A quotient of the ring of symmetric functions generalizing quantum cohomology

Darij Grinberg

7 December 2018
Massachusetts Institute of Technology, Cambridge, MA

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/mit2018.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf

What is this about?

- From a modern point of view, **Schubert calculus** is about two cohomology rings:

$$\mathrm{H}^* \left(\underbrace{\mathrm{Gr}\,(k,n)}_{\text{Grassmannian}} \right) \text{ and } \mathrm{H}^* \left(\underbrace{\mathrm{Fl}\,(n)}_{\text{flag variety}} \right)$$

(both varieties over $\mathbb{C}$).

What is this about?

- From a modern point of view, **Schubert calculus** is about two cohomology rings:

$$\mathrm{H}^* \left(\underbrace{\mathrm{Gr}\,(k, n)}_{\text{Grassmannian}} \right) \text{ and } \mathrm{H}^* \left(\underbrace{\mathrm{Fl}\,(n)}_{\text{flag variety}} \right)$$

(both varieties over $\mathbb{C}$).
- In this talk, we are concerned with the first.

What is this about?

- From a modern point of view, **Schubert calculus** is about two cohomology rings:

$$\mathrm{H}^*\left(\underbrace{\mathrm{Gr}(k,n)}_{\text{Grassmannian}}\right) \quad \text{and} \quad \mathrm{H}^*\left(\underbrace{\mathrm{Fl}(n)}_{\text{flag variety}}\right)$$

(both varieties over $\mathbb{C}$).

- In this talk, we are concerned with the first.
- Classical result: as rings,

$$\begin{aligned} &\mathrm{H}^*(\mathrm{Gr}(k,n)) \\ &\cong (\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}) \\ &\qquad \diagup (h_{n-k+1}, h_{n-k+2}, \ldots, h_n)_{\text{ideal}} \end{aligned}$$

(where the h_i are complete homogeneous symmetric polynomials).

- (Small) **Quantum cohomology** is a deformation of cohomology from the 1980–90s. For the Grassmannian, it is

$$\begin{aligned}&\mathrm{QH}^*\left(\mathrm{Gr}\left(k,n\right)\right)\\&\cong \left(\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}\left[q\right]\right)\\&\qquad \diagup \left(h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n + \left(-1\right)^k q\right)_{\text{ideal}}.\end{aligned}$$

- For comparison, the **classical cohomology** of the Grassmannian is

$$\begin{aligned} & \mathrm{H}^* (\operatorname{Gr}(k, n)) \\ & \cong (\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}) \\ & \qquad \diagup (h_{n-k+1}, h_{n-k+2}, \ldots, h_n)_{\text{ideal}} \end{aligned}$$

(where the h_i are complete homogeneous symmetric polynomials).

Quantum cohomology of $\operatorname{Gr}(k, n)$

- (Small) **Quantum cohomology** is a deformation of cohomology from the 1980–90s. For the Grassmannian, it is

$$\begin{aligned} &\operatorname{QH}^*(\operatorname{Gr}(k,n)) \\ &\cong (\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}[q]) \\ &\qquad / \left(h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n + (-1)^k q \right)_{\text{ideal}} . \end{aligned}$$

- Many properties of classical cohomology still hold here. In particular: $\operatorname{QH}^*(\operatorname{Gr}(k,n))$ has a $\mathbb{Z}[q]$-module basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of (projected) Schur polynomials, with λ ranging over all partitions with $\leq k$ parts and each part $\leq n-k$. The structure constants are the **Gromov–Witten invariants**.
- References:
  - Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, 1999.
  - Alexander Postnikov, *Affine approach to quantum Schubert calculus*, 2005.

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- We will now deform $\mathrm{H}^*(\mathrm{Gr}(k,n))$ using k parameters instead of one, generalizing $\mathrm{QH}^*(\mathrm{Gr}(k,n))$.

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- We will now deform $\mathrm{H}^*(\mathrm{Gr}(k, n))$ using k parameters instead of one, generalizing $\mathrm{QH}^*(\mathrm{Gr}(k, n))$.
- Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $n \geq k \geq 0$ be integers.

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- We will now deform $\mathrm{H}^*(\mathrm{Gr}(k,n))$ using k parameters instead of one, generalizing $\mathrm{QH}^*(\mathrm{Gr}(k,n))$.
- Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $n \geq k \geq 0$ be integers.
- Let $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$.

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- We will now deform $\operatorname{H}^*(\operatorname{Gr}(k, n))$ using k parameters instead of one, generalizing $\operatorname{QH}^*(\operatorname{Gr}(k, n))$.
- Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $n \geq k \geq 0$ be integers.
- Let $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$.
- For each $\alpha \in \mathbb{N}^k$ and each $i \in \{1, 2, \ldots, k\}$, let α_i be the i-th entry of α. Same for infinite sequences (like partitions).

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- We will now deform $\mathrm{H}^*(\mathrm{Gr}(k,n))$ using k parameters instead of one, generalizing $\mathrm{QH}^*(\mathrm{Gr}(k,n))$.
- Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $n \geq k \geq 0$ be integers.
- Let $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$.
- For each $\alpha \in \mathbb{N}^k$ and each $i \in \{1, 2, \ldots, k\}$, let α_i be the i-th entry of α. Same for infinite sequences (like partitions).
- For each $\alpha \in \mathbb{N}^k$, let x^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$, and let $|\alpha|$ be the degree $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ of this monomial.

