Critical groups for Hopf algebra modules

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joint work with Victor Reiner (UMN) and Jia Huang (UNK)

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slides:
http://www.cip.ifi.lmu.de/~grinberg/algebra/madison17.pdf
paper:
http://www.cip.ifi.lmu.de/~grinberg/algebra/McKayTensor.pdf or arXiv:1704.03778v1

1

Chip-firing on digraphs

References:

- Alexander E. Holroyd, Lionel Levine, Karola Mészáros, Yuval Peres, James Propp, David B. Wilson, *Chip-Firing and Rotor-Routing on Directed Graphs*, arXiv:0801.3306.
- Georgia Benkart, Caroline Klivans, Victor Reiner, *Chip firing on Dynkin diagrams and McKay quivers*, arXiv:1601.06849.

- *Chip-firing* on a loopless digraph D is a "solitaire game" (rigorously: rewriting system, or finite state machine). A brief definition:
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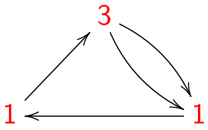
- **Example**:
  Start with



(The vertices drawn in red are the ones that can be fired.)
Let us fire the top vertex.

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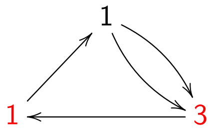

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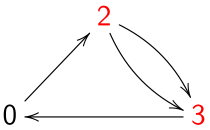

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Let us fire the bottom right vertex thrice.

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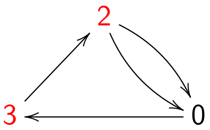

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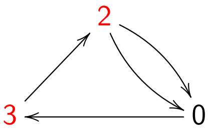

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And so on... this game can (and will) go on forever.

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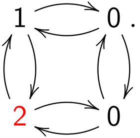

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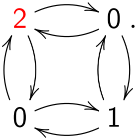

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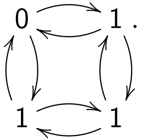

No more firing is possible here; the game has *terminated*.

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- We see that the chip-firing game will sometimes terminate after finitely many steps, but sometimes never will. There are some nontrivial results (Björner, Lovasz, Shor and others):
  - Whether it terminates depends only on the starting configuration (not on the choices of vertices to fire).
  - If it terminates, the configuration obtained in the end depends only on the starting configuration.

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- A neater situation is obtained if we fix a "global sink" q (a vertex reachable from every vertex), and disallow firing q. Then, the game **always** terminates. Again, there are remarkable properties (see Holroyd et al., arXiv:0801.3306):
  - The configuration obtained in the end depends only on the starting configuration.
  - "Sandpile monoid" and "sandpile group".
  - Relations to Eulerian walks and to spanning trees.

- We can describe chip-firing on a loopless digraph D via the Laplacian of D.
- Label the vertices of D by $1, 2, \ldots, n$.
- The *Laplacian* of D is the $n \times n$-matrix L whose (i, j)-th entry is

$$L_{i,j} = \begin{cases} \deg^+ i, & \text{if } j = i; \\ -a_{i,j}, & \text{if } j \neq i \end{cases},$$

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- The same holds for the variant where we fix a global sink q and never fire it...

- We can describe chip-firing on a loopless digraph D with a global sink q via the **reduced** Laplacian of D.
- Label the vertices of D by $1, 2, \ldots, n$ in such a way that the global sink q is n.
- The *reduced Laplacian* of D is the $(n-1) \times (n-1)$-matrix $\overline{L}$ obtained from L by removing the last row and the last column.

Chip-firing on digraphs: the matrix point of view, 2

- We can describe chip-firing on a loopless digraph D with a global sink q via the **reduced** Laplacian of D.
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Nonsingular M-matrices, 1

- Restating everything in terms of the Laplacian L and forgetting about the digraph allows us to crystallize the important parts of the argument and gain further generality.

Nonsingular M-matrices, 2

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- **Theorem (Gabrielov, Benkart, Klivans, Reiner, ...?):** For a Z-matrix C, the following are equivalent:
  - C is a nonsingular M-matrix.
  - C^T is a nonsingular M-matrix.
  - There exists a column vector $x \in \mathbb{Q}^\ell$ with $x > 0$ and $Cx > 0$. (Again, entrywise.)
  - The "generalized chip-firing game" in which we start with a row vector $r \geq 0$ and keep subtracting rows of C while keeping the vector ≥ 0 is confluent (i.e., terminates, and the final state depends only on the starting state).

