otimes \leq(p-1)} + J + \mathfrak{g}^{\otimes(p-1)}\mathfrak{h} + N^{\otimes p}} \cdot \mathfrak{g} \subseteq (\mathfrak{g}^{\otimes \leq(p-1)} + J + \mathfrak{g}^{\otimes(p-1)}\mathfrak{h} + N^{\otimes p}) \cdot \mathfrak{g} \\
&= \mathfrak{g}^{\otimes \leq(p-1)} \cdot \underbrace{\mathfrak{g}}_{=\mathfrak{g}^{\otimes 1} \subseteq \mathfrak{g}^{\otimes \leq 1}} + J \cdot \underbrace{\mathfrak{g}}_{\subseteq \otimes \mathfrak{g}} + \underbrace{\mathfrak{g}^{\otimes(p-1)}\mathfrak{h}\mathfrak{g}}_{\subseteq J + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p} \text{ (by (69))}} + N^{\otimes p} \cdot \underbrace{\mathfrak{g}}_{=\mathfrak{h} \oplus N = \mathfrak{h} + N} \\
&\subseteq \underbrace{\mathfrak{g}^{\otimes \leq(p-1)} \cdot \mathfrak{g}^{\otimes \leq 1}}_{\subseteq \mathfrak{g}^{\otimes \leq((p-1)+1)}} + \underbrace{J \cdot (\otimes \mathfrak{g})}_{=J \text{ (by (65))}} + J + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p} + \underbrace{N^{\otimes p} \cdot (\mathfrak{h} + N)}_{=N^{\otimes p} \cdot \mathfrak{h} + N^{\otimes p} \cdot N} \\
&\quad \text{(by Proposition 1.95 (b), applied to } \mathfrak{g}, p-1 \text{ and 1 instead of } V, n \text{ and } m\text{)} \\
&\subseteq \underbrace{\mathfrak{g}^{\otimes \leq((p-1)+1)}}_{=\mathfrak{g}^{\otimes \leq p}} + \underbrace{J + J}_{=J \text{ (since } J \text{ is a } k\text{-module)}} + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p} + \underbrace{N^{\otimes p} \cdot \mathfrak{h}}_{\subseteq \mathfrak{g}^{\otimes p}} + N^{\otimes p} \cdot \underbrace{N}_{=N^{\otimes 1}} \\
&\subseteq \mathfrak{g}^{\otimes \leq p} + J + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p} + \mathfrak{g}^{\otimes p}\mathfrak{h} + \underbrace{N^{\otimes p} \cdot N^{\otimes 1}}_{=N^{\otimes(p+1)} \text{ (by Proposition 1.95 (a), applied to } N, p \text{ and 1 instead of } V, i \text{ and } j\text{)}} \\
&= \mathfrak{g}^{\otimes \leq p} + J + \mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes \leq p} + \mathfrak{g}^{\otimes p}\mathfrak{h} + N^{\otimes(p+1)} \\
&= \underbrace{\mathfrak{g}^{\otimes \leq p} + \mathfrak{g}^{\otimes \leq p}}_{=\mathfrak{g}^{\otimes \leq p} \text{ (since } \mathfrak{g}^{\otimes \leq p} \text{ is a } k\text{-module)}} + J + \underbrace{\mathfrak{g}^{\otimes p}\mathfrak{h} + \mathfrak{g}^{\otimes p}\mathfrak{h}}_{=\mathfrak{g}^{\otimes p}\mathfrak{h} \text{ (since } \mathfrak{g}^{\otimes p}\mathfrak{h} \text{ is a } k\text{-module)}} + N^{\otimes(p+1)} \\
&= \underbrace{\mathfrak{g}^{\otimes \leq p}}_{=\mathfrak{g}^{\otimes \leq((p+1)-1)}} + J + \underbrace{\mathfrak{g}^{\otimes p}}_{=\mathfrak{g}^{\otimes((p+1)-1)}} \mathfrak{h} + N^{\otimes(p+1)} = \mathfrak{g}^{\otimes \leq((p+1)-1)} + J + \mathfrak{g}^{\otimes((p+1)-1)}\mathfrak{h} + N^{\otimes(p+1)}.
\end{aligned}$$

In other words, (70) holds for $q = p + 1$. This completes the induction step, and thus (70) is proven.

Now, our next (and, for this proof, our last) claim is:

$$\mathfrak{g}^{\otimes \leq p} \subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \quad \text{for every } p \in \mathbb{N}. \quad (71)$$

Proof of (71). We are going to verify (71) by induction over p :

Induction base: For $p = 0$, we have $\mathfrak{g}^{\otimes \leq p} = \mathfrak{g}^{\otimes \leq 0} = \bigoplus_{i=0}^0 \mathfrak{g}^{\otimes i} = \mathfrak{g}^{\otimes 0} = k \subseteq \otimes N \subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$. Thus, (71) holds for $p = 0$. This settles the induction base.

Induction step: Let $q \in \mathbb{N}_+$. Assume that (71) is already proven for $p = q - 1$. We now must prove (71) for $p = q$.

Since we assumed that (71) is already proven for $p = q - 1$, we know that $\mathfrak{g}^{\otimes \leq(q-1)} \subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$.

Now, (70) yields

$$\begin{aligned}
\mathfrak{g}^{\otimes q} &\subseteq \underbrace{\mathfrak{g}^{\otimes \leq(q-1)}}_{\subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)} + J + \underbrace{\mathfrak{g}^{\otimes(q-1)}\mathfrak{h}}_{\subseteq \otimes \mathfrak{g}} + \underbrace{N^{\otimes q}}_{\subseteq \otimes N} \subseteq (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)) + J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \\
&= (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)) + (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)) = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \\
&\quad \left(\begin{array}{l} \text{since } J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \text{ is a } k\text{-module} \\ \text{(because } J, (\otimes \mathfrak{g}) \cdot \mathfrak{h} \text{ and } \otimes N \text{ are } k\text{-modules)} \end{array} \right).
\end{aligned}$$

Now, the definition of $\mathfrak{g}^{\otimes \leq q}$ says $\mathfrak{g}^{\otimes \leq q} = \bigoplus_{i=0}^q \mathfrak{g}^{\otimes i}$, and the definition of $\mathfrak{g}^{\otimes \leq (q-1)}$ says

$$\mathfrak{g}^{\otimes \leq (q-1)} = \bigoplus_{i=0}^{q-1} \mathfrak{g}^{\otimes i}. \text{ Therefore,}$$

$$\begin{aligned} \mathfrak{g}^{\otimes \leq q} &= \bigoplus_{i=0}^q \mathfrak{g}^{\otimes i} = \underbrace{\left(\bigoplus_{i=0}^{q-1} \mathfrak{g}^{\otimes i} \right)}_{=\mathfrak{g}^{\otimes \leq (q-1)}} \oplus \mathfrak{g}^{\otimes q} = \mathfrak{g}^{\otimes \leq (q-1)} \oplus \mathfrak{g}^{\otimes q} = \underbrace{\mathfrak{g}^{\otimes \leq (q-1)}}_{\subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)} + \underbrace{\mathfrak{g}^{\otimes q}}_{\subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)} \\ &\subseteq (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)) + (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)) \subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \\ &\quad \left(\begin{array}{l} \text{since } J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \text{ is a } k\text{-module} \\ \text{(because } J, (\otimes \mathfrak{g}) \cdot \mathfrak{h} \text{ and } \otimes N \text{ are } k\text{-modules)} \end{array} \right). \end{aligned}$$

In other words, (71) holds for $p = q$. This completes the induction step, and thus the induction proof of (71) is done.

Now we are ready to make short shrift of Lemma 2.19: We have

$$\begin{aligned} \otimes \mathfrak{g} &= \bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i} = \bigoplus_{p \in \mathbb{N}} \mathfrak{g}^{\otimes p} \quad (\text{here we substituted } p \text{ for } i \text{ in the sum}) \\ &= \sum_{p \in \mathbb{N}} \underbrace{\mathfrak{g}^{\otimes p}}_{\substack{\subseteq \mathfrak{g}^{\otimes \leq p} \subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \\ \text{(according to (71))}}} \quad (\text{since direct sums are sums}) \\ &\subseteq \sum_{p \in \mathbb{N}} (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)) \subseteq J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \\ &\quad \left(\begin{array}{l} \text{since } J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \text{ is a } k\text{-module} \\ \text{(because } J, (\otimes \mathfrak{g}) \cdot \mathfrak{h} \text{ and } \otimes N \text{ are } k\text{-modules)} \end{array} \right). \end{aligned}$$

Together with $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N) \subseteq \otimes \mathfrak{g}$ (which is obvious), this yields $\otimes \mathfrak{g} = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$. This proves Lemma 2.19.

2.8. The factor map $\bar{\varphi}$

Before we start proving Proposition 2.18, let us show that $(F_n)_{n \geq 0}$ is a filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ (this is part of the claim of Theorem 2.1 (b)):

Every $n \in \mathbb{N}$ satisfies $\mathfrak{g}^{\otimes \leq n} \subseteq \mathfrak{g}^{\otimes \leq (n+1)}$ (since $(\mathfrak{g}^{\otimes \leq n})_{n \geq 0}$ is a filtration of $\otimes \mathfrak{g}$), and thus $\zeta(\mathfrak{g}^{\otimes \leq n}) \subseteq \zeta(\mathfrak{g}^{\otimes \leq (n+1)})$. Since $F_n = \zeta(\mathfrak{g}^{\otimes \leq n})$ (by the definition of F_n) and $F_{n+1} = \zeta(\mathfrak{g}^{\otimes \leq (n+1)})$ (by the definition of F_{n+1}), we thus have $F_n = \zeta(\mathfrak{g}^{\otimes \leq n}) \subseteq \zeta(\mathfrak{g}^{\otimes \leq (n+1)}) = F_{n+1}$. Since this holds for all $n \in \mathbb{N}$, we thus have $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$. On the other hand, $\bigcup_{n \in \mathbb{N}} \mathfrak{g}^{\otimes \leq n} = \otimes \mathfrak{g}$ (since $(\mathfrak{g}^{\otimes \leq n})_{n \geq 0}$ is a filtration of $\otimes \mathfrak{g}$). Now,

$$\bigcup_{n \in \mathbb{N}} \underbrace{F_n}_{=\zeta(\mathfrak{g}^{\otimes \leq n})} = \bigcup_{n \in \mathbb{N}} \zeta(\mathfrak{g}^{\otimes \leq n}) = \zeta \left(\underbrace{\bigcup_{n \in \mathbb{N}} \mathfrak{g}^{\otimes \leq n}}_{=\otimes \mathfrak{g}} \right) = \zeta(\otimes \mathfrak{g}) = (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$$

(since ζ is the canonical projection on $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$). Together with $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$, this yields that $(F_n)_{n \geq 0}$ is a filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$.

We are now able to show the first four parts of Proposition 2.18:

Proof of Proposition 2.18. **(a)** For every $U \in \otimes N$, we have $U = \varphi(U)$ (because we know from Proposition 2.13 that $\varphi|_{\otimes N} = \text{id}_{\otimes N}$, and now $U \in \otimes N$ yields $\varphi(U) = \underbrace{(\varphi|_{\otimes N})}_{=\text{id}_{\otimes N}}(U) = \text{id}_{\otimes N}(U) = U$). Thus, for every $U \in \otimes N$, we have $U \in \varphi(\otimes \mathfrak{g})$. In other words, φ is surjective.

Next we must prove that $\text{Ker } \varphi = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$.

First of all, we know that

$$\begin{aligned} \varphi(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) &= \underbrace{\varphi(J)}_{=0 \text{ (by Proposition 2.16)}} + \underbrace{\varphi((\otimes \mathfrak{g}) \cdot \mathfrak{h})}_{=0 \text{ (by Proposition 2.17)}} \quad (\text{since } \varphi \text{ is } k\text{-linear}) \\ &= 0 + 0 = 0. \end{aligned}$$

Lemma 2.19 yields $\otimes \mathfrak{g} = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} + (\otimes N)$. In other words, $\otimes \mathfrak{g} = (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) + (\otimes N)$.

We are going to strengthen this to $\otimes \mathfrak{g} = (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \oplus (\otimes N)$ now:

Any element U of $(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \cap (\otimes N)$ satisfies $\varphi(U) = 0$ (because $U \in (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \cap (\otimes N)$ yields $U \in J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$ and thus $\varphi(U) \in \varphi(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$, so that $\varphi(U) = 0$). But on the other hand, any element U of $(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \cap (\otimes N)$ satisfies $\varphi(U) = U$ (since $U \in (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \cap (\otimes N)$ yields $U \in \otimes N$, so that $\varphi(U) = \underbrace{(\varphi|_{\otimes N})}_{=\text{id}_{\otimes N} \text{ (by Proposition 2.13)}}(U) = \text{id}_{\otimes N}(U) = U$). Thus, any element U of $(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \cap (\otimes N)$ satisfies $U = \varphi(U) = 0$.

Hence, $(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \cap (\otimes N) = 0$. Thus, the sum $(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) + (\otimes N)$ is a direct sum; hence, $\otimes \mathfrak{g} = (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) + (\otimes N)$ becomes $\otimes \mathfrak{g} = (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \oplus (\otimes N)$. Thus, there exists a k -linear projection φ' of $\otimes \mathfrak{g}$ on $\otimes N$ with kernel $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$. We are now going to show that $\varphi = \varphi'$.

In fact,

$$(\varphi - \varphi')(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = \underbrace{\varphi(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})}_{=0} - \underbrace{\varphi'(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})}_{=0 \text{ (since } \varphi' \text{ is a projection with kernel } J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})} = 0 - 0 = 0$$

and thus $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} \in \text{Ker}(\varphi - \varphi')$. On the other hand, every $U \in \otimes N$ satisfies

$$\varphi(U) = \underbrace{(\varphi|_{\otimes N})}_{=\text{id}_{\otimes N} \text{ (by Proposition 2.13)}}(U) = \text{id}_{\otimes N}(U) = U = \varphi'(U)$$

(since φ' is a projection on $\otimes N$, whereas U is an element of $\otimes N$)

and thus $(\varphi - \varphi')(U) = \underbrace{\varphi(U)}_{=\varphi'(U)} - \varphi'(U) = 0$, so that $U \in \text{Ker}(\varphi - \varphi')$. So we have

shown that every $U \in \otimes N$ satisfies $U \in \text{Ker}(\varphi - \varphi')$. Thus, $\otimes N \subseteq \text{Ker}(\varphi - \varphi')$. Now,

$$\begin{aligned} \otimes \mathfrak{g} &= \underbrace{J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}}_{\subseteq \text{Ker}(\varphi - \varphi')} + \underbrace{(\otimes N)}_{\subseteq \text{Ker}(\varphi - \varphi')} \subseteq \text{Ker}(\varphi - \varphi') + \text{Ker}(\varphi - \varphi') \\ &= \text{Ker}(\varphi - \varphi') \quad (\text{since } \text{Ker}(\varphi - \varphi') \text{ is a } k\text{-module (since } \varphi - \varphi' \text{ is } k\text{-linear)}), \end{aligned}$$

so that $\varphi - \varphi' = 0$. Thus, $\varphi = \varphi'$. Thus, $\text{Ker } \varphi' = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$ (since φ' is a projection with kernel $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$) rewrites as $\text{Ker } \varphi = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$.

By the isomorphism theorem, the k -module homomorphism $\varphi : \otimes \mathfrak{g} \rightarrow \otimes N$ induces a k -module isomorphism $\bar{\varphi} : (\otimes \mathfrak{g}) / \text{Ker } \varphi \rightarrow \varphi(\otimes \mathfrak{g})$ which satisfies $\varphi = \bar{\varphi} \circ \zeta'$, where ζ' is the canonical projection of $\otimes \mathfrak{g}$ onto $(\otimes \mathfrak{g}) / \text{Ker } \varphi$. Since $\text{Ker } \varphi = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$ and $\varphi(\otimes \mathfrak{g}) = \otimes N$ (because φ is surjective), this is therefore a k -module isomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes N$. Moreover, ζ' is the canonical projection of $\otimes \mathfrak{g}$ onto $(\otimes \mathfrak{g}) / \text{Ker } \varphi$, and therefore equal to the canonical projection of $\otimes \mathfrak{g}$ onto $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ (since $\text{Ker } \varphi = J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}$). Since the canonical projection of $\otimes \mathfrak{g}$ onto $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is ζ , this yields that ζ' is equal to ζ . Therefore, $\varphi = \bar{\varphi} \circ \zeta'$ rewrites as $\varphi = \bar{\varphi} \circ \zeta$. This completes the proof of Proposition 2.18 (a).

(b) Let $n \in \mathbb{N}$ be arbitrary. Since $F_n = \zeta(\mathfrak{g}^{\otimes \leq n})$, we have

$$\bar{\varphi}(F_n) = \bar{\varphi}(\zeta(\mathfrak{g}^{\otimes \leq n})) = \underbrace{(\bar{\varphi} \circ \zeta)}_{=\varphi}(\mathfrak{g}^{\otimes \leq n}) = \varphi(\mathfrak{g}^{\otimes \leq n}) \subseteq N^{\otimes \leq n}$$

(according to Proposition 2.12).

In other words, the isomorphism $\bar{\varphi}$ respects the filtration. This proves Proposition 2.18 (b).

(c) The homomorphism η respects the filtration (because $\eta = \otimes(\pi|_N)$, and because Proposition 1.104 (applied to $\pi|_N$, N and \mathfrak{n} instead of f , V and W) yields that $\otimes(\pi|_N)$ respects the filtration). Everything else claimed in Proposition 2.18 (c) is trivial. Thus, Proposition 2.18 (c) is proven.

(d) We have defined $\iota : \otimes N \rightarrow \otimes \mathfrak{g}$ as the canonical inclusion of $\otimes N$ in $\otimes \mathfrak{g}$. Therefore, ι clearly respects the filtration. Also, we know that ζ respects the filtration, because the

filtration on $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ was defined to be $\left(\underbrace{F_n}_{=\zeta(\mathfrak{g}^{\otimes \leq n})} \right)_{n \geq 0} = (\zeta(\mathfrak{g}^{\otimes \leq n}))_{n \geq 0}$.

Since ζ and ι respect the filtration, the composition $\zeta \circ \iota$ respects the filtration as well (by Proposition 1.99 (b)).

Besides, $\bar{\varphi} \circ (\zeta \circ \iota) = \underbrace{\bar{\varphi} \circ \zeta}_{=\varphi} \circ \iota = \varphi \circ \iota = \text{id}_{\otimes N}$ (because every $U \in \otimes N$ satisfies

$$\begin{aligned} (\varphi \circ \iota)(U) &= \varphi \left(\underbrace{\iota(U)}_{=U \text{ (since } \iota \text{ is the inclusion map)}} \right) = \varphi(U) = \underbrace{(\varphi|_{\otimes N})}_{=\text{id}_{\otimes N}}(U) \quad (\text{since } U \in \otimes N) \\ &= \text{id}_{\otimes N}(U) \end{aligned} \quad (\text{due to Proposition 2.13})$$

). Hence, $\zeta \circ \iota = \bar{\varphi}^{-1}$ (here we know that $\bar{\varphi}^{-1}$ exists, because $\bar{\varphi}$ is a k -module isomorphism). Thus, $\bar{\varphi}^{-1}$ respects the filtration (since $\zeta \circ \iota$ respects the filtration). This proves Proposition 2.18 (d).

2.9. φ and $\bar{\varphi}$ on the associated graded objects

Before we can verify Proposition 2.18 (e), we are going to show the next result, which will also help us in explicitly describing the associated graded morphism of $\bar{\varphi}$ later:

Proposition 2.20. Consider the k -module homomorphism $\otimes t : \otimes \mathfrak{g} \rightarrow \otimes N$ induced by the k -module homomorphism $t : \mathfrak{g} \rightarrow N$. This homomorphism $\otimes t$ respects the filtration and satisfies

$$(\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq n}) \subseteq N^{\otimes \leq (n-1)} \quad \text{for every } n \in \mathbb{N}. \quad (72)$$

Proof of Proposition 2.20. Proposition 1.104 (applied to t , \mathfrak{g} and N instead of f , V and W) yields that $\otimes t$ respects the filtration. Thus, the only thing that remains to be done for the proof of Proposition 2.20 is proving the relation (72).

We are going to prove (72) by induction over n :

Induction base: Every $\lambda \in \mathfrak{g}^{\otimes \leq 0}$ satisfies $\lambda \in k$ (since $\mathfrak{g}^{\otimes \leq 0} = \bigoplus_{i=0}^0 \mathfrak{g}^{\otimes i} = \mathfrak{g}^{\otimes 0} = k$).

Therefore, every $\lambda \in \mathfrak{g}^{\otimes \leq 0}$ satisfies $\varphi(\lambda) = \lambda$ (according to (52), since $\lambda \in k$) and $(\otimes t)(\lambda) = \lambda$ (by the definition of $\otimes t$, since $\lambda \in k$), so that $(\varphi - (\otimes t))(\lambda) = \underbrace{\varphi(\lambda)}_{=\lambda} - \underbrace{(\otimes t)(\lambda)}_{=\lambda} = 0$ and thus $\lambda \in \text{Ker}(\varphi - (\otimes t))$. We have thus shown that every $\lambda \in \mathfrak{g}^{\otimes \leq 0}$ satisfies $\lambda \in \text{Ker}(\varphi - (\otimes t))$. Hence, $\mathfrak{g}^{\otimes \leq 0} \subseteq \text{Ker}(\varphi - (\otimes t))$, so that $(\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq 0}) = 0 \subseteq N^{\otimes \leq (0-1)}$. In other words, (72) holds for $n = 0$. This completes the induction base.

Induction step: Let $p \in \mathbb{N}$. Assume that (72) holds for $n = p$. We now must show that (72) also holds for $n = p + 1$.

Since (72) holds for $n = p$, we have $(\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq p}) \subseteq N^{\otimes \leq (p-1)}$.

We are now going to show that $(\varphi - (\otimes t))(\mathfrak{g}^{\otimes (p+1)}) \subseteq N^{\otimes \leq p}$. This will quickly yield $(\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq (p+1)}) \subseteq N^{\otimes \leq p}$, which will bring us to the end of the induction step.

Proposition 2.11 (applied to $p + 1$ and \mathfrak{g} instead of p and V) yields $\mathfrak{g}^{\otimes (p+1)} = \langle \mathfrak{g}_{\text{lind}}^{\otimes (p+1)} \rangle$.

Note that $\varphi - (\otimes t)$ is a k -linear map (since φ and $\otimes t$ are k -linear), and thus $\text{Ker}(\varphi - (\otimes t))$ is a k -module.

Now we are going to prove that $\mathfrak{g}_{\text{lind}}^{\otimes (p+1)} \subseteq (\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$. Indeed, let $V \in \mathfrak{g}_{\text{lind}}^{\otimes (p+1)}$ be arbitrary. Then, V is a left-induced tensor in $\mathfrak{g}^{\otimes (p+1)}$ (since $V \in \mathfrak{g}_{\text{lind}}^{\otimes (p+1)}$), and thus there exist $u \in \mathfrak{g}$ and $U \in \mathfrak{g}^{\otimes p}$ such that $V = u \otimes U$. Consider these u and U . Since $u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $U \in \mathfrak{g}^{\otimes p}$, we have $u \cdot U = u \otimes U$ (due to (31), applied to u , U , 1 and p instead of a , b , n and m).

Since $U \in \mathfrak{g}^{\otimes p}$ and since $\mathfrak{g}^{\otimes p}$ is a \mathfrak{g} -module, we have $s(u) \rightarrow U \in \mathfrak{g}^{\otimes p} \subseteq \mathfrak{g}^{\otimes \leq p}$. Thus $\varphi(s(u) \rightarrow U) \in \varphi(\mathfrak{g}^{\otimes \leq p}) \subseteq N^{\otimes \leq p}$ (according to Proposition 2.12), so that $\varphi(s(u) \rightarrow U) \equiv 0 \pmod{N^{\otimes \leq p}}$.

On the other hand, $U \in \mathfrak{g}^{\otimes p} \subseteq \mathfrak{g}^{\otimes \leq p}$ yields $(\varphi - (\otimes t))(U) \in (\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq p}) \subseteq N^{\otimes \leq (p-1)}$. Since $(\varphi - (\otimes t))(U) = \varphi(U) - (\otimes t)(U)$, this rewrites as $\varphi(U) - (\otimes t)(U) \in N^{\otimes \leq (p-1)}$. On the other hand, $\otimes t$ is a k -algebra homomorphism, so that

$$\begin{aligned} (\otimes t)(u \cdot U) &= \underbrace{(\otimes t)(u)}_{=t(u) \text{ (by the definition of } \otimes t, \text{ since } u \in \mathfrak{g}^{\otimes 1})} \cdot (\otimes t)(U) = t(u) \cdot (\otimes t)(U). \end{aligned}$$

Now,

$$\begin{aligned}
& t(u) \cdot \varphi(U) - (\otimes t) \left(\underbrace{V}_{=u \otimes U = u \cdot U} \right) \\
&= t(u) \cdot \varphi(U) - \underbrace{(\otimes t)(u \cdot U)}_{=t(u) \cdot (\otimes t)(U)} = t(u) \cdot \varphi(U) - t(u) \cdot (\otimes t)(U) \\
&= \underbrace{t(u)}_{\in N = N^{\otimes 1} \subseteq N^{\otimes \leq 1}} \cdot \underbrace{(\varphi(U) - (\otimes t)(U))}_{\in N^{\otimes \leq (p-1)}} \in N^{\otimes \leq 1} \cdot N^{\otimes \leq (p-1)} \\
&\subseteq N^{\otimes \leq (1+(p-1))} \quad \left(\begin{array}{l} \text{according to Proposition 1.95 (b), applied} \\ \text{to } N, 1 \text{ and } p-1 \text{ instead of } V, n \text{ and } m \end{array} \right) \\
&= N^{\otimes \leq p}.
\end{aligned}$$

In other words, $t(u) \cdot \varphi(U) \equiv (\otimes t)(V) \pmod{N^{\otimes \leq p}}$.

Now, $V = u \otimes U = u \cdot U$ yields

$$\begin{aligned}
\varphi(V) &= \varphi(u \cdot U) = \underbrace{t(u) \cdot \varphi(U)}_{\equiv (\otimes t)(V) \pmod{N^{\otimes \leq p}}} + \underbrace{\varphi(s(u) \dashv U)}_{\equiv 0 \pmod{N^{\otimes \leq p}}} \quad (\text{by (54)}) \\
&\equiv (\otimes t)(V) + 0 = (\otimes t)(V) \pmod{N^{\otimes \leq p}}.
\end{aligned}$$

In other words, $\varphi(V) - (\otimes t)(V) \in N^{\otimes \leq p}$. Since $\varphi(V) - (\otimes t)(V) = (\varphi - (\otimes t))(V)$, this rewrites as $(\varphi - (\otimes t))(V) \in N^{\otimes \leq p}$. In other words, $V \in (\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$.

So we have proven that every $V \in \mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$ satisfies $V \in (\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$. In other words, $\mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \subseteq (\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$. Therefore, Proposition 1.29 (a) (applied to $\mathfrak{g}^{\otimes(p+1)}$, $\mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$ and $(\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$ instead of M, S and Q) yields $\langle \mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \rangle \subseteq (\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$ (since $(\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$ is a k -module). Altogether we now have $\mathfrak{g}^{\otimes(p+1)} = \langle \mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \rangle \subseteq (\varphi - (\otimes t))^{-1}(N^{\otimes \leq p})$, so that $(\varphi - (\otimes t))(\mathfrak{g}^{\otimes(p+1)}) \subseteq N^{\otimes \leq p}$.

Now, the definition of $\mathfrak{g}^{\otimes \leq (p+1)}$ is $\mathfrak{g}^{\otimes \leq (p+1)} = \bigoplus_{i=0}^{p+1} \mathfrak{g}^{\otimes i}$, while the definition of $\mathfrak{g}^{\otimes \leq p} = \bigoplus_{i=0}^p \mathfrak{g}^{\otimes i}$. Thus,

$$\mathfrak{g}^{\otimes \leq (p+1)} = \bigoplus_{i=0}^{p+1} \mathfrak{g}^{\otimes i} = \underbrace{\left(\bigoplus_{i=0}^p \mathfrak{g}^{\otimes i} \right)}_{=\mathfrak{g}^{\otimes \leq p}} \oplus \mathfrak{g}^{\otimes (p+1)} = \mathfrak{g}^{\otimes \leq p} \oplus \mathfrak{g}^{\otimes (p+1)} = \mathfrak{g}^{\otimes \leq p} + \mathfrak{g}^{\otimes (p+1)}$$

(since direct sums are sums). Thus,

$$\begin{aligned}
(\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq (p+1)}) &= (\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq p} + \mathfrak{g}^{\otimes (p+1)}) \\
&= \underbrace{(\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq p})}_{\substack{\subseteq N^{\otimes \leq (p-1)} \subseteq N^{\otimes \leq p} \\ \text{(since } (N^{\otimes \leq n})_{n \geq 0} \text{ is a filtration)}}} + \underbrace{(\varphi - (\otimes t))(\mathfrak{g}^{\otimes (p+1)})}_{\subseteq N^{\otimes \leq p}} \\
&\quad \text{(since } \varphi - (\otimes t) \text{ is } k\text{-linear)} \\
&\subseteq N^{\otimes \leq p} + N^{\otimes \leq p} = N^{\otimes \leq p} \quad \text{(since } N^{\otimes \leq p} \text{ is a } k\text{-module)} \\
&= N^{\otimes \leq ((p+1)-1)}.
\end{aligned}$$

In other words, (72) holds for $n = p + 1$. This completes the induction step. Thus, (72) is proven for all $n \in \mathbb{N}$. In other words, Proposition 2.20 is proven.

Now let us finally prove Proposition 2.18 (e):

Both maps $\bar{\varphi}$ and η respect the filtration. Hence, their composition $\eta \circ \bar{\varphi} : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ respects the filtration as well (by Proposition 1.99 (b)). In order to prove Proposition 2.18 (e), it thus only remains to show that the homomorphism $\text{gr}_p(\eta \circ \bar{\varphi}) : \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_p(\otimes \mathfrak{n})$ is an \mathfrak{h} -module isomorphism for every $p \in \mathbb{N}$. In order to do this, we must show two things: we must show that it is a k -module isomorphism, and that it is an \mathfrak{h} -module homomorphism.

First, we notice $\eta \circ \bar{\varphi}$ is a k -module isomorphism, since both η and $\bar{\varphi}$ are k -module isomorphisms. But this alone is not enough to conclude that $\text{gr}_p(\eta \circ \bar{\varphi})$ is a k -module isomorphism (see Warning 1.100). However, we can get around this as follows:

We also know that $\eta^{-1} = (\otimes (\pi|_N))^{-1} = \otimes ((\pi|_N)^{-1})$, so that η^{-1} respects the filtration (by Proposition 1.104). Since $\bar{\varphi}^{-1}$ and η^{-1} respect the filtration, the composition $\bar{\varphi}^{-1} \circ \eta^{-1}$ respects the filtration as well (by Proposition 1.99 (b)). Since $\bar{\varphi}^{-1} \circ \eta^{-1} = (\eta \circ \bar{\varphi})^{-1}$, this means that $(\eta \circ \bar{\varphi})^{-1}$ respects the filtration.

Now, Proposition 1.101 (applied to $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$, $\otimes \mathfrak{n}$, $(F_n)_{n \geq 0}$, $(N^{\otimes \leq n})_{n \geq 0}$ and $\eta \circ \bar{\varphi}$ instead of V , W , $(V_n)_{n \geq 0}$, $(W_n)_{n \geq 0}$ and f) yields that $\text{gr}_p(\eta \circ \bar{\varphi}) : \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_p(\otimes \mathfrak{n})$ is a k -module isomorphism for every $p \in \mathbb{N}$.

It remains now to prove that it is an \mathfrak{h} -module homomorphism. In order to do this, we make some preparations:

We have $(\pi|_N) \circ t = \pi$ ³⁰. Thus, $\otimes((\pi|_N) \circ t) = \otimes \pi$. Since $\otimes((\pi|_N) \circ t) = \underbrace{(\otimes (\pi|_N))}_{= \eta} \circ (\otimes t) = \eta \circ (\otimes t)$, this becomes $\eta \circ (\otimes t) = \otimes \pi$.

Thus,

$$\eta \circ \varphi - \underbrace{(\otimes \pi)}_{= \eta \circ (\otimes t)} = \eta \circ \varphi - \eta \circ (\otimes t) = \eta \circ (\varphi - (\otimes t)).$$

³⁰*Proof.* Let $v \in \mathfrak{g}$ be arbitrary. Then, $v \in \mathfrak{g} = \mathfrak{h} \oplus N$. Thus, $v = h + n$ for some $h \in \mathfrak{h}$ and $n \in N$.

Consider these h and n . Then, $v = h + n$ yields

$$\begin{aligned} t(v) = t(h + n) &= \underbrace{t(h)}_{=0 \text{ (since } h \in \mathfrak{h}, \text{ while } t \text{ is a projection with kernel } \mathfrak{h})} + \underbrace{t(n)}_{=n \text{ (since } n \in N, \text{ while } t \text{ is a projection on } N)} \quad (\text{since } t \text{ is } k\text{-linear}) \\ &= n \end{aligned}$$

But

$$((\pi|_N) \circ t)(v) = (\pi|_N)(t(v)) = \pi \left(\underbrace{t(v)}_{=n} \right) = \pi(n).$$

Compared with

$$\begin{aligned} \pi \left(\underbrace{v}_{=h+n} \right) &= \pi(h + n) = \underbrace{\pi(h)}_{=0 \text{ (since } h \in \mathfrak{h}, \text{ while } \pi \text{ is a projection with kernel } \mathfrak{h})} + \pi(n) \quad (\text{since } \pi \text{ is } k\text{-linear}) \\ &= \pi(n), \end{aligned}$$

this yields $((\pi|_N) \circ t)(v) = \pi(v)$. We have thus shown that every $v \in \mathfrak{g}$ satisfies $((\pi|_N) \circ t)(v) = \pi(v)$. Thus, $(\pi|_N) \circ t = \pi$.

Both maps φ and $\otimes t$ respect the filtration. Thus, their difference $\varphi - (\otimes t)$ also respects the filtration (by Proposition 1.99 (c)). Together with the fact that η respects the filtration, this yields that the composition $\eta \circ (\varphi - (\otimes t))$ also respects the filtration (by Proposition 1.99 (b)). Since $\eta \circ (\varphi - (\otimes t)) = \eta \circ \varphi - (\otimes \pi)$, we have thus proven that the map $\eta \circ \varphi - (\otimes \pi)$ respects the filtration.

Now let $p \in \mathbb{N}$. Proposition 2.20 (applied to $n = p$) yields $(\varphi - (\otimes t))(\mathfrak{g}^{\otimes \leq p}) \subseteq N^{\otimes \leq (p-1)}$. Now,

$$\begin{aligned} \left(\underbrace{\eta \circ \varphi - (\otimes \pi)}_{= \eta \circ (\varphi - (\otimes t))} \right) (\mathfrak{g}^{\otimes \leq p}) &= (\eta \circ (\varphi - (\otimes t))) (\mathfrak{g}^{\otimes \leq p}) = \eta \left(\underbrace{(\varphi - (\otimes t)) (\mathfrak{g}^{\otimes \leq p})}_{\subseteq N^{\otimes \leq (p-1)}} \right) \\ &\subseteq \eta (N^{\otimes \leq (p-1)}) \subseteq \mathfrak{n}^{\otimes \leq (p-1)} \end{aligned} \quad (73)$$

(since η respects the filtration). Using this relation, Proposition 1.103 (applied to $\otimes \mathfrak{g}$, $\otimes \mathfrak{n}$, $(\mathfrak{g}^{\otimes \leq n})_{n \geq 0}$, $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$ and $\eta \circ (\varphi - (\otimes t))$ instead of V , W , $(V_n)_{n \geq 0}$, $(W_n)_{n \geq 0}$ and f) yields that $\text{gr}_p(\eta \circ \varphi - (\otimes \pi)) = 0$. Thus, $0 = \text{gr}_p(\eta \circ \varphi - (\otimes \pi)) = \text{gr}_p(\eta \circ \varphi) - \text{gr}_p(\otimes \pi)$ (by Proposition 1.99 (c)), so that $\text{gr}_p(\eta \circ \varphi) = \text{gr}_p(\otimes \pi)$.

Now, F_p was defined by $F_p = \zeta(\mathfrak{g}^{\otimes \leq p})$. Now, $\text{gr}_p \zeta : \text{gr}_p(\otimes \mathfrak{g}) \rightarrow \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}))$ is an \mathfrak{h} -module homomorphism (because ζ is an \mathfrak{h} -module homomorphism) and is surjective (according to Proposition 1.103 (applied to $\otimes \mathfrak{g}$, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$, $(\mathfrak{g}^{\otimes \leq n})_{n \geq 0}$, $(F_n)_{n \geq 0}$ and ζ instead of V , W , $(V_n)_{n \geq 0}$, $(W_n)_{n \geq 0}$ and f), because $\zeta(\mathfrak{g}^{\otimes \leq p}) = F_p$ (by the definition of F_p)).

Now, Proposition 1.99 (b) yields $\text{gr}_p((\eta \circ \bar{\varphi}) \circ \zeta) = \text{gr}_p(\eta \circ \bar{\varphi}) \circ \text{gr}_p \zeta$. Thus,

$$\text{gr}_p(\eta \circ \bar{\varphi}) \circ \text{gr}_p \zeta = \text{gr}_p((\eta \circ \bar{\varphi}) \circ \zeta) = \text{gr}_p \left(\underbrace{\eta \circ \bar{\varphi} \circ \zeta}_{=\varphi} \right) = \text{gr}_p(\eta \circ \varphi) = \text{gr}_p(\otimes \pi). \quad (74)$$

Since $\text{gr}_p(\otimes \pi)$ is an \mathfrak{h} -module homomorphism (because π is an \mathfrak{h} -module homomorphism, and thus $\otimes \pi$ is an \mathfrak{h} -module homomorphism), this yields that $\text{gr}_p(\eta \circ \bar{\varphi}) \circ \text{gr}_p \zeta$ is an \mathfrak{h} -module homomorphism. Thus, Lemma 1.121 (applied to $\text{gr}_p(\otimes \mathfrak{g})$, $\text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}))$, $\text{gr}_p(\otimes \mathfrak{n})$, $\text{gr}_p \zeta$ and $\text{gr}_p(\eta \circ \bar{\varphi})$ instead of A , B , C , f and g) yields that $\text{gr}_p(\eta \circ \bar{\varphi})$ is an \mathfrak{h} -module homomorphism. Since we know that $\text{gr}_p(\eta \circ \bar{\varphi})$ is a k -module isomorphism, we can thus conclude that $\text{gr}_p(\eta \circ \bar{\varphi})$ is an \mathfrak{h} -module isomorphism (due to Proposition 1.14). This completes the proof of Proposition 2.18 (e). Thus, Proposition 2.18 is finished.

Now we can finish off Theorem 2.1:

Proof of Theorem 2.1. (a) We have already proven Theorem 2.1 (a).

(b) We have already proven farther above that $(F_n)_{n \geq 0}$ is an \mathfrak{h} -module filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$.

Let $p \in \mathbb{N}$ be arbitrary. According to Proposition 2.18 (e), there exists an \mathfrak{h} -module isomorphism $\text{gr}_p(\eta \circ \bar{\varphi}) : \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_p(\otimes \mathfrak{n})$. Thus, $\text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \cong \text{gr}_p(\otimes \mathfrak{n})$ as \mathfrak{h} -modules. Since $\text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) = F_p / F_{p-1}$ (because the fil-

tration on $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is given by $(F_n)_{n \geq 0}$ and

$$\begin{aligned} & \text{gr}_p(\otimes \mathfrak{n}) \\ &= \mathfrak{n}^{\otimes \leq p} / \mathfrak{n}^{\otimes \leq (p-1)} \quad \left(\text{since the filtration on } \otimes \mathfrak{n} \text{ is } (\mathfrak{n}^{\otimes \leq n})_{n \geq 0} \right) \\ &= \left(\bigoplus_{i=0}^p \mathfrak{n}^{\otimes i} \right) / \left(\bigoplus_{i=0}^{p-1} \mathfrak{n}^{\otimes i} \right) \quad \left(\begin{array}{l} \text{since } \mathfrak{n}^{\otimes \leq p} = \bigoplus_{i=0}^p \mathfrak{n}^{\otimes i} \text{ by the definition of } \mathfrak{n}^{\otimes \leq p} \\ \text{and since } \mathfrak{n}^{\otimes \leq (p-1)} = \bigoplus_{i=0}^{p-1} \mathfrak{n}^{\otimes i} \text{ by the definition of } \mathfrak{n}^{\otimes \leq (p-1)} \end{array} \right) \\ &\cong \mathfrak{n}^{\otimes p}, \end{aligned}$$

this becomes $F_p / F_{p-1} \cong \mathfrak{n}^{\otimes p}$ as \mathfrak{h} -modules. Renaming p as n , we conclude that $F_n / F_{n-1} \cong \mathfrak{n}^{\otimes n}$ as \mathfrak{h} -modules for every $n \in \mathbb{N}$. This completes the proof of Theorem 2.1 (b).

(c) Every $n \in \mathbb{N}$ satisfies $\text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) = F_n / F_{n-1}$ (because the filtration on $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is given by $(F_n)_{n \geq 0}$).

Now let $n \in \mathbb{N}$ be arbitrary.

Proposition 2.18 (e) yields that $\text{gr}_p(\eta \circ \bar{\varphi})$ is an \mathfrak{h} -module isomorphism for every $p \in \mathbb{N}$. Applying this to $p = n$, we conclude that $\text{gr}_n(\eta \circ \bar{\varphi})$ is an \mathfrak{h} -module isomorphism.

During the proof of Proposition 2.18 (e), we have showed that every $p \in \mathbb{N}$ satisfies $\text{gr}_p(\eta \circ \bar{\varphi}) \circ \text{gr}_p \zeta = \text{gr}_p(\otimes \pi)$ (due to (74)). Applying this to $p = n$, we obtain $\text{gr}_n(\eta \circ \bar{\varphi}) \circ \text{gr}_n \zeta = \text{gr}_n(\otimes \pi)$. In other words,

$$\left(\text{the diagram (44) commutes if } \Omega_n = \text{gr}_n(\eta \circ \bar{\varphi}) \right). \quad (75)$$

Now, let us prove that

$$\left(\begin{array}{l} \text{if } \Omega_n : F_n / F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n}) \text{ is a } k\text{-module homomorphism} \\ \text{for which the diagram (44) commutes, then } \Omega_n = \text{gr}_n(\eta \circ \bar{\varphi}) \end{array} \right) \quad (76)$$

Proof of (76). Let $\Omega_n : F_n / F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ be a k -module homomorphism for which the diagram (44) commutes. Then, the diagram (44) commutes, so that $\Omega_n \circ \text{gr}_n \zeta = \text{gr}_n(\otimes \pi)$. Combining this with $\text{gr}_n(\eta \circ \bar{\varphi}) \circ \text{gr}_n \zeta = \text{gr}_n(\otimes \pi)$, we obtain $\Omega_n \circ \text{gr}_n \zeta = \text{gr}_n(\eta \circ \bar{\varphi}) \circ \text{gr}_n \zeta$.

So now we know that $\Omega_n \circ \text{gr}_n \zeta = \text{gr}_n(\eta \circ \bar{\varphi}) \circ \text{gr}_n \zeta$, and we want to show that $\Omega_n = \text{gr}_n(\eta \circ \bar{\varphi})$.

During the proof of Proposition 2.18 (e), we have showed that $\text{gr}_p \zeta$ is surjective for every $p \in \mathbb{N}$. Applying this to $p = n$, we obtain that $\text{gr}_n \zeta$ is surjective. Now let us show that $\Omega_n = \text{gr}_n(\eta \circ \bar{\varphi})$:

Let $v \in F_n / F_{n-1}$ be arbitrary. Since $\text{gr}_n \zeta$ is surjective, there exists some $w \in \text{gr}_n(\otimes \mathfrak{g})$ such that $v = (\text{gr}_n \zeta)(w)$. Consider this w . Then,

$$\begin{aligned} \Omega_n \left(\underbrace{v}_{=(\text{gr}_n \zeta)(w)} \right) &= \Omega_n((\text{gr}_n \zeta)(w)) = \left(\underbrace{\Omega_n \circ \text{gr}_n \zeta}_{=\text{gr}_n(\eta \circ \bar{\varphi}) \circ \text{gr}_n \zeta} \right)(w) = (\text{gr}_n(\eta \circ \bar{\varphi}) \circ \text{gr}_n \zeta)(w) \\ &= (\text{gr}_n(\eta \circ \bar{\varphi})) \left(\underbrace{(\text{gr}_n \zeta)(w)}_{=v} \right) = (\text{gr}_n(\eta \circ \bar{\varphi}))(v). \end{aligned}$$

Since this holds for every $v \in F_n/F_{n-1}$, we thus have proven that $\Omega_n = \text{gr}_n(\eta \circ \bar{\varphi})$. This proves (76).

From (76), we see that every k -module homomorphism $\Omega_n : F_n/F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ for which the diagram (44) commutes must be equal to $\text{gr}_n(\eta \circ \bar{\varphi})$. Hence, there exists at most one k -module homomorphism $\Omega_n : F_n/F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ for which the diagram (44) commutes. But since we also know that there exists at least one k -module homomorphism $\Omega_n : F_n/F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ for which the diagram (44) commutes (namely, the homomorphism $\text{gr}_n(\eta \circ \bar{\varphi})$, because of (75)), we thus conclude that there exists one and only one k -module homomorphism $\Omega_n : F_n/F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ for which the diagram (44) commutes. This proves part of Theorem 2.1 (c).

According to Theorem 2.1 (c), we define ω_n as the k -module homomorphism $\Omega_n : F_n/F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ for which the diagram (44) commutes (the existence and uniqueness of this homomorphism Ω_n was already proven above). This definition immediately yields that the diagram (45) commutes. In other words, the diagram (44) commutes if $\Omega_n = \omega_n$. Thus, (76) (applied to $\Omega_n = \omega_n$) yields that $\omega_n = \text{gr}_n(\eta \circ \bar{\varphi})$. Thus, ω_n is an \mathfrak{h} -module isomorphism (because we know that $\text{gr}_n(\eta \circ \bar{\varphi})$ is an \mathfrak{h} -module isomorphism).

Now, all nontrivial statements in Theorem 2.1 (c) are proven. This completes the proof of Theorem 2.1.

2.10. Independency of the splitting

As a bonus from the above proof of Theorem 2.1, we obtain the following strengthening of this theorem:

Proposition 2.21. In the context of Theorem 2.1 (b), for every $n \in \mathbb{N}$, there exists an \mathfrak{h} -module isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$ which is independent of the choice of N .

First proof of Proposition 2.21. According to Theorem 2.1 (c), the map $\text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n$ (where $\text{grad}_{\mathfrak{n},n}$ and ω_n are defined in Theorem 2.1 (c)) is an \mathfrak{h} -module isomorphism $F_n/F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$. This isomorphism is clearly independent of the choice of N (since the definitions of $\text{grad}_{\mathfrak{n},n}$ and ω_n are independent of the choice of N). This proves Proposition 2.21.

We can also show Proposition 2.21 without reference to Theorem 2.1 (c):

Second proof of Proposition 2.21. Let $p \in \mathbb{N}$ be arbitrary.

In our above proof of Theorem 2.1 (b), we have constructed an \mathfrak{h} -module isomorphism $\text{gr}_p(\eta \circ \bar{\varphi}) : \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_p(\otimes \mathfrak{n})$. Let us now prove that this isomorphism $\text{gr}_p(\eta \circ \bar{\varphi})$ is independent of the choice of N . (Note that this is absolutely not trivial from its definition, because $\bar{\varphi}$ does depend on N .)

In fact, let us show that

$$\left(\begin{array}{l} \text{for every } v \in \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})), \text{ the image} \\ (\text{gr}_p(\eta \circ \bar{\varphi}))(v) \text{ is independent of the choice of } N \end{array} \right). \quad (77)$$

Proof of (77). Let $v \in \text{gr}_p((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}))$ be arbitrary.

In the proof of Proposition 2.18 (e), we showed that $\text{gr}_p \zeta$ is surjective. Hence, there exists some $w \in \text{gr}_p(\otimes \mathfrak{g})$ such that $v = (\text{gr}_p \zeta)(w)$. Thus

$$(\text{gr}_p(\eta \circ \bar{\varphi}))(v) = (\text{gr}_p(\eta \circ \bar{\varphi}))((\text{gr}_p \zeta)(w)) = \underbrace{(\text{gr}_p(\eta \circ \bar{\varphi}) \circ \text{gr}_p \zeta)}_{=\text{gr}_p(\otimes \pi) \text{ (due to (74))}}(w) = (\text{gr}_p(\otimes \pi))(w).$$

Since $(\text{gr}_p(\otimes\pi))(w)$ is obviously independent of the choice of N , it thus follows that $(\text{gr}_p(\eta \circ \bar{\varphi}))(v)$ is independent of the choice of N . This proves (77).

Now, (77) yields that the map $\text{gr}_p(\eta \circ \bar{\varphi})$ is independent of the choice of N . Since $\text{gr}_p(\eta \circ \bar{\varphi})$ is an \mathfrak{h} -module isomorphism $\text{gr}_p((\otimes\mathfrak{g}) / (J + (\otimes\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_p(\otimes\mathfrak{n})$, we have thus found an \mathfrak{h} -module isomorphism $\text{gr}_p((\otimes\mathfrak{g}) / (J + (\otimes\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_p(\otimes\mathfrak{n})$ independent of the choice of N . Since $\text{gr}_p((\otimes\mathfrak{g}) / (J + (\otimes\mathfrak{g}) \cdot \mathfrak{h})) = F_p / F_{p-1}$ (because the filtration on $(\otimes\mathfrak{g}) / (J + (\otimes\mathfrak{g}) \cdot \mathfrak{h})$ is given by $(F_n)_{n \geq 0}$) and $\text{gr}_p(\otimes\mathfrak{n}) = \mathfrak{n}^{\otimes \leq p} / \mathfrak{n}^{\otimes \leq (p-1)}$ (since the filtration on $\otimes\mathfrak{n}$ is $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$), this can be rewritten as follows: We have found an \mathfrak{h} -module isomorphism $F_p / F_{p-1} \rightarrow \mathfrak{n}^{\otimes \leq p} / \mathfrak{n}^{\otimes \leq (p-1)}$ independent of the choice of N . Composing it with the canonical \mathfrak{h} -module isomorphism $\mathfrak{n}^{\otimes \leq p} / \mathfrak{n}^{\otimes \leq (p-1)} \rightarrow \mathfrak{n}^{\otimes p}$ ³¹, which is also independent of the choice of N , we thus obtain an \mathfrak{h} -module isomorphism $F_p / F_{p-1} \rightarrow \mathfrak{n}^{\otimes p}$ which is independent of the choice of N .

Thus, we have shown that for every $p \in \mathbb{N}$, there exists an \mathfrak{h} -module isomorphism $F_p / F_{p-1} \rightarrow \mathfrak{n}^{\otimes p}$ which is independent of the choice of N . Renaming p into n here, we obtain the assertion of Proposition 2.21. Thus, Proposition 2.21 is proven.

3. $(\mathfrak{g}, \mathfrak{h})$ -semimodules

Before we prove some more interesting results, we are going to introduce a notion - that of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. This notion will be defined for every commutative ring k , every k -Lie algebra \mathfrak{g} and every Lie subalgebra \mathfrak{h} of \mathfrak{g} . It will be a kind of intermediate link between the notion of a \mathfrak{g} -module and that of an \mathfrak{h} -module. Here is the definition:

3.1. $(\mathfrak{g}, \mathfrak{h})$ -semimodules: the definition

Definition 3.1. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be a k -module. Let $\mu : \mathfrak{g} \times V \rightarrow V$ be a k -bilinear map. We say that (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule if and only if

$$(\mu([a, b], v) = \mu(a, \mu(b, v)) - \mu(b, \mu(a, v)) \text{ for every } a \in \mathfrak{h}, b \in \mathfrak{g} \text{ and } v \in V). \quad (78)$$

If (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, then the k -bilinear map $\mu : \mathfrak{g} \times V \rightarrow V$ is called the *Lie action* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V .

Often, when the map μ is obvious from the context, we abbreviate the term $\mu(a, v)$ by $a \rightharpoonup v$ for any $a \in \mathfrak{g}$ and $v \in V$. Using this notation, the relation (78) rewrites as

$$([a, b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v) \text{ for every } a \in \mathfrak{h}, b \in \mathfrak{g} \text{ and } v \in V). \quad (79)$$

³¹This isomorphism is constructed as follows:

$$\begin{aligned} & \mathfrak{n}^{\otimes \leq p} / \mathfrak{n}^{\otimes \leq (p-1)} \\ &= \left(\bigoplus_{i=0}^p \mathfrak{n}^{\otimes i} \right) / \left(\bigoplus_{i=0}^{p-1} \mathfrak{n}^{\otimes i} \right) \quad \left(\begin{array}{l} \text{since } \mathfrak{n}^{\otimes \leq p} = \bigoplus_{i=0}^p \mathfrak{n}^{\otimes i} \text{ by the definition of } \mathfrak{n}^{\otimes \leq p} \\ \text{and since } \mathfrak{n}^{\otimes \leq (p-1)} = \bigoplus_{i=0}^{p-1} \mathfrak{n}^{\otimes i} \text{ by the definition of } \mathfrak{n}^{\otimes \leq (p-1)} \end{array} \right) \\ &\cong \mathfrak{n}^{\otimes p}. \end{aligned}$$

Also, an abuse of notation allows us to write " V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule" instead of " (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule" if the map μ is clear from the context or has not been introduced yet.

Besides, when (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, we will say that μ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on V . In other words, if V is a k -module, then a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on V means a map $\mu : \mathfrak{g} \times V \rightarrow V$ such that (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. (Thus, in order to make a k -module into a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, we must define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on it.)

