The Pelletier–Ressayre hidden symmetry for Littlewood–Richardson coefficients

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January 2, 2022
(Extended version of a paper published in:
Combinatorial Theory **1** (2021), #16.)

Abstract. We prove an identity for Littlewood–Richardson coefficients conjectured by Pelletier and Ressayre. The proof relies on a novel birational involution defined over any semifield.

Keywords: symmetric functions, Schur functions, Schur polynomials, Littlewood–Richardson coefficients, birational combinatorics, detropicalization, partitions.

MSC2010 Mathematics Subject Classifications: 05E05.

Contents

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One of the central concepts in the theory of symmetric functions are the *Littlewood–Richardson coefficients* $c^{\lambda}_{\mu,\nu}$: the coefficients when a product $s_\mu s_\nu$ of two Schur functions is expanded back in the Schur basis $(s_\lambda)_{\lambda\in\operatorname{Par}}$. Various properties of these coefficients have been found, among them combinatorial interpretations, vanishing results, bounds and symmetries (i.e., equalities between $c^{\lambda}_{\mu,\nu}$ for different λ,μ,ν). A recent overview of the latter can be found in [BriRos20].

In [PelRes20], Pelletier and Ressayre conjectured a further symmetry of Littlewood–Richardson coefficients. Unless the classical ones, it is a partial symmetry (i.e., it does not cover every Littlewood–Richardson coefficient); it is furthermore much less simple to state, to the extent that Pelletier and Ressayre have conjectured its existence while leaving open the question which exact coefficients it matches up. In this paper, we answer this question and prove the conjecture thus concretized.

The conjecture, in its original (unconcrete) form, can be stated as follows: Let $n \geq 2$, and consider the set $\operatorname{Par}[n]$ of all partitions having length $\leq n$. Let a and b be two nonnegative integers, and define the two partitions $\alpha = (a+b, a^{n-2})$ and $\beta = (a+b, b^{n-2})$ (where c^{n-2} means $\underbrace{c, c, \ldots, c}_{n-2 \text{ times}}$, as usual in partition combinatorics). Fix another partition $\mu \in \operatorname{Par}[n]$. Then, the families $\left(c^{\omega}_{\alpha,\mu}\right)_{\omega\in\operatorname{Par}[n]}$ and $\left(c^{\omega}_{\beta,\mu}\right)_{\omega\in\operatorname{Par}[n]}$ of Littlewood–Richardson coefficients seem to be identical up to permutation. (We can restate this in terms of Schur polynomials in the n variables $x_1, x_2, \ldots, x_n$; this then becomes the claim that the products $s_\alpha(x_1, x_2, \ldots, x_n) \cdot s_\mu(x_1, x_2, \ldots, x_n)$ and $s_\beta(x_1, x_2, \ldots, x_n) \cdot s_\mu(x_1, x_2, \ldots, x_n)$, when expanded in the basis of Schur polynomials, have the same multiset of coefficients.)

Pelletier and Ressayre have proved this conjecture for $n = 3$ (see [PelRes20, Corollary 2]) and in some further cases. We shall prove it in full generality, and construct what is essentially a bijection $\varphi : \operatorname{Par}[n] \to \operatorname{Par}[n]$ that makes it explicit (i.e., that satisfies $c^{\omega}_{\alpha,\mu} = c^{\varphi(\omega)}_{\beta,\mu}$ for each $\omega \in \operatorname{Par}[n]$). To be fully precise, φ will not be a bijection $\operatorname{Par}[n] \to \operatorname{Par}[n]$, but rather a bijection from $\mathbb{Z}^n$ to $\mathbb{Z}^n$, and it will satisfy $c^{\omega}_{\alpha,\mu} = c^{\varphi(\omega)}_{\beta,\mu}$ with the understanding that $c^{\omega}_{\alpha,\mu} = c^{\omega}_{\beta,\mu} = 0$ when $\omega \notin \operatorname{Par}[n]$. (Here, $\operatorname{Par}[n]$ is understood to be a subset of $\mathbb{Z}^n$ by identifying each partition $\lambda \in \operatorname{Par}[n]$ with the n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.)

We will define this bijection φ by explicit (if somewhat intricate) formulas that "mingle" the entries of the partition it is being applied to with those of μ (as well as a and b) using the min and $+$ operators. These formulas are best understood in the *birational picture*, in which these min and $+$ operators are generalized to the addition and the multiplication of an arbitrary semifield. (Our proof does not require this generality, but the birational picture has the advantage of greater familiarity and better notational support. It also reveals a connection with a known birational map known as a "birational R-matrix" (see Section 5.4 for details), which throws some light on the otherwise rather mysterious bijection.)

Another ingredient of our proof is an explicit formula for $s_\alpha(x_1, x_2, \ldots, x_n)$ for the above-mentioned partition α.

Acknowledgments

I thank Sunita Chepuri, Grigori Olshanski, Pasha Pylyavskyy and Tom Roby for interesting and helpful conversations.

Parts of this paper were written at the Mathematisches Forschungsinstitut Oberwolfach, where I was staying as a Leibniz fellow in Summer 2020. The SageMath computer algebra system [SageMath] has been used in discovering some of the results.

Remark on alternative versions

This paper also has a detailed version [Grinbe20], which includes some proofs that have been omitted from the present version (mostly basic properties of symmetric functions).

An older version of this paper appeared as Oberwolfach Preprint OWP-2020-18.

1. Notations

We will use the following notations (most of which are also used in [GriRei20, §2.1]):

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- We fix a commutative ring $\mathbf{k}$; we will use this $\mathbf{k}$ as the base ring in what follows.

- A *weak composition* means an infinite sequence of nonnegative integers that contains only finitely many nonzero entries (i.e., a sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ such that all but finitely many $i \in \{1, 2, 3, \ldots\}$ satisfy $\alpha_i = 0$).
- We let WC denote the set of all weak compositions.
- For any weak composition α and any positive integer i, we let α_i denote the i-th entry of α (so that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$). More generally, we use this notation whenever α is an infinite sequence of any kind of objects.
- The *size* $|\alpha|$ of a weak composition α is defined to be $\alpha_1 + \alpha_2 + \alpha_3 + \cdots \in \mathbb{N}$.
- A *partition* means a weak composition whose entries weakly decrease (i.e., a weak composition α satisfying $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$).
- We let Par denote the set of all partitions.
- We will sometimes omit trailing zeroes from partitions: i.e., a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ will be identified with the k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whenever $k \in \mathbb{N}$ satisfies $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$. For example, $(3, 2, 1, 0, 0, 0, \ldots) = (3, 2, 1) = (3, 2, 1, 0)$.

  As a consequence of this, an n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ (for any given $n \in \mathbb{N}$) is a partition if and only if it satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.
- A *part* of a partition λ means a nonzero entry of λ. For example, the parts of the partition $(3, 1, 1) = (3, 1, 1, 0, 0, 0, \ldots)$ are $3, 1, 1$.
- The *length* of a partition λ means the smallest $k \in \mathbb{N}$ such that $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$. Equivalently, the length of a partition λ is the number of parts of λ (counted with multiplicity). This length is denoted by $\ell(\lambda)$. For example, $\ell((4, 2, 0, 0)) = \ell((4, 2)) = 2$ and $\ell((5, 1, 1)) = 3$.
- We will use the notation m^k for "$\underbrace{m, m, \ldots, m}_{k \text{ times}}$" in partitions and tuples (whenever $m \in \mathbb{N}$ and $k \in \mathbb{N}$). (For example, $(2, 1^4) = (2, 1, 1, 1, 1)$.)
- We let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. This is a subring of the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series. To be more specific, Λ consists of all power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that are symmetric (i.e., invariant under permutations of the variables) and of bounded degree (see [GriRei20, §2.1] for the precise meaning of this).
- A *monomial* shall mean a formal expression of the form $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ with $\alpha \in \mathrm{WC}$. Formal power series are formal infinite $\mathbf{k}$-linear combinations of such monomials.
- For any weak composition α, we let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$.

- The *degree* of a monomial $\mathbf{x}^{\alpha}$ is defined to be $|\alpha|$.

We shall use the symmetric functions h_n and s_λ in Λ as defined in [GriRei20, Sections 2.1 and 2.2]. Let us briefly recall how they are defined:

- For each $n \in \mathbb{Z}$, we define the *complete homogeneous symmetric function* $h_n \in \Lambda$ by
$$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = n}} \mathbf{x}^{\alpha}.$$
Thus, $h_0 = 1$ and $h_n = 0$ for all $n < 0$.

- For each partition λ, we define the *Schur function* $s_\lambda \in \Lambda$ by
$$s_\lambda = \sum \mathbf{x}_T,$$
where the sum ranges over all semistandard tableaux T of shape λ, and where $\mathbf{x}_T$ denotes the monomial obtained by multiplying the x_i for all entries i of T. We refer the reader to [GriRei20, Definition 2.2.1] or to [Stanle01, §7.10] for the details of this definition and further descriptions of the Schur functions.

  The family $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module Λ, and is known as the *Schur basis*. It is easy to see that each $n \in \mathbb{N}$ satisfies $s_{(n)} = h_n$.

- We shall use the *Littlewood–Richardson coefficients* $c^{\lambda}_{\mu,\nu}$ (for $\lambda, \mu, \nu \in \mathrm{Par}$), as defined in [GriRei20, Definition 2.5.8], in [Stanle01, §7.15] or in [Egge19, Chapter 10]. One of their defining properties is the following fact (see, e.g., [GriRei20, (2.5.6)] or [Stanle01, (7.64)] or [Egge19, (10.1)]): Any two partitions $\mu, \nu \in \mathrm{Par}$ satisfy
$$s_\mu s_\nu = \sum_{\lambda \in \mathrm{Par}} c^{\lambda}_{\mu,\nu} s_\lambda. \tag{1}$$

2. The theorem

Convention 2.1.

(a) For the rest of this paper, we fix a positive integer n.

(b) Let $\mathrm{Par}[n]$ be the set of all partitions having length $\leq n$. In other words,
$$\begin{aligned} \mathrm{Par}[n] &= \{\lambda \in \mathrm{Par} \mid \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\} = \mathrm{Par} \cap \mathbb{N}^n \\ &= \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\} \end{aligned}$$
(where we are using our convention that trailing zeroes can be omitted from partitions, so that a partition of length $\leq n$ can always be identified with an n-tuple).

(c) A family $(u_i)_{i\in\mathbb{Z}}$ of objects (e.g., of numbers) is said to be *n-periodic* if each $j \in \mathbb{Z}$ satisfies $u_j = u_{j+n}$. Equivalently, a family $(u_i)_{i\in\mathbb{Z}}$ of objects is n-periodic if and only if it has the property that

$$\left(u_j = u_{j'} \text{ whenever } j \text{ and } j' \text{ are two integers satisfying } j \equiv j' \bmod n\right).$$

Thus, an n-periodic family $(u_i)_{i\in\mathbb{Z}}$ is uniquely determined by the n entries $u_1, u_2, \ldots, u_n$ (because for any integer j, we have $u_j = u_{j'}$, where j' is the unique element of $\{1, 2, \ldots, n\}$ that is congruent to j modulo n).

Example 2.2. If $n = 3$, then both partitions $(3, 2)$ and $(3, 2, 2)$ belong to $\operatorname{Par}[n]$, while the partition $(3, 2, 2, 2)$ does not. The n-tuples $(4, 2, 1)$ and $(3, 3, 0)$ are partitions, while the n-tuples $(1, 0, -1)$ and $(2, 0, 1)$ are not.