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- Actually, "depends only on the starting state" follows from "Z-matrix", but termination requires "nonsingular M-matrix".

The sandpile monoid

- Given a digraph D with a chosen global sink q, we can define a finite abelian monoid as follows:
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  - The *sandpile monoid* of (D, q) is the monoid of all stable configurations, with monoid operation given by $(f, g) \mapsto (f + g)^\circ$.

The critical group

- Given a digraph D with a chosen global sink q, we can define a finite abelian group as follows:
  - If M is a finite abelian monoid, then the intersection of all (nonempty) ideals of M is a group. (Neat exercise.)
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- But again, we can also define this in terms of the Laplacian: Namely, the critical group of (D, q) is $K(D, q) = \operatorname{coker}\left(\overline{L}^T\right) = \mathbb{Z}^{n-1} / \left(\overline{L}^T \mathbb{Z}^{n-1}\right)$.

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- When D is Eulerian, this group does not depend on q (up to iso). Thus, we call it just $K(D)$.
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- When D is Eulerian, we have $\operatorname{coker}\left(L^T\right) = \underbrace{\mathbb{Z}}_{\text{free part}} \oplus \underbrace{K(D)}_{\text{torsion part}}$.
- Much of chip-firing theory doesn't need a digraph. A square matrix over $\mathbb{Z}$ is enough... and a nonsingular M-matrix is particularly helpful.

2

The critical group of a group character

References:

- Georgia Benkart, Caroline Klivans, Victor Reiner, *Chip firing on Dynkin diagrams and McKay quivers*, arXiv:1601.06849.
- Christian Gaetz, *Critical groups of McKay-Cartan matrices*, honors thesis 2016.
- Victor Reiner's talk slides.

The McKay matrix of a representation, 1

- Where else can we get nonsingular M-matrices from?

The McKay matrix of a representation, 1

- Let G be a finite group.
  Let $S_1, S_2, \ldots, S_{\ell+1}$ be the irreps (= irreducible representations) of G over $\mathbb{C}$. Let $\chi_1, \chi_2, \ldots, \chi_{\ell+1}$ be their characters.

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- Fix any representation V of G over $\mathbb{C}$ (not necessarily irreducible), and let χ_V be its character. Set $n = \dim V = \chi_V(e)$.
- The *McKay matrix* of V is the $(\ell+1) \times (\ell+1)$-matrix M_V whose (i,j)-th entry is the coefficient $m_{i,j}$ in the expansion

$$\chi_{S_i \otimes V} = \chi_i \chi_V = \sum_{j=1}^{\ell+1} m_{i,j} \chi_j .$$

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- Fix any representation V of G over $\mathbb{C}$ (not necessarily irreducible), and let χ_V be its character. Set $n = \dim V = \chi_V(e)$.
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$$\chi_{S_i \otimes V} = \chi_i \chi_V = \sum_{j=1}^{\ell+1} m_{i,j} \chi_j.$$

- We define a further $(\ell+1) \times (\ell+1)$-matrix L_V (our "Laplacian") by $L_V = nI - M_V$.
  Warning: Unlike the digraph case, the matrix L_V neither has row sums 0 nor has column sums 0!

- **Example:** The symmetric group $\mathfrak{S}_4$ has 5 irreps S_1, S_2, S_3, S_4, S_5, corresponding to the partitions $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$, respectively. We shall just call them $D^4, D^{31}, D^{22}, D^{211}, D^{1111}$ for clarity.
  Their characters
  $\chi_0 = \chi_{D^4}, \ \chi_1 = \chi_{D^{31}}, \ \chi_2 = \chi_{D^{22}}, \ \chi_3 = \chi_{D^{211}}, \ \chi_4 = \chi_{D^{1111}}$
  are the rows of the following character table:

$$\begin{array}{c} \\ \chi_{D^4} \\ \chi_{D^{31}} \\ \chi_{D^{22}} \\ \chi_{D^{211}} \\ \chi_{D^{1111}} \end{array}
\begin{array}{c} \begin{array}{ccccc} e & (ij) & (ij)(kl) & (ijk) & (ijkl) \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & -1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & -1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

  (these are given by weighted counting of rim hook tableaux, according to the Murnaghan-Nakayama rule).