This definition is very similar to the Definition 1.9. We will see that this similarity is not just superficial, and that most properties of \mathfrak{g} -modules have their analogues concerning $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

But first let us notice that:

Proposition 3.2. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Then, every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

In fact, this proposition follows trivially from comparing Definition 1.9 with Definition 3.1. The converse of this proposition does not hold. However, a \mathfrak{g} -module is exactly the same as a $(\mathfrak{g}, \mathfrak{g})$ -semimodule, i. e., we have:

Proposition 3.3. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a k -module. Let $\mu : \mathfrak{g} \times V \rightarrow V$ be a map. Then, (V, μ) is a \mathfrak{g} -module if and only if (V, μ) is a $(\mathfrak{g}, \mathfrak{g})$ -semimodule.

This is again clear from comparing Definition 1.9 with Definition 3.1.

Proposition 3.3 shows that the notion of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule is a generalization of the notion of a \mathfrak{g} -module. Much of this Section 3 will be devoted to formulating some properties of $(\mathfrak{g}, \mathfrak{h})$ -semimodules which are analogous to the well-known properties of \mathfrak{g} -modules which we collected in Section 1. We are *not* going to prove all of these properties anew, because the proofs will often be almost identical to the corresponding proofs for \mathfrak{g} -modules done in Section 1. We will only point out which changes must be made to those proofs in order to make them apply to $(\mathfrak{g}, \mathfrak{h})$ -semimodules rather than just to \mathfrak{g} -modules.

In this Section 3, we are also going to define several notions related to $(\mathfrak{g}, \mathfrak{h})$ -semimodules, such as the notion of a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule, and the notion of the tensor product of two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. **All the definitions that we are going to give will be analogous to the definitions of the corresponding notions for \mathfrak{g} -modules given in Section 1, and therefore will not conflict with these latter notions.** For example, Definition 3.25 (the definition of the tensor product of two $(\mathfrak{g}, \mathfrak{h})$ -semimodules) will be analogous to Definition 1.31 (the definition of the tensor product of two \mathfrak{g} -modules). Therefore, if we have a commutative ring k , some k -Lie algebra \mathfrak{g} , some Lie subalgebra \mathfrak{h} of \mathfrak{g} , and two \mathfrak{g} -modules V and W , then the tensor product of the \mathfrak{g} -modules V and W (as defined in Definition 1.31) will be the same as the tensor product of the $(\mathfrak{g}, \mathfrak{h})$ -semimodules V and W (as defined in Definition 3.25). Similarly, the direct sum of the \mathfrak{g} -modules V and W (as defined in Definition 1.22) will be the same as the direct sum of the $(\mathfrak{g}, \mathfrak{h})$ -semimodules V and W (as defined in Definition 3.17).

We begin with a convention:

Convention 3.4. We are going to use the notation $a \rightarrow v$ as a universal notation for the Lie action of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. This means that whenever we have some Lie algebra \mathfrak{g} , some Lie subalgebra \mathfrak{h} of \mathfrak{g} and some $(\mathfrak{g}, \mathfrak{h})$ -semimodule V (they need not be actually called \mathfrak{g} , \mathfrak{h} and V ; I only refer to them as \mathfrak{g} , \mathfrak{h} and V here in this Convention), and we are given two elements $a \in \mathfrak{g}$ and $v \in V$ (they need not be actually called a and v ; I only refer to them by a and v here in this Convention), we will denote by $a \rightarrow v$ the Lie action of V applied to (a, v) (unless we explicitly stated that the notation $a \rightarrow v$ means something different).

This convention is, of course, just the extension of Convention 1.10 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

Warning 3.5. We know from Proposition 3.2 that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Thus, when k is a commutative ring, \mathfrak{g} is a k -Lie algebra, \mathfrak{h} is a Lie subalgebra of \mathfrak{g} , and V is some \mathfrak{g} -module, then V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule as well, and thus, the notation $a \rightarrow v$ (where $a \in \mathfrak{g}$ and $v \in V$) is overloaded: It can be interpreted according to Convention 1.10, but it can also be interpreted according to Convention 3.4. However, fortunately these two conventions give the same definition for $a \rightarrow v$, and thus they do not conflict. So when we have a \mathfrak{g} -module V , then we do not have to worry about Convention 1.10 and Convention 3.4 leading to different interpretations $a \rightarrow v$; they don't.

However, Convention 3.4 can conflict with Convention 1.10 in two other cases: The first case is when we have a \mathfrak{g} -module structure and a **different** $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure defined on one and the same k -module; the second case is when we have an \mathfrak{h} -module structure and a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure defined on one and the same k -module. The first of these cases will not appear in our studies; however, the second will appear. In such a case, we will *not* be allowed to use Convention 3.4 until we verify that, for every $a \in \mathfrak{h}$ and $v \in V$, the meaning of $a \rightarrow v$ according to Convention 1.10 (that is, the Lie action of the \mathfrak{h} -module V applied to (a, v)) equals to the meaning of $a \rightarrow v$ according to Convention 3.4 (that is, the Lie action of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V applied to (a, v)), so that the value of $a \rightarrow v$ does not depend on which of the two Conventions we are using.

Fortunately, within this paper, this will always be fulfilled and easy to verify. In fact, within this paper, each time when we have an \mathfrak{h} -module structure and a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure defined on one and the same k -module, the \mathfrak{h} -module will be the restriction of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h} (see Definition 3.11 for the definition of "restriction"), and thus we *will* be allowed to use Convention 3.4 (according to Remark 3.13).

Now that we have defined a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, let us do the next logical step and define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism:

Definition 3.6. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Let $f : V \rightarrow W$ be a k -linear map. Then, f is said to be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism if and only if

$$(f(a \rightarrow v) = a \rightarrow (f(v)) \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V).$$

Often, we will use the words " $(\mathfrak{g}, \mathfrak{h})$ -semimodule map" or the words "homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules" or the words " $(\mathfrak{g}, \mathfrak{h})$ -semilinear map" as synonyms for " $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism".

This Definition 3.6 is the analogue of Definition 1.12 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Therefore, we have:

Proposition 3.7. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two \mathfrak{g} -modules. Let $f : V \rightarrow W$ be a map. Then, f is a \mathfrak{g} -module homomorphism if and only if f is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism. (Here, it makes sense to say that " f is a \mathfrak{g} -module homomorphism" since V and W are $(\mathfrak{g}, \mathfrak{h})$ -semimodules (which is because Proposition 3.2 yields that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule).)

It is easy to see that for every commutative ring k , for every k -Lie algebra \mathfrak{g} , and for every Lie subalgebra \mathfrak{h} of \mathfrak{g} , there is a category whose objects are $(\mathfrak{g}, \mathfrak{h})$ -semimodules and whose morphisms are $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphisms. We further define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism as an isomorphism in this category; this is equivalent to the following definition:

Definition 3.8. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Let $f : V \rightarrow W$ be a k -linear map. Then, f is said to be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism if and only if f is an invertible $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism whose inverse f^{-1} is also a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism.

We can easily prove that this definition is somewhat redundant, viz., the condition that f^{-1} be also a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism can be omitted:

Proposition 3.9. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Let $f : V \rightarrow W$ be a k -linear map. Then, f is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism if and only if f is an invertible $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism. In other words, f is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism if and only if f is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism and a k -module isomorphism at the same time.

The proof of this proposition is identical with the proof of Proposition 1.14.

Clearly, Definition 3.8 is the analogue of Definition 1.13 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Thus:

Proposition 3.10. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two \mathfrak{g} -modules. Let $f : V \rightarrow W$ be a map. Then, f is a \mathfrak{g} -module isomorphism if and only if f is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism. (Here, it makes sense to say that " f is a \mathfrak{g} -module isomorphism" since V and W are $(\mathfrak{g}, \mathfrak{h})$ -semimodules (which is because Proposition 3.2 yields that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule).)

Also, Proposition 3.9 is the analogue of Proposition 1.14 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

3.2. Restriction of $(\mathfrak{g}, \mathfrak{h})$ -semimodules

If \mathfrak{h} is a Lie subalgebra of a k -Lie algebra \mathfrak{g} , then we can canonically make every $(\mathfrak{g}, \mathfrak{h})$ -semimodule into an \mathfrak{h} -module according to the following definition:

Definition 3.11. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Then, every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V canonically becomes an \mathfrak{h} -module (by restricting its Lie action $\mu : \mathfrak{g} \times V \rightarrow V$ to $\mathfrak{h} \times V$). This \mathfrak{h} -module is called the *restriction of V to \mathfrak{h}* , and denoted by $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V$. However, when there is no possibility of confusion, we will denote this \mathfrak{h} -module by V , and we will distinguish it from the original $(\mathfrak{g}, \mathfrak{h})$ -semimodule V by means of referring to the former one as "the \mathfrak{h} -module V " and referring to the latter one as "the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V ".

This Definition 3.11 is the analogue of Definition 1.15 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Therefore:

Proposition 3.12. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be a \mathfrak{g} -module. Then, the restriction of V to \mathfrak{h} defined in Definition 1.15 is the same \mathfrak{h} -module as the restriction of V to \mathfrak{h} defined in Definition 3.11 (which is well-defined since V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the restriction of V to \mathfrak{h} " (or simply of "the \mathfrak{h} -module V ") without having to worry whether it is understood according to Definition 1.15 or according to Definition 3.11 (because it doesn't matter, as both definitions yield the same result).

Remark 3.13. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Then, Definition 3.11 makes V into an \mathfrak{h} -module. Hence, V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule and an \mathfrak{h} -module at the same time. As we know from Warning 3.5, we are normally *not* allowed to use Convention 3.4 when we have a k -module V which is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule and an \mathfrak{h} -module at the same time, because in this case each of the two Conventions 3.4 and 1.10 defines $a \rightarrow v$ for $a \in \mathfrak{h}$ and $v \in V$, and these definitions might conflict. However, in *our* case (the case when the \mathfrak{h} -module V is obtained from the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V according to Definition 3.11), these definitions cannot conflict, because every $a \in \mathfrak{h}$ and $v \in V$ satisfy

$$\begin{aligned} & \text{(the meaning of the term } a \rightarrow v \text{ according to Convention 3.4)} \\ & = \text{(the meaning of the term } a \rightarrow v \text{ according to Convention 1.10)} \end{aligned}$$

³², so that both Conventions 3.4 and 1.10 define $a \rightarrow v$ to mean one and the same value. Therefore, we can use Convention 3.4 in our case (the case when the \mathfrak{h} -module V is obtained from the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V according to Definition 3.11) without worrying that it might conflict with Convention 1.10.

In Definition 3.11, we have defined the restriction of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule to an \mathfrak{h} -module. We could also define a more general kind of restriction, namely restriction

³²*Proof.* Let us denote by μ the Lie action of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V . Then, the Lie action of the \mathfrak{h} -module V is defined to be $\mu|_{\mathfrak{h} \times V}$ (in fact, this is how the \mathfrak{h} -module V was defined in Definition

of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule to a $(\mathfrak{g}', \mathfrak{h}')$ -semimodule (where \mathfrak{g}' is a Lie subalgebra of \mathfrak{g} and where \mathfrak{h}' is a Lie subalgebra of \mathfrak{h}), but we will not find need for this form of restriction in the following and therefore overlook its (trivial) definition.

3.3. Subsemimodules, factors and direct sums of $(\mathfrak{g}, \mathfrak{h})$ -semimodules

From Proposition 3.2, we know that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (where \mathfrak{g} is a k -Lie algebra, and \mathfrak{h} is a Lie subalgebra of \mathfrak{g}). This gives us plenty of examples of $(\mathfrak{g}, \mathfrak{h})$ -semimodules: For instance, we know (from Definition 1.17) that \mathfrak{g} itself canonically is a \mathfrak{g} -module, and thus we conclude that \mathfrak{g} is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Also we know (from Definition 1.19) that k canonically is a \mathfrak{g} -module, and thus we conclude that k is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

To obtain more $(\mathfrak{g}, \mathfrak{h})$ -semimodule structures, we can factor existing $(\mathfrak{g}, \mathfrak{h})$ -semimodules by submodules:

Definition 3.14. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

(a) A k -submodule W of V is said to be a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of V if and only if

$$(a \rightarrow w \in W \text{ for every } a \in \mathfrak{g} \text{ and } w \in W).$$

In other words, a k -submodule W of V is said to be a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of V if and only if $\mu(\mathfrak{g} \times W) \subseteq W$, where μ denotes the Lie action of V . (We remind ourselves that the Lie action of V means the k -bilinear map $\mu : \mathfrak{g} \times V \rightarrow V$ from Definition 3.1.)

(b) If W is a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of V , then the quotient k -module V/W becomes a $(\mathfrak{g}, \mathfrak{h})$ -semimodule by setting

$$(a \rightarrow \bar{v} = \overline{a \rightarrow v} \text{ for every } a \in \mathfrak{g} \text{ and } v \in V)$$

(where \bar{u} denotes the residue class of u modulo W for every $u \in V$). (This $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure is indeed well-defined, as can be easily seen.)

This Definition 3.14 is the analogue of Definition 1.20 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. This immediately yields:

3.11). Now,

$$\begin{aligned} & \text{(the meaning of the term } a \rightarrow v \text{ according to Convention 3.4)} \\ &= \text{(the Lie action of the } (\mathfrak{g}, \mathfrak{h})\text{-semimodule } V \text{ applied to } (a, v)) \quad \text{(according to Convention 3.4)} \\ &= \mu(a, v) \quad \text{(since the Lie action of the } (\mathfrak{g}, \mathfrak{h})\text{-semimodule } V \text{ is } \mu) \\ &= (\mu|_{\mathfrak{h} \times V})(a, v) \quad \text{(since } a \in \mathfrak{h} \text{ and } v \in V, \text{ so that } (a, v) \in \mathfrak{h} \times V) \\ &= \text{(the Lie action of the } \mathfrak{h}\text{-module } V \text{ applied to } (a, v)) \\ & \quad \text{(because } \mu|_{\mathfrak{h} \times V} \text{ is the Lie action of the } \mathfrak{h}\text{-module } V) \\ &= \text{(the meaning of the term } a \rightarrow v \text{ according to Convention 1.10),} \end{aligned}$$

qed.

Proposition 3.15. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

(a) Let V be a \mathfrak{g} -module. Let W be a subset of V . Then, W is a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of V if and only if W is a \mathfrak{g} -submodule of V . (Here, the words " W is a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule" make sense because V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (which is because Proposition 3.2 yields that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule).)

(b) Let V be a \mathfrak{g} -module, and let W be a \mathfrak{g} -submodule of V . Then, the quotient $(\mathfrak{g}, \mathfrak{h})$ -semimodule V/W defined in Definition 3.14 (b) is identical with the quotient \mathfrak{g} -module V/W defined in Definition 1.20 (b). This allows us to speak of the \mathfrak{g} -module V/W without having to worry whether it is understood according to Definition 3.14 or according to Definition 1.20 (because it doesn't matter, as both definitions yield the same result).

We can also add $(\mathfrak{g}, \mathfrak{h})$ -semimodules via the direct sum:

Proposition 3.16. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Define a map $\mu_{V \oplus W} : \mathfrak{g} \times (V \oplus W) \rightarrow V \oplus W$ by

$$(\mu_{V \oplus W}(a, (v, w))) = (a \rightarrow v, a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W). \quad (80)$$

Then, this map $\mu_{V \oplus W}$ is k -bilinear, and $(V \oplus W, \mu_{V \oplus W})$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule satisfying

$$a \rightarrow (v, w) = (a \rightarrow v, a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W. \quad (81)$$

This proposition is straightforward to prove, so we are not going to elaborate on its proof. Anyway it allows a definition:

Definition 3.17. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

The $(\mathfrak{g}, \mathfrak{h})$ -semimodule $(V \oplus W, \mu_{V \oplus W})$ constructed in Proposition 3.16 is called the *direct sum* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodules V and W . We are going to denote this $(\mathfrak{g}, \mathfrak{h})$ -semimodule $(V \oplus W, \mu_{V \oplus W})$ simply by $V \oplus W$.

Note that Proposition 3.16 is the analogue of Proposition 1.21 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Also, Definition 3.17 is the analogue of Definition 1.22 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. This yields:

Proposition 3.18. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two \mathfrak{g} -modules. Then, the \mathfrak{g} -module $V \oplus W$ defined in Definition 1.22 is identical with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V \oplus W$ defined in Definition 3.17 (which is well-defined since V and W are $(\mathfrak{g}, \mathfrak{h})$ -semimodules (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the \mathfrak{g} -module $V \oplus W$ " without having to worry whether it is understood according to Definition 1.22 or according to Definition 3.17 (because it doesn't matter, as both definitions yield the same result).

We can similarly define the direct sum of several (not necessarily just two) $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.19. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let S be a set. For every $s \in S$, let V_s be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

Define a map $\mu_{\oplus} : \mathfrak{g} \times \left(\bigoplus_{s \in S} V_s \right) \rightarrow \bigoplus_{s \in S} V_s$ by

$$\left(\mu_{\oplus} (a, (v_s)_{s \in S}) = (a \rightharpoonup v_s)_{s \in S} \quad \text{for every } a \in \mathfrak{g} \text{ and every family } (v_s)_{s \in S} \in \bigoplus_{s \in S} V_s \right). \quad (82)$$

Then, this map μ_{\oplus} is k -bilinear, and $\left(\bigoplus_{s \in S} V_s, \mu_{\oplus} \right)$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule satisfying

$$a \rightharpoonup (v_s)_{s \in S} = (a \rightharpoonup v_s)_{s \in S} \quad \text{for every } a \in \mathfrak{g} \text{ and every family } (v_s)_{s \in S} \in \bigoplus_{s \in S} V_s. \quad (83)$$

Definition 3.20. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let S be a set. For every $s \in S$, let V_s be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

The $(\mathfrak{g}, \mathfrak{h})$ -semimodule $\left(\bigoplus_{s \in S} V_s, \mu_{\oplus} \right)$ constructed in Proposition 3.19 is called the *direct sum* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_s over all $s \in S$. We are going to denote this

$(\mathfrak{g}, \mathfrak{h})$ -semimodule $\left(\bigoplus_{s \in S} V_s, \mu_{\oplus} \right)$ simply by $\bigoplus_{s \in S} V_s$.

Again, there is nothing substantial to prove here. Notice that if $S = \emptyset$, then $\bigoplus_{s \in S} V_s$ is to be understood as 0.

Note that Proposition 3.19 is the analogue of Proposition 1.23 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules, and that Definition 3.20 is the analogue of Definition 1.24 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. This yields:

Proposition 3.21. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let S be a set. For every $s \in S$, let V_s be a \mathfrak{g} -module.

Then, the \mathfrak{g} -module $\bigoplus_{s \in S} V_s$ defined in Definition 1.24 is identical with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $\bigoplus_{s \in S} V_s$ defined in Definition 3.20 (which is well-defined since V_s is a

$(\mathfrak{g}, \mathfrak{h})$ -semimodule for every $s \in S$ (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the \mathfrak{g} -module $\bigoplus_{s \in S} V_s$ "

without having to worry whether it is understood according to Definition 1.24 or according to Definition 3.20 (because it doesn't matter, as both definitions yield the same result).

Proposition 1.26 also has an analogue for $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.22. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let S be a set. For every $s \in S$, let V_s be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. In Convention 1.25, we have identified the k -module V_t with the image of V_t under the canonical injection $V_t \rightarrow \bigoplus_{s \in S} V_s$ for every $t \in S$. Thus, by means of this identification, V_t becomes a k -submodule of the direct sum $\bigoplus_{s \in S} V_s$. But actually, something stronger holds: By means of this identification, V_t becomes a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of the direct sum $\bigoplus_{s \in S} V_s$.

There is also a natural generalization of Proposition 1.27 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.23. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

(a) If V and W are two $(\mathfrak{g}, \mathfrak{h})$ -semimodules, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \oplus W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \oplus (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V \oplus W$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \oplus W)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \oplus (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \oplus W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \oplus (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$).

(b) If S is a set, and if V_t is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule for every $t \in S$, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}\left(\bigoplus_{s \in S} V_s\right) = \bigoplus_{s \in S} (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_s)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $\bigoplus_{s \in S} V_s$ " without

having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}\left(\bigoplus_{s \in S} V_s\right)$ or $\bigoplus_{s \in S} (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_s)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}\left(\bigoplus_{s \in S} V_s\right) = \bigoplus_{s \in S} (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_s)$).

(c) The \mathfrak{h} -module k is identical with the restriction $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} k$ of the \mathfrak{g} -module k to \mathfrak{h} .

(d) If V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, and if W is a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of V , then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W$ is an \mathfrak{h} -submodule of the \mathfrak{h} -module $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V$ and satisfies $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V/W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) / (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module V/W " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V/W)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) / (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V/W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) / (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$).

Just as Proposition 1.27, this generalization will often be tacitly used.

3.4. Tensor products of two $(\mathfrak{g}, \mathfrak{h})$ -semimodules

We can define tensor products of $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.24. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Then, there exists one and only one k -bilinear map $m : \mathfrak{g} \times (V \otimes W) \rightarrow V \otimes W$ which satisfies

$$(m(a, v \otimes w) = (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W). \quad (84)$$

If we denote this map m by $\mu_{V \otimes W}$, then $(V \otimes W, \mu_{V \otimes W})$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. This $(\mathfrak{g}, \mathfrak{h})$ -semimodule satisfies

$$a \rightharpoonup (v \otimes w) = (a \rightharpoonup v) \otimes w + v \otimes (a \rightharpoonup w) \quad \text{for every } a \in \mathfrak{g}, v \in V \text{ and } w \in W. \quad (85)$$

Definition 3.25. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

The $(\mathfrak{g}, \mathfrak{h})$ -semimodule $(V \otimes W, \mu_{V \otimes W})$ constructed in Proposition 3.24 is called the *tensor product* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodules V and W . We are going to denote this $(\mathfrak{g}, \mathfrak{h})$ -semimodule $(V \otimes W, \mu_{V \otimes W})$ simply by $V \otimes W$.

Obviously, Proposition 3.24 is a generalization of Proposition 1.30 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules, and Definition 3.25 is a generalization of Definition 1.31 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules. This yields:

Proposition 3.26. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two \mathfrak{g} -modules. Then, the \mathfrak{g} -module $V \otimes W$ defined in Definition 1.31 is identical with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V \otimes W$ defined in Definition 3.25 (which is well-defined since V and W are $(\mathfrak{g}, \mathfrak{h})$ -semimodules (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the \mathfrak{g} -module $V \otimes W$ " without having to worry whether it is understood according to Definition 1.31 or according to Definition 3.25 (because it doesn't matter, as both definitions yield the same result).

Proof of Proposition 3.24. In order to obtain a proof of Proposition 3.24, it is enough to apply the following changes to the proof of Proposition 1.30:

- Replace "Proposition 1.30" by "Proposition 3.24".
- Replace "Definition 1.9" by "Definition 3.1".
- Replace the words " \mathfrak{g} -module" by " $(\mathfrak{g}, \mathfrak{h})$ -semimodule".
- Replace the references to (14) by references to (84).
- Replace the references to (15) by references to (85).
- In the proof of Assertion γ , replace " $a \in \mathfrak{g}$ " by " $a \in \mathfrak{h}$ " (except in the formula (19)).
- Replace the references to (8) by references to (79).

This completes the proof of Proposition 3.24.

The next proposition is a generalization of Proposition 1.34 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.27. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

(a) Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Then, the k -linear map

$$V \rightarrow k \otimes V, \quad v \mapsto 1 \otimes v$$

is a canonical isomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules. (Here, as usual, k denotes the \mathfrak{g} -module k defined in Definition 1.19.)

(b) Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Then, the k -linear map

$$V \rightarrow V \otimes k, \quad v \mapsto v \otimes 1$$

is a canonical isomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules. (Here, as usual, k denotes the \mathfrak{g} -module k defined in Definition 1.19.)

(c) Let U, V and W be $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Then, the k -linear map

$$(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

is a canonical isomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

Proof of Proposition 3.27. In order to obtain a proof of Proposition 3.27, it is enough to apply the following changes to the proof of Proposition 1.34:

- Replace "Proposition 1.34" by "Proposition 3.27".
- Replace the words " \mathfrak{g} -module" by " $(\mathfrak{g}, \mathfrak{h})$ -semimodule".
- Replace the references to (15) by references to (85).

This completes the proof of Proposition 3.27.

We also have:

Corollary 3.28. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Then, the canonical k -module isomorphism $k \otimes V \rightarrow V$ (this is the k -module homomorphism that sends $\lambda \otimes v$ to λv for all $\lambda \in k$ and $v \in V$) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism.

This Corollary 3.28 is the generalization of Corollary 1.36 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

Proof of Corollary 3.28. In order to obtain a proof of Corollary 3.28, it is enough to apply the following changes to the proof of Corollary 1.36:

- Replace "Proposition 1.34" by "Proposition 3.27".
- Replace "Corollary 1.36" by "Corollary 3.28".
- Replace the words " \mathfrak{g} -module" by " $(\mathfrak{g}, \mathfrak{h})$ -semimodule".

This completes the proof of Corollary 3.28.

Next we formulate a generalization of Proposition 1.38 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.29. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let V, W, V' and W' be four $(\mathfrak{g}, \mathfrak{h})$ -semimodules, and let $f : V \rightarrow V'$ and $g : W \rightarrow W'$ be two $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphisms. Then, $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism.

Proof of Proposition 3.29. In order to obtain a proof of Proposition 3.29, it is enough to apply the following changes to the proof of Proposition 1.38:

- Replace "Proposition 1.38" by "Proposition 3.29".
- Replace the references to (15) by references to (85).
- Replace the words " \mathfrak{g} -module" by " $(\mathfrak{g}, \mathfrak{h})$ -semimodule".

This completes the proof of Proposition 3.29.

We can also generalize Proposition 1.39 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules, obtaining the following proposition:

Proposition 3.30. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

If V and W are two $(\mathfrak{g}, \mathfrak{h})$ -semimodules, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \otimes W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V \otimes W$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \otimes W)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V \otimes W) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} W)$).

This follows from the definitions.

3.5. Tensor products of several $(\mathfrak{g}, \mathfrak{h})$ -semimodules

We now define the tensor product of several $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Definition 3.31. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$.

Now, by induction over n , we are going to define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_n :

Induction base: For $n = 0$, we define $V_1 \otimes V_2 \otimes \dots \otimes V_n$ as the \mathfrak{g} -module k defined in Definition 1.19.

Induction step: Let $p \in \mathbb{N}$. Assuming that we have defined a $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_p$ for any p arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_p , we now define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_{p+1}$ for any $p + 1$ arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_{p+1} by the equation

$$V_1 \otimes V_2 \otimes \dots \otimes V_{p+1} = V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}). \quad (86)$$

Here, $V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_{p+1})$ is to be understood as the tensor product of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V_1 with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}$ (note that the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_2 \otimes V_3 \otimes \dots \otimes V_{p+1}$ is already defined because we assumed that we have defined a $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_p$ for any p arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_p). This completes the inductive definition.

Thus we have defined a $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_n for any $n \in \mathbb{N}$. This $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ is called the *tensor product* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_n .

This Definition 3.31 is the obvious generalization of Definition 1.42 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Therefore:

Proposition 3.32. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. Let V_1, V_2, \dots, V_n be n arbitrary \mathfrak{g} -modules. Then, the \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 1.42 is identical with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 3.31 (which is well-defined since V_1, V_2, \dots, V_n are $(\mathfrak{g}, \mathfrak{h})$ -semimodules (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the \mathfrak{g} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ " without having to worry whether it is understood according to Definition 1.42 or according to Definition 3.31 (because it doesn't matter, as both definitions yield the same result).

Remark 3.33. (a) In Definition 3.31, we could have replaced the equation (86) by

$$V_1 \otimes V_2 \otimes \dots \otimes V_{p+1} = (V_1 \otimes V_2 \otimes \dots \otimes V_p) \otimes V_{p+1}.$$

This would have given us a *different* $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ for any n arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_n for any $n \in \mathbb{N}$ than the one defined in Definition 3.31. However, this $(\mathfrak{g}, \mathfrak{h})$ -semimodule would still be *canonically isomorphic* to the one defined in Definition 3.31 (we will prove this and actually something more general in Proposition 3.34), and thus it is commonly considered to be "more or less the same $(\mathfrak{g}, \mathfrak{h})$ -semimodule".

(b) Definition 3.31, applied to $n = 1$, defines the tensor product of *one* $(\mathfrak{g}, \mathfrak{h})$ -semimodule V_1 as $V_1 \otimes k$. This takes some getting used to, since it seems more natural to define the tensor product of one $(\mathfrak{g}, \mathfrak{h})$ -semimodule V_1 simply as V_1 . But this isn't really different because Proposition 3.27 (b) gives a canonical isomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules $V_1 \cong V_1 \otimes k$, so most people consider V_1 to be "more or less the same $(\mathfrak{g}, \mathfrak{h})$ -semimodule" as $V_1 \otimes k$.

(c) Definition 3.31 does not conflict with Definition 1.40, because the underlying k -module of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 3.31 is indeed the k -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 1.40. (This is trivial by induction.)

Now we formulate the generalization of Proposition 1.45 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.34. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$.

Then, for any n arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules V_1, V_2, \dots, V_n and every $i \in \{0, 1, \dots, n\}$, the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_n) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_n$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism.³³

Proof of Proposition 3.34. In order to obtain a proof of Proposition 3.34, it is enough to apply the following changes to the proof of Proposition 1.45:

³³Here and in the following, whenever I speak of "the canonical k -module isomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_n) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_n$ ", I mean the k -module homomorphism $(V_1 \otimes V_2 \otimes \dots \otimes V_i) \otimes (V_{i+1} \otimes V_{i+2} \otimes \dots \otimes V_n) \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_n$ which sends $(v_1 \otimes v_2 \otimes \dots \otimes v_i) \otimes (v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n)$ to $v_1 \otimes v_2 \otimes \dots \otimes v_n$ for every $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$. This homomorphism is known (from linear algebra) to exist, be unique and be a k -module isomorphism. (This is independent of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule structures on V_1, V_2, \dots, V_n .)

- Replace "Proposition 1.45" by "Proposition 3.34".
- Replace "Proposition 1.34" by "Proposition 3.27".
- Replace "Proposition 1.38" by "Proposition 3.29".
- Replace "Proposition 1.14" by "Proposition 3.9".
- Replace the words "g-module" by "(g, h)-semimodule".

This completes the proof of Proposition 3.34.

In Definition 3.31, we defined the (g, h)-semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ by induction over n . It turns out that we can also easily describe the (g, h)-semimodule structure on $V_1 \otimes V_2 \otimes \dots \otimes V_n$ explicitly:

Proposition 3.35. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. Let V_1, V_2, \dots, V_n be n arbitrary (g, h)-semimodules. Then, the (g, h)-semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 3.31 satisfies

$$a \rightarrow (v_1 \otimes v_2 \otimes \dots \otimes v_n) = \sum_{i=1}^n v_1 \otimes v_2 \otimes \dots \otimes \boxed{\begin{array}{c} a \rightarrow v_i \\ v_i \end{array}} \otimes \dots \otimes v_n$$

for every $a \in \mathfrak{g}$ and every $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$. Here, we are using Convention 1.46.

This Proposition 3.35 is the straightforward generalization of Proposition 1.47 to (g, h)-semimodules.

We are going to prove this later. First an obvious lemma, which generalizes Proposition 1.48 to (g, h)-semimodules:

Proposition 3.36. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}_+$. Let V_1, V_2, \dots, V_n be n arbitrary (g, h)-semimodules. Then, the (g, h)-semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ defined in Definition 3.31 satisfies

$$a \rightarrow (v \otimes T) = (a \rightarrow v) \otimes T + v \otimes (a \rightarrow T) \quad (87)$$

for every $a \in \mathfrak{g}$, $v \in V_1$ and $T \in V_2 \otimes V_3 \otimes \dots \otimes V_n$.

Proof of Proposition 3.36. According to Definition 3.31, the (g, h)-semimodule $V_1 \otimes V_2 \otimes \dots \otimes V_n$ is defined as the tensor product $V_1 \otimes (V_2 \otimes V_3 \otimes \dots \otimes V_n)$. Hence, applying (85) to $V = V_1$ and $W = V_2 \otimes V_3 \otimes \dots \otimes V_n$, we see that

$$a \rightarrow (v \otimes w) = (a \rightarrow v) \otimes w + v \otimes (a \rightarrow w)$$

for every $a \in \mathfrak{g}$, $v \in V_1$ and $w \in V_2 \otimes V_3 \otimes \dots \otimes V_n$

(because the tensor product $V \otimes W$ of two (g, h)-semimodules V and W always satisfies (85)). If we rename w as T , this rewrites as follows:

$$a \rightarrow (v \otimes T) = (a \rightarrow v) \otimes T + v \otimes (a \rightarrow T)$$

for every $a \in \mathfrak{g}$, $v \in V_1$ and $T \in V_2 \otimes V_3 \otimes \dots \otimes V_n$.

This proves Proposition 3.36.

Proof of Proposition 3.35. In order to obtain a proof of Proposition 3.35, it is enough to apply the following changes to the proof of Proposition 1.47:

- Replace "Proposition 1.47" by "Proposition 3.35".
- Replace "Proposition 1.48" by "Proposition 3.36".
- Replace the words " \mathfrak{g} -module" by " $(\mathfrak{g}, \mathfrak{h})$ -semimodule".

This completes the proof of Proposition 3.35.

The next proposition is a multi-factor version of Proposition 3.29, and generalizes Proposition 1.49 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.37. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let $n \in \mathbb{N}$. Let V_1, V_2, \dots, V_n be n arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Let V'_1, V'_2, \dots, V'_n be n arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Let $f_i : V_i \rightarrow V'_i$ be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism for every $i \in \{1, 2, \dots, n\}$. Then, $f_1 \otimes f_2 \otimes \dots \otimes f_n : V_1 \otimes V_2 \otimes \dots \otimes V_n \rightarrow V'_1 \otimes V'_2 \otimes \dots \otimes V'_n$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism.

The next proposition is a multi-factor version of Proposition 3.30, and generalizes Proposition 1.50 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.38. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$.

If V_1, V_2, \dots, V_n are n arbitrary $(\mathfrak{g}, \mathfrak{h})$ -semimodules, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2 \otimes \dots \otimes V_n) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_1) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_2) \otimes \dots \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_n)$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V_1 \otimes V_2 \otimes \dots \otimes V_n$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2 \otimes \dots \otimes V_n)$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_1) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_2) \otimes \dots \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_n)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2 \otimes \dots \otimes V_n) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_1) \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_2) \otimes \dots \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V_n)$).

3.6. Tensor powers of $(\mathfrak{g}, \mathfrak{h})$ -semimodules

Next we define a particular case of tensor products of $(\mathfrak{g}, \mathfrak{h})$ -semimodules, namely the tensor powers. Their definition is analogous to the definition of the tensor powers of a k -module (Definition 1.51) and to the definition of the tensor powers of a \mathfrak{g} -module (Definition 1.53), but proceeds from a $(\mathfrak{g}, \mathfrak{h})$ -semimodule:

Definition 3.39. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. For any $(\mathfrak{g}, \mathfrak{h})$ -semimodule V , we define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V^{\otimes n}$ by $V^{\otimes n} = \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}$. This $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V^{\otimes n}$ is

called the n -th tensor power of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V .

Since this Definition 3.39 is the analogue of Definition 1.53 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules, we have:

Proposition 3.40. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. Let V be any \mathfrak{g} -module. Then, the \mathfrak{g} -module $V^{\otimes n}$ defined in Definition 1.53 is identic with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V^{\otimes n}$ defined in Definition 3.39 (which is well-defined since V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the \mathfrak{g} -module $V^{\otimes n}$ " without having to worry whether it is understood according to Definition 1.53 or according to Definition 3.39 (because it doesn't matter, as both definitions yield the same result).

Remark 3.41. Let k be a commutative ring, let \mathfrak{g} be a k -Lie algebra, let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} , and let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Then, $V^{\otimes 0} = k$ (as $(\mathfrak{g}, \mathfrak{h})$ -semimodules) and $V^{\otimes 1} = V$ (as $(\mathfrak{g}, \mathfrak{h})$ -semimodules), where we identify the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V \otimes k$ with V . This is proven the same way as Remark 1.52.

As a consequence of Proposition 3.37, we now have:

Proposition 3.42. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. Let V and V' be $(\mathfrak{g}, \mathfrak{h})$ -semimodules, and let $f : V \rightarrow V'$ be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism. Then, $f^{\otimes n} : V^{\otimes n} \rightarrow V'^{\otimes n}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism. (Here, we are using Convention 1.56.)

This Proposition 3.42 is the analogue of Proposition 1.55 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules. The following proposition generalizes Proposition 1.57 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Proposition 3.43. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. Then, for any $(\mathfrak{g}, \mathfrak{h})$ -semimodule V and every $i \in \{0, 1, \dots, n\}$, the canonical k -module isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule isomorphism.³⁴

This proposition follows directly from applying Proposition 3.34 to V, V, \dots, V instead of V_1, V_2, \dots, V_n .

The following convention extends Convention 1.58 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Convention 3.44. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . For every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V , every $n \in \mathbb{N}$ and every $i \in \{0, 1, \dots, n\}$, we are going to identify the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V^{\otimes i} \otimes V^{\otimes(n-i)}$ with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V^{\otimes n}$ (this is allowed because of Proposition 3.43). In other words, for every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V , every $a \in \mathbb{N}$ and every $b \in \mathbb{N}$, we are going to identify the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V^{\otimes a} \otimes V^{\otimes b}$ with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $V^{\otimes(a+b)}$.

Finally, Proposition 3.38 yields:

³⁴Here and in the following, whenever I speak of "the canonical k -module isomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$ ", I mean the k -module homomorphism $V^{\otimes i} \otimes V^{\otimes(n-i)} \rightarrow V^{\otimes n}$ which sends $(v_1 \otimes v_2 \otimes \dots \otimes v_i) \otimes (v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n)$ to $v_1 \otimes v_2 \otimes \dots \otimes v_n$ for every $(v_1, v_2, \dots, v_n) \in V^n$. This homomorphism is known (from linear algebra) to exist, be unique and be a k -module isomorphism. It is actually the canonical k -module isomorphism $\underbrace{V \otimes V \otimes \dots \otimes V}_{i \text{ times}} \otimes \underbrace{V \otimes V \otimes \dots \otimes V}_{n-i \text{ times}} \rightarrow$

$$\underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}.$$

Proposition 3.45. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$.

If V is any $(\mathfrak{g}, \mathfrak{h})$ -semimodule, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V^{\otimes n}) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)^{\otimes n}$ as \mathfrak{h} -modules. This allows us to speak of "the \mathfrak{h} -module $V^{\otimes n}$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V^{\otimes n})$ or $(\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)^{\otimes n}$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(V^{\otimes n}) = (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)^{\otimes n}$).

This Proposition 3.45 generalizes Proposition 1.60 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

3.7. The tensor algebra of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule

In Definition 1.66, we have defined a canonical \mathfrak{g} -module structure on $\otimes V$ for every \mathfrak{g} -module V . Here is a definition which extends this notion to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Definition 3.46. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Since $V^{\otimes i}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule for all $i \in \mathbb{N}$ (by Definition 3.39), the direct sum $\bigoplus_{i \in \mathbb{N}} V^{\otimes i}$ is also a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (by Definition 3.20). In other words, the tensor algebra $\otimes V$ thus becomes a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (since $\otimes V = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$). This $(\mathfrak{g}, \mathfrak{h})$ -semimodule $\otimes V$ is called the *tensor $(\mathfrak{g}, \mathfrak{h})$ -semimodule* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V .

Whenever we will speak of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $\otimes V$, we will be meaning this $(\mathfrak{g}, \mathfrak{h})$ -semimodule (although there might be many different $(\mathfrak{g}, \mathfrak{h})$ -semimodule structures on the k -module $\otimes V$).

Since this Definition 3.46 is analogous to Definition 1.66, we have:

Proposition 3.47. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be any \mathfrak{g} -module. Then, the \mathfrak{g} -module $\otimes V$ defined in Definition 1.66 is identical with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $\otimes V$ defined in Definition 3.46 (which is well-defined since V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the \mathfrak{g} -module $\otimes V$ " without having to worry whether it is understood according to Definition 1.66 or according to Definition 3.46 (because it doesn't matter, as both definitions yield the same result).

It is very easy to see that:

Proposition 3.48. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules, and let $f : V \rightarrow W$ be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism. Then, $\otimes f : \otimes V \rightarrow \otimes W$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule homomorphism.

This Proposition 3.48 is the analogue of Proposition 1.68 for $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

Definition 3.49. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Then, according to Proposition 3.22, we consider $V^{\otimes n}$ as a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of the direct sum $\bigoplus_{i \in \mathbb{N}} V^{\otimes i} = \otimes V$ for every $n \in \mathbb{N}$. In particular, $k = V^{\otimes 0}$ and $V = V^{\otimes 1}$ become $(\mathfrak{g}, \mathfrak{h})$ -subsemimodules of $\otimes V$ this way.

This Definition 3.49 is completely analogous to Definition 1.69. Therefore:

Proposition 3.50. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be any \mathfrak{g} -module. Let $n \in \mathbb{N}$. Then, we can identify $V^{\otimes n}$ with a \mathfrak{g} -submodule of $\otimes V$ (according to Definition 1.69), but we can also identify $V^{\otimes n}$ with a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of $\otimes V$ (according to Definition 3.49, because V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). These two identifications do **not** conflict with each other, because they both identify $V^{\otimes n}$ with **one and the same** subset of $\otimes V$.

3.8. $(\mathfrak{g}, \mathfrak{h})$ -semialgebras

Just as we introduced \mathfrak{g} -algebras in Section 1, we can define $(\mathfrak{g}, \mathfrak{h})$ -semialgebras:

Definition 3.51. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . A $(\mathfrak{g}, \mathfrak{h})$ -semialgebra will mean a k -algebra A equipped with a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure such that

$$(a \rightharpoonup (uv)) = (a \rightharpoonup u) \cdot v + u \cdot (a \rightharpoonup v) \text{ for every } a \in \mathfrak{g}, u \in A \text{ and } v \in A). \quad (88)$$

Remark 3.52. In Definition 3.51, when we speak of "a k -algebra A equipped with a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure", the words "a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure" mean "a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on the underlying k -module of the k -algebra A ". This $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure must therefore be k -bilinear with respect to the underlying k -module structure of the k -algebra A .

Remark 3.53. Definition 3.51 is often rewritten as follows:

Definition 3.54. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . A $(\mathfrak{g}, \mathfrak{h})$ -semialgebra will mean a k -algebra A equipped with a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure such that \mathfrak{g} acts on A by means of derivations. Here, we say that " \mathfrak{g} acts on A by means of derivations" if and only if the map $A \rightarrow A, u \mapsto (a \rightharpoonup u)$ is a derivation for every $a \in \mathfrak{g}$.

This Definition 3.54 is indeed equivalent to Definition 3.51 because the condition that \mathfrak{g} acts on A by means of derivations is equivalent to (88) (as can be easily seen).

The above Definition 3.51 is a weakening of Definition 1.70, in the sense that we have:

Proposition 3.55. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Then, every \mathfrak{g} -algebra is a $(\mathfrak{g}, \mathfrak{h})$ -semialgebra.

This proposition is completely trivial (since every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule). It is easy to see that, when \mathfrak{h} is a Lie subalgebra of a k -Lie algebra \mathfrak{g} , every $(\mathfrak{g}, \mathfrak{h})$ -semialgebra canonically can be made into an \mathfrak{h} -algebra:

Definition 3.56. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Then, every $(\mathfrak{g}, \mathfrak{h})$ -semialgebra A canonically becomes an \mathfrak{h} -algebra. (In fact, the $(\mathfrak{g}, \mathfrak{h})$ -semimodule A canonically becomes an \mathfrak{h} -module according to Definition 3.11, and thus A is a k -algebra equipped with an \mathfrak{h} -module structure which satisfies (33) with \mathfrak{g} replaced by \mathfrak{h} , so that A thus is an \mathfrak{h} -algebra.). This \mathfrak{h} -algebra is called the *restriction of A to \mathfrak{h}* , and denoted by $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} A$. However, when there is no possibility of confusion, we will denote this \mathfrak{h} -algebra by A , and we will distinguish it from the original $(\mathfrak{g}, \mathfrak{h})$ -semialgebra A by means of referring to the former one as "the \mathfrak{h} -algebra A " and referring to the latter one as "the $(\mathfrak{g}, \mathfrak{h})$ -semialgebra A ".

This Definition 3.56 is a generalization of Definition 1.74 to $(\mathfrak{g}, \mathfrak{h})$ -semialgebras. The following proposition generalizes Proposition 1.75 to $(\mathfrak{g}, \mathfrak{h})$ -semialgebras:

Proposition 3.57. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let A be a $(\mathfrak{g}, \mathfrak{h})$ -semialgebra. Let P and Q be two $(\mathfrak{g}, \mathfrak{h})$ -subsemimodules of A . Then, $P \cdot Q$ is a $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule of A .

Remark 3.58. Here, $P \cdot Q$ is to be understood as according to Convention 1.63 (a).

Proof of Proposition 3.57. In order to obtain a proof of Proposition 3.57, it is enough to apply the following changes to the proof of Proposition 1.75:

- Replace "Proposition 1.75" by "Proposition 3.57".
- Replace the reference to (33) by a reference to (88).
- Replace the words " \mathfrak{g} -submodule" by " $(\mathfrak{g}, \mathfrak{h})$ -subsemimodule".

This completes the proof of Proposition 3.57.

3.9. $\otimes V$ is a $(\mathfrak{g}, \mathfrak{h})$ -semialgebra

The following proposition is a generalization of Proposition 1.77 to $(\mathfrak{g}, \mathfrak{h})$ -semialgebras:

Proposition 3.59. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. If we equip the k -algebra $\otimes V$ (this k -algebra was defined in Definition 1.61 (a)) with the $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure defined in Definition 3.46, we obtain a $(\mathfrak{g}, \mathfrak{h})$ -semialgebra.

Definition 3.60. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. The $(\mathfrak{g}, \mathfrak{h})$ -semialgebra $\otimes V$ defined in Proposition 3.59 is called the *tensor $(\mathfrak{g}, \mathfrak{h})$ -semialgebra* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V . Whenever we will speak of the $(\mathfrak{g}, \mathfrak{h})$ -semialgebra $\otimes V$, we will be meaning this tensor $(\mathfrak{g}, \mathfrak{h})$ -semialgebra $\otimes V$ (unless we explicitly say that we are talking about a different $(\mathfrak{g}, \mathfrak{h})$ -semialgebra structure on $\otimes V$).

This Definition 3.60 is a generalization of Definition 1.78 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Therefore:

Proposition 3.61. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be any \mathfrak{g} -module. Then, the \mathfrak{g} -algebra $\otimes V$ defined in Definition 1.78 is identic with the $(\mathfrak{g}, \mathfrak{h})$ -semialgebra $\otimes V$ defined in Definition 3.60 (which is well-defined since V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (which is because Proposition 3.2 says that every \mathfrak{g} -module is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). This allows us to speak of "the \mathfrak{g} -algebra $\otimes V$ " without having to worry whether it is understood according to Definition 1.78 or according to Definition 3.60 (because it doesn't matter, as both definitions yield the same result).

Proof of Proposition 3.59. In order to obtain a proof of Proposition 3.59, it is enough to apply the following changes to the proof of Proposition 1.77:

- Replace "Proposition 1.77" by "Proposition 3.59".
- Replace "Proposition 1.57" by "Proposition 3.43".
- Replace "Definition 1.70" by "Definition 3.51".
- Replace the words " \mathfrak{g} -module" by " $(\mathfrak{g}, \mathfrak{h})$ -semimodule".
- Replace the words " \mathfrak{g} -algebra" by " $(\mathfrak{g}, \mathfrak{h})$ -semialgebra".
- Replace the references to (13) by references to (83).
- Replace the references to (15) by references to (85).
- Replace the references to (33) by references to (88).

This completes the proof of Proposition 3.59.

We notice that the $(\mathfrak{g}, \mathfrak{h})$ -semialgebra $\otimes V$ behaves under restriction as we would want it to:

Proposition 3.62. Let k be a commutative ring. Let \mathfrak{g} be a Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

If V is any $(\mathfrak{g}, \mathfrak{h})$ -semimodule, then $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(\otimes V) = \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)$ as \mathfrak{h} -algebras. This allows us to speak of "the \mathfrak{h} -algebra $\otimes V$ " without having to worry whether we mean $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(\otimes V)$ or $\otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)$ (because it does not matter, since $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}(\otimes V) = \otimes (\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} V)$).

This Proposition 3.62 is the generalization of Proposition 1.80 to $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

3.10. Semimodules and \mathfrak{h} -module homomorphisms

We have devoted a great part of Section 3 to formulating properties of $(\mathfrak{g}, \mathfrak{h})$ -semimodules which are analogous to some known properties of \mathfrak{g} -modules. Of course, we have barely scratched the surface - there are many more such properties. In the present Subsection 3.10, as well as in Subsection 3.11 further below, we are going to present some different viewpoints on $(\mathfrak{g}, \mathfrak{h})$ -semimodules.

The following result is not an analogue of a result from Section 1 anymore, but instead rewrites the definition of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule in terms of the notion of \mathfrak{h} -module homomorphisms:

Proposition 3.63. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let V be an \mathfrak{h} -module. Let $\beta : \mathfrak{g} \times V \rightarrow V$ denote a k -bilinear map such that $\beta|_{\mathfrak{h} \times V}$ is the Lie action of the \mathfrak{h} -module V . By the universal property of the tensor product, this k -bilinear map $\beta : \mathfrak{g} \times V \rightarrow V$ gives rise to a k -linear map $\tilde{\beta} : \mathfrak{g} \otimes V \rightarrow V$ which satisfies

$$\left(\tilde{\beta}(a \otimes v) = \beta(a, v) \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V \right).$$

Then, (V, β) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule if and only if $\tilde{\beta} : \mathfrak{g} \otimes V \rightarrow V$ is an \mathfrak{h} -module homomorphism.

We are neither going to use, nor going to prove this (something the reader could easily do), but we remark that this can be applied to $\mathfrak{h} = \mathfrak{g}$ and results in a property of \mathfrak{g} -modules:

Proposition 3.64. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $\beta : \mathfrak{g} \times V \rightarrow V$ denote the Lie action of the \mathfrak{g} -module V . By the universal property of the tensor product, this k -bilinear map $\beta : \mathfrak{g} \times V \rightarrow V$ gives rise to a k -linear map $\tilde{\beta} : \mathfrak{g} \otimes V \rightarrow V$ which satisfies

$$\left(\tilde{\beta}(a \otimes v) = \beta(a, v) \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V \right).$$

Then, $\tilde{\beta} : \mathfrak{g} \otimes V \rightarrow V$ is a \mathfrak{g} -module homomorphism.

3.11. $(\mathfrak{g}, \mathfrak{h})$ -semimodules as modules

In this Subsection 3.11, we are going to relate the notion of $(\mathfrak{g}, \mathfrak{h})$ -semimodules to the notion of \mathfrak{g} -modules (but for a different Lie algebra \mathfrak{g}) and to the notion of A -modules for an (associative) algebra A . This subsection is not relevant to the rest of the present paper, but it sheds a different light on the notion of $(\mathfrak{g}, \mathfrak{h})$ -semimodules (more concretely, it provides alternative definitions of the notion of $(\mathfrak{g}, \mathfrak{h})$ -semimodules, although I consider Definition 3.1 to be the simplest and most explanatory one).

First, let us give an analogue of Definition 1.64 tailored to $(\mathfrak{g}, \mathfrak{h})$ -semimodules:

Definition 3.65. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . We define the algebra $U(\mathfrak{g}, \mathfrak{h})$ to be the factor algebra $(\otimes \mathfrak{g}) / I_{\mathfrak{g}, \mathfrak{h}}$, where $I_{\mathfrak{g}, \mathfrak{h}}$ is the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

of the algebra $\otimes \mathfrak{g}$.

Remark 3.66. In Definition 3.65, the term $\langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle$ is to be understood according to Convention 1.28, and the multiplication sign \cdot in

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

is to be understood according to Convention 1.63. We note that, although the multiplication in $\otimes \mathfrak{g}$ is related to the tensor product by (31), the product $(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$ has nothing to do with the tensor product $(\otimes \mathfrak{g}) \otimes \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \otimes (\otimes \mathfrak{g})$!

Remark 3.67. In Definition 3.65, the ideal $I_{\mathfrak{g}, \mathfrak{h}}$ is the same as the ideal J from Theorem 2.1.

Now, the following generalization of Proposition 1.83 allows us to consider $(\mathfrak{g}, \mathfrak{h})$ -semimodules as A -modules for $A = U(\mathfrak{g}, \mathfrak{h})$:

Proposition 3.68. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Consider the algebra $U(\mathfrak{g}, \mathfrak{h})$ and the ideal $I_{\mathfrak{g}, \mathfrak{h}}$ defined in Definition 3.65.

(a) For every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V , there is one and only one $U(\mathfrak{g}, \mathfrak{h})$ -module structure on V satisfying

$$(\bar{a} \cdot v = a \rightharpoonup v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V)$$

(where \bar{a} denotes the projection of $a \in \mathfrak{g} \subseteq \otimes \mathfrak{g}$ on $(\otimes \mathfrak{g}) / I_{\mathfrak{g}, \mathfrak{h}} = U(\mathfrak{g}, \mathfrak{h})$). This $U(\mathfrak{g}, \mathfrak{h})$ -module structure is canonical. Thus, every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V canonically becomes a $U(\mathfrak{g}, \mathfrak{h})$ -module.