If ζ is an n-th root of unity, then the family $\left(\zeta^i\right)_{i\in\mathbb{Z}}$ of complex numbers is n-periodic.

We can now state our main theorem, which is a concretization of [PelRes20, Conjecture 1]:

Theorem 2.3. Assume that $n \geq 2$. Let $a, b \in \mathbb{N}$.

Define the two partitions $\alpha = \left(a + b, a^{n-2}\right)$ and $\beta = \left(a + b, b^{n-2}\right)$.

Fix any partition $\mu \in \operatorname{Par}[n]$.

Define a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ as follows:

Let $\omega \in \mathbb{Z}^n$. Define an n-tuple $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$ by

$$\nu_i = \omega_i - a \qquad \text{for each } i \in \{1, 2, \ldots, n\},$$

where ω_i means the i-th entry of ω.

For each $i \in \mathbb{Z}$, we let $i\#$ denote the unique element of $\{1, 2, \ldots, n\}$ congruent to i modulo n.

For each $j \in \mathbb{Z}$, set

$$\begin{aligned}\tau_j = \min \Big\{ &\left(\nu_{(j+1)\#} + \nu_{(j+2)\#} + \cdots + \nu_{(j+k)\#}\right) \\ &+ \left(\mu_{(j+k+1)\#} + \mu_{(j+k+2)\#} + \cdots + \mu_{(j+n-1)\#}\right) \\ &\mid\ k \in \{0, 1, \ldots, n-1\} \Big\}.\end{aligned}$$

Define an n-tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{Z}^n$ by setting

$$\eta_i = \mu_{i\#} + \left(\mu_{(i-1)\#} + \tau_{(i-1)\#}\right) - \left(\nu_{(i+1)\#} + \tau_{(i+1)\#}\right) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Let $\varphi(\omega)$ be the n-tuple $(\eta_1 + b, \eta_2 + b, \ldots, \eta_n + b) \in \mathbb{Z}^n$. Thus, we have defined a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$.

Then:

(a) The map φ is a bijection.

(b) We have

$$c_{\alpha,\mu}^{\omega} = c_{\beta,\mu}^{\varphi(\omega)} \qquad \text{for each } \omega \in \mathbb{Z}^n.$$

Here, we are using the convention that every n-tuple $\omega \in \mathbb{Z}^n$ that is not a partition satisfies $c_{\alpha,\mu}^{\omega} = 0$ and $c_{\beta,\mu}^{\omega} = 0$.

This theorem will be proved at the end of this paper, after we have shown several (often seemingly unrelated, yet eventually useful) results.

Example 2.4. Let $n = 4$ and $a = 1$ and $b = 4$. The partitions α and β defined in Theorem 2.3 then take the forms $\alpha = \left(1+4, 1^2\right) = (5,1,1)$ and $\beta = \left(1+4, 4^2\right) = (5,4,4)$.

Let $\mu \in \operatorname{Par}[n]$ be the partition $(2,1) = (2,1,0,0)$. Let $\omega \in \operatorname{Par}[n]$ be the partition $(5,3,2) = (5,3,2,0)$. We shall compute the n-tuple $\varphi(\omega)$ defined in Theorem 2.3.

Indeed, the n-tuple ν from Theorem 2.3 is

$$\nu = (\omega_1 - a, \omega_2 - a, \omega_3 - a, \omega_4 - a) = (5-1, 3-1, 2-1, 0-1) = (4,2,1,-1).$$

The integers $i\#$ from Theorem 2.3 form an n-periodic family

$$(i\#)_{i\in\mathbb{Z}} = (\ldots, 0\#, 1\#, 2\#, 3\#, 4\#, 5\#, 6\#, 7\#, \ldots) = (\ldots, 4, 1, 2, 3, 4, 1, 2, 3, \ldots).$$

The integers τ_j (for $j \in \mathbb{Z}$) from Theorem 2.3 are given by

$$\begin{aligned}
\tau_1 &= \min\left\{\left(\nu_{2\#} + \nu_{3\#} + \cdots + \nu_{(k+1)\#}\right) + \left(\mu_{(k+2)\#} + \mu_{(k+3)\#} + \cdots + \mu_{4\#}\right) \right.\\
&\qquad\qquad \left. \mid\ k \in \{0,1,2,3\}\right\}\\
&= \min\left\{\mu_{2\#} + \mu_{3\#} + \mu_{4\#},\quad \nu_{2\#} + \mu_{3\#} + \mu_{4\#},\quad \nu_{2\#} + \nu_{3\#} + \mu_{4\#},\quad \nu_{2\#} + \nu_{3\#} + \nu_{4\#}\right\}\\
&= \min\left\{\mu_2 + \mu_3 + \mu_4,\quad \nu_2 + \mu_3 + \mu_4,\quad \nu_2 + \nu_3 + \mu_4,\quad \nu_2 + \nu_3 + \nu_4\right\}\\
&= \min\left\{1+0+0,\quad 2+0+0,\quad 2+1+0,\quad 2+1+(-1)\right\}\\
&= \min\{1,2,3,2\} = 1
\end{aligned}$$

and

$$\begin{aligned}
\tau_2 &= \min\left\{\left(\nu_{3\#} + \nu_{4\#} + \cdots + \nu_{(k+2)\#}\right) + \left(\mu_{(k+3)\#} + \mu_{(k+4)\#} + \cdots + \mu_{5\#}\right) \right.\\
&\qquad\qquad \left. \mid\ k \in \{0,1,2,3\}\right\}\\
&= \min\left\{\mu_{3\#} + \mu_{4\#} + \mu_{5\#},\quad \nu_{3\#} + \mu_{4\#} + \mu_{5\#},\quad \nu_{3\#} + \nu_{4\#} + \mu_{5\#},\quad \nu_{3\#} + \nu_{4\#} + \nu_{5\#}\right\}\\
&= \min\left\{\mu_3 + \mu_4 + \mu_1,\quad \nu_3 + \mu_4 + \mu_1,\quad \nu_3 + \nu_4 + \mu_1,\quad \nu_3 + \nu_4 + \nu_1\right\}\\
&= \min\left\{0+0+2,\quad 1+0+2,\quad 1+(-1)+2,\quad 1+(-1)+4\right\}\\
&= \min\{2,3,2,4\} = 2
\end{aligned}$$

and

$$\begin{aligned}
\tau_3 &= \min\Big\{\Big(\nu_{4\#}+\nu_{5\#}+\cdots+\nu_{(k+3)\#}\Big)+\Big(\mu_{(k+4)\#}+\mu_{(k+5)\#}+\cdots+\mu_{6\#}\Big)\\
&\qquad\qquad\mid\ k\in\{0,1,2,3\}\Big\}\\
&= \min\{\mu_{4\#}+\mu_{5\#}+\mu_{6\#},\quad \nu_{4\#}+\mu_{5\#}+\mu_{6\#},\quad \nu_{4\#}+\nu_{5\#}+\mu_{6\#},\quad \nu_{4\#}+\nu_{5\#}+\nu_{6\#}\}\\
&= \min\{\mu_4+\mu_1+\mu_2,\quad \nu_4+\mu_1+\mu_2,\quad \nu_4+\nu_1+\mu_2,\quad \nu_4+\nu_1+\nu_2\}\\
&= \min\{0+2+1,\quad (-1)+2+1,\quad (-1)+4+1,\quad (-1)+4+2\}\\
&= \min\{3,2,4,5\} = 2
\end{aligned}$$

and

$$\begin{aligned}
\tau_4 &= \min\Big\{\Big(\nu_{5\#}+\nu_{6\#}+\cdots+\nu_{(k+4)\#}\Big)+\Big(\mu_{(k+5)\#}+\mu_{(k+6)\#}+\cdots+\mu_{7\#}\Big)\\
&\qquad\qquad\mid\ k\in\{0,1,2,3\}\Big\}\\
&= \min\{\mu_{5\#}+\mu_{6\#}+\mu_{7\#},\quad \nu_{5\#}+\mu_{6\#}+\mu_{7\#},\quad \nu_{5\#}+\nu_{6\#}+\mu_{7\#},\quad \nu_{5\#}+\nu_{6\#}+\nu_{7\#}\}\\
&= \min\{\mu_1+\mu_2+\mu_3,\quad \nu_1+\mu_2+\mu_3,\quad \nu_1+\nu_2+\mu_3,\quad \nu_1+\nu_2+\nu_3\}\\
&= \min\{2+1+0,\quad 4+1+0,\quad 4+2+0,\quad 4+2+1\}\\
&= \min\{3,5,6,7\} = 3
\end{aligned}$$

and

$$\tau_j = \tau_{j'} \qquad \text{whenever } j \equiv j' \bmod 4$$

(the latter equality follows from the n-periodicity of the family $(i\#)_{i\in\mathbb{Z}}$). Thus, the n-tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ from Theorem 2.3 is given by

$$\eta_1 = \underbrace{\mu_{1\#}}_{=\mu_1=2} + \left(\underbrace{\mu_{0\#}}_{=\mu_4=0} + \underbrace{\tau_{0\#}}_{=\tau_4=3}\right) - \left(\underbrace{\nu_{2\#}}_{=\nu_2=2} + \underbrace{\tau_{2\#}}_{=\tau_2=2}\right) = 2+(0+3)-(2+2) = 1$$

and

$$\eta_2 = \underbrace{\mu_{2\#}}_{=\mu_2=1} + \left(\underbrace{\mu_{1\#}}_{=\mu_1=2} + \underbrace{\tau_{1\#}}_{=\tau_1=1}\right) - \left(\underbrace{\nu_{3\#}}_{=\nu_3=1} + \underbrace{\tau_{3\#}}_{=\tau_3=2}\right) = 1+(2+1)-(1+2) = 1$$

and

$$\eta_3 = \underbrace{\mu_{3\#}}_{=\mu_3=0} + \left(\underbrace{\mu_{2\#}}_{=\mu_2=1} + \underbrace{\tau_{2\#}}_{=\tau_2=2}\right) - \left(\underbrace{\nu_{4\#}}_{=\nu_4=-1} + \underbrace{\tau_{4\#}}_{=\tau_4=3}\right) = 0+(1+2)-((-1)+3) = 1$$

and

$$\eta_4 = \underbrace{\mu_{4\#}}_{=\mu_4=0} + \left(\underbrace{\mu_{3\#}}_{=\mu_3=0} + \underbrace{\tau_{3\#}}_{=\tau_3=2}\right) - \left(\underbrace{\nu_{5\#}}_{=\nu_1=4} + \underbrace{\tau_{5\#}}_{=\tau_1=1}\right) = 0+(0+2)-(4+1) = -3,$$

so $\eta = (1,1,1,-3)$. Hence, $\varphi(\omega) = (1+b, 1+b, 1+b, -3+b) = (5,5,5,1)$ (since $b = 4$). This is a partition. Theorem 2.3 **(b)** now yields $c^{\omega}_{\alpha,\mu} = c^{\varphi(\omega)}_{\beta,\mu}$, that is, $c^{(5,3,2)}_{(5,1,1),(2,1)} = c^{(5,5,5,1)}_{(5,4,4),(2,1)}$. And indeed, this equality holds (both of its sides being equal to 1).

Question 2.5. Can the bijection φ in Theorem 2.3 be defined in a more "intuitive" way, similar to (e.g.) jeu-de-taquin or the RSK correspondence? (Of course, there is no tableau being transformed here, just a partition, but this should make this construction easier.)