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- We will now deform $\mathrm{H}^*(\mathrm{Gr}(k,n))$ using k parameters instead of one, generalizing $\mathrm{QH}^*(\mathrm{Gr}(k,n))$.
- Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $n \geq k \geq 0$ be integers.
- Let $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$.
- For each $\alpha \in \mathbb{N}^k$ and each $i \in \{1, 2, \ldots, k\}$, let α_i be the i-th entry of α. Same for infinite sequences (like partitions).
- For each $\alpha \in \mathbb{N}^k$, let x^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$, and let $|\alpha|$ be the degree $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ of this monomial.
- Let $\mathcal{S}$ denote the ring of **symmetric** polynomials in $\mathcal{P}$.

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- We will now deform $\mathrm{H}^*\left(\mathrm{Gr}\left(k,n\right)\right)$ using k parameters instead of one, generalizing $\mathrm{QH}^*\left(\mathrm{Gr}\left(k,n\right)\right)$.
- Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $n \geq k \geq 0$ be integers.
- Let $\mathcal{P} = \mathbf{k}\left[x_1, x_2, \ldots, x_k\right]$.
- For each $\alpha \in \mathbb{N}^k$ and each $i \in \{1, 2, \ldots, k\}$, let α_i be the i-th entry of α. Same for infinite sequences (like partitions).
- For each $\alpha \in \mathbb{N}^k$, let x^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$, and let $|\alpha|$ be the degree $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ of this monomial.
- Let $\mathcal{S}$ denote the ring of **symmetric** polynomials in $\mathcal{P}$.
- **Theorem (Artin $\leq$1944):** The $\mathcal{S}$-module $\mathcal{P}$ is free with basis

$$\left(x^\alpha\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}.$$

- The ring $\mathcal{S}$ of symmetric polynomials in $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$ has several bases, usually indexed by certain sets of (integer) partitions.
  We need the following ones:

Reminders on symmetric polynomials: the e-basis

- For each $m \in \mathbb{Z}$, we let e_m denote the m-th **elementary symmetric polynomial**:

$$e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

(Thus, $e_0 = 1$, and $e_m = 0$ when $m < 0$.)

Reminders on symmetric polynomials: the e-basis

- For each $m \in \mathbb{Z}$, we let e_m denote the m-th **elementary symmetric polynomial**:

$$e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

  (Thus, $e_0 = 1$, and $e_m = 0$ when $m < 0$.)
- For each $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ (e.g., a partition), set

$$e_\nu = e_{\nu_1} e_{\nu_2} \cdots e_{\nu_\ell} \in \mathcal{S}.$$

Reminders on symmetric polynomials: the e-basis

- For each $m \in \mathbb{Z}$, we let e_m denote the m-th **elementary symmetric polynomial**:

$$e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

(Thus, $e_0 = 1$, and $e_m = 0$ when $m < 0$.)

- For each $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ (e.g., a partition), set

$$e_\nu = e_{\nu_1} e_{\nu_2} \cdots e_{\nu_\ell} \in \mathcal{S}.$$

- Then, $(e_\lambda)_{\lambda \text{ is a partition with } \lambda_1 \leq k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$. (Gauss's theorem.)

Reminders on symmetric polynomials: the e-basis

- For each $m \in \mathbb{Z}$, we let e_m denote the m-th **elementary symmetric polynomial**:

$$e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

  (Thus, $e_0 = 1$, and $e_m = 0$ when $m < 0$.)
- For each $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ (e.g., a partition), set

$$e_\nu = e_{\nu_1} e_{\nu_2} \cdots e_{\nu_\ell} \in \mathcal{S}.$$

- Then, $(e_\lambda)_{\lambda \text{ is a partition with } \lambda_1 \leq k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$. (Gauss's theorem.)
- Note that $e_m = 0$ when $m > k$.

Reminders on symmetric polynomials: the h-bases

- For each $m \in \mathbb{Z}$, we let h_m denote the m-th **complete homogeneous symmetric polynomial**:

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

(Thus, $h_0 = 1$, and $h_m = 0$ when $m < 0$.)

- For each $m \in \mathbb{Z}$, we let h_m denote the m-th **complete homogeneous symmetric polynomial**:

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

  (Thus, $h_0 = 1$, and $h_m = 0$ when $m < 0$.)

- For each $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ (e.g., a partition), set

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell} \in \mathcal{S}.$$

Reminders on symmetric polynomials: the h-bases

- For each $m \in \mathbb{Z}$, we let h_m denote the m-th **complete homogeneous symmetric polynomial**:

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

  (Thus, $h_0 = 1$, and $h_m = 0$ when $m < 0$.)

- For each $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ (e.g., a partition), set

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell} \in \mathcal{S}.$$

- Then, $(h_\lambda)_{\lambda \text{ is a partition with } \lambda_1 \leq k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$.

Reminders on symmetric polynomials: the h-bases

- For each $m \in \mathbb{Z}$, we let h_m denote the m-th **complete homogeneous symmetric polynomial**:

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

  (Thus, $h_0 = 1$, and $h_m = 0$ when $m < 0$.)

- For each $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ (e.g., a partition), set

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell} \in \mathcal{S}.$$

- Then, $(h_\lambda)_{\lambda \text{ is a partition with } \lambda_1 \leq k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$.
- Also, $(h_\lambda)_{\lambda \text{ is a partition with } \ell(\lambda) \leq k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$.

  Here, $\ell(\lambda)$ is the length of λ, that is, the number of parts ($=$ nonzero entries) of λ.