(b) Conversely, for every $U(\mathfrak{g}, \mathfrak{h})$ -module V , we can define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on V by

$$(a \rightharpoonup v = \bar{a} \cdot v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V)$$

(where \bar{a} denotes the projection of $a \in \mathfrak{g} \subseteq \otimes \mathfrak{g}$ on $(\otimes \mathfrak{g}) / I_{\mathfrak{g}, \mathfrak{h}} = U(\mathfrak{g}, \mathfrak{h})$). This $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure is canonical. Thus, every $U(\mathfrak{g}, \mathfrak{h})$ -module V canonically becomes a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

(c) Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Then, according to Proposition 3.68

(a), each of V and W canonically becomes a $U(\mathfrak{g}, \mathfrak{h})$ -module. Let $f : V \rightarrow W$ be a map. Then, f is a homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules if and only if f is a homomorphism of $U(\mathfrak{g}, \mathfrak{h})$ -modules.

(d) Let V and W be two $U(\mathfrak{g}, \mathfrak{h})$ -modules. Then, according to Proposition 3.68

(b), each of V and W canonically becomes a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Let $f : V \rightarrow W$ be a map. Then, f is a homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules if and only if f is a homomorphism of $U(\mathfrak{g}, \mathfrak{h})$ -modules.

(e) We can define a functor U_1 from the category of $(\mathfrak{g}, \mathfrak{h})$ -semimodules to the category of $U(\mathfrak{g}, \mathfrak{h})$ -modules as follows: For every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V , let $U_1(V)$ be the $U(\mathfrak{g}, \mathfrak{h})$ -module V defined in Proposition 3.68 (a). For every homomorphism f between $(\mathfrak{g}, \mathfrak{h})$ -semimodules, let $U_1(f)$ be the same homomorphism f , but considered as a homomorphism between $U(\mathfrak{g}, \mathfrak{h})$ -modules this time (this is legitimate due to Proposition 3.68 (c)).

(f) We can define a functor U_2 from the category of $U(\mathfrak{g}, \mathfrak{h})$ -modules to the category of $(\mathfrak{g}, \mathfrak{h})$ -semimodules as follows: For every $U(\mathfrak{g}, \mathfrak{h})$ -module V , let $U_2(V)$ be the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V defined in Proposition 3.68 (b). For every homomorphism f between $U(\mathfrak{g}, \mathfrak{h})$ -modules, let $U_2(f)$ be the same homomorphism f , but considered as a homomorphism between $(\mathfrak{g}, \mathfrak{h})$ -semimodules this time (this is legitimate due to Proposition 3.68 (d)).

(g) The two functors U_1 and U_2 defined in Proposition 3.68 (e) and (f) are mutually inverse.

(h) Both functors U_1 and U_2 are additive, exact and preserve kernels, cokernels and direct sums.

This proposition provides for an easy way to obtain results about $(\mathfrak{g}, \mathfrak{h})$ -semimodules from results about A -modules (over every associative algebra A), at least when the latter results are basic enough to remain valid under an invertible, additive and exact functor which preserves kernels, cokernels and direct sums. We can also add tensor products to this list by making $U(\mathfrak{g}, \mathfrak{h})$ into a Hopf algebra (in analogy to Proposition 1.84) and proving an analogue of Proposition 1.85. We will not delve into details here as these analogues are straightforward to obtain and automatic to prove (by taking the proofs of Propositions 1.84 and 1.85 and applying the obvious changes).

Thus we have found a way to see $(\mathfrak{g}, \mathfrak{h})$ -semimodules as A -modules for an associative algebra A . One could wonder whether we can also see them as \mathfrak{g} -modules for some *other* Lie algebra \mathfrak{g} . The answer is, again, positive:

Proposition 3.69. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} .

Let $\text{FreeLie } \mathfrak{g}$ denote the free Lie algebra on the k -module \mathfrak{g} , and let $\iota : \mathfrak{g} \rightarrow \text{FreeLie } \mathfrak{g}$ be the corresponding homomorphism. (A definition of a "free Lie algebra" along with the corresponding homomorphism is given in various sources, e. g., in [11, §1.11.2]. For us it is only important that it satisfies the following universal property: For every k -Lie algebra \mathfrak{u} and every k -module homomorphism $p : \mathfrak{g} \rightarrow \mathfrak{u}$, there exists one and only one k -Lie algebra homomorphism $P : \text{FreeLie } \mathfrak{g} \rightarrow \mathfrak{u}$ satisfying $P \circ \iota = p$.) Let \mathfrak{i} denote the Lie ideal of $\text{FreeLie } \mathfrak{g}$ generated by the k -submodule $\langle [\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g} \rangle$ of $\text{FreeLie } \mathfrak{g}$. Let $\mathfrak{h}^{(1)}$ denote the k -Lie algebra $(\text{FreeLie } \mathfrak{g}) / \mathfrak{i}$.

(a) For every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V , there is one and only one $\mathfrak{h}^{(1)}$ -module structure on V satisfying

$$\left(\overline{\iota(a)} \rightarrow v = a \rightarrow v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V \right) \quad (89)$$

(where $\overline{\iota(a)}$ denotes the projection of $\iota(a) \in \text{FreeLie } \mathfrak{g}$ on $(\text{FreeLie } \mathfrak{g})/\mathfrak{i} = \mathfrak{h}^{(1)}$). This $\mathfrak{h}^{(1)}$ -module structure is canonical. Thus, every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V canonically becomes an $\mathfrak{h}^{(1)}$ -module.

(b) Conversely, for every $\mathfrak{h}^{(1)}$ -module V , we can define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on V by

$$\left(a \rightharpoonup v = \overline{\iota(a)} \rightharpoonup v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V \right)$$

(where $\overline{\iota(a)}$ denotes the projection of $\iota(a) \in \text{FreeLie } \mathfrak{g}$ on $(\text{FreeLie } \mathfrak{g})/\mathfrak{i} = \mathfrak{h}^{(1)}$). This $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure is canonical. Thus, every $\mathfrak{h}^{(1)}$ -module V canonically becomes a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

(c) Let V and W be two $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Then, according to Proposition 3.69 (a), each of V and W canonically becomes an $\mathfrak{h}^{(1)}$ -module. Let $f : V \rightarrow W$ be a map. Then, f is a homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules if and only if f is a homomorphism of $\mathfrak{h}^{(1)}$ -modules.

(d) Let V and W be two $\mathfrak{h}^{(1)}$ -modules. Then, according to Proposition 3.69 (b), each of V and W canonically becomes a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Let $f : V \rightarrow W$ be a map. Then, f is a homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules if and only if f is a homomorphism of $\mathfrak{h}^{(1)}$ -modules.

(e) We can define a functor U_1 from the category of $(\mathfrak{g}, \mathfrak{h})$ -semimodules to the category of $\mathfrak{h}^{(1)}$ -modules as follows: For every $(\mathfrak{g}, \mathfrak{h})$ -semimodule V , let $U_1(V)$ be the $\mathfrak{h}^{(1)}$ -module V defined in Proposition 3.69 (a). For every homomorphism f between $(\mathfrak{g}, \mathfrak{h})$ -semimodules, let $U_1(f)$ be the same homomorphism f , but considered as a homomorphism between $\mathfrak{h}^{(1)}$ -modules this time (this is legitimate due to Proposition 3.69 (c)).

(f) We can define a functor U_2 from the category of $\mathfrak{h}^{(1)}$ -modules to the category of $(\mathfrak{g}, \mathfrak{h})$ -semimodules as follows: For every $\mathfrak{h}^{(1)}$ -module V , let $U_2(V)$ be the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V defined in Proposition 3.69 (b). For every homomorphism f between $\mathfrak{h}^{(1)}$ -modules, let $U_2(f)$ be the same homomorphism f , but considered as a homomorphism between $(\mathfrak{g}, \mathfrak{h})$ -semimodules this time (this is legitimate due to Proposition 3.69 (d)).

(g) The two functors U_1 and U_2 defined in Proposition 3.69 (e) and (f) are mutually inverse.

(h) Both functors U_1 and U_2 are additive, exact and preserve kernels, cokernels and direct sums.

Additionally:

Proposition 3.70. Consider the situation of Proposition 3.69.

(a) The composition

$$\mathfrak{h} \xrightarrow{\text{canonical inclusion}} \mathfrak{g} \xrightarrow{\iota} \text{FreeLie } \mathfrak{g} \xrightarrow{\text{canonical projection}} \mathfrak{h}^{(1)}$$

is an injective k -Lie algebra homomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^{(1)}$.

(b) Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Then, according to Proposition 3.69 (a), this V canonically becomes an $\mathfrak{h}^{(1)}$ -module. The restriction of this $\mathfrak{h}^{(1)}$ -module to \mathfrak{h} (by means of the injective k -Lie algebra homomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^{(1)}$ constructed in part (a)) is identic with the restriction of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V to \mathfrak{h} .

As a consequence, an \mathfrak{h} -module is the restriction of some $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h} if and only if it is the restriction of some $\mathfrak{h}^{(1)}$ -module to \mathfrak{h} .

The paper [2] mostly views $(\mathfrak{g}, \mathfrak{h})$ -semimodules through the lens of Proposition 3.69, thus considering them as $\mathfrak{h}^{(1)}$ -modules. The notion of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule does not even occur in this paper.

We can view $(\mathfrak{g}, \mathfrak{h})$ -semimodules as $U(\mathfrak{g}, \mathfrak{h})$ -modules (by Proposition 3.68), but on the other hand we can also view them as $\mathfrak{h}^{(1)}$ -modules (by Proposition 3.69) and therefore as $U(\mathfrak{h}^{(1)})$ -modules (by Proposition 1.83, applied to $\mathfrak{h}^{(1)}$ instead of \mathfrak{g}). This leaves us wondering whether there is a relation between $U(\mathfrak{g}, \mathfrak{h})$ and $U(\mathfrak{h}^{(1)})$. The answer is as simple as it could be: there exists an isomorphism:

Proposition 3.71. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Consider the k -algebra $U(\mathfrak{g}, \mathfrak{h})$ defined in Definition 3.65, and consider the k -Lie algebra $\mathfrak{h}^{(1)}$ defined in Proposition 3.69. Then, there exists a unique k -algebra homomorphism $P : U(\mathfrak{g}, \mathfrak{h}) \rightarrow U(\mathfrak{h}^{(1)})$ which satisfies $(P(\bar{a}) = \overline{\iota(a)}$ for every $a \in \mathfrak{g}$). This homomorphism P is a k -algebra isomorphism.

Before we continue, let us contour the proof of Proposition 3.69. It needs the following fact about Lie algebra modules:

Proposition 3.72. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

(a) Let V be a \mathfrak{g} -module. For every $a \in \mathfrak{g}$, let $\beta_a : V \rightarrow V$ be the map which sends every $v \in V$ to $a \rightarrow v \in V$. Then, $\beta_a \in \text{End } V$. Now, define a map $\text{cur}_V : \mathfrak{g} \rightarrow \text{End } V$ by

$$(\text{cur}_V(a) = \beta_a \quad \text{for every } a \in \mathfrak{g}).$$

Then, cur_V is a Lie algebra homomorphism. (Here, $\text{End } V$ is a k -Lie algebra with the commutator of endomorphisms as Lie bracket.)

(b) Let V be a k -module, and let $P : \mathfrak{g} \rightarrow \text{End } V$ be a Lie algebra homomorphism. Then, we can make V into a \mathfrak{g} -module by setting

$$(a \rightarrow v = (P(a))(v) \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V).$$

(c) Let V be a k -module. Part (a) of this proposition assigns a Lie algebra homomorphism $\text{cur}_V : \mathfrak{g} \rightarrow \text{End } V$ to any \mathfrak{g} -module structure on V , whereas part (b) of this proposition assigns a \mathfrak{g} -module structure on V to any Lie algebra homomorphism $P : \mathfrak{g} \rightarrow \text{End } V$. These two assignments are mutually inverse. This means that if we start with a \mathfrak{g} -module structure on V , then assign a Lie algebra homomorphism $\text{cur}_V : \mathfrak{g} \rightarrow \text{End } V$ to it according to part (a), and then assign a \mathfrak{g} -module structure to it according to part (b) (applied to $P = \text{cur}_V$), then we get our original \mathfrak{g} -module structure on V back, and the same holds the other way round.

This yields that if V is a k -module, then Lie algebra homomorphisms $\mathfrak{g} \rightarrow \text{End } V$ stand in 1-to-1 correspondence with \mathfrak{g} -module structures on V .

This is a known fact, and has an even better known counterpart about associative algebras (which is proven similarly), so we see no need to prove it here. Moreover, we need the following lemma:

Lemma 3.73. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let S be a subset of \mathfrak{g} such that the Lie algebra \mathfrak{g} is generated (as a Lie algebra) by the subset S .

Let V be a k -module, and let $\mu_1 : \mathfrak{g} \times V \rightarrow V$ and $\mu_2 : \mathfrak{g} \times V \rightarrow V$ be two maps such that (V, μ_1) and (V, μ_2) are two \mathfrak{g} -modules. Assume that every $s \in S$ and every $v \in V$ satisfy $\mu_1(s, v) = \mu_2(s, v)$. Then, $\mu_1 = \mu_2$.

Proof of Lemma 3.73. First, we notice that μ_1 and μ_2 are two k -bilinear maps (since (V, μ_1) and (V, μ_2) are two \mathfrak{g} -modules).

Let \mathfrak{h} be the subset

$$\{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\}$$

of \mathfrak{g} .

We note that every $s \in S$ satisfies $\mu_1(s, v) = \mu_2(s, v)$ for every $v \in V$. In other words, every $s \in S$ satisfies $s \in \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\} = \mathfrak{h}$. This yields $S \subseteq \mathfrak{h}$.

Then,

$$(\text{every } x \in \mathfrak{h} \text{ and every } \lambda \in k \text{ satisfy } \lambda x \in \mathfrak{h}). \quad (90)$$

Proof of (90). Let $x \in \mathfrak{h}$ and $\lambda \in k$ be arbitrary. Then,

$$x \in \mathfrak{h} = \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\}.$$

Thus, $\mu_1(x, v) = \mu_2(x, v)$ for every $v \in V$. Thus,

$$\begin{aligned} \mu_1(\lambda x, v) &= \lambda \mu_1(x, v) && (\text{since } \mu_1 \text{ is } k\text{-bilinear}) \\ &= \lambda \mu_2(x, v) && (\text{since } \mu_1(x, v) = \mu_2(x, v)) \\ &= \mu_2(\lambda x, v) && (\text{since } \mu_2 \text{ is } k\text{-bilinear}) \end{aligned}$$

for every $v \in V$. This yields that $\lambda x \in \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\} = \mathfrak{h}$. This proves (90).

Besides,

$$(\text{every } x \in \mathfrak{h} \text{ and every } y \in \mathfrak{h} \text{ satisfy } x + y \in \mathfrak{h}). \quad (91)$$

Proof of (91). Let $x \in \mathfrak{h}$ and $y \in \mathfrak{h}$ be arbitrary. Then, $\mu_1(x, v) = \mu_2(x, v)$ for every $v \in V$ (because $x \in \mathfrak{h} = \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\}$). On the other hand, $\mu_1(y, v) = \mu_2(y, v)$ for every $v \in V$ (because $y \in \mathfrak{h} = \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\}$). Thus,

$$\begin{aligned} \mu_1(x + y, v) &= \underbrace{\mu_1(x, v)}_{=\mu_2(x, v)} + \underbrace{\mu_1(y, v)}_{=\mu_2(y, v)} && (\text{since } \mu_1 \text{ is } k\text{-bilinear}) \\ &= \mu_2(x, v) + \mu_2(y, v) = \mu_2(x + y, v) && (\text{since } \mu_2 \text{ is } k\text{-bilinear}) \end{aligned}$$

for every $v \in V$. This yields that $x + y \in \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\} = \mathfrak{h}$. This proves (91).

Finally,

$$(\text{every } x \in \mathfrak{h} \text{ and every } y \in \mathfrak{h} \text{ satisfy } [x, y] \in \mathfrak{h}). \quad (92)$$

Proof of (92). Let $x \in \mathfrak{h}$ and $y \in \mathfrak{h}$ be arbitrary. Then,

$$\mu_1(x, v) = \mu_2(x, v) \quad \text{for every } v \in V \quad (93)$$

(because $x \in \mathfrak{h} = \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\}$). On the other hand,

$$\mu_1(y, v) = \mu_2(y, v) \quad \text{for every } v \in V \quad (94)$$

(because $y \in \mathfrak{h} = \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\}$). Thus, every $v \in V$ satisfies

$$\begin{aligned} \mu_1([x, y], v) &= \underbrace{\mu_1(x, \mu_1(y, v))}_{\substack{= \mu_2(x, \mu_1(y, v)) \\ \text{(due to (93)),} \\ \text{applied to } \mu_1(y, v) \text{ instead of } v}} - \underbrace{\mu_1(y, \mu_1(x, v))}_{\substack{= \mu_2(y, \mu_1(x, v)) \\ \text{(due to (94)),} \\ \text{applied to } \mu_1(x, v) \text{ instead of } v}} \\ &\quad \left(\begin{array}{c} \text{due to (7) (applied to } \mu_1, x \text{ and } y \text{ instead of } \mu, a \text{ and } b), \\ \text{since } (V, \mu_1) \text{ is a } \mathfrak{g}\text{-module} \end{array} \right) \\ &= \mu_2 \left(x, \underbrace{\mu_1(y, v)}_{\substack{= \mu_2(y, v) \\ \text{(due to (94))}}} \right) - \mu_2 \left(y, \underbrace{\mu_1(x, v)}_{\substack{= \mu_2(x, v) \\ \text{(due to (93))}}} \right) \\ &= \mu_2(x, \mu_2(y, v)) - \mu_2(y, \mu_2(x, v)). \end{aligned}$$

Comparing this with

$$\begin{aligned} \mu_2([x, y], v) &= \mu_2(x, \mu_2(y, v)) - \mu_2(y, \mu_2(x, v)) \\ &\quad \left(\begin{array}{c} \text{due to (7) (applied to } \mu_2, x \text{ and } y \text{ instead of } \mu, a \text{ and } b), \\ \text{since } (V, \mu_2) \text{ is a } \mathfrak{g}\text{-module} \end{array} \right), \end{aligned}$$

we conclude that $\mu_1([x, y], v) = \mu_2([x, y], v)$ for every $v \in V$. In other words, $[x, y] \in \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\} = \mathfrak{h}$. This proves (92).

From (90) and (91), we conclude that \mathfrak{h} is a k -submodule of \mathfrak{g} .

Since we now know that \mathfrak{h} is a k -submodule of \mathfrak{g} which satisfies (92), we conclude that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . Since $S \subseteq \mathfrak{h}$, this shows that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} which contains S as a subset.

But we know that the Lie algebra \mathfrak{g} is generated (as a Lie algebra) by the subset S . This means that \mathfrak{g} is identical with the Lie subalgebra of \mathfrak{g} generated by S . So we have

$$\begin{aligned} \mathfrak{g} &= (\text{the Lie subalgebra of } \mathfrak{g} \text{ generated by } S) \\ &= (\text{the smallest Lie subalgebra of } \mathfrak{g} \text{ which contains } S \text{ as a subset}) \\ &\subseteq \mathfrak{h} \quad (\text{since } \mathfrak{h} \text{ is a Lie subalgebra of } \mathfrak{g} \text{ which contains } S \text{ as a subset}). \end{aligned}$$

Thus, every $x \in \mathfrak{g}$ satisfies $\mu_1(x, v) = \mu_2(x, v)$ for every $v \in V$ (because $x \in \mathfrak{g} \subseteq \mathfrak{h} = \{a \in \mathfrak{g} \mid \mu_1(a, v) = \mu_2(a, v) \text{ for every } v \in V\}$). In other words, every $(x, v) \in \mathfrak{g} \times V$ satisfies $\mu_1(x, v) = \mu_2(x, v)$. This yields that $\mu_1 = \mu_2$. Lemma 3.73 is now proven.

Our next lemma is proven in the same spirit:

Lemma 3.74. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let S be a subset of \mathfrak{g} such that the Lie algebra \mathfrak{g} is generated (as a Lie algebra) by the subset S .

Let V and W be two \mathfrak{g} -modules, and let $f : V \rightarrow W$ be a k -module homomorphism. Assume that every $s \in S$ and every $v \in V$ satisfy $f(s \rightarrow v) = s \rightarrow (f(v))$.

Then, f is a \mathfrak{g} -module homomorphism.

Proof of Lemma 3.74. Let \mathfrak{h} be the subset

$$\{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\}$$

of \mathfrak{g} .

We note that every $s \in S$ satisfies $f(s \rightarrow v) = s \rightarrow (f(v))$ for every $v \in V$. In other words, every $s \in S$ satisfies $s \in \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\} = \mathfrak{h}$. This yields $S \subseteq \mathfrak{h}$.

Then,

$$(\text{every } x \in \mathfrak{h} \text{ and every } \lambda \in k \text{ satisfy } \lambda x \in \mathfrak{h}). \quad (95)$$

Proof of (95). Let $x \in \mathfrak{h}$ and $\lambda \in k$ be arbitrary. Then,

$$x \in \mathfrak{h} = \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\}.$$

Thus, $f(x \rightarrow v) = x \rightarrow (f(v))$ for every $v \in V$. Thus,

$$\begin{aligned} f \left(\begin{array}{c} (\lambda x) \rightarrow v \\ \underbrace{= \lambda(x \rightarrow v)} \\ \text{(since the Lie action} \\ \text{on } V \text{ is } k\text{-bilinear)} \end{array} \right) &= f(\lambda(x \rightarrow v)) = \lambda \underbrace{f(x \rightarrow v)}_{=x \rightarrow (f(v))} \quad (\text{since } f \text{ is } k\text{-linear}) \\ &= \lambda(x \rightarrow (f(v))) = (\lambda x) \rightarrow (f(v)) \\ &\quad (\text{since the Lie action on } W \text{ is } k\text{-bilinear}) \end{aligned}$$

for every $v \in V$. This yields that $\lambda x \in \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\} = \mathfrak{h}$. This proves (95).

Besides,

$$(\text{every } x \in \mathfrak{h} \text{ and every } y \in \mathfrak{h} \text{ satisfy } x + y \in \mathfrak{h}). \quad (96)$$

Proof of (96). Let $x \in \mathfrak{h}$ and $y \in \mathfrak{h}$ be arbitrary. Then, $f(x \rightarrow v) = x \rightarrow (f(v))$ for every $v \in V$ (because $x \in \mathfrak{h} = \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\}$). On the other hand, $f(y \rightarrow v) = y \rightarrow (f(v))$ for every $v \in V$ (because $y \in \mathfrak{h} = \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\}$). Thus,

$$\begin{aligned} f \left(\begin{array}{c} (x + y) \rightarrow v \\ \underbrace{= x \rightarrow v + y \rightarrow v} \\ \text{(since the Lie action} \\ \text{on } V \text{ is } k\text{-bilinear)} \end{array} \right) &= f(x \rightarrow v + y \rightarrow v) = \underbrace{f(x \rightarrow v)}_{=x \rightarrow (f(v))} + \underbrace{f(y \rightarrow v)}_{=y \rightarrow (f(v))} \quad (\text{since } f \text{ is } k\text{-linear}) \\ &= x \rightarrow (f(v)) + y \rightarrow (f(v)) = (x + y) \rightarrow (f(v)) \\ &\quad (\text{since the Lie action on } W \text{ is } k\text{-bilinear}) \end{aligned}$$

for every $v \in V$. This yields that $x+y \in \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\} = \mathfrak{h}$. This proves (96).

Finally,

$$(\text{every } x \in \mathfrak{h} \text{ and every } y \in \mathfrak{h} \text{ satisfy } [x, y] \in \mathfrak{h}). \quad (97)$$

Proof of (97). Let $x \in \mathfrak{h}$ and $y \in \mathfrak{h}$ be arbitrary. Then,

$$f(x \rightarrow v) = x \rightarrow (f(v)) \quad \text{for every } v \in V \quad (98)$$

(because $x \in \mathfrak{h} = \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\}$). On the other hand,

$$f(y \rightarrow v) = y \rightarrow (f(v)) \quad \text{for every } v \in V \quad (99)$$

(because $y \in \mathfrak{h} = \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\}$). Thus, every $v \in V$ satisfies

$$\begin{aligned} f([x, y] \rightarrow v) &= f(x \rightarrow (y \rightarrow v) - y \rightarrow (x \rightarrow v)) \\ &= \left(\begin{array}{c} \text{since (8) (applied to } x \text{ and } y \text{ instead of } a \text{ and } b) \\ \text{yields } [x, y] \rightarrow v = x \rightarrow (y \rightarrow v) - y \rightarrow (x \rightarrow v) \\ \text{(since } V \text{ is a } \mathfrak{g}\text{-module)} \end{array} \right) \\ &= \underbrace{f(x \rightarrow (y \rightarrow v))}_{\substack{=x \rightarrow (f(y \rightarrow v)) \\ \text{(due to (98)), \\ \text{applied to } y \rightarrow v \text{ instead of } v}} - \underbrace{f(y \rightarrow (x \rightarrow v))}_{\substack{=y \rightarrow (f(x \rightarrow v)) \\ \text{(due to (99)), \\ \text{applied to } x \rightarrow v \text{ instead of } v}} \quad (\text{since } f \text{ is } k\text{-linear}) \\ &= x \rightarrow \underbrace{(f(y \rightarrow v))}_{\substack{=y \rightarrow (f(v)) \\ \text{(due to (98))}}} - y \rightarrow \underbrace{(f(x \rightarrow v))}_{\substack{=x \rightarrow (f(v)) \\ \text{(due to (99))}}} \\ &= x \rightarrow (y \rightarrow (f(v))) - y \rightarrow (x \rightarrow (f(v))). \end{aligned}$$

Comparing this with

$$[x, y] \rightarrow (f(v)) = x \rightarrow (y \rightarrow (f(v))) - y \rightarrow (x \rightarrow (f(v))) \left(\begin{array}{c} \text{due to (8) (applied to } W, f(v), x \text{ and } y \text{ instead of } V, v, a \text{ and } b), \\ \text{since } W \text{ is a } \mathfrak{g}\text{-module} \end{array} \right),$$

we conclude that $f([x, y] \rightarrow v) = [x, y] \rightarrow (f(v))$ for every $v \in V$. In other words, $[x, y] \in \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\} = \mathfrak{h}$. This proves (97).

From (95) and (96), we conclude that \mathfrak{h} is a k -submodule of \mathfrak{g} .

Since we now know that \mathfrak{h} is a k -submodule of \mathfrak{g} which satisfies (97), we conclude that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . Since $S \subseteq \mathfrak{h}$, this shows that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} which contains S as a subset.

But we know that the Lie algebra \mathfrak{g} is generated (as a Lie algebra) by the subset S . This means that \mathfrak{g} is identical with the Lie subalgebra of \mathfrak{g} generated by S . So we have

$$\begin{aligned} \mathfrak{g} &= (\text{the Lie subalgebra of } \mathfrak{g} \text{ generated by } S) \\ &= (\text{the smallest Lie subalgebra of } \mathfrak{g} \text{ which contains } S \text{ as a subset}) \\ &\subseteq \mathfrak{h} \quad (\text{since } \mathfrak{h} \text{ is a Lie subalgebra of } \mathfrak{g} \text{ which contains } S \text{ as a subset}). \end{aligned}$$

Thus, every $x \in \mathfrak{g}$ satisfies $f(x \rightarrow v) = x \rightarrow (f(v))$ for every $v \in V$ (because $x \in \mathfrak{g} \subseteq \mathfrak{h} = \{a \in \mathfrak{g} \mid f(a \rightarrow v) = a \rightarrow (f(v)) \text{ for every } v \in V\}$). In other words, every

$(x, v) \in \mathfrak{g} \times V$ satisfies $f(x \rightarrow v) = x \rightarrow (f(v))$. This yields that f is a \mathfrak{g} -module homomorphism. Lemma 3.74 is now proven.

Finally:

Lemma 3.75. Let k be a commutative ring. Let \mathfrak{g} and \mathfrak{h} be two k -Lie algebras. Let $f : \mathfrak{g} \rightarrow \mathfrak{h}$ be a k -Lie algebra homomorphism. Then, $\text{Ker } f$ is a Lie ideal of \mathfrak{g} .

Proof of Lemma 3.75. Let $v \in \mathfrak{g}$ and $x \in \text{Ker } f$ be arbitrary. Then, $f(x) = 0$ (since $x \in \text{Ker } f$) and

$$\begin{aligned} f([v, x]) &= \left[f(v), \underbrace{f(x)}_{=0} \right] && \text{(since } f \text{ is a } k\text{-Lie algebra homomorphism)} \\ &= [f(v), 0] = 0 && \text{(since the Lie bracket on } \mathfrak{h} \text{ is } k\text{-bilinear)}. \end{aligned}$$

Thus, $[v, x] \in \text{Ker } f$.

We have thus proven that $[v, x] \in \text{Ker } f$ for every $v \in \mathfrak{g}$ and $x \in \text{Ker } f$. Since we know that $\text{Ker } f$ is a k -submodule of \mathfrak{g} (because f is k -linear), this yields that $\text{Ker } f$ is a Lie ideal of \mathfrak{g} .

Proof of Proposition 3.69 (sketched). We record the universal property of the free Lie algebra $\text{FreeLie } \mathfrak{g}$ on \mathfrak{g} : It claims that

$$\left(\begin{array}{l} \text{for every } k\text{-Lie algebra } \mathfrak{u} \text{ and every } k\text{-module homomorphism } p : \mathfrak{g} \rightarrow \mathfrak{u}, \\ \text{there exists one and only one } k\text{-Lie algebra homomorphism} \\ P : \text{FreeLie } \mathfrak{g} \rightarrow \mathfrak{u} \text{ satisfying } P \circ \iota = p \end{array} \right). \quad (100)$$

According to the construction of the free Lie algebra $\text{FreeLie } \mathfrak{g}$, every element of $\text{FreeLie } \mathfrak{g}$ can be obtained by repeated addition, scalar multiplication (i. e., multiplication with elements of k) and forming the Lie bracket from elements of the form $\iota(a)$ with $a \in \mathfrak{g}$. Since $\mathfrak{h}^{(1)}$ is a quotient of this Lie algebra $\text{FreeLie } \mathfrak{g}$, we thus conclude that every element of $\mathfrak{h}^{(1)}$ can be obtained by repeated addition, scalar multiplication (i. e., multiplication with elements of k) and forming the Lie bracket from elements of the form $\iota(a)$ with $a \in \mathfrak{g}$. In other words, the k -Lie algebra $\mathfrak{h}^{(1)}$ is generated (as a Lie algebra) by the elements $\overline{\iota(a)}$ with $a \in \mathfrak{g}$. In other words, the k -Lie algebra $\mathfrak{h}^{(1)}$ is generated (as a Lie algebra) by the subset S , where $S = \left\{ \overline{\iota(a)} \mid a \in \mathfrak{g} \right\}$.

(a) Let V be a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. We are now going to prove the following two assertions:

Assertion \mathcal{X} : There exists a canonical $\mathfrak{h}^{(1)}$ -module structure on V satisfying (89).

Assertion \mathcal{Y} : There exists at most one $\mathfrak{h}^{(1)}$ -module structure on V satisfying (89).

Proof of Assertion \mathcal{X} : For every $a \in \mathfrak{g}$, let $\beta_a : V \rightarrow V$ be the map which sends every $v \in V$ to $a \rightarrow v \in V$. Then, $\beta_a \in \text{End } V$. Now, define a map $\beth : \mathfrak{g} \rightarrow \text{End } V$ by

$$(\beth(a) = \beta_a \quad \text{for every } a \in \mathfrak{g}).$$

We consider $\text{End } V$ as a k -Lie algebra, with the commutator of endomorphisms as its

Lie bracket. Every $(a, b) \in \mathfrak{h} \times \mathfrak{g}$ and every $v \in V$ satisfy

$$\begin{aligned}
& \left(\underbrace{[\underbrace{\beth(a), \beth(b)}_{=(\beth(a)) \circ (\beth(b)) - (\beth(b)) \circ (\beth(a))}] - \beth([a, b])}_{=0} \right) (v) \\
&= ((\beth(a)) \circ (\beth(b)) - (\beth(b)) \circ (\beth(a)) - \beth([a, b])) (v) \\
&= \underbrace{(\beth(a))}_{=\beta_a} \left(\underbrace{(\beth(b))}_{=\beta_b} (v) \right) - \underbrace{(\beth(b))}_{=\beta_b} \left(\underbrace{(\beth(a))}_{=\beta_a} (v) \right) - \underbrace{(\beth([a, b]))}_{=\beta_{[a,b]}} (v) \\
&= \beta_a \left(\underbrace{\beta_b(v)}_{=b \rightarrow v} \right) - \beta_b \left(\underbrace{\beta_a(v)}_{=a \rightarrow v} \right) - \underbrace{\beta_{[a,b]}(v)}_{=[a,b] \rightarrow v} = \underbrace{\beta_a(b \rightarrow v)}_{=a \rightarrow (b \rightarrow v)} - \underbrace{\beta_b(a \rightarrow v)}_{=b \rightarrow (a \rightarrow v)} - \underbrace{[a, b] \rightarrow v}_{=a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v)} \\
& \hspace{15em} \text{(by (79))} \\
&= a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v) - (a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v)) = 0.
\end{aligned}$$

This means that

$$\text{every } (a, b) \in \mathfrak{h} \times \mathfrak{g} \text{ satisfies } [\beth(a), \beth(b)] - \beth([a, b]) = 0. \quad (101)$$

Applying (100) to $\mathbf{u} = \text{End } V$ and $p = \beth$, we see that there exists one and only one k -Lie algebra homomorphism $P : \text{FreeLie } \mathfrak{g} \rightarrow \text{End } V$ satisfying $P \circ \iota = \beth$. Let P_1 be this homomorphism. Then, $P_1 \circ \iota = \beth$.

We now are going to show that $P_1(\mathfrak{i}) = 0$.

Since P_1 is a k -Lie algebra homomorphism, Lemma 3.75 (applied to $\text{FreeLie } \mathfrak{g}$, $\text{End } V$ and P_1 instead of \mathfrak{g} , \mathfrak{h} and f) yields that $\text{Ker } P_1$ is a Lie ideal of $\text{FreeLie } \mathfrak{g}$.

Now, let T be the k -submodule $\langle [\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g} \rangle$ of $\text{FreeLie } \mathfrak{g}$. Then, \mathfrak{i} is the Lie ideal of $\text{FreeLie } \mathfrak{g}$ generated by the subset T (since \mathfrak{i} was defined as the Lie ideal of $\text{FreeLie } \mathfrak{g}$ generated by the k -submodule $\langle [\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g} \rangle$ of $\text{FreeLie } \mathfrak{g}$). In other words, \mathfrak{i} is the smallest Lie ideal of $\text{FreeLie } \mathfrak{g}$ which contains T as a subset.

Now,

$$\begin{aligned}
& P_1(\{[\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g}\}) \\
&= P_1(\{[\iota(a), \iota(b)] - \iota([a, b]) \mid (a, b) \in \mathfrak{h} \times \mathfrak{g}\}) = 0,
\end{aligned}$$

because every $(a, b) \in \mathfrak{h} \times \mathfrak{g}$ satisfies

$$\begin{aligned}
P_1([\iota(a), \iota(b)] - \iota([a, b])) &= \underbrace{P_1([\iota(a), \iota(b)])}_{\substack{=[P_1(\iota(a)), P_1(\iota(b))] \\ \text{(since } P_1 \text{ is a } k\text{-Lie algebra} \\ \text{homomorphism)}}} - \underbrace{P_1(\iota([a, b]))}_{=(P_1 \circ \iota)([a, b])} \\
&= \left[\underbrace{P_1(\iota(a))}_{=(P_1 \circ \iota)(a)}, \underbrace{P_1(\iota(b))}_{=(P_1 \circ \iota)(b)} \right] - (P_1 \circ \iota)([a, b]) \\
&= \left[\underbrace{(P_1 \circ \iota)(a)}_{=\beth(a)}, \underbrace{(P_1 \circ \iota)(b)}_{=\beth(b)} \right] - \underbrace{(P_1 \circ \iota)([a, b])}_{=\beth([a, b])} \\
&= [\beth(a), \beth(b)] - \beth([a, b]) = 0 \quad \text{(by (101)).}
\end{aligned}$$

Now,

$$\begin{aligned}
P_1(T) &= P_1(\langle [\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g} \rangle) \\
&\quad (\text{since } T = \langle [\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g} \rangle) \\
&= P_1(\langle \{[\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g}\} \rangle) \\
&= \left\langle \underbrace{P_1(\{[\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g}\})}_{=0} \right\rangle \quad (\text{since } P_1 \text{ is } k\text{-linear}) \\
&= \langle 0 \rangle = 0,
\end{aligned}$$

so that $T \subseteq \text{Ker } P_1$. Since $\text{Ker } P_1$ is a Lie ideal of $\text{FreeLie } \mathfrak{g}$, we thus conclude that $\text{Ker } P_1$ is a Lie ideal of $\text{FreeLie } \mathfrak{g}$ which contains T as a subset. Thus,

$$\begin{aligned}
\text{Ker } P_1 &\supseteq (\text{the smallest Lie ideal of } \text{FreeLie } \mathfrak{g} \text{ which contains } T \text{ as a subset}) \\
&= \mathfrak{i}.
\end{aligned}$$

In other words, $P_1(\mathfrak{i}) = 0$.

We thus have constructed a Lie algebra homomorphism $P_1 : \text{FreeLie } \mathfrak{g} \rightarrow \text{End } V$ satisfying $P_1 \circ \iota = \beth$, and we have shown that $P_1(\mathfrak{i}) = 0$. Hence, by the homomorphism theorem (for Lie algebras), the homomorphism P_1 factors through the Lie algebra $(\text{FreeLie } \mathfrak{g}) / \mathfrak{i} = \mathfrak{h}^{(1)}$. That is, there exists a Lie algebra homomorphism $P_2 : \mathfrak{h}^{(1)} \rightarrow \text{End } V$ satisfying $P_2 \circ z = P_1$, where z denotes the canonical projection $\text{FreeLie } \mathfrak{g} \rightarrow \mathfrak{h}^{(1)}$. Consider this P_2 . Combining $P_2 \circ z = P_1$ with $P_1 \circ \iota = \beth$, we obtain $\underbrace{P_2 \circ z \circ \iota}_{=P_1} = P_1 \circ \iota = \beth$.

Now, Proposition 3.72 (c) (applied to $\mathfrak{h}^{(1)}$ instead of \mathfrak{g}) shows that k -Lie algebra homomorphisms $\mathfrak{h}^{(1)} \rightarrow \text{End } V$ stand in 1-to-1 correspondence with $\mathfrak{h}^{(1)}$ -module structures on V . Therefore, the Lie algebra homomorphism $P_2 : \mathfrak{h}^{(1)} \rightarrow \text{End } V$ gives rise to an $\mathfrak{h}^{(1)}$ -module structure on V . This structure satisfies

$$x \rightarrow v = (P_2(x))(v) \quad \text{for every } x \in \mathfrak{h}^{(1)} \text{ and } v \in V.$$

Therefore, this structure satisfies

$$\begin{aligned}
\overline{\iota(a)} \rightarrow v &= \left(P_2 \left(\overline{\iota(a)} \right) \right) (v) = (P_2(z(\iota(a))))(v) \\
&\quad \left(\text{since } z \text{ is the canonical projection } \text{FreeLie } \mathfrak{g} \rightarrow \mathfrak{h}^{(1)}, \text{ and thus } \overline{\iota(a)} = z(\iota(a)) \right) \\
&= \left(\underbrace{(P_2 \circ z \circ \iota)}_{=\beth} (a) \right) (v) = \underbrace{(\beth(a))}_{=\beta_a} (v) = \beta_a(v) = a \rightarrow v
\end{aligned}$$

for every $a \in \mathfrak{g}$ and $v \in V$.

We have thus proven that there exists a canonical $\mathfrak{h}^{(1)}$ -module structure on V satisfying (89). This proves Assertion \mathcal{X} .

Proof of Assertion \mathcal{Y} : Let μ_1 and μ_2 be two $\mathfrak{h}^{(1)}$ -module structures on V which both satisfy (89). We are now going to show that $\mu_1 = \mu_2$.

In fact, we have assumed that the $\mathfrak{h}^{(1)}$ -module structure μ_1 satisfies (89). In other words, we have

$$\left(\overline{\iota(a)} \right) \rightarrow v = a \rightarrow v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V$$

if we understand $\overline{\iota(a)} \rightarrow v$ to mean $\mu_1(\overline{\iota(a)}, v)$. In other words, we have

$$\left(\mu_1(\overline{\iota(a)}, v) = a \rightarrow v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V \right).$$

The same argument, applied to μ_2 instead of μ_1 , shows that

$$\left(\mu_2(\overline{\iota(a)}, v) = a \rightarrow v \quad \text{for every } a \in \mathfrak{g} \text{ and } v \in V \right).$$

Thus, every $s \in S$ satisfies $\mu_1(s, v) = \mu_2(s, v)$ for every $v \in V$. ³⁵

But (V, μ_1) and (V, μ_2) are two $\mathfrak{h}^{(1)}$ -modules (since μ_1 and μ_2 are two $\mathfrak{h}^{(1)}$ -module structures on V). Thus, Lemma 3.73 (applied to $\mathfrak{h}^{(1)}$ instead of \mathfrak{g}) yields that $\mu_1 = \mu_2$.

We have thus shown that whenever μ_1 and μ_2 are two $\mathfrak{h}^{(1)}$ -module structures on V which both satisfy (89), we must necessarily have $\mu_1 = \mu_2$. In other words, we have shown that any two $\mathfrak{h}^{(1)}$ -module structures on V which both satisfy (89) must be equal to each other. In other words, we have proven Assertion \mathcal{Y} .

(There is also an alternative proof of Assertion \mathcal{Y} , which proceeds by tracking down the universal properties used in the above proof of Assertion \mathcal{X} , and applying the uniqueness assertions of these properties.)

Combining Assertions \mathcal{X} and \mathcal{Y} , we see that there is one and only one $\mathfrak{h}^{(1)}$ -module structure on V satisfying (89), and that this structure is canonical. This proves Proposition 3.69 (a).

(b) The only thing we must prove to show the validity of Proposition 3.69 (b) is that

$$[a, b] \rightarrow v = a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v) \quad \text{for every } (a, b) \in \mathfrak{h} \times \mathfrak{g}.$$

But this can be seen by simple computation: In fact, every $(a, b) \in \mathfrak{h} \times \mathfrak{g}$ satisfies $[\iota(a), \iota(b)] - \iota([a, b]) \in \mathfrak{i}$ (since \mathfrak{i} is the Lie ideal of $\text{FreeLie } \mathfrak{g}$ generated by $\langle [\iota(v), \iota(w)] - \iota([v, w]) \mid (v, w) \in \mathfrak{h} \times \mathfrak{g} \rangle$, and therefore contains $[\iota(v), \iota(w)] - \iota([v, w])$ for every $(v, w) \in \mathfrak{h} \times \mathfrak{g}$). Thus, every $(a, b) \in \mathfrak{h} \times \mathfrak{g}$ satisfies $[\iota(a), \iota(b)] = \iota([a, b])$ in $(\text{FreeLie } \mathfrak{g}) / \mathfrak{i}$. Consequently, every $(a, b) \in \mathfrak{h} \times \mathfrak{g}$ satisfies

$$\begin{aligned} & a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v) \\ &= \overline{\iota(a)} \rightarrow (\overline{\iota(b)} \rightarrow v) - \overline{\iota(b)} \rightarrow (\overline{\iota(a)} \rightarrow v) = \underbrace{[\overline{\iota(a)}, \overline{\iota(b)}]}_{= [\iota(a), \iota(b)] = \iota([a, b])} \rightarrow v \\ & \quad \left(\begin{array}{c} \text{due to (8) (applied to } \overline{\iota(a)}, \overline{\iota(b)} \text{ and } \mathfrak{h}^{(1)} \text{ instead of } a, b \text{ and } \mathfrak{g}), \\ \text{since } V \text{ is an } \mathfrak{h}^{(1)}\text{-module} \end{array} \right) \\ &= \overline{\iota([a, b])} \rightarrow v = [a, b] \rightarrow v, \end{aligned}$$

qed.

³⁵*Proof.* Let $s \in S$ be arbitrary. Then, $s \in S = \{ \overline{\iota(a)} \mid a \in \mathfrak{g} \}$. Hence, there exists some $a \in \mathfrak{g}$ such

$$\text{that } s = \overline{\iota(a)}. \text{ Consider this } a. \text{ Then, } \mu_1 \left(\underbrace{s}_{=\overline{\iota(a)}}, v \right) = \mu_1(\overline{\iota(a)}, v) = a \rightarrow v = \mu_2 \left(\underbrace{\overline{\iota(a)}}_{=s}, v \right) =$$

$\mu_2(s, v)$ for every $v \in V$, qed.

(c) It is trivial that if f is a homomorphism of $\mathfrak{h}^{(1)}$ -modules, then f is a homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules. What is less trivial is the converse direction: that if f is a homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules, then f is a homomorphism of $\mathfrak{h}^{(1)}$ -modules. Here is a fast way to see this: Assume that f is a homomorphism of $(\mathfrak{g}, \mathfrak{h})$ -semimodules. Then, every $a \in \mathfrak{g}$ and every $v \in V$ satisfy

$$\begin{aligned} f(\overline{\iota(a)} \rightarrow v) &= f(a \rightarrow v) \\ &= a \rightarrow (f(v)) && \text{(since } f \text{ is a homomorphism of } (\mathfrak{g}, \mathfrak{h})\text{-semimodules)} \\ &= \overline{\iota(a)} \rightarrow (f(v)). \end{aligned}$$

In other words, every $s \in S$ and every $v \in V$ satisfy $f(s \rightarrow v) = s \rightarrow (f(v))$ (because $s \in S = \{\overline{\iota(a)} \mid a \in \mathfrak{g}\}$, so that for every $s \in S$ there exists some $a \in \mathfrak{g}$ such that $s = \overline{\iota(a)}$). According to Lemma 3.74 (applied to $\mathfrak{h}^{(1)}$ instead of \mathfrak{g}), this yields that f is a homomorphism of $\mathfrak{h}^{(1)}$ -modules, qed.

(d) This is proven exactly in the same way as (c) (and is actually equivalent to (c) in light of part (g), which we prove below).

(e) and (f) are trivial.

(g) It is clear that $U_2 \circ U_1 = \text{id}$. That $U_1 \circ U_2 = \text{id}$ can be shown easily:

- For every $\mathfrak{h}^{(1)}$ -module V , we have $V = (U_1 \circ U_2)(V)$ as k -modules (trivially, because neither U_1 nor U_2 change the underlying k -module), and the $\mathfrak{h}^{(1)}$ -module structures of these $\mathfrak{h}^{(1)}$ -modules V and $(U_1 \circ U_2)(V)$ are identical (because both of them are $\mathfrak{h}^{(1)}$ -module structures on $U_2(V)$ satisfying (89), but the Assertion \mathcal{Y} that we showed above (in the proof of (a)) shows that there exists at most one $\mathfrak{h}^{(1)}$ -module structure on $U_2(V)$ satisfying (89)). Thus, for every $\mathfrak{h}^{(1)}$ -module V , we have $V = (U_1 \circ U_2)(V)$ as $\mathfrak{h}^{(1)}$ -modules. In other words, $U_1 \circ U_2 = \text{id}$ on objects.
- We also have $U_1 \circ U_2 = \text{id}$ on morphisms (because both functors U_1 and U_2 leave morphisms unchanged).

Therefore, $U_1 \circ U_2 = \text{id}$ is shown, qed.

(h) This is left to the reader.

4. The splitting of the filtration of $(\otimes \mathfrak{g}) \not\! / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$

4.1. Statement of the theorem

In this section we are going to show a certain strengthening of Theorem 2.1 under an additional condition:

Theorem 4.1. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra, and let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Assume that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion (but not necessarily as an \mathfrak{h} -module inclusion).

Let J be the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

of the k -algebra $\otimes \mathfrak{g}$.

As we know, \mathfrak{g} is a \mathfrak{g} -module, and thus also an \mathfrak{h} -module (by Definition 1.15). Let $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$. This \mathfrak{n} is an \mathfrak{h} -module (because both \mathfrak{g} and \mathfrak{h} are \mathfrak{h} -modules). **Assume that this \mathfrak{h} -module \mathfrak{n} is actually the restriction of some $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h} .**

Let $\pi : \mathfrak{g} \rightarrow \mathfrak{n}$ be the canonical projection with kernel \mathfrak{h} . Obviously, π is an \mathfrak{h} -module homomorphism. Thus, $\otimes \pi : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ is also an \mathfrak{h} -module homomorphism (according to Proposition 1.68).

We consider \mathfrak{h} as an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ by means of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow \otimes \mathfrak{g}$.

(a) Both J and $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$. Thus, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is an \mathfrak{h} -module. Let $\zeta : \otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ be the canonical projection. Then, ζ is an \mathfrak{h} -module homomorphism.

(b) For every $n \in \mathbb{N}$, let F_n be the \mathfrak{h} -submodule $\zeta(\mathfrak{g}^{\otimes \leq n})$ of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$. (That F_n indeed is an \mathfrak{h} -submodule was proven in Theorem 2.1 already.) Then, $(F_n)_{n \geq 0}$ is an \mathfrak{h} -module filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ and satisfies $F_n \cong \mathfrak{n}^{\otimes \leq n}$ as \mathfrak{h} -modules for every $n \in \mathbb{N}$.

(c) There exists an \mathfrak{h} -module isomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ such that for every $n \in \mathbb{N}$, the image of F_n under this isomorphism is $\mathfrak{n}^{\otimes \leq n}$.

(d) The filtration $(F_n)_{n \geq 0}$ of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is \mathfrak{h} -split.

This theorem provides the main ingredient of Lemma 3.9 in [2].

4.2. Preparations for the proof

Before we embark upon the proof of this fact (which will be shorter than that of Theorem 2.1, since we have already paved some of the way), let us fix a convention that we are going to use throughout the rest of Section 4:

Convention 4.2. (a) Throughout the whole Section 4, we are going to work under the conditions of Theorem 4.1.

(b) According to the conditions of Theorem 4.1, the \mathfrak{h} -module \mathfrak{n} is actually the restriction of some $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h} . Let us fix one such $(\mathfrak{g}, \mathfrak{h})$ -semimodule for the rest of Section 4. We are going to denote this $(\mathfrak{g}, \mathfrak{h})$ -semimodule by \mathfrak{n} . (This is allowed since this $(\mathfrak{g}, \mathfrak{h})$ -semimodule is equal to \mathfrak{n} as an \mathfrak{h} -module, because its restriction to \mathfrak{h} is \mathfrak{n}).

Since \mathfrak{n} is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, Proposition 3.59 (applied to $V = \mathfrak{n}$) yields that $\otimes \mathfrak{n}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semialgebra. In particular, $\otimes \mathfrak{n}$ is thus a $(\mathfrak{g}, \mathfrak{h})$ -semimodule.

We are going to use Convention 3.4 as liberally as we can. This means that when a is an element of \mathfrak{g} whereas v is an element of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule V (for example, V can be one of the $(\mathfrak{g}, \mathfrak{h})$ -semimodules \mathfrak{g} , \mathfrak{n} or $\otimes \mathfrak{n}$), we will denote by $a \rightharpoonup v$ the Lie action of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V applied to (a, v) .

4.3. Definitions and basic properties of γ

We will now construct a homomorphism from $\otimes \mathfrak{g}$ to $\otimes \mathfrak{n}$ which will later give rise to the required \mathfrak{h} -module isomorphism from F_n to $\mathfrak{n}^{\otimes \leq n}$:

Definition 4.3. Consider the situation of Theorem 4.1.

(a) Let us define a k -linear map $\gamma_p : \mathfrak{g}^{\otimes p} \rightarrow \otimes \mathfrak{n}$ for every $p \in \mathbb{N}$. We are going to define this map γ_p by induction over p :

Induction base: For $p = 0$, define the map $\gamma_p : \mathfrak{g}^{\otimes p} \rightarrow \otimes \mathfrak{n}$ by

$$(\gamma_0(\lambda) = \lambda \quad \text{for every } \lambda \in \mathfrak{g}^{\otimes 0}) \quad (102)$$

(this definition makes sense since $\mathfrak{g}^{\otimes 0} = k \subseteq \otimes \mathfrak{n}$).

Induction step: For any $p > 0$, we assume that the map $\gamma_{p-1} : \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes \mathfrak{n}$ is already defined, and now we define a map $\gamma_p : \mathfrak{g}^{\otimes p} \rightarrow \otimes \mathfrak{n}$ as follows: The map

$$\mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes \mathfrak{n}, \quad (u, U) \mapsto \pi(u) \cdot \gamma_{p-1}(U) + u \curvearrowright (\gamma_{p-1}(U))$$

³⁶ is k -bilinear (because the maps γ_{p-1} and π are k -linear and the Lie action of $\otimes \mathfrak{n}$ is k -bilinear). Thus, by the universal property of the tensor product, this map gives rise to a k -linear map $\mathfrak{g} \otimes \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes \mathfrak{n}$ which sends $u \otimes U$ to $\pi(u) \cdot \gamma_{p-1}(U) + u \curvearrowright (\gamma_{p-1}(U))$ for every $(u, U) \in \mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)}$. This k -linear map is going to be denoted by γ_p . It is a map from $\mathfrak{g}^{\otimes p}$ to $\otimes \mathfrak{n}$ because $\mathfrak{g} \otimes \mathfrak{g}^{\otimes(p-1)} = \mathfrak{g}^{\otimes p}$.

This completes the inductive definition of γ_p for every $p \in \mathbb{N}$.

(b) Now, we define a k -linear map $\gamma : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ as follows: The sum $\sum_{i \in \mathbb{N}} \gamma_i$ of the maps $\gamma_i : \mathfrak{g}^{\otimes i} \rightarrow \otimes \mathfrak{n}$ is a map from $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$ to $\otimes \mathfrak{n}$. Since $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i} = \otimes \mathfrak{g}$, the sum $\sum_{i \in \mathbb{N}} \gamma_i$ of the maps $\gamma_i : \mathfrak{g}^{\otimes i} \rightarrow \otimes \mathfrak{n}$ is thus a map from $\otimes \mathfrak{g}$ to $\otimes \mathfrak{n}$. Denote this map by γ .