3. A birational involution

The leading role in our proof of Theorem 2.3 will be played by a certain piecewise-linear involution (which is similar to the bijection φ in Theorem 2.3, but without the shifting by $-a$ and b). For the sake of convenience, we prefer to study this involution in a more general setting, in which the operations min, $+$ and $-$ are replaced by the structure operations $+$, $\cdot$ and $/$ of a semifield. This kind of generalization is called *detropicalization* (or *birational lifting*, or *tropicalization* in the older combinatorial literature); see, e.g., [Kirill01], [NouYam02], [EinPro13, Sections 5 and 9] or [Roby15, §4.2] for examples of this procedure (although our use of it will be conceptually simpler).

3.1. Semifields

We recall some basic definitions from basic abstract algebra (mostly to avoid confusion arising from slight terminological differences):

- A *semigroup* means a pair $(S, *)$, where S is a set and where $*$ is an associative binary operation on S. We do not require this operation $*$ to have a neutral element. We usually write the operation $*$ infix (i.e., we write $a * b$ instead of $*(a, b)$ when $a, b \in S$).
- A semigroup $(S, *)$ is said to be *abelian* if the operation $*$ is commutative (i.e., we have $a * b = b * a$ for all $a, b \in S$).
- A *monoid* means a triple $(S, *, e)$, where $(S, *)$ is a semigroup and where e is a neutral element for the operation $*$ (that is, e is an element of S that satisfies $e * a = a * e = a$ for each $a \in S$). Usually, the monoid $(S, *, e)$ is equated with the semigroup $(S, *)$ because the neutral element is uniquely determined by S and $*$.
- If $(S, *, e)$ is a monoid and a is an element of S, then an *inverse* of a (with respect to $*$) means an element b of S satisfying $a * b = b * a = e$. Such an inverse of a is always unique when it exists.

- A *group* means a monoid $(S, *, e)$ such that each element of S has an inverse (with respect to $*$).

We next recall the definition of a semifield (more precisely, the one we will be using, as there are many competing ones):

Definition 3.1. A *semifield* means a set $\mathbb{K}$ endowed with

- two binary operations called "addition" and "multiplication", and denoted by $+$ and $\cdot$, respectively, and both written infix (i.e., we write $a+b$ and $a \cdot b$ instead of $+(a,b)$ and $\cdot(a,b)$), and
- an element called "unity" and denoted by 1

such that $(\mathbb{K}, +)$ is an abelian semigroup and $(\mathbb{K}, \cdot, 1)$ is an abelian group, and such that the following axiom is satisfied:

- *Distributivity:* We have $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(a+b) \cdot c = (a \cdot c) + (b \cdot c)$ for all $a \in \mathbb{K}$, $b \in \mathbb{K}$ and $c \in \mathbb{K}$.

Thus, a semifield is similar to a field, except that it has no additive inverses and no zero element, but, on the other hand, has multiplicative inverses for all its elements (not just the nonzero ones).

Example 3.2. Let $\mathbb{Q}_+$ be the set of all positive rational numbers. Then, $\mathbb{Q}_+$ (endowed with its standard addition and multiplication and the number 1) is a semifield.

Example 3.3. Let $(\mathbb{A}, *, e)$ be any totally ordered abelian group (whose operation is $*$ and whose neutral element is e). Then, $\mathbb{A}$ becomes a semifield if we endow it with the "addition" min (that is, we set $a + b := \min\{a, b\}$ for all $a, b \in \mathbb{A}$), the "multiplication" $*$ (that is, we set $a \cdot b := a * b$ for all $a, b \in \mathbb{A}$), and the "unity" e. This semifield $(\mathbb{A}, \min, *, e)$ is called the *min tropical semifield* of $(\mathbb{A}, *, e)$.

Convention 3.4. All conventions that are typically used for fields will be used for semifields as well, to the extent they apply. Specifically:

- If $\mathbb{K}$ is a semifield, and if $a, b \in \mathbb{K}$, then $a \cdot b$ shall be abbreviated by ab.
- We shall use the standard "PEMDAS" convention that multiplication-like operations have higher precedence than addition-like operations; thus, e.g., the expression "$ab + ac$" must be understood as "$(ab) + (ac)$" (and not, for example, as "$a(b+a)c$").
- If $\mathbb{K}$ is a semifield, then the inverse of any element $b \in \mathbb{K}$ in the abelian group $(\mathbb{K}, \cdot, 1)$ will be denoted by b^{-1}. Note that this inverse is always defined (unlike when $\mathbb{K}$ is a field).

- If $\mathbb{K}$ is a semifield, and if $a, b \in \mathbb{K}$, then the product ab^{-1} will be denoted by a/b and by $\frac{a}{b}$. Note that this is always defined (unlike when $\mathbb{K}$ is a field).
- Finite products $\prod_{i \in I} a_i$ of elements of a semifield are defined in the same way as in commutative rings. The same applies to finite sums $\sum_{i \in I} a_i$ as long as they are nonempty (i.e., as long as $I \neq \varnothing$). The empty sum is not defined in a semifield, since there is no zero element.

3.2. The birational involution

For the rest of Section 3, we agree to the following two conventions:

Convention 3.5. We fix a positive integer n and a semifield $\mathbb{K}$. We also fix an n-tuple $u \in \mathbb{K}^n$.

Convention 3.6. If $a \in \mathbb{K}^n$ is an n-tuple, and if $i \in \mathbb{Z}$, then a_i shall denote the i'-th entry of a, where i' is the unique element of $\{1, 2, \ldots, n\}$ satisfying $i' \equiv i \bmod n$. Thus, each n-tuple $a \in \mathbb{K}^n$ satisfies $a = (a_1, a_2, \ldots, a_n)$ and $a_i = a_{i+n}$ for each $i \in \mathbb{Z}$. Therefore, if $a \in \mathbb{K}^n$ is any n-tuple, then the family $(a_i)_{i \in \mathbb{Z}}$ is n-periodic.

We shall soon use the letter x for an n-tuple in $\mathbb{K}^n$; thus, $x_1, x_2, \ldots, x_n$ will be the entries of this n-tuple. This has nothing to do with the indeterminates $x_1, x_2, x_3, \ldots$ from Section 1 (that unfortunately use the same letters); we actually **forget all conventions from Section 1** (apart from $\mathbb{N} = \{0, 1, 2, \ldots\}$) for the entire Section 3.

The following is obvious:

Lemma 3.7. If $a \in \mathbb{K}^n$ is any n-tuple, then $a_{k+1} a_{k+2} \cdots a_{k+n} = a_1 a_2 \cdots a_n$ for each $k \in \mathbb{Z}$.

Definition 3.8. We define a map $\mathbf{f}_u : \mathbb{K}^n \to \mathbb{K}^n$ as follows:

Let $x \in \mathbb{K}^n$ be an n-tuple. For each $j \in \mathbb{Z}$ and $r \in \mathbb{N}$, define an element $t_{r,j} \in \mathbb{K}$ by

$$t_{r,j} = \sum_{k=0}^{r} \underbrace{x_{j+1} x_{j+2} \cdots x_{j+k}}_{= \prod_{i=1}^{k} x_{j+i}} \cdot \underbrace{u_{j+k+1} u_{j+k+2} \cdots u_{j+r}}_{= \prod_{i=k+1}^{r} u_{j+i}} .$$

Define $y \in \mathbb{K}^n$ by setting

$$y_i = u_i \cdot \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\} .$$

Set $\mathbf{f}_u(x) = y$.

Example 3.9. Set $n = 4$ for this example. Let $x \in \mathbb{K}^n$ be an n-tuple; thus, $x = (x_1, x_2, x_3, x_4)$. Let us see what the definition of $\mathbf{f}_u(x)$ in Definition 3.8 boils down to in this case.

Let us first compute the elements $t_{n-1,j} = t_{3,j}$ from Definition 3.8. The definition of $t_{3,0}$ yields

$$\begin{aligned} t_{3,0} &= \sum_{k=0}^{3} x_{0+1}x_{0+2}\cdots x_{0+k} \cdot u_{0+k+1}u_{0+k+2}\cdots u_{0+3} \\ &= \sum_{k=0}^{3} x_1x_2\cdots x_k \cdot u_{k+1}u_{k+2}\cdots u_3 \\ &= u_1u_2u_3 + x_1u_2u_3 + x_1x_2u_3 + x_1x_2x_3. \end{aligned}$$

Similarly,

$$\begin{aligned} t_{3,1} &= u_2u_3u_4 + x_2u_3u_4 + x_2x_3u_4 + x_2x_3x_4; \\ t_{3,2} &= u_3u_4u_5 + x_3u_4u_5 + x_3x_4u_5 + x_3x_4x_5 \\ &= u_3u_4u_1 + x_3u_4u_1 + x_3x_4u_1 + x_3x_4x_1 \\ &\qquad (\text{since } u_5 = u_1 \text{ and } x_5 = x_1); \\ t_{3,3} &= u_4u_5u_6 + x_4u_5u_6 + x_4x_5u_6 + x_4x_5x_6 \\ &= u_4u_1u_2 + x_4u_1u_2 + x_4x_1u_2 + x_4x_1x_2 \\ &\qquad (\text{since } u_5 = u_1 \text{ and } x_5 = x_1 \text{ and } u_6 = u_2 \text{ and } x_6 = x_2). \end{aligned}$$

We don't need to compute any further $t_{3,j}$'s, since we can easily see that

$$t_{3,j} = t_{3,j'} \text{ for any integers } j \text{ and } j' \text{ satisfying } j \equiv j' \bmod 4. \tag{2}$$

Thus, in particular, $t_{3,4} = t_{3,0}$ and $t_{3,5} = t_{3,1}$.

Now, let us compute the 4-tuple $y \in \mathbb{K}^n = \mathbb{K}^4$ from Definition 3.8. By its definition, we have

$$\begin{aligned} y_1 &= u_1 \cdot \frac{u_{1-1}t_{3,1-1}}{x_{1+1}t_{3,1+1}} = u_1 \cdot \frac{u_0t_{3,0}}{x_2t_{3,2}} = u_1 \cdot \frac{u_4t_{3,0}}{x_2t_{3,2}} \\ &\qquad (\text{since } u_0 = u_4) \\ &= u_1 \cdot \frac{u_4(u_1u_2u_3 + x_1u_2u_3 + x_1x_2u_3 + x_1x_2x_3)}{x_2(u_3u_4u_1 + x_3u_4u_1 + x_3x_4u_1 + x_3x_4x_1)} \end{aligned}$$

(by our formulas for $t_{3,0}$ and $t_{3,2}$). Similar computations lead to

$$\begin{aligned} y_2 &= u_2 \cdot \frac{u_1(u_2u_3u_4 + x_2u_3u_4 + x_2x_3u_4 + x_2x_3x_4)}{x_3(u_4u_1u_2 + x_4u_1u_2 + x_4x_1u_2 + x_4x_1x_2)}; \\ y_3 &= u_3 \cdot \frac{u_2(u_3u_4u_1 + x_3u_4u_1 + x_3x_4u_1 + x_3x_4x_1)}{x_4(u_1u_2u_3 + x_1u_2u_3 + x_1x_2u_3 + x_1x_2x_3)}; \\ y_4 &= u_4 \cdot \frac{u_3(u_4u_1u_2 + x_4u_1u_2 + x_4x_1u_2 + x_4x_1x_2)}{x_1(u_2u_3u_4 + x_2u_3u_4 + x_2x_3u_4 + x_2x_3x_4)}. \end{aligned}$$

Of course, knowing one of these four equalities is enough; the expression for y_{i+1} is obtained from the expression for y_i by shifting all indices (other than the "3"s that were originally "$n-1$"s) forward by 1.

