# Noncommutative Abel-like identities

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## 1. Introduction

In this (self-contained) note, we are going to prove three identities that hold in arbitrary noncommutative rings, and generalize some well-known combinatorial identities (known as the *Abel-Hurwitz identities*).

In their simplest and least general versions, the identities we are generalizing are equalities between polynomials in $\mathbb{Z}[X,Y,Z]$; namely, they state that

$$\sum_{k=0}^{n}\binom{n}{k}(X+kZ)^{k}(Y-kZ)^{n-k}=\sum_{k=0}^{n}\frac{n!}{k!}(X+Y)^{k}Z^{n-k}; \tag{1}$$

$$\sum_{k=0}^{n}\binom{n}{k}X(X+kZ)^{k-1}(Y-kZ)^{n-k}=(X+Y)^{n}; \tag{2}$$

$$\sum_{k=0}^{n}\binom{n}{k}X(X+kZ)^{k-1}Y(Y+(n-k)Z)^{n-k-1}=(X+Y)(X+Y+nZ)^{n-1} \tag{3}$$

for every nonnegative integer $n$. [1] These identities have a long history; for example, (2) goes back to Abel [Abel26], who observed that it is a generalization of the binomial formula (obtained by specializing $Z$ to 0). The equality (1) is ascribed to Cauchy in Riordan's text [Riorda68, §1.5, Cauchy's identity] (at least in the specialization $Z=1$; but the general version can be recovered from this specialization by dehomogenization). The equality (3) is also well-known in combinatorics, and tends to appear in the context of tree enumeration (see, e.g., [Grinbe17, Theorem 2]) and of umbral calculus (see, e.g., [Roman84, Section 2.6, Example 3]).

The identities (1), (2) and (3) have been generalized by various authors in different directions. The most famous generalization is due to Hurwitz [Hurwit02], who replaced $Z$ by $n$ commuting indeterminates $Z_1, Z_2, \ldots, Z_n$. More precisely, the equalities (IV), (II) and (III) in [Hurwit02] say (in a more modern language) that if $n$ is a nonnegative integer and $V$ denotes the set $\{1,2,\ldots,n\}$, then

$$\sum_{S\subseteq V}\left(X+\sum_{s\in S}Z_s\right)^{|S|}\left(Y-\sum_{s\in S}Z_s\right)^{n-|S|}=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}}(X+Y)^{n-k}Z_{i_1}Z_{i_2}\cdots Z_{i_k}; \tag{4}$$

$$\sum_{S\subseteq V}X\left(X+\sum_{s\in S}Z_s\right)^{|S|-1}\left(Y-\sum_{s\in S}Z_s\right)^{n-|S|}=(X+Y)^{n}; \tag{5}$$

$$\sum_{S\subseteq V}X\left(X+\sum_{s\in S}Z_s\right)^{|S|-1}Y\left(Y+\sum_{s\in V\setminus S}Z_s\right)^{n-|S|-1}=(X+Y)\left(X+Y+\sum_{s\in V}Z_s\right)^{n-1} \tag{6}$$

[1] The pedantic reader will have observed that two of these identities contain "fractional" terms like $X^{-1}$ and $Y^{-1}$ and thus should be regarded as identities in the function field $\mathbb{Q}(X,Y,Z)$ rather than in the polynomial ring $\mathbb{Z}[X,Y,Z]$. However, this is a false alarm, because all these "fractional" terms are cancelled. For example, the addend for $k=0$ in the sum on the left hand side of (2) contains the "fractional" term $(X+0Z)^{0-1}=X^{-1}$, but this term is cancelled by the factor $X$ directly to its left. Similarly, all the other "fractional" terms disappear. Thus, all three identities are actually identities in $\mathbb{Z}[X,Y,Z]$.

in the polynomial ring $\mathbb{Z}[X, Y, Z_1, Z_2, \ldots, Z_n]$. [2] It is easy to see that setting all indeterminates $Z_1, Z_2, \ldots, Z_n$ equal to a single indeterminate $Z$ transforms these three identities (4), (5) and (6) into the original three identities (1), (2) and (3).

In this note, we shall show that the three identities (4), (5) and (6) can be further generalized to a noncommutative setting: Namely, the commuting indeterminates $X, Y, Z_1, Z_2, \ldots, Z_n$ can be replaced by arbitrary elements $X, Y, x_1, x_2, \ldots, x_n$ of any noncommutative ring $\mathbb{L}$, provided that a centrality assumption holds (for the identities (4) and (5), the sum $X + Y$ needs to lie in the center of $\mathbb{L}$, whereas for (6), the sum $X + Y + \sum\limits_{s \in V} x_s$ needs to lie in the center of $\mathbb{L}$), and provided that the product $Y \left( Y + \sum\limits_{s \in V \setminus S} Z_s \right)^{n - |S| - 1}$ in (6) is replaced by $\left( Y + \sum\limits_{s \in V \setminus S} Z_s \right)^{n - |S| - 1} Y$. These generalized versions of (4), (5) and (6) are Theorem 2.2, Theorem 2.4 and Theorem 2.7 below, and will be proven by a not-too-complicated induction on $n$.

## Acknowledgments

This note was prompted by an enumerative result of Gjergji Zaimi [Zaimi17]. The computer algebra SageMath [SageMath] (specifically, its `FreeAlgebra` class) was used to make conjectures. Thanks to Dennis Stanton for making me aware of [Johns96].

# 2. The identities

Let us now state our results.