- For each partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, we let s_λ be the λ-**th Schur polynomial**:

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard tableau} \\ \text{of shape } \lambda \text{ with entries } 1,2,\ldots,k}} \prod_{i=1}^{k} x_i^{(\text{number of } i\text{'s in } T)}$$

$$= \det\left((h_{\lambda_i - i + j})_{1 \leq i \leq \ell(\lambda),\ 1 \leq j \leq \ell(\lambda)} \right) \qquad \text{(Jacobi-Trudi)}\,.$$

Reminders on symmetric polynomials: the s-basis

- For each partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, we let s_λ be the λ-**th Schur polynomial**:

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard tableau} \\ \text{of shape } \lambda \text{ with entries } 1,2,\ldots,k}} \prod_{i=1}^{k} x_i^{(\text{number of } i\text{'s in } T)}$$

$$= \det\left(\left(h_{\lambda_i - i + j} \right)_{1 \leq i \leq \ell(\lambda),\ 1 \leq j \leq \ell(\lambda)} \right) \qquad \text{(Jacobi-Trudi)}.$$

- If $\ell(\lambda) > k$, then $s_\lambda = 0$.
- If $\ell(\lambda) \leq k$, then

$$s_\lambda = \frac{\det\left(\left(x_i^{\lambda_j + k - j} \right)_{1 \leq i \leq k,\ 1 \leq j \leq k} \right)}{\det\left(\left(x_i^{k-j} \right)_{1 \leq i \leq k,\ 1 \leq j \leq k} \right)} \qquad \text{(alternant formula)}.$$

Reminders on symmetric polynomials: the s-basis

- For each partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, we let s_λ be the λ-**th Schur polynomial**:

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard tableau} \\ \text{of shape } \lambda \text{ with entries } 1,2,\ldots,k}} \prod_{i=1}^{k} x_i^{(\text{number of } i\text{'s in } T)}$$

$$= \det\left(\left(h_{\lambda_i - i + j} \right)_{1 \leq i \leq \ell(\lambda),\ 1 \leq j \leq \ell(\lambda)} \right) \qquad \text{(Jacobi-Trudi)}\,.$$

- If $\ell(\lambda) > k$, then $s_\lambda = 0$.
- If $\ell(\lambda) \leq k$, then

$$s_\lambda = \frac{\det\left(\left(x_i^{\lambda_j + k - j} \right)_{1 \leq i \leq k,\ 1 \leq j \leq k} \right)}{\det\left(\left(x_i^{k-j} \right)_{1 \leq i \leq k,\ 1 \leq j \leq k} \right)} \qquad \text{(alternant formula)}\,.$$

- Now, $(s_\lambda)_{\lambda \text{ is a partition with } \ell(\lambda) \leq k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$.

A more general setting: $a_1, a_2, \ldots, a_k$ and J

- Let $a_1, a_2, \ldots, a_k \in \mathcal{P}$ such that $\deg a_i < n - k + i$ for all i. (For example, this holds if $a_i \in \mathbf{k}$.)

A more general setting: $a_1, a_2, \ldots, a_k$ and J

- Let $a_1, a_2, \ldots, a_k \in \mathcal{P}$ such that $\deg a_i < n - k + i$ for all i. (For example, this holds if $a_i \in \mathbf{k}$.)
- Let J be the ideal of $\mathcal{P}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

A more general setting: $a_1, a_2, \ldots, a_k$ and J

- Let $a_1, a_2, \ldots, a_k \in \mathcal{P}$ such that $\deg a_i < n - k + i$ for all i. (For example, this holds if $a_i \in \mathbf{k}$.)
- Let J be the ideal of $\mathcal{P}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{P} / J$ is free with basis

$$\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i},$$

  where the overline $\overline{}$ means "projection" onto whatever quotient we need (here: from $\mathcal{P}$ onto $\mathcal{P} / J$).
  (This basis has $n(n-1)\cdots(n-k+1)$ elements.)

A slightly less general setting: symmetric $a_1, a_2, \ldots, a_k$ and J

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathcal{S}$.

A slightly less general setting: symmetric $a_1, a_2, \ldots, a_k$ and J

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathcal{S}$.
- Let I be the ideal of $\mathcal{S}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

  (Same differences as for J, but we are generating an ideal of $\mathcal{S}$ now.)

A slightly less general setting: symmetric $a_1, a_2, \ldots, a_k$ and J

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathcal{S}$.
- Let I be the ideal of $\mathcal{S}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

  (Same differences as for J, but we are generating an ideal of $\mathcal{S}$ now.)
- For each partition λ, let $s_\lambda \in \mathcal{S}$ be the corresponding Schur polynomial.

A slightly less general setting: symmetric $a_1, a_2, \ldots, a_k$ and J

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathcal{S}$.
- Let I be the ideal of $\mathcal{S}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

  (Same differences as for J, but we are generating an ideal of $\mathcal{S}$ now.)
- For each partition λ, let $s_\lambda \in \mathcal{S}$ be the corresponding Schur polynomial.
- Let

$$P_{k,n} = \{\lambda \text{ is a partition} \mid \lambda_1 \leq n-k \text{ and } \ell(\lambda) \leq k\} \\ = \{\text{partitions } \lambda \subseteq \omega\},$$

  where $\omega = \underbrace{(n-k, n-k, \ldots, n-k)}_{k \text{ entries}}$.