Remark 3.10. Instead of assuming $\mathbb{K}$ to be a semifield, we could have assumed that $\mathbb{K}$ is an infinite field. In that case, the $\mathbf{f}_u$ in Definition 3.8 would be a birational map instead of a map in the usual sense of this word, since the denominators $x_{i+1}t_{n-1,i+1}$ in the definition of y can be zero. Everything we say below about $\mathbf{f}_u$ would nevertheless still hold on the level of birational maps.

The map $\mathbf{f}_u$ we just defined has the following properties:

Theorem 3.11.

(a) The map $\mathbf{f}_u$ is an involution (i.e., we have $\mathbf{f}_u \circ \mathbf{f}_u = \mathrm{id}$).

(b) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}_u(x)$. Then,

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 .$$

(c) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}_u(x)$. Then,

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right)$$

for each $i \in \mathbb{Z}$.

(d) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}_u(x)$. Then,

$$\prod_{i=1}^{n} \frac{u_i + x_i}{x_i} = \prod_{i=1}^{n} \frac{u_i + y_i}{u_i}.$$

Theorem 3.11 will be crucial for us; but before we can prove it, we will need a few lemmas.

Lemma 3.12. Let $x \in \mathbb{K}^n$ be an n-tuple. Let $t_{r,j}$ and y be as in Definition 3.8. Then:

(a) We have $t_{r,j} = t_{r,j'}$ for any $r \in \mathbb{N}$ and any two integers j and j' satisfying $j \equiv j' \bmod n$. In other words, for each $r \in \mathbb{N}$, the family $(t_{r,j})_{j\in\mathbb{Z}}$ is n-periodic.

(b) We have $t_{0,j} = 1$ for each $j \in \mathbb{Z}$.

(c) For each $r \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have

$$x_j t_{r,j} + u_j u_{j+1} \cdots u_{j+r} = t_{r+1,j-1}.$$

(d) For each $r \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have

$$u_{j+r+1} t_{r,j} + x_{j+1} x_{j+2} \cdots x_{j+r+1} = t_{r+1,j}.$$

(e) For each $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, we have

$$x_a t_{n-1,a} + u_{b-1} t_{n-1,b-1} = x_b t_{n-1,b} + u_{a-1} t_{n-1,a-1}.$$

(f) For each $i \in \mathbb{Z}$, we have

$$x_{i+1} t_{n-1,i+1} + u_{i-1} t_{n-1,i-1} = (x_i + u_i)\, t_{n-1,i}.$$

(g) For each $j \in \mathbb{Z}$ and each positive integer q, we have

$$t_{n-1,j+q+1} \cdot x_{j+2} x_{j+3} \cdots x_{j+q+1} + u_j t_{n-1,j} t_{q-1,j+1} = t_{n-1,j+1} t_{q,j}.$$

(h) For each $i \in \mathbb{Z}$, we have

$$y_i = u_i \cdot \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}}.$$

Now, for each $j \in \mathbb{Z}$ and $r \in \mathbb{N}$, let us define an element $t'_{r,j} \in \mathbb{K}$ by

$$t'_{r,j} = \sum_{k=0}^{r} \underbrace{y_{j+1} y_{j+2} \cdots y_{j+k}}_{= \prod_{i=1}^{k} y_{j+i}} \cdot \underbrace{u_{j+k+1} u_{j+k+2} \cdots u_{j+r}}_{= \prod_{i=k+1}^{r} u_{j+i}}.$$

(This is precisely how $t_{r,j}$ was defined, except that we are using y in place of x now.) Then:

(i) For each $j \in \mathbb{Z}$ and $q \in \mathbb{N}$, we have

$$\frac{t'_{q,j}}{u_{j+1} u_{j+2} \cdots u_{j+q}} = \frac{t_{n-1,j+1}}{t_{n-1,j+q+1}} \cdot \frac{t_{q,j}}{x_{j+2} x_{j+3} \cdots x_{j+q+1}}.$$

(j) For each $j \in \mathbb{Z}$, we have

$$\frac{t'_{n-1,j} u_j}{u_1 u_2 \cdots u_n} = \frac{t_{n-1,j+1} x_{j+1}}{x_1 x_2 \cdots x_n}.$$

(k) For each $i \in \mathbb{Z}$, we have

$$x_i = u_i \cdot \frac{u_{i-1} t'_{n-1,i-1}}{y_{i+1} t'_{n-1,i+1}}.$$

Proof of Lemma 3.12. The proof of this lemma is long but unsophisticated: Each part follows by rather straightforward computations (and, in the cases of parts **(g)** and **(i)**, an induction on q) from the previously proven parts. We shall show the details, but a computationally inclined reader may have a better time reconstructing them independently.[1]

(a) Let $r \in \mathbb{N}$. The definition of $t_{r,j}$ shows that $t_{r,j} = t_{r,j+n}$ for each $j \in \mathbb{Z}$ (since each $i \in \mathbb{Z}$ satisfies $u_i = u_{i+n}$ and $x_i = x_{i+n}$). Thus, the family $\left(t_{r,j}\right)_{j\in\mathbb{Z}}$ is n-periodic. This yields the claim of Lemma 3.12 **(a)**.

(b) Trivial consequence of the definition of $t_{0,j}$.

(c) Let $r \in \mathbb{N}$ and $j \in \mathbb{Z}$. Then, the definition of $t_{r+1,j-1}$ yields

$$\begin{aligned}
&t_{r+1,j-1} \\
&= \sum_{k=0}^{r+1} x_j x_{j+1} \cdots x_{j+k-1} \cdot u_{j+k} u_{j+k+1} \cdots u_{j+r} \\
&= \underbrace{x_j x_{j+1} \cdots x_{j-1}}_{=(\text{empty product})=1} \cdot u_j u_{j+1} \cdots u_{j+r} + \sum_{k=1}^{r+1} \underbrace{x_j x_{j+1} \cdots x_{j+k-1}}_{=x_j \cdot x_{j+1} x_{j+2} \cdots x_{j+k-1}} \cdot u_{j+k} u_{j+k+1} \cdots u_{j+r} \\
&\qquad \text{(here, we have split off the addend for } k=0 \text{ from the sum)} \\
&= u_j u_{j+1} \cdots u_{j+r} + \underbrace{\sum_{k=1}^{r+1} x_j \cdot x_{j+1} x_{j+2} \cdots x_{j+k-1} \cdot u_{j+k} u_{j+k+1} \cdots u_{j+r}}_{= x_j \sum_{k=1}^{r+1} x_{j+1} x_{j+2} \cdots x_{j+k-1} \cdot u_{j+k} u_{j+k+1} \cdots u_{j+r}} \\
&= u_j u_{j+1} \cdots u_{j+r} + x_j \underbrace{\sum_{k=1}^{r+1} x_{j+1} x_{j+2} \cdots x_{j+k-1} \cdot u_{j+k} u_{j+k+1} \cdots u_{j+r}}_{\substack{= \sum_{k=0}^{r} x_{j+1} x_{j+2} \cdots x_{j+k} \cdot u_{j+k+1} u_{j+k+2} \cdots u_{j+r} \\ \text{(here, we have substituted } k \text{ for } k-1 \text{ in the sum)}}} \\
&= u_j u_{j+1} \cdots u_{j+r} + x_j \underbrace{\sum_{k=0}^{r} x_{j+1} x_{j+2} \cdots x_{j+k} \cdot u_{j+k+1} u_{j+k+2} \cdots u_{j+r}}_{\substack{= t_{r,j} \\ \text{(by the definition of } t_{r,j})}} \\
&= u_j u_{j+1} \cdots u_{j+r} + x_j t_{r,j} = x_j t_{r,j} + u_j u_{j+1} \cdots u_{j+r}.
\end{aligned}$$

This proves Lemma 3.12 **(c)**.

[1] We note that the hardest parts of the proof – namely, the proofs of parts **(g)**, **(i)**, **(j)** and **(k)** – can be sidestepped entirely, as these parts will only be used in the proof of Theorem 3.11 **(a)**, but we will give an alternative proof of Theorem 3.11 **(a)** later on (in Remark 3.16), which avoids using them.

(d) Let $r \in \mathbb{N}$ and $j \in \mathbb{Z}$. Then, the definition of $t_{r+1,j}$ yields

$$\begin{aligned}
t_{r+1,j} &= \sum_{k=0}^{r+1} x_{j+1}x_{j+2}\cdots x_{j+k} \cdot u_{j+k+1}u_{j+k+2}\cdots u_{j+r+1} \\
&= \sum_{k=0}^{r} x_{j+1}x_{j+2}\cdots x_{j+k} \cdot \underbrace{u_{j+k+1}u_{j+k+2}\cdots u_{j+r+1}}_{=u_{j+k+1}u_{j+k+2}\cdots u_{j+r}\cdot u_{j+r+1}} \\
&\qquad + x_{j+1}x_{j+2}\cdots x_{j+r+1} \cdot \underbrace{u_{j+(r+1)+1}u_{j+(r+1)+2}\cdots u_{j+r+1}}_{=(\text{empty product})=1} \\
&\qquad (\text{here, we have split off the addend for } k = r+1 \text{ from the sum}) \\
&= \underbrace{\sum_{k=0}^{r} x_{j+1}x_{j+2}\cdots x_{j+k} \cdot u_{j+k+1}u_{j+k+2}\cdots u_{j+r} \cdot u_{j+r+1}}_{=u_{j+r+1}\sum_{k=0}^{r} x_{j+1}x_{j+2}\cdots x_{j+k}\cdot u_{j+k+1}u_{j+k+2}\cdots u_{j+r}} + x_{j+1}x_{j+2}\cdots x_{j+r+1} \\
&= u_{j+r+1} \underbrace{\sum_{k=0}^{r} x_{j+1}x_{j+2}\cdots x_{j+k} \cdot u_{j+k+1}u_{j+k+2}\cdots u_{j+r}}_{\substack{=t_{r,j} \\ (\text{by the definition of } t_{r,j})}} + x_{j+1}x_{j+2}\cdots x_{j+r+1} \\
&= u_{j+r+1}t_{r,j} + x_{j+1}x_{j+2}\cdots x_{j+r+1}.
\end{aligned}$$

This proves Lemma 3.12 **(d)**.

(e) We WLOG assume that $n \neq 1$, since otherwise the claim is easy to check by hand. Thus, $n \geq 2$, so that $n-2 \in \mathbb{N}$.