**Convention 2.1.** Let $\mathbb{L}$ be a noncommutative ring with unity.

We claim that the following four theorems hold:[3]

**Theorem 2.2.** Let $V$ be a finite set. Let $n = |V|$. For each $s \in V$, let $x_s$ be an element of $\mathbb{L}$. Let $X$ and $Y$ be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center

[2] The sum on the right hand side of (4) ranges over all nonnegative integers $k$ and all $k$-tuples $(i_1, i_2, \ldots, i_k)$ of distinct elements of $V$. This includes the case of $k = 0$ and the empty 0-tuple (which contributes the addend $(X + Y)^{n-0}$ (empty product) $= (X + Y)^n$). Notice that many of the addends in this sum will be equal (indeed, if two $k$-tuples $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_k)$ are permutations of each other, then they produce equal addends).

Once again, "fractional" terms appear in two of these identities, but are all cancelled.

[3] We promised three identities, but we are stating four theorems. This is not a mistake, since Theorem 2.7 is just an equivalent version of Theorem 2.6 (more precisely, it is obtained from Theorem 2.6 by replacing $Y$ with $Y + \sum\limits_{s \in V} x_s$) and so should not be considered a separate identity. We are stating these two theorems on an equal footing since we have no opinion on which of them is the "better" one.

of $\mathbb{L}$. Then,

$$\sum_{S\subseteq V}\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|} = \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}} (X+Y)^{n-k}\, x_{i_1}x_{i_2}\cdots x_{i_k}.$$

(Here, the sum on the right hand side ranges over all nonnegative integers $k$ and all $k$-tuples $(i_1, i_2, \ldots, i_k)$ of distinct elements of $V$. In particular, it has an addend corresponding to $k = 0$ and $(i_1, i_2, \ldots, i_k) = ()$ (the empty 0-tuple); this addend is $\underbrace{(X+Y)^{n-0}}_{=(X+Y)^n}\cdot\underbrace{(\text{empty product})}_{=1} = (X+Y)^n$.)

**Example 2.3.** In the case when $V = \{1, 2\}$, the claim of Theorem 2.2 takes the following form (for any two elements $x_1$ and $x_2$ of $\mathbb{L}$, and any two elements $X$ and $Y$ of $L$ such that $X + Y$ lies in the center of $\mathbb{L}$):

$$\begin{aligned}
&X^0Y^2 + (X+x_1)^1(Y-x_1)^1 + (X+x_2)^1(Y-x_2)^1\\
&\qquad + (X+x_1+x_2)^2(Y-(x_1+x_2))^0\\
&= (X+Y)^2 + (X+Y)^1 x_1 + (X+Y)^1 x_2 + (X+Y)^0 x_1x_2 + (X+Y)^0 x_2x_1.
\end{aligned}$$

If we try to verify this identity by subtracting the right hand side from the left hand side and expanding, we can quickly realize that it boils down to

$$[x_1 + x_2 + X, X + Y] = 0,$$

where $[a, b]$ denotes the commutator of two elements $a$ and $b$ of $\mathbb{L}$ (that is, $[a, b] = ab - ba$). Since $X + Y$ is assumed to lie in the center of $\mathbb{L}$, this equality is correct. This example shows that the requirement that $X + Y$ should lie in the center of $\mathbb{L}$ cannot be lifted from Theorem 2.2.

This example might suggest that we can replace this requirement by the weaker condition that $\left[\sum_{s\in V} x_s + X, X+Y\right] = 0$; but this would not suffice for $n = 3$.

**Theorem 2.4.** Let $V$ be a finite set. Let $n = |V|$. For each $s \in V$, let $x_s$ be an element of $\mathbb{L}$. Let $X$ and $Y$ be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center of $\mathbb{L}$. Then,

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|} = (X+Y)^n.$$

(Here, the product $X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}$ has to be interpreted as 1 when $S = \varnothing$.)

**Example 2.5.** In the case when $V = \{1, 2\}$, the claim of Theorem 2.4 takes the following form (for any two elements $x_1$ and $x_2$ of $\mathbb{L}$, and any two elements $X$ and $Y$ of $L$ such that $X + Y$ lies in the center of $\mathbb{L}$):

$$\begin{aligned} & XX^{-1}Y^2 + X(X + x_1)^0 (Y - x_1)^1 + X(X + x_2)^0 (Y - x_2)^1 \\ & \qquad + X(X + x_1 + x_2)^1 (Y - (x_1 + x_2))^0 \\ & = (X + Y)^2 . \end{aligned}$$

(As explained in Theorem 2.4, we should interpret the product $XX^{-1}$ as 1, so we don't need $X$ to be invertible.) This identity boils down to $XY = YX$, which is a consequence of $X + Y$ lying in the center of $\mathbb{L}$. Computations with $n \geq 3$ show that merely assuming $XY = YX$ (without requiring that $X + Y$ lie in the center of $\mathbb{L}$) is not sufficient.

**Theorem 2.6.** Let $V$ be a finite set. Let $n = |V|$. For each $s \in V$, let $x_s$ be an element of $\mathbb{L}$. Let $X$ and $Y$ be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center of $\mathbb{L}$. Then,

$$\begin{aligned} & \sum_{S \subseteq V} X \left( X + \sum_{s \in S} x_s \right)^{|S|-1} \left( Y - \sum_{s \in S} x_s \right)^{n-|S|-1} \left( Y - \sum_{s \in V} x_s \right) \\ & = \left( X + Y - \sum_{s \in V} x_s \right) (X + Y)^{n-1} . \end{aligned}$$

(Here,

- the product $X \left( X + \sum_{s \in S} x_s \right)^{|S|-1}$ has to be interpreted as 1 when $S = \varnothing$;
- the product $\left( Y - \sum_{s \in S} x_s \right)^{n-|S|-1} \left( Y - \sum_{s \in V} x_s \right)$ has to be interpreted as 1 when $|S| = n$;
- the product $\left( X + Y - \sum_{s \in V} x_s \right) (X + Y)^{n-1}$ has to be interpreted as 1 when $n = 0$.)