A slightly less general setting: symmetric $a_1, a_2, \ldots, a_k$ and J

- **FROM NOW ON, assume that $a_1, a_2, \ldots, a_k \in \mathcal{S}$.**
- Let I be the ideal of $\mathcal{S}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

  (Same differences as for J, but we are generating an ideal of $\mathcal{S}$ now.)
- For each partition λ, let $s_\lambda \in \mathcal{S}$ be the corresponding Schur polynomial.
- Let

$$P_{k,n} = \{\lambda \text{ is a partition} \mid \lambda_1 \leq n-k \text{ and } \ell(\lambda) \leq k\}$$
$$= \{\text{partitions } \lambda \subseteq \omega\},$$

  where $\omega = (\underbrace{n-k, n-k, \ldots, n-k}_{k \text{ entries}})$.
- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{S} / I$ is free with basis

$$(\overline{s_\lambda})_{\lambda \in P_{k,n}}.$$

An even less general setting: constant $a_1, a_2, \ldots, a_k$

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathbf{k}$.

An even less general setting: constant $a_1, a_2, \ldots, a_k$

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathbf{k}$.
- This setting still is general enough to encompass several that we know:
  - If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S} \diagup I$ becomes the cohomology ring $\mathrm{H}^* (\mathrm{Gr}(k, n))$; the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ corresponds to the Schubert classes.
  - If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then $\mathcal{S} \diagup I$ becomes the quantum cohomology ring $\mathrm{QH}^* (\mathrm{Gr}(k, n))$.

An even less general setting: constant $a_1, a_2, \ldots, a_k$

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathbf{k}$.
- This setting still is general enough to encompass several that we know:
  - If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S} \diagup I$ becomes the cohomology ring $\mathrm{H}^* \left(\mathrm{Gr} \left(k, n \right) \right)$; the basis $\left(\overline{s_\lambda} \right)_{\lambda \in P_{k,n}}$ corresponds to the Schubert classes.
  - If $\mathbf{k} = \mathbb{Z} \left[q \right]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = - \left(-1 \right)^k q$, then $\mathcal{S} \diagup I$ becomes the quantum cohomology ring $\mathrm{QH}^* \left(\mathrm{Gr} \left(k, n \right) \right)$.
- The above theorem lets us work in these rings (and more generally) without relying on geometry.

- Recall that $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the **k**-module $\mathcal{S} \diagup I$.

S_3-symmetry of the Gromov–Witten invariants

- Recall that $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the **k**-module $\mathcal{S} \diagup I$.
  For each $\mu \in P_{k,n}$, let $\operatorname{coeff}_\mu : \mathcal{S} \diagup I \to \mathbf{k}$ send each element to its $\overline{s_\mu}$-coordinate wrt this basis.

S_3-symmetry of the Gromov–Witten invariants

- Recall that $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the **k**-module $\mathcal{S} / I$.
  For each $\mu \in P_{k,n}$, let $\operatorname{coeff}_\mu : \mathcal{S} / I \to \mathbf{k}$ send each element to its $\overline{s_\mu}$-coordinate wrt this basis.
- For every partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we define

$$\nu^\vee := (n - k - \nu_k, n - k - \nu_{k-1}, \ldots, n - k - \nu_1) \in P_{k,n}.$$

  This partition $\nu^\vee$ is called the *complement* of ν.

- Recall that $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S} / I$.
  For each $\mu \in P_{k,n}$, let $\operatorname{coeff}_\mu : \mathcal{S} / I \to \mathbf{k}$ send each element to its $\overline{s_\mu}$-coordinate wrt this basis.
- For every partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we define

$$\nu^\vee := (n - k - \nu_k, n - k - \nu_{k-1}, \ldots, n - k - \nu_1) \in P_{k,n}.$$

  This partition $\nu^\vee$ is called the *complement* of ν.
- For any three partitions $\alpha, \beta, \gamma \in P_{k,n}$, let

$$g_{\alpha,\beta,\gamma} := \operatorname{coeff}_{\gamma^\vee} \left(\overline{s_\alpha s_\beta} \right) \in \mathbf{k}.$$

  These generalize the Littlewood–Richardson numbers and (3-point) Gromov–Witten invariants.

S_3-symmetry of the Gromov–Witten invariants

- Recall that $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S} / I$.
  For each $\mu \in P_{k,n}$, let $\operatorname{coeff}_\mu : \mathcal{S} / I \to \mathbf{k}$ send each element to its $\overline{s_\mu}$-coordinate wrt this basis.
- For every partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we define

$$\nu^\vee := (n - k - \nu_k, n - k - \nu_{k-1}, \ldots, n - k - \nu_1) \in P_{k,n}.$$

  This partition $\nu^\vee$ is called the *complement* of ν.
- For any three partitions $\alpha, \beta, \gamma \in P_{k,n}$, let

$$g_{\alpha,\beta,\gamma} := \operatorname{coeff}_{\gamma^\vee} \left(\overline{s_\alpha s_\beta} \right) \in \mathbf{k}.$$

  These generalize the Littlewood–Richardson numbers and (3-point) Gromov–Witten invariants.
- **Theorem (G.):** For any $\alpha, \beta, \gamma \in P_{k,n}$, we have

$$g_{\alpha,\beta,\gamma} = g_{\alpha,\gamma,\beta} = g_{\beta,\alpha,\gamma} = g_{\beta,\gamma,\alpha} = g_{\gamma,\alpha,\beta} = g_{\gamma,\beta,\alpha} = \operatorname{coeff}_\omega \left(\overline{s_\alpha s_\beta s_\gamma} \right).$$