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. Then, Lemma 3.12 **(c)** (applied to $r = n-2$ and $j = a$) yields

$$x_a t_{n-2,a} + u_a u_{a+1}\cdots u_{a+n-2} = t_{(n-2)+1,a-1} = t_{n-1,a-1}$$

(since $(n-2)+1 = n-1$). Multiplying both sides of this equality by u_{a-1}, we obtain

$$u_{a-1}\left(x_a t_{n-2,a} + u_a u_{a+1}\cdots u_{a+n-2}\right) = u_{a-1}t_{n-1,a-1}.$$

Hence,

$$\begin{aligned}
u_{a-1}t_{n-1,a-1} &= u_{a-1}\left(x_a t_{n-2,a} + u_a u_{a+1}\cdots u_{a+n-2}\right) \\
&= \underbrace{u_{a-1}x_a}_{=x_a u_{a-1}} t_{n-2,a} + \underbrace{u_{a-1}\cdot u_a u_{a+1}\cdots u_{a+n-2}}_{\substack{=u_{a-1}u_a\cdots u_{a+n-2} \\ =u_{(a-2)+1}u_{(a-2)+2}\cdots u_{(a-2)+n} \\ =u_1u_2\cdots u_n \\ (\text{by Lemma 3.7})}} \\
&= x_a u_{a-1} t_{n-2,a} + u_1 u_2\cdots u_n.
\end{aligned} \tag{3}$$

Also, Lemma 3.12 **(d)** (applied to $r = n-2$ and $j = b$) yields

$$u_{b+(n-2)+1}t_{n-2,b} + x_{b+1}x_{b+2}\cdots x_{b+(n-2)+1} = t_{(n-2)+1,b}.$$

In view of $(n-2)+1 = n-1$, this rewrites as

$$u_{b+n-1}t_{n-2,b} + x_{b+1}x_{b+2}\cdots x_{b+n-1} = t_{n-1,b}.$$

Multiplying both sides of this equality by x_b, we obtain

$$x_b\left(u_{b+n-1}t_{n-2,b} + x_{b+1}x_{b+2}\cdots x_{b+n-1}\right) = x_b t_{n-1,b},$$

so that

$$\begin{aligned} x_b t_{n-1,b} &= x_b\left(u_{b+n-1}t_{n-2,b} + x_{b+1}x_{b+2}\cdots x_{b+n-1}\right) \\ &= x_b \underbrace{u_{b+n-1}}_{=u_{(b-1)+n}=u_{b-1}} t_{n-2,b} + \underbrace{x_b \cdot x_{b+1}x_{b+2}\cdots x_{b+n-1}}_{\substack{=x_b x_{b+1}\cdots x_{b+n-1} \\ =x_{(b-1)+1}x_{(b-1)+2}\cdots x_{(b-1)+n} \\ =x_1x_2\cdots x_n \\ \text{(by Lemma 3.7)}}} \\ &= x_b u_{b-1} t_{n-2,b} + x_1x_2\cdots x_n. \end{aligned}$$

Adding (3) to this equality, we obtain

$$\begin{aligned} & x_b t_{n-1,b} + u_{a-1}t_{n-1,a-1} \\ & = x_b u_{b-1}t_{n-2,b} + x_1x_2\cdots x_n + x_a u_{a-1}t_{n-2,a} + u_1u_2\cdots u_n \\ & = x_a u_{a-1}t_{n-2,a} + x_1x_2\cdots x_n + x_b u_{b-1}t_{n-2,b} + u_1u_2\cdots u_n. \end{aligned} \tag{4}$$

The same argument (applied to b and a instead of a and b) yields

$$\begin{aligned} & x_a t_{n-1,a} + u_{b-1}t_{n-1,b-1} \\ & = x_b u_{b-1}t_{n-2,b} + x_1x_2\cdots x_n + x_a u_{a-1}t_{n-2,a} + u_1u_2\cdots u_n. \end{aligned}$$

Comparing this with (4), we obtain $x_a t_{n-1,a} + u_{b-1}t_{n-1,b-1} = x_b t_{n-1,b} + u_{a-1}t_{n-1,a-1}$. This proves Lemma 3.12 **(e)**.

(f) Applying Lemma 3.12 **(e)** to $a = i+1$ and $b = i$, we obtain

$$\begin{aligned} x_{i+1}t_{n-1,i+1} + u_{i-1}t_{n-1,i-1} &= x_i t_{n-1,i} + \underbrace{u_{i+1-1}}_{=u_i}\underbrace{t_{n-1,i+1-1}}_{=t_{n-1,i}} \\ &= x_i t_{n-1,i} + u_i t_{n-1,i} = \left(x_i + u_i\right)t_{n-1,i}. \end{aligned}$$

This proves Lemma 3.12 **(f)**.

(g) We shall prove Lemma 3.12 **(g)** by induction on q:

Induction base: Let us show that Lemma 3.12 **(g)** holds for $q = 1$.

Indeed, let $j \in \mathbb{Z}$. The definition of $t_{1,j}$ yields

$$t_{1,j} = \sum_{k=0}^{1} x_{j+1}x_{j+2}\cdots x_{j+k} \cdot u_{j+k+1}u_{j+k+2}\cdots u_{j+1} = x_{j+1} + u_{j+1}.$$

Hence,

$$t_{n-1,j+1}\underbrace{t_{1,j}}_{=x_{j+1}+u_{j+1}} = t_{n-1,j+1}\left(x_{j+1}+u_{j+1}\right) = \left(x_{j+1}+u_{j+1}\right)t_{n-1,j+1}.$$

Comparing this with

$$\underbrace{t_{n-1,j+1+1}}_{=t_{n-1,j+2}} \cdot \underbrace{x_{j+2}x_{j+3}\cdots x_{j+1+1}}_{\substack{=x_{j+2}x_{j+3}\cdots x_{j+2} \\ =x_{j+2}}} + u_j t_{n-1,j} \qquad \underbrace{t_{1-1,j+1}}_{\substack{=t_{0,j+1}=1 \\ \text{(by Lemma 3.12 \textbf{(b)})}}}$$
$$= t_{n-1,j+2}x_{j+2} + u_j t_{n-1,j} = x_{j+2}t_{n-1,j+2} + u_j t_{n-1,j} = \left(x_{j+1} + u_{j+1}\right) t_{n-1,j+1}$$
$$\text{(by Lemma 3.12 \textbf{(f)}, applied to } i = j+1\text{)},$$

we obtain

$$t_{n-1,j+1+1} \cdot x_{j+2}x_{j+3}\cdots x_{j+1+1} + u_j t_{n-1,j} t_{1-1,j+1} = t_{n-1,j+1} t_{1,j}.$$

Now, forget that we fixed j. We thus have proved that

$$t_{n-1,j+1+1} \cdot x_{j+2}x_{j+3}\cdots x_{j+1+1} + u_j t_{n-1,j} t_{1-1,j+1} = t_{n-1,j+1} t_{1,j}$$

for each $j \in \mathbb{Z}$. In other words, Lemma 3.12 **(g)** holds for $q = 1$. This completes the induction base.[2]

Induction step: Fix a positive integer p. Assume (as induction hypothesis) that Lemma 3.12 **(g)** holds for $q = p$. We must now show that Lemma 3.12 **(g)** holds for $q = p + 1$.

We have assumed that Lemma 3.12 **(g)** holds for $q = p$. In other words, each $j \in \mathbb{Z}$ satisfies

$$t_{n-1,j+p+1} \cdot x_{j+2}x_{j+3}\cdots x_{j+p+1} + u_j t_{n-1,j} t_{p-1,j+1} = t_{n-1,j+1} t_{p,j}. \tag{5}$$

Now, let $j \in \mathbb{Z}$ be arbitrary. Then, Lemma 3.12 **(d)** (applied to $r = p$) yields

$$u_{j+p+1} t_{p,j} + x_{j+1}x_{j+2}\cdots x_{j+p+1} = t_{p+1,j}.$$

Multiplying both sides of this equality by $t_{n-1,j+1}$, we obtain

$$t_{n-1,j+1}\left(u_{j+p+1} t_{p,j} + x_{j+1}x_{j+2}\cdots x_{j+p+1}\right) = t_{n-1,j+1} t_{p+1,j}.$$

[2] We could have simplified this part of the proof by taking $q = 0$ as induction base instead. But this would have required extending the semifield $\mathbb{K}$ to a semiring $\mathbb{K} \sqcup \{0\}$ by adjoining a zero (since $t_{-1,j}$ would be an empty sum). It is not hard to do this, but we prefer computations to technicalities.

Hence,

$$
\begin{aligned}
& t_{n-1,j+1} t_{p+1,j} \\
&= t_{n-1,j+1} \left(u_{j+p+1} t_{p,j} + x_{j+1} x_{j+2} \cdots x_{j+p+1} \right) \\
&= \underbrace{t_{n-1,j+1} u_{j+p+1}}_{= u_{j+p+1} t_{n-1,j+1}} t_{p,j} + t_{n-1,j+1} \cdot x_{j+1} x_{j+2} \cdots x_{j+p+1} \\
&= u_{j+p+1} \underbrace{t_{n-1,j+1} t_{p,j}}_{\substack{= t_{n-1,j+p+1} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} + u_j t_{n-1,j} t_{p-1,j+1} \\ \text{(by (5))}}} + t_{n-1,j+1} \cdot \underbrace{x_{j+1} x_{j+2} \cdots x_{j+p+1}}_{= x_{j+1} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1}} \\
&= u_{j+p+1} \left(t_{n-1,j+p+1} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} + u_j t_{n-1,j} t_{p-1,j+1} \right) \\
&\qquad + t_{n-1,j+1} \cdot x_{j+1} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} \\
&= u_{j+p+1} t_{n-1,j+p+1} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} + u_{j+p+1} u_j t_{n-1,j} t_{p-1,j+1} \\
&\qquad + t_{n-1,j+1} \cdot x_{j+1} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} \\
&= \underbrace{\left(u_{j+p+1} t_{n-1,j+p+1} + t_{n-1,j+1} \cdot x_{j+1} \right)}_{= x_{j+1} t_{n-1,j+1} + u_{j+p+1} t_{n-1,j+p+1}} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} + u_{j+p+1} u_j t_{n-1,j} t_{p-1,j+1} \\
&= \underbrace{\left(x_{j+1} t_{n-1,j+1} + u_{j+p+1} t_{n-1,j+p+1} \right)}_{\substack{= x_{j+p+2} t_{n-1,j+p+2} + u_j t_{n-1,j} \\ \text{(by Lemma 3.12 \textbf{(e)}, applied to } a = j+1 \text{ and } b = j+p+2)}} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} + u_{j+p+1} u_j t_{n-1,j} t_{p-1,j+1} \\
&= \left(x_{j+p+2} t_{n-1,j+p+2} + u_j t_{n-1,j} \right) \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} + u_{j+p+1} u_j t_{n-1,j} t_{p-1,j+1} \\
&= x_{j+p+2} t_{n-1,j+p+2} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} + u_j t_{n-1,j} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} \\
&\qquad + u_{j+p+1} u_j t_{n-1,j} t_{p-1,j+1} \\
&= u_j t_{n-1,j} \underbrace{\left(u_{j+p+1} t_{p-1,j+1} + x_{j+2} x_{j+3} \cdots x_{j+p+1} \right)}_{\substack{= t_{p,j+1} \\ \text{(by Lemma 3.12 \textbf{(d)}, applied to } j+1 \text{ and } p-1 \text{ instead of } j \text{ and } r)}} \\
&\qquad + \underbrace{x_{j+p+2} t_{n-1,j+p+2} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1}}_{= t_{n-1,j+p+2} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+1} \cdot x_{j+p+2}} \\
&= u_j t_{n-1,j} t_{p,j+1} + t_{n-1,j+p+2} \cdot \underbrace{x_{j+2} x_{j+3} \cdots x_{j+p+1} \cdot x_{j+p+2}}_{= x_{j+2} x_{j+3} \cdots x_{j+p+2}} \\
&= u_j t_{n-1,j} t_{p,j+1} + t_{n-1,j+p+2} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+2} \\
&= t_{n-1,j+p+2} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+2} + u_j t_{n-1,j} t_{p,j+1}.
\end{aligned}
$$

In other words,

$$t_{n-1,j+p+2} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+2} + u_j t_{n-1,j} t_{p,j+1} = t_{n-1,j+1} t_{p+1,j}.$$

Now, forget that we fixed j. We thus have proved that each $j \in \mathbb{Z}$ satisfies

$$t_{n-1,j+p+2} \cdot x_{j+2} x_{j+3} \cdots x_{j+p+2} + u_j t_{n-1,j} t_{p,j+1} = t_{n-1,j+1} t_{p+1,j}.$$

In other words, Lemma 3.12 **(g)** holds for $q = p + 1$. This completes the induction step. Hence, Lemma 3.12 **(g)** is proved by induction.