**Theorem 2.7.** Let $V$ be a finite set. Let $n = |V|$. For each $s \in V$, let $x_s$ be an element of $\mathbb{L}$. Let $X$ and $Y$ be two elements of $\mathbb{L}$ such that $X + Y + \sum_{s \in V} x_s$ lies in the center of $\mathbb{L}$. Then,

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y+\sum_{s\in V\setminus S} x_s\right)^{n-|S|-1} Y$$
$$=(X+Y)\left(X+Y+\sum_{s\in V} x_s\right)^{n-1}.$$

(Here,

- the product $X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}$ has to be interpreted as $1$ when $S=\varnothing$;
- the product $\left(Y+\sum_{s\in V\setminus S} x_s\right)^{n-|S|-1} Y$ has to be interpreted as $1$ when $|S|=n$;
- the product $(X+Y)\left(X+Y+\sum_{s\in V} x_s\right)^{n-1}$ has to be interpreted as $1$ when $n=0$.)

Before we prove these theorems, let us cite some appearances of their particular cases in the literature:

- Theorem 2.2 generalizes [Grinbe09, Problem 4] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[X,Y]$ and $x_s=1$) and [Riorda68, §1.5, Cauchy's identity] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[X,Y]$ and $X=x$ and $Y=y+n$ and $x_s=1$).
- Theorem 2.4 generalizes [Comtet74, Chapter III, Theorem B] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[X,Y]$ and $x_s=z$) and [Grinbe09, Theorem 4] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[X,Y]$ and $x_s=1$) and [Kalai79, (11)] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[x,y]$ and $X=x$ and $Y=n+y$) and [KelPos08, 1.3] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[z,y,x(a)\mid a\in V]$ and $X=y$ and $Y=z+x(V)$ and $x_s=x(s)$) and "Hurwitz's formula" in [Knuth97, solution to Section 1.2.6, Exercise 51] (which is obtained by setting $V=\{1,2,\ldots,n\}$ and $X=x$ and $Y=y$ and $x_s=z_s$) and [Riorda68, §1.5, (13)] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[X,Y,a]$ and $X=x$ and $Y=y+na$ and $x_s=a$) and [Stanle99, Exercise 5.31 **b**] (which is obtained by setting $\mathbb{L}=\mathbb{Z}[x_1,x_2,\ldots,x_{n+2}]$ and $X=x_{n+1}$ and $Y=\sum_{i=1}^{n} x_i+x_{n+2}$).
- Theorem 2.7 generalizes [Comtet74, Chapter III, Exercise 20] (which is obtained when $\mathbb{L}$ is commutative) and [KelPos08, 1.2] (which is obtained by

setting $\mathbb{L} = \mathbb{Z}[z, y, x(a) \mid a \in V]$ and $X = y$ and $Y = z$ and $x_s = x(s)$) and [Knuth97, Section 2.3.4.4, Exercise 30] (which is obtained by setting $V = \{1, 2, \ldots, n\}$ and $X = x$ and $Y = y$ and $x_s = z_s$).

# 3. The proofs

We now come to the proofs of the identities stated above.

**Convention 3.1.** We shall use the notation $\mathbb{N}$ for the set $\{0, 1, 2, \ldots\}$.

## 3.1. Proofs of Theorems 2.2 and 2.4

*Proof of Theorem 2.2 and Theorem 2.4.* We shall prove Theorem 2.2 and Theorem 2.4 together, by a simultaneous induction. The induction base (the case $n = 0$) is left to the reader.

For the *induction step*, we fix a positive integer $n$, and we assume (as the induction hypothesis) that both Theorem 2.2 and Theorem 2.4 are proven for $n-1$ instead of $n$. We shall now prove Theorem 2.2 and Theorem 2.4 for our number $n$. So let $V$, $x_s$, $X$ and $Y$ be as in Theorem 2.2 and Theorem 2.4.

Fix $t \in V$.

The induction hypothesis shows that Theorem 2.4 is proven for $n-1$ instead of $n$. We can thus apply Theorem 2.4 to $V \setminus \{t\}$ instead of $V$ (since the finite set $V \setminus \{t\}$ has size $|V \setminus \{t\}| = n-1$). Thus, we obtain

$$\sum_{S \subseteq V \setminus \{t\}} X \left( X + \sum_{s \in S} x_s \right)^{|S|-1} \left( Y - \sum_{s \in S} x_s \right)^{n-1-|S|} = (X+Y)^{n-1}. \tag{7}$$

But the induction hypothesis also shows that Theorem 2.2 is proven for $n-1$ instead of $n$. Thus, we can apply Theorem 2.2 to $V \setminus \{t\}$ instead of $V$ (since the finite set $V \setminus \{t\}$ has size $|V \setminus \{t\}| = n-1$). Thus, we obtain

$$\begin{aligned} & \sum_{S \subseteq V \setminus \{t\}} \left( X + \sum_{s \in S} x_s \right)^{|S|} \left( Y - \sum_{s \in S} x_s \right)^{n-1-|S|} \\ &= \sum_{\substack{i_1, i_2, \ldots, i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V \setminus \{t\}}} (X+Y)^{n-1-k} x_{i_1} x_{i_2} \cdots x_{i_k}. \end{aligned} \tag{8}$$