The McKay matrix of a representation, 2b: example

- **Example (cont'd):** Let $V = D^{31}$. Then, the McKay matrix M_V is

$$M_V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(these are Kronecker coefficients, since D^{31} too is irreducible).

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For example, the second row is because

$$\chi_{D^{31} \otimes D^{31}} = 1\chi_{D^4} + 1\chi_{D^{31}} + 1\chi_{D^{22}} + 1\chi_{D^{211}} + 0\chi_{D^{1111}}.$$

- **Example (cont'd):** Let $V = D^{31}$. Then, the McKay matrix M_V is

$$M_V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For example, the third row is because

$$\chi_{D^{22} \otimes D^{31}} = 0\chi_{D^4} + 1\chi_{D^{31}} + 0\chi_{D^{22}} + 1\chi_{D^{211}} + 0\chi_{D^{1111}}.$$

- **Example (cont'd):** Let $V = D^{31}$. Then, the McKay matrix M_V is

$$M_V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

  Hence,

$$L_V = \underbrace{n}_{=\dim V = 3} I - M_V = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

The critical group of a representation

- Let $\overline{L_V}$ be the matrix L_V with its row and column corresponding to the trivial irrep removed. This is an $\ell \times \ell$-matrix.
- Define the *critical group* $K(V)$ of V by $K(V) = \operatorname{coker}\left(\overline{L_V}\right)$.
- Also, $\operatorname{coker}(L_V) \cong \mathbb{Z} \oplus K(V)$.
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- In our above **example**,

$$L_V = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix} \Longrightarrow \overline{L_V} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}.$$

(Here, we removed the 1-st row and 1-st column, since they index the trivial irrep.)

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  Hence, $K(V) = \operatorname{coker}\left(\overline{L_V}\right) \cong \mathbb{Z}/4\mathbb{Z}$.
- (Recall that the cokernel of a square matrix $M \in \mathbb{Z}^{N \times N}$ is $\cong \bigoplus_i (\mathbb{Z}/m_i\mathbb{Z})$, where the m_i are the diagonal entries in the Smith normal form of M. This is how the above was computed.)

Facts on the critical group

- **Theorems (Benkart, Klivans, Reiner, Gaetz):**
  - The column vector $\mathbf{s} = (\dim S_1, \dim S_2, \ldots, \dim S_{\ell+1})^T$ belongs to $\ker(L_V)$.
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    It spans the $\mathbb{Z}$-module $\ker(L_V)$ if and only if the G-representation V is faithful.
  - Actually, M_V and L_V can be diagonalized:
    For each $g \in G$, the vector $\mathbf{s}(g) = \left(\chi_{S_1}(g), \chi_{S_2}(g), \ldots, \chi_{S_{\ell+1}}(g)\right)^T$ (a column of the character table of G) is an eigenvector of M_V (with eigenvalue $\chi_V(g)$) and of L_V (with eigenvalue $n - \chi_V(g)$).

Facts on the critical group

- **Theorems (Benkart, Klivans, Reiner, Gaetz):**
  - If the G-representation V is faithful, then $\overline{L_V}$ is a nonsingular M-matrix.

    (Hence, a theory of "chip-firing" exists. Benkart, Klivans and Reiner have further results on this, but much is still unexplored.

    For **some** groups G and representations V, this "chip-firing" is equivalent to actual chip-firing on certain specific digraphs. See Benkart-Klivans-Reiner paper.)

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$$\# K(V) = \frac{1}{\# G} \prod_{G\text{-conjugacy class } [g] \neq [e]} (n - \chi_V(g)).$$

  - For the regular G-representation $\mathbb{C}G$, we have

$$K(\mathbb{C}G) \cong (\mathbb{Z}/n\mathbb{Z})^{\ell-1}.$$

    Here, $n = \dim(\mathbb{C}G) = \# G$ and
    $\ell = (\text{number of } G\text{-conjugacy classes}) - 1$.

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  - algebraically closed field $\mathbb{C} \to$ any field.
  - finite-dimensional $\to$ arbitrary dimension.
  - finite group $\to$ finite-dimensional Hopf algebra.
- We shall only study the two blue directions. (The others are interesting, too!)

3

The critical group of a Hopf algebra module

References:

- Darij Grinberg, Jia Huang, Victor Reiner, *Critical groups for Hopf algebra modules*, arXiv:1704.03778.