Convention 4.4. Throughout the rest of Section 4, we are going to work in the situation of Definition 4.3. So, for example, when we refer to \mathfrak{h} , we mean the Lie subalgebra \mathfrak{h} of Theorem 4.1, and when we refer to γ , we mean the map γ of Definition 4.3.

Remark 4.5. (a) As a consequence of the inductive step in the definition of γ_p (in Definition 4.3 (a)), we know that for every $p > 0$, the map γ_p is the k -linear map $\mathfrak{g} \otimes \mathfrak{g}^{\otimes(p-1)} \rightarrow \otimes \mathfrak{n}$ which sends $u \otimes U$ to $\pi(u) \cdot \gamma_{p-1}(U) + u \curvearrowright (\gamma_{p-1}(U))$ for every $(u, U) \in \mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)}$. In other words,

$$\gamma_p(u \otimes U) = \pi(u) \cdot \gamma_{p-1}(U) + u \curvearrowright (\gamma_{p-1}(U)) \quad \text{for every } (u, U) \in \mathfrak{g} \times \mathfrak{g}^{\otimes(p-1)}. \quad (103)$$

(b) According to Definition 4.3 (b), the map $\gamma : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ is the sum $\sum_{i \in \mathbb{N}} \gamma_i$ of the maps $\gamma_i : \mathfrak{g}^{\otimes i} \rightarrow \otimes \mathfrak{n}$. Hence,

$$\gamma(T) = \left(\sum_{i \in \mathbb{N}} \gamma_i \right) (T) = \gamma_p(T) \quad \text{for every } p \in \mathbb{N} \text{ and every } T \in \mathfrak{g}^{\otimes p}. \quad (104)$$

(c) Every $\lambda \in k$ satisfies

$$\begin{aligned} \gamma(\lambda) &= \gamma_0(\lambda) && \text{(by (104) (applied to } p = 0 \text{ and } T = \lambda), \text{ because } \lambda \in k = \mathfrak{g}^{\otimes 0}) \\ &= \lambda && \text{(by (102)).} \end{aligned} \quad (105)$$

³⁶Here, according to Convention 4.2 (b), the term $u \curvearrowright (\gamma_{p-1}(U))$ denotes the Lie action of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule $\otimes \mathfrak{n}$, applied to $(u, \gamma_{p-1}(U))$.

This yields, in particular, that $\gamma(k) = k$.

(d) Every $u \in \mathfrak{g}$ satisfies

$$\begin{aligned}
\gamma(u) &= \gamma_1(u) && \text{(by (104) (applied to } p = 1 \text{ and } T = u\text{), because } u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}\text{)} \\
&= \gamma_1(u \otimes 1) && \text{(since } u = u \otimes 1 \text{ under the identification } \mathfrak{g} \cong \mathfrak{g} \otimes k\text{)} \\
&= \pi(u) \cdot \underbrace{\gamma_{1-1}(1)}_{=\gamma_0} + u \rightarrow \left(\underbrace{\gamma_{1-1}(1)}_{=\gamma_0} \right) \\
&\quad \text{(by (103), applied to } p = 1 \text{ and } U = 1\text{)} \\
&= \pi(u) \cdot \underbrace{\gamma_0(1)}_{\substack{=1 \text{ (by (102),} \\ \text{applied to } \lambda=1)}} + u \rightarrow \left(\underbrace{\gamma_0(1)}_{\substack{=1 \text{ (by (102),} \\ \text{applied to } \lambda=1)}} \right) \\
&= \pi(u) \cdot 1 + \underbrace{u \rightarrow 1}_{\substack{=0 \text{ (since the Lie} \\ \text{action of } k \text{ is 0)}}} = \pi(u) \cdot 1 + 0 = \pi(u). \tag{106}
\end{aligned}$$

This yields $\gamma(\mathfrak{g}) = \pi(\mathfrak{g}) = \mathfrak{n}$ (since π is a projection of \mathfrak{g} on \mathfrak{n}).

(e) We now know that $\gamma(k) = k$ and $\gamma(\mathfrak{g}) \subseteq \mathfrak{n}$. But in general, we cannot generalize this to $\gamma(\mathfrak{g}^{\otimes p}) \subseteq \mathfrak{n}^{\otimes p}$ for all $p \in \mathbb{N}$. However, Proposition 4.8 will give us a weaker result that is actually true.

The next proposition (a kind of analogue of Proposition 2.7 for γ instead of φ) generalizes (103) to arbitrary U :

Proposition 4.6. Every $u \in \mathfrak{g}$ and $U \in \otimes \mathfrak{g}$ satisfy

$$\gamma(u \cdot U) = \pi(u) \cdot \gamma(U) + u \rightarrow (\gamma(U)). \tag{107}$$

Proof of Proposition 4.6. Let $u \in \mathfrak{g}$ and $U \in \otimes \mathfrak{g}$ be arbitrary. Since $U \in \otimes \mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$, we can write U as a family $U = (U_i)_{i \in \mathbb{N}}$, where $(U_i \in \mathfrak{g}^{\otimes i}$ for every $i \in \mathbb{N}$). Taking into account that we consider $\mathfrak{g}^{\otimes p}$ as a k -submodule of $\bigoplus_{i \in \mathbb{N}} \mathfrak{g}^{\otimes i}$ for every $p \in \mathbb{N}$, we have

$$(U_i)_{i \in \mathbb{N}} = \sum_{i \in \mathbb{N}} U_i, \text{ and thus } U = (U_i)_{i \in \mathbb{N}} = \sum_{i \in \mathbb{N}} U_i.$$

On the other hand, every $i \in \mathbb{N}$ satisfies $\underbrace{u}_{\in \mathfrak{g}} \otimes \underbrace{U_i}_{\in \mathfrak{g}^{\otimes i}} \in \mathfrak{g} \otimes \mathfrak{g}^{\otimes i} = \mathfrak{g}^{\otimes(i+1)}$. Also, since

$\gamma = \sum_{i \in \mathbb{N}} \gamma_i = \sum_{p \in \mathbb{N}} \gamma_p$ is the sum of the maps $\gamma_p : \mathfrak{g}^{\otimes p} \rightarrow \otimes \mathfrak{n}$ for all $p \in \mathbb{N}$, we have

$$\gamma(v) = \gamma_i(v) \quad \text{for every } v \in \mathfrak{g}^{\otimes i}. \tag{108}$$

Similarly,

$$\gamma(v) = \gamma_{i+1}(v) \quad \text{for every } v \in \mathfrak{g}^{\otimes(i+1)}. \tag{109}$$

Every $i \in \mathbb{N}$ satisfies $\gamma(u \otimes U_i) = \gamma_{i+1}(u \otimes U_i)$ according to (109) (applied to $u \otimes U_i$ instead of v), because $u \otimes U_i \in \mathfrak{g}^{\otimes(i+1)}$. Every $i \in \mathbb{N}$ satisfies $\gamma(U_i) = \gamma_i(U_i)$ according

to (108) (applied to U_i instead of v), since $U_i \in \mathfrak{g}^{\otimes i}$. Now, $U = \sum_{i \in \mathbb{N}} U_i$ leads to

$$\begin{aligned}
& u \cdot U \\
&= u \cdot \sum_{i \in \mathbb{N}} U_i = \sum_{i \in \mathbb{N}} u \cdot U_i = \sum_{i \in \mathbb{N}} u \otimes U_i \\
&\quad \left(\begin{array}{c} \text{since every } i \in \mathbb{N} \text{ satisfies } u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1} \text{ and } U_i \in \mathfrak{g}^{\otimes i} \text{ and therefore} \\ u \cdot U_i = u \otimes U_i \text{ (due to (31), applied to } u, U_i, 1 \text{ and } i \text{ instead of } a, b, n \text{ and } m) \end{array} \right),
\end{aligned}$$

and therefore

$$\begin{aligned}
\gamma(u \cdot U) &= \gamma \left(\sum_{i \in \mathbb{N}} u \otimes U_i \right) = \sum_{i \in \mathbb{N}} \underbrace{\gamma(u \otimes U_i)}_{=\gamma_{i+1}(u \otimes U_i)} \quad (\text{since } \gamma \text{ is } k\text{-linear}) \\
&= \sum_{i \in \mathbb{N}} \gamma_{i+1}(u \otimes U_i) = \sum_{i \in \mathbb{N}} \left(\pi(u) \cdot \underbrace{\gamma_{(i+1)-1}(U_i)}_{=\gamma_i} + u \rightarrow \left(\underbrace{\gamma_{(i+1)-1}(U_i)}_{=\gamma_i} \right) \right) \\
&\quad \left(\begin{array}{c} \text{since every } i \in \mathbb{N} \text{ satisfies} \\ \gamma_{i+1}(u \otimes U_i) = \pi(u) \cdot \gamma_{(i+1)-1}(U_i) + u \rightarrow (\gamma_{(i+1)-1}(U_i)) \\ \text{according to (103) (applied to } i+1 \text{ and } U_i \text{ instead of } p \text{ and } U) \end{array} \right) \\
&= \sum_{i \in \mathbb{N}} (\pi(u) \cdot \gamma_i(U_i) + u \rightarrow (\gamma_i(U_i))) = \sum_{i \in \mathbb{N}} \pi(u) \cdot \underbrace{\gamma_i(U_i)}_{=\gamma(U_i)} + \sum_{i \in \mathbb{N}} u \rightarrow \left(\underbrace{\gamma_i(U_i)}_{=\gamma(U_i)} \right) \\
&= \sum_{i \in \mathbb{N}} \pi(u) \cdot \gamma(U_i) + \sum_{i \in \mathbb{N}} u \rightarrow (\gamma(U_i)).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{i \in \mathbb{N}} \pi(u) \cdot \gamma(U_i) &= \pi(u) \cdot \underbrace{\sum_{i \in \mathbb{N}} \gamma(U_i)}_{=\gamma\left(\sum_{i \in \mathbb{N}} U_i\right)} = \pi(u) \cdot \gamma \left(\underbrace{\sum_{i \in \mathbb{N}} U_i}_{=U} \right) = \pi(u) \cdot \gamma(U) \\
&\quad (\text{since } \gamma \text{ is } k\text{-linear})
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i \in \mathbb{N}} u \rightarrow (\gamma(U_i)) &= u \rightarrow \underbrace{\left(\sum_{i \in \mathbb{N}} \gamma(U_i) \right)}_{=\gamma\left(\sum_{i \in \mathbb{N}} U_i\right)} \quad (\text{since the Lie action of } \otimes \mathfrak{n} \text{ is linear}) \\
&\quad (\text{since } \gamma \text{ is } k\text{-linear}) \\
&= u \rightarrow \left(\gamma \left(\underbrace{\sum_{i \in \mathbb{N}} U_i}_{=U} \right) \right) = u \rightarrow (\gamma(U)),
\end{aligned}$$

this becomes

$$\gamma(u \cdot U) = \underbrace{\sum_{i \in \mathbb{N}} \pi(u) \cdot \gamma(U_i)}_{=\pi(u) \cdot \gamma(U)} + \underbrace{\sum_{i \in \mathbb{N}} u \rightharpoonup (\gamma(U_i))}_{=u \rightharpoonup (\gamma(U))} = \pi(u) \cdot \gamma(U) + u \rightharpoonup (\gamma(U)).$$

This proves Proposition 4.6.

Now we come to a corollary of Proposition 4.6 which is similar to Proposition 2.8 (note that an analogue to Corollary 2.9 does not seem to exist):

Corollary 4.7. Every $u \in \mathfrak{h}$ and $U \in \otimes \mathfrak{g}$ satisfy

$$\gamma(u \cdot U) = u \rightharpoonup (\gamma(U)). \quad (110)$$

Proof of Corollary 4.7. Since $u \in \mathfrak{h}$, we have $\pi(u) = 0$ (since π is a projection with kernel \mathfrak{h}). Thus, (107) yields

$$\gamma(u \cdot U) = \underbrace{\pi(u)}_{=0} \cdot \gamma(U) + u \rightharpoonup (\gamma(U)) = \underbrace{0 \cdot \gamma(U)}_{=0} + u \rightharpoonup (\gamma(U)) = u \rightharpoonup (\gamma(U)).$$

This proves Corollary 4.7.

Here is an analogue of Proposition 2.12:

Proposition 4.8. The map $\gamma : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ respects the filtration. Here, the filtration on $\otimes \mathfrak{g}$ is the degree filtration $(\mathfrak{g}^{\otimes \leq n})_{n \geq 0}$, and the filtration on $\otimes \mathfrak{n}$ is the degree filtration $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$.

Proof of Proposition 4.8. Let us use Convention 2.10.

We are going to show that

$$\gamma(\mathfrak{g}^{\otimes p}) \subseteq \mathfrak{n}^{\otimes \leq p} \text{ for every } p \in \mathbb{N}. \quad (111)$$

Proof of (111). We will prove (111) by induction over p :

Induction base: We know that $\mathfrak{g}^{\otimes 0} = k$ and $\mathfrak{n}^{\otimes \leq 0} = \bigoplus_{i=0}^0 \mathfrak{n}^{\otimes i} = \mathfrak{n}^{\otimes 0} = k$, and we know (from Remark 4.5 (c)) that $\gamma(k) = k$. Thus, $\mathfrak{g}^{\otimes 0} = k$ yields $\gamma(\mathfrak{g}^{\otimes 0}) = \gamma(k) = k = \mathfrak{n}^{\otimes \leq 0}$. In other words, (111) holds for $p = 0$. This completes the induction base.

Induction step: Let $q \in \mathbb{N}_+$. Assume that (111) holds for $p = q - 1$. Now we must prove that (111) holds for $p = q$.

We have assumed that (111) holds for $p = q - 1$. In other words, $\gamma(\mathfrak{g}^{\otimes (q-1)}) \subseteq \mathfrak{n}^{\otimes \leq (q-1)}$.

Now we are going to prove that every $U \in \gamma(\mathfrak{g}_{\text{lind}}^{\otimes q})$ satisfies $U \in \mathfrak{n}^{\otimes \leq q}$.

In fact, let $U \in \gamma(\mathfrak{g}_{\text{lind}}^{\otimes q})$ be arbitrary. Then, there exists a $U' \in \mathfrak{g}_{\text{lind}}^{\otimes q}$ such that $U = \gamma(U')$ (since $U \in \gamma(\mathfrak{g}_{\text{lind}}^{\otimes q})$). Consider this U' . This U' is a left-induced tensor in $\mathfrak{g}^{\otimes q}$ (since $U' \in \mathfrak{g}_{\text{lind}}^{\otimes q}$); this means that there exist some $v \in \mathfrak{g}$ and $T \in \mathfrak{g}^{\otimes (q-1)}$ such that $U' = v \otimes T$. Consider these v and T . Since $v \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $T \in \mathfrak{g}^{\otimes (q-1)}$, we have $v \cdot T = v \otimes T$ (due to (31), applied to $v, T, 1$ and $q - 1$ instead of a, b, n and m).

We notice that $\gamma \left(\underbrace{T}_{\in \mathfrak{g}^{\otimes(q-1)}} \right) \in \gamma(\mathfrak{g}^{\otimes(q-1)}) \subseteq \mathfrak{n}^{\otimes \leq(q-1)}$, and thus $v \mapsto (\gamma(T)) \in \mathfrak{n}^{\otimes \leq(q-1)}$ (since $\mathfrak{n}^{\otimes \leq(q-1)}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, which is because \mathfrak{n} is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule).
Now,

$$\begin{aligned} U &= \gamma \left(\underbrace{U'}_{=v \otimes T = v \cdot T} \right) = \gamma(v \cdot T) = \underbrace{\pi(v)}_{\in \mathfrak{n}^{\otimes 1} \subseteq \mathfrak{n}^{\otimes \leq 1}} \cdot \gamma \left(\underbrace{T}_{\in \mathfrak{g}^{\otimes(q-1)}} \right) + \underbrace{v \mapsto (\gamma(T))}_{\in \mathfrak{n}^{\otimes \leq(q-1)}} \\ &\quad \text{(by (107), applied to } v \text{ and } T \text{ instead of } u \text{ and } U) \\ &\in \mathfrak{n}^{\otimes \leq 1} \cdot \underbrace{\gamma(\mathfrak{g}^{\otimes(q-1)})}_{\subseteq \mathfrak{n}^{\otimes \leq(q-1)}} + \underbrace{\mathfrak{n}^{\otimes \leq(q-1)}}_{\subseteq \mathfrak{n}^{\otimes \leq q}} \subseteq \mathfrak{n}^{\otimes \leq 1} \cdot \mathfrak{n}^{\otimes \leq(q-1)} + \mathfrak{n}^{\otimes \leq q}. \end{aligned}$$

(since $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$ is a filtration)

Since Proposition 1.95 (b) (applied to \mathfrak{n} , 1 and $q-1$ instead of V , n and m) yields $\mathfrak{n}^{\otimes \leq 1} \cdot \mathfrak{n}^{\otimes \leq(q-1)} \subseteq \mathfrak{n}^{\otimes \leq(1+(q-1))} = \mathfrak{n}^{\otimes \leq q}$, this becomes

$$U \in \underbrace{\mathfrak{n}^{\otimes \leq 1} \cdot \mathfrak{n}^{\otimes \leq(q-1)}}_{\subseteq \mathfrak{n}^{\otimes \leq q}} + \mathfrak{n}^{\otimes \leq q} \subseteq \mathfrak{n}^{\otimes \leq q} + \mathfrak{n}^{\otimes \leq q} \subseteq \mathfrak{n}^{\otimes \leq q} \quad (\text{since } \mathfrak{n}^{\otimes \leq q} \text{ is a } k\text{-module}).$$

We have thus proven that every $U \in \gamma(\mathfrak{g}_{\text{lind}}^{\otimes q})$ satisfies $U \in \mathfrak{n}^{\otimes \leq q}$. In other words, $\gamma(\mathfrak{g}_{\text{lind}}^{\otimes q}) \subseteq \mathfrak{n}^{\otimes \leq q}$.

Now, Proposition 2.11 (applied to q and \mathfrak{g} instead of p and V) says $\mathfrak{g}^{\otimes q} = \langle \mathfrak{g}_{\text{lind}}^{\otimes q} \rangle$, and thus

$$\begin{aligned} \gamma(\mathfrak{g}^{\otimes q}) &= \gamma(\langle \mathfrak{g}_{\text{lind}}^{\otimes q} \rangle) = \langle \gamma(\mathfrak{g}_{\text{lind}}^{\otimes q}) \rangle \quad \left(\begin{array}{l} \text{by Proposition 1.29 (b) (applied to} \\ \mathfrak{g}^{\otimes q}, \mathfrak{g}_{\text{lind}}^{\otimes q}, \otimes \mathfrak{n} \text{ and } \gamma \text{ instead of } M, S, R \text{ and } f) \end{array} \right) \\ &\subseteq \mathfrak{n}^{\otimes \leq q} \quad \left(\begin{array}{l} \text{by Proposition 1.29 (a) (applied to } \otimes \mathfrak{n}, \gamma(\mathfrak{g}_{\text{lind}}^{\otimes q}) \text{ and } \mathfrak{n}^{\otimes \leq q} \\ \text{instead of } M, S \text{ and } Q), \text{ since } \gamma(\mathfrak{g}_{\text{lind}}^{\otimes q}) \subseteq \mathfrak{n}^{\otimes \leq q} \end{array} \right). \end{aligned}$$

In other words, (111) holds for $p = q$. This completes the induction step, and thus the induction proof of (111) is done.

Now, let $m \in \mathbb{N}$. The definition of $\mathfrak{g}^{\otimes \leq m}$ says $\mathfrak{g}^{\otimes \leq m} = \bigoplus_{i=0}^m \mathfrak{g}^{\otimes i}$, so that $\mathfrak{g}^{\otimes \leq m} = \bigoplus_{i=0}^m \mathfrak{g}^{\otimes i}$ (since direct sums are sums) and thus

$$\begin{aligned} \gamma(\mathfrak{g}^{\otimes \leq m}) &= \gamma \left(\sum_{i=0}^m \mathfrak{g}^{\otimes i} \right) = \sum_{i=0}^m \underbrace{\gamma(\mathfrak{g}^{\otimes i})}_{\subseteq \mathfrak{n}^{\otimes \leq i}} \quad (\text{since } \gamma \text{ is } k\text{-linear}) \\ &\quad \text{(by (111), applied to } i \text{ instead of } p) \\ &\subseteq \sum_{i=0}^m \underbrace{\mathfrak{n}^{\otimes \leq i}}_{\subseteq \mathfrak{n}^{\otimes \leq m} \text{ (since } i \leq m \text{ and since } (\mathfrak{n}^{\otimes \leq n})_{n \geq 0} \text{ is a filtration)}} \subseteq \sum_{i=0}^m \mathfrak{n}^{\otimes \leq m} \subseteq \mathfrak{n}^{\otimes \leq m} \quad (\text{since } \mathfrak{n}^{\otimes \leq m} \text{ is a } k\text{-module}). \end{aligned}$$

Since this holds for every $m \in \mathbb{N}$, this yields that γ respects the filtration. This proves Proposition 4.8.

4.4. γ is an \mathfrak{h} -module map

Next we will show a property of γ which distinguishes it from the map φ of Section 2:

Proposition 4.9. The map $\gamma : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ is an \mathfrak{h} -module homomorphism.

Proof of Proposition 4.9. We know that $\otimes \mathfrak{n}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (because \mathfrak{n} is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule). Thus, (according to Definition 3.1) the relation (79) with V replaced by $\otimes \mathfrak{n}$ holds.

Let $a \in \mathfrak{h}$. Then, the map $\otimes \mathfrak{g} \rightarrow \otimes \mathfrak{g}$, $T \mapsto a \rightharpoonup T$ is k -linear (because the Lie action of $\otimes \mathfrak{g}$ is k -bilinear), and the map $\otimes \mathfrak{n} \rightarrow \otimes \mathfrak{n}$, $T \mapsto a \rightharpoonup T$ is k -linear (because the Lie action of $\otimes \mathfrak{n}$ is k -bilinear).

Let $Z : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ be the map defined by

$$(Z(T) = a \rightharpoonup (\gamma(T)) - \gamma(a \rightharpoonup T)) \quad \text{for every } T \in \otimes \mathfrak{g}. \quad (112)$$

This map Z is k -linear (since the map $\otimes \mathfrak{g} \rightarrow \otimes \mathfrak{g}$, $T \mapsto a \rightharpoonup T$, the map $\otimes \mathfrak{n} \rightarrow \otimes \mathfrak{n}$, $T \mapsto a \rightharpoonup T$ and the map γ are all k -linear). Thus, $\text{Ker } Z$ is a k -submodule of $\otimes \mathfrak{g}$.

Now we are going to prove that

$$\mathfrak{g}^{\otimes n} \subseteq \text{Ker } Z \quad \text{for every } n \in \mathbb{N}. \quad (113)$$

Proof of (113). We are going to prove (113) by induction over n :

Induction base: Every $\lambda \in \mathfrak{g}^{\otimes \leq 0}$ satisfies $\lambda \in k$ (since $\mathfrak{g}^{\otimes \leq 0} = \bigoplus_{i=0}^0 \mathfrak{g}^{\otimes i} = \mathfrak{g}^{\otimes 0} = k$).

Therefore, every $\lambda \in \mathfrak{g}^{\otimes \leq 0}$ satisfies $\gamma(\lambda) = \lambda$ (according to (105), since $\lambda \in k$) and thus

$$\begin{aligned} Z(\lambda) &= a \rightharpoonup \left(\underbrace{\gamma(\lambda)}_{=\lambda} \right) - \gamma(a \rightharpoonup \lambda) \quad (\text{by (112), applied to } T = \lambda) \\ &= \underbrace{a \rightharpoonup \lambda}_{=0 \text{ (since the Lie action of } k \text{ is 0)}} - \gamma \left(\underbrace{a \rightharpoonup \lambda}_{=0 \text{ (since the Lie action of } k \text{ is 0)}} \right) = 0 - \underbrace{\gamma(0)}_{=0 \text{ (since } \gamma \text{ is } k\text{-linear)}} = 0 - 0 = 0. \end{aligned}$$

We have thus shown that every $\lambda \in \mathfrak{g}^{\otimes \leq 0}$ satisfies $Z(\lambda) = 0$. In other words, every $\lambda \in \mathfrak{g}^{\otimes \leq 0}$ satisfies $\lambda \in \text{Ker } Z$. Hence, $\mathfrak{g}^{\otimes \leq 0} \subseteq \text{Ker } Z$. In other words, (113) holds for $n = 0$. This completes the induction base.

Induction step: Let $p \in \mathbb{N}$. Assume that (113) holds for $n = p$. We now must show that (113) also holds for $n = p + 1$.

Since (113) holds for $n = p$, we have $\mathfrak{g}^{\otimes p} \subseteq \text{Ker } Z$.

We are now going to show that $\mathfrak{g}^{\otimes(p+1)} \subseteq \text{Ker } Z$.

Proposition 2.11 (applied to $p+1$ and \mathfrak{g} instead of p and V) yields $\mathfrak{g}^{\otimes(p+1)} = \left\langle \mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \right\rangle$ (where we are using Convention 2.10).

Now we are going to prove that $\mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \subseteq \text{Ker } Z$. Indeed, let $V \in \mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$ be arbitrary. Then, V is a left-induced tensor in $\mathfrak{g}^{\otimes(p+1)}$ (since $V \in \mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$), and thus there exist $u \in \mathfrak{g}$ and $U \in \mathfrak{g}^{\otimes p}$ such that $V = u \otimes U$. Consider these u and U . Since $u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $U \in \mathfrak{g}^{\otimes p}$, we have $u \cdot U = u \otimes U$ (due to (31), applied to $u, U, 1$ and p instead of a, b, n and m).

Since $U \in \mathfrak{g}^{\otimes p} \subseteq \text{Ker } Z$, we have $Z(U) = 0$. But applying (112) to $T = U$, we obtain $Z(U) = a \rightarrow (\gamma(U)) - \gamma(a \rightarrow U)$. Thus, $a \rightarrow (\gamma(U)) - \gamma(a \rightarrow U) = Z(U) = 0$, so that $a \rightarrow (\gamma(U)) = \gamma(a \rightarrow U)$.

We know that the relation (79) with V replaced by $\otimes \mathfrak{n}$ holds. Thus, we can apply this relation (79) to u , $\gamma(U)$ and $\otimes \mathfrak{n}$ instead of b , v and V , and thus we obtain

$$[a, u] \rightarrow (\gamma(U)) = a \rightarrow (u \rightarrow (\gamma(U))) - u \rightarrow (a \rightarrow (\gamma(U))).$$

Now, $V = u \otimes U = u \cdot U$, so that

$$a \rightarrow V = a \rightarrow (u \cdot U) = (a \rightarrow u) \cdot U + u \cdot (a \rightarrow U) \\ \left(\begin{array}{l} \text{by (33) (applied to } \otimes \mathfrak{g} \text{ and } U \text{ instead of } A \text{ and } v), \\ \text{because } \otimes \mathfrak{g} \text{ is a } \mathfrak{g}\text{-algebra} \end{array} \right).$$

Hence,

$$\begin{aligned} & \gamma(a \rightarrow V) \\ &= \gamma((a \rightarrow u) \cdot U + u \cdot (a \rightarrow U)) \\ &= \underbrace{\gamma((a \rightarrow u) \cdot U)}_{\substack{=\pi(a \rightarrow u) \cdot \gamma(U) + (a \rightarrow u) \rightarrow (\gamma(U)) \\ \text{(by (107), applied to } a \rightarrow u \text{ instead of } u)}} + \underbrace{\gamma(u \cdot (a \rightarrow U))}_{\substack{=\pi(u) \cdot \gamma(a \rightarrow U) + u \rightarrow (\gamma(a \rightarrow U)) \\ \text{(by (107), applied to } a \rightarrow U \text{ instead of } U)}} \quad (\text{since } \gamma \text{ is linear}) \\ &= \underbrace{\pi(a \rightarrow u)}_{\substack{=a \rightarrow (\pi(u)) \\ \text{(since } \pi \text{ is an } \mathfrak{h}\text{-module} \\ \text{homomorphism, while } a \in \mathfrak{h})}} \cdot \gamma(U) + \underbrace{(a \rightarrow u)}_{\substack{=[a, u] \\ \text{(due to (9), applied to} \\ \text{ } a \text{ and } u \text{ instead of } v \text{ and } w)}} \rightarrow (\gamma(U)) \\ & \quad + \pi(u) \cdot \gamma(a \rightarrow U) + u \rightarrow \underbrace{(\gamma(a \rightarrow U))}_{=a \rightarrow (\gamma(U))} \\ &= (a \rightarrow (\pi(u))) \cdot \gamma(U) + [a, u] \rightarrow (\gamma(U)) + \pi(u) \cdot \gamma(a \rightarrow U) + u \rightarrow (a \rightarrow (\gamma(U))). \end{aligned} \tag{114}$$

On the other hand, $V = u \cdot U$ leads to

$$\begin{aligned} a \rightarrow (\gamma(V)) &= a \rightarrow (\gamma(u \cdot U)) = a \rightarrow (\pi(u) \cdot \gamma(U) + u \rightarrow (\gamma(U))) \quad (\text{by (107)}) \\ &= \underbrace{a \rightarrow (\pi(u) \cdot \gamma(U))}_{\substack{=(a \rightarrow (\pi(u))) \cdot \gamma(U) + \pi(u) \cdot (a \rightarrow (\gamma(U))) \\ \text{(by (107), applied to } \pi(u) \text{ and } \gamma(U) \text{ instead of } u \text{ and } v)}} + a \rightarrow (u \rightarrow (\gamma(U))) \\ & \quad (\text{since the Lie action of } \otimes \mathfrak{n} \text{ is } k\text{-bilinear}) \\ &= (a \rightarrow (\pi(u))) \cdot \gamma(U) + \pi(u) \cdot \underbrace{(a \rightarrow (\gamma(U)))}_{= \gamma(a \rightarrow U)} + a \rightarrow (u \rightarrow (\gamma(U))) \\ &= (a \rightarrow (\pi(u))) \cdot \gamma(U) + \pi(u) \cdot \gamma(a \rightarrow U) + a \rightarrow (u \rightarrow (\gamma(U))). \end{aligned}$$

Subtracting (114) from this equation, we obtain

$$\begin{aligned}
& a \rightarrow (\gamma(V)) - \gamma(a \rightarrow V) \\
&= ((a \rightarrow (\pi(u))) \cdot \gamma(U) + \pi(u) \cdot \gamma(a \rightarrow U) + a \rightarrow (u \rightarrow (\gamma(U)))) \\
&\quad - ((a \rightarrow (\pi(u))) \cdot \gamma(U) + [a, u] \rightarrow (\gamma(U)) \\
&\quad\quad + \pi(u) \cdot \gamma(a \rightarrow U) + u \rightarrow (a \rightarrow (\gamma(U)))) \\
&= a \rightarrow (u \rightarrow (\gamma(U))) - u \rightarrow (a \rightarrow (\gamma(U))) - \underbrace{[a, u] \rightarrow (\gamma(U))}_{=a \rightarrow (u \rightarrow (\gamma(U))) - u \rightarrow (a \rightarrow (\gamma(U)))} \\
&\quad \text{(after some cancellation of terms)} \\
&= (a \rightarrow (u \rightarrow (\gamma(U))) - u \rightarrow (a \rightarrow (\gamma(U)))) - (a \rightarrow (u \rightarrow (\gamma(U))) - u \rightarrow (a \rightarrow (\gamma(U)))) \\
&= 0.
\end{aligned}$$

But (112) (applied to $T = V$) yields

$$Z(V) = a \rightarrow (\gamma(V)) - \gamma(a \rightarrow V) = 0.$$

Thus, $V \in \text{Ker } Z$.

We have thus shown that every $V \in \mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$ satisfies $V \in \text{Ker } Z$. In other words, $\mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \subseteq \text{Ker } Z$.

Consequently, Proposition 1.29 (a) (applied to $\otimes \mathfrak{g}$, $\mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$ and $\text{Ker } Z$ instead of M , S and Q) yields $\langle \mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \rangle \subseteq \text{Ker } Z$. Since $\mathfrak{g}^{\otimes(p+1)} = \langle \mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \rangle$, this becomes $\mathfrak{g}^{\otimes(p+1)} \subseteq \text{Ker } Z$. In other words, (113) holds for $n = p + 1$. This completes the induction step, and thus the induction proof of (113) is done.

Now that (113) is proven, we notice that

$$\begin{aligned}
\otimes \mathfrak{g} &= \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n} = \sum_{n \in \mathbb{N}} \underbrace{\mathfrak{g}^{\otimes n}}_{\substack{\subseteq \text{Ker } Z \\ \text{(by (113))}}} \quad \text{(since direct sums are sums)} \\
&\subseteq \sum_{n \in \mathbb{N}} \text{Ker } Z \subseteq \text{Ker } Z \quad \text{(since Ker } Z \text{ is a } k\text{-module)}.
\end{aligned}$$

In other words, $Z = 0$. Thus, $Z(T) = 0$ for every $T \in \otimes \mathfrak{g}$. Using (112), this rewrites as

$$a \rightarrow (\gamma(T)) - \gamma(a \rightarrow T) = 0 \quad \text{for every } T \in \otimes \mathfrak{g}.$$

In other words, $\gamma(a \rightarrow T) = a \rightarrow (\gamma(T))$ for every $T \in \otimes \mathfrak{g}$. Since this holds for every $a \in \mathfrak{h}$, this shows that γ is an \mathfrak{h} -module homomorphism. This proves Proposition 4.9.

4.5. A lemma on γ and k -submodules of $\otimes \mathfrak{g}$

We now set forth for a proof of $\gamma(J) = 0$ and $\gamma((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$. The first step will be a "little brother" of Lemma 2.15 (but notice that it has a weaker condition than Lemma 2.15):

Lemma 4.10. Let C be a k -submodule of $\otimes \mathfrak{g}$ satisfying $\gamma(C) = 0$. Then, $\gamma((\otimes \mathfrak{g}) \cdot C) = 0$.

Proof of Lemma 4.10. We are going to prove that

$$\gamma(\mathfrak{g}^{\otimes p} \cdot C) = 0 \quad \text{for every } p \in \mathbb{N}. \quad (115)$$

Proof of (115). We will prove (115) by induction over p :

Induction base: We have $\mathfrak{g}^{\otimes 0} = k$ and thus $\mathfrak{g}^{\otimes 0} \cdot C = k \cdot C = C$ (since C is a k -submodule), so that $\gamma(\mathfrak{g}^{\otimes 0} \cdot C) = \gamma(C) = 0$. Thus, (115) holds for $p = 0$. This completes the induction base.

Induction step: Let $q \in \mathbb{N}_+$. Assume that (115) holds for $p = q - 1$. Now we must prove that (115) holds for $p = q$.

We have assumed that (115) holds for $p = q - 1$. In other words, $\gamma(\mathfrak{g}^{\otimes(q-1)} \cdot C) = 0$.

Let $D = \mathfrak{g}^{\otimes(q-1)} \cdot C$. Then, $D = \mathfrak{g}^{\otimes(q-1)} \cdot C$ yields $\gamma(D) = \gamma(\mathfrak{g}^{\otimes(q-1)} \cdot C) = 0$.

On the other hand, Proposition 1.95 (a) (applied to $V = \mathfrak{g}$, $i = 1$ and $j = q - 1$) yields $\mathfrak{g}^{\otimes 1} \cdot \mathfrak{g}^{\otimes(q-1)} = \mathfrak{g}^{\otimes(1+(q-1))} = \mathfrak{g}^{\otimes q}$. Thus,

$$\underbrace{\mathfrak{g}^{\otimes q}}_{=\mathfrak{g}^{\otimes 1} \cdot \mathfrak{g}^{\otimes(q-1)}} \cdot C = \underbrace{\mathfrak{g}^{\otimes 1}}_{=\mathfrak{g}} \cdot \underbrace{\mathfrak{g}^{\otimes(q-1)} \cdot C}_{=D} = \mathfrak{g} \cdot D.$$

We are now in the following situation: We know that D is a k -submodule of $\otimes \mathfrak{g}$ satisfying $\gamma(D) = 0$. We want to prove that $\gamma(\mathfrak{g} \cdot D) = 0$.

Let E be the subset $\{u \cdot U \mid (u, U) \in \mathfrak{g} \times D\}$ of $\otimes \mathfrak{g}$. Then,

$$\mathfrak{g} \cdot D = \langle u \cdot U \mid (u, U) \in \mathfrak{g} \times D \rangle = \left\langle \underbrace{\{u \cdot U \mid (u, U) \in \mathfrak{g} \times D\}}_{=E} \right\rangle = \langle E \rangle.$$

Now we are going to show that $E \subseteq \text{Ker } \gamma$.

In fact, let $T \in E$ be arbitrary. Then, $T \in E = \{u \cdot U \mid (u, U) \in \mathfrak{g} \times D\}$, so that there exists some $(u, U) \in \mathfrak{g} \times D$ such that $T = u \cdot U$. Consider this (u, U) . Clearly, $u \in \mathfrak{g}$ and $U \in D$. Obviously $U \in D$ leads to $\gamma(U) = 0$ (since $\gamma(D) = 0$). Now, (107) yields

$$\gamma(u \cdot U) = \underbrace{\pi(u)}_{=0} \cdot \underbrace{\gamma(U)}_{=0} + u \cdot \underbrace{\gamma(U)}_{=0} = \underbrace{\pi(u)}_{=0} \cdot 0 + \underbrace{u \cdot 0}_{=0 \text{ (since the Lie action of } \otimes \mathfrak{n} \text{ is } k\text{-bilinear)}} = 0 + 0 = 0.$$

Since $u \cdot U = T$, this rewrites as $\gamma(T) = 0$. Thus, $T \in \text{Ker } \gamma$.

We have therefore shown that every $T \in E$ satisfies $T \in \text{Ker } \gamma$. Thus, $E \subseteq \text{Ker } \gamma$.

Consequently, Proposition 1.29 (a) (applied to $\otimes \mathfrak{g}$, E and $\text{Ker } \gamma$ instead of M , S and Q) yields $\langle E \rangle \subseteq \text{Ker } \gamma$ (since $\text{Ker } \gamma$ is a k -submodule of $\otimes \mathfrak{g}$, since γ is k -linear). Since $\langle E \rangle = \mathfrak{g} \cdot D$, this rewrites as $\mathfrak{g} \cdot D \subseteq \text{Ker } \gamma$, and thus $\gamma(\mathfrak{g} \cdot D) = 0$. Since $\mathfrak{g}^{\otimes q} \cdot C = \mathfrak{g} \cdot D$, this becomes $\gamma(\mathfrak{g}^{\otimes q} \cdot C) = 0$. In other words, (115) holds for $p = q$. This completes the induction step, and thus the induction proof of (115) is done.

Now, $\otimes \mathfrak{g} = \bigoplus_{p \in \mathbb{N}} \mathfrak{g}^{\otimes p} = \sum_{p \in \mathbb{N}} \mathfrak{g}^{\otimes p}$ (since direct sums are sums) yields $(\otimes \mathfrak{g}) \cdot C =$

$\left(\sum_{p \in \mathbb{N}} \mathfrak{g}^{\otimes p}\right) \cdot C = \sum_{p \in \mathbb{N}} (\mathfrak{g}^{\otimes p} \cdot C)$ and thus

$$\begin{aligned} \gamma((\otimes \mathfrak{g}) \cdot C) &= \gamma\left(\sum_{p \in \mathbb{N}} (\mathfrak{g}^{\otimes p} \cdot C)\right) = \sum_{p \in \mathbb{N}} \underbrace{\gamma(\mathfrak{g}^{\otimes p} \cdot C)}_{=0 \text{ (due to (115))}} \quad (\text{since } \gamma \text{ is } k\text{-linear}) \\ &= \sum_{p \in \mathbb{N}} 0 = 0. \end{aligned}$$

This proves Lemma 4.10.

4.6. $\gamma(J) = 0$ and $\gamma((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$

We are now ready to prove the following facts (which are similar to Propositions 2.16 and 2.17, respectively):

■ **Proposition 4.11.** We have $\gamma(J) = 0$.

■ **Proposition 4.12.** We have $\gamma((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$.

Proof of Proposition 4.11. Consider the k -submodule J_0 defined in Proposition 2.3 (b).

We are going to prove that $\gamma(J_0 \cdot (\otimes \mathfrak{g})) = 0$ now (this will quickly yield $\gamma(J) = 0$ then, due to $J = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g})$ and Lemma 4.10).

We know that $\otimes \mathfrak{n}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. Therefore, (according to Definition 3.1) the relation (79) with V replaced by $\otimes \mathfrak{n}$ holds. In other words, we have

$$([a, b] \rightharpoonup v = a \rightharpoonup (b \rightharpoonup v) - b \rightharpoonup (a \rightharpoonup v) \text{ for every } a \in \mathfrak{h}, b \in \mathfrak{g} \text{ and } v \in \otimes \mathfrak{n}). \quad (116)$$

Let S_0 denote the subset $\{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}$ of $\otimes \mathfrak{g}$. Then, we know from the definition of J_0 that

$$\begin{aligned} J_0 &= \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \\ &= \left\langle \underbrace{\{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}}_{=S_0} \right\rangle = \langle S_0 \rangle. \end{aligned}$$

Now, let $T \in \otimes \mathfrak{g}$ be arbitrary. Let $\varrho'_T : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ be the map defined by

$$(\varrho'_T(U) = \gamma(U \cdot T) \quad \text{for every } U \in \otimes \mathfrak{g}).$$

This map ϱ'_T is k -linear (since it is the composition of the two k -linear maps γ and $\otimes \mathfrak{g} \rightarrow \otimes \mathfrak{g}, U \mapsto U \cdot T$). Thus, $\text{Ker } \varrho'_T$ is a k -submodule of $\otimes \mathfrak{g}$.

We are now going to prove that every $s \in S_0$ satisfy $s \in \text{Ker } \varrho'_T$.

In fact, let $s \in S_0$ be arbitrary. Since $s \in S_0 = \{v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h}\}$, there exists some $(v, w) \in \mathfrak{g} \times \mathfrak{h}$ such that $s = v \otimes w - w \otimes v - [v, w]$. So we have $v \in \mathfrak{g}$ and $w \in \mathfrak{h}$. Note that $v \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $w \in \mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ yield $v \cdot w = v \otimes w$ (by

(31), applied to 1 and 1 instead of n and m), and similarly $w \cdot v = w \otimes v$. Using these observations, we see that

$$s = \underbrace{v \otimes w}_{=v \cdot w} - \underbrace{w \otimes v}_{=w \cdot v} - [v, w] = v \cdot w - w \cdot v - [v, w].$$

Now, we are going to prove that $\varrho'_T(s) = 0$. In order to do this, we will compute $\varrho'_T(v \cdot w)$, $\varrho'_T(w \cdot v)$ and $\varrho'_T([v, w])$. First let us compute $\gamma(v \cdot T)$ and $\gamma(w \cdot T)$.

We have $\gamma(v \cdot T) = \pi(v) \cdot \gamma(T) + v \rightharpoonup (\gamma(T))$ (according to (107), applied to $u = v$ and $U = T$) and $\gamma(w \cdot T) = w \rightharpoonup (\gamma(T))$ (according to (110), applied to $u = w$ and $U = T$).

The definition of ϱ'_T yields

$$\begin{aligned} \varrho'_T(v \cdot w) &= \gamma(v \cdot w \cdot T) = \pi(v) \cdot \underbrace{\gamma(w \cdot T)}_{=w \rightharpoonup (\gamma(T))} + v \rightharpoonup \left(\underbrace{\gamma(w \cdot T)}_{=w \rightharpoonup (\gamma(T))} \right) \\ &\quad \text{(according to (107), applied to } u = v \text{ and } U = w \cdot T) \\ &= \pi(v) \cdot (w \rightharpoonup (\gamma(T))) + v \rightharpoonup (w \rightharpoonup (\gamma(T))). \end{aligned} \quad (117)$$

The definition of ϱ'_T yields

$$\begin{aligned} \varrho'_T(w \cdot v) &= \gamma(w \cdot v \cdot T) = w \rightharpoonup \left(\underbrace{\gamma(v \cdot T)}_{=\pi(v) \cdot \gamma(T) + v \rightharpoonup (\gamma(T))} \right) \\ &\quad \text{(by (110), applied to } u = w \text{ and } U = v \cdot T) \\ &= w \rightharpoonup (\pi(v) \cdot \gamma(T) + v \rightharpoonup (\gamma(T))) \\ &= \underbrace{w \rightharpoonup (\pi(v) \cdot \gamma(T))}_{=\underbrace{(w \rightharpoonup (\pi(v))) \cdot \gamma(T)}_{\substack{\text{(by (88) (applied to } \otimes \mathfrak{n}, w, \pi(v) \text{ and } \gamma(T)) \\ \text{instead of } A, a, u \text{ and } v), \text{ because } \otimes \mathfrak{n} \text{ is a } (\mathfrak{g}, \mathfrak{h})\text{-semialgebra}}}} + w \rightharpoonup (v \rightharpoonup (\gamma(T))) \\ &\quad \text{(since the Lie action on } \otimes \mathfrak{n} \text{ is } k\text{-linear}) \\ &= (w \rightharpoonup (\pi(v))) \cdot \gamma(T) + \pi(v) \cdot (w \rightharpoonup (\gamma(T))) + w \rightharpoonup (v \rightharpoonup (\gamma(T))). \end{aligned} \quad (118)$$

Finally, the definition of ϱ'_T yields

$$\begin{aligned} \varrho'_T([v, w]) &= \gamma \left(\underbrace{[v, w]}_{\substack{=-[w, v] \\ \text{(due to (5))}}} \cdot T \right) = \gamma(-[w, v] \cdot T) \\ &= - \underbrace{\gamma([w, v] \cdot T)}_{\substack{=\pi([w, v]) \cdot \gamma(T) + [w, v] \rightharpoonup (\gamma(T)) \\ \text{(according to (107), applied to} \\ \text{ } u=[w, v] \text{ and } U=T)}} \quad \text{(since } \gamma \text{ is } k\text{-linear}) \\ &= -(\pi([w, v]) \cdot \gamma(T) + [w, v] \rightharpoonup (\gamma(T))) \end{aligned}$$

Since

$$\pi \left(\begin{array}{c} [w, v] \\ \underbrace{=w \rightarrow v \text{ (since (9) (applied to } v \text{ and } w \\ \text{instead of } w \text{ and } v) \text{ yields } w \rightarrow v = [w, v])} \end{array} \right) = \pi(w \rightarrow v) = w \rightarrow (\pi(v))$$

(since π is an \mathfrak{h} -module map, while $w \in \mathfrak{h}$) and

$$\begin{aligned} [w, v] \rightarrow (\gamma(T)) &= w \rightarrow (v \rightarrow (\gamma(T))) - v \rightarrow (w \rightarrow (\gamma(T))) \\ &\quad \text{(by (116), applied to } w, v \text{ and } \gamma(T) \text{ instead of } a, b \text{ and } v), \end{aligned}$$

this rewrites as

$$\begin{aligned} \varrho'_T([v, w]) &= - \left(\begin{array}{c} \underbrace{\pi([w, v]) \cdot \gamma(T)}_{=w \rightarrow (\pi(v))} + \underbrace{[w, v] \rightarrow (\gamma(T))}_{=w \rightarrow (v \rightarrow (\gamma(T))) - v \rightarrow (w \rightarrow (\gamma(T)))} \end{array} \right) \\ &= - ((w \rightarrow (\pi(v))) \cdot \gamma(T) + w \rightarrow (v \rightarrow (\gamma(T))) - v \rightarrow (w \rightarrow (\gamma(T))))). \end{aligned} \tag{119}$$

Now, $s = v \cdot w - w \cdot v - [v, w]$ yields

$$\begin{aligned} \varrho'_T(s) &= \varrho'_T(v \cdot w - w \cdot v - [v, w]) \\ &= \underbrace{\varrho'_T(v \cdot w)}_{=\pi(v) \cdot (w \rightarrow (\gamma(T))) + v \rightarrow (w \rightarrow (\gamma(T))) \text{ (due to (117))}} - \underbrace{\varrho'_T(w \cdot v)}_{=(w \rightarrow (\pi(v))) \cdot \gamma(T) + \pi(v) \cdot (w \rightarrow (\gamma(T))) + w \rightarrow (v \rightarrow (\gamma(T))) \text{ (due to (118))}} \\ &\quad - \underbrace{\varrho'_T([v, w])}_{=-((w \rightarrow (\pi(v))) \cdot \gamma(T) + w \rightarrow (v \rightarrow (\gamma(T))) - v \rightarrow (w \rightarrow (\gamma(T)))) \text{ (due to (119))}} \\ &\quad \text{(since } \varrho'_T \text{ is } k\text{-linear)} \\ &= (\pi(v) \cdot (w \rightarrow (\gamma(T))) + v \rightarrow (w \rightarrow (\gamma(T)))) \\ &\quad - ((w \rightarrow (\pi(v))) \cdot \gamma(T) + \pi(v) \cdot (w \rightarrow (\gamma(T))) + w \rightarrow (v \rightarrow (\gamma(T)))) \\ &\quad - (-((w \rightarrow (\pi(v))) \cdot \gamma(T) + w \rightarrow (v \rightarrow (\gamma(T))) - v \rightarrow (w \rightarrow (\gamma(T)))))) \\ &= 0 \quad \text{(because all terms in the sum cancel out),} \end{aligned}$$

so that $s \in \text{Ker } \varrho'_T$.

We have thus proven that every $s \in S_0$ satisfies $s \in \text{Ker } \varrho'_T$. Thus, $S_0 \subseteq \text{Ker } \varrho'_T$. Since $\text{Ker } \varrho'_T$ is a k -module, Proposition 1.29 (a) (applied to $\otimes \mathfrak{g}$, S_0 and $\text{Ker } \varrho'_T$ instead of M , S and Q) now yields $\langle S_0 \rangle \subseteq \text{Ker } \varrho'_T$. Since $\langle S_0 \rangle = J_0$, this rewrites as $J_0 \subseteq \text{Ker } \varrho'_T$. Thus, $\varrho'_T(J_0) = 0$. In other words, every $j \in J_0$ satisfies $\varrho'_T(j) = 0$. Since $\varrho'_T(j) = \gamma(j \cdot T)$ (by the definition of ϱ'_T), this rewrites as $\gamma(j \cdot T) = 0$, and thus $j \cdot T \in \text{Ker } \gamma$ for every $j \in J_0$.

We have thus shown that every $j \in J_0$ and $T \in \otimes \mathfrak{g}$ satisfy $j \cdot T \in \text{Ker } \gamma$. In other words, every $(j, T) \in J_0 \times (\otimes \mathfrak{g})$ satisfies $j \cdot T \in \text{Ker } \gamma$. In other words, $\{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \subseteq \text{Ker } \gamma$.

Now, applying Proposition 1.29 (a) to $\otimes \mathfrak{g}$, $\{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\}$ and $\text{Ker } \gamma$ instead of M , S and Q , we see that $\langle \{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \rangle \subseteq \text{Ker } \gamma$ (since $\{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \subseteq \text{Ker } \gamma$ and since $\text{Ker } \gamma$ is a k -module). Now

$$J_0 \cdot (\otimes \mathfrak{g}) = \langle j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g}) \rangle = \langle \{j \cdot T \mid (j, T) \in J_0 \times (\otimes \mathfrak{g})\} \rangle \subseteq \text{Ker } \gamma$$

and thus $\gamma(J_0 \cdot (\otimes \mathfrak{g})) = 0$.

Now, Lemma 4.10 (applied to $C = J_0 \cdot (\otimes \mathfrak{g})$) yields that $\gamma((\otimes \mathfrak{g}) \cdot (J_0 \cdot (\otimes \mathfrak{g}))) = 0$ (because $\gamma(J_0 \cdot (\otimes \mathfrak{g})) = 0$). Since $(\otimes \mathfrak{g}) \cdot (J_0 \cdot (\otimes \mathfrak{g})) = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g}) = J$ (according to Proposition 2.3 (b)), this rewrites as $\gamma(J) = 0$. Thus, Proposition 4.11 is proven.

Proof of Proposition 4.12. Every $u \in \mathfrak{h}$ satisfies

$$\begin{aligned} \gamma(u) &= \pi(u) && \text{(according to (106), since } u \in \mathfrak{h} \subseteq \mathfrak{g}) \\ &= 0 && \text{(since } u \in \mathfrak{h}, \text{ while } \pi \text{ is a projection with kernel } \mathfrak{h}). \end{aligned}$$

Thus, $\gamma(\mathfrak{h}) = 0$. We can thus follow from Lemma 4.10 (applied to $C = \mathfrak{h}$) that $\gamma((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$ (because $\gamma(\mathfrak{h}) = 0$). Thus, Proposition 4.12 is proven.

4.7. The homomorphism $\bar{\gamma}$

We now formulate our further procedure:

Proposition 4.13. (a) The \mathfrak{h} -module homomorphism $\gamma : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ satisfies $J + (\otimes \mathfrak{g}) \cdot \mathfrak{h} \subseteq \text{Ker } \gamma$. Thus, γ induces an \mathfrak{h} -module homomorphism $\bar{\gamma} : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ which satisfies $\gamma = \bar{\gamma} \circ \zeta$ (where ζ denotes the canonical projection $\otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ as in Theorem 2.1).

(b) The homomorphism $\bar{\gamma}$ respects the filtration. Here, the filtration on $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is given by $(F_n)_{n \geq 0}$, and the filtration on $\otimes \mathfrak{n}$ is given by $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$.

(c) The homomorphism $\bar{\gamma}$ is an \mathfrak{h} -module isomorphism, and its inverse $\bar{\gamma}^{-1}$ also respects the filtration.