Likewise, we can apply Theorem 2.2 to $V \setminus \{t\}$, $X + x_t$ and $Y - x_t$ instead of $V$, $X$ and $Y$ (because the finite set $V \setminus \{t\}$ has size $|V \setminus \{t\}| = n-1$, and because the sum $(X + x_t) + (Y - x_t) = X + Y$ lies in the center of $\mathbb{L}$). We thus obtain

$$\begin{aligned}
&\sum_{S\subseteq V\setminus\{t\}} \left(X + x_t + \sum_{s\in S} x_s\right)^{|S|} \left(Y - x_t - \sum_{s\in S} x_s\right)^{n-1-|S|} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} \left(\underbrace{(X + x_t) + (Y - x_t)}_{=X+Y}\right)^{n-1-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} (X+Y)^{n-1-k} x_{i_1} x_{i_2} \cdots x_{i_k}. \qquad (9)
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{\substack{S\subseteq V; \\ t\in S}} \left(X + \sum_{s\in S} x_s\right)^{|S|-1} \left(Y - \sum_{s\in S} x_s\right)^{n-|S|} \\
&= \underbrace{\sum_{\substack{S\subseteq V; \\ t\notin S}}}_{=\sum\limits_{S\subseteq V\setminus\{t\}}} \underbrace{\left(X + \sum_{s\in S\cup\{t\}} x_s\right)^{|S\cup\{t\}|-1}}_{\substack{=\left(X+x_t+\sum\limits_{s\in S} x_s\right)^{|S|} \\ \left(\text{since } t\notin S \text{ yields } \sum\limits_{s\in S\cup\{t\}} x_s = x_t + \sum\limits_{s\in S} x_s \right. \\ \left. \text{and } |S\cup\{t\}| = |S|+1\right)}} \underbrace{\left(Y - \sum_{s\in S\cup\{t\}} x_s\right)^{n-|S\cup\{t\}|}}_{\substack{=\left(Y-x_t-\sum\limits_{s\in S} x_s\right)^{n-1-|S|} \\ \left(\text{since } t\notin S \text{ yields } \sum\limits_{s\in S\cup\{t\}} x_s = x_t + \sum\limits_{s\in S} x_s \right. \\ \left. \text{and } |S\cup\{t\}| = |S|+1\right)}} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } S\cup\{t\} \text{ for } S \text{ in the sum, since} \\ \text{the map } \{S\subseteq V \mid t\notin S\} \to \{S\subseteq V \mid t\in S\},\ S\mapsto S\cup\{t\} \\ \text{is a bijection} \end{array}\right) \\
&= \sum_{S\subseteq V\setminus\{t\}} \left(X + x_t + \sum_{s\in S} x_s\right)^{|S|} \left(Y - x_t - \sum_{s\in S} x_s\right)^{n-1-|S|} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} (X+Y)^{n-1-k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad (\text{by } (9)) \qquad (10) \\
&= \sum_{S\subseteq V\setminus\{t\}} \left(X + \sum_{s\in S} x_s\right)^{|S|} \left(Y - \sum_{s\in S} x_s\right)^{n-1-|S|} \qquad (11)
\end{aligned}$$

(by (8)). Multiplying both sides of this equality by $X$, we obtain

$$\sum_{\substack{S\subseteq V;\\ t\in S}} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|} = \sum_{S\subseteq V\setminus\{t\}} X\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(Y-\sum_{s\in S}x_s\right)^{n-1-|S|}. \tag{12}$$

Now,

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}$$
$$= \underbrace{\sum_{\substack{S\subseteq V;\\ t\in S}} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}_{\substack{=\sum_{S\subseteq V\setminus\{t\}} X\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(Y-\sum_{s\in S}x_s\right)^{n-1-|S|}\\ \text{(by (12))}}}$$
$$+ \underbrace{\sum_{\substack{S\subseteq V;\\ t\notin S}}}_{=\sum_{S\subseteq V\setminus\{t\}}} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\underbrace{\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}_{=\left(Y-\sum_{s\in S}x_s\right)\left(Y-\sum_{s\in S}x_s\right)^{n-1-|S|}}$$

$$
\begin{aligned}
&= \sum_{S\subseteq V\setminus\{t\}} X \underbrace{\left(X+\sum_{s\in S} x_s\right)^{|S|}}_{\substack{=\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(X+\sum_{s\in S} x_s\right) \\ \text{(it is easy to check that this makes sense} \\ \text{even for } S=\varnothing \text{, due to the } X \text{ factor)}}} \left(Y-\sum_{s\in S} x_s\right)^{n-1-|S|} \\
&\qquad + \sum_{S\subseteq V\setminus\{t\}} X \left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(Y-\sum_{s\in S} x_s\right) \left(Y-\sum_{s\in S} x_s\right)^{n-1-|S|} \\
&= \sum_{S\subseteq V\setminus\{t\}} X \left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(X+\sum_{s\in S} x_s\right) \left(Y-\sum_{s\in S} x_s\right)^{n-1-|S|} \\
&\qquad + \sum_{S\subseteq V\setminus\{t\}} X \left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(Y-\sum_{s\in S} x_s\right) \left(Y-\sum_{s\in S} x_s\right)^{n-1-|S|} \\
&= \sum_{S\subseteq V\setminus\{t\}} X \left(X+\sum_{s\in S} x_s\right)^{|S|-1} \underbrace{\left(\left(X+\sum_{s\in S} x_s\right)+\left(Y-\sum_{s\in S} x_s\right)\right)}_{=X+Y} \left(Y-\sum_{s\in S} x_s\right)^{n-1-|S|} \\
&= \sum_{S\subseteq V\setminus\{t\}} \underbrace{X \left(X+\sum_{s\in S} x_s\right)^{|S|-1} (X+Y)}_{\substack{=(X+Y)X\left(X+\sum_{s\in S} x_s\right)^{|S|-1} \\ \text{(since } X+Y \text{ lies in the center of } \mathbb{L}\text{)}}} \left(Y-\sum_{s\in S} x_s\right)^{n-1-|S|} \\
&= (X+Y) \underbrace{\sum_{S\subseteq V\setminus\{t\}} X \left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(Y-\sum_{s\in S} x_s\right)^{n-1-|S|}}_{\substack{=(X+Y)^{n-1} \\ \text{(by (7))}}} \\
&= (X+Y)(X+Y)^{n-1} = (X+Y)^n .
\end{aligned}
\tag{13}
$$

Now, forget that we fixed $t$. We thus have proven the equalities (10), (9) and (13) for each $t \in V$.