(h) For $i \in \{1, 2, \ldots, n\}$, the claim of Lemma 3.12 **(h)** follows from the definition of y. Thus, it also holds for all $i \in \mathbb{Z}$, since any integers j and j' satisfying $j \equiv j' \operatorname{mod} n$ satisfy $x_j = x_{j'}$ and $u_j = u_{j'}$ and $t_{n-1,j} = t_{n-1,j'}$ (by Lemma 3.12 **(a)**) and $y_j = y_{j'}$. Thus, Lemma 3.12 **(h)** is proved.

(i) We shall prove Lemma 3.12 **(i)** by induction on q:

Induction base: For each $j \in \mathbb{Z}$, we have

$$\frac{t'_{0,j}}{u_{j+1}u_{j+2}\cdots u_j} = \frac{t_{n-1,j+1}}{t_{n-1,j+1}} \cdot \frac{t_{0,j}}{x_{j+2}x_{j+3}\cdots x_{j+1}}$$

[3]. In other words, Lemma 3.12 **(i)** holds for $q = 0$. This completes the induction base.

Induction step: Fix $r \in \mathbb{N}$. Assume (as induction hypothesis) that Lemma 3.12 **(i)** holds for $q = r$. We must now show that Lemma 3.12 **(i)** holds for $q = r + 1$.

We have assumed that Lemma 3.12 **(i)** holds for $q = r$. In other words, each $j \in \mathbb{Z}$ satisfies

$$\frac{t'_{r,j}}{u_{j+1}u_{j+2}\cdots u_{j+r}} = \frac{t_{n-1,j+1}}{t_{n-1,j+r+1}} \cdot \frac{t_{r,j}}{x_{j+2}x_{j+3}\cdots x_{j+r+1}}. \tag{6}$$

Now, let $j \in \mathbb{Z}$ be arbitrary. Then, (6) (applied to $j+1$ instead of j) yields

$$\frac{t'_{r,j+1}}{u_{j+2}u_{j+3}\cdots u_{j+r+1}} = \frac{t_{n-1,j+2}}{t_{n-1,j+r+2}} \cdot \frac{t_{r,j+1}}{x_{j+3}x_{j+4}\cdots x_{j+r+2}}. \tag{7}$$

But Lemma 3.12 **(c)** (applied to $j+1$ instead of j) yields

$$x_{j+1}t_{r,j+1} + u_{j+1}u_{j+2}\cdots u_{j+r+1} = t_{r+1,j}.$$

Hence,

$$t_{r+1,j} = x_{j+1}t_{r,j+1} + u_{j+1}u_{j+2}\cdots u_{j+r+1} = u_{j+1}u_{j+2}\cdots u_{j+r+1} + x_{j+1}t_{r,j+1}.$$

[3]*Proof.* Let $j \in \mathbb{Z}$. Lemma 3.12 **(b)** yields $t_{0,j} = 1$. Similarly, $t'_{0,j} = 1$. From this equality, and from $u_{j+1}u_{j+2}\cdots u_j = (\text{empty product}) = 1$, we obtain

$$\frac{t'_{0,j}}{u_{j+1}u_{j+2}\cdots u_j} = \frac{1}{1}.$$

Comparing this with

$$\frac{t_{n-1,j+1}}{t_{n-1,j+1}} \cdot \frac{t_{0,j}}{x_{j+2}x_{j+3}\cdots x_{j+1}} = \frac{t_{0,j}}{x_{j+2}x_{j+3}\cdots x_{j+1}} = \frac{1}{1}$$
$$\left(\text{since } t_{0,j} = 1 \text{ and } x_{j+2}x_{j+3}\cdots x_{j+1} = (\text{empty product}) = 1\right),$$

we obtain $\dfrac{t'_{0,j}}{u_{j+1}u_{j+2}\cdots u_j} = \dfrac{t_{n-1,j+1}}{t_{n-1,j+1}} \cdot \dfrac{t_{0,j}}{x_{j+2}x_{j+3}\cdots x_{j+1}}$. Qed.

The same reasoning (applied to y and $t'_{r,j}$ instead of x and $t_{r,j}$) yields

$$t'_{r+1,j} = u_{j+1}u_{j+2}\cdots u_{j+r+1} + y_{j+1}t'_{r,j+1}.$$

Hence,

$$\begin{aligned}
\frac{t'_{r+1,j}}{u_{j+1}u_{j+2}\cdots u_{j+r+1}} &= \frac{u_{j+1}u_{j+2}\cdots u_{j+r+1} + y_{j+1}t'_{r,j+1}}{u_{j+1}u_{j+2}\cdots u_{j+r+1}} \\
&= 1 + \underbrace{\frac{y_{j+1}t'_{r,j+1}}{u_{j+1}u_{j+2}\cdots u_{j+r+1}}}_{\substack{=\dfrac{y_{j+1}t'_{r,j+1}}{u_{j+1}\cdot u_{j+2}u_{j+3}\cdots u_{j+r+1}} \\ =\dfrac{y_{j+1}}{u_{j+1}}\cdot\dfrac{t'_{r,j+1}}{u_{j+2}u_{j+3}\cdots u_{j+r+1}}}} \\
&= 1 + \frac{y_{j+1}}{u_{j+1}} \cdot \underbrace{\frac{t'_{r,j+1}}{u_{j+2}u_{j+3}\cdots u_{j+r+1}}}_{\substack{=\dfrac{t_{n-1,j+2}}{t_{n-1,j+r+2}}\cdot\dfrac{t_{r,j+1}}{x_{j+3}x_{j+4}\cdots x_{j+r+2}} \\ \text{(by (7))}}} \\
&= 1 + \frac{y_{j+1}}{u_{j+1}} \cdot \frac{t_{n-1,j+2}}{t_{n-1,j+r+2}} \cdot \frac{t_{r,j+1}}{x_{j+3}x_{j+4}\cdots x_{j+r+2}}.
\end{aligned} \tag{8}$$

But Lemma 3.12 **(h)** (applied to $i = j+1$) yields

$$y_{j+1} = u_{j+1} \cdot \frac{u_j t_{n-1,j}}{x_{j+2}t_{n-1,j+2}}.$$

Dividing this equality by u_{j+1}, we find

$$\frac{y_{j+1}}{u_{j+1}} = \frac{u_j t_{n-1,j}}{x_{j+2}t_{n-1,j+2}}.$$

Hence, (8) becomes

$$\frac{t'_{r+1,j}}{u_{j+1}u_{j+2}\cdots u_{j+r+1}} = 1 + \underbrace{\frac{y_{j+1}}{u_{j+1}}}_{=\frac{u_j t_{n-1,j}}{x_{j+2}t_{n-1,j+2}}} \cdot \frac{t_{n-1,j+2}}{t_{n-1,j+r+2}} \cdot \frac{t_{r,j+1}}{x_{j+3}x_{j+4}\cdots x_{j+r+2}}$$
$$= 1 + \frac{u_j t_{n-1,j}}{x_{j+2}t_{n-1,j+2}} \cdot \frac{t_{n-1,j+2}}{t_{n-1,j+r+2}} \cdot \frac{t_{r,j+1}}{x_{j+3}x_{j+4}\cdots x_{j+r+2}}$$
$$= 1 + \frac{u_j t_{n-1,j}}{t_{n-1,j+r+2}} \cdot \frac{t_{r,j+1}}{x_{j+2}\cdot x_{j+3}x_{j+4}\cdots x_{j+r+2}}$$
$$= 1 + \frac{u_j t_{n-1,j}}{t_{n-1,j+r+2}} \cdot \frac{t_{r,j+1}}{x_{j+2}x_{j+3}\cdots x_{j+r+2}}$$
$$\left(\text{since } x_{j+2}\cdot x_{j+3}x_{j+4}\cdots x_{j+r+2} = x_{j+2}x_{j+3}\cdots x_{j+r+2}\right)$$
$$= \frac{t_{n-1,j+r+2}\cdot x_{j+2}x_{j+3}\cdots x_{j+r+2} + u_j t_{n-1,j}t_{r,j+1}}{t_{n-1,j+r+2}\cdot x_{j+2}x_{j+3}\cdots x_{j+r+2}}. \tag{9}$$

But Lemma 3.12 **(g)** (applied to $q = r+1$) yields

$$t_{n-1,j+r+2}\cdot x_{j+2}x_{j+3}\cdots x_{j+r+2} + u_j t_{n-1,j}t_{r,j+1} = t_{n-1,j+1}t_{r+1,j}. \tag{10}$$

Hence, (9) becomes

$$\frac{t'_{r+1,j}}{u_{j+1}u_{j+2}\cdots u_{j+r+1}} = \frac{t_{n-1,j+r+2}\cdot x_{j+2}x_{j+3}\cdots x_{j+r+2} + u_j t_{n-1,j}t_{r,j+1}}{t_{n-1,j+r+2}\cdot x_{j+2}x_{j+3}\cdots x_{j+r+2}}$$
$$= \frac{t_{n-1,j+1}t_{r+1,j}}{t_{n-1,j+r+2}\cdot x_{j+2}x_{j+3}\cdots x_{j+r+2}} \qquad (\text{by } (10))$$
$$= \frac{t_{n-1,j+1}}{t_{n-1,j+r+2}} \cdot \frac{t_{r+1,j}}{x_{j+2}x_{j+3}\cdots x_{j+r+2}}.$$

Forget that we fixed j. We thus have shown that each $j \in \mathbb{Z}$ satisfies

$$\frac{t'_{r+1,j}}{u_{j+1}u_{j+2}\cdots u_{j+r+1}} = \frac{t_{n-1,j+1}}{t_{n-1,j+r+2}} \cdot \frac{t_{r+1,j}}{x_{j+2}x_{j+3}\cdots x_{j+r+2}}.$$

In other words, Lemma 3.12 **(i)** holds for $q = r+1$. This completes the induction step. Hence, Lemma 3.12 **(i)** is proved by induction.