We notice that the first two of the three parts of this proposition are trivial:

Proof of Proposition 4.13 (a). Since γ is k -linear, it is clear that $\text{Ker } \gamma$ is a k -module.

We know from Proposition 4.9 that γ is an \mathfrak{h} -module homomorphism. Further, $J \subseteq \text{Ker } \gamma$ (due to Proposition 4.11) and $(\otimes \mathfrak{g}) \cdot \mathfrak{h} \subseteq \text{Ker } \gamma$ (due to Proposition 4.12). Thus, $\underbrace{J}_{\subseteq \text{Ker } \gamma} + \underbrace{(\otimes \mathfrak{g}) \cdot \mathfrak{h}}_{\subseteq \text{Ker } \gamma} \subseteq \text{Ker } \gamma + \text{Ker } \gamma \subseteq \text{Ker } \gamma$ (since $\text{Ker } \gamma$ is a k -module). Thus,

by the homomorphism theorem, we see that γ induces an \mathfrak{h} -module homomorphism $\bar{\gamma} : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ which satisfies $\gamma = \bar{\gamma} \circ \zeta$ (where ζ denotes the canonical projection $\otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ as in Theorem 2.1). This proves Proposition 4.13 (a).

Proof of Proposition 4.13 (b). Let $n \in \mathbb{N}$ be arbitrary. Since $F_n = \zeta(\mathfrak{g}^{\otimes \leq n})$, we have

$$\begin{aligned} \bar{\gamma}(F_n) &= \bar{\gamma}(\zeta(\mathfrak{g}^{\otimes \leq n})) = \underbrace{(\bar{\gamma} \circ \zeta)}_{=\gamma}(\mathfrak{g}^{\otimes \leq n}) = \gamma(\mathfrak{g}^{\otimes \leq n}) \subseteq \mathfrak{n}^{\otimes \leq n} \\ &\quad \text{(according to Proposition 4.8).} \end{aligned}$$

In other words, the homomorphism $\bar{\gamma}$ respects the filtration. This proves Proposition 4.13 (b).

The rest of Section 4 will now be devoted to proving Proposition 4.13 (c).

4.8. Approximating γ by $\otimes\pi$

As an auxiliary result, we need the following analogue of Proposition 2.20:

Proposition 4.14. Consider the k -module homomorphism $\otimes\pi : \otimes\mathfrak{g} \rightarrow \otimes\mathfrak{n}$ induced by the k -module homomorphism $\pi : \mathfrak{g} \rightarrow \mathfrak{n}$. This homomorphism $\otimes\pi$ respects the filtration and satisfies

$$(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq n}) \subseteq \mathfrak{n}^{\otimes\leq(n-1)} \quad \text{for every } n \in \mathbb{N}. \quad (120)$$

Proof of Proposition 4.14. Proposition 1.104 (applied to π , \mathfrak{g} and \mathfrak{n} instead of f , V and W) yields that $\otimes\pi$ respects the filtration. Thus, the only thing that remains to be done for the proof of Proposition 4.14 is proving the relation (120).

We are going to prove (120) by induction over n :

Induction base: Every $\lambda \in \mathfrak{g}^{\otimes\leq 0}$ satisfies $\lambda \in k$ (since $\mathfrak{g}^{\otimes\leq 0} = \bigoplus_{i=0}^0 \mathfrak{g}^{\otimes i} = \mathfrak{g}^{\otimes 0} = k$).

Therefore, every $\lambda \in \mathfrak{g}^{\otimes\leq 0}$ satisfies $\gamma(\lambda) = \lambda$ (according to (105), since $\lambda \in k$) and $(\otimes\pi)(\lambda) = \lambda$ (by the definition of $\otimes\pi$, since $\lambda \in k$), so that $(\gamma - (\otimes\pi))(\lambda) = \underbrace{\gamma(\lambda)}_{=\lambda} - \underbrace{(\otimes\pi)(\lambda)}_{=\lambda} = 0$ and thus $\lambda \in \text{Ker}(\gamma - (\otimes\pi))$. We have thus shown that every $\lambda \in \mathfrak{g}^{\otimes\leq 0}$ satisfies $\lambda \in \text{Ker}(\gamma - (\otimes\pi))$. Hence, $\mathfrak{g}^{\otimes\leq 0} \subseteq \text{Ker}(\gamma - (\otimes\pi))$, so that $(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq 0}) = 0 \subseteq \mathfrak{n}^{\otimes\leq(0-1)}$. In other words, (120) holds for $n = 0$. This completes the induction base.

Induction step: Let $p \in \mathbb{N}$. Assume that (120) holds for $n = p$. We now must show that (120) also holds for $n = p + 1$.

Since (120) holds for $n = p$, we have $(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq p}) \subseteq \mathfrak{n}^{\otimes\leq(p-1)}$.

We are now going to show that $(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes(p+1)}) \subseteq \mathfrak{n}^{\otimes\leq p}$. This will quickly yield $(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq(p+1)}) \subseteq \mathfrak{n}^{\otimes\leq p}$, which will bring us to the end of the induction step.

Proposition 2.11 (applied to $p + 1$ and \mathfrak{g} instead of p and V) yields $\mathfrak{g}^{\otimes(p+1)} = \langle \mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \rangle$, where we are using Convention 2.10.

Note that $\gamma - (\otimes\pi)$ is a k -linear map (since γ and $\otimes\pi$ are k -linear), and thus $\text{Ker}(\gamma - (\otimes\pi))$ is a k -module.

Now we are going to prove that $\mathfrak{g}_{\text{lind}}^{\otimes(p+1)} \subseteq (\gamma - (\otimes\pi))^{-1}(\mathfrak{n}^{\otimes\leq p})$. Indeed, let $V \in \mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$ be arbitrary. Then, V is a left-induced tensor in $\mathfrak{g}^{\otimes(p+1)}$ (since $V \in \mathfrak{g}_{\text{lind}}^{\otimes(p+1)}$), and thus there exist $u \in \mathfrak{g}$ and $U \in \mathfrak{g}^{\otimes p}$ such that $V = u \otimes U$. Consider these u and U . Since $u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1}$ and $U \in \mathfrak{g}^{\otimes p}$, we have $u \cdot U = u \otimes U$ (due to (31), applied to u , U , 1 and p instead of a , b , n and m).

Since $U \in \mathfrak{g}^{\otimes p} \subseteq \mathfrak{g}^{\otimes\leq p}$ and since $\gamma(\mathfrak{g}^{\otimes\leq p}) \subseteq \mathfrak{n}^{\otimes\leq p}$ (because the map γ respects the

filtration), we have $\gamma\left(\underbrace{U}_{\in \mathfrak{g}^{\otimes\leq p}}\right) \in \gamma(\mathfrak{g}^{\otimes\leq p}) \subseteq \mathfrak{n}^{\otimes\leq p}$. Thus, $u \mapsto (\gamma(U)) \in \mathfrak{n}^{\otimes\leq p}$ (since $\mathfrak{n}^{\otimes\leq p}$ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule (which is because \mathfrak{n} is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule)). In other words, $u \mapsto (\gamma(U)) \equiv 0 \pmod{\mathfrak{n}^{\otimes\leq p}}$.

On the other hand, $U \in \mathfrak{g}^{\otimes p} \subseteq \mathfrak{g}^{\otimes\leq p}$ yields $(\gamma - (\otimes\pi))(U) \in (\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq p}) \subseteq \mathfrak{n}^{\otimes\leq(p-1)}$. Since $(\gamma - (\otimes\pi))(U) = \gamma(U) - (\otimes\pi)(U)$, this rewrites as $\gamma(U) - (\otimes\pi)(U) \in$

$\mathfrak{n}^{\otimes \leq (p-1)}$. On the other hand, $\otimes \pi$ is a k -algebra homomorphism, so that

$$\begin{aligned} (\otimes \pi)(u \cdot U) &= \underbrace{(\otimes \pi)(u)}_{=\pi(u) \text{ (by the definition of } \otimes \pi, \text{ since } u \in \mathfrak{g} = \mathfrak{g}^{\otimes 1})} \cdot (\otimes \pi)(U) = \pi(u) \cdot (\otimes \pi)(U). \end{aligned}$$

Now,

$$\begin{aligned} & \pi(u) \cdot \gamma(U) - (\otimes \pi) \left(\underbrace{V}_{=u \otimes U = u \cdot U} \right) \\ &= \pi(u) \cdot \gamma(U) - \underbrace{(\otimes \pi)(u \cdot U)}_{=\pi(u) \cdot (\otimes \pi)(U)} = \pi(u) \cdot \gamma(U) - \pi(u) \cdot (\otimes \pi)(U) \\ &= \underbrace{\pi(u)}_{\in \mathfrak{n} = \mathfrak{n}^{\otimes 1} \subseteq \mathfrak{n}^{\otimes \leq 1}} \cdot \underbrace{(\gamma(U) - (\otimes \pi)(U))}_{\in \mathfrak{n}^{\otimes \leq (p-1)}} \in \mathfrak{n}^{\otimes \leq 1} \cdot \mathfrak{n}^{\otimes \leq (p-1)} \\ &\subseteq \mathfrak{n}^{\otimes \leq (1+(p-1))} \quad \left(\begin{array}{l} \text{according to Proposition 1.95 (b), applied} \\ \text{to } \mathfrak{n}, 1 \text{ and } p-1 \text{ instead of } V, n \text{ and } m \end{array} \right) \\ &= \mathfrak{n}^{\otimes \leq p}. \end{aligned}$$

In other words, $\pi(u) \cdot \gamma(U) \equiv (\otimes \pi)(V) \pmod{\mathfrak{n}^{\otimes \leq p}}$.

Now, $V = u \otimes U = u \cdot U$ yields

$$\begin{aligned} \gamma(V) = \gamma(u \cdot U) &= \underbrace{\pi(u) \cdot \gamma(U)}_{\equiv (\otimes \pi)(V) \pmod{\mathfrak{n}^{\otimes \leq p}}} + \underbrace{u \cdot \gamma(U)}_{\equiv 0 \pmod{\mathfrak{n}^{\otimes \leq p}}} \quad (\text{by (107)}) \\ &\equiv (\otimes \pi)(V) + 0 = (\otimes \pi)(V) \pmod{\mathfrak{n}^{\otimes \leq p}}. \end{aligned}$$

In other words, $\gamma(V) - (\otimes \pi)(V) \in \mathfrak{n}^{\otimes \leq p}$. Since $\gamma(V) - (\otimes \pi)(V) = (\gamma - (\otimes \pi))(V)$, this rewrites as $(\gamma - (\otimes \pi))(V) \in \mathfrak{n}^{\otimes \leq p}$. In other words, $V \in (\gamma - (\otimes \pi))^{-1}(\mathfrak{n}^{\otimes \leq p})$.

So we have proven that every $V \in \mathfrak{g}_{\text{lind}}^{\otimes (p+1)}$ satisfies $V \in (\gamma - (\otimes \pi))^{-1}(\mathfrak{n}^{\otimes \leq p})$. In other words, $\mathfrak{g}_{\text{lind}}^{\otimes (p+1)} \subseteq (\gamma - (\otimes \pi))^{-1}(\mathfrak{n}^{\otimes \leq p})$. Therefore, Proposition 1.29 (a) (applied to $\mathfrak{g}^{\otimes (p+1)}$, $\mathfrak{g}_{\text{lind}}^{\otimes (p+1)}$ and $(\gamma - (\otimes \pi))^{-1}(\mathfrak{n}^{\otimes \leq p})$ instead of M, S and Q) yields $\langle \mathfrak{g}_{\text{lind}}^{\otimes (p+1)} \rangle \subseteq (\gamma - (\otimes \pi))^{-1}(\mathfrak{n}^{\otimes \leq p})$ (since $(\gamma - (\otimes \pi))^{-1}(\mathfrak{n}^{\otimes \leq p})$ is a k -module). Altogether we now have $\mathfrak{g}^{\otimes (p+1)} = \langle \mathfrak{g}_{\text{lind}}^{\otimes (p+1)} \rangle \subseteq (\gamma - (\otimes \pi))^{-1}(\mathfrak{n}^{\otimes \leq p})$, so that $(\gamma - (\otimes \pi))(\mathfrak{g}^{\otimes (p+1)}) \subseteq \mathfrak{n}^{\otimes \leq p}$.

Now, the definition of $\mathfrak{g}^{\otimes \leq (p+1)}$ is $\mathfrak{g}^{\otimes \leq (p+1)} = \bigoplus_{i=0}^{p+1} \mathfrak{g}^{\otimes i}$, while the definition of $\mathfrak{g}^{\otimes \leq p} =$

$\bigoplus_{i=0}^p \mathfrak{g}^{\otimes i}$. Thus,

$$\mathfrak{g}^{\otimes \leq (p+1)} = \bigoplus_{i=0}^{p+1} \mathfrak{g}^{\otimes i} = \underbrace{\left(\bigoplus_{i=0}^p \mathfrak{g}^{\otimes i} \right)}_{=\mathfrak{g}^{\otimes \leq p}} \oplus \mathfrak{g}^{\otimes (p+1)} = \mathfrak{g}^{\otimes \leq p} \oplus \mathfrak{g}^{\otimes (p+1)} = \mathfrak{g}^{\otimes \leq p} + \mathfrak{g}^{\otimes (p+1)}$$

(since direct sums are sums). Thus,

$$\begin{aligned}
(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq(p+1)}) &= (\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq p} + \mathfrak{g}^{\otimes(p+1)}) \\
&= \underbrace{(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq p})}_{\substack{\subseteq \mathfrak{n}^{\otimes\leq(p-1)} \subseteq \mathfrak{n}^{\otimes\leq p} \\ \text{(since } (\mathfrak{n}^{\otimes\leq n})_{n \geq 0} \text{ is a filtration)}}} + \underbrace{(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes(p+1)})}_{\subseteq \mathfrak{n}^{\otimes\leq p}} \\
&\quad \text{(since } \gamma - (\otimes\pi) \text{ is } k\text{-linear)} \\
&\subseteq \mathfrak{n}^{\otimes\leq p} + \mathfrak{n}^{\otimes\leq p} = \mathfrak{n}^{\otimes\leq p} \quad \text{(since } \mathfrak{n}^{\otimes\leq p} \text{ is a } k\text{-module)} \\
&= \mathfrak{n}^{\otimes\leq((p+1)-1)}.
\end{aligned}$$

In other words, (120) holds for $n = p + 1$. This completes the induction step. Thus, (120) is proven for all $n \in \mathbb{N}$. In other words, Proposition 4.14 is proven.

4.9. Finishing the proof

Proposition 4.14 provides us with the following consequence:

Corollary 4.15. Consider the maps $\bar{\varphi} : (\otimes\mathfrak{g}) / (J + (\otimes\mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes N$ and $\pi|_N : N \rightarrow \mathfrak{n}$ defined in Proposition 2.18. Also, let $\eta = \otimes(\pi|_N)$. Then,

$$(\bar{\gamma} - \eta \circ \bar{\varphi})(F_n) \subseteq \mathfrak{n}^{\otimes\leq(n-1)} \quad \text{for every } n \in \mathbb{N}.$$

Proof of Corollary 4.15. Let $x \in F_n$. Since $F_n = \zeta(\mathfrak{g}^{\otimes\leq n})$, this becomes $x \in \zeta(\mathfrak{g}^{\otimes\leq n})$, so that there exists some $y \in \mathfrak{g}^{\otimes\leq n}$ such that $x = \zeta(y)$. Consider this y . It satisfies

$$\begin{aligned}
&(\bar{\gamma} - \eta \circ \bar{\varphi})\left(\underbrace{x}_{=\zeta(y)}\right) \\
&= (\bar{\gamma} - \eta \circ \bar{\varphi})(\zeta(y)) = \underbrace{\bar{\gamma}(\zeta(y))}_{=(\bar{\gamma} \circ \zeta)(y)} - \underbrace{(\eta \circ \bar{\varphi})(\zeta(y))}_{=(\eta \circ \bar{\varphi} \circ \zeta)(y)} \\
&= \underbrace{(\bar{\gamma} \circ \zeta)}_{=\gamma}(y) - \left(\underbrace{\eta \circ \bar{\varphi} \circ \zeta}_{=\varphi}\right)(y) = \gamma(y) - (\eta \circ \varphi)(y) \\
&= \underbrace{(\gamma(y) - (\otimes\pi)(y))}_{=(\gamma - (\otimes\pi))(y)} - \underbrace{((\eta \circ \varphi)(y) - (\otimes\pi)(y))}_{=(\eta \circ \varphi - (\otimes\pi))(y)} \\
&= (\gamma - (\otimes\pi))\left(\underbrace{y}_{\in \mathfrak{g}^{\otimes\leq n}}\right) - (\eta \circ \varphi - (\otimes\pi))\left(\underbrace{y}_{\in \mathfrak{g}^{\otimes\leq n}}\right) \\
&\in \underbrace{(\gamma - (\otimes\pi))(\mathfrak{g}^{\otimes\leq n})}_{\substack{\subseteq \mathfrak{n}^{\otimes\leq(n-1)} \\ \text{(according to (120))}}} - \underbrace{(\eta \circ \varphi - (\otimes\pi))(\mathfrak{g}^{\otimes\leq n})}_{\substack{\subseteq \mathfrak{n}^{\otimes\leq(n-1)} \\ \text{(according to (73), applied} \\ \text{to } n \text{ instead of } p)}} \\
&\subseteq \mathfrak{n}^{\otimes\leq(n-1)} + \mathfrak{n}^{\otimes\leq(n-1)} \subseteq \mathfrak{n}^{\otimes\leq(n-1)} \quad \text{(since } \mathfrak{n}^{\otimes\leq(n-1)} \text{ is a } k\text{-module)}.
\end{aligned}$$

We have thus shown that $(\bar{\gamma} - \eta \circ \bar{\varphi})(x) \in \mathfrak{n}^{\otimes \leq (n-1)}$ for every $x \in F_n$. In other words, $(\bar{\gamma} - \eta \circ \bar{\varphi})(F_n) \subseteq \mathfrak{n}^{\otimes \leq (n-1)}$. This proves Corollary 4.15.

A further corollary from this corollary:

Corollary 4.16. Consider the maps $\bar{\varphi} : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes N$ and $\pi|_N : N \rightarrow \mathfrak{n}$ defined in Proposition 2.18. Also, let $\eta = \otimes(\pi|_N)$. Then, $\eta \circ \bar{\varphi}$ is a k -module isomorphism and satisfies

$$(\text{id} - \bar{\gamma} \circ (\eta \circ \bar{\varphi})^{-1})(\mathfrak{n}^{\otimes \leq n}) \subseteq \mathfrak{n}^{\otimes \leq (n-1)} \quad \text{for every } n \in \mathbb{N}.$$

Proof of Corollary 4.16. Since η and $\bar{\varphi}$ are k -module isomorphisms, their composition $\eta \circ \bar{\varphi}$ must also be a k -module isomorphism.

Besides, $\eta = \otimes(\pi|_N)$, so that $\eta^{-1} = (\otimes(\pi|_N))^{-1} = \otimes((\pi|_N)^{-1})$, and thus η^{-1} respects the filtration (by Proposition 1.104). On the other hand, $\bar{\varphi}^{-1}$ respects the filtration (according to Proposition 2.18 (d)). Thus, Proposition 1.99 (b) (applied to $\otimes \mathfrak{n}$, $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$, $\otimes N$, $(N^{\otimes \leq n})_{n \geq 0}$, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$, $(F_n)_{n \geq 0}$, η^{-1} and $\bar{\varphi}^{-1}$ instead of U , $(U_n)_{n \geq 0}$, V , $(V_n)_{n \geq 0}$, W , $(W_n)_{n \geq 0}$, f and g) yields that the composition $\bar{\varphi}^{-1} \circ \eta^{-1}$ also respects the filtration. Since $\bar{\varphi}^{-1} \circ \eta^{-1} = (\eta \circ \bar{\varphi})^{-1}$, this rewrites as follows: The homomorphism $(\eta \circ \bar{\varphi})^{-1}$ respects the filtration. In other words, $(\eta \circ \bar{\varphi})^{-1}(\mathfrak{n}^{\otimes \leq n}) \subseteq F_n$ for every $n \in \mathbb{N}$.

Now, every $n \in \mathbb{N}$ satisfies

$$\begin{aligned} & \left(\underbrace{\text{id}}_{=(\eta \circ \bar{\varphi}) \circ (\eta \circ \bar{\varphi})^{-1}} - \bar{\gamma} \circ (\eta \circ \bar{\varphi})^{-1} \right) (\mathfrak{n}^{\otimes \leq n}) \\ &= \left(\underbrace{(\eta \circ \bar{\varphi}) \circ (\eta \circ \bar{\varphi})^{-1} - \bar{\gamma} \circ (\eta \circ \bar{\varphi})^{-1}}_{=(\eta \circ \bar{\varphi} - \bar{\gamma}) \circ (\eta \circ \bar{\varphi})^{-1}} \right) (\mathfrak{n}^{\otimes \leq n}) \\ &= ((\eta \circ \bar{\varphi} - \bar{\gamma}) \circ (\eta \circ \bar{\varphi})^{-1}) (\mathfrak{n}^{\otimes \leq n}) = (\eta \circ \bar{\varphi} - \bar{\gamma}) \left(\underbrace{(\eta \circ \bar{\varphi})^{-1} (\mathfrak{n}^{\otimes \leq n})}_{\subseteq F_n} \right) \\ &\subseteq \underbrace{(\eta \circ \bar{\varphi} - \bar{\gamma}) (F_n)}_{=-(\bar{\gamma} - \eta \circ \bar{\varphi})} = - \underbrace{(\bar{\gamma} - \eta \circ \bar{\varphi}) (F_n)}_{\subseteq \mathfrak{n}^{\otimes \leq (n-1)} \text{ (by Corollary 4.15)}} \\ &\subseteq -\mathfrak{n}^{\otimes \leq (n-1)} \subseteq \mathfrak{n}^{\otimes \leq (n-1)} \quad (\text{since } \mathfrak{n}^{\otimes \leq (n-1)} \text{ is a } k\text{-module}). \end{aligned}$$

This proves Corollary 4.16.

Now finally we can come to the *proof of Proposition 4.13*:

We have already proven parts (a) and (b) of Proposition 4.13. Now to part (c):

In the proof of Corollary 4.16, we showed that the map $\eta \circ \bar{\varphi}$ respects the filtration.

We now know that:

- The filtration on $\otimes \mathfrak{n}$ is $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$, and $\mathfrak{n}^{\otimes \leq (-1)} = 0$.
- The map $(\eta \circ \bar{\varphi})^{-1}$ is a k -module isomorphism (because it has an inverse, namely $\eta \circ \bar{\varphi}$).

- The map $(\eta \circ \bar{\varphi})^{-1}$ respects the filtration (this was proven in the proof of Corollary 4.16).
- The map $((\eta \circ \bar{\varphi})^{-1})^{-1}$ respects the filtration (because $((\eta \circ \bar{\varphi})^{-1})^{-1} = \eta \circ \bar{\varphi}$, while we know that the map $\eta \circ \bar{\varphi}$ respects the filtration).
- We have $(\text{id} - \bar{\gamma} \circ (\eta \circ \bar{\varphi})^{-1})(\mathfrak{n}^{\otimes \leq n}) \subseteq \mathfrak{n}^{\otimes \leq (n-1)}$ for every $n \in \mathbb{N}$ (according to Corollary 4.16).

Thus, we can apply Corollary 1.111 to $\otimes \mathfrak{n}$, $(\mathfrak{n}^{\otimes \leq n})_{n \geq 0}$, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$, $(F_n)_{n \geq 0}$, $(\eta \circ \bar{\varphi})^{-1}$ and $\bar{\gamma}$ instead of V , $(V_n)_{n \geq 0}$, W , $(W_n)_{n \geq 0}$, f and g . This yields that the k -module homomorphism $\bar{\gamma}$ is an isomorphism (according to Corollary 1.111 (a)) and that each of the maps $\bar{\gamma}$ and $\bar{\gamma}^{-1}$ respects the filtration (according to Corollary 1.111 (b)).

We now know that $\bar{\gamma}$ is a k -module isomorphism, but we also know that $\bar{\gamma}$ is an \mathfrak{h} -module homomorphism (due to Proposition 4.9). Thus, $\bar{\gamma}$ is an \mathfrak{h} -module isomorphism (by an application of Proposition 1.14). This proves Proposition 4.13 (c), and thus completes the proof of Proposition 4.13.

We can now easily obtain Theorem 4.1:

Proof of Theorem 4.1. (a) Theorem 4.1 (a) is identic with Theorem 2.1 (a). Thus, Theorem 4.1 (a) is already proven (as we have proven Theorem 2.1 (a) in Section 2).

(b) We already know from Theorem 2.1 (b) that $(F_n)_{n \geq 0}$ is an \mathfrak{h} -module filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$.

Now, let $n \in \mathbb{N}$. Since $\bar{\gamma}$ respects the filtration, we have $\bar{\gamma}(F_n) \subseteq \mathfrak{n}^{\otimes \leq n}$. Since the homomorphism $\bar{\gamma}^{-1}$ respects the filtration, we have $\bar{\gamma}^{-1}(\mathfrak{n}^{\otimes \leq n}) \subseteq F_n$. Thus, $F_n \supseteq \bar{\gamma}^{-1}(\mathfrak{n}^{\otimes \leq n})$, so that $\bar{\gamma}(F_n) \supseteq \bar{\gamma}(\bar{\gamma}^{-1}(\mathfrak{n}^{\otimes \leq n})) = \mathfrak{n}^{\otimes \leq n}$ (since $\bar{\gamma}$ is an isomorphism). Combining this with $\bar{\gamma}(F_n) \subseteq \mathfrak{n}^{\otimes \leq n}$, we obtain $\bar{\gamma}(F_n) = \mathfrak{n}^{\otimes \leq n}$. Since $\bar{\gamma}$ is an \mathfrak{h} -module isomorphism, it thus follows that $\bar{\gamma}$ induces an \mathfrak{h} -module isomorphism $F_n \rightarrow \mathfrak{n}^{\otimes \leq n}$. Hence, $F_n \cong \mathfrak{n}^{\otimes \leq n}$ as \mathfrak{h} -modules. This completes the proof of Theorem 4.1 (b).

(c) The map $\bar{\gamma}$ is an \mathfrak{h} -module isomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$, and we know that for every $n \in \mathbb{N}$, the image of F_n under this isomorphism is $\mathfrak{n}^{\otimes \leq n}$ (because $\bar{\gamma}(F_n) = \mathfrak{n}^{\otimes \leq n}$). Thus, Theorem 4.1 (c) must hold.

(d) Since the map $\bar{\gamma}$ is an \mathfrak{h} -module isomorphism, its inverse $\bar{\gamma}^{-1}$ must be an \mathfrak{h} -module isomorphism as well.

For every $n \in \mathbb{N}$, define an \mathfrak{h} -submodule W_n of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ by $W_n = \bar{\gamma}^{-1}(\mathfrak{n}^{\otimes n})$ (here, we use that $\bar{\gamma}^{-1}(\mathfrak{n}^{\otimes n})$ is an \mathfrak{h} -submodule of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$, because $\bar{\gamma}^{-1}$ is an \mathfrak{h} -module isomorphism).

We have $\otimes \mathfrak{n} = \bigoplus_{n \in \mathbb{N}} \mathfrak{n}^{\otimes n}$ and thus

$$\begin{aligned} \bar{\gamma}^{-1}(\otimes \mathfrak{n}) &= \bar{\gamma}^{-1}\left(\bigoplus_{n \in \mathbb{N}} \mathfrak{n}^{\otimes n}\right) = \bigoplus_{n \in \mathbb{N}} \underbrace{\bar{\gamma}^{-1}(\mathfrak{n}^{\otimes n})}_{=W_n} && \text{(since } \bar{\gamma}^{-1} \text{ is an } \mathfrak{h}\text{-module isomorphism)} \\ &= \bigoplus_{n \in \mathbb{N}} W_n. \end{aligned}$$

Now, since $\bar{\gamma}^{-1}$ is an isomorphism, we have $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = \bar{\gamma}^{-1}(\otimes \mathfrak{n}) = \bigoplus_{n \in \mathbb{N}} W_n$.

Every $n \in \mathbb{N}$ satisfies $\bar{\gamma}(F_n) = \mathfrak{n}^{\otimes \leq n}$ (as we saw during the proof of Theorem 4.1 (b)). Renaming n as p in this result, we obtain: Every $p \in \mathbb{N}$ satisfies $\bar{\gamma}(F_p) = \mathfrak{n}^{\otimes \leq p}$. Now, every $p \in \mathbb{N}$ satisfies

$$\begin{aligned}
F_p &= \bar{\gamma}^{-1} \left(\underbrace{\bar{\gamma}(F_p)}_{=\mathfrak{n}^{\otimes \leq p}} \right) && \text{(since } \bar{\gamma} \text{ is an isomorphism)} \\
&= \bar{\gamma}^{-1} \left(\underbrace{\mathfrak{n}^{\otimes \leq p}}_{=\bigoplus_{n=0}^p \mathfrak{n}^{\otimes n}} \right) = \bar{\gamma}^{-1} \left(\bigoplus_{n=0}^p \mathfrak{n}^{\otimes n} \right) \\
&= \bigoplus_{n=0}^p \underbrace{\bar{\gamma}^{-1}(\mathfrak{n}^{\otimes n})}_{=W_n} && \text{(since } \bar{\gamma}^{-1} \text{ is an } \mathfrak{h}\text{-module isomorphism)} \\
&= \bigoplus_{n=0}^p W_n.
\end{aligned}$$

Thus we have found a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{h} -submodules of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ such that $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $F_p = \bigoplus_{n=0}^p W_n$.

Now, Proposition 1.113 (applied to \mathfrak{h} , $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ and $(F_n)_{n \geq 0}$ instead of \mathfrak{g} , V and $(V_n)_{n \geq 0}$) yields that the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split if and only if there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{h} -submodules of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ such that $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $F_p = \bigoplus_{n=0}^p W_n$. Since we know that there exists a family $(W_n)_{n \in \mathbb{N}}$ of \mathfrak{h} -submodules of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ such that $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = \bigoplus_{n \in \mathbb{N}} W_n$ and such that every $p \in \mathbb{N}$ satisfies $F_p = \bigoplus_{n=0}^p W_n$, we can thus conclude that the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split. This proves Theorem 4.1 (d).

Hence, the whole Theorem 4.1 is proven.

5. The Poincaré-Birkhoff-Witt theorem

5.1. The symmetric powers of a module

We recall the definition of the n -th symmetric power of a k -module:

Definition 5.1. Let k be a commutative ring. Let V be a k -module. Let $n \in \mathbb{N}$. Let $K_n(V)$ be the k -submodule

$$\langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n \rangle$$

of the k -module $V^{\otimes n}$ (where we are using Convention 1.28, and are denoting the n -th symmetric group by S_n).

The factor k -module $V^{\otimes n} / K_n(V)$ is called the n -th symmetric power of the k -module V and will be denoted by $\text{Sym}^n V$. We denote by $\text{sym}_{V,n}$ the canonical

projection $V^{\otimes n} \rightarrow V^{\otimes n}/K_n(V) = \text{Sym}^n V$. Clearly, this map $\text{sym}_{V,n}$ is a surjective k -module homomorphism.

Here is an alternative description of the module $K_n(V)$ from this definition:

Proposition 5.2. Let k be a commutative ring. Let V be a k -module. Let $n \in \mathbb{N}$. Then,

$$K_n(V) = \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \dots, v_n) \in V^n \rangle,$$

where τ_i denotes the transposition $(i, i+1) \in S_n$.

Proof of Proposition 5.2. Let T denote the subset

$$\{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\}$$

of $V^{\otimes n}$. Then,

$$\begin{aligned} \langle T \rangle &= \langle \{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\} \rangle \\ &= \langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n \rangle \\ &= K_n(V) \quad (\text{by Definition 5.1}). \end{aligned}$$

On the other hand,

$$\begin{aligned} v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} &\in T \\ \text{for every } ((v_1, v_2, \dots, v_n), \sigma) &\in V^n \times S_n \end{aligned} \quad (121)$$

(since

$$T = \{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\}$$

).

On the other hand, let Z denote the k -submodule

$$\sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \dots, v_n) \in V^n \rangle$$

of $V^{\otimes n}$. Then,

$$\langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau(1)} \otimes v_{\tau(2)} \otimes \dots \otimes v_{\tau(n)} \mid (v_1, v_2, \dots, v_n) \in V^n \rangle \subseteq Z \quad (122)$$

for every $\tau \in \{1, 2, \dots, n-1\}$. Now,

$$\begin{aligned} &\{w_1 \otimes w_2 \otimes \dots \otimes w_n - w_{\tau(1)} \otimes w_{\tau(2)} \otimes \dots \otimes w_{\tau(n)} \mid (w_1, w_2, \dots, w_n) \in V^n\} \\ &= \{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau(1)} \otimes v_{\tau(2)} \otimes \dots \otimes v_{\tau(n)} \mid (v_1, v_2, \dots, v_n) \in V^n\} \\ &\quad (\text{here, we renamed } (w_1, w_2, \dots, w_n) \text{ as } (v_1, v_2, \dots, v_n)) \\ &\subseteq \langle \{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau(1)} \otimes v_{\tau(2)} \otimes \dots \otimes v_{\tau(n)} \mid (v_1, v_2, \dots, v_n) \in V^n\} \rangle \\ &= \langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau(1)} \otimes v_{\tau(2)} \otimes \dots \otimes v_{\tau(n)} \mid (v_1, v_2, \dots, v_n) \in V^n \rangle \subseteq Z \end{aligned}$$

for every $\mathbf{I} \in \{1, 2, \dots, n-1\}$. Thus,

$$w_1 \otimes w_2 \otimes \dots \otimes w_n - w_{\tau_{\mathbf{I}}(1)} \otimes w_{\tau_{\mathbf{I}}(2)} \otimes \dots \otimes w_{\tau_{\mathbf{I}}(n)} \in Z$$

for every $(w_1, w_2, \dots, w_n) \in V^n$ and every $\mathbf{I} \in \{1, 2, \dots, n-1\}$. (123)

We are now going to show that $Z = \langle T \rangle$.

First, let us prove that $Z \subseteq \langle T \rangle$. In fact, every $i \in \{1, 2, \dots, n-1\}$ satisfies

$$\{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \dots, v_n) \in V^n\} \subseteq \langle T \rangle$$

(since every $(v_1, v_2, \dots, v_n) \in V^n$ satisfies

$$\begin{aligned} & v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)} \\ & \in T \quad (\text{due to (121), applied to } ((v_1, v_2, \dots, v_n), \tau_i) \text{ instead of } ((v_1, v_2, \dots, v_n), \sigma)) \\ & \subseteq \langle T \rangle \end{aligned}$$

). Now,

$$\begin{aligned} Z &= \sum_{i=1}^{n-1} \underbrace{\langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \dots, v_n) \in V^n \rangle}_{\subseteq \langle T \rangle} \\ &\subseteq \sum_{i=1}^{n-1} \langle T \rangle \subseteq \langle T \rangle \quad (\text{since } \langle T \rangle \text{ is a } k\text{-module}). \end{aligned}$$

Now, let us show that $\langle T \rangle \subseteq Z$. To that aim, we will show that $T \subseteq Z$.

In fact, let $((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n$ be arbitrary. Then, $(v_1, v_2, \dots, v_n) \in V^n$ and $\sigma \in S_n$. Now, it is known that every element of the symmetric group S_n can be written as a product of some transpositions from the set $\{\tau_1, \tau_2, \dots, \tau_{n-1}\}$. Applying this to the element $\sigma \in S_n$, we conclude that σ can be written as a product of some transpositions from the set $\{\tau_1, \tau_2, \dots, \tau_{n-1}\}$. In other words, there exists a natural number $m \in \mathbb{N}$ and a sequence $(i_1, i_2, \dots, i_m) \in \{1, 2, \dots, n-1\}^m$ such that $\sigma = \tau_{i_1} \tau_{i_2} \dots \tau_{i_m}$. Consider this m and this (i_1, i_2, \dots, i_m) . For every $j \in \{0, 1, \dots, m\}$, let σ_j denote the permutation $\tau_{i_1} \tau_{i_2} \dots \tau_{i_j} \in S_n$. Then, $\sigma_0 = \tau_{i_1} \tau_{i_2} \dots \tau_{i_0} = (\text{empty product}) = \text{id}$ and $\sigma_m = \tau_{i_1} \tau_{i_2} \dots \tau_{i_m} = \sigma$. Moreover, every $j \in \{1, 2, \dots, m\}$ satisfies $v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)} - v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)} \in Z$.³⁷ Thus,

$$\sum_{j=1}^m \underbrace{(v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)} - v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)})}_{\in Z} \in \sum_{j=1}^m Z \subseteq Z$$

³⁷*Proof.* Let $j \in \{1, 2, \dots, m\}$ be arbitrary. Then,

$$\begin{aligned} \tau_{i_j} &= \tau_{i_1} \tau_{i_2} \dots \tau_{i_{j-1}} \tau_{i_j} = \tau_{i_1} \tau_{i_2} \dots \tau_{i_j} = \sigma_j, \\ & \underbrace{\sigma_{j-1}}_{= \tau_{i_1} \tau_{i_2} \dots \tau_{i_{j-1}}} \\ & \text{(by the formula } \sigma_j = \tau_{i_1} \tau_{i_2} \dots \tau_{i_j}, \\ & \text{applied to } j-1 \text{ instead of } j) \end{aligned}$$

so that $\sigma_j = \sigma_{j-1} \tau_{i_j}$. Denote i_j by \mathbf{I} . Then, $\sigma_j = \sigma_{j-1} \tau_{i_j}$ rewrites as $\sigma_j = \sigma_{j-1} \tau_{\mathbf{I}}$.

Now, define an n -tuple $(w_1, w_2, \dots, w_n) \in V^n$ by $(w_\rho = v_{\sigma_{j-1}(\rho)})$ for every $\rho \in \{1, 2, \dots, n\}$. Then, $(w_1, w_2, \dots, w_n) = (v_{\sigma_{j-1}(1)}, v_{\sigma_{j-1}(2)}, \dots, v_{\sigma_{j-1}(n)})$, so that $w_1 \otimes w_2 \otimes \dots \otimes w_n = v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)}$.

(since Z is a k -module). Since

$$\begin{aligned} & \sum_{j=1}^m (v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)} - v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}) \\ &= v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \end{aligned} \quad (124)$$

³⁸, this rewrites as

$$v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \in Z.$$

We have thus shown that every $((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n$ satisfies $v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \in Z$. Thus,

$$\{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\} \subseteq Z.$$

Since $\{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\} = T$, this rewrites as $T \subseteq Z$. Proposition 1.29 (a) (applied to $V^{\otimes n}$, T and Z instead of M , S and Q) thus yields that $\langle T \rangle \subseteq Z$. Combined with $Z \subseteq \langle T \rangle$, this yields $Z = \langle T \rangle$.

We now have

$$\begin{aligned} & \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \dots, v_n) \in V^n \rangle \\ &= Z = \langle T \rangle = K_n(V). \end{aligned}$$

This proves Proposition 5.2.

Just like the n -th tensor power, $\text{Sym}^n V$ becomes a \mathfrak{g} -module when V is a \mathfrak{g} -module:

Definition 5.3. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $n \in \mathbb{N}$.

Consider the \mathfrak{g} -module $V^{\otimes n}$ (the n -th tensor power of the \mathfrak{g} -module V). Then, the k -module $K_n(V)$ introduced in Definition 5.1 is a \mathfrak{g} -submodule of the \mathfrak{g} -module

$\dots \otimes v_{\sigma_{j-1}(n)}$. On the other hand, every $\xi \in \{1, 2, \dots, n\}$ satisfies

$$\begin{aligned} w_{\tau_1(\xi)} &= v_{\sigma_{j-1}(\tau_1(\xi))} && \text{(by the formula } w_\rho = v_{\sigma_{j-1}(\rho)}, \text{ applied to } \rho = \tau_1(\xi)) \\ &= v_{\sigma_j(\xi)} && \left(\text{since } \sigma_{j-1}(\tau_1(\xi)) = \underbrace{(\sigma_{j-1}\tau_1)}_{=\sigma_j}(\xi) = \sigma_j(\xi) \right). \end{aligned}$$

Thus, $(w_{\tau_1(1)}, w_{\tau_1(2)}, \dots, w_{\tau_1(n)}) = (v_{\sigma_j(1)}, v_{\sigma_j(2)}, \dots, v_{\sigma_j(n)})$, so that $w_{\tau_1(1)} \otimes w_{\tau_1(2)} \otimes \dots \otimes w_{\tau_1(n)} = v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}$.

Now,

$$\begin{aligned} & \underbrace{v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)}}_{=w_1 \otimes w_2 \otimes \dots \otimes w_n} - \underbrace{v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}}_{=w_{\tau_1(1)} \otimes w_{\tau_1(2)} \otimes \dots \otimes w_{\tau_1(n)}} \\ &= w_1 \otimes w_2 \otimes \dots \otimes w_n - w_{\tau_1(1)} \otimes w_{\tau_1(2)} \otimes \dots \otimes w_{\tau_1(n)} \in Z \\ & \quad \text{(by (123)),} \end{aligned}$$

qed.

³⁸ Proof of (124). We distinguish between two cases: the case when $m > 0$, and the case when $m = 0$.

$V^{\otimes n}$ (according to Proposition 5.4 below). Thus, the factor k -module $V^{\otimes n}/K_n(V)$ becomes a \mathfrak{g} -module. Since $V^{\otimes n}/K_n(V) = \text{Sym}^n V$, this means that the k -module $\text{Sym}^n V$ becomes a \mathfrak{g} -module. Whenever we will speak of "the \mathfrak{g} -module $\text{Sym}^n V$ ", we are going to mean the \mathfrak{g} -module $\text{Sym}^n V$ just defined.

It is clear that the map $\text{sym}_{V,n}$ (being the canonical projection $V^{\otimes n} \rightarrow V^{\otimes n}/K_n(V)$) is a \mathfrak{g} -module homomorphism.

In the case when $m > 0$, we have

$$\begin{aligned}
& \sum_{j=1}^m (v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)} - v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}) \\
&= \sum_{j=1}^m v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)} - \sum_{j=1}^m v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)} \\
&= \underbrace{\sum_{j=0}^{m-1} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}}_{=v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \dots \otimes v_{\sigma_0(n)} + \sum_{j=1}^{m-1} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}} \\
&\quad - \underbrace{\sum_{j=1}^m v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}}_{= \sum_{j=1}^{m-1} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)} + v_{\sigma_m(1)} \otimes v_{\sigma_m(2)} \otimes \dots \otimes v_{\sigma_m(n)}} \\
&\quad \text{(here, we substituted } j \text{ for } j-1 \text{ in the first sum)} \\
&= \left(v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \dots \otimes v_{\sigma_0(n)} + \sum_{j=1}^{m-1} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)} \right) \\
&\quad - \left(\sum_{j=1}^{m-1} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)} + v_{\sigma_m(1)} \otimes v_{\sigma_m(2)} \otimes \dots \otimes v_{\sigma_m(n)} \right) \\
&= \underbrace{v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \dots \otimes v_{\sigma_0(n)}}_{\substack{=v_1 \otimes v_2 \otimes \dots \otimes v_n \\ \text{(since } \sigma_0 = \text{id yields} \\ v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \dots \otimes v_{\sigma_0(n)} = v_{\text{id}(1)} \otimes v_{\text{id}(2)} \otimes \dots \otimes v_{\text{id}(n)} \\ = v_1 \otimes v_2 \otimes \dots \otimes v_n)}} \quad - \underbrace{v_{\sigma_m(1)} \otimes v_{\sigma_m(2)} \otimes \dots \otimes v_{\sigma_m(n)}}_{\substack{=v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \\ \text{(since } \sigma_m = \sigma)}} \\
&= v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)},
\end{aligned}$$

so that (124) is proven in the case when $m > 0$.

In the case when $m = 0$, we have

$$\begin{aligned}
& \sum_{j=1}^m (v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \dots \otimes v_{\sigma_{j-1}(n)} - v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \dots \otimes v_{\sigma_j(n)}) \\
&= (\text{empty sum}) = 0 \\
&= \underbrace{v_{\text{id}(1)} \otimes v_{\text{id}(2)} \otimes \dots \otimes v_{\text{id}(n)}}_{=v_1 \otimes v_2 \otimes \dots \otimes v_n} - \underbrace{v_{\text{id}(1)} \otimes v_{\text{id}(2)} \otimes \dots \otimes v_{\text{id}(n)}}_{\substack{=v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \\ \text{(since } m=0 \text{ leads to } \sigma_m = \sigma_0, \text{ so that} \\ \text{id} = \sigma_0 = \sigma_m = \sigma)}} \\
&= v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)},
\end{aligned}$$

so that (124) is proven in the case when $m = 0$.

Thus, (124) is proven in both possible cases. This completes the proof of (124).

Here, we are using the following standard fact:

Proposition 5.4. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let V be a \mathfrak{g} -module. Let $n \in \mathbb{N}$.

Consider the \mathfrak{g} -module $V^{\otimes n}$ (the n -th tensor power of the \mathfrak{g} -module V). Then, the k -module $K_n(V)$ introduced in Definition 5.1 is a \mathfrak{g} -submodule of the \mathfrak{g} -module $V^{\otimes n}$.

Proof of Proposition 5.4. Let T denote the subset

$$\{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\}$$

of $V^{\otimes n}$. Then, clearly,

$$w_1 \otimes w_2 \otimes \dots \otimes w_n - w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \dots \otimes w_{\sigma(n)} \in T \quad (125)$$

for every $((w_1, w_2, \dots, w_n), \sigma) \in V^n \times S_n$. We will now show that

$$a \rightarrow t \in K_n(V) \quad \text{for every } a \in \mathfrak{g} \text{ and } t \in T. \quad (126)$$

Proof of (126). We will be using Convention 1.46 during the following proof of (126).

Also, let us introduce another notation: If α and β are two integers such that $\alpha \leq \beta + 1$, and if $p_\alpha, p_{\alpha+1}, \dots, p_\beta$ are some elements of V , then $\bigotimes_{j=\alpha}^{\beta} p_j$ will mean the tensor $p_\alpha \otimes p_{\alpha+1} \otimes \dots \otimes p_\beta \in V^{\otimes(\beta-\alpha+1)}$. (This tensor means 1 if $\alpha = \beta + 1$, and it means p_α if $\alpha = \beta$.)

We notice that every $i \in \{1, 2, \dots, n\}$, every $(v_1, v_2, \dots, v_n) \in V^n$ and every $w_i \in V$ satisfy

$$v_1 \otimes v_2 \otimes \dots \otimes \begin{array}{|c|} \hline w_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n = \left(\bigotimes_{j=1}^{\overleftarrow{i-1}} v_j \right) \otimes w_i \otimes \left(\bigotimes_{j=i+1}^{\overrightarrow{n}} v_j \right). \quad (127)$$

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Let $a \in \mathfrak{g}$ and $t \in T$ be arbitrary. Then,

$$t \in T = \{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\}.$$

Hence, there exists some $((u_1, u_2, \dots, u_n), \sigma) \in V^n \times S_n$ such that $t = u_1 \otimes u_2 \otimes \dots \otimes u_n - u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(n)}$. Consider this $((u_1, u_2, \dots, u_n), \sigma)$. Then, $(u_1, u_2, \dots, u_n) \in V^n$ and $\sigma \in S_n$ (since $((u_1, u_2, \dots, u_n), \sigma) \in V^n \times S_n$).

³⁹This is because Convention 1.46 (applied to $V_j = V$ and $W_i = V$) yields

$$\begin{aligned} v_1 \otimes v_2 \otimes \dots \otimes \begin{array}{|c|} \hline w_i \\ \hline v_i \\ \hline \end{array} \otimes \dots \otimes v_n &= \underbrace{v_1 \otimes v_2 \otimes \dots \otimes v_{i-1}}_{\text{tensor product of the first } i-1 \text{ vectors } v_\ell} \otimes w_i \otimes \underbrace{v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n}_{\text{tensor product of the last } n-i \text{ vectors } v_\ell} \\ &= \underbrace{v_1 \otimes v_2 \otimes \dots \otimes v_{i-1}}_{\substack{\overleftarrow{i-1} \\ = \bigotimes_{j=1} v_j}} \otimes w_i \otimes \underbrace{v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n}_{\substack{\overrightarrow{n} \\ = \bigotimes_{j=i+1} v_j}} \\ &= \left(\bigotimes_{j=1}^{\overleftarrow{i-1}} v_j \right) \otimes w_i \otimes \left(\bigotimes_{j=i+1}^{\overrightarrow{n}} v_j \right). \end{aligned}$$

For every $i \in \{1, 2, \dots, n\}$, let us define a tensor $M_i \in V^{\otimes n}$ by

$$M_i = u_1 \otimes u_2 \otimes \dots \otimes \frac{a \dashv u_i}{u_i} \otimes \dots \otimes u_n. \quad (128)$$

For every $i \in \{1, 2, \dots, n\}$, let us define a tensor $N_i \in V^{\otimes n}$ by

$$N_i = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes \frac{a \dashv u_{\sigma(i)}}{u_{\sigma(i)}} \otimes \dots \otimes u_{\sigma(n)}. \quad (129)$$

Proposition 1.47 (applied to $V_i = V$ and $v_i = u_i$) yields

$$\begin{aligned} a \dashv (u_1 \otimes u_2 \otimes \dots \otimes u_n) &= \sum_{i=1}^n \underbrace{u_1 \otimes u_2 \otimes \dots \otimes \frac{a \dashv u_i}{u_i} \otimes \dots \otimes u_n}_{=M_i} = \sum_{i \in \{1, 2, \dots, n\}} M_i \\ &= \sum_{i \in \{1, 2, \dots, n\}} \end{aligned}$$

On the other hand, Proposition 1.47 (applied to $V_i = V$ and $v_i = u_{\sigma(i)}$) yields

$$\begin{aligned} a \dashv (u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(n)}) &= \sum_{i=1}^n \underbrace{u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes \frac{a \dashv u_{\sigma(i)}}{u_{\sigma(i)}} \otimes \dots \otimes u_{\sigma(n)}}_{=N_i} = \sum_{i \in \{1, 2, \dots, n\}} N_i = \sum_{i \in \{1, 2, \dots, n\}} N_{\sigma^{-1}(i)} \\ &= \sum_{i \in \{1, 2, \dots, n\}} \left(\begin{array}{l} \text{here, we substituted } \sigma^{-1}(i) \text{ for } i \text{ in the sum, since} \\ \sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\} \text{ is a bijection (because } \sigma \in S_n \text{)} \end{array} \right). \end{aligned}$$

Since $t = u_1 \otimes u_2 \otimes \dots \otimes u_n - u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(n)}$, we have

$$\begin{aligned} a \dashv t &= a \dashv (u_1 \otimes u_2 \otimes \dots \otimes u_n - u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(n)}) \\ &= \underbrace{a \dashv (u_1 \otimes u_2 \otimes \dots \otimes u_n)}_{= \sum_{i \in \{1, 2, \dots, n\}} M_i} - \underbrace{a \dashv (u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(n)})}_{= \sum_{i \in \{1, 2, \dots, n\}} N_{\sigma^{-1}(i)}} \\ &\quad \text{(since the Lie action of } V^{\otimes n} \text{ is } k\text{-bilinear)} \\ &= \sum_{i \in \{1, 2, \dots, n\}} M_i - \sum_{i \in \{1, 2, \dots, n\}} N_{\sigma^{-1}(i)} = \sum_{i \in \{1, 2, \dots, n\}} (M_i - N_{\sigma^{-1}(i)}). \end{aligned}$$

Now, every $i \in \{1, 2, \dots, n\}$ satisfies

$$M_i - N_{\sigma^{-1}(i)} \in T. \quad (130)$$

Proof of (130). Let $i \in \{1, 2, \dots, n\}$ be arbitrary. Let $r = \sigma^{-1}(i)$. This is well-defined since σ is a bijection (because $\sigma \in S_n$). Clearly, $\sigma(r) = i$.