S_3-symmetry of the Gromov–Witten invariants

- Recall that $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the **k**-module $\mathcal{S} / I$.
  For each $\mu \in P_{k,n}$, let $\operatorname{coeff}_\mu : \mathcal{S} / I \to \mathbf{k}$ send each element to its $\overline{s_\mu}$-coordinate wrt this basis.
- For every partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we define

$$\nu^\vee := (n - k - \nu_k, n - k - \nu_{k-1}, \ldots, n - k - \nu_1) \in P_{k,n}.$$

  This partition $\nu^\vee$ is called the *complement* of ν.
- For any three partitions $\alpha, \beta, \gamma \in P_{k,n}$, let

$$g_{\alpha,\beta,\gamma} := \operatorname{coeff}_{\gamma^\vee}\left(\overline{s_\alpha s_\beta}\right) \in \mathbf{k}.$$

  These generalize the Littlewood–Richardson numbers and (3-point) Gromov–Witten invariants.
- **Theorem (G.):** For any $\alpha, \beta, \gamma \in P_{k,n}$, we have

$$g_{\alpha,\beta,\gamma} = g_{\alpha,\gamma,\beta} = g_{\beta,\alpha,\gamma} = g_{\beta,\gamma,\alpha} = g_{\gamma,\alpha,\beta} = g_{\gamma,\beta,\alpha} = \operatorname{coeff}_\omega\left(\overline{s_\alpha s_\beta s_\gamma}\right).$$

- **Equivalent restatement:** Each $\nu \in P_{k,n}$ and $f \in \mathcal{S} / I$ satisfy $\operatorname{coeff}_\omega\left(\overline{s_\nu} f\right) = \operatorname{coeff}_{\nu^\vee}(f)$.

The h-basis

- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{S} / I$ is free with basis

$$\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}.$$

The h-basis

- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{S} / I$ is free with basis

$$\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}.$$

- The transfer matrix between the two bases $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ and $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is unitriangular wrt the "size-then-anti-dominance" order, but seems hard to describe.

- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis
$$\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}.$$
- The transfer matrix between the two bases $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ and $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is unitriangular wrt the "size-then-anti-dominance" order, but seems hard to describe.
- **Proposition (G.):** Let m be a positive integer. Then,
$$\overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j \, a_{k-j} \overline{s_{(m,1^j)}},$$
where $\left(m, 1^j\right) := (m, \underbrace{1, 1, \ldots, 1}_{j \text{ ones}})$ (a hook-shaped partition).

A Pieri rule

- **Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu / \lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i \, a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{(n-k-j+1, 1^{i-1}), \nu} \overline{s_\nu},$$

where $c^{\gamma}_{\alpha,\beta}$ are the usual Littlewood–Richardson coefficients.

A Pieri rule

- **Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^\lambda_{(n-k-j+1, 1^{i-1}), \nu} \overline{s_\nu},$$

  where $c^\gamma_{\alpha,\beta}$ are the usual Littlewood–Richardson coefficients.
- This generalizes the Bertram/Ciocan-Fontanine/Fulton Pieri rule, but note that $c^\lambda_{(n-k-j+1, 1^{i-1}), \nu}$ may be > 1.

A Pieri rule

- **Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{(n-k-j+1, 1^{i-1}), \nu} \overline{s_\nu},$$

where $c^{\gamma}_{\alpha,\beta}$ are the usual Littlewood–Richardson coefficients.

- This generalizes the Bertram/Ciocan-Fontanine/Fulton Pieri rule, but note that $c^{\lambda}_{(n-k-j+1,1^{i-1}),\nu}$ may be > 1.
- Example:

$$\begin{aligned}\overline{s_{(4,3,2)} h_2} = \overline{s_{(4,4,3)}} + a_1 \left(\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}\right) \\ - a_2 \left(\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}\right) \\ + a_3 \left(\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}\right).\end{aligned}$$

A Pieri rule

- **Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{(n-k-j+1, 1^{i-1}), \nu} \overline{s_\nu},$$

where $c^{\gamma}_{\alpha,\beta}$ are the usual Littlewood–Richardson coefficients.

- This generalizes the Bertram/Ciocan-Fontanine/Fulton Pieri rule, but note that $c^{\lambda}_{(n-k-j+1, 1^{i-1}), \nu}$ may be > 1.
- Example:

$$\begin{aligned} \overline{s_{(4,3,2)} h_2} = \overline{s_{(4,4,3)}} &+ a_1 \left(\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}} \right) \\ &- a_2 \left(\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}} \right) \\ &+ a_3 \left(\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}} \right). \end{aligned}$$

- Multiplying by e_j appears harder:

$$\overline{s_{(2,2,1)} e_2} = a_1 \overline{s_{(2,2)}} - 2 a_2 \overline{s_{(2,1)}} + a_3 \left(\overline{s_{(2)}} + \overline{s_{(1,1)}} \right) + a_1^2 \overline{s_{(1)}} - 2 a_1 a_2 \overline{s_{()}}.$$

Positivity?

- **Conjecture:** Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1, 2, \ldots, k\}$. Let $\lambda, \mu, \nu \in P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|} \operatorname{coeff}_\nu \left(\overline{s_\lambda s_\mu}\right)$ is a polynomial in $b_1, b_2, \ldots, b_k$ with coefficients in $\mathbb{N}$.
- Verified for all $n \leq 7$ using SageMath.
- This would generalize positivity of Gromov–Witten invariants.

More questions

- **Question:** Does $\mathcal{S}/I$ have a geometric meaning? If not, why does it behave so nicely?