(j) Let $j \in \mathbb{Z}$. Then, $u_{j+n} = u_j$ (by Convention 3.6). Lemma 3.12 **(i)** (applied to $q = n-1$) yields

$$\frac{t'_{n-1,j}}{u_{j+1}u_{j+2}\cdots u_{j+n-1}} = \frac{t_{n-1,j+1}}{t_{n-1,j+(n-1)+1}} \cdot \frac{t_{n-1,j}}{x_{j+2}x_{j+3}\cdots x_{j+(n-1)+1}}$$
$$= \frac{t_{n-1,j+1}}{t_{n-1,j+n}} \cdot \frac{t_{n-1,j}}{x_{j+2}x_{j+3}\cdots x_{j+n}} \tag{11}$$

(since $(n-1)+1=n$). But $j \equiv j+n \mod n$; hence, Lemma 3.12 **(a)** (applied to $r=n-1$ and $j'=j+n$) yields $t_{n-1,j} = t_{n-1,j+n}$. Hence, (11) becomes

$$\begin{aligned}
\frac{t'_{n-1,j}}{u_{j+1}u_{j+2}\cdots u_{j+n-1}} &= \frac{t_{n-1,j+1}}{t_{n-1,j+n}} \cdot \frac{t_{n-1,j}}{x_{j+2}x_{j+3}\cdots x_{j+n}} = \frac{t_{n-1,j+1}}{t_{n-1,j+n}} \cdot \frac{t_{n-1,j+n}}{x_{j+2}x_{j+3}\cdots x_{j+n}} \\
&\qquad \left(\text{since } t_{n-1,j} = t_{n-1,j+n}\right) \\
&= \frac{t_{n-1,j+1}}{x_{j+2}x_{j+3}\cdots x_{j+n}} = \frac{t_{n-1,j+1}x_{j+1}}{x_{j+2}x_{j+3}\cdots x_{j+n}\cdot x_{j+1}} = \frac{t_{n-1,j+1}x_{j+1}}{x_1x_2\cdots x_n}
\end{aligned}$$

(since $x_{j+2}x_{j+3}\cdots x_{j+n}\cdot x_{j+1} = x_{j+1}\cdot x_{j+2}x_{j+3}\cdots x_{j+n} = x_{j+1}x_{j+2}\cdots x_{j+n} = x_1x_2\cdots x_n$ (by Lemma 3.7)). Hence,

$$\begin{aligned}
\frac{t_{n-1,j+1}x_{j+1}}{x_1x_2\cdots x_n} &= \frac{t'_{n-1,j}}{u_{j+1}u_{j+2}\cdots u_{j+n-1}} = \frac{t'_{n-1,j}u_{j+n}}{u_{j+1}u_{j+2}\cdots u_{j+n-1}\cdot u_{j+n}} = \frac{t'_{n-1,j}u_{j+n}}{u_{j+1}u_{j+2}\cdots u_{j+n}} \\
&\qquad \left(\text{since } u_{j+1}u_{j+2}\cdots u_{j+n-1}\cdot u_{j+n} = u_{j+1}u_{j+2}\cdots u_{j+n}\right) \\
&= \frac{t'_{n-1,j}u_j}{u_{j+1}u_{j+2}\cdots u_{j+n}} \qquad \left(\text{since } u_{j+n} = u_j\right) \\
&= \frac{t'_{n-1,j}u_j}{u_1u_2\cdots u_n}
\end{aligned}$$

(since $u_{j+1}u_{j+2}\cdots u_{j+n} = u_1u_2\cdots u_n$ (by Lemma 3.7)). This proves Lemma 3.12 **(j)**.

(k) Let $i \in \mathbb{Z}$. Applying Lemma 3.12 **(j)** to $j = i-1$, we obtain

$$\frac{t'_{n-1,i-1}u_{i-1}}{u_1u_2\cdots u_n} = \frac{t_{n-1,i}x_i}{x_1x_2\cdots x_n}.$$

Applying Lemma 3.12 **(j)** to $j = i+1$, we obtain

$$\frac{t'_{n-1,i+1}u_{i+1}}{u_1u_2\cdots u_n} = \frac{t_{n-1,i+2}x_{i+2}}{x_1x_2\cdots x_n}.$$

Dividing the former equality by the latter, we obtain

$$\frac{t'_{n-1,i-1}u_{i-1}}{u_1u_2\cdots u_n} \Big/ \frac{t'_{n-1,i+1}u_{i+1}}{u_1u_2\cdots u_n} = \frac{t_{n-1,i}x_i}{x_1x_2\cdots x_n} \Big/ \frac{t_{n-1,i+2}x_{i+2}}{x_1x_2\cdots x_n}.$$

This rewrites as

$$\frac{t'_{n-1,i-1}u_{i-1}}{t'_{n-1,i+1}u_{i+1}} = \frac{t_{n-1,i}x_i}{t_{n-1,i+2}x_{i+2}}.$$

Multiplying both sides of this equality by u_{i+1}, we obtain

$$\frac{t'_{n-1,i-1}u_{i-1}}{t'_{n-1,i+1}} = u_{i+1}\cdot\frac{t_{n-1,i}x_i}{t_{n-1,i+2}x_{i+2}}. \tag{12}$$

Now,

$$u_i \cdot \frac{u_{i-1} t'_{n-1,i-1}}{y_{i+1} t'_{n-1,i+1}} = \underbrace{\frac{t'_{n-1,i-1} u_{i-1}}{t'_{n-1,i+1}}}_{\substack{= u_{i+1} \cdot \frac{t_{n-1,i} x_i}{t_{n-1,i+2} x_{i+2}} \\ \text{(by (12))}}} \cdot u_i / \underbrace{y_{i+1}}_{\substack{= u_{i+1} \cdot \frac{u_i t_{n-1,i}}{x_{i+2} t_{n-1,i+2}} \\ \text{(by Lemma 3.12 } \textbf{(h)}\text{,} \\ \text{applied to } i+1 \text{ instead of } i\text{)}}}$$

$$= u_{i+1} \cdot \frac{t_{n-1,i} x_i}{t_{n-1,i+2} x_{i+2}} \cdot u_i / \left(u_{i+1} \cdot \frac{u_i t_{n-1,i}}{x_{i+2} t_{n-1,i+2}} \right) = x_i.$$

This proves Lemma 3.12 **(k)**. □

Lemma 3.13. Let $x \in \mathbb{K}^n$ be an n-tuple. For each $j \in \mathbb{Z}$, let

$$q_j = \sum_{k=0}^{n-1} \underbrace{x_{j+1} x_{j+2} \cdots x_{j+k}}_{= \prod_{i=1}^{k} x_{j+i}} \cdot \underbrace{u_{j+k+1} u_{j+k+2} \cdots u_{j+n-1}}_{= \prod_{i=k+1}^{n-1} u_{j+i}}.$$

Let $z \in \mathbb{K}^n$ be such that

$$z_i = u_i \cdot \frac{u_{i-1} q_{i-1}}{x_{i+1} q_{i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Then, $\mathbf{f}_u(x) = z$.

Proof of Lemma 3.13. Let $t_{r,j}$ and y be as in Definition 3.8. Then, $t_{n-1,j} = q_j$ for each $j \in \mathbb{Z}$ (by comparing the definitions of $t_{n-1,j}$ and q_j). Hence, $z_i = y_i$ for each $i \in \{1, 2, \ldots, n\}$ (by comparing the definitions of z_i and y_i). Hence, $z = y = \mathbf{f}_u(x)$ (since $\mathbf{f}_u(x)$ was defined to be y). This proves Lemma 3.13. □

For future convenience, let us restate Lemma 3.13 with different labels:

Lemma 3.14. Let $y \in \mathbb{K}^n$ be an n-tuple. For each $j \in \mathbb{Z}$, let

$$r_j = \sum_{k=0}^{n-1} \underbrace{y_{j+1} y_{j+2} \cdots y_{j+k}}_{= \prod_{i=1}^{k} y_{j+i}} \cdot \underbrace{u_{j+k+1} u_{j+k+2} \cdots u_{j+n-1}}_{= \prod_{i=k+1}^{n-1} u_{j+i}}.$$

Let $x \in \mathbb{K}^n$ be such that

$$x_i = u_i \cdot \frac{u_{i-1} r_{i-1}}{y_{i+1} r_{i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Then, $\mathbf{f}_u(y) = x$.

Proof of Lemma 3.14. Lemma 3.14 is just Lemma 3.13, with x, q_j and z renamed as y, r_j and x. $\square$

We are now ready for the proof of Theorem 3.11:

Proof of Theorem 3.11. **(a)** Let $x \in \mathbb{K}^n$. We shall prove that $(\mathbf{f}_u \circ \mathbf{f}_u)(x) = x$.

Let $t_{r,j}$ and y be as in Definition 3.8. Then, $\mathbf{f}_u(x) = y$ (by the definition of $\mathbf{f}_u$), so that $y = \mathbf{f}_u(x)$. Let $t'_{r,j}$ (for each $r \in \mathbb{N}$ and $j \in \mathbb{Z}$) be as in Lemma 3.12. The definition of $t'_{n-1,j}$ shows that

$$t'_{n-1,j} = \sum_{k=0}^{n-1} \underbrace{y_{j+1}y_{j+2}\cdots y_{j+k}}_{=\prod_{i=1}^{k} y_{j+i}} \cdot \underbrace{u_{j+k+1}u_{j+k+2}\cdots u_{j+n-1}}_{=\prod_{i=k+1}^{n-1} u_{j+i}}$$

for each $j \in \mathbb{Z}$. Lemma 3.12 **(k)** shows that

$$x_i = u_i \cdot \frac{u_{i-1}t'_{n-1,i-1}}{y_{i+1}t'_{n-1,i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Thus, Lemma 3.14 (applied to $r_j = t'_{n-1,j}$) yields that $\mathbf{f}_u(y) = x$. Hence, $x = \mathbf{f}_u\left(\underbrace{y}_{=\mathbf{f}_u(x)}\right) = \mathbf{f}_u(\mathbf{f}_u(x)) = (\mathbf{f}_u \circ \mathbf{f}_u)(x)$. In other words, $(\mathbf{f}_u \circ \mathbf{f}_u)(x) = x$.

Forget that we fixed x. We thus have proved that $(\mathbf{f}_u \circ \mathbf{f}_u)(x) = x$ for each $x \in \mathbb{K}^n$. In other words, $\mathbf{f}_u \circ \mathbf{f}_u = \mathrm{id}$. In other words, $\mathbf{f}_u$ is an involution. This proves Theorem 3.11 **(a)**.

(b) Let $t_{r,j}$ be as in Definition 3.8. Note that the y from Definition 3.8 is precisely the y in Theorem 3.11 **(b)** (because both y's satisfy $\mathbf{f}_u(x) = y$).