The set $V$ is nonempty (since $|V| = n$ is positive). Hence, there exists some $q \in V$. Consider this $q$. Applying (13) to $t = q$, we obtain

$$\sum_{S\subseteq V} X \left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(Y-\sum_{s\in S} x_s\right)^{n-|S|} = (X+Y)^n . \tag{14}$$

Therefore, Theorem 2.4 is proven for our $n$. It remains to prove Theorem 2.2 for our $n$.

From (14), we obtain

$$\begin{aligned}(X+Y)^n &= \sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|} \\ &= Y^n + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|} \qquad (15)\end{aligned}$$

(here, we have split off the addend for $S=\varnothing$ from the sum).

We have

$$\begin{aligned}
&\sum_{S\subseteq V}\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}\\
&=Y^n+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}\underbrace{\left(X+\sum_{s\in S}x_s\right)^{|S|}}_{=\left(X+\sum_{s\in S}x_s\right)\left(X+\sum_{s\in S}x_s\right)^{|S|-1}}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}\\
&\qquad\left(\text{here, we have split off the addend for } S=\varnothing \text{ from the sum}\right)\\
&=Y^n+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}\left(X+\underbrace{\sum_{s\in S}x_s}_{=\sum_{t\in S}x_t}\right)\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}\\
&=Y^n+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}\underbrace{\left(X+\sum_{t\in S}x_t\right)\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}_{=X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}+\sum_{t\in S}x_t\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}\\
&=\underbrace{Y^n+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}_{\substack{=(X+Y)^n\\ \text{(by (15))}}}\\
&\qquad+\underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}\sum_{t\in S}}_{=\sum_{t\in V}\sum_{\substack{S\subseteq V;\\ t\in S}}}x_t\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}\\
&=(X+Y)^n+\sum_{t\in V}x_t\underbrace{\sum_{\substack{S\subseteq V;\\ t\in S}}\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}_{\substack{=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}}(X+Y)^{n-1-k}x_{i_1}x_{i_2}\cdots x_{i_k}\\ \text{(by (10))}}}\\
&=(X+Y)^n+\sum_{t\in V}x_t\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}}(X+Y)^{n-1-k}x_{i_1}x_{i_2}\cdots x_{i_k}.
\end{aligned}$$

Compared with

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= \underbrace{(X+Y)^{n-0}}_{=(X+Y)^n} \underbrace{(\text{empty product})}_{=1} + \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k>0}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend for } k=0 \text{ and } (i_1,i_2,\ldots,i_k)=() \\ \text{from the sum}\end{array}\right) \\
&= (X+Y)^n + \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k>0}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= (X+Y)^n + \sum_{\substack{i_1,i_2,\ldots,i_{k+1} \text{ are} \\ \text{distinct} \\ \text{elements of } V}} \underbrace{(X+Y)^{n-(k+1)}}_{=(X+Y)^{n-1-k}} x_{i_1} x_{i_2} \cdots x_{i_{k+1}} \\
&\qquad (\text{here, we have substituted } k+1 \text{ for } k \text{ in the sum}) \\
&= (X+Y)^n + \sum_{\substack{i_1,i_2,\ldots,i_{k+1} \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-1-k} x_{i_1} x_{i_2} \cdots x_{i_{k+1}} \\
&= (X+Y)^n + \underbrace{\sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}}}_{=\sum\limits_{t\in V} \sum\limits_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}}} \underbrace{(X+Y)^{n-1-k} x_t}_{\substack{=x_t (X+Y)^{n-1-k} \\ (\text{since } X+Y \text{ lies in the center of } \mathbb{L}, \\ \text{and therefore so does } (X+Y)^{n-1-k})}} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&\qquad \left(\begin{array}{c}\text{here, we have renamed the summation index } (i_1,i_2,\ldots,i_{k+1}) \\ \text{as } (t,i_1,i_2,\ldots,i_k)\end{array}\right) \\
&= (X+Y)^n + \sum_{t\in V} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} x_t (X+Y)^{n-1-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= (X+Y)^n + \sum_{t\in V} x_t \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} (X+Y)^{n-1-k} x_{i_1} x_{i_2} \cdots x_{i_k},
\end{aligned}$$

this yields

$$\sum_{S\subseteq V} \left(X + \sum_{s\in S} x_s\right)^{|S|} \left(Y - \sum_{s\in S} x_s\right)^{n-|S|} = \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Therefore, Theorem 2.2 is proven for our $n$. This completes the induction step. Hence, both Theorem 2.2 and Theorem 2.4 are proven. $\square$

## 3.2. Proofs of Theorems 2.6 and 2.7

*Proof of Theorem 2.6.* We have

$$\begin{aligned}
&\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\underbrace{\left(Y-\sum_{s\in V}x_s\right)}_{=\left(Y-\sum_{s\in S}x_s\right)-\sum_{t\in V\setminus S}x_t}\\
&=\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\left(\left(Y-\sum_{s\in S}x_s\right)-\sum_{t\in V\setminus S}x_t\right)\\
&=\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\underbrace{\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\left(Y-\sum_{s\in S}x_s\right)}_{=\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}\\
&\qquad-\underbrace{\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\sum_{t\in V\setminus S}x_t}_{=\sum_{S\subseteq V}\sum_{t\in V\setminus S}X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}x_t}\\
&=\underbrace{\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|}}_{\substack{=(X+Y)^n\\ \text{(by Theorem 2.4)}}}\\
&\qquad-\underbrace{\sum_{S\subseteq V}\sum_{t\in V\setminus S}}_{=\sum_{t\in V}\sum_{S\subseteq V\setminus\{t\}}} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\underbrace{\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}}_{=\left(Y-\sum_{s\in S}x_s\right)^{n-1-|S|}}x_t
\end{aligned}$$