More questions

- **Question:** Does $\mathcal{S}/I$ have a geometric meaning? If not, why does it behave so nicely?
- **Question:** What other bases does $\mathcal{S}/I$ have? Monomial symmetric? Power-sum?

More questions

- **Question:** Does $\mathcal{S} / I$ have a geometric meaning? If not, why does it behave so nicely?
- **Question:** What other bases does $\mathcal{S} / I$ have? Monomial symmetric? Power-sum?
- **Question:** Do other properties of $QH^*(Gr(k, n))$ (such as "curious duality" and "cyclic hidden symmetry") generalize to $\mathcal{S} / I$?

  (The $Gr(k, n) \to Gr(n-k, n)$ duality isomorphism does not exist in general: If $\mathbf{k} = \mathbb{C}$ and $a_1 = 6$ and $a_2 = 16$, then $(\mathcal{S} / I)_{k=2,\ n=3,\ a_1=6,\ a_2=16} \cong \mathbb{C}[x] / ((x-10)(x+2)^2)$, which can never be a $(\mathcal{S} / I)_{k=1,\ n=3}$, since $(\mathcal{S} / I)_{k=1,\ n=3} \cong \mathbb{C}[x] / (x^3 - a_1)$.)

More questions

- **Question:** Does $\mathcal{S} / I$ have a geometric meaning? If not, why does it behave so nicely?
- **Question:** What other bases does $\mathcal{S} / I$ have? Monomial symmetric? Power-sum?
- **Question:** Do other properties of $\mathrm{QH}^* (\mathrm{Gr}(k, n))$ (such as "curious duality" and "cyclic hidden symmetry") generalize to $\mathcal{S} / I$?

  (The $\mathrm{Gr}(k, n) \to \mathrm{Gr}(n-k, n)$ duality isomorphism does not exist in general: If $\mathbf{k} = \mathbb{C}$ and $a_1 = 6$ and $a_2 = 16$, then $(\mathcal{S} / I)_{k=2,\ n=3,\ a_1=6,\ a_2=16} \cong \mathbb{C}[x] / ((x-10)(x+2)^2)$, which can never be a $(\mathcal{S} / I)_{k=1,\ n=3}$, since $(\mathcal{S} / I)_{k=1,\ n=3} \cong \mathbb{C}[x] / (x^3 - a_1)$.)
- **Question:** Is there an analogous generalization of $\mathrm{QH}^* (\mathrm{Fl}(n))$? Is it connected to Fulton's "universal Schubert polynomials"?

More questions

- **Question:** Does $\mathcal{S} / I$ have a geometric meaning? If not, why does it behave so nicely?
- **Question:** What other bases does $\mathcal{S} / I$ have? Monomial symmetric? Power-sum?
- **Question:** Do other properties of $\mathrm{QH}^*(\mathrm{Gr}(k,n))$ (such as "curious duality" and "cyclic hidden symmetry") generalize to $\mathcal{S} / I$?

  (The $\mathrm{Gr}(k,n) \to \mathrm{Gr}(n-k,n)$ duality isomorphism does not exist in general: If $\mathbf{k} = \mathbb{C}$ and $a_1 = 6$ and $a_2 = 16$, then $(\mathcal{S} / I)_{k=2,\ n=3,\ a_1=6,\ a_2=16} \cong \mathbb{C}[x] / ((x-10)(x+2)^2)$, which can never be a $(\mathcal{S} / I)_{k=1,\ n=3}$, since $(\mathcal{S} / I)_{k=1,\ n=3} \cong \mathbb{C}[x] / (x^3 - a_1)$.)
- **Question:** Is there an analogous generalization of $\mathrm{QH}^*(\mathrm{Fl}(n))$? Is it connected to Fulton's "universal Schubert polynomials"?
- **Question:** Is there an equivariant analogue?

More questions

- **Question:** Does $\mathcal{S} / I$ have a geometric meaning? If not, why does it behave so nicely?
- **Question:** What other bases does $\mathcal{S} / I$ have? Monomial symmetric? Power-sum?
- **Question:** Do other properties of $QH^*(Gr(k, n))$ (such as "curious duality" and "cyclic hidden symmetry") generalize to $\mathcal{S} / I$?

  (The $Gr(k, n) \to Gr(n - k, n)$ duality isomorphism does not exist in general: If $\mathbf{k} = \mathbb{C}$ and $a_1 = 6$ and $a_2 = 16$, then $(\mathcal{S} / I)_{k=2,\ n=3,\ a_1=6,\ a_2=16} \cong \mathbb{C}[x] / ((x - 10)(x + 2)^2)$, which can never be a $(\mathcal{S} / I)_{k=1,\ n=3}$, since $(\mathcal{S} / I)_{k=1,\ n=3} \cong \mathbb{C}[x] / (x^3 - a_1)$.)
- **Question:** Is there an analogous generalization of $QH^*(Fl(n))$? Is it connected to Fulton's "universal Schubert polynomials"?
- **Question:** Is there an equivariant analogue?
- **Question:** "Straightening rules" for $\overline{s_\lambda}$ when $\lambda \notin P_{k,n}$, similar to the Bertram/Ciocan-Fontanine/Fulton "rim hook algorithm"?

S_k-module structure

- The symmetric group S_k acts on $\mathcal{P}$, with invariant ring $\mathcal{S}$.
- What is the S_k-module structure on $\mathcal{P} \diagup J$?