For every $j \in \{1, 2, \dots, n\}$, define an element w_j of V by $w_j = \begin{cases} u_j, & \text{if } j \neq i; \\ a \dashv u_j, & \text{if } j = i. \end{cases}$

Thus we have defined an n -tuple $(w_1, w_2, \dots, w_n) \in V^n$. Every $j \in \{1, 2, \dots, n\}$ satisfying $j \neq i$ satisfies $w_j = \begin{cases} u_j, & \text{if } j \neq i; \\ a \dashv u_j, & \text{if } j = i \end{cases} = u_j$ (since $j \neq i$). In other words,

$$\text{every } j \in \{1, 2, \dots, n\} \text{ satisfying } j \neq i \text{ satisfies } u_j = w_j. \quad (131)$$

Hence,

$$\text{every } j \in \{1, 2, \dots, n\} \text{ satisfying } j \neq r \text{ satisfies } u_{\sigma(j)} = w_{\sigma(j)} \quad (132)$$

(by (131) (applied to $\sigma(j)$ instead of j), since $\sigma(j) \neq i$ (because $j \neq r = \sigma^{-1}(i)$ and because σ is a bijection)).

$$\text{Also, the definition of } w_i \text{ yields } w_i = \begin{cases} u_i, & \text{if } i \neq i; \\ a \rightarrow u_i, & \text{if } i = i \end{cases} = a \rightarrow u_i \text{ (since } i = i).$$

Now,

$$\begin{aligned} M_i &= u_1 \otimes u_2 \otimes \dots \otimes \boxed{\begin{array}{c} a \rightarrow u_i \\ u_i \end{array}} \otimes \dots \otimes u_n \\ &= \left(\begin{array}{c} \xrightarrow{i-1} \\ \bigotimes_{j=1}^{i-1} \\ \underbrace{u_j}_{=w_j} \\ \text{(by (131),} \\ \text{since } j < i \text{ yields } j \neq i) \end{array} \right) \otimes \underbrace{(a \rightarrow u_i)}_{=w_i} \otimes \left(\begin{array}{c} \xrightarrow{n} \\ \bigotimes_{j=i+1}^n \\ \underbrace{u_j}_{=w_j} \\ \text{(by (131),} \\ \text{since } j > i \text{ yields } j \neq i) \end{array} \right) \\ &\quad \text{(due to (127), applied to } u_j \text{ and } a \rightarrow u_i \text{ instead of } v_j \text{ and } w_i) \\ &= \underbrace{\left(\begin{array}{c} \xrightarrow{i-1} \\ \bigotimes_{j=1}^{i-1} w_j \end{array} \right)}_{=w_1 \otimes w_2 \otimes \dots \otimes w_{i-1}} \otimes w_i \otimes \underbrace{\left(\begin{array}{c} \xrightarrow{n} \\ \bigotimes_{j=i+1}^n w_j \end{array} \right)}_{=w_{i+1} \otimes w_{i+2} \otimes \dots \otimes w_n} \\ &= (w_1 \otimes w_2 \otimes \dots \otimes w_{i-1}) \otimes w_i \otimes (w_{i+1} \otimes w_{i+2} \otimes \dots \otimes w_n) = w_1 \otimes w_2 \otimes \dots \otimes w_n \end{aligned}$$

and

$$\begin{aligned} N_{\sigma^{-1}(i)} &= N_r \quad \text{(since } \sigma^{-1}(i) = r) \\ &= u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes \boxed{\begin{array}{c} a \rightarrow u_{\sigma(r)} \\ u_{\sigma(r)} \end{array}} \otimes \dots \otimes u_{\sigma(n)} \quad \text{(by (129), applied to } r \text{ instead of } i) \\ &= \left(\begin{array}{c} \xrightarrow{r-1} \\ \bigotimes_{j=1}^{r-1} \\ \underbrace{u_{\sigma(j)}}_{=w_{\sigma(j)}} \\ \text{(by (132)} \\ \text{since } j < r \text{ yields } j \neq r) \end{array} \right) \otimes \underbrace{(a \rightarrow u_{\sigma(r)})}_{=a \rightarrow u_i \text{ (since } \sigma(r) = i)} \otimes \left(\begin{array}{c} \xrightarrow{n} \\ \bigotimes_{j=r+1}^n \\ \underbrace{u_{\sigma(j)}}_{=w_{\sigma(j)}} \\ \text{(by (132)} \\ \text{since } j > r \text{ yields } j \neq r) \end{array} \right) \\ &\quad \text{(due to (127), applied to } r, u_{\sigma(j)} \text{ and } a \rightarrow u_{\sigma(r)} \text{ instead of } i, v_j \text{ and } w_i) \\ &= \underbrace{\left(\begin{array}{c} \xrightarrow{r-1} \\ \bigotimes_{j=1}^{r-1} w_{\sigma(j)} \end{array} \right)}_{=w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \dots \otimes w_{\sigma(r-1)}} \otimes \underbrace{(a \rightarrow u_i)}_{=w_i = w_{\sigma(r)} \text{ (since } i = \sigma(r))} \otimes \underbrace{\left(\begin{array}{c} \xrightarrow{n} \\ \bigotimes_{j=r+1}^n w_{\sigma(j)} \end{array} \right)}_{=w_{\sigma(r+1)} \otimes w_{\sigma(r+2)} \otimes \dots \otimes w_{\sigma(n)}} \\ &= (w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \dots \otimes w_{\sigma(r-1)}) \otimes w_{\sigma(r)} \otimes (w_{\sigma(r+1)} \otimes w_{\sigma(r+2)} \otimes \dots \otimes w_{\sigma(n)}) \\ &= w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \dots \otimes w_{\sigma(n)}. \end{aligned}$$

Hence,

$$\begin{aligned} & \underbrace{M_i}_{=w_1 \otimes w_2 \otimes \dots \otimes w_n} - \underbrace{N_{\sigma^{-1}(i)}}_{=w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \dots \otimes w_{\sigma(n)}} \\ & = w_1 \otimes w_2 \otimes \dots \otimes w_n - w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \dots \otimes w_{\sigma(n)} \in T \end{aligned} \quad (\text{by (125)}),$$

and thus (130) is proven.

Let us now continue proving (126): We have

$$a \rightharpoonup t = \sum_{i \in \{1, 2, \dots, n\}} \underbrace{(M_i - N_{\sigma^{-1}(i)})}_{\in T \text{ (by (130))}} \in \langle T \rangle.$$

This proves (126).

Now, let $a \in \mathfrak{g}$ be arbitrary. Define a map $\partial_a : V^{\otimes n} \rightarrow V^{\otimes n}$ by

$$(\partial_a(x) = a \rightharpoonup x \text{ for every } x \in V^{\otimes n}).$$

Then, ∂_a is a k -linear map (since the Lie action of $V^{\otimes n}$ is k -bilinear). Proposition 1.29 (b) (applied to $V^{\otimes n}$, T , $V^{\otimes n}$ and ∂_a instead of M , S , R and f) thus yields that $\partial_a(\langle T \rangle) = \langle \partial_a(T) \rangle$. But since

$$\begin{aligned} \partial_a(T) &= \left\{ \underbrace{\partial_a(x)}_{=a \rightharpoonup x} \mid x \in T \right\} = \{a \rightharpoonup x \mid x \in T\} = \{a \rightharpoonup t \mid t \in T\} \\ & \quad (\text{here, we substituted } t \text{ for } x) \\ & \subseteq K_n(V) \quad (\text{due to (126)}), \end{aligned}$$

Proposition 1.29 (a) (applied to $V^{\otimes n}$, T and $K_n(V)$ instead of M , S and Q) yields $\langle \partial_a(T) \rangle \subseteq K_n(V)$.

Now, let $x \in K_n(V)$ be arbitrary. Then,

$$\begin{aligned} x & \in K_n(V) \\ & = \langle v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n \rangle \\ & = \left\langle \underbrace{\{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \dots, v_n), \sigma) \in V^n \times S_n\}}_{=T} \right\rangle \\ & = \langle T \rangle, \end{aligned}$$

so that $\partial_a(x) \in \partial_a(\langle T \rangle) = \langle \partial_a(T) \rangle \subseteq K_n(V)$. Since $\partial_a(x) = a \rightharpoonup x$, this rewrites as $a \rightharpoonup x \in K_n(V)$.

We have thus proven that $a \rightharpoonup x \in K_n(V)$ for every $a \in \mathfrak{g}$ and every $x \in K_n(V)$. In other words, $K_n(V)$ is a \mathfrak{g} -submodule of $V^{\otimes n}$. This proves Proposition 5.4.

5.2. The PBW map

The next proposition sets the stage for the Poincaré-Birkhoff-Witt theorem:

Proposition 5.5. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Consider the universal enveloping algebra $U(\mathfrak{g})$ defined in Definition 1.64.

Let $\psi : \otimes \mathfrak{g} \rightarrow U(\mathfrak{g})$ be the canonical projection (which is well-defined since $U(\mathfrak{g})$ is defined as a factor algebra of $\otimes \mathfrak{g}$). Clearly, ψ is a surjective k -algebra homomorphism.

For every $n \in \mathbb{N}$, let $U_{\leq n}(\mathfrak{g})$ be the k -submodule $\psi(\mathfrak{g}^{\otimes \leq n})$ of $U(\mathfrak{g})$.

(a) Then, $(U_{\leq n}(\mathfrak{g}))_{n \geq 0}$ is a \mathfrak{g} -module filtration of the \mathfrak{g} -module $U(\mathfrak{g})$.⁴⁰

(b) The k -module homomorphism $\psi : \otimes \mathfrak{g} \rightarrow U(\mathfrak{g})$ respects the filtration. Thus, for every $n \in \mathbb{N}$, we have a k -module homomorphism $\text{gr}_n \psi : \text{gr}_n(\otimes \mathfrak{g}) \rightarrow \text{gr}_n(U(\mathfrak{g}))$. Composing this homomorphism with the canonical k -module isomorphism $\text{grad}_{\mathfrak{g},n} : \mathfrak{g}^{\otimes n} \rightarrow \text{gr}_n(\otimes \mathfrak{g})$, we obtain a k -module homomorphism $\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n} : \mathfrak{g}^{\otimes n} \rightarrow \text{gr}_n(U(\mathfrak{g}))$. Explicitly this homomorphism looks as follows:

$$\begin{aligned} & (\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n})(v_1 \otimes v_2 \otimes \dots \otimes v_n) \\ &= \left(\text{the residue class of } \overline{v_1 \otimes v_2 \otimes \dots \otimes v_n} \in U_{\leq n}(\mathfrak{g}) \text{ modulo } U_{\leq (n-1)}(\mathfrak{g}) \right) \end{aligned}$$

for every $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$.

(c) For every $n \in \mathbb{N}$, there exists a unique k -module homomorphism $\mathbf{P} : \text{Sym}^n \mathfrak{g} \rightarrow \text{gr}_n(U(\mathfrak{g}))$ such that $\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n} = \mathbf{P} \circ \text{sym}_{\mathfrak{g},n}$. This homomorphism \mathbf{P} is surjective.

We are not going to prove this proposition in detail, as every text on the Poincaré-Birkhoff-Witt theorem does it. Here is a rough outline:

Proof of Proposition 5.5 (sketched). (a) Clear, since ψ is surjective and a \mathfrak{g} -module homomorphism.

(b) Clear from the definition of ψ .

(c) Since $\text{sym}_{\mathfrak{g},n}$ is surjective, it is enough to prove that $\text{Ker } \text{sym}_{\mathfrak{g},n} \subseteq \text{Ker } (\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n})$ (because when this is proven, Proposition 5.5 (c) follows from the homomorphism theorem). But $\text{Ker } \text{sym}_{\mathfrak{g},n} = K_n(\mathfrak{g})$, so we must check that $K_n(\mathfrak{g}) \subseteq \text{Ker } (\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n})$. Due to Proposition 5.2 (applied to $V = \mathfrak{g}$), this only requires showing that

$$v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)} \in \text{Ker } (\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n})$$

for every $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$ and every $i \in \{1, 2, \dots, n-1\}$. In other words, we have to show that

$$(\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n})(v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)}) = 0$$

for every $(v_1, v_2, \dots, v_n) \in \mathfrak{g}^n$ and every $i \in \{1, 2, \dots, n-1\}$. But this is easy: From (b), we have

$$\begin{aligned} & (\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n})(v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)}) \\ &= \left(\text{the residue class of } \overline{v_1 \otimes v_2 \otimes \dots \otimes v_n} \in U_{\leq n}(\mathfrak{g}) \text{ modulo } U_{\leq (n-1)}(\mathfrak{g}) \right) \\ &\quad - \left(\text{the residue class of } \overline{v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)}} \in U_{\leq n}(\mathfrak{g}) \text{ modulo } U_{\leq (n-1)}(\mathfrak{g}) \right) \\ &= \left(\text{the residue class of } \overline{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)}} \in U_{\leq n}(\mathfrak{g}) \text{ modulo } U_{\leq (n-1)}(\mathfrak{g}) \right), \end{aligned}$$

⁴⁰Let us remind ourselves that here (and in the following), "the \mathfrak{g} -module $U(\mathfrak{g})$ " is to be understood according to Remark 1.67. In other words, "the \mathfrak{g} -module $U(\mathfrak{g})$ " means the \mathfrak{g} -module obtained by applying Definition 1.66 to $V = \mathfrak{g}$ and setting $U(\mathfrak{g}) = (\otimes \mathfrak{g}) / I_{\mathfrak{g}}$, not the \mathfrak{g} -module structure given by (32).

so we must prove that $\overline{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)}} \in U_{\leq(n-1)}(\mathfrak{g})$. But since τ_i is the transposition $(i, i+1)$, we have

$$\begin{aligned}
& \overline{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \dots \otimes v_{\tau_i(n)}} \\
&= \overline{v_1 \otimes v_2 \otimes \dots \otimes v_n - v_1 \otimes v_2 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes v_{i+3} \otimes \dots \otimes v_n} \\
&= \overline{v_1 \otimes v_2 \otimes \dots \otimes v_{i-1} \otimes (v_i \otimes v_{i+1} - v_{i+1} \otimes v_i) \otimes v_{i+2} \otimes v_{i+3} \otimes \dots \otimes v_n} \\
&= \overline{v_1 \otimes v_2 \otimes \dots \otimes v_{i-1}} \cdot \underbrace{\overline{v_i \otimes v_{i+1} - v_{i+1} \otimes v_i}}_{\substack{=[v_i, v_{i+1}] \\ \text{(since we are in } U(\mathfrak{g}))}} \cdot \overline{v_{i+2} \otimes v_{i+3} \otimes \dots \otimes v_n} \\
&= \overline{v_1 \otimes v_2 \otimes \dots \otimes v_{i-1}} \cdot \overline{[v_i, v_{i+1}]} \cdot \overline{v_{i+2} \otimes v_{i+3} \otimes \dots \otimes v_n} \\
&= \overline{v_1 \otimes v_2 \otimes \dots \otimes v_{i-1} \otimes [v_i, v_{i+1}] \otimes v_{i+2} \otimes v_{i+3} \otimes \dots \otimes v_n} \in U_{\leq(n-1)}(\mathfrak{g}).
\end{aligned}$$

This completes the proof of Proposition 5.5.

Definition 5.6. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$.

(a) According to Proposition 5.5 (c), there exists a unique k -module homomorphism $\mathbf{P} : \text{Sym}^n \mathfrak{g} \rightarrow \text{gr}_n(U(\mathfrak{g}))$ such that $\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n} = \mathbf{P} \circ \text{sym}_{\mathfrak{g},n}$ (where the notations used are those of Proposition 5.5). This homomorphism will be denoted by $\text{PBW}_{\mathfrak{g},n}$ and called the n -PBW homomorphism of the Lie algebra \mathfrak{g} .

(b) We say that the Lie algebra \mathfrak{g} satisfies the n -PBW condition if the n -PBW homomorphism $\text{PBW}_{\mathfrak{g},n}$ is a k -module isomorphism.

Remark 5.7. The terminology "n-PBW isomorphism" and "n-PBW condition" is peculiar to the author. Most people only speak of the "PBW isomorphism" (which is the direct sum of the n -PBW isomorphisms over all $n \in \mathbb{N}$) and of the "PBW condition" (which is the conjunction of the n -PBW conditions over all $n \in \mathbb{N}$), as it is most often enough to consider all n together. However, in considering restricted Lie algebras in characteristic p it is often necessary to consider only the n -PBW conditions for $n < p$, so we prefer to have a way to refer to the individual n -PBW conditions separately.

Note that "PBW" is an abbreviation for Poincaré-Birkhoff-Witt, and "BW" is an abbreviation for "Birkhoff-Witt". Some older literature writes "BW" instead of "PBW", as Poincaré's discovery of universal enveloping algebras was forgotten for a long time.

5.3. The Poincaré-Birkhoff-Witt theorem

There are several interrelated (but still different and non-equivalent) facts referred to as Poincaré-Birkhoff-Witt theorems in literature. All of them are similar in that each of them has a condition (such as: k is a \mathbb{Q} -algebra, or: k is a field, or: \mathfrak{g} is a free k -module), under which they claim that the universal enveloping algebra $U(\mathfrak{g})$ of a k -Lie algebra \mathfrak{g} is "similar" in a certain way to the symmetric algebra $\text{Sym} \mathfrak{g}$ of the k -module \mathfrak{g} as a k -module, as a \mathfrak{g} -module, or as a k -algebra. What this "similar" means depends on which of the various Poincaré-Birkhoff-Witt theorems we consider. There is usually no isomorphism of k -algebras, but there is often an isomorphism of k -modules, sometimes one of \mathfrak{g} -modules, and often one between the associated graded

k -algebras. Here we are interested in the latter. The fact we are going to use is the following one:

Proposition 5.8. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra.

(a) Assume that the Lie algebra \mathfrak{g} satisfies the n -PBW condition for every $n \in \mathbb{N}$. Then, the associated graded k -algebra $\text{gr}(U(\mathfrak{g}))$ is isomorphic to the symmetric algebra $\text{Sym } \mathfrak{g} = \bigoplus_{p \in \mathbb{N}} \text{Sym}^p \mathfrak{g}$ as a k -algebra. (Here, the *associated graded k -algebra* $\text{gr}(U(\mathfrak{g}))$ is defined as the direct sum $\bigoplus_{p \in \mathbb{N}} \text{gr}_p(U(\mathfrak{g}))$, with multiplication defined by

$$\begin{aligned} (\overline{u_p})_{p \in \mathbb{N}} \cdot (\overline{v_p})_{p \in \mathbb{N}} &= \left(\sum_{i=0}^p u_i \cdot v_{p-i} \right)_{p \in \mathbb{N}} \\ &\text{for every } (\overline{u_p})_{p \in \mathbb{N}} \in \bigoplus_{p \in \mathbb{N}} \text{gr}_p(U(\mathfrak{g})) \\ &\text{and } (\overline{v_p})_{p \in \mathbb{N}} \in \bigoplus_{p \in \mathbb{N}} \text{gr}_p(U(\mathfrak{g})), \end{aligned}$$

where $\overline{u_p}$, $\overline{v_p}$ and $\overline{\sum_{i=0}^p u_i \cdot v_{p-i}}$ are to be understood as residue classes of certain elements of $U_{\leq p}(\mathfrak{g})$ modulo $U_{\leq (p-1)}(\mathfrak{g})$.)

(b) Let $n \in \mathbb{N}$. Assume that the Lie algebra \mathfrak{g} satisfies the n -PBW condition. Then, $\text{Ker}(\text{gr}_n \psi) = \text{grad}_{\mathfrak{g},n}(K_n(\mathfrak{g}))$. Here, ψ is defined as in Proposition 5.5, and $K_n(\mathfrak{g})$ is defined as in Definition 5.1 (applied to $V = \mathfrak{g}$).

Proof of Proposition 5.8. (a) We assumed that the Lie algebra \mathfrak{g} satisfies the n -PBW condition for every $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, the map $\text{PBW}_{\mathfrak{g},n} : \text{Sym}^n \mathfrak{g} \rightarrow \text{gr}_n(U(\mathfrak{g}))$ is a k -module isomorphism. The direct sum $\bigoplus_{p \in \mathbb{N}} \text{PBW}_{\mathfrak{g},p}$ of these maps is thus a k -module isomorphism from $\bigoplus_{p \in \mathbb{N}} \text{Sym}^p \mathfrak{g} = \text{Sym } \mathfrak{g}$ to $\bigoplus_{p \in \mathbb{N}} \text{gr}_p(U(\mathfrak{g})) = \text{gr}(U(\mathfrak{g}))$.

It only remains to show that this is a k -algebra isomorphism. This is straightforward and left to the reader (especially given that we will not be using this fact anyway).

(b) We assumed that the Lie algebra \mathfrak{g} satisfies the n -PBW condition. This means that the map $\text{PBW}_{\mathfrak{g},n} : \text{Sym}^n \mathfrak{g} \rightarrow \text{gr}_n(U(\mathfrak{g}))$ is a k -module isomorphism. Thus, $\text{Ker}(\text{PBW}_{\mathfrak{g},n} \circ \text{sym}_{\mathfrak{g},n}) = \text{Ker } \text{sym}_{\mathfrak{g},n} = K_n(\mathfrak{g})$ (since $\text{sym}_{\mathfrak{g},n}$ is the projection of $\mathfrak{g}^{\otimes n}$ onto $\mathfrak{g}^{\otimes n} / K_n(\mathfrak{g})$). But the definition of $\text{PBW}_{\mathfrak{g},n}$ yields $\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n} = \text{PBW}_{\mathfrak{g},n} \circ \text{sym}_{\mathfrak{g},n}$. Thus, $\text{Ker}(\text{PBW}_{\mathfrak{g},n} \circ \text{sym}_{\mathfrak{g},n}) = K_n(\mathfrak{g})$ becomes $\text{Ker}(\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n}) = K_n(\mathfrak{g})$. But $\text{Ker}(\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n}) = \text{grad}_{\mathfrak{g},n}^{-1}(\text{Ker}(\text{gr}_n \psi))$ (since $\text{grad}_{\mathfrak{g},n}$ is a k -module isomorphism). Thus we get

$$K_n(\mathfrak{g}) = \text{Ker}(\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n}) = \text{grad}_{\mathfrak{g},n}^{-1}(\text{Ker}(\text{gr}_n \psi)),$$

so that $\text{Ker}(\text{gr}_n \psi) = \text{grad}_{\mathfrak{g},n}(K_n(\mathfrak{g}))$ (again since $\text{grad}_{\mathfrak{g},n}$ is an isomorphism). This proves Proposition 5.8 (b).

To get anything useful out of Proposition 5.8, we need to know some simple conditions under which the n -PBW condition is guaranteed to hold (the n -PBW condition

itself is rather hard to check in most cases, particularly if we want to check it *for all* $n \in \mathbb{N}$ *at once*). These are the essence of the Poincaré-Birkhoff-Witt theorems:⁴¹

- Theorem 5.9.** Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $n \in \mathbb{N}$.
- (a) If \mathfrak{g} is a free k -module, then \mathfrak{g} satisfies the n -PBW condition.
 - (b) If k is a \mathbb{Q} -algebra, then \mathfrak{g} satisfies the n -PBW condition.
 - (c) If \mathfrak{g} is a projective k -module, then \mathfrak{g} satisfies the n -PBW condition.
 - (d) If the (additive) abelian group \mathfrak{g} is torsion-free, then \mathfrak{g} satisfies the n -PBW condition.
 - (e) If k is a Dedekind domain, then \mathfrak{g} satisfies the n -PBW condition.
 - (f) If the k -module \mathfrak{g} is the direct sum of cyclic modules (where a module is said to be *cyclic* if it is generated by one element), then \mathfrak{g} satisfies the n -PBW condition.
 - (g) If \mathfrak{g} is a flat k -module, then \mathfrak{g} satisfies the n -PBW condition.

Calling this theorem "Poincaré-Birkhoff-Witt theorem" is an anachronism, although a rather convenient one. The first to discover anything related to this result was apparently Poincaré in 1900; it was a weak version of Theorem 5.9 which required k to be a field of characteristic 0 and claimed that if (v_1, v_2, \dots, v_m) is a basis of the k -vector space \mathfrak{g} , then $(\overline{v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in \{1, 2, \dots, m\}^n; i_1 \leq i_2 \leq \dots \leq i_n}$ is a basis of the k -vector space $U(\mathfrak{g})$ (see Theorem 5.10 for why this is weaker than Theorem 5.9). This can be shown to be equivalent to the claim that \mathfrak{g} satisfies the n -PBW condition for every $n \in \mathbb{N}$. Whether Poincaré's proof of this is correct is still a matter of controversy. More historical details, along with a modernized version of Poincaré's original alleged proof, can be found in [20].

A comprehensive proof of Theorem 5.9 can be found in Higgins's paper [10, Theorems 6 and 7 and Corollary 2] (where he proves all parts except for (d) and (g), but (d) can be derived from (b) by tensoring with \mathbb{Q} , and (g) can be derived from his Theorem 8 combined with Lazard's theorem that any flat module is a direct limit of free modules). However, its parts (a) and (b) are proven more frequently in different sources: Theorem 5.9 (a) is shown in most references which consider the Poincaré-Birkhoff-Witt theorem: for example, [1, Theorem 1.3.1], [13, Theorem 8.2.2], [5, §17.3], [4, §2.1], [11, §1.9], [21, Theorem 5.15], [8, §2.7, Théorème 1], [7, Theorem 3.3.1], [12, Part I, Chapter III, Theorem 4.3], [16, Theorem 2], [15, Theorem 6.5] give proofs (and while most of these sources superficially require k to be a field, the only condition they actually use is that \mathfrak{g} be a free k -module). [24, Part 1, Chapter 1, §1.3.7] proves Theorem 5.9 (b) as well, and so do [22, §2.5] and [23, §5.5]. Theorem 5.9 (e) and (f) were also shown by Pierre Cartier in [28].

In the case when \mathfrak{g} is a free k -module, the following is also known as "the Poincaré-Birkhoff-Witt theorem", as it is an equivalent version of Theorem 5.9 (a):

- Theorem 5.10.** Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Assume that the k -module \mathfrak{g} has a basis $(e_i)_{i \in I}$, where I is a totally ordered set. Then, $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; i_1 \leq i_2 \leq \dots \leq i_n}$ is a basis of the k -module $U(\mathfrak{g})$.

⁴¹Note that Theorem 5.9 (at least part (a), but probably some of the other parts as well) depends on the axiom of choice (or at least its proof does). So do some of its consequences which we are going to use.

Convention 5.11. Here and in the following, we use the notation $U_{\leq n}(\mathfrak{g})$ for every $n \in \mathbb{N}$ as defined in Proposition 5.5. This means that for every $n \in \mathbb{N}$, we denote by $U_{\leq n}(\mathfrak{g})$ the k -submodule $\psi(\mathfrak{g}^{\otimes \leq n})$ of $U(\mathfrak{g})$.

Theorem 5.10 appears, for instance, in literature as [30, Theorem 5.1.1], [17, Theorem 3.1], [3, Section V.2, Theorem 3], [16, Theorem 5], [31, Chapter XIII, Theorem 3.1], [8, §2.7, Corollaire 3], [27, Chapter III, Theorem 3.8]⁴² or [7, Theorem 3.2.2]. It can be easily derived from Theorem 5.9 (a), but the other direction is more standard: Almost all proofs of Theorem 5.9 (a) proceed by deriving Theorem 5.10 first ([4, §2.1], [5] and [11, §1.9] are very explicit about doing so - for example, Theorem 5.10 is [4, Lemma 2.1.8] and [5, §17.3, Corollary C]). Also, one of the most translucent proofs for Theorem 5.10 is given in [6] and in [29, §7.1].

Remark 5.12. Some sources prove Theorem 5.9 (a) only in the case when I is well-ordered (and not just totally ordered). However, once Theorem 5.9 (a) is proven in this case, we can easily see that Theorem 5.9 (a) holds for any totally ordered set I . Here is why: The only difficult part of Theorem 5.9 (a) is the linear independency of the family $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n}$. To prove this independency, it is enough to show that the family $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in J^n; i_1 \leq i_2 \leq \dots \leq i_n}$ is linearly independent for every finite subset J of I . But for every finite subset J of I , we can extend the ordering on J to a well-ordering of I (without changing the order of the elements of J), and apply the proof from the well-ordered case.

This all requires the axiom of choice, but then again, in all applications of Poincaré-Birkhoff-Witt I have seen, the basis is countable and thus a well-ordering is easy to find.

We note that a part of Theorem 5.10 holds more generally:

Proposition 5.13. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $(e_i)_{i \in I}$ be a generating set of the k -module \mathfrak{g} , where I is a totally ordered set. Then, $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; i_1 \leq i_2 \leq \dots \leq i_n}$ is a generating set of the k -module $U(\mathfrak{g})$.

This is actually the easier part of Theorem 5.10, and is proven by induction in almost every text on Lie algebras. The same argument shows the following strengthening of this proposition:

Proposition 5.14. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $(e_i)_{i \in I}$ be a generating set of the k -module \mathfrak{g} , where I is a totally ordered set. Let $m \in \mathbb{N}$. Then, the family $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; i_1 \leq i_2 \leq \dots \leq i_n; n \leq m}$ is a generating set of the k -module $U_{\leq m}(\mathfrak{g})$.

This proposition appears, e. g., in [4, Lemma 2.1.6] and [12, Part I, Chapter III, Lemma 4.4] (although in an unnecessarily restrictive version).

We notice the following slight strengthening of Theorem 5.10:

⁴²Note that [27, Chapter III, Theorem 3.8] requires k to be \mathbb{C} , but this requirement is neither necessary for the theorem nor used in the proof. The proof works just as well for the general case.

Corollary 5.15. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Assume that the k -module \mathfrak{g} has a basis $(e_i)_{i \in I}$, where I is a totally ordered set. Let $m \in \mathbb{N}$. Then, the family $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{\substack{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; \\ i_1 \leq i_2 \leq \dots \leq i_n; n \leq m}}$ is a basis of the k -module $U_{\leq m}(\mathfrak{g})$.

Proof of Corollary 5.15. The family $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{\substack{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; \\ i_1 \leq i_2 \leq \dots \leq i_n; n \leq m}}$ is a generating set of the k -module $U_{\leq m}(\mathfrak{g})$ (by Proposition 5.14) and linearly independent (by Theorem 5.10). Therefore it is a basis of the k -module $U_{\leq m}(\mathfrak{g})$. This proves Corollary 5.15.

Note that while most literature does not explicitly mention Corollary 5.15, it often tacitly uses it (for example, when deriving Theorem 5.9 from Theorem 5.10).

Our next results are concerned with the case when \mathfrak{h} is a free k -module and satisfies $\mathfrak{g} = \mathfrak{h} \oplus N$ for some free k -module N . This requirement is harsh in comparison to what we have required in previous sections, but it still encompasses the situation when k is a field, and besides is satisfied for many standard cases such as $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{gl}_n k, \mathfrak{sl}_n k)$ even if k is just a commutative ring with 1.

We are going to prove the following consequence of Poincaré-Birkhoff-Witt:

Proposition 5.16. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $m \in \mathbb{N}$.

Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} such that \mathfrak{h} is a free k -module and such that there exists a free k -submodule N of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus N$.

Then, $U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) = U_{\leq (m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$. (Here, we are using the notation of Definition 5.11, and we are abbreviating the k -submodule $U(\mathfrak{g}) \cdot \psi(\mathfrak{h})$ of $U(\mathfrak{g})$ by $U(\mathfrak{g}) \cdot \mathfrak{h}$.)

Before we sketch a proof of Proposition 5.16, let us recall a basic from linear algebra:

Lemma 5.17. Let k be a commutative ring. Let V be a free k -module with basis $(\mathbf{e}_\kappa)_{\kappa \in K}$, where K is a set. Let X and Y be two subsets of K . Then,

$$\langle \mathbf{e}_\kappa \mid \kappa \in X \rangle \cap \langle \mathbf{e}_\kappa \mid \kappa \in Y \rangle = \langle \mathbf{e}_\kappa \mid \kappa \in X \cap Y \rangle.$$

(Here we are using Convention 1.28.)

Proof of Proposition 5.16. Let $(e_i)_{i \in P}$ be a basis of the k -module \mathfrak{h} , and let $(e_i)_{i \in Q}$ be a basis of the k -module N . Assume WLOG that the sets P and Q are disjoint. Let $I = P \cup Q$. Choose a well-ordering on I such that every element of Q is smaller than any element of P . Then, $(e_i)_{i \in I}$ is a basis of the k -module $\mathfrak{h} \oplus N = \mathfrak{g}$. Proposition 5.10 thus yields that $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{\substack{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; \\ i_1 \leq i_2 \leq \dots \leq i_n}}$ is a basis of the k -module $U(\mathfrak{g})$.

Let I^* be the disjoint union of the sets I^n for all $n \in \mathbb{N}$. In other words, let I^* be the set of all finite sequences of elements of I . In particular, the empty sequence (i. e., the only element of I^0) is an element of I^* .

For every two elements $\kappa \in I^*$ and $\kappa' \in I^*$, we define an element $\kappa \cdot \kappa' \in I^*$ as follows: Write κ in the form $\kappa = (i_1, i_2, \dots, i_n)$ and write κ' in the form $\kappa' = (j_1, j_2, \dots, j_m)$; then set $\kappa \cdot \kappa' = (i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_m)$.

Let K be the set

$$\begin{aligned} & \{(i_1, i_2, \dots, i_n) \mid (i_1, i_2, \dots, i_n) \in I^*; i_1 \leq i_2 \leq \dots \leq i_n\} \\ & = \bigcup_{n \in \mathbb{N}} \{(i_1, i_2, \dots, i_n) \mid (i_1, i_2, \dots, i_n) \in I^n; i_1 \leq i_2 \leq \dots \leq i_n\}. \end{aligned}$$

In other words, K is the set of all increasing finite sequences of elements of I .

Let us note that every $q \in K \cap Q^*$ and $p \in K \cap P^*$ satisfy $q \cdot p \in K$. (This is because we have chosen a well-ordering on I such that every element of Q is smaller than any element of P .)

For every $\kappa \in I^*$, define an element \mathbf{e}_κ of $U(\mathfrak{g})$ by

$$\mathbf{e}_\kappa = \overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}}, \quad \text{where } (i_1, i_2, \dots, i_n) \text{ is such that } \kappa = (i_1, i_2, \dots, i_n).$$

Then, $(\mathbf{e}_\kappa)_{\kappa \in K}$ is a basis of the k -module $U(\mathfrak{g})$ (since this is just another way to state our knowledge that $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; i_1 \leq i_2 \leq \dots \leq i_n}$ is a basis of the k -module $U(\mathfrak{g})$). Thus, $U(\mathfrak{g}) = \langle \mathbf{e}_\kappa \mid \kappa \in K \rangle$.

Let X be the subset $\{(i_1, i_2, \dots, i_n) \in K \mid i_n \in P\}$ of K .

Let Y be the subset $\{(i_1, i_2, \dots, i_n) \in K \mid n \leq m\}$ of K .

Clearly, $X \cap Y$ is the subset $\{(i_1, i_2, \dots, i_n) \in K \mid i_n \in P; n \leq m\}$ of K .

We notice that $(\mathbf{e}_\kappa)_{\kappa \in Y}$ is a basis of the k -module $U_{\leq m}(\mathfrak{g})$ (since this is just another way to state our knowledge that the family $(\overline{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}})_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; i_1 \leq i_2 \leq \dots \leq i_n; n \leq m}$ is a basis of the k -module $U_{\leq m}(\mathfrak{g})$).

The definition of \mathbf{e}_κ readily yields

$$\mathbf{e}_\kappa \cdot \mathbf{e}_{\kappa'} = \mathbf{e}_{\kappa \cdot \kappa'} \quad \text{for every } \kappa \in I^* \text{ and } \kappa' \in I^*. \quad (133)$$

Also note that

$$\mathbf{e}_{(i)} = e_i \quad \text{for every } i \in I, \quad (134)$$

where e_i means $\overline{e_i} = \psi(e_i) \in U(\mathfrak{g})$ on the right hand side (this is a slight abuse of notation, but legitimate in view of the Poincaré-Birkhoff-Witt theorem).

Now let us show that $U(\mathfrak{g}) \cdot \mathfrak{h} = \langle \mathbf{e}_\kappa \mid \kappa \in X \rangle$. In fact,

$$\begin{aligned} & \underbrace{U(\mathfrak{g})}_{= \langle \mathbf{e}_\kappa \mid \kappa \in K \rangle} \cdot \underbrace{\mathfrak{h}}_{\substack{= \langle e_i \mid i \in P \rangle \\ \text{(since } (e_i)_{i \in P} \text{ is a basis of } \mathfrak{h})}} = \langle \mathbf{e}_\kappa \mid \kappa \in K \rangle \cdot \langle e_i \mid i \in P \rangle \\ & = \langle \mathbf{e}_\kappa e_i \mid (\kappa, i) \in K \times P \rangle. \end{aligned} \quad (135)$$

Hence, in order to prove that $U(\mathfrak{g}) \cdot \mathfrak{h} = \langle \mathbf{e}_\kappa \mid \kappa \in X \rangle$, it is enough to show that $\langle \mathbf{e}_\kappa e_i \mid (\kappa, i) \in K \times P \rangle = \langle \mathbf{e}_\kappa \mid \kappa \in X \rangle$. In order to do so, we must prove that

$$\langle \mathbf{e}_\kappa \mid \kappa \in X \rangle \subseteq \langle \mathbf{e}_\kappa e_i \mid (\kappa, i) \in K \times P \rangle \quad (136)$$

and

$$\langle \mathbf{e}_\kappa e_i \mid (\kappa, i) \in K \times P \rangle \subseteq \langle \mathbf{e}_\kappa \mid \kappa \in X \rangle. \quad (137)$$

Proof of (136). Every $\kappa \in X$ can be written in the form (i_1, i_2, \dots, i_n) for some $n \in \mathbb{N}$ and $(i_1, i_2, \dots, i_n) \in K$ satisfying $i_n \in P$. Thus,

$$\begin{aligned} \mathbf{e}_\kappa &= \mathbf{e}_{(i_1, i_2, \dots, i_n)} = \mathbf{e}_{(i_1, i_2, \dots, i_{n-1}) \cdot (i_n)} = \underbrace{\mathbf{e}_{(i_1, i_2, \dots, i_{n-1})}}_{\in U(\mathfrak{g})} \underbrace{\mathbf{e}_{(i_n)}}_{=e_{i_n} \text{ (by (134))}} && \text{(by (133))} \\ &\in U(\mathfrak{g}) \underbrace{e_{i_n}}_{\in \mathfrak{h} \text{ (since } i_n \in P)} \subseteq U(\mathfrak{g}) \cdot \mathfrak{h}. \end{aligned}$$

We have thus shown that every $\kappa \in X$ satisfies $\mathbf{e}_\kappa \in U(\mathfrak{g}) \cdot \mathfrak{h}$. Therefore,

$$\langle \mathbf{e}_\kappa \mid \kappa \in X \rangle \subseteq U(\mathfrak{g}) \cdot \mathfrak{h}. \quad (138)$$

Combined with (135), this proves (136).

Proof of (137). Let $(\kappa, i) \in K \times P$ be arbitrary. We only have to prove that $\mathbf{e}_\kappa e_i \in \langle \mathbf{e}_{\kappa'} \mid \kappa' \in X \rangle$ (because once this is shown for every $(\kappa, i) \in K \times P$, it will become clear that $\langle \mathbf{e}_\kappa e_i \mid (\kappa, i) \in K \times P \rangle \subseteq \langle \mathbf{e}_{\kappa'} \mid \kappa' \in X \rangle = \langle \mathbf{e}_\kappa \mid \kappa \in X \rangle$, and this will prove (137)).

Since $\kappa \in K$, we can write κ in the form (i_1, i_2, \dots, i_n) for some $n \in \mathbb{N}$ and $(i_1, i_2, \dots, i_n) \in I^n$ satisfying $i_1 \leq i_2 \leq \dots \leq i_n$. Let $i_{n+1} = i$. Let ν be the smallest integer in $\{1, 2, \dots, n+1\}$ such that $i_\nu \in P$ (such a ν exists since $i_{n+1} = i \in P$). Then, $i_1, i_2, \dots, i_{\nu-1}$ lie in Q whereas $i_\nu, i_{\nu+1}, \dots, i_n$ lie in P (this is because $i_1 \leq i_2 \leq \dots \leq i_n$ and because every element of Q is smaller than any element of P). Since $i_{n+1} = i$ also lies in P , we conclude that all of the elements $i_\nu, i_{\nu+1}, \dots, i_{n+1}$ lie in P . This yields $(i_\nu, i_{\nu+1}, \dots, i_{n+1}) \in P^*$.

Since $i_1, i_2, \dots, i_{\nu-1}$ lie in Q , we have $(i_1, i_2, \dots, i_{\nu-1}) \in Q^*$. Combined with $(i_1, i_2, \dots, i_{\nu-1}) \in K$ (since $i_1 \leq i_2 \leq \dots \leq i_{\nu-1}$), this yields $(i_1, i_2, \dots, i_{\nu-1}) \in K \cap Q^*$.

Now we have

$$\begin{aligned} \mathbf{e}_\kappa e_i &= \mathbf{e}_{(i_1, i_2, \dots, i_n)} e_i && \text{(since } \kappa = (i_1, i_2, \dots, i_n)) \\ &= \mathbf{e}_{(i_1, i_2, \dots, i_n)} \underbrace{e_{i_{n+1}}}_{=e_{(i_{n+1})} \text{ (by (134))}} && \text{(since } i = i_{n+1}) \\ &= \mathbf{e}_{(i_1, i_2, \dots, i_n)} \mathbf{e}_{(i_{n+1})} = \mathbf{e}_{(i_1, i_2, \dots, i_n) \cdot (i_{n+1})} && \text{(by (133))} \\ &= \mathbf{e}_{(i_1, i_2, \dots, i_{n+1})} = \mathbf{e}_{(i_1, i_2, \dots, i_{\nu-1}) \cdot (i_\nu, i_{\nu+1}, \dots, i_{n+1})} = \mathbf{e}_{(i_1, i_2, \dots, i_{\nu-1})} \mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})} && \text{(by (133)).} \end{aligned}$$

But we can identify the universal enveloping algebra $U(\mathfrak{h})$ with a k -submodule of $U(\mathfrak{g})$ - namely, with the k -submodule $\langle \mathbf{e}_\kappa \mid \kappa \in P^* \rangle$ of $U(\mathfrak{g})$. The element $\mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})}$ of $U(\mathfrak{g})$ lies in this submodule $U(\mathfrak{h})$ (because $(i_\nu, i_{\nu+1}, \dots, i_{n+1}) \in P^*$).

Applying Proposition 5.13 to \mathfrak{h} and $(e_i)_{i \in P}$ instead of \mathfrak{g} and $(e_i)_{i \in I}$, we see that $\overline{(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n})}_{n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in P^n; i_1 \leq i_2 \leq \dots \leq i_n}$ is a generating set of the k -module $U(\mathfrak{h})$. In other words,

$$\begin{aligned} U(\mathfrak{h}) &= \left\langle \underbrace{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}}_{=e_{(i_1, i_2, \dots, i_n)}} \mid n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in P^n; \underbrace{i_1 \leq i_2 \leq \dots \leq i_n}_{\text{this is equivalent to } (i_1, i_2, \dots, i_n) \in K} \right\rangle \\ &= \langle \mathbf{e}_{(i_1, i_2, \dots, i_n)} \mid n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in P^n; (i_1, i_2, \dots, i_n) \in K \rangle \\ &= \langle \mathbf{e}_{(i_1, i_2, \dots, i_n)} \mid n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in K \cap P^n \rangle = \langle \mathbf{e}_\kappa \mid \kappa \in K \cap P^* \rangle. \end{aligned}$$

Therefore, $\mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})} = \sum_{\lambda \in K \cap P^*} \rho_\lambda \mathbf{e}_\lambda$ for some scalars $\rho_\lambda \in k$ (because $\mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})} \in U(\mathfrak{h})$). Consider these scalars ρ_λ . It is easily seen that $\rho_{()} = 0$, where $()$ denotes the empty sequence (in fact, if the map $\varepsilon_{U(\mathfrak{g})} : U(\mathfrak{g}) \rightarrow k$ is defined as in Proposition 1.86, then Proposition 1.86 (b) yields that $\varepsilon_{U(\mathfrak{g})}(\mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})}) = 0$ (because $(i_\nu, i_{\nu+1}, \dots, i_{n+1})$ is not the empty sequence) on one hand but $\varepsilon_{U(\mathfrak{g})}\left(\sum_{\lambda \in K \cap P^*} \rho_\lambda \mathbf{e}_\lambda\right) = \rho_{()}$ on the other, so that we obtain $0 = \rho_{()})$. Therefore, $\mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})} = \sum_{\lambda \in K \cap P^*} \rho_\lambda \mathbf{e}_\lambda$ can be rewritten as

$$\mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})} = \sum_{\substack{\lambda \in K \cap P^*; \\ \lambda \neq ()}} \rho_\lambda \mathbf{e}_\lambda$$

(here we removed the $\lambda = ()$ addend, since $\rho_{()} = 0$).

Now,

$$\begin{aligned} \mathbf{e}_\kappa e_i &= \mathbf{e}_{(i_1, i_2, \dots, i_{\nu-1})} \underbrace{\mathbf{e}_{(i_\nu, i_{\nu+1}, \dots, i_{n+1})}}_{\substack{= \sum_{\substack{\lambda \in K \cap P^*; \\ \lambda \neq ()}} \rho_\lambda \mathbf{e}_\lambda}} = \sum_{\substack{\lambda \in K \cap P^*; \\ \lambda \neq ()}} \rho_\lambda \underbrace{\mathbf{e}_{(i_1, i_2, \dots, i_{\nu-1})} \mathbf{e}_\lambda}_{= \mathbf{e}_{(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda} \text{ (by (133))}} \\ &= \sum_{\substack{\lambda \in K \cap P^*; \\ \lambda \neq ()}} \rho_\lambda \mathbf{e}_{(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda}. \end{aligned}$$

Now, every $\lambda \in K \cap P^*$ satisfies $(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda \in K$ (because $(i_1, i_2, \dots, i_{\nu-1}) \in K \cap Q^*$ and $\lambda \in K \cap P^*$, and because every $q \in K \cap Q^*$ and $p \in K \cap P^*$ satisfy $q \cdot p \in K$). Therefore, every $\lambda \in K \cap P^*$ such that $\lambda \neq ()$ satisfies $(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda \in X$ (because $(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda \in K$ on the one hand, but on the other hand the last element of the sequence $(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda$ lies in P ⁴³). Thus,

$$\mathbf{e}_\kappa e_i = \sum_{\substack{\lambda \in K \cap P^*; \\ \lambda \neq ()}} \rho_\lambda \mathbf{e}_{(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda} \in \langle \mathbf{e}_{\kappa'} \mid \kappa' \in X \rangle$$

(since $(i_1, i_2, \dots, i_{\nu-1}) \cdot \lambda \in X$ for every $\lambda \in K \cap P^*$ such that $\lambda \neq ()$). This proves (137).

We could also prove $U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h} = \langle \mathbf{e}_\kappa \mid \kappa \in X \cap Y \rangle$ by a similar argument, but this would be a slight overkill. Instead we only need the weaker result that $\langle \mathbf{e}_\kappa \mid \kappa \in X \cap Y \rangle \subseteq U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$, which can be seen exactly the same way as we have shown (138).

Now, combining $U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h} \subseteq (U(\mathfrak{g}) \cdot \mathfrak{h}) \cap U_{\leq m}(\mathfrak{g})$ (this is because $\underbrace{U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}}_{\subseteq U(\mathfrak{g})} \subseteq$

$U(\mathfrak{g}) \cdot \mathfrak{h}$ and $U_{\leq(m-1)}(\mathfrak{g}) \cdot \underbrace{\mathfrak{h}}_{\subseteq \mathfrak{g}} \subseteq U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{g} \subseteq U_{\leq m}(\mathfrak{g})$) with

$$\begin{aligned} \underbrace{(U(\mathfrak{g}) \cdot \mathfrak{h})}_{= \langle \mathbf{e}_\kappa \mid \kappa \in X \rangle} \cap \underbrace{U_{\leq m}(\mathfrak{g})}_{= \langle \mathbf{e}_\kappa \mid \kappa \in Y \rangle} &= \langle \mathbf{e}_\kappa \mid \kappa \in X \rangle \cap \langle \mathbf{e}_\kappa \mid \kappa \in Y \rangle \\ &= \langle \mathbf{e}_\kappa \mid \kappa \in X \cap Y \rangle \quad (\text{by Lemma 5.17}) \\ &\subseteq U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}, \end{aligned}$$

⁴³This is because $\lambda \neq ()$ and $\lambda \in P^*$.

we obtain $(U(\mathfrak{g}) \cdot \mathfrak{h}) \cap U_{\leq m}(\mathfrak{g}) = U_{\leq (m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$. In other words, Proposition 5.16 is proven.

5.4. The kernel of

$$\mathrm{gr}_n \tau : \mathrm{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \mathrm{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}))$$

Next we prove a result that generalizes Lemma 4.3 of [2]:

Theorem 5.18. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Let $n \in \mathbb{N}$. Assume that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion (but not necessarily as an \mathfrak{h} -module inclusion). This means that there exists a k -submodule N of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus N$.

Let us work with the notations introduced in Theorem 2.1 and in Definition 1.64. Let also ψ denote the canonical projection from the k -algebra $\otimes \mathfrak{g}$ to the factor algebra $(\otimes \mathfrak{g}) / I_{\mathfrak{g}} = U(\mathfrak{g})$. Clearly, ψ is a k -algebra homomorphism.

Let us abbreviate the k -submodule $U(\mathfrak{g}) \cdot \psi(\mathfrak{h})$ of $U(\mathfrak{g})$ by $U(\mathfrak{g}) \cdot \mathfrak{h}$.

Let ρ be the canonical k -module projection $U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$.

(a) There exists one and only one map $\theta : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ for which the diagram

$$\begin{array}{ccc} \otimes \mathfrak{g} & \xrightarrow{\psi} & U(\mathfrak{g}) \\ \zeta \downarrow & & \downarrow \rho \\ (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) & \xrightarrow{\theta} & U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \end{array} \quad (139)$$

commutes. Denote this map θ by τ .

(b) This τ is a surjective \mathfrak{h} -module homomorphism. Also, it satisfies $\tau \circ \zeta = \rho \circ \psi$. In other words, the diagram

$$\begin{array}{ccc} \otimes \mathfrak{g} & \xrightarrow{\psi} & U(\mathfrak{g}) \\ \zeta \downarrow & & \downarrow \rho \\ (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) & \xrightarrow{\tau} & U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \end{array} \quad (140)$$

commutes.

(c) Every of the four corners of the commutative square (140) is endowed with a filtration - namely as follows:

- The filtration on $\otimes \mathfrak{g}$ is the degree filtration $(\mathfrak{g}^{\otimes \leq n})_{n \geq 0}$.
- The filtration on $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is the filtration $(F_n)_{n \geq 0}$ defined in Theorem 2.1 (b) by $F_n = \zeta(\mathfrak{g}^{\otimes \leq n})$.
- The filtration on $U(\mathfrak{g})$ is the filtration $(U_{\leq n}(\mathfrak{g}))_{n \geq 0}$ defined in Convention 5.11 by $U_{\leq n}(\mathfrak{g}) = \psi(\mathfrak{g}^{\otimes \leq n})$.
- The filtration on $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ is the filtration $(W_n)_{n \geq 0}$ defined by $W_n = (\tau \circ \zeta)(\mathfrak{g}^{\otimes \leq n})$.

Then, $W_n = (\rho \circ \psi)(\mathfrak{g}^{\otimes \leq n}) = \rho(U_{\leq n}(\mathfrak{g})) = \tau(F_n)$. Also, the maps ψ , ρ , ζ and τ all respect the filtration. Therefore, for every $n \in \mathbb{N}$, we can apply the functor gr_n to

the diagram (140), and obtain the commutative diagram

$$\begin{array}{ccc}
 \mathrm{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\mathrm{gr}_n \psi} & \mathrm{gr}_n(U(\mathfrak{g})) \\
 \mathrm{gr}_n \zeta \downarrow & & \downarrow \mathrm{gr}_n \rho \\
 \mathrm{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) & \xrightarrow{\mathrm{gr}_n \tau} & \mathrm{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}))
 \end{array} \quad . \quad (141)$$

(d) Assume that both \mathfrak{h} and N are free k -modules. Let $n \in \mathbb{N}$. Then, $(\mathrm{grad}_{n,n}^{-1} \circ \omega_n)(\mathrm{Ker}(\mathrm{gr}_n \tau)) = K_n(\mathfrak{n})$. (For the definition of $K_n(\mathfrak{n})$, see Definition 5.1, applied to $V = \mathfrak{n}$. For the definition of $\mathrm{grad}_{n,n}$ and ω_n , see Theorem 2.1 (c).)

The condition that both \mathfrak{h} and N are free in (d) is somewhat restrictive - we will partly lift it in Subsection 6.4.

Proof of Theorem 5.18. (a) Let J_0 denote the k -submodule

$$\langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle$$

of $\otimes \mathfrak{g}$. Then, $J = (\otimes \mathfrak{g}) \cdot J_0 \cdot (\otimes \mathfrak{g})$ (by Proposition 2.3 (b)). On the other hand,

$$\begin{aligned}
 J_0 &= \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \\
 &\subseteq \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle
 \end{aligned}$$

(since $\mathfrak{g} \times \mathfrak{h} \subseteq \mathfrak{g} \times \mathfrak{g}$). Now

$$\begin{aligned}
 J &= (\otimes \mathfrak{g}) \cdot \underbrace{J_0}_{\subseteq \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle} \cdot (\otimes \mathfrak{g}) \\
 &\subseteq (\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{g} \rangle \cdot (\otimes \mathfrak{g}) = I_{\mathfrak{g}}.
 \end{aligned}$$

With the help of

$$\begin{aligned}
 \psi(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) &= \psi\left(\underbrace{J}_{\subseteq I_{\mathfrak{g}}}\right) + \underbrace{\psi((\otimes \mathfrak{g}) \cdot \mathfrak{h})}_{=\psi(\otimes \mathfrak{g}) \cdot \psi(\mathfrak{h})} \subseteq \underbrace{\psi(I_{\mathfrak{g}})}_{=0 \text{ (since } \psi \text{ is the projection on } (\otimes \mathfrak{g})/I_{\mathfrak{g}})} + \underbrace{\psi(\otimes \mathfrak{g}) \cdot \psi(\mathfrak{h})}_{\subseteq U(\mathfrak{g})} \\
 &\subseteq 0 + U(\mathfrak{g}) \cdot \psi(\mathfrak{h}) = U(\mathfrak{g}) \cdot \psi(\mathfrak{h}) = U(\mathfrak{g}) \cdot \mathfrak{h},
 \end{aligned}$$

we obtain

$$(\rho \circ \psi)(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = \rho\left(\underbrace{\psi(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})}_{\subseteq U(\mathfrak{g}) \cdot \mathfrak{h}}\right) \subseteq \rho(U(\mathfrak{g}) \cdot \mathfrak{h}) = 0$$

(since ρ is the projection on $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$). In other words, $(\rho \circ \psi)(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) = 0$.

Thus, by the homomorphism theorem, the map $\rho \circ \psi : \otimes \mathfrak{g} \rightarrow U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ factors through the factor map $\zeta : \otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$. In other words, there exists one and only one map $\theta : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ satisfying

$\theta \circ \zeta = \rho \circ \psi$, and this map θ is a k -module homomorphism. Since $\theta \circ \zeta = \rho \circ \psi$ is equivalent to the commutativity of the diagram (139), this result rewrites as follows: There exists one and only one map $\theta : (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ for which the diagram (139) commutes, and this map θ is a k -module homomorphism. This proves Theorem 5.18 (a), and even a part of Theorem 5.18 (b) (namely, the part saying that τ is a k -module homomorphism).

(b) We have already proven that τ is a k -module homomorphism. Now, we need only show that τ is surjective and an \mathfrak{h} -module homomorphism.