$$= (X+Y)^n - \sum_{t\in V} \underbrace{\sum_{S\subseteq V\setminus\{t\}} X \left(X + \sum_{s\in S} x_s\right)^{|S|-1} \left(Y - \sum_{s\in S} x_s\right)^{n-1-|S|}}_{\substack{=(X+Y)^{n-1} \\ \text{(by Theorem 2.4 (applied to } V\setminus\{t\} \text{ and } n-1 \text{ instead of } V \text{ and } n))}} x_t$$

$$= (X+Y)^n - \underbrace{\sum_{t\in V} (X+Y)^{n-1} x_t}_{=(X+Y)^{n-1}\sum_{t\in V} x_t} = (X+Y)^n - \underbrace{(X+Y)^{n-1} \sum_{t\in V} x_t}_{\substack{=\left(\sum_{t\in V} x_t\right)(X+Y)^{n-1} \\ \text{(since } X+Y \text{ lies in the center of } \mathbb{L})}}$$

$$= (X+Y)^n - \underbrace{\left(\sum_{t\in V} x_t\right)}_{=\sum_{s\in V} x_s} (X+Y)^{n-1} = (X+Y)^n - \left(\sum_{s\in V} x_s\right) (X+Y)^{n-1}$$

$$= \left(X + Y - \sum_{s\in V} x_s\right) (X+Y)^{n-1}.$$

This proves Theorem 2.6. ☐

*Proof of Theorem 2.7.* Apply Theorem 2.6 to $Y + \sum_{s\in V} x_s$ instead of $Y$. ☐

# 4. Applications

## 4.1. Polarization identities

Let us show how a rather classical identity in noncommutative rings follows as a particular case from Theorem 2.2. Namely, we shall prove the following *polarization identity*:

**Corollary 4.1.** Let $V$ be a finite set. Let $n = |V|$. For each $s \in V$, let $x_s$ be an element of $\mathbb{L}$. Let $X \in \mathbb{L}$. Then,

$$\sum_{S\subseteq V} (-1)^{n-|S|} \left(X + \sum_{s\in S} x_s\right)^n = \sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list} \\ \text{of all elements of } V \\ \text{(with no repetitions)}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

*Proof of Corollary 4.1.* Apply Theorem 2.2 to $Y = -X$, and notice how all addends on the right hand side having $k < n$ vanish (since $(X + (-X))^{n-k} = 0$ for $k < n$), whereas the remaining addends are precisely the addends of the sum

$$\sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list} \\ \text{of all elements of } V \\ \text{(with no repetitions)}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$ ☐

Corollary 4.1 has a companion result:

**Corollary 4.2.** Let $V$ be a finite set. Let $n = |V|$. For each $s \in V$, let $x_s$ be an element of $\mathbb{L}$. Let $X \in \mathbb{L}$. Let $m \in \mathbb{N}$ be such that $m < n$. Then,

$$\sum_{S \subseteq V} (-1)^{n-|S|} \left( X + \sum_{s \in S} x_s \right)^m = 0.$$

*Proof of Corollary 4.2.* For any subset $W$ of $V$, we define an element $s(W) \in \mathbb{L}$ by

$$s(W) = \sum_{\substack{(i_1, i_2, \ldots, i_n) \text{ is a list} \\ \text{of all elements of } W \\ \text{(with no repetitions)}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{16}$$

If $W$ is any subset of $V$, and if $Y$ is any element of $\mathbb{L}$, then we define an element $r(Y, W) \in \mathbb{L}$ by

$$r(Y, W) = \sum_{S \subseteq W} (-1)^{|W|-|S|} \left( Y + \sum_{s \in S} x_s \right)^m. \tag{17}$$

Now, from Corollary 4.1, we can easily deduce the following claim:

*Claim 1:* Let $W$ be any subset of $V$ satisfying $|W| = m$. Let $Y \in \mathbb{L}$. Then, $r(Y, W) = s(W)$.

[*Proof of Claim 1:* We have $m = |W|$. Hence, Corollary 4.1 (applied to $W$, $Y$ and $m$ instead of $V$, $X$ and $n$) yields

$$\sum_{S \subseteq W} (-1)^{m-|S|} \left( X + \sum_{s \in S} x_s \right)^m = \sum_{\substack{(i_1, i_2, \ldots, i_n) \text{ is a list} \\ \text{of all elements of } W \\ \text{(with no repetitions)}}} x_{i_1} x_{i_2} \cdots x_{i_n} = s(W)$$

(by (16)). Thus, (17) becomes

$$\begin{aligned} r(Y, W) &= \sum_{S \subseteq W} \underbrace{(-1)^{|W|-|S|}}_{\substack{=(-1)^{m-|S|} \\ \text{(since } |W|=m)}} \left( Y + \sum_{s \in S} x_s \right)^m \\ &= \sum_{S \subseteq W} (-1)^{m-|S|} \left( X + \sum_{s \in S} x_s \right)^m = s(W). \end{aligned}$$

This proves Claim 1.]