Introducing the Hopf algebra A

- Let $\mathbb{F}$ be any algebraically closed field of any characteristic.
- All $\mathbb{F}$-vector spaces in the following are finite-dimensional.
  dim always means $\mathbb{F}$-vector space dimension.
  $\otimes$ always means $\otimes_{\mathbb{F}}$.
- Let A be a finite-dimensional Hopf algebra over $\mathbb{F}$. This means:
  - First of all, A is an $\mathbb{F}$-algebra.
  - Also, A is finite-dimensional as an $\mathbb{F}$-vector space.
  - Also, A is equipped with
    - a comultiplication $\Delta : A \to A \otimes A$,
    - a counit $\epsilon : A \to \mathbb{F}$,
    - an antipode $\alpha : A \to A$

    satisfying certain axioms.

- In the following, "A-module" means "left A-module".
- Classical results on representations of A:
  - There are finitely many simple A-modules
    $S_1, S_2, \ldots, S_{\ell+1}$,
    and finitely many indecomposable projective A-modules
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  - The left-regular A-module A decomposes as a direct sum

$$A \cong \bigoplus_{i=1}^{\ell+1} P_i^{\dim S_i}.$$

- For an A-module V, if $[V : S_i]$ denotes the multiplicity of S_i as a composition factor of V, then

$$[V : S_i] = \dim \mathrm{Hom}_A(P_i, V).$$

Representations of A: tensor category structure

- So far we have just used the $\mathbb{F}$-algebra structure on A (and the algebraic closedness of $\mathbb{F}$).
  What if we take into account the Hopf algebra structure too?

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    and thus in particular define a "dual A-module" V^* of an A-module V (without switching sides).

Hopf algebra examples, 1: the group algebra

- **Example 1:** Let A be the group algebra $\mathbb{F}G$ of a finite group G.
  This becomes a Hopf algebra by setting
  $$\begin{aligned} \epsilon(g) &= 1, \\ \Delta(g) &= g \otimes g, \\ \alpha(g) &= g^{-1} \end{aligned}$$
  for all $g \in G$.

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  Note that if $\operatorname{char} \mathbb{F} = 0$, then A is semisimple, so that the theory dramatically simplifies (e.g., we have $P_i = S_i$ for all i).

- **"Example 0":** This example does not really fit into our framework (yet?), but is too good to omit:
  Let A be the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.
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$$\begin{aligned} \epsilon(x) &= 0, \\ \Delta(x) &= x \otimes 1 + 1 \otimes x, \\ \alpha(x) &= -x \end{aligned}$$

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  The A-modules are precisely the representations of $\mathfrak{g}$; the notions of tensor product, trivial module, etc. are the ones we know from Lie algebra representation theory.
  Sadly, A is usually infinite-dimensional, and our theory is not ready for this. (Restricted universal enveloping algebras in characteristic p do work, though.)

Hopf algebra examples, 2: the generalized Taft Hopf algebra

- **Example 2:** Fix integers $m \geq 0$ and $n > 0$ with $m \mid n$. Fix a primitive n-th root of unity $\omega \in \mathbb{F}$. (Recall we assumed $\mathbb{F}$ algebraically closed!)

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  As an $\mathbb{F}$-algebra, the *generalized Taft Hopf algebra* $A = H_{n,m}$ is given by

$$\begin{aligned} \text{generators} \quad & g, x; \\ \text{relations} \quad & g^n = 1, \qquad x^m = 0, \qquad xg = \omega g x. \end{aligned}$$

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  More conceptual definition: A is a skew group ring

  $$H_{n,m} = \mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \ltimes \mathbb{F}[x]/(x^m)$$

  for the cyclic group $\mathbb{Z}/n\mathbb{Z} = \{e, g, g^2, \ldots, g^{n-1}\}$ acting on coefficients in a truncated polynomial algebra $\mathbb{F}[x]/(x^m)$, via $gxg^{-1} = \omega^{-1}x$.

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  This A has $\mathbb{F}$-basis $\{g^i x^j : 0 \leq i < n \text{ and } 0 \leq j < m\}$, whence $\dim A = mn$.
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$$\begin{array}{rclrcl} \epsilon(g) & = & 1, & \epsilon(x) & = & 0, \\ \Delta(g) & = & g \otimes g, & \Delta(x) & = & 1 \otimes x + x \otimes g, \\ \alpha(g) & = & g^{-1}, & \alpha(x) & = & -\omega^{-1} g^{-1} x. \end{array}$$

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  This A has n projective indecomposable modules, each of dimension m, whereas its n simple modules are all 1-dimensional.