By the definition of τ , the diagram (139) commutes for $\theta = \tau$. That is, $\tau \circ \zeta = \rho \circ \psi$. Since the maps ρ and ψ are surjective (because ρ and ψ are projections), their composition $\rho \circ \psi$ is surjective. Thus, $\tau \circ \zeta = \rho \circ \psi$ is surjective, so that τ must too be surjective.

We know that ζ is surjective, and that $\tau \circ \zeta$ is an \mathfrak{h} -module homomorphism (since $\tau \circ \zeta = \rho \circ \psi$, and since both ρ and ψ are \mathfrak{h} -module homomorphisms). Applying Lemma 1.121 to $\otimes \mathfrak{g}$, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$, $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$, ζ and τ instead of A , B , C , f and g , we thus obtain that τ is an \mathfrak{h} -module homomorphism.

This completes the proof of Theorem 5.18 (b).

(c) It is pretty much trivial that $(W_n)_{n \geq 0}$ is indeed a filtration of $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ (all we use here is that $\tau \circ \zeta$ is surjective). Now, $W_n = (\tau \circ \zeta)(\mathfrak{g}^{\otimes \leq n})$ yields $W_n = (\rho \circ \psi)(\mathfrak{g}^{\otimes \leq n})$ because of $\tau \circ \zeta = \rho \circ \psi$. Also, $W_n = (\tau \circ \zeta)(\mathfrak{g}^{\otimes \leq n}) = \tau \left(\underbrace{\zeta(\mathfrak{g}^{\otimes \leq n})}_{=F_n} \right) =$

$\tau(F_n)$ and $W_n = (\rho \circ \psi)(\mathfrak{g}^{\otimes \leq n}) = \rho \left(\underbrace{\psi(\mathfrak{g}^{\otimes \leq n})}_{=U_{\leq n}(\mathfrak{g})} \right) = \rho(U_{\leq n}(\mathfrak{g}))$. It is now absolutely

obvious that the maps ψ , ρ , ζ and τ all respect the filtration. Theorem 5.18 (c) is now proven.

(d) Assume that both \mathfrak{h} and N are free k -modules. This, of course, yields that $\mathfrak{g} = \mathfrak{h} \oplus N$ must also be free, and thus \mathfrak{g} satisfies the n -PBW condition (by Theorem 5.9 (a)). Thus, Proposition 5.8 (b) yields $\text{Ker}(\text{gr}_n \psi) = \text{grad}_{\mathfrak{g},n}(K_n(\mathfrak{g}))$. In other words, $\text{grad}_{\mathfrak{g},n}^{-1}(\text{Ker}(\text{gr}_n \psi)) = K_n(\mathfrak{g})$ (since $\text{grad}_{\mathfrak{g},n}$ is an isomorphism).

On the other hand, the construction of $K_n(V)$ for a k -module V yields that whenever $f : A \rightarrow B$ is a surjective k -linear map between two k -modules A and B , then $f^{\otimes n}(K_n(A)) = K_n(B)$. Applying this to $A = \mathfrak{g}$, $B = \mathfrak{n}$ and $f = \pi$, we obtain $\pi^{\otimes n}(K_n(\mathfrak{g})) = K_n(\mathfrak{n})$.

Our goal is to show that $(\text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n)(\text{Ker}(\text{gr}_n \tau)) = K_n(\mathfrak{n})$. This will be done once we have proven that

$$\text{Ker}(\text{gr}_n \tau) = (\text{gr}_n \zeta)(\text{Ker}(\text{gr}_n \psi)). \quad (142)$$

In fact, once (142) is shown, we have

$$\begin{aligned}
& (\text{grad}_{n,n}^{-1} \circ \omega_n) \left(\underbrace{\text{Ker}(\text{gr}_n \tau)}_{=(\text{gr}_n \zeta)(\text{Ker}(\text{gr}_n \psi))} \right) \\
&= (\text{grad}_{n,n}^{-1} \circ \omega_n) ((\text{gr}_n \zeta)(\text{Ker}(\text{gr}_n \psi))) = \left(\text{grad}_{n,n}^{-1} \circ \underbrace{\omega_n \circ \text{gr}_n \zeta}_{\substack{=\text{gr}_n(\otimes\pi) \\ \text{(since the diagram} \\ \text{(45) commutes)}}} \right) (\text{Ker}(\text{gr}_n \psi)) \\
&= \underbrace{(\text{grad}_{n,n}^{-1} \circ \text{gr}_n(\otimes\pi))}_{=\pi^{\otimes n} \circ \text{grad}_{\mathfrak{g},n}^{-1}} (\text{Ker}(\text{gr}_n \psi)) = (\pi^{\otimes n} \circ \text{grad}_{\mathfrak{g},n}^{-1})(\text{Ker}(\text{gr}_n \psi)) \\
&\quad \text{(this follows easily from the} \\
&\quad \text{definitions of } \text{grad}_{\mathfrak{g},n} \text{ and } \text{grad}_{n,n} \text{)} \\
&= \pi^{\otimes n} \left(\underbrace{\text{grad}_{\mathfrak{g},n}^{-1}(\text{Ker}(\text{gr}_n \psi))}_{=K_n(\mathfrak{g})} \right) = \pi^{\otimes n}(K_n(\mathfrak{g})) = K_n(\mathfrak{n}),
\end{aligned}$$

which proves Theorem 5.18 **(d)**. So, in order to complete the proof of Theorem 5.18 **(d)**, the only thing we need to do is verify (142).

According to Proposition 1.122 (applied to the diagram (141) instead of the diagram (43)), the equality (142) will follow once we can show that $\text{gr}_n \zeta$ is surjective and that

$$\text{Ker}(\text{gr}_n \rho) \subseteq (\text{gr}_n \psi)(\text{Ker}(\text{gr}_n \zeta)). \quad (143)$$

But it is easy to see that $\text{gr}_n \zeta$ is surjective (and it was proven in detail during the proof of Proposition 2.18 **(e)**). Thus, we only have to prove (143) now.

Due to the commutative diagram (45), and due to the fact that ω_n is an isomorphism, we have $\text{Ker}(\text{gr}_n \zeta) = \text{Ker}(\text{gr}_n(\otimes\pi))$.

Let $s \in \text{Ker}(\text{gr}_n \rho)$ be arbitrary. Then, $s \in \text{gr}_n(U(\mathfrak{g})) = (U_{\leq n}(\mathfrak{g})) / (U_{\leq (n-1)}(\mathfrak{g}))$, so that there exists some $S \in U_{\leq n}(\mathfrak{g})$ such that $s = \overline{S}$.⁴⁴ Consider this S . Since $s \in \text{Ker}(\text{gr}_n \rho)$, we have $(\text{gr}_n \rho)(s) = 0$. But $s = \overline{S}$ shows that $(\text{gr}_n \rho)(s) = (\text{gr}_n \rho)(\overline{S}) = \overline{\rho(S)}$, so that $(\text{gr}_n \rho)(s) = 0$ becomes $\overline{\rho(S)} = 0$. In other words, $\rho(S) \in W_{n-1} = \rho(U_{\leq (n-1)}(\mathfrak{g}))$. This means that there exists some $S' \in U_{\leq (n-1)}(\mathfrak{g})$ such that $\rho(S) = \rho(S')$. Consider this S' . Then, $\rho(S) = \rho(S')$ yields $0 = \rho(S) - \rho(S') = \rho(S - S')$, so that $S - S' \in \text{Ker} \rho = U(\mathfrak{g}) \cdot \mathfrak{h}$. On the other hand, $S \in U_{\leq n}(\mathfrak{g})$ and $S' \in U_{\leq (n-1)}(\mathfrak{g}) \subseteq U_{\leq n}(\mathfrak{g})$ lead to $S - S' \in U_{\leq n}(\mathfrak{g})$ (since $U_{\leq n}(\mathfrak{g})$ is a k -module). Combining this with $S - S' \in U(\mathfrak{g}) \cdot \mathfrak{h}$, we obtain

$$\begin{aligned}
S - S' &\in U_{\leq n}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) = U_{\leq (n-1)}(\mathfrak{g}) \cdot \mathfrak{h} \\
&\quad \text{(by Proposition 5.16, applied to } n \text{ instead of } m\text{)} \\
&= \psi(\mathfrak{g}^{\otimes \leq (n-1)} \cdot \mathfrak{h}).
\end{aligned}$$

⁴⁴Henceforth until the end of this proof of Theorem 5.18, the term \overline{S} always denotes the residue class of S modulo $U_{\leq (n-1)}(\mathfrak{g})$. Similarly, the terms $\overline{\rho(S)}$ and $\overline{S - S'}$ will have to be interpreted.

Thus,

$$\begin{aligned}
\overline{S - S'} &\in \overline{\psi(\mathfrak{g}^{\otimes \leq (n-1)} \cdot \mathfrak{h})} \\
&\left(\begin{array}{c} \text{here, } \overline{\psi(\mathfrak{g}^{\otimes \leq (n-1)} \cdot \mathfrak{h})} \text{ means the image of } \psi(\mathfrak{g}^{\otimes \leq (n-1)} \cdot \mathfrak{h}) \\ \text{under the canonical projection } U_{\leq n}(\mathfrak{g}) \rightarrow \text{gr}_n(U(\mathfrak{g})) \end{array} \right) \\
&= (\text{gr}_n \psi) \left(\begin{array}{c} \overline{\mathfrak{g}^{\otimes \leq (n-1)} \cdot \mathfrak{h}} \\ \subseteq \text{Ker}(\text{gr}_n(\otimes \pi)) = \text{Ker}(\text{gr}_n \zeta) \end{array} \right) \\
&\left(\begin{array}{c} \text{here, } \overline{\mathfrak{g}^{\otimes \leq (n-1)} \cdot \mathfrak{h}} \text{ means the image of } \mathfrak{g}^{\otimes \leq (n-1)} \cdot \mathfrak{h} \\ \text{under the canonical projection } \mathfrak{g}^{\otimes \leq n} \rightarrow \text{gr}_n(\otimes \mathfrak{g}) \end{array} \right) \\
&\subseteq (\text{gr}_n \psi)(\text{Ker}(\text{gr}_n \zeta)).
\end{aligned}$$

Now,

$$\begin{aligned}
s &= \overline{S} = \overline{S - S'} \quad (\text{since } S' \in U_{\leq (n-1)}(\mathfrak{g})) \\
&\in (\text{gr}_n \psi)(\text{Ker}(\text{gr}_n \zeta)).
\end{aligned}$$

Since this is shown for every $s \in \text{Ker}(\text{gr}_n \rho)$, we thus conclude that $\text{Ker}(\text{gr}_n \rho) \subseteq (\text{gr}_n \psi)(\text{Ker}(\text{gr}_n \zeta))$. Thus, (143) is shown. This completes the proof of Theorem 5.18 (d).

5.5. The associated graded object of $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$

Here an important consequence of Theorem 5.18:

Corollary 5.19. Consider the situation of Theorem 5.18 (d). Let $n \in \mathbb{N}$.

Consider the canonical projection $\text{sym}_{\mathfrak{n},n} : \mathfrak{n}^{\otimes n} \rightarrow \text{Sym}^n \mathfrak{n}$.

(a) The map $\text{gr}_n \tau : \text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}))$ is surjective.

(b) There exists one and only one k -module homomorphism $\Xi : \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ for which the diagram

$$\begin{array}{ccc}
\text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) & \xrightarrow{\text{gr}_n \tau} & \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) & (144) \\
\omega_n \downarrow \cong & & \downarrow \Xi & \\
\text{gr}_n(\otimes \mathfrak{n}) & & & \\
\text{grad}_{\mathfrak{n},n}^{-1} \downarrow \cong & & & \\
\mathfrak{n}^{\otimes n} & \xrightarrow{\text{sym}_{\mathfrak{n},n}} & \text{Sym}^n \mathfrak{n} &
\end{array}$$

commutes.

(c) Let us denote this map Ξ by Θ_n . Then, Θ_n is an \mathfrak{h} -module isomorphism $\text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ which satisfies $\text{sym}_{\mathfrak{n},n} \circ \text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n = \Theta_n \circ \text{gr}_n \tau$.

(d) Let π be the canonical projection $\mathfrak{g} \rightarrow \mathfrak{g} / \mathfrak{h} = \mathfrak{n}$. The diagram

$$\begin{array}{ccc}
\text{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\text{gr}_n(\rho \circ \psi)} & \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) & (145) \\
\text{gr}_n(\otimes \pi) \downarrow & & \downarrow \Theta_n & \\
\text{gr}_n(\otimes \mathfrak{n}) & & & \\
\text{grad}_{\mathfrak{n},n}^{-1} \downarrow \cong & & & \\
\mathfrak{n}^{\otimes n} & \xrightarrow{\text{sym}_{\mathfrak{n},n}} & \text{Sym}^n \mathfrak{n} &
\end{array}$$

■ commutes.

Proof of Corollary 5.19. **(a)** This follows from Proposition 1.102, because $\tau(F_n) = W_n$.

(b) We have $\text{Ker sym}_{\mathfrak{n},\mathfrak{n}} = K_n(\mathfrak{n})$ (since $\text{sym}_{\mathfrak{n},\mathfrak{n}}$ is the projection from $\mathfrak{n}^{\otimes n}$ on $\mathfrak{n}^{\otimes n}/K_n(\mathfrak{n})$).

Since $\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n$ is a k -module isomorphism (which is because $\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1}$ and ω_n are k -module isomorphisms), we have

$$\begin{aligned} \text{Ker}(\text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n) &= (\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n)^{-1} \left(\underbrace{\text{Ker sym}_{\mathfrak{n},\mathfrak{n}}}_{=K_n(\mathfrak{n})} \right) = (\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n)^{-1} (K_n(\mathfrak{n})) \\ &= \text{Ker}(\text{gr}_n \tau) \end{aligned}$$

(because Theorem 5.18 **(d)** says that $(\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n)(\text{Ker}(\text{gr}_n \tau)) = K_n(\mathfrak{n})$, and because $\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n$ is an isomorphism). In particular, this yields that $\text{Ker}(\text{gr}_n \tau) \subseteq \text{Ker}(\text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n)$. Since $\text{gr}_n \tau$ is surjective, the homomorphism theorem thus yields that there exists one and only one k -module homomorphism $\Xi : \text{gr}_n(U(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ which satisfies $\Xi \circ \text{gr}_n \tau = \text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n$. In other words, there exists one and only one k -module homomorphism $\Xi : \text{gr}_n(U(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ for which the diagram (144) commutes. This proves Corollary 5.19 **(b)**.

(c) The map $\text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n : \text{gr}_n((\otimes \mathfrak{g})/(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ is surjective (since $\text{sym}_{\mathfrak{n},\mathfrak{n}}$ is surjective while $\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1}$ and ω_n are isomorphisms). Since $\text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n = \Theta_n \circ \text{gr}_n \tau$ (because the map Θ_n is defined as the map Ξ for which the diagram (144) commutes), this yields that the map $\Theta_n \circ \text{gr}_n \tau$ is surjective. Hence, the map Θ_n is surjective.

On the other hand, let i be an arbitrary element of $\text{Ker} \Theta_n$. Then, we can write i in the form $i = (\text{gr}_n \tau)(i')$ for some $i' \in \text{gr}_n((\otimes \mathfrak{g})/(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}))$ (since $\text{gr}_n \tau$ is surjective). Now, $i \in \text{Ker} \Theta_n$ yields $\Theta_n(i) = 0$, so that

$$\left(\underbrace{\text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n}_{=\Theta_n \circ \text{gr}_n \tau} \right) (i') = (\Theta_n \circ \text{gr}_n \tau)(i') = \Theta_n \left(\underbrace{(\text{gr}_n \tau)(i')}_{=i} \right) = \Theta_n(i) = 0,$$

thus $i' \in \text{Ker}(\text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n) = \text{Ker}(\text{gr}_n \tau)$, so that $0 = (\text{gr}_n \tau)(i') = i$. We have thus shown that every $i \in \text{Ker} \Theta_n$ satisfies $i = 0$. Thus, $\text{Ker} \Theta_n = 0$, so that Θ_n is injective.

Since Θ_n is surjective and injective and a k -module homomorphism, we conclude that Θ_n is a k -module isomorphism.

We know that $\text{gr}_n \tau$ is a surjective \mathfrak{h} -module homomorphism (since $\text{gr}_n \tau$ is surjective according to part **(a)**, and is an \mathfrak{h} -module homomorphism since τ is an \mathfrak{h} -module homomorphism (by Theorem 5.18 **(b)**)), and we know that $\Theta_n \circ \text{gr}_n \tau$ is an \mathfrak{h} -module homomorphism (since $\Theta_n \circ \text{gr}_n \tau = \text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n$, and since all of the maps $\text{sym}_{\mathfrak{n},\mathfrak{n}}$, $\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1}$ and ω_n are \mathfrak{h} -module homomorphisms). Applying Lemma 1.121 to $\text{gr}_n(\otimes \mathfrak{g}/(J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}))$, $\text{gr}_n(U(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{h}))$, $\text{Sym}^n \mathfrak{n}$, $\text{gr}_n \tau$ and Θ_n instead of A , B , C , f and g , we thus conclude that Θ_n is an \mathfrak{h} -module homomorphism. Combining

this with the fact that Θ_n is a k -module isomorphism, we conclude that Θ_n is an \mathfrak{h} -module isomorphism. This proves Corollary 5.19 (c).

(d) By the functoriality of gr_n , we have $\text{gr}_n(\tau \circ \zeta) = \text{gr}_n \tau \circ \text{gr}_n \zeta$. On the other hand, the commutative diagram (45) yields $\omega_n \circ \text{gr}_n \zeta = \text{gr}_n(\otimes \pi)$. Now,

$$\begin{aligned} \Theta_n \circ \text{gr}_n \left(\begin{array}{c} \underbrace{\rho \circ \psi}_{=\tau \circ \zeta} \\ \text{(by Theorem 5.18 (b))} \end{array} \right) &= \Theta_n \circ \underbrace{\text{gr}_n(\tau \circ \zeta)}_{=\text{gr}_n \tau \circ \text{gr}_n \zeta} = \underbrace{\Theta_n \circ \text{gr}_n \tau}_{=\text{sym}_{\mathfrak{n},n} \circ \text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n} \circ \text{gr}_n \zeta \\ &\quad \text{(by Corollary 5.19 (c))} \\ &= \text{sym}_{\mathfrak{n},n} \circ \text{grad}_{\mathfrak{n},n}^{-1} \circ \underbrace{\omega_n \circ \text{gr}_n \zeta}_{=\text{gr}_n(\otimes \pi)} = \text{sym}_{\mathfrak{n},n} \circ \text{grad}_{\mathfrak{n},n}^{-1} \circ \text{gr}_n(\otimes \pi). \end{aligned}$$

In other words, the diagram (145) commutes. This proves Corollary 5.19 (d).

Note that Corollary 5.19 can be used to prove Theorem 5.9 (a). This is not particularly surprising and not particularly useful, as we have used Theorem 5.9 (a) in our proof of Corollary 5.19. But let us give the proof for the sake of completeness:

Proof of Theorem 5.9 (a) using Corollary 5.19: Let $n \in \mathbb{N}$. Let \mathfrak{g} be a k -Lie algebra which is a free k -module. Let $\mathfrak{h} = 0$ and $N = \mathfrak{g}$. Then, \mathfrak{h} is a Lie subalgebra of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus N$. Moreover, both \mathfrak{h} and N are free k -modules. Moreover, the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion. Thus, we can apply Corollary 5.19 (d) and obtain that the diagram (145) commutes (where we are using the notations of Corollary 5.19, of course). But since we are in a situation where $\mathfrak{h} = 0$ and $\mathfrak{n} = \mathfrak{g}/\mathfrak{h} = \mathfrak{g}$, this diagram simplifies to

$$\begin{array}{ccc} \text{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\text{gr}_n \psi} & \text{gr}_n(U(\mathfrak{g})) \\ \text{id} \downarrow & & \downarrow \Theta_n \\ \text{gr}_n(\otimes \mathfrak{g}) & & \\ \text{grad}_{\mathfrak{g},n}^{-1} \cong \downarrow & & \\ \mathfrak{g}^{\otimes n} & \xrightarrow{\text{sym}_{\mathfrak{g},n}} & \text{Sym}^n \mathfrak{g} \end{array}$$

(since $\pi = \text{id}$, $\otimes \pi = \text{id}$, $\text{gr}_n(\otimes \pi) = \text{id}$ and $\rho = \text{id}$). Thus, $\Theta_n \circ \text{gr}_n \psi = \text{sym}_{\mathfrak{g},n} \circ \text{grad}_{\mathfrak{g},n}^{-1}$. Thus, $\Theta_n \circ \text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n} = \text{sym}_{\mathfrak{g},n}$, so that

$$\text{sym}_{\mathfrak{g},n} = \Theta_n \circ \underbrace{\text{gr}_n \psi \circ \text{grad}_{\mathfrak{g},n}}_{=\text{PBW}_{\mathfrak{g},n} \circ \text{sym}_{\mathfrak{g},n}} = \Theta_n \circ \text{PBW}_{\mathfrak{g},n} \circ \text{sym}_{\mathfrak{g},n}.$$

(since this is how we defined $\text{PBW}_{\mathfrak{g},n}$)

Since $\text{sym}_{\mathfrak{g},n}$ is surjective, this yields $\text{id} = \Theta_n \circ \text{PBW}_{\mathfrak{g},n}$. Thus, $\text{PBW}_{\mathfrak{g},n}$ is the inverse of the \mathfrak{h} -module isomorphism Θ_n (here, we are using the fact that Θ_n is an \mathfrak{h} -module isomorphism; this follows from Corollary 5.19 (c)). This yields that $\text{PBW}_{\mathfrak{g},n}$ itself is an \mathfrak{h} -module isomorphism. In other words, \mathfrak{g} satisfies the n -PBW condition, and our proof of Theorem 5.9 (a) is complete. As already explained, this proof does not take us far, as the proof of Corollary 5.19 given above made substantial use of Theorem 5.9 (a); but at least it shows that Corollary 5.19 is indeed a generalization of Theorem 5.9 (a).

5.6. The splitting of the filtration

The next theorem encompasses a part of [2, Theorem 4.5], and (together with the theorems we have already proven) will (almost) complete the proof of the main result of [2] ("almost" because the converse direction will still be missing, but it is rather easy and straightforward).

Theorem 5.20. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Assume that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion (but not necessarily as an \mathfrak{h} -module inclusion). This means that there exists a k -submodule N of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus N$.

Assume that both \mathfrak{h} and N are free k -modules.

Let us consider the \mathfrak{g} -module $U(\mathfrak{g})$.⁴⁵

Let us also use the notations introduced in Theorem 2.1, in particularly the \mathfrak{h} -module \mathfrak{n} .

We also consider the \mathfrak{h} -module $\text{Sym}^n \mathfrak{n}$ (see Definition 5.3 for its definition) for every $n \in \mathbb{N}$.

Assume that this \mathfrak{h} -module \mathfrak{n} is actually the restriction of some $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h} .

Now let us assume the following statement, which we call the **symmetric splitting assumption: The canonical projection $\text{sym}_{\mathfrak{n}, \mathfrak{n}} : \mathfrak{n}^{\otimes n} \rightarrow \text{Sym}^n \mathfrak{n}$ (defined in Definition 5.1) splits as an \mathfrak{h} -module projection for every $n \in \mathbb{N}$.**

(Note that the symmetric splitting assumption is automatically satisfied in the case when every positive integer is invertible in the ring k , because in this case we can split the projection $\text{sym}_{\mathfrak{n}, \mathfrak{n}} : \mathfrak{n}^{\otimes n} \rightarrow \text{Sym}^n \mathfrak{n}$ by the map $\text{Sym}^n \mathfrak{n} \rightarrow \mathfrak{n}^{\otimes n}$ which sends

$\overline{v_1 \otimes v_2 \otimes \dots \otimes v_n}$ to $\frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)}$ for every $(v_1, v_2, \dots, v_n) \in \mathfrak{n}^n$. But

this is not the only case in which the symmetric splitting assumption holds.)

Define a filtration $(W_n)_{n \geq 0}$ on $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ as in Theorem 5.18 (c).

Then, the filtration $(W_n)_{n \geq 0}$ is an \mathfrak{h} -split \mathfrak{h} -module filtration.

More precisely, we can construct a splitting for the \mathfrak{h} -module projection $W_n \rightarrow W_n / W_{n-1}$ for every $n \geq 1$ as follows:

Consider the map \mathfrak{h} -module isomorphism $\Theta_n : \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ from Corollary 5.19 (c). Since $\text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) = W_n / W_{n-1}$, this Θ_n is thus an \mathfrak{h} -module isomorphism $W_n / W_{n-1} \rightarrow \text{Sym}^n \mathfrak{n}$.

Let $\mathbf{I}_n : \text{Sym}^n \mathfrak{n} \rightarrow \mathfrak{n}^{\otimes n}$ be an \mathfrak{h} -module homomorphism such that $\text{sym}_{\mathfrak{n}, \mathfrak{n}} \circ \mathbf{I}_n = \text{id}$.

(In other words, let $\mathbf{I}_n : \text{Sym}^n \mathfrak{n} \rightarrow \mathfrak{n}^{\otimes n}$ be an \mathfrak{h} -module homomorphism which splits the projection $\text{sym}_{\mathfrak{n}, \mathfrak{n}} : \mathfrak{n}^{\otimes n} \rightarrow \text{Sym}^n \mathfrak{n}$. Such an \mathbf{I}_n exists due to the symmetric splitting assumption. If every positive integer is invertible in the ring k , then we can even find a *canonical* \mathbf{I}_n .)

Since the filtration $(F_n)_{n \geq 0}$ is \mathfrak{h} -split (by Theorem 4.1), the exact sequence

$$0 \longrightarrow F_{n-1} \xrightarrow{\text{inclusion}} F_n \xrightarrow{\text{projection}} F_n / F_{n-1} \longrightarrow 0$$

is \mathfrak{h} -split. So there exists some \mathfrak{h} -module homomorphism $\vartheta_n : F_n / F_{n-1} \rightarrow F_n$ which splits the projection $F_n \rightarrow F_n / F_{n-1}$. Consider this ϑ_n .

Let $\tau|_{F_n}^{W_n}$ denote the map $F_n \rightarrow W_n$ obtained by restricting τ to F_n (since we know that $\tau(F_n) = W_n$).

Then, the map $(\tau|_{F_n}^{W_n}) \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n}, \mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n^{-1} : W_n / W_{n-1} \rightarrow W_n$ is an \mathfrak{h} -module

homomorphism which splits the projection $W_n \rightarrow W_n/W_{n-1}$.

Proof of Theorem 5.20. First, let us make sure that the map $(\tau|_{F_n}^{W_n}) \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n : W_n/W_{n-1} \rightarrow W_n$ is well-defined at all. This is since $\Theta_n : W_n/W_{n-1} \rightarrow \text{Sym}^n \mathfrak{n}$ (in fact, we know that $\Theta_n : \text{gr}_n(U(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$, but by definition of gr_n we have $\text{gr}_n(U(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{h})) = W_n/W_{n-1}$), since $\mathbf{I}_n : \text{Sym}^n \mathfrak{n} \rightarrow \mathfrak{n}^{\otimes n}$, since $\text{grad}_{\mathfrak{n},\mathfrak{n}} : \mathfrak{n}^{\otimes n} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$, since $\omega_n^{-1} : \text{gr}_n(\otimes \mathfrak{n}) \rightarrow F_n/F_{n-1}$ (because $\omega_n : F_n/F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$), since $\vartheta_n : F_n/F_{n-1} \rightarrow F_n$ and since $\tau|_{F_n}^{W_n} : F_n \rightarrow W_n$.

Second, this map $(\tau|_{F_n}^{W_n}) \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n$ is indeed an \mathfrak{h} -module homomorphism, since it is a composition of six \mathfrak{h} -module homomorphisms.

We must now prove that this map $(\tau|_{F_n}^{W_n}) \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n$ splits the projection $W_n \rightarrow W_n/W_{n-1}$. In fact, let us denote the projection $W_n \rightarrow W_n/W_{n-1}$ by Γ_n . Let us denote the projection $F_n \rightarrow F_n/F_{n-1}$ by Γ'_n . Then, $\Gamma_n \circ (\tau|_{F_n}^{W_n}) = \text{gr}_n \tau \circ \Gamma'_n$ follows easily from the definition of gr_n . On the other hand, $\Gamma'_n \circ \vartheta_n = \text{id}$ (since ϑ_n is defined as a splitting of the projection $F_n \rightarrow F_n/F_{n-1}$, but that projection $F_n \rightarrow F_n/F_{n-1}$ is Γ'_n). Thus,

$$\begin{aligned} & \Gamma_n \circ \left((\tau|_{F_n}^{W_n}) \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n \right) \\ &= \underbrace{\Gamma_n \circ (\tau|_{F_n}^{W_n})}_{=\text{gr}_n \tau \circ \Gamma'_n} \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n \\ &= \text{gr}_n \tau \circ \underbrace{\Gamma'_n \circ \vartheta_n}_{=\text{id}} \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n = \text{gr}_n \tau \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n \\ &= \Theta_n^{-1} \circ \text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \underbrace{\omega_n \circ \omega_n^{-1}}_{=\text{id}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n \\ & \quad \left(\begin{array}{l} \text{since } \text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n = \Theta_n \circ \text{gr}_n \tau \text{ (by Corollary 5.19 (c))}, \\ \text{and thus } \text{gr}_n \tau = \Theta_n^{-1} \circ \text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \omega_n \text{ (since } \Theta_n \text{ is an isomorphism)} \end{array} \right) \\ &= \Theta_n^{-1} \circ \text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \underbrace{\text{grad}_{\mathfrak{n},\mathfrak{n}}^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}}}_{=\text{id}} \circ \mathbf{I}_n \circ \Theta_n = \Theta_n^{-1} \circ \underbrace{\text{sym}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n}_{=\text{id}} \circ \Theta_n = \Theta_n^{-1} \circ \Theta_n = \text{id}. \end{aligned}$$

In other words, the map $(\tau|_{F_n}^{W_n}) \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n$ splits the projection $W_n \rightarrow W_n/W_{n-1}$ (because Γ_n is the projection $W_n \rightarrow W_n/W_{n-1}$). Therefore, the exact sequence $0 \longrightarrow W_{n-1} \xrightarrow{\text{inclusion}} W_n \xrightarrow{\text{projection}} W_n/W_{n-1} \longrightarrow 0$ is \mathfrak{h} -split (since the map $(\tau|_{F_n}^{W_n}) \circ \vartheta_n \circ \omega_n^{-1} \circ \text{grad}_{\mathfrak{n},\mathfrak{n}} \circ \mathbf{I}_n \circ \Theta_n$ is an \mathfrak{h} -module homomorphism). Since we have proven this for each $n \in \mathbb{N}$, we thus conclude that the filtration $(W_n)_{n \geq 0}$ is \mathfrak{h} -split.

This proves Theorem 5.20.

5.7. Non-canonical isomorphisms

Corollary 5.19 (c) gave us a canonical \mathfrak{h} -module isomorphism $\text{gr}_n(U(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^n \mathfrak{n}$ for every $n \in \mathbb{N}$. Therefore,

$$\bigoplus_{n \in \mathbb{N}} \text{gr}_n(U(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{h})) \cong \bigoplus_{n \in \mathbb{N}} \text{Sym}^n \mathfrak{n} = \text{Sym} \mathfrak{n}$$

⁴⁵Again, let us remind ourselves that "the \mathfrak{g} -module $U(\mathfrak{g})$ " is to be understood in accordance to Remark 1.67 here.

by a canonical isomorphism. Now, the standard intuition for the direct sum $\bigoplus_{n \in \mathbb{N}} \text{gr}_n V$ (where V is some filtered k -module) is that this sum is a kind of "approximation" for V , which is usually simpler than V itself (for example, $U(\mathfrak{g})$ is generally a non-commutative algebra, while $\bigoplus_{n \in \mathbb{N}} \text{gr}_n(U(\mathfrak{g}))$ is a commutative one). So now, knowing that the "approximation" $\bigoplus_{n \in \mathbb{N}} \text{gr}_n(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}))$ of $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ is isomorphic to $\text{Sym } \mathfrak{n}$, we can ask ourselves what $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$ itself is isomorphic to. It turns out that in the situation of Theorem 5.20, we have the same answer, but the isomorphism is not canonical anymore:

Proposition 5.21. Consider the situation of Theorem 5.20.

We have $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \cong \text{Sym } \mathfrak{n}$ as \mathfrak{h} -modules.

More precisely, there is an \mathfrak{h} -module isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \text{Sym } \mathfrak{n}$ which respects the filtration such that the inverse of this isomorphism also respects the filtration.

Proof of Proposition 5.21. We know from Theorem 5.20 that the filtration $(W_n)_{n \geq 0}$ is \mathfrak{h} -split. Thus, Proposition 1.118 (applied to $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})$, $(W_n)_{n \geq 0}$ and \mathfrak{h} instead of V , $(V_n)_{n \geq 0}$ and \mathfrak{g}) yields that there exists a bifiltered \mathfrak{h} -module isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}))$.

On the other hand, every $p \in \mathbb{N}$ satisfies $\text{gr}_p(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \cong \text{Sym}^p \mathfrak{n}$ as \mathfrak{h} -modules (because Corollary 5.19 (c) (applied to p instead of n) shows that Θ_p is an \mathfrak{h} -module isomorphism $\text{gr}_p(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \text{Sym}^p \mathfrak{n}$). Therefore, there exists a bifiltered \mathfrak{h} -module isomorphism $\bigoplus_{p \in \mathbb{N}} \text{gr}_p(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h})) \rightarrow \bigoplus_{p \in \mathbb{N}} \text{Sym}^p \mathfrak{n} = \text{Sym } \mathfrak{n}$. Composing this isomorphism with our bifiltered \mathfrak{h} -module isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \bigoplus_{p \in \mathbb{N}} \text{gr}_p(U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}))$, we obtain a bifiltered \mathfrak{h} -module isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \text{Sym } \mathfrak{n}$. By the definition of "bifiltered", this is an isomorphism which respects the filtration such that the inverse of this isomorphism also respects the filtration. This proves Proposition 5.21.

Even if the (somewhat restrictive) conditions of Theorem 5.20 are not satisfied, we can still obtain a k -module isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \cong \text{Sym } \mathfrak{n}$ under somewhat weaker conditions:

Proposition 5.22. Consider the situation of Theorem 5.18. Assume that \mathfrak{h} and N are free k -modules.

We have $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \cong \text{Sym } \mathfrak{n}$ as k -modules.

More precisely, there is a k -module isomorphism $U(\mathfrak{g}) / (U(\mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \text{Sym } \mathfrak{n}$ which respects the filtration such that the inverse of this isomorphism also respects the filtration.

Proof of Proposition 5.22. The idea of the proof of Proposition 5.22 is to proceed as in the proof of Proposition 5.21, but to read "k-split", "k-module" and "k-module homomorphism" instead of each " \mathfrak{h} -split", " \mathfrak{h} -module" and " \mathfrak{h} -module homomorphism", respectively. Do the same modifications in the proof of Theorem 5.20.

Of course, it is not really that easy, because Theorem 5.20 has some more conditions than we have assumed in Proposition 5.22. Here is how to deal with them:

- Since we read "k-module" instead of "h-module", the symmetric splitting assumption (which is needed for Theorem 5.20 to hold) now takes the following form: "The canonical projection $\text{sym}_{n,n} : \mathfrak{n}^{\otimes n} \rightarrow \text{Sym}^n \mathfrak{n}$ (defined in Definition 5.1) splits as a k-module projection for every $n \in \mathbb{N}$." But this is obviously satisfied, because $\mathfrak{n} \cong N$ (as k-modules) is a free k-module.
- The assumption that the h-module \mathfrak{n} be the restriction of some $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h} is not granted anymore. Fortunately, we only use it to make sure that the filtration $(F_n)_{n \geq 0}$ is h-split. Since we read "k-split" instead of "h-split", we therefore just need a new argument for why the filtration $(F_n)_{n \geq 0}$ is k-split. We have more or less done this in Section 2 already: From Proposition 2.18 (b) and (d), the map $\bar{\varphi}$ (constructed in Proposition 2.18 (a)) is a bifiltered isomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes N$. Since the filtration $(N^{\otimes \leq n})_{n \geq 0}$ of $\otimes N$ is k-split, we can now easily conclude that so is the filtration $(F_n)_{n \geq 0}$ of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$.

This proves Proposition 5.22.

6. Generalizations, improvements and analogues

While the results we gave above were already somewhat more general than those of [2], some of them can be extended even further, and/or shown to have analogues. These extensions and analogues have never been studied in detail, and neither am I going to do so in the present paper, but I will discuss them in brief in this Section 6.

6.1. When \mathfrak{g} is not a Lie algebra

The results of Sections 2 and 4 can be substantially generalized once we notice the following: All results of Sections 2 and 4 were formulated in the following setup:

\mathfrak{g} is a k-Lie algebra, and \mathfrak{h} is a Lie subalgebra of \mathfrak{g} .

We consider \mathfrak{g} as an h-module (by restricting the \mathfrak{g} -module \mathfrak{g}).

However, we have never used this setup in its full glory in Sections 2 and 4, and everything done in these Sections can be extended to the case when this setup is replaced by the following one:

\mathfrak{h} is a k-Lie algebra, and \mathfrak{g} is an h-module which happens to contain the h-module \mathfrak{h} as an h-submodule.

For example, Theorem 2.1 takes the following form in this case:

Theorem 6.1. Let k be a commutative ring. Let \mathfrak{h} be a k-Lie algebra, and let \mathfrak{g} be an h-module (not necessarily a Lie algebra itself!). Assume that the h-module \mathfrak{h} itself (this h-module \mathfrak{h} is defined according to Definition 1.17, applied to \mathfrak{h} instead of \mathfrak{g}) is an h-submodule of \mathfrak{g} .

Assume that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k-module inclusion (but not necessarily

as an \mathfrak{h} -module inclusion).

Let J be the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v + w \rightharpoonup v \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

of the k -algebra $\otimes \mathfrak{g}$.

Let $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$. This \mathfrak{n} is an \mathfrak{h} -module (because both \mathfrak{g} and \mathfrak{h} are \mathfrak{h} -modules).

Let $\pi : \mathfrak{g} \rightarrow \mathfrak{n}$ be the canonical projection with kernel \mathfrak{h} . Obviously, π is an \mathfrak{h} -module homomorphism. Thus, $\otimes \pi : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ is also an \mathfrak{h} -module homomorphism (according to Proposition 1.68).

We consider \mathfrak{h} as an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ by means of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow \otimes \mathfrak{g}$.

(a) Both J and $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$. Thus, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is an \mathfrak{h} -module. Let $\zeta : \otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ be the canonical projection. Then, ζ is an \mathfrak{h} -module homomorphism.

(b) For every $n \in \mathbb{N}$, let F_n be the \mathfrak{h} -submodule $\zeta(\mathfrak{g}^{\otimes \leq n})$ of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$.⁴⁶ Also define an \mathfrak{h} -submodule F_{-1} of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ by $F_{-1} = 0$. Then, $(F_n)_{n \geq 0}$ is an \mathfrak{h} -module filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ and satisfies $F_n / F_{n-1} \cong \mathfrak{n}^{\otimes n}$ as \mathfrak{h} -modules for every $n \in \mathbb{N}$.

(c) Let $n \in \mathbb{N}$. There exists one and only one k -module homomorphism $\Omega_n : F_n / F_{n-1} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$ for which the diagram

$$\begin{array}{ccc} \text{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\text{gr}_n(\otimes \pi)} & \\ \text{gr}_n \zeta \downarrow & & \\ \text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) = F_n / F_{n-1} & \xrightarrow{\Omega_n} & \text{gr}_n(\otimes \mathfrak{n}) \end{array}$$

commutes. Denote this homomorphism Ω_n by ω_n . Then, ω_n is an \mathfrak{h} -module isomorphism, and the diagram

$$\begin{array}{ccc} \text{gr}_n(\otimes \mathfrak{g}) & \xrightarrow{\text{gr}_n(\otimes \pi)} & \\ \text{gr}_n \zeta \downarrow & & \\ \text{gr}_n((\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})) = F_n / F_{n-1} & \xrightarrow{\omega_n} & \text{gr}_n(\otimes \mathfrak{n}) \end{array}$$

commutes.

Applying Definition 1.105 to \mathfrak{n} and n instead of V and p , we obtain a map $\text{grad}_{\mathfrak{n},n} : \mathfrak{n}^{\otimes n} \rightarrow \text{gr}_n(\otimes \mathfrak{n})$. According to Proposition 1.108 (applied to \mathfrak{h} , n and \mathfrak{n} instead of \mathfrak{g} , p and V), this map $\text{grad}_{\mathfrak{n},n}$ is a canonical \mathfrak{h} -module isomorphism. Thus, its inverse $\text{grad}_{\mathfrak{n},n}^{-1}$ is an \mathfrak{h} -module isomorphism as well. The composition $\text{grad}_{\mathfrak{n},n}^{-1} \circ \omega_n : F_n / F_{n-1} \rightarrow \mathfrak{n}^{\otimes n}$ is an \mathfrak{h} -module isomorphism (because ω_n and $\text{grad}_{\mathfrak{n},n}^{-1}$ are \mathfrak{h} -module isomorphisms).

This is indeed a generalization of Theorem 2.1, because in the situation of Theorem

⁴⁶In fact, $\zeta(\mathfrak{g}^{\otimes \leq n})$ is indeed an \mathfrak{h} -submodule (because ζ is an \mathfrak{h} -module homomorphism and because $\mathfrak{g}^{\otimes \leq n}$ is an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$).

2.1, we have:

$$\left(\begin{array}{l} \text{the ideal } J \text{ defined in Theorem 2.1 is identical} \\ \text{with the ideal } J \text{ defined in Theorem 6.1} \end{array} \right). \quad (146)$$

Proof of (146). Every $(v, w) \in \mathfrak{g} \times \mathfrak{h}$ satisfies $[v, w] = -[w, v]$ (due to (5)) and $w \rightharpoonup v = [w, v]$ (due to (9), applied to w and v instead of v and w). Thus, every $(v, w) \in \mathfrak{g} \times \mathfrak{h}$ satisfies

$$\begin{aligned} v \otimes w - w \otimes v - \underbrace{[v, w]}_{=-[w, v]} &= v \otimes w - w \otimes v + \underbrace{[w, v]}_{=w \rightharpoonup v} \\ &= v \otimes w - w \otimes v + w \rightharpoonup v. \end{aligned} \quad (147)$$

Now,

$$\begin{aligned} & \text{(the ideal } J \text{ defined in Theorem 6.1)} \\ &= (\otimes \mathfrak{g}) \cdot \left\langle \underbrace{v \otimes w - w \otimes v + w \rightharpoonup v}_{=v \otimes w - w \otimes v - [v, w]} \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \right\rangle \cdot (\otimes \mathfrak{g}) \\ &= (\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v - [v, w] \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g}) \\ &= \text{(the ideal } J \text{ defined in Theorem 2.1)}. \end{aligned}$$

This proves (146).

To give a proof of Theorem 6.1, we just have to repeat the proof of Theorem 2.1 that we gave in Section 2 (including all the auxiliary facts we showed in Section 2) up to the following changes:

- Replace the words "g-module" by "h-module".
- Replace the words "g-algebra" by "h-algebra".
- Replace the words "g-submodule" by "h-submodule".
- Whenever a term of the form $[x, y]$ for some $x \in \mathfrak{g}$ and $y \in \mathfrak{g}$ appears in Section 2, proceed by the following rules:
 - If x is known to lie in \mathfrak{h} , replace this term by $x \rightharpoonup y$.
 - If y is known to lie in \mathfrak{h} , replace this term by $-y \rightharpoonup x$.

(If both x and y are known to lie in \mathfrak{h} , then it does not matter which of these two rules is being followed, because $x \rightharpoonup y = [x, y] = -[y, x] = -y \rightharpoonup x$ for any $x \in \mathfrak{h}$ and $y \in \mathfrak{h}$.)

Fortunately, all terms of the form $[x, y]$ that appear in Section 2 have either x or y lying in \mathfrak{h} , so that after these replacements, no terms of the form $[x, y]$ remain anymore.

Most results in Section 3 can be generalized as soon as we extend Definition 3.1 (the definition of a $(\mathfrak{g}, \mathfrak{h})$ -semimodule) as follows:

Definition 6.2. Let k be a commutative ring. Let \mathfrak{h} be a k -Lie algebra, and let \mathfrak{g} be an \mathfrak{h} -module (not necessarily a Lie algebra itself!). Assume that the \mathfrak{h} -module \mathfrak{h} itself (this \mathfrak{h} -module \mathfrak{h} is defined according to Definition 1.17, applied to \mathfrak{h} instead of \mathfrak{g}) is an \mathfrak{h} -submodule of \mathfrak{g} .

Let V be a k -module. Let $\mu : \mathfrak{g} \times V \rightarrow V$ be a k -bilinear map. We say that (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule if and only if

$$(\mu(a \rightarrow b, v) = \mu(a, \mu(b, v)) - \mu(b, \mu(a, v)) \text{ for every } a \in \mathfrak{h}, b \in \mathfrak{g} \text{ and } v \in V). \quad (148)$$

If (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, then the k -bilinear map $\mu : \mathfrak{g} \times V \rightarrow V$ is called the *Lie action* of the $(\mathfrak{g}, \mathfrak{h})$ -semimodule V .

Often, when the map μ is obvious from the context, we abbreviate the term $\mu(a, v)$ by $a \rightarrow v$ for any $a \in \mathfrak{g}$ and $v \in V$. Using this notation, the relation (148) rewrites as

$$((a \rightarrow b) \rightarrow v = a \rightarrow (b \rightarrow v) - b \rightarrow (a \rightarrow v) \text{ for every } a \in \mathfrak{h}, b \in \mathfrak{g} \text{ and } v \in V).$$

Also, an abuse of notation allows us to write " V is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule" instead of " (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule" if the map μ is clear from the context or has not been introduced yet.

Besides, when (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, we will say that μ is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on V . In other words, if V is a k -module, then a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on V means a map $\mu : \mathfrak{g} \times V \rightarrow V$ such that (V, μ) is a $(\mathfrak{g}, \mathfrak{h})$ -semimodule. (Thus, in order to make a k -module into a $(\mathfrak{g}, \mathfrak{h})$ -semimodule, we must define a $(\mathfrak{g}, \mathfrak{h})$ -semimodule structure on it.)

Theorem 4.1 generalizes as follows:

Theorem 6.3. Let k be a commutative ring. Let \mathfrak{h} be a k -Lie algebra, and let \mathfrak{g} be an \mathfrak{h} -module (not necessarily a Lie algebra itself!). Assume that the \mathfrak{h} -module \mathfrak{h} itself (this \mathfrak{h} -module \mathfrak{h} is defined according to Definition 1.17, applied to \mathfrak{h} instead of \mathfrak{g}) is an \mathfrak{h} -submodule of \mathfrak{g} .

Assume that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a k -module inclusion (but not necessarily as an \mathfrak{h} -module inclusion).

Let J be the two-sided ideal

$$(\otimes \mathfrak{g}) \cdot \langle v \otimes w - w \otimes v + w \rightarrow v \mid (v, w) \in \mathfrak{g} \times \mathfrak{h} \rangle \cdot (\otimes \mathfrak{g})$$

of the k -algebra $\otimes \mathfrak{g}$.

Let $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$. This \mathfrak{n} is an \mathfrak{h} -module (because both \mathfrak{g} and \mathfrak{h} are \mathfrak{h} -modules). **Assume that this \mathfrak{h} -module \mathfrak{n} is actually the restriction of some $(\mathfrak{g}, \mathfrak{h})$ -semimodule to \mathfrak{h}** (where " $(\mathfrak{g}, \mathfrak{h})$ -semimodule" is to be understood according to Definition 6.2).

Let $\pi : \mathfrak{g} \rightarrow \mathfrak{n}$ be the canonical projection with kernel \mathfrak{h} . Obviously, π is an \mathfrak{h} -module homomorphism. Thus, $\otimes \pi : \otimes \mathfrak{g} \rightarrow \otimes \mathfrak{n}$ is also an \mathfrak{h} -module homomorphism (according to Proposition 1.68).

We consider \mathfrak{h} as an \mathfrak{h} -submodule of $\otimes \mathfrak{g}$ by means of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow \otimes \mathfrak{g}$.

(a) Both J and $(\otimes \mathfrak{g}) \cdot \mathfrak{h}$ are \mathfrak{h} -submodules of $\otimes \mathfrak{g}$. Thus, $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is an

\mathfrak{h} -module. Let $\zeta : \otimes \mathfrak{g} \rightarrow (\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ be the canonical projection. Then, ζ is an \mathfrak{h} -module homomorphism.

(b) For every $n \in \mathbb{N}$, let F_n be the \mathfrak{h} -submodule $\zeta(\mathfrak{g}^{\otimes \leq n})$ of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$. (That F_n indeed is an \mathfrak{h} -submodule was proven in Theorem 2.1 already.) Then, $(F_n)_{n \geq 0}$ is an \mathfrak{h} -module filtration of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ and satisfies $F_n \cong \mathfrak{n}^{\otimes \leq n}$ as \mathfrak{h} -modules for every $n \in \mathbb{N}$.

(c) There exists an \mathfrak{h} -module isomorphism $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h}) \rightarrow \otimes \mathfrak{n}$ such that for every $n \in \mathbb{N}$, the image of F_n under this isomorphism is $\mathfrak{n}^{\otimes \leq n}$.

(d) The filtration $(F_n)_{n \geq 0}$ of $(\otimes \mathfrak{g}) / (J + (\otimes \mathfrak{g}) \cdot \mathfrak{h})$ is \mathfrak{h} -split.

Again, the proof of this theorem is a repetition of the proof of Theorem 4.1 with the same replacements as we had to do to obtain a proof of Theorem 6.1.

The results of Section 5 probably cannot be generalized in a similar fashion.

A further surprise seems to be that the proofs of Theorems 6.1 and 6.3 apparently never use the axioms (3) and (4) for the k -Lie algebra \mathfrak{h} , instead only using

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]] \quad \text{for all } u \in \mathfrak{h}, v \in \mathfrak{h} \text{ and } w \in \mathfrak{h}.$$

”Apparently” because I have not had enough time to check that this indeed is the case. If it is, this means that Theorems 6.1 and 6.3 extend to Leibniz algebras in lieu of Lie algebras. I am not aware of a similar extension of the Poincaré-Birkhoff-Witt theorem.

6.2. The case of Lie superalgebras

The notion of a *Lie superalgebra* (also known under the name *super Lie algebra* and studied in [14], [24]) is one of the most well-understood generalizations of that of a Lie algebra. While classification results for Lie superalgebras are significantly harder than their non-super counterparts, most ”purely algebraic” properties of Lie algebras tend to have their analogues for Lie superalgebras, which usually are even proven in more or less the same manner. This has to do with the fact that Lie superalgebras are just Lie algebras in the category of super- k -modules; however there is also a much more pedestrian approach to proving properties of Lie superalgebras by re-reading the proofs of the corresponding facts about Lie algebras and adding signs via the Koszul rule.

Different sources disagree about the correct way to define the notion of a Lie superalgebra. This might have to do with the fact that the primary interest lies in Lie superalgebras over a field of characteristic 0 (rather than an arbitrary field, let alone a commutative ring), and all the definitions of a Lie superalgebra are equivalent to each other if we are over a field of characteristic 0. As I am interested in the general case, let me give the following definition of a Lie superalgebra (which is one of the most restrictive ones, but not as restrictive as [13, Definition 8.1.1]):

Definition 6.4. Let k be a commutative ring. A *k -Lie superalgebra* will mean a super- k -module \mathfrak{g} (see Definition 6.5 below) together with a k -bilinear map $\beta :$

$\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the conditions

$$(\beta(v, v) = 0 \text{ for every } v \in \mathfrak{g}_0); \quad (149)$$

$$\left(\begin{array}{l} (-1)^{i\ell} \beta(u, \beta(v, w)) + (-1)^{ji} \beta(v, \beta(w, u)) + (-1)^{\ell j} \beta(w, \beta(u, v)) = 0 \\ \text{for every } i \in \mathbb{Z}/2\mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z} \text{ and } \ell \in \mathbb{Z}/2\mathbb{Z} \\ \text{and every } u \in \mathfrak{g}_i, v \in \mathfrak{g}_j \text{ and } w \in \mathfrak{g}_\ell \end{array} \right); \quad (150)$$

$$\left(\begin{array}{l} \beta(v, w) = -(-1)^{ij} \beta(w, v) \\ \text{for every } i \in \mathbb{Z}/2\mathbb{Z} \text{ and } j \in \mathbb{Z}/2\mathbb{Z} \text{ and every } v \in \mathfrak{g}_i \text{ and } w \in \mathfrak{g}_j \end{array} \right); \quad (151)$$

$$(\beta(v, \beta(v, v)) = 0 \text{ for every } v \in \mathfrak{g}_1); \quad (152)$$

$$(\beta(\mathfrak{g}_i \times \mathfrak{g}_j) \subseteq \mathfrak{g}_{i+j} \text{ for every } i \in \mathbb{Z}/2\mathbb{Z} \text{ and } j \in \mathbb{Z}/2\mathbb{Z}). \quad (153)$$

This k -bilinear map $\beta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ will be called the *Lie bracket* of the k -Lie superalgebra \mathfrak{g} . We will often use the square brackets notation for β , which means that we are going to abbreviate $\beta(v, w)$ by $[v, w]$ for any $v \in \mathfrak{g}$ and $w \in \mathfrak{g}$. Using this notation, the equations (149), (150), (151), (152) and (153) rewrite as

$$([v, v] = 0 \text{ for every } v \in \mathfrak{g}_0); \quad (154)$$

$$\left(\begin{array}{l} (-1)^{i\ell} [u, [v, w]] + (-1)^{ji} [v, [w, u]] + (-1)^{\ell j} [w, [u, v]] = 0 \\ \text{for every } i \in \mathbb{Z}/2\mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z} \text{ and } \ell \in \mathbb{Z}/2\mathbb{Z} \\ \text{and every } u \in \mathfrak{g}_i, v \in \mathfrak{g}_j \text{ and } w \in \mathfrak{g}_\ell \end{array} \right); \quad (155)$$

$$\left(\begin{array}{l} [v, w] = -(-1)^{ij} [w, v] \\ \text{for every } i \in \mathbb{Z}/2\mathbb{Z} \text{ and } j \in \mathbb{Z}/2\mathbb{Z} \text{ and every } v \in \mathfrak{g}_i \text{ and } w \in \mathfrak{g}_j \end{array} \right); \quad (156)$$

$$([v, [v, v]] = 0 \text{ for every } v \in \mathfrak{g}_1); \quad (157)$$

$$([\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \text{ for every } i \in \mathbb{Z}/2\mathbb{Z} \text{ and } j \in \mathbb{Z}/2\mathbb{Z}) \quad (158)$$

(where $[\mathfrak{g}_i, \mathfrak{g}_j]$ means the k -linear span $\langle [v, w] \mid (v, w) \in \mathfrak{g}_i \times \mathfrak{g}_j \rangle$).