A more interesting claim is the following:

*Claim 2:* Let $W$ be a subset of $V$. Let $t \in W$. Let $Y \in \mathbb{L}$. Then,

$$r(Y, W) = r(Y + x_t, W \setminus \{t\}) - r(Y, W \setminus \{t\}).$$

[*Proof of Claim 2:* The definition of $r(Y, W \setminus \{t\})$ yields

$$\begin{aligned} r(Y, W \setminus \{t\}) &= \underbrace{\sum_{S \subseteq W \setminus \{t\}}}_{= \sum_{\substack{S \subseteq W; \\ t \notin S}}} \underbrace{(-1)^{|W \setminus \{t\}| - |S|}}_{\substack{=(-1)^{|W|-1-|S|} \\ \text{(since } |W \setminus \{t\}| = |W|-1 \\ \text{(since } t \in W))}} \left(Y + \sum_{s \in S} x_s\right)^m \\ &= \sum_{\substack{S \subseteq W; \\ t \notin S}} (-1)^{|W|-1-|S|} \left(Y + \sum_{s \in S} x_s\right)^m. \end{aligned} \tag{18}$$

The same argument (applied to $Y + x_t$ instead of $Y$) yields

$$\begin{aligned} r(Y + x_t, W \setminus \{t\}) &= \sum_{\substack{S \subseteq W; \\ t \notin S}} \underbrace{(-1)^{|W|-1-|S|}}_{\substack{=(-1)^{|W|-|S \cup \{t\}|} \\ \text{(since } |S| = |S \cup \{t\}| - 1 \\ \text{(because } t \notin S))}} \left(Y + \underbrace{x_t + \sum_{s \in S} x_s}_{\substack{= \sum_{s \in S \cup \{t\}} x_s \\ \text{(since } t \notin S)}}\right)^m \\ &= \sum_{\substack{S \subseteq W; \\ t \notin S}} (-1)^{|W|-|S \cup \{t\}|} \left(Y + \sum_{s \in S \cup \{t\}} x_s\right)^m. \end{aligned} \tag{19}$$

But the definition of $r(Y,W)$ yields

$$\begin{aligned}
& r(Y,W) \\
&= \sum_{S\subseteq W} (-1)^{|W|-|S|} \left(Y + \sum_{s\in S} x_s\right)^m \\
&= \underbrace{\sum_{\substack{S\subseteq W;\\ t\in S}} (-1)^{|W|-|S|} \left(Y + \sum_{s\in S} x_s\right)^m}_{\substack{= \sum_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-|S\cup\{t\}|} \left(Y+ \sum_{s\in S\cup\{t\}} x_s\right)^m \\ \text{(here, we have substituted } S\cup\{t\} \text{ for } S \text{ in the sum,} \\ \text{since the map } \{S\subseteq W \mid t\notin S\}\to\{S\subseteq W \mid t\in S\},\ S\mapsto S\cup\{t\} \\ \text{is a bijection)}}} + \sum_{\substack{S\subseteq W;\\ t\notin S}} \underbrace{(-1)^{|W|-|S|}}_{\substack{=-(-1)^{|W|-|S|-1}\\ =-(-1)^{|W|-1-|S|}}} \left(Y + \sum_{s\in S} x_s\right)^m \\
&\qquad \left(\begin{array}{c}\text{since each subset } S \text{ of } W \text{ satisfies either } t\in S \text{ or } t\notin S \\ \text{(but not both)}\end{array}\right) \\
&= \underbrace{\sum_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-|S\cup\{t\}|} \left(Y + \sum_{s\in S\cup\{t\}} x_s\right)^m}_{\substack{=r(Y+x_t,W\setminus\{t\})\\ \text{(by (19))}}} - \underbrace{\sum_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-1-|S|} \left(Y + \sum_{s\in S} x_s\right)^m}_{\substack{=r(Y,W\setminus\{t\})\\ \text{(by (18))}}} \\
&= r(Y+x_t, W\setminus\{t\}) - r(Y, W\setminus\{t\}).
\end{aligned}$$

This proves Claim 2.]

Now, the following is easy to show by induction:

> *Claim 3:* Let $W$ be a subset of $V$ satisfying $|W| > m$. Let $Y \in \mathbb{L}$. Then, $r(Y,W) = 0$.

[*Proof of Claim 3:* We shall prove Claim 3 by strong induction over $|W|$:

*Induction step:* Let $k \in \mathbb{N}$. Assume that Claim 3 is proven in the case when $|W| < k$. We must show that Claim 3 holds in the case when $|W| = k$.

We have assumed that Claim 3 is proven in the case when $|W| < k$. In other words,

$$\left(\begin{array}{c}\text{if } W \text{ is any subset of } V \text{ satisfying } |W| > m \text{ and } |W| < k, \\ \text{and if } Y\in\mathbb{L}, \text{ then } r(Y,W) = 0\end{array}\right). \tag{20}$$

Now, let $W$ be any subset of $V$ satisfying $|W| > m$ and $|W| = k$. Let $Y \in \mathbb{L}$. We shall show that $r(Y,W) = 0$.

We have $|W| > m \geq 0$. Hence, there exists some $t \in W$. Consider this $t$.

From $t \in W$, we obtain $|W\setminus\{t\}| = \underbrace{|W|}_{>m} - 1 > m-1$, so that $|W\setminus\{t\}| \geq m$.

Thus, we are in one of the following two cases:

*Case 1:* We have $|W \setminus \{t\}| = m$.

*Case 2:* We have $|W \setminus \{t\}| > m$.