- The symmetric group S_k acts on $\mathcal{P}$, with invariant ring $\mathcal{S}$.
- What is the S_k-module structure on $\mathcal{P} / J$?
- **Almost-theorem (G., needs to be checked):** Assume that $\mathbf{k}$ is a $\mathbb{Q}$-algebra. Then, as S_k-modules,

$$\mathcal{P} / J \cong \left(\mathcal{P} / \mathcal{P}\mathcal{S}^+\right)^{\times \binom{n}{k}} \cong \left(\underbrace{\mathbf{k} S_k}_{\text{regular rep}}\right)^{\times \binom{n}{k}},$$

where $\mathcal{P}\mathcal{S}^+$ is the ideal of $\mathcal{P}$ generated by symmetric polynomials with constant term 0.

Deforming symmetric functions, 1

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

$$\mathcal{S} := \{\text{symmetric polynomials in } x_1, x_2, \ldots, x_k\};$$
$$\Lambda := \{\text{symmetric functions in } x_1, x_2, x_3, \ldots\}.$$

Deforming symmetric functions, 1

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

$$\mathcal{S} := \{\text{symmetric polynomials in } x_1, x_2, \ldots, x_k\};$$
$$\Lambda := \{\text{symmetric functions in } x_1, x_2, x_3, \ldots\}.$$

- We use standard notations for symmetric functions, but in boldface:

$$\mathbf{e} = \text{elementary symmetric},$$
$$\mathbf{h} = \text{complete homogeneous},$$
$$\mathbf{s} = \text{Schur (or skew Schur)}.$$

Deforming symmetric functions, 1

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

$$\mathcal{S} := \{\text{symmetric polynomials in } x_1, x_2, \ldots, x_k\};$$
$$\Lambda := \{\text{symmetric functions in } x_1, x_2, x_3, \ldots\}.$$

- We use standard notations for symmetric functions, but in boldface:

$$\mathbf{e} = \text{elementary symmetric},$$
$$\mathbf{h} = \text{complete homogeneous},$$
$$\mathbf{s} = \text{Schur (or skew Schur)}.$$

- We have

$$\mathcal{S} \cong \Lambda / (\mathbf{e}_{k+1}, \quad \mathbf{e}_{k+2}, \quad \mathbf{e}_{k+3}, \quad \ldots)_{\text{ideal}}, \qquad \text{thus}$$
$$\mathcal{S} / I \cong \Lambda / (\mathbf{h}_{n-k+1} - a_1, \quad \mathbf{h}_{n-k+2} - a_2, \quad \ldots, \quad \mathbf{h}_n - a_k, \quad \mathbf{e}_{k+1}, \quad \mathbf{e}_{k+2}, \quad \mathbf{e}_{k+3}, \quad \ldots)_{\text{ideal}}.$$

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

$$\mathcal{S} := \{\text{symmetric polynomials in } x_1, x_2, \ldots, x_k\};$$
$$\Lambda := \{\text{symmetric functions in } x_1, x_2, x_3, \ldots\}.$$

- We use standard notations for symmetric functions, but in boldface:

$$\mathbf{e} = \text{elementary symmetric},$$
$$\mathbf{h} = \text{complete homogeneous},$$
$$\mathbf{s} = \text{Schur (or skew Schur)}.$$

- We have

$$\mathcal{S} \cong \Lambda / (\mathbf{e}_{k+1}, \ \mathbf{e}_{k+2}, \ \mathbf{e}_{k+3}, \ \ldots)_{\text{ideal}}, \qquad \text{thus}$$
$$\mathcal{S} / I \cong \Lambda / (\mathbf{h}_{n-k+1} - a_1, \ \mathbf{h}_{n-k+2} - a_2, \ \ldots, \ \mathbf{h}_n - a_k, \ \mathbf{e}_{k+1}, \ \mathbf{e}_{k+2}, \ \mathbf{e}_{k+3}, \ \ldots)_{\text{ideal}}.$$

- So why not replace the $\mathbf{e}_j$ by $\mathbf{e}_j - b_j$ too?

- **Theorem (G.):** Assume that $a_1, a_2, \ldots, a_k$ as well as $b_1, b_2, b_3, \ldots$ are elements of $\mathbf{k}$. Then,

$$\Lambda \diagup (\mathbf{h}_{n-k+1} - a_1, \quad \mathbf{h}_{n-k+2} - a_2, \quad \ldots, \quad \mathbf{h}_n - a_k, \quad \mathbf{e}_{k+1} - b_1, \quad \mathbf{e}_{k+2} - b_2, \quad \mathbf{e}_{k+3} - b_3, \quad \ldots)_{\text{ideal}}$$

is a free $\mathbf{k}$-module with basis $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda \in P_{k,n}}$.

On the proofs, 1

- Proofs of all the above (except for the S_k-action) can be found in
  - Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, `http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf` .

- Proofs of all the above (except for the S_k-action) can be found in
  - Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf .
- **Main ideas:**
  - Use Gröbner bases to show that $\mathcal{P} \diagup J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$.
    (This was already outlined in Aldo Conca, Christian Krattenthaler, Junzo Watanabe, *Regular Sequences of Symmetric Polynomials*, 2009.)

On the proofs, 1

- Proofs of all the above (except for the S_k-action) can be found in
  - Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, `http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf` .
- **Main ideas:**
  - Use Gröbner bases to show that $\mathcal{P} \diagup J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^k;\ \alpha_i<n-k+i \text{ for each } i}$.
    (This was already outlined in Aldo Conca, Christian Krattenthaler, Junzo Watanabe, *Regular Sequences of Symmetric Polynomials*, 2009.)
  - Using that + Jacobi–Trudi, show that $\mathcal{S} \diagup I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$.