The equation (150) (or its equivalent version (155)) is called the *super-Jacobi identity*.

Here we have used the following definition:

Definition 6.5. Let k be a commutative ring. A *super- k -module* will mean a k -module V together with a pair (V_0, V_1) of k -submodules of V such that $V = V_0 \oplus V_1$. Here, 0 and 1 are considered not as integers, but as elements of $\mathbb{Z}/2\mathbb{Z}$ (so that $1 + 1 = 0$). This sounds like a useless requirement, but it helps us in handling super- k -modules notationally; for example, the equation (153) would not make sense if 0 and 1 would be considered as integers (because in the case $i = 1$ and $j = 1$, we would have $i + j = 2$, but there is no such thing as \mathfrak{g}_2 unless 2 is treated as an element of $\mathbb{Z}/2\mathbb{Z}$).

The k -submodule V_0 of V is called the *even part* of V . The k -submodule V_1 of V is called the *odd part* of V .

An element of V is said to be *homogeneous* if it lies in V_0 or in V_1 .

Convention 6.6. We are going to use the notation V_0 as a universal notation for the even part of a super- k -module V . This means that whenever we have some super- k -module V (it needs not be actually called V ; I only refer to it by V here in this Convention), the even part of V will be called V_0 .

Similarly, we are going to use the notation V_1 as a universal notation for the odd part of a super- k -module V .

Remark 6.7. Our Definition 6.4 differs from some definitions of a Lie superalgebra given in literature by having the axioms (149) and (152). These axioms are indeed dispensable when one is only interested in the case of k being a field of characteristic 0 (or, more generally, of k being a commutative ring in which 2 and 3 are invertible)⁴⁷. However, for the sake of generality, we keep these axioms in.

Just as the notion of Lie algebras gives birth to that of \mathfrak{g} -modules, we can define the notion of a \mathfrak{g} -supermodule (or just \mathfrak{g} -module) over a Lie superalgebra \mathfrak{g} :

Definition 6.8. Let k be a commutative ring. Let \mathfrak{g} be a Lie superalgebra. (According to Convention 6.6, this automatically entails that we denote by \mathfrak{g}_0 the even part of \mathfrak{g} , and denote by \mathfrak{g}_1 the odd part of \mathfrak{g} .)

Let V be a k -supermodule. (According to Convention 6.6, this automatically entails that we denote by V_0 the even part of V , and denote by V_1 the odd part of V .)

Let $\mu : \mathfrak{g} \times V \rightarrow V$ be a k -bilinear map. We say that (V, μ) is a \mathfrak{g} -supermodule if and only if

$$\left(\begin{array}{l} \mu([a, b], v) = \mu(a, \mu(b, v)) - (-1)^{ij} \mu(b, \mu(a, v)) \\ \text{for every } i \in \mathbb{Z}/2\mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z} \text{ and every } a \in \mathfrak{g}_i, b \in \mathfrak{g}_j \text{ and } v \in V \end{array} \right) \quad (159)$$

and

$$(\mu(\mathfrak{g}_i \times V_j) \subseteq V_{i+j} \text{ for every } i \in \mathbb{Z}/2\mathbb{Z} \text{ and } j \in \mathbb{Z}/2\mathbb{Z}).$$

If (V, μ) is a \mathfrak{g} -supermodule, then the k -bilinear map $\mu : \mathfrak{g} \times V \rightarrow V$ is called the *Lie action* of the \mathfrak{g} -supermodule V .

(This definition seems to be agreed on in most references. I have not seen any conflicting definitions as in the case of Definition 6.4.)

While I have not checked the details, I am convinced that all results of Sections 2, 3 and 4 (and Subsection 6.1) carry over to Lie superalgebras (and Lie supermodules) as long as 2 is invertible in the ground ring k . Even the invertibility of 2 might actually be redundant for most of these results (and it seems that the reason for its redundancy is the fact that most of the results still hold for Leibniz algebras - but, as I already said, this is not thoroughly checked). As for Section 5, trouble might come from the Poincaré-Birkhoff-Witt theorem (Theorem 5.9), whose validity in the Lie superalgebra case has not been studied to the extent it has been studied in the original, Lie algebraic case. However, there *are* two known results:

⁴⁷In fact,

- axiom (149) follows from axiom (151) if 2 is invertible in k ;
- axiom (152) follows from axiom (150) if 3 is invertible in k .

Theorem 6.9. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie superalgebra. Let $n \in \mathbb{N}$.

(a) If \mathfrak{g}_0 and \mathfrak{g}_1 are free k -modules, and if 2 is invertible in the ring k , then \mathfrak{g} satisfies the n -PBW condition.

(b) If k is a \mathbb{Q} -algebra, then \mathfrak{g} satisfies the n -PBW condition.

A proof of Theorem 6.9 (b) was given in [24, Part 1, Chapter 1, §1.3.7] and [22, §2.5]; a proof of Theorem 6.9 (a) can be found in [14, §2.3, Theorem 1]. Note that whoever claims that 3 must be invertible in the ring k in order for Theorem 6.9 (a) to hold is probably using a definition of Lie superalgebra which does not contain the axiom (152). *However*, even having the axiom (149) does not prevent us from having to require the invertibility of 2, unless we replace our definition of a Lie superalgebra by a significantly more complicated one ([13, Definition 8.1.1]), in which case we can indeed drop the invertibility of 2 ([13, Theorem 8.2.2]). Having said this, we are *not* going to use [13, Definition 8.1.1] as the definition of a Lie superalgebra in this paper; instead we will keep understanding a Lie superalgebra according to Definition 6.4. As a consequence, we will not be able to get rid of the condition that 2 be invertible in k in the Poincaré-Birkhoff-Witt theorem and its consequences.

The correct analogue of Theorem 5.10 now says:

Theorem 6.10. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie superalgebra. Assume that 2 is invertible in the ring k . Also assume that the k -module \mathfrak{g} has a basis $(e_i)_{i \in I}$, where I is a totally ordered set, and where e_i is homogeneous for every $i \in I$. Then,

$$\overline{(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n})} \quad \begin{array}{l} n \in \mathbb{N}; (i_1, i_2, \dots, i_n) \in I^n; \\ i_1 \leq i_2 \leq \dots \leq i_n; \\ \text{every } p \text{ which satisfies } (e_{i_p} \in \mathfrak{g}_1 \text{ and } e_{i_{p+1}} \in \mathfrak{g}_1) \text{ satisfies } i_p < i_{p+1} \end{array}$$

is a basis of the k -module $U(\mathfrak{g})$.

A proof of Theorem 6.10 in the case when k is a field of characteristic $\neq 2$ and $\neq 3$ can be found in [25, Theorem 6.1.1].

These changes in the formulation of the Poincaré-Birkhoff-Witt theorem(s) don't seem to keep Proposition 5.16 from retaining its validity in the case of \mathfrak{g} being a Lie superalgebra, at least as long as 2 is assumed to be invertible in k and we assume the even and the odd parts of \mathfrak{h} and N to be free k -modules (and not just \mathfrak{h} and N themselves). As a consequence, nothing speaks against the other results of Section 5 holding in this case, although this has yet to be verified more accurately.

6.3. Poincaré-Birkhoff-Witt type theorems for Clifford algebras

6.3.1. Clifford algebras

There is an analogy between Lie algebras and quadratic spaces, with universal enveloping algebras of Lie algebras on the one side corresponding to Clifford algebras of quadratic spaces on the other. This analogy, however, is marred by an imbalance: Numerous results which hold in high generality on the quadratic spaces side require additional assumptions or weakenings on the Lie algebras side. As a basic example,

let me show the quadratic-spaces counterpart of the Poincaré-Birkhoff-Witt theorem. First, the relevant definitions (I am not really working with quadratic spaces, but rather with spaces with bilinear forms):

Definition 6.11. Let k be a commutative ring. Let L be a k -module.

(a) Let $f : L \times L \rightarrow k$ be a k -bilinear form on L . We define the *Clifford algebra* $\text{Cl}(L, f)$ to be the k -algebra $(\otimes L) / I_f$, where I_f is the two-sided ideal

$$(\otimes L) \cdot \langle v \otimes v - f(v, v) \mid v \in L \rangle \cdot (\otimes L)$$

of the k -algebra $\otimes L$.

(b) We denote by $\wedge L$ the exterior algebra of the k -module L . Clearly, $\wedge L = \text{Cl}(L, \mathbf{0})$, where $\mathbf{0}$ denotes the bilinear form $L \times L \rightarrow k$ which sends every pair $(x, y) \in L \times L$ to 0.

Remark 6.12. You are reading right: In this Definition 6.11, the form f is not required to be symmetric, but only the values of $f(v, v)$ for $v \in L$ are actually used. Over a field of characteristic $\neq 2$ (and more generally, over a ring where 2 is invertible), every bilinear form $f : L \times L \rightarrow k$ has a "symmetrization", which means a symmetric bilinear form \tilde{f} satisfying $(\tilde{f}(v, v) = f(v, v)$ for every $v \in L$). But, in my experience [35], restricting one's attention to symmetric bilinear forms is not really of much use in the theory of Clifford algebras; most important facts don't require this.

Most texts define Clifford algebras for quadratic forms rather than bilinear forms. Unfortunately, I was not able to spot an undisputed definition of what a quadratic form on an arbitrary k -module is. If one uses the definition of a quadratic form given in [8, §3, n°4, Définition 2], then the main result I want to state (Theorem 6.13) is not valid for Clifford algebras of quadratic forms (that is, the Clifford algebra of a quadratic form on a k -module L is not necessarily isomorphic to $\wedge L$; see [34] for a counterexample). It is still valid when the k -module is free, but this should not come as a surprise: For free k -modules, every quadratic form can be written in the form $v \mapsto f(v, v)$ for some bilinear form f (but not necessarily a symmetric bilinear form f ; this is yet another reason not to require f to be symmetric in Definition 6.11); this shows that Definition 6.11 encompasses the notion of the Clifford algebra of a quadratic form at least for free k -modules.

6.3.2. Poincaré-Birkhoff-Witt for Clifford algebras

Now, how would an analogue of the Poincaré-Birkhoff-Witt theorem for Clifford algebras look like?

We can consider a commutative ring k , some k -module L with a k -bilinear form $f : L \times L \rightarrow k$, and some $n \in \mathbb{N}$. Let $\psi : \otimes L \rightarrow \text{Cl}(L, f)$ be the canonical projection. Consider the canonical filtration on $\text{Cl}(L, f)$ (that is, the filtration obtained from the degree filtration on $\otimes L$ via the projection ψ). Let $\wedge^n L$ denote the n -th exterior power of the k -module L , and let $\wedge_{L,n}$ be the canonical projection $L^{\otimes n} \rightarrow \wedge^n L$. Then, it can be easily seen that there exists a unique k -module homomorphism $\mathbf{P} : \wedge^n L \rightarrow \text{gr}_n(\text{Cl}(L, f))$ such that $\text{gr}_n \psi \circ \text{grad}_{L,n} = \mathbf{P} \circ \wedge_{L,n}$. In analogy to Definition 5.6 (a),

we can call this map \mathbf{P} the *n-PBW homomorphism* of the pair (L, f) . In analogy to Definition 5.6 (b), we could now say that the pair (L, f) satisfies *the n-PBW condition* if the *n-PBW homomorphism* of (L, f) is a k -module isomorphism.

We could now expect that an analogue of Theorem 5.9 would state several conditions under which the pair (L, f) satisfies the *n-PBW condition*. But things turn out simpler this time: The *n-PBW condition* is a tautological condition, since the *n-PBW homomorphism* of (L, f) always is an isomorphism! But actually things are even simpler: Not only do we have an isomorphism $\wedge^n L \cong \text{gr}_n(\text{Cl}(L, f))$ (the *n-PBW homomorphism*), but also this isomorphism has the form $\text{gr}_n(\underline{\alpha}_0^f)$ for an isomorphism $\underline{\alpha}_0^f : \wedge L \rightarrow \text{Cl}(L, f)$ (a k -module isomorphism, not a k -algebra isomorphism, of course). This means that not only the associated graded modules of $\wedge L$ and $\text{Cl}(L, f)$ are isomorphic, but also the k -modules $\wedge L$ and $\text{Cl}(L, f)$ are isomorphic themselves. We record this as a theorem:

Theorem 6.13. Let k be a commutative ring. Let L be a k -module. Let $f : L \times L \rightarrow k$ be a k -bilinear form on L . Then, there exists an isomorphism $\text{Cl}(L, f) \cong \wedge L$ of filtered k -modules.

This theorem is not new. It is an obvious consequence of combining [9, §9, n°3, Proposition 3] and [9, §9, n°3, Lemme 4]. It is also stated in [32, Theorem (2.16)] for the case when L is a finitely-generated projective k -module, but the proof uses neither the finite generation nor the projectivity assumption.

The particular case of Theorem 6.13 when k is a field is a rather well-known fact, which is unfortunately usually proven in ways which don't extend to the general case. In 2010, I rediscovered Theorem 6.13 as a generalization of this fact, and wrote down a proof in [35, Theorem 1], unaware of the result already been known.

6.3.3. A very rough outline of the proof

The proofs of Theorem 6.13 given in [9], [32] and [35] are essentially one and the same argument (but vary in notation and in the level of detail). I will sketch this argument⁴⁸, because it was the archetype for my construction of the map φ in Definition 2.4 and for my construction of the map γ in Definition 4.3.

The proof of Theorem 6.13 proceeds in a purely computational way by recursively constructing both an isomorphism $\underline{\alpha}_0^f : \wedge L \rightarrow \text{Cl}(L, f)$ (which actually turns out to be induced by an automorphism $\alpha^f : \otimes L \rightarrow \otimes L$ ⁴⁹) and its inverse (which turns out to be induced by an automorphism $\alpha^{-f} : \otimes L \rightarrow \otimes L$ which is constructed in the same way as $\alpha^f : \otimes L \rightarrow \otimes L$ except that it is based on the form $-f$ rather than f). Note that the isomorphism $\underline{\alpha}_0^f : \wedge L \rightarrow \text{Cl}(L, f)$ is called the *quantization map* in [23, §2.5], while its inverse is called the *symbol map*. Here is how these isomorphisms are constructed:

The construction starts off by defining a tensor $v \lrcorner^f U \in \otimes L$ for every $v \in L$ and

⁴⁸The notations I will be using in the following are those of [35, Theorem 1].

⁴⁹This automorphism α^f would be called λ_{-f} in the notations of [9, §9, n°2], and would be called $\widehat{-f}$ in the notations of [32, Chapter 2, §2].

$U \in \otimes L$.⁵⁰ This tensor $v \lrcorner U$ is defined in such a way that it bilinearly depends on (v, U) , and satisfies

$$v \lrcorner (u_1 \otimes u_2 \otimes \dots \otimes u_p) = \sum_{i=1}^p (-1)^{i-1} f(v, u_i) \cdot u_1 \otimes u_2 \otimes \dots \otimes \widehat{u_i} \otimes \dots \otimes u_p$$

for every $v \in L$ and $u_1, u_2, \dots, u_p \in L$

(where the hat over u_i means "omit the tensorand u_i from this tensor product"). (This is easily seen to be well-defined. The definition given in [35] is slightly different, but easily shown equivalent to the one given here.)

Now, we define a k -linear map $\alpha^f : \otimes L \rightarrow \otimes L$ by

$$\begin{aligned} \alpha^f(\lambda) &= \lambda && \text{for every } \lambda \in k = L^{\otimes 0}; \\ \alpha^f(u \cdot U) &= u \cdot \alpha^f(U) - u \lrcorner \alpha^f(U) && \text{for every } n \in \mathbb{N}, u \in L \text{ and } U \in L^{\otimes n}. \end{aligned}$$

After a bit of work, we see that this map α^f is well-defined and respects the degree filtration of $\otimes L$.

We can write down explicit formulae for α^f in low degrees:

$$\begin{aligned} \alpha^f(\lambda) &= \lambda && \text{for every } \lambda \in k = L^{\otimes 0}; \\ \alpha^f(u) &= u && \text{for any } u \in L; \\ \alpha^f(u \otimes v) &= u \otimes v - f(u, v) && \text{for any } u, v \in L; \\ \alpha^f(u \otimes v \otimes w) &= u \otimes v \otimes w - f(v, w)u + f(u, w)v - f(u, v)w && \text{for any } u, v, w \in L; \\ \alpha^f(u \otimes v \otimes w \otimes t) &= u \otimes v \otimes w \otimes t - f(w, t)u \otimes v + f(v, t)u \otimes w - f(v, w)u \otimes t \\ &\quad - f(u, v)w \otimes t + f(u, w)v \otimes t - f(u, t)v \otimes w \\ &\quad + f(w, t)f(u, v) - f(v, t)f(u, w) + f(v, w)f(u, t) \\ &&& \text{for any } u, v, w, t \in L. \end{aligned}$$

See also [35, §5] for a general (but rather unwieldy) combinatorial expression for $\alpha^f(u_1 \otimes u_2 \otimes \dots \otimes u_n)$ for arbitrary n .

The maps α^f for various bilinear forms f satisfy some surprising properties: First, $\alpha^0 = \text{id}$ and $\alpha^f \circ \alpha^g = \alpha^{f+g}$ for any two bilinear forms f and g (see [35, Theorem 32]). (This means that $f \mapsto \alpha^f$ defines a representation of the additive group $\{f : L \times L \rightarrow k \mid f \text{ is } k\text{-bilinear}\}$ on $\otimes L$. In how far this can be related to the general representation theory of Lie/algebraic groups is unclear to me at the moment.) As a consequence, α^f is an automorphism of $\otimes L$, and α^{-f} is its inverse.

Another property that can be showed by computation ([35, Theorem 31]) is that $\alpha^f(I_g) = I_{f+g}$ for any two k -bilinear forms f and g . In particular, this yields $\alpha^f(I_0) = I_f$ and $\alpha^{-f}(I_f) = I_0$. Since $\alpha^f(I_0) = I_f$, the map α^f induces a k -module homomorphism from $(\otimes L) / I_0 = \wedge L$ to $(\otimes L) / I_f = \text{Cl}(L, f)$. Similarly, the map α^{-f} induces the inverse of this k -module homomorphism. So we have constructed our isomorphism between $\wedge L$ and $\text{Cl}(L, f)$.

⁵⁰What I call $v \lrcorner U$ would be called $i_v^f(U)$ in the notations of [9, §9, n°2], and would be called $\overline{f}_v(U)$ in the notations of [32, Chapter 2, §2].

All steps of this argument can be found in the detailed version of [35]. However, the reader will probably be able to reconstruct them on her own using from the sketch given above, since most of what has been omitted is straightforward computation and induction arguments.

When k is a field of characteristic $\neq 2$, much shorter proofs of Theorem 6.13 abound (for example, a standard proof proceeds by symmetrization of the bilinear form f and subsequent construction of a Gram-Schmidt orthogonal basis of L , which gives a "canonical" form for the Clifford algebra). Probably because most users of Clifford algebras come from a geometrical or physical background and have little use for the luxury of allowing k to be an arbitrary commutative ring (or a field of characteristic 2), the general case of Theorem 6.13 appears to be little known to the mathematical community. However, it is the general, computational proof of Theorem 6.13 which, by its inductive construction of the map α^f , motivated my arguments in Sections 2 and 4 of the present paper. In fact, compare the above inductive definition

$$\begin{aligned} \alpha^f(\lambda) &= \lambda && \text{for every } \lambda \in k = L^{\otimes 0}; \\ \alpha^f(u \cdot U) &= u \cdot \alpha^f(U) - u \lrcorner^f \alpha^f(U) && \text{for every } n \in \mathbb{N}, u \in L \text{ and } U \in L^{\otimes n} \end{aligned}$$

of the map α^f in [35] with the inductive definition

$$\begin{aligned} \varphi(\lambda) &= \lambda && \text{for every } \lambda \in k = \mathfrak{g}^{\otimes 0}; \\ \varphi(u \cdot U) &= t(u) \cdot \varphi(U) + \varphi(s(u) \rightharpoonup U) && \text{for every } n \in \mathbb{N}, u \in \mathfrak{g} \text{ and } U \in \mathfrak{g}^{\otimes n} \end{aligned}$$

of the map φ in Section 2 of the present paper (this is not exactly the way we defined φ in Definition 2.4, but it is easily seen to be equivalent) and with the inductive definition

$$\begin{aligned} \gamma(\lambda) &= \lambda && \text{for every } \lambda \in k = \mathfrak{g}^{\otimes 0}; \\ \gamma(u \cdot U) &= \pi(u) \cdot \gamma(U) + u \rightharpoonup (\gamma(U)) && \text{for every } n \in \mathbb{N}, u \in \mathfrak{g} \text{ and } U \in \mathfrak{g}^{\otimes n} \end{aligned}$$

of the map γ in Section 4 of the present paper (this is not exactly the way we defined γ in Definition 4.3, but it is easily seen to be equivalent). The similarity between the terms $u \lrcorner^f \alpha^f(U)$ and $u \rightharpoonup (\gamma(U))$ is particularly obvious, since \lrcorner^f is a quadratic-space analogue of the \mathfrak{g} -action \rightharpoonup .

6.3.4. The heuristics of the proof

What idea was behind the recursive definition of the map α^f in [35]? It was inspired by the standard construction of the quantization map in characteristic 0, which, I think, goes back to Chevalley. This construction gives the following formula for the quantization map $q : \wedge L \rightarrow \text{Cl}(L, f)$ (this q is my $\underline{\alpha}_0^f$) when k is a field of characteristic 0 and f is a symmetric bilinear form:

$$q(u_1 \wedge u_2 \wedge \dots \wedge u_n) = \frac{1}{n!} \sum_{s \in S_n} (-1)^s u_{s(1)} u_{s(2)} \dots u_{s(n)} \quad \text{for every } u_1, u_2, \dots, u_n \in L$$

(the right hand side is to be understood as a product in $\text{Cl}(L, f)$). This appears, e. g., in [23, Chapter 2, Proposition 2.9]. I tried to transform this formula for q into an equivalent form which did not require k to have characteristic 0 anymore (i. e., which

did not contain the $\frac{1}{n!}$ in front of the sum). By trial and error, I came up with the following:

$$\begin{aligned}
q(u_1) &= u_1 && \text{(this is already okay);} \\
q(u_1 \wedge u_2) &= \frac{1}{2}(u_1u_2 - u_2u_1) = \frac{1}{2}(u_1u_2 + u_1u_2 - 2f(u_1, u_2)) \\
&\quad \left(\begin{array}{l} \text{since } u_1u_2 + u_2u_1 = (u_1 + u_2)^2 - u_1^2 - u_2^2 \\ = f(u_1 + u_2, u_1 + u_2) - f(u_1, u_1) - f(u_2, u_2) = 2f(u_1, u_2) \\ \text{in } \text{Cl}(L, f) \text{ (because } f \text{ is symmetric)} \end{array} \right) \\
&= u_1u_2 - f(u_1, u_2); \\
q(u_1 \wedge u_2 \wedge u_3) &= \frac{1}{6}(u_1u_2u_3 - u_1u_3u_2 - u_2u_1u_3 - u_3u_2u_1 + u_2u_3u_1 + u_3u_1u_2) \\
&= \dots && \left(\begin{array}{l} \text{another computation using the symmetry of } f \text{ and} \\ \text{the identity } v^2 = f(v, v) \text{ in } \text{Cl}(L, f) \text{ omitted} \end{array} \right) \\
&= u_1u_2u_3 - f(u_2, u_3)u_1 + f(u_1, u_3)u_2 - f(u_1, u_2)u_3.
\end{aligned}$$

These results suggested me the recursive equation $q(u \wedge U) = u \cdot q(U) - u \llcorner^f q(U)$ (where $u \llcorner^f q(U)$ makes sense because $u \llcorner I_f \subseteq I_f$). Once this equation was found, the next obvious step was to lift the map $q : \wedge L \rightarrow \text{Cl}(L, f)$ to a map $\alpha^f : \otimes L \rightarrow \otimes L$ because tensors are easier to deal with than elements of $\wedge L$. The most straightforward approach to construct such a lifting is by lifting the recursive equation $q(u \wedge U) = u \cdot q(U) - u \llcorner^f q(U)$ to $\otimes L$; so, I defined a map $\alpha^f : \otimes L \rightarrow \otimes L$ by

$$\begin{aligned}
\alpha^f(\lambda) &= \lambda && \text{for every } \lambda \in k = L^{\otimes 0}; \\
\alpha^f(u \cdot U) &= u \cdot \alpha^f(U) - u \llcorner^f \alpha^f(U) && \text{for every } n \in \mathbb{N}, u \in L \text{ and } U \in L^{\otimes n}.
\end{aligned}$$

It turned out that this map α^f is an isomorphism (due to Proposition 1.109) and that $\alpha^f(I_0) \subseteq I_f$ (by computation). Yet, this did not yet prove that q is an isomorphism; in fact, the latter would require showing that $\alpha^f(I_0) = I_f$, and not only $\alpha^f(I_0) \subseteq I_f$. Again, the most straightforward (to a constructivist) approach to this problem was to construct the inverse of α^f by recursion. Some experimentation showed that its inverse $(\alpha^f)^{-1}$ satisfies exactly the same recursive equation as α^f , up to a sign change:

$$(\alpha^f)^{-1}(u \cdot U) = u \cdot (\alpha^f)^{-1}(U) + u \llcorner^f (\alpha^f)^{-1}(U).$$

This means this inverse is α^{-f} . Searching for a reason why α^{-f} is the inverse of α^f , I began to suspect the $\alpha^f \circ \alpha^g = \alpha^{f+g}$ identity, and it did not take long for this identity to be proven (as everything is defined recursively, making induction easy). The rest was automatic. At the end of the journey, I was met by the surprising realization that f was nowhere required to be symmetric.

Note that [35, Theorem 38] shows that my map $\underline{\alpha}_0^f$ is indeed the same as q as long as the form f is symmetric.

6.3.5. A relative Poincaré-Birkhoff-Witt for Clifford algebras

Just like the standard Poincaré-Birkhoff-Witt theorem, the relative Poincaré-Birkhoff-Witt theorem (for example, in the avatar of Lemma 0.5), too, has a quadratic-space analogue with weaker conditions and a stronger assertion:

Theorem 6.14. Let k be a commutative ring. Let L be a k -module. Let $f : L \times L \rightarrow k$ be a k -bilinear form on L . Let M be a k -submodule of L such that $f(M \times M) = 0$ and such that the k -module inclusion $M \hookrightarrow L$ splits. Then, there exists a k -module isomorphism $\text{Cl}(L, f) \cong \wedge L$ which maps $\text{Cl}(L, f) \cdot M$ to $(\wedge L) \cdot M$. Therefore, $(\text{Cl}(L, f)) / (\text{Cl}(L, f) \cdot M) \cong (\wedge L) / ((\wedge L) \cdot M) \cong \wedge(L/M)$.

A proof of this theorem can be found in [35, Theorem 61 (b)]. The condition $f(M \times M) = 0$ is a quadratic-space analogue of the condition that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} ; it cannot be improved. The condition that the k -module inclusion $M \hookrightarrow L$ splits cannot be dropped either, but possibly can be weakened.

6.3.6. Remark on Weyl algebras

It is a known fact that if we extend the notion of Clifford algebras to k -supermodules rather than k -modules only, then we obtain the tensor product of the Clifford algebra of the even part and the Weyl algebra of the odd part - however, at the price of requiring that 2 is invertible in k . As long as we are ready to pay this price, all of our results on Clifford algebras carry over to Weyl algebras.

6.3.7. A relative version of Theorem 2.1

In the spirit of the above quadratic-space versions, here is an analogue of Theorem 2.1:

Theorem 6.15. Let k be a commutative ring. Let L be a k -module, and let M be a k -submodule of L . Assume that the inclusion $M \hookrightarrow L$ splits. Let N be the k -module L/M .

Let $f : L \times M \rightarrow k$ be a k -bilinear form such that $f(M \times M) = 0$.

Let J be the two-sided ideal

$$(\otimes L) \cdot \langle v \otimes w - w \otimes v - f(v, w) \mid (v, w) \in L \times M \rangle \cdot (\otimes L)$$

of the k -algebra $\otimes L$.

The k -module $(\otimes L) / (J + (\otimes L) \cdot M)$ is isomorphic to the k -module $\otimes N$. (More detailed assertions are left to the reader.)

6.4. Flat modules

Our formulation of Theorem 5.20 (the main result of this paper) easily provokes the question whether some of its many conditions can be lifted or, at least, weakened. The latter is indeed the case:

Theorem 6.16. Theorem 5.20 still holds if we replace the sentence "Assume that both \mathfrak{h} and N are free k -modules" by "Assume that \mathfrak{g} and N are flat k -modules".

In order to prove this Theorem 6.16, we notice that most steps of our proof of Theorem 5.20 did not use the condition that both \mathfrak{h} and N are free k -modules. The only steps that did were the ones that used the n -PBW condition, and the one that used Proposition 5.16. As for the n -PBW condition, it still holds under the weakened

assumption ("Assume that \mathfrak{g} and N are flat k -modules") due to Theorem 5.9 (\mathfrak{g}), so there is no trouble to expect from its direction. As for Proposition 5.16, we have to generalize it as follows:

Proposition 6.17. Let k be a commutative ring. Let \mathfrak{g} be a k -Lie algebra. Let $m \in \mathbb{N}$.

Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} such that there exists a flat k -submodule N of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus N$. Assume further that the k -Lie algebra \mathfrak{g} itself satisfies the n -PBW condition for every $n \in \mathbb{N}$.

Then, $U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) = U_{\leq (m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$. (Here, we are using the notation of Definition 5.11, and we are abbreviating the k -submodule $U(\mathfrak{g}) \cdot \psi(\mathfrak{h})$ of $U(\mathfrak{g})$ by $U(\mathfrak{g}) \cdot \mathfrak{h}$.)

For the proof of this proposition, we need a lemma that was proven by Thomas Goodwillie [33]:

Lemma 6.18 (Goodwillie). Let k be a commutative ring. Let A be some k -module, and let B be a k -submodule of A such that the k -module A/B is flat.

Let $i \in \mathbb{N}$ be such that $i \geq 1$.

Let \mathbf{m}_1 denote the canonical map $K_i(A) \otimes B \rightarrow A^{\otimes i} \otimes B$.

Let \mathbf{m}_2 denote the canonical map $A^{\otimes i} \otimes B \rightarrow A^{\otimes i} \otimes A \xrightarrow{\cong} A^{\otimes (i+1)}$.

Let \mathbf{m}_3 denote the canonical map $A^{\otimes (i-1)} \otimes K_2(B) \rightarrow A^{\otimes (i-1)} \otimes B^{\otimes 2} \rightarrow A^{\otimes (i-1)} \otimes A^{\otimes 2} \xrightarrow{\cong} A^{\otimes (i+1)}$.

Then,

$$K_{i+1}(A) \cap \mathbf{m}_2(A^{\otimes i} \otimes B) = \mathbf{m}_2(\mathbf{m}_1(K_i(A) \otimes B)) + \mathbf{m}_3(A^{\otimes (i-1)} \otimes K_2(B)). \quad (160)$$

Remark 6.19. All three maps \mathbf{m}_1 , \mathbf{m}_2 , \mathbf{m}_3 in Lemma 6.18 are obtained by tensoring some inclusions with identity maps and composing. (For example, \mathbf{m}_1 is obtained by tensoring the inclusion $K_i(A) \rightarrow A^{\otimes i}$ with the identity map $B \rightarrow B$.) This yields that these maps are injective whenever k is a field (or at least some flatness conditions are satisfied). Therefore, when k is a field, these three maps are often regarded as inclusions and thus suppressed from the equality (160) (so that this equality takes the simple-looking form $K_{i+1}(A) \cap (A^{\otimes i} \otimes B) = K_i(A) \otimes B + A^{\otimes (i-1)} \otimes K_2(B)$). However, we are considering a more general case here, and I do not believe that these maps \mathbf{m}_1 , \mathbf{m}_2 , \mathbf{m}_3 are always injective in our case; thus, suppressing these maps from (160) is not justified for us.

Note that Lemma 6.18 does not involve any Lie algebras; it is a purely module-theoretical lemma and probably has its right place in homological algebra. We refer to [33] for a proof of this lemma (where i was called $m - 1$).

Let us rewrite Lemma 6.18 with the help of the tensor algebra first:

Lemma 6.20 (Goodwillie). Let k be a commutative ring. Let A be some k -module, and let B be a k -submodule of A such that the k -module A/B is flat.

Let $i \in \mathbb{N}$ be such that $i \geq 1$.

Then, the following equality of subsets of the tensor algebra $\otimes A$ holds:

$$K_{i+1}(A) \cap (A^{\otimes i} \cdot B) = K_i(A) \cdot B + A^{\otimes(i-1)} \cdot K_{2,A}(B).$$

Here, we are identifying B with a submodule of $\otimes A$ (due to $B \subseteq A \subseteq \otimes A$), and denoting by $K_{2,A}(B)$ the image of $K_2(B)$ under the canonical map $\otimes B \rightarrow \otimes A$.

Proof of Lemma 6.20. With the notations of Lemma 6.18, we have $\mathbf{m}_2(A^{\otimes i} \otimes B) = A^{\otimes i} \cdot B$, $\mathbf{m}_2(\mathbf{m}_1(K_i(A) \otimes B)) = K_i(A) \cdot B$ and $\mathbf{m}_3(A^{\otimes(i-1)} \otimes K_2(B)) = A^{\otimes(i-1)} \cdot K_{2,A}(B)$. Therefore, Lemma 6.20 follows from Lemma 6.18.

Proof of Proposition 6.17. Since it is trivial that $U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h} \subseteq U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h})$ (just as in the proof of Proposition 5.16), we only have to prove that $U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) \subseteq U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$.

Let us prove that

$$(\text{every integer } i \geq m \text{ satisfies } U_{\leq m}(\mathfrak{g}) \cap (U_{\leq i}(\mathfrak{g}) \cdot \mathfrak{h}) \subseteq U_{\leq m}(\mathfrak{g}) \cap (U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h})). \quad (161)$$

Why prove this? Because once it is proven, Proposition 6.17 follows by a simple induction argument (which we are going to show in more details after we have proven (161)).

Proof of (161). We assume WLOG that $i \geq 1$ (because otherwise, $i = 0$ and $i \geq m$ lead to $m = 0$, and the whole statement of (161) boils down to a triviality).

Let $x \in U_{\leq m}(\mathfrak{g}) \cap (U_{\leq i}(\mathfrak{g}) \cdot \mathfrak{h})$ be arbitrary. Then, $x \in U_{\leq m}(\mathfrak{g})$ and $x \in U_{\leq i}(\mathfrak{g}) \cdot \mathfrak{h}$.

The projection $\psi : \otimes \mathfrak{g} \rightarrow U(\mathfrak{g})$ clearly satisfies $U_{\leq i}(\mathfrak{g}) \cdot \mathfrak{h} = \psi(\mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h})$. Thus, $x \in U_{\leq i}(\mathfrak{g}) \cdot \mathfrak{h} = \psi(\mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h})$, so that there exists some $y \in \mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h}$ such that $x = \psi(y)$. Consider this y .

Let $n = i + 1$. Then, $i = n - 1$.

Now, $y \in \mathfrak{g}^{\otimes \leq i} \cdot \underbrace{\mathfrak{h}}_{\subseteq \mathfrak{g}} \subseteq \mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{\otimes \leq(i+1)} = \mathfrak{g}^{\otimes \leq n}$ (since $i + 1 = n$), so that we

can speak of the element $\bar{y} \in \text{gr}_n(\otimes \mathfrak{g})$. This element satisfies $(\text{gr}_n \psi)(\bar{y}) = \overline{\psi(y)} = 0$ (since $\psi(y) = x \in U_{\leq m}(\mathfrak{g}) \subseteq U_{\leq(n-1)}(\mathfrak{g})$ (because $m \leq i = n - 1$)). Thus, $\bar{y} \in \text{Ker}(\text{gr}_n \psi) = \text{grad}_{\mathfrak{g},n}^{-1}(K_n(\mathfrak{g}))$ (by Proposition 5.8 (b)). Let $z = \text{grad}_{\mathfrak{g},n}^{-1}(\bar{y})$. Then, $\bar{y} \in \text{grad}_{\mathfrak{g},n}(K_n(\mathfrak{g}))$ leads to $z \in K_n(\mathfrak{g})$.

On the other hand, $\text{grad}_{\mathfrak{g},n}^{-1}(\bar{y})$ is the n -th graded component of the tensor $y \in \otimes \mathfrak{g}$ (in fact, for every tensor $T \in \mathfrak{g}^{\otimes \leq n}$, it is clear that $\text{grad}_{\mathfrak{g},n}^{-1}(\overline{T})$ is the n -th graded component of T). Since $z = \text{grad}_{\mathfrak{g},n}^{-1}(\bar{y})$, this means that z is the n -th graded component of the tensor $y \in \otimes \mathfrak{g}$. Since $n = i + 1$, this yields that z is the $(i + 1)$ -th graded component of the tensor $y \in \otimes \mathfrak{g}$. Thus, $z \in \mathfrak{g}^{\otimes i} \cdot \mathfrak{h}$ because of $y \in \mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h}$ (since the $(i + 1)$ -th graded component of a tensor in $\mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h}$ must always lie in $\mathfrak{g}^{\otimes i} \cdot \mathfrak{h}$). Combined with $z \in K_n(\mathfrak{g}) = K_{i+1}(\mathfrak{g})$ (since $n = i + 1$), this yields $z \in K_{i+1}(\mathfrak{g}) \cap (\mathfrak{g}^{\otimes i} \cdot \mathfrak{h})$.

Now, let us recall that $y \in \mathfrak{g}^{\otimes \leq n}$, and that z is the n -th graded component of the tensor $y \in \otimes \mathfrak{g}$. Thus, $y - z \in \mathfrak{g}^{\otimes \leq(n-1)} = \mathfrak{g}^{\otimes \leq i}$ (since $n - 1 = i$). Combined with

$y - z \in \mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h}$ (since $y \in \mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h}$ and $z \in \mathfrak{g}^{\otimes i} \cdot \mathfrak{h} \subseteq \mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h}$), this yields

$$y - z \in \mathfrak{g}^{\otimes \leq i} \cap \left(\underbrace{\mathfrak{g}^{\otimes \leq i} \cdot \mathfrak{h}}_{\subseteq \otimes \mathfrak{g}} \right) \subseteq \mathfrak{g}^{\otimes \leq i} \cap ((\otimes \mathfrak{g}) \cdot \mathfrak{h}) = \mathfrak{g}^{\otimes \leq (i-1)} \cdot \mathfrak{h}$$

$$\left(\begin{array}{l} \text{because } (\otimes \mathfrak{g}) \cdot \mathfrak{h} \text{ is a } \textit{homogeneous} \text{ right ideal of } \otimes \mathfrak{g}, \\ \text{whose } p\text{-th graded component is } \mathfrak{g}^{\otimes (p-1)} \cdot \mathfrak{h} \text{ for every } p \in \mathbb{N} \end{array} \right).$$

Thus, $\psi(y - z) \in \psi(\mathfrak{g}^{\otimes \leq (i-1)} \cdot \mathfrak{h}) = U_{\leq (i-1)}(\mathfrak{g}) \cdot \mathfrak{h}$.

Since the k -module $\mathfrak{g}/\mathfrak{h}$ is flat (since $\mathfrak{g} = \mathfrak{h} \oplus N$ yields $\mathfrak{g}/\mathfrak{h} \cong N$, and we know that N is flat), we can apply Lemma 6.20 to \mathfrak{g} and \mathfrak{h} instead of A and B (it is here that we use $i \geq 1$), and obtain

$$K_{i+1}(\mathfrak{g}) \cap (\mathfrak{g}^{\otimes i} \cdot \mathfrak{h}) = K_i(\mathfrak{g}) \cdot \mathfrak{h} + \mathfrak{g}^{\otimes (i-1)} \cdot K_{2,\mathfrak{g}}(\mathfrak{h}). \quad (162)$$

Now, let us look at $\psi(K_i(\mathfrak{g}))$ and $\psi(K_{2,\mathfrak{g}}(\mathfrak{h}))$ more closely.

Proposition 5.8 (b) (applied to i instead of n) yields $\text{Ker}(\text{gr}_i \psi) = \text{grad}_{\mathfrak{g},i}(K_i(\mathfrak{g}))$. This yields $(\text{gr}_i \psi)(\text{grad}_{\mathfrak{g},i}(K_i(\mathfrak{g}))) = 0$ (although this is clear from much simpler reasons). Thus,

$$0 = (\text{gr}_i \psi)(\text{grad}_{\mathfrak{g},i}(K_i(\mathfrak{g}))) = (\text{gr}_i \psi \circ \text{grad}_{\mathfrak{g},i})(K_i(\mathfrak{g})).$$

Now let inc_i be the canonical inclusion $\mathfrak{g}^{\otimes i} \rightarrow \mathfrak{g}^{\otimes \leq i}$. Furthermore, let $\psi_i : \mathfrak{g}^{\otimes \leq i} \rightarrow U_{\leq i}(\mathfrak{g})$ be the homomorphism obtained from ψ by restricting the domain to $\mathfrak{g}^{\otimes \leq i}$ and the codomain to $U_{\leq i}(\mathfrak{g})$ (this is well-defined since $\psi(\mathfrak{g}^{\otimes \leq i}) \subseteq U_{\leq i}(\mathfrak{g})$). Further, let proj_i be the canonical projection $U_{\leq i}(\mathfrak{g}) \rightarrow (U_{\leq i}(\mathfrak{g})) / (U_{\leq (i-1)}(\mathfrak{g})) = \text{gr}_i(U(\mathfrak{g}))$. Then, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}^{\otimes i} & \xrightarrow{\text{grad}_{\mathfrak{g},i}} & \text{gr}_i(\otimes \mathfrak{g}) \\ \psi_i \circ \text{inc}_i \downarrow & & \downarrow \text{gr}_i \psi \\ U_{\leq i}(\mathfrak{g}) & \xrightarrow{\text{proj}_i} & \text{gr}_i(U(\mathfrak{g})) \end{array}$$

(this is clear from the definitions of the arrows involved). In other words, $\text{gr}_i \psi \circ \text{grad}_{\mathfrak{g},i} = \text{proj}_i \circ \psi_i \circ \text{inc}_i$. Thus, every $p \in K_i(\mathfrak{g})$ satisfies

$$\begin{aligned} 0 &= \left(\underbrace{\text{gr}_i \psi \circ \text{grad}_{\mathfrak{g},i}}_{=\text{proj}_i \circ \psi_i \circ \text{inc}_i} \right) (p) && \text{(since } 0 = (\text{gr}_i \psi \circ \text{grad}_{\mathfrak{g},i})(K_i(\mathfrak{g}))) \\ &= (\text{proj}_i \circ \psi_i \circ \text{inc}_i)(p) = (\text{proj}_i \circ \psi_i) \left(\underbrace{\text{inc}_i(p)}_{=p \text{ (since } \text{inc}_i \text{ is an inclusion map)}} \right) \\ &= (\text{proj}_i \circ \psi_i)(p) = \text{proj}_i \left(\underbrace{\psi_i(p)}_{=\psi(p)} \right) && \text{(by the definition of } \psi_i) \\ &= \text{proj}_i(\psi(p)), \end{aligned}$$

so that $\psi(p) \in \text{Ker}(\text{proj}_i) = U_{\leq(i-1)}(\mathfrak{g})$. In other words,

$$\psi(K_i(\mathfrak{g})) \subseteq U_{\leq(i-1)}(\mathfrak{g}). \quad (163)$$

On the other hand, Proposition 5.2 (applied to \mathfrak{h} and 2 instead of V and n) yields

$$K_2(\mathfrak{h}) = \sum_{i=1}^1 \langle v_1 \otimes v_2 - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \text{ (seen as a tensor in } \otimes \mathfrak{h}) \mid (v_1, v_2) \in \mathfrak{h}^2 \rangle,$$

where τ_1 is the transposition $(1, 2) \in S_2$. This immediately simplifies to

$$\begin{aligned} K_2(\mathfrak{h}) &= \left\langle v_1 \otimes v_2 - \underbrace{v_{\tau_1(1)}}_{=v_2} \otimes \underbrace{v_{\tau_1(2)}}_{=v_1} \text{ (seen as a tensor in } \otimes \mathfrak{h}) \mid (v_1, v_2) \in \mathfrak{h}^2 \right\rangle \\ &= \langle v_1 \otimes v_2 - v_2 \otimes v_1 \text{ (seen as a tensor in } \otimes \mathfrak{h}) \mid (v_1, v_2) \in \mathfrak{h}^2 \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} &K_{2,\mathfrak{g}}(\mathfrak{h}) \\ &= \text{(the image of } K_2(\mathfrak{h}) \text{ under the canonical map } \otimes \mathfrak{h} \rightarrow \otimes \mathfrak{g}) \\ &= \text{(the image of } \langle v_1 \otimes v_2 - v_2 \otimes v_1 \text{ (seen as a tensor in } \otimes \mathfrak{h}) \mid (v_1, v_2) \in \mathfrak{h}^2 \rangle \\ &\quad \text{under the canonical map } \otimes \mathfrak{h} \rightarrow \otimes \mathfrak{g}) \\ &\quad \left(\begin{array}{c} \text{because} \\ K_2(\mathfrak{h}) = \langle v_1 \otimes v_2 - v_2 \otimes v_1 \text{ (seen as a tensor in } \otimes \mathfrak{h}) \mid (v_1, v_2) \in \mathfrak{h}^2 \rangle \end{array} \right) \\ &= \langle v_1 \otimes v_2 - v_2 \otimes v_1 \text{ (seen as a tensor in } \otimes \mathfrak{g}) \mid (v_1, v_2) \in \mathfrak{h}^2 \rangle, \end{aligned}$$

so that

$$\begin{aligned} \psi(K_{2,\mathfrak{g}}(\mathfrak{h})) &= \psi(\langle v_1 \otimes v_2 - v_2 \otimes v_1 \text{ (seen as a tensor in } \otimes \mathfrak{g}) \mid (v_1, v_2) \in \mathfrak{h}^2 \rangle) \\ &= \left\langle \underbrace{\psi(v_1 \otimes v_2 - v_2 \otimes v_1 \text{ (seen as a tensor in } \otimes \mathfrak{g}))}_{= \psi([v_1, v_2])} \mid (v_1, v_2) \in \mathfrak{h}^2 \right\rangle \\ &\quad \text{(since } v_1 \otimes v_2 - v_2 \otimes v_1 - [v_1, v_2] \in I_{\mathfrak{g}} = \text{Ker } \psi) \\ &= \langle \psi([v_1, v_2]) \mid (v_1, v_2) \in \mathfrak{h}^2 \rangle \\ &= \psi \left(\left\langle \underbrace{[v_1, v_2]}_{\substack{\in \mathfrak{h} \text{ (since } (v_1, v_2) \in \mathfrak{h}^2 \\ \text{and since } \mathfrak{h} \text{ is a Lie subalgebra of } \mathfrak{g})}} \mid (v_1, v_2) \in \mathfrak{h}^2 \right\rangle \right) \\ &\subseteq \psi \left(\underbrace{\langle \mathfrak{h} \rangle}_{=\mathfrak{h}} \right) = \psi(\mathfrak{h}). \end{aligned} \quad (164)$$

Since $z \in K_{i+1}(\mathfrak{g}) \cap (\mathfrak{g}^{\otimes i} \cdot \mathfrak{h}) = K_i(\mathfrak{g}) \cdot \mathfrak{h} + \mathfrak{g}^{\otimes(i-1)} \cdot K_{2,\mathfrak{g}}(\mathfrak{h})$ (by (162)), we have

$$\begin{aligned}
\psi(z) &\in \psi(K_i(\mathfrak{g}) \cdot \mathfrak{h} + \mathfrak{g}^{\otimes(i-1)} \cdot K_{2,\mathfrak{g}}(\mathfrak{h})) \\
&\subseteq \underbrace{\psi(K_i(\mathfrak{g})) \cdot \psi(\mathfrak{h})}_{\substack{\subseteq U_{\leq(i-1)}(\mathfrak{g}) \\ \text{(by (163))}}} + \underbrace{\psi(\mathfrak{g}^{\otimes(i-1)}) \cdot \psi(K_{2,\mathfrak{g}}(\mathfrak{h}))}_{\substack{\subseteq U_{\leq(i-1)}(\mathfrak{g}) \\ \text{(by (164))}}} \\
&\subseteq \underbrace{U_{\leq(i-1)}(\mathfrak{g}) \cdot \psi(\mathfrak{h})}_{=U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h}} + \underbrace{U_{\leq(i-1)}(\mathfrak{g}) \cdot \psi(\mathfrak{h})}_{=U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h}} = U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} + U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} \\
&= U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} \quad (\text{since } U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} \text{ is a } k\text{-module}).
\end{aligned}$$

But let us recall that we have shown that $\psi(y - z) \in U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h}$. Now,

$$\begin{aligned}
x = \psi(y) &= \psi((y - z) + z) = \underbrace{\psi(y - z)}_{\in U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h}} + \underbrace{\psi(z)}_{\in U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h}} \quad (\text{since } \psi \text{ is } k\text{-linear}) \\
&\in U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} + U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} = U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} \quad (\text{since } U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h} \text{ is a } k\text{-module}).
\end{aligned}$$

Combined with $x \in U_{\leq m}(\mathfrak{g})$, this yields $x \in U_{\leq m}(\mathfrak{g}) \cap (U_{\leq(i-1)}(\mathfrak{g}) \cdot \mathfrak{h})$. Since we have shown this for every $x \in U_{\leq m}(\mathfrak{g}) \cap (U_{\leq i}(\mathfrak{g}) \cdot \mathfrak{h})$, we have therefore proven (161).

Now let us finish verifying $U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) \subseteq U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$:

Let $x \in U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h})$ be arbitrary. We want to prove that $x \in U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$. We have

$$\begin{aligned}
x \in U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) &\subseteq \underbrace{U(\mathfrak{g})}_{=\bigcup_{i \in \mathbb{N}} U_{\leq i}(\mathfrak{g})} \cdot \mathfrak{h} = \left(\bigcup_{i \in \mathbb{N}} U_{\leq i}(\mathfrak{g}) \right) \cdot \mathfrak{h} \\
&= \bigcup_{i \in \mathbb{N}} (U_{\leq i}(\mathfrak{g}) \cdot \mathfrak{h}) \quad \left(\begin{array}{l} \text{this is because } \bigcup_{i \in \mathbb{N}} U_{\leq i}(\mathfrak{g}) \text{ is an increasing union,} \\ \text{and increasing unions commute with multiplication} \end{array} \right).
\end{aligned}$$

Thus, there exists some $j \in \mathbb{N}$ such that $x \in U_{\leq j}(\mathfrak{g}) \cdot \mathfrak{h}$. Consider this j .

If $j \leq m - 1$, then $x \in U_{\leq j}(\mathfrak{g}) \cdot \mathfrak{h}$ leads to $x \in U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$, which is exactly what we want to have. So let us assume that $j > m - 1$. Then, $j \geq m$. Now, combining $x \in U_{\leq j}(\mathfrak{g}) \cdot \mathfrak{h}$ with $x \in U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) \subseteq U_{\leq m}(\mathfrak{g})$, we obtain $x \in U_{\leq m}(\mathfrak{g}) \cap (U_{\leq j}(\mathfrak{g}) \cdot \mathfrak{h})$. Thus,

$$\begin{aligned}
x &\in U_{\leq m}(\mathfrak{g}) \cap (U_{\leq j}(\mathfrak{g}) \cdot \mathfrak{h}) \\
&\subseteq U_{\leq m}(\mathfrak{g}) \cap (U_{\leq(j-1)}(\mathfrak{g}) \cdot \mathfrak{h}) \quad (\text{due to (161), applied to } i = j) \\
&\subseteq U_{\leq m}(\mathfrak{g}) \cap (U_{\leq(j-2)}(\mathfrak{g}) \cdot \mathfrak{h}) \quad (\text{due to (161), applied to } i = j - 1) \\
&\subseteq U_{\leq m}(\mathfrak{g}) \cap (U_{\leq(j-3)}(\mathfrak{g}) \cdot \mathfrak{h}) \quad (\text{due to (161), applied to } i = j - 2) \\
&\subseteq \dots \\
&\subseteq U_{\leq m}(\mathfrak{g}) \cap (U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}) \quad \left(\begin{array}{l} \text{we must stop this chain of inclusions at } m - 1, \\ \text{because (161) holds only for } i \geq m \end{array} \right) \\
&\subseteq U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h},
\end{aligned}$$

which is exactly what we wanted to prove.

Thus, $x \in U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$ has been shown to hold for every $x \in U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h})$. This means that $U_{\leq m}(\mathfrak{g}) \cap (U(\mathfrak{g}) \cdot \mathfrak{h}) \subseteq U_{\leq(m-1)}(\mathfrak{g}) \cdot \mathfrak{h}$. Proposition 6.17 is now proven.

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