Let us first consider Case 1. In this case, we have $|W \setminus \{t\}| = m$. Hence, Claim 1 (applied to $W \setminus \{t\}$ instead of $W$) yields $r(Y, W \setminus \{t\}) = s(W \setminus \{t\})$. Also, Claim 1 (applied to $W \setminus \{t\}$ and $Y + x_t$ instead of $W$ and $Y$) yields $r(Y + x_t, W \setminus \{t\}) = s(W \setminus \{t\})$. Now, Claim 2 yields

$$r(Y, W) = \underbrace{r(Y + x_t, W \setminus \{t\})}_{=s(W \setminus \{t\})} - \underbrace{r(Y, W \setminus \{t\})}_{=s(W \setminus \{t\})} = s(W \setminus \{t\}) - s(W \setminus \{t\}) = 0.$$

Thus, $r(Y, W) = 0$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $|W \setminus \{t\}| > m$. Also, $|W \setminus \{t\}| = |W| - 1 < |W| = k$. Hence, (20) (applied to $W \setminus \{t\}$ instead of $W$) yields $r(Y, W \setminus \{t\}) = 0$. Also, (20) (applied to $W \setminus \{t\}$ and $Y + x_t$ instead of $W$ and $Y$) yields $r(Y + x_t, W \setminus \{t\}) = 0$. Now, Claim 2 yields

$$r(Y, W) = \underbrace{r(Y + x_t, W \setminus \{t\})}_{=0} - \underbrace{r(Y, W \setminus \{t\})}_{=0} = 0 - 0 = 0.$$

Thus, $r(Y, W) = 0$ is proven in Case 2.

We have now proven $r(Y, W) = 0$ in each of the two Cases 1 and 2. Hence, $r(Y, W) = 0$ always holds.

Now, let us forget that we fixed $W$ and $Y$. We thus have proven that if $W$ is any subset of $V$ satisfying $|W| > m$ and $|W| = k$, and if $Y \in \mathbb{L}$, then $r(Y, W) = 0$. In other words, Claim 3 holds in the case when $|W| = k$. This completes the induction step. Thus, Claim 3 is proven by strong induction.]

Now, recall that $V$ is a subset of $V$ satisfying $|V| = n > m$ (since $m < n$). Hence, Claim 3 (applied to $W = V$ and $Y = X$) yields $r(X, V) = 0$. But the definition of $r(X, V)$ yields

$$r(X, V) = \sum_{S \subseteq V} \underbrace{(-1)^{|V| - |S|}}_{\substack{=(-1)^{n-|S|} \\ (\text{since } |V| = n)}} \left(X + \sum_{s \in S} x_s\right)^m = \sum_{S \subseteq V} (-1)^{n-|S|} \left(X + \sum_{s \in S} x_s\right)^m.$$

Hence,

$$\sum_{S \subseteq V} (-1)^{n-|S|} \left(X + \sum_{s \in S} x_s\right)^m = r(X, V) = 0.$$

This proves Corollary 4.2. □

# 5. Questions

The above results are not the first generalizations of the classical Abel-Hurwitz identities; there are various others. In particular, generalizations appear in [Strehl92], [Johns96], [Kalai79], [Pitman02], [Riorda68, §1.6] (see the end of [Grinbe09] for some of these) and [KelPos08]. We have not tried to lift these generalizations into our noncommutative setting, but we suspect that this is possible.

# References

[Abel26] N. H. Abel, *Beweis eines Ausdrucks von welchem die Binomial-Formel ein einzelner Fall ist*, Journal für die reine und angewandte Mathematik **1** (1826), pp. 159–160.
`https://eudml.org/doc/183023`

[Comtet74] Louis Comtet, *Advanced Combinatorics*, Revised and enlarged edition, D. Reidel, 1974.

[Grinbe09] Darij Grinberg, *6th QEDMO 2009, Problem 4 (the Cauchy identity)*.
`http://www.cip.ifi.lmu.de/~grinberg/QEDMO6P4long.pdf`

[Grinbe17] Darij Grinberg, *MathOverflow post #273459 (answer to: Identity with binomial coefficients and k^k)*.
`https://mathoverflow.net/q/273459`

[Hurwit02] A. Hurwitz, *Über Abel's Verallgemeinerung der binomischen Formel*, Acta Math. **26** (1902), pp. 199–203.
`https://doi.org/10.1007/BF02415491`

[Johns96] Warren P. Johnson, *q-Extensions of identities of Abel-Rothe type*, Discrete Mathematics **159** (1996), pp. 161–177.
`https://doi.org/10.1016/0012-365X(95)00108-9`

[Kalai79] Gill Kalai, *A Note on an Evaluation of Abel Sums*, Journal of Combinatorial Theory, Series A, Volume 27, Issue 2, September 1979, pp. 213–217.
`https://doi.org/10.1016/0097-3165(79)90047-5`

[KelPos08] Alexander Kelmans, Alexander Postnikov, *Generalizations of Abel's and Hurwitz's identities*, European Journal of Combinatorics **29** (2008), pp. 1535–1543.
`https://doi.org/10.1016/j.ejc.2007.12.003`

[Knuth97] Donald Ervin Knuth, *The Art of Computer Programming, Volume 1 / Fundamental Algorithms*, 3rd edition, Addison Wesley 1997.

[Pitman02] Jim Pitman, *Forest Volume Decompositions and Abel-Cayley-Hurwitz Multinomial Expansions*, Journal of Combinatorial Theory, Series A **98**, Issue 1, April 2002, pp. 175–191.
`https://doi.org/10.1006/jcta.2001.3238`

[Riorda68] John Riordan, *Combinatorial Identities*, John Wiley & Sons, 1968.

[Roman84] Steven Roman, *The Umbral Calculus*, Academic Press 1984.

[SageMath] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 8.0)*, 2017.

[Stanle99] Richard P. Stanley, *Enumerative Combinatorics, volume 2*, Cambridge University Press 1999.

[Strehl92] Volker Strehl, *Identities of Rothe-Abel-Schläfli-Hurwitz-type*, Discrete Mathematics, Volume 99, Issues 1–3, 2 April 1992, pp. 321–340.
`https://doi.org/10.1016/0012-365X(92)90379-T`

[Zaimi17] Gjergji Zaimi, *MathOverflow post #275383 (Counting some binary trees with lots of extra stucture)*.
`https://mathoverflow.net/q/275383`