On the proofs, 1

- Proofs of all the above (except for the S_k-action) can be found in
  - Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, `http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf` .
- **Main ideas:**
  - Use Gröbner bases to show that $\mathcal{P} \diagup J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$.
    (This was already outlined in Aldo Conca, Christian Krattenthaler, Junzo Watanabe, *Regular Sequences of Symmetric Polynomials*, 2009.)
  - Using that + Jacobi–Trudi, show that $\mathcal{S} \diagup I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$.
  - As for the rest, compute in Λ... a lot.

On the proofs, 2: the Gröbner basis argument

- The Gröbner basis argument relies on the easy identity

$$h_p(x_{i..k}) = \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) h_{p-t}(x_{1..k})$$

  for all $i \in \{1, 2, \ldots, k+1\}$ and $p \in \mathbb{N}$.
  Here, $x_{a..b}$ means $x_a, x_{a+1}, \ldots, x_b$.

- Use it to show that

$$\left(h_{n-k+i}(x_{i..k}) - \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) a_{i-t} \right)_{i \in \{1,2,\ldots,k\}}$$

  is a Gröbner basis of the ideal J wrt the degree-lexicographic term order, where the variables are ordered by $x_1 > x_2 > \cdots > x_k$.

- This Gröbner basis leads to a basis of $\mathcal{P} / J$, which is precisely our $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$.

On the proofs, 3: the first basis of $\mathcal{S} \diagup I$

- How to prove that $\mathcal{S} \diagup I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.
- On the other hand, $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ as an $\mathcal{S}$-module (Artin).

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow \left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.
- On the other hand, $\left(x^\alpha\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ as an $\mathcal{S}$-module (Artin).
- Combining these yields that $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P} / I\mathcal{P} = \mathcal{P} / J$.

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.
- On the other hand, $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ as an $\mathcal{S}$-module (Artin).
- Combining these yields that $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P} / I\mathcal{P} = \mathcal{P} / J$.
- But we also know that the family $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$ is a basis of $\mathcal{P} / J$.

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.
- On the other hand, $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ as an $\mathcal{S}$-module (Artin).
- Combining these yields that $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P} / I\mathcal{P} = \mathcal{P} / J$.
- But we also know that the family $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$ is a basis of $\mathcal{P} / J$.
- What can you say if a **k**-module has a basis $(a_v)_{v \in V}$ and a spanning family $(b_u)_{u \in U}$ of the same finite size $(|U| = |V| < \infty)$?

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow \left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.
- On the other hand, $\left(x^\alpha\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ as an $\mathcal{S}$-module (Artin).
- Combining these yields that $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P} / I\mathcal{P} = \mathcal{P} / J$.
- But we also know that the family $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$ is a basis of $\mathcal{P} / J$.
- What can you say if a $\mathbf{k}$-module has a basis $\left(a_v\right)_{v \in V}$ and a spanning family $\left(b_u\right)_{u \in U}$ of the same finite size ($|U| = |V| < \infty$)?
  Easy exercise: You can say that $\left(b_u\right)_{u \in U}$ is also a basis.

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.
- On the other hand, $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ as an $\mathcal{S}$-module (Artin).
- Combining these yields that $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P} / I\mathcal{P} = \mathcal{P} / J$.
- But we also know that the family $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$ is a basis of $\mathcal{P} / J$.
- What can you say if a **k**-module has a basis $(a_v)_{v \in V}$ and a spanning family $(b_u)_{u \in U}$ of the same finite size $(|U| = |V| < \infty)$?
  Easy exercise: You can say that $(b_u)_{u \in U}$ is also a basis.
- Thus, $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ is a basis of $\mathcal{P} / J$.

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?
- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
  $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans $\mathcal{S} / I$.
- On the other hand, $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}$ as an $\mathcal{S}$-module (Artin).
- Combining these yields that $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans $\mathcal{P} / I\mathcal{P} = \mathcal{P} / J$.
- But we also know that the family $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$ is a basis of $\mathcal{P} / J$.
- What can you say if a **k**-module has a basis $(a_v)_{v \in V}$ and a spanning family $(b_u)_{u \in U}$ of the same finite size ($|U| = |V| < \infty$)?
  Easy exercise: You can say that $(b_u)_{u \in U}$ is also a basis.
- Thus, $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ is a basis of $\mathcal{P} / J$.
- $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of $\mathcal{S} / I$.

On the proofs, 4: Bernstein's identity

- The rest of the proofs are long computations inside Λ, using various identities for symmetric functions.

- The rest of the proofs are long computations inside Λ, using various identities for symmetric functions.
- Maybe the most important one:
  Bernstein's identity: Let λ be a partition. Let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Then,

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{m+i} (\mathbf{e}_i)^{\perp} \mathbf{s}_\lambda = \mathbf{s}_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)}.$$

  Here, $\mathbf{f}^{\perp}\mathbf{g}$ means "$\mathbf{g}$ skewed by $\mathbf{f}$" (so that $(\mathbf{s}_\mu)^{\perp} \mathbf{s}_\lambda = \mathbf{s}_{\lambda/\mu}$).

Thank you

- **Sasha Postnikov** for the invitation and the paper that gave rise to this project 5 years ago.
- **Victor Reiner, Tom Roby, Mark Shimozono, Josh Swanson, Kaisa Taipale, and Anders Thorup** for enlightening discussions.
- **you** for your patience.