- **Example 3:** Fix integers $m \geq 0$ and $n > 0$ such that n is even and n lies in $\mathbb{F}^\times$. Fix a primitive n-th root of unity $\omega \in \mathbb{F}$. (Recall we assumed $\mathbb{F}$ algebraically closed!)

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  for the cyclic group $\mathbb{Z}/n\mathbb{Z} = \{e, g, g^2, \ldots, g^{n-1}\}$ acting this time on coefficients in an exterior algebra $\bigwedge_{\mathbb{F}}[x_1, \ldots, x_m]$, via $g x_i g^{-1} = \omega x_i$.

Hopf algebra examples, 3: a weird one from Radford

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  This A has $\mathbb{F}$-basis $\left\{ g^i x_J : 0 \leq i < n, \ J \subseteq \{1, 2, \ldots, m\} \right\}$ (where $x_J := x_{j_1} x_{j_2} \cdots x_{j_k}$ if $J = \{j_1 < j_2 < \cdots < j_k\}$), whence $\dim A = n 2^m$.
  It becomes a Hopf algebra by setting

$$\begin{array}{ll} \epsilon(g) = 1, & \epsilon(x_i) = 0, \\ \Delta(g) = g \otimes g, & \Delta(x_i) = 1 \otimes x_i + x_i \otimes g^{n/2}, \\ \alpha(g) = g^{-1}, & \alpha(x_i) = -x_i g^{n/2}. \end{array}$$

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- The $\mathbb{Z}$-module $G_0(A)$ becomes a ring (not always commutative) by setting $[V] \cdot [W] = [V \otimes W]$ and $1 = [\epsilon]$.

The McKay matrix of an A-module, 1

- Now, let us generalize the McKay matrix of a group representation to the case of an A-module.

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  In other words, its entries are chosen to satisfy

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More conceptually: The matrix M_V represents the right multiplication by $[V]$ as an endomorphism of $G_0\,(A)$ with respect to the basis $([S_1], \ldots, [S_{\ell+1}])$.

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  In other words, its entries are chosen to satisfy

$$[S_i \otimes V] = [S_i]\,[V] = \sum_{j=1}^{\ell+1} (M_V)_{i,j}\,[S_j]\,.$$

  More conceptually: The matrix M_V represents the right multiplication by $[V]$ as an endomorphism of $G_0(A)$ with respect to the basis $([S_1], \ldots, [S_{\ell+1}])$.
- We define a further $(\ell+1) \times (\ell+1)$-matrix L_V (our "Laplacian") by $L_V = nI - M_V$.

The critical group of an A-module

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  How do we pick up a canonical complement to $\mathbb{Z}$?
- It is tempting to again define $\overline{L_V}$ by removing the trivial row and trivial column from L_V, and set $K(V) = \operatorname{coker}\left(\overline{L_V}\right)$. But that does not work.

The critical group of an A-module

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  How do we pick up a canonical complement to $\mathbb{Z}$?
- In matrix terms:
  Consider the column vector
  $\mathbf{s} = (\dim S_1, \dim S_2, \ldots, \dim S_{\ell+1})^T$. Then, set
  $K(V) = \mathbf{s}^{\perp} / \operatorname{Im}(L_V)$, where $\mathbf{s}^{\perp} = \{\mathbf{x} \in \mathbb{Z}^{\ell+1} \mid \mathbf{s}^T \mathbf{x} = 0\}$.

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- More conceptually:
  There is an *augmentation map* on $G_0(A) \to \mathbb{Z}$. This is the ring homomorphism sending each $[V]$ to $\dim V$.
  Let I be its kernel.
- Set $K(V) = I / G_0(A)(n - [V])$.
  Note that multiplication by $n - [V]$ corresponds to the action of M_V, so this makes sense.

- **Theorems (G., Huang, Reiner):**
  - $\mathbf{p} = (\dim P_1, \dim P_2, \ldots, \dim P_{\ell+1})^T \in \ker(L_V)$, whereas $\mathbf{s} = (\dim S_1, \dim S_2, \ldots, \dim S_{\ell+1})^T \in \ker\left((L_V)^T\right)$.
    These vectors span the respective kernels over $\mathbb{Q}$ if and only if the A-module V is *tensor-rich* (which is our way to say that each simple A-module appears in a composition series of $V^{\otimes k}$ for at least one k).

- **Theorems (G., Huang, Reiner):**
  - If $A = \mathbb{F}G$ is a group algebra, then M_V and L_V can be diagonalized:
    - Fix a *Brauer character* χ_W for each A-module W.
    - Given a p-regular element $g \in G$, let $\mathbf{s}(g) = \left(\chi_{S_1}(g), \ldots, \chi_{S_{\ell+1}}(g)\right)^T$ be the Brauer character values of the simple $\mathbb{F}G$-modules at g. Then, $\mathbf{s}(g)$ is an eigenvector of $(M_V)^T$ (with eigenvalue $\chi_V(g)$) and of $(L_V)^T$ (with eigenvalue $n - \chi_V(g)$).
    - Given a p-regular element $g \in G$, let $\mathbf{p}(g) = \left(\chi_{P_1}(g), \ldots, \chi_{P_{\ell+1}}(g)\right)^T$ be the Brauer character values of the indecomposable projective $\mathbb{F}G$-modules at g. Then, $\mathbf{p}(g)$ is an eigenvector of M_V (with eigenvalue $\chi_V(g)$) and of L_V (with eigenvalue $n - \chi_V(g)$).

Facts on the critical group

- **Theorems (G., Huang, Reiner):**
  - If the A-module V is tensor-rich, then $\overline{L_V}$ is a nonsingular M-matrix.
    (Hence, a theory of "chip-firing" exists. We have not studied it. It is complicated by the fact that $K(V)$ is not generally isomorphic to coker $\left(\overline{(L_V)^T}\right)$ any more.)

- **Theorems (G., Huang, Reiner):**
  - If the A-module V is tensor-rich, then $\overline{L_V}$ is a nonsingular M-matrix.
  - Actually, the following are equivalent:
    - **(i)** The matrix $\overline{L_V}$ (obtained from L_V by removing the row and the column corresponding to the trivial A-module ε) is a nonsingular M-matrix.
    - **(ii)** The matrix $\overline{L_V}$ is nonsingular.
    - **(iii)** L_V has rank ℓ, so nullity 1.
    - **(iv)** The critical group $K(V)$ is finite.
    - **(v)** The A-module V is tensor-rich.

Facts on the critical group

- **Theorems (G., Huang, Reiner):**
  - If the A-module V is tensor-rich, then $\overline{L_V}$ is a nonsingular M-matrix.
  - If the A-module V is tensor-rich, then

    $$\#K(V) = \left| \frac{\gamma}{\dim A} \left(\text{product of the nonzero eigenvalues of } L_V\right) \right|,$$

    where $\gamma = \gcd(\dim P_1, \dim P_2, \ldots, \dim P_{\ell+1})$.

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where $\gamma = \gcd(\dim P_1, \dim P_2, \ldots, \dim P_{\ell+1})$.
If $A = \mathbb{F}G$ is a group algebra, then this can be rewritten in a more explicit way using Brauer characters as well:

$$\#K(V) = \frac{\gamma}{\#G} \prod_{\substack{[g] \neq [e] \text{ is a } p\text{-regular} \\ \text{conjugacy class in } G}} (n - \chi_V(g)).$$

Also, γ is the size of the p-Sylow subgroups of G if $\mathbb{F}$ has characteristic $p > 0$.

Facts on the critical group

- **Theorems (G., Huang, Reiner):**
  - If the A-module V is tensor-rich, then $\overline{L_V}$ is a nonsingular M-matrix.
  - For the regular A-module A, we have

    $$K(A) \cong (\mathbb{Z}/\gamma\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})^{\ell-1}.$$

    Here, $n = \dim A$, $\gamma = \gcd(\dim P_1, \dim P_2, \ldots, \dim P_{\ell+1})$ and $\ell =$ (number of simple A-modules) $- 1$.

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  Is there a notion of "Sylow subalgebras" (presumably subalgebras of dimension γ, which have the form $\mathbb{F} \oplus$ (nilpotents) and over which A is free as a module)?

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- What can we say about A and V if V is not tensor-rich? Does $A \to \text{End}\, V$ factor through a quotient Hopf algebra then?
- Does the theory extend to infinite-dimensional A ? (Think of universal enveloping algebras – the finite-dimensional A-modules can be fairly tame.)

Thank you!

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