Lemma 3.12 **(a)** yields $t_{n-1,0} = t_{n-1,n}$ (since $0 \equiv n \bmod n$) and $t_{n-1,1} = t_{n-1,n+1}$ (since $1 \equiv n+1 \bmod n$). Multiplying these two equalities, we obtain $t_{n-1,0}t_{n-1,1} = t_{n-1,n}t_{n-1,n+1}$, whence

$$\frac{t_{n-1,0}t_{n-1,1}}{t_{n-1,n}t_{n-1,n+1}} = 1. \tag{13}$$

We have

$$y_1 y_2 \cdots y_n = \prod_{i=1}^{n} \underbrace{y_i}_{\substack{= u_i \cdot \dfrac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \\ \text{(by the definition of } y \\ \text{in Definition 3.8)}}} = \prod_{i=1}^{n} \left(u_i \cdot \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \right)$$

$$= \left(\prod_{i=1}^{n} u_i \right) \cdot \frac{\left(\prod_{i=1}^{n} u_{i-1} \right) \cdot \left(\prod_{i=1}^{n} t_{n-1,i-1} \right)}{\left(\prod_{i=1}^{n} x_{i+1} \right) \cdot \left(\prod_{i=1}^{n} t_{n-1,i+1} \right)}$$

$$= \underbrace{\left(\prod_{i=1}^{n} u_i \right)}_{= u_1 u_2 \cdots u_n} \cdot \underbrace{\left(\prod_{i=1}^{n} u_{i-1} \right)}_{\substack{= u_0 u_1 \cdots u_{n-1} \\ = u_1 u_2 \cdots u_n \\ \text{(by Lemma 3.7)}}} \cdot \underbrace{\left(\prod_{i=1}^{n} t_{n-1,i-1} \right)}_{\substack{= t_{n-1,0} t_{n-1,1} \cdots t_{n-1,n-1} \\ = \dfrac{t_{n-1,0} t_{n-1,1} \cdots t_{n-1,n+1}}{t_{n-1,n} t_{n-1,n+1}}}}$$

$$/ \left(\underbrace{\left(\prod_{i=1}^{n} x_{i+1} \right)}_{\substack{= x_2 x_3 \cdots x_{n+1} \\ = x_1 x_2 \cdots x_n \\ \text{(by Lemma 3.7)}}} \cdot \underbrace{\left(\prod_{i=1}^{n} t_{n-1,i+1} \right)}_{\substack{= t_{n-1,2} t_{n-1,3} \cdots t_{n-1,n+1} \\ = \dfrac{t_{n-1,0} t_{n-1,1} \cdots t_{n-1,n+1}}{t_{n-1,0} t_{n-1,1}}}} \right)$$

$$= (u_1 u_2 \cdots u_n) \cdot (u_1 u_2 \cdots u_n) \cdot \frac{t_{n-1,0} t_{n-1,1} \cdots t_{n-1,n+1}}{t_{n-1,n} t_{n-1,n+1}}$$

$$/ \left((x_1 x_2 \cdots x_n) \cdot \frac{t_{n-1,0} t_{n-1,1} \cdots t_{n-1,n+1}}{t_{n-1,0} t_{n-1,1}} \right)$$

$$= \frac{(u_1 u_2 \cdots u_n)^2}{x_1 x_2 \cdots x_n} \cdot \underbrace{\frac{t_{n-1,0} t_{n-1,1}}{t_{n-1,n} t_{n-1,n+1}}}_{\substack{=1 \\ \text{(by (13))}}} = \frac{(u_1 u_2 \cdots u_n)^2}{x_1 x_2 \cdots x_n},$$

so that

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 .$$

This proves Theorem 3.11 **(b)**.

(c) Let $t_{r,j}$ be as in Definition 3.8. Note that the y from Definition 3.8 is precisely the y in Theorem 3.11 **(c)** (because both y's satisfy $\mathbf{f}_u(x) = y$).

Let $i \in \mathbb{Z}$. Then,

$$\begin{aligned}
u_i + \underbrace{y_i}_{\substack{= u_i \cdot \dfrac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \\ \text{(by Lemma 3.12 \textbf{(h)})}}}
&= u_i + u_i \cdot \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} = u_i \left(1 + \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}}\right) = u_i \cdot \frac{x_{i+1} t_{n-1,i+1} + u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \\
&= \frac{u_i}{x_{i+1} t_{n-1,i+1}} \underbrace{\left(x_{i+1} t_{n-1,i+1} + u_{i-1} t_{n-1,i-1}\right)}_{\substack{= (x_i + u_i) t_{n-1,i} \\ \text{(by Lemma 3.12 \textbf{(f)})}}} \\
&= \frac{u_i}{x_{i+1} t_{n-1,i+1}} \left(x_i + u_i\right) t_{n-1,i}. \qquad (14)
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{1}{u_i} + \frac{1}{y_i} &= \underbrace{(u_i + y_i)}_{\substack{= \dfrac{u_i}{x_{i+1} t_{n-1,i+1}} (x_i + u_i) t_{n-1,i} \\ \text{(by (14))}}} / \left(u_i \cdot y_i\right) = \frac{u_i}{x_{i+1} t_{n-1,i+1}} \left(x_i + u_i\right) t_{n-1,i} / \left(u_i \cdot y_i\right) \\
&= \frac{1}{x_{i+1} t_{n-1,i+1}} \left(x_i + u_i\right) t_{n-1,i} / \underbrace{y_i}_{\substack{= u_i \cdot \dfrac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \\ \text{(by Lemma 3.12 \textbf{(h)})}}} \\
&= \frac{1}{x_{i+1} t_{n-1,i+1}} \left(x_i + u_i\right) t_{n-1,i} / \left(u_i \cdot \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}}\right) = \frac{\left(x_i + u_i\right) t_{n-1,i}}{u_i u_{i-1} t_{n-1,i-1}}.
\end{aligned}$$

The same argument (applied to $i+1$ instead of i) yields

$$\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}} = \frac{\left(x_{i+1} + u_{i+1}\right) t_{n-1,i+1}}{u_{i+1} u_{(i+1)-1} t_{n-1,(i+1)-1}} = \frac{\left(x_{i+1} + u_{i+1}\right) t_{n-1,i+1}}{u_{i+1} u_i t_{n-1,i}}$$

(since $(i+1) - 1 = i$). Multiplying (14) with this equality, we obtain

$$\begin{aligned}
\left(u_i + y_i\right) \left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) &= \frac{u_i}{x_{i+1} t_{n-1,i+1}} \left(x_i + u_i\right) t_{n-1,i} \cdot \frac{\left(x_{i+1} + u_{i+1}\right) t_{n-1,i+1}}{u_{i+1} u_i t_{n-1,i}} \\
&= \underbrace{\left(x_i + u_i\right)}_{= u_i + x_i} \cdot \underbrace{\frac{x_{i+1} + u_{i+1}}{x_{i+1} u_{i+1}}}_{= \dfrac{1}{u_{i+1}} + \dfrac{1}{x_{i+1}}} = \left(u_i + x_i\right) \left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right).
\end{aligned}$$

This proves Theorem 3.11 **(c)**.

(d) Let $t_{r,j}$ be as in Definition 3.8. Note that the y from Definition 3.8 is precisely the y in Theorem 3.11 **(d)** (because both y's satisfy $\mathbf{f}_u(x) = y$).

We have

$$\begin{aligned} \prod_{i=1}^{n} x_{i+1} &= x_2 x_3 \cdots x_{n+1} = x_1 x_2 \cdots x_n \qquad \text{(by Lemma 3.7)} \\ &= \prod_{i=1}^{n} x_i \end{aligned} \tag{15}$$

and

$$\begin{aligned} \prod_{i=1}^{n} t_{n-1,i+1} &= \prod_{i=2}^{n+1} t_{n-1,i} \qquad \text{(here, we have substituted } i \text{ for } i+1 \text{ in the product)} \\ &= \underbrace{t_{n-1,n+1}}_{\substack{=t_{n-1,1} \\ \text{(by Lemma 3.12 (a),} \\ \text{since } n+1\equiv 1 \bmod n)}} \prod_{i=2}^{n} t_{n-1,i} = t_{n-1,1} \prod_{i=2}^{n} t_{n-1,i} \\ &= \prod_{i=1}^{n} t_{n-1,i}. \end{aligned} \tag{16}$$

Every $i \in \mathbb{Z}$ satisfies (14) (as we have shown in the proof of Theorem 3.11 **(c)** above). Hence,

$$\begin{aligned} \prod_{i=1}^{n} \underbrace{(u_i + y_i)}_{\substack{= \frac{u_i}{x_{i+1} t_{n-1,i+1}} (x_i+u_i) t_{n-1,i} \\ \text{(by (14))}}} &= \prod_{i=1}^{n} \left(\frac{u_i}{x_{i+1} t_{n-1,i+1}} (x_i + u_i) \, t_{n-1,i} \right) \\ &= \frac{\prod_{i=1}^{n} u_i}{\left(\prod_{i=1}^{n} x_{i+1} \right) \left(\prod_{i=1}^{n} t_{n-1,i+1} \right)} \left(\prod_{i=1}^{n} (x_i + u_i) \right) \left(\prod_{i=1}^{n} t_{n-1,i} \right) \\ &= \frac{\prod_{i=1}^{n} u_i}{\left(\prod_{i=1}^{n} x_i \right) \left(\prod_{i=1}^{n} t_{n-1,i} \right)} \left(\prod_{i=1}^{n} (x_i + u_i) \right) \left(\prod_{i=1}^{n} t_{n-1,i} \right) \\ &\qquad \text{(by (15) and (16))} \\ &= \left(\prod_{i=1}^{n} u_i \right) \underbrace{\frac{\prod_{i=1}^{n} (x_i + u_i)}{\prod_{i=1}^{n} x_i}}_{= \prod_{i=1}^{n} \frac{x_i + u_i}{x_i}} = \left(\prod_{i=1}^{n} u_i \right) \prod_{i=1}^{n} \frac{x_i + u_i}{x_i}. \end{aligned}$$

Thus,

$$\frac{\prod_{i=1}^{n}(u_i+y_i)}{\prod_{i=1}^{n}u_i}=\prod_{i=1}^{n}\underbrace{\frac{x_i+u_i}{x_i}}_{=\frac{u_i+x_i}{x_i}}=\prod_{i=1}^{n}\frac{u_i+x_i}{x_i},$$

so that

$$\prod_{i=1}^{n}\frac{u_i+x_i}{x_i}=\frac{\prod_{i=1}^{n}(u_i+y_i)}{\prod_{i=1}^{n}u_i}=\prod_{i=1}^{n}\frac{u_i+y_i}{u_i}.$$

This proves Theorem 3.11 **(d)**. □

Let us observe one more property of the involution $\mathbf{f}_u$ (even though we will have no use for it):

Proposition 3.15. Let $x \in \mathbb{K}^n$ be such that $x_1x_2\cdots x_n = u_1u_2\cdots u_n$. Then, $\mathbf{f}_u(x) = x$.

Proof of Proposition 3.15. Let $t_{r,j}$ and y be as in Definition 3.8. Then, $\mathbf{f}_u(x) = y$ (by the definition of $\mathbf{f}_u$).

Let $i \in \mathbb{Z}$. We shall first show that $u_{i-1}t_{n-1,i-1} = x_i t_{n-1,i}$.

Indeed, the definition of $t_{n-1,i}$ yields

$$t_{n-1,i}=\sum_{k=0}^{n-1}x_{i+1}x_{i+2}\cdots x_{i+k}\cdot u_{i+k+1}u_{i+k+2}\cdots u_{i+n-1}.$$

Multiplying both sides of this equality by x_i, we find

$$\begin{aligned}
x_i t_{n-1,i} &= x_i \sum_{k=0}^{n-1} x_{i+1} x_{i+2} \cdots x_{i+k} \cdot u_{i+k+1} u_{i+k+2} \cdots u_{i+n-1} \\
&= \sum_{k=0}^{n-1} \underbrace{x_i \cdot x_{i+1} x_{i+2} \cdots x_{i+k}}_{= x_i x_{i+1} \cdots x_{i+k}} \cdot u_{i+k+1} u_{i+k+2} \cdots u_{i+n-1} \\
&= \sum_{k=0}^{n-1} x_i x_{i+1} \cdots x_{i+k} \cdot u_{i+k+1} u_{i+k+2} \cdots u_{i+n-1} \\
&= \underbrace{x_i x_{i+1} \cdots x_{i+n-1}}_{\substack{= x_1 x_2 \cdots x_n \\ \text{(by Lemma 3.7)}}} \cdot \underbrace{u_{i+n} u_{i+n+1} \cdots u_{i+n-1}}_{= (\text{empty product}) = 1} \\
&\qquad + \sum_{k=0}^{n-2} x_i x_{i+1} \cdots x_{i+k} \cdot u_{i+k+1} u_{i+k+2} \cdots u_{i+n-1} \\
&\qquad\qquad (\text{here, we have split off the addend for } k = n-1 \text{ from the sum}) \\
&= \underbrace{x_1 x_2 \cdots x_n}_{= u_1 u_2 \cdots u_n} + \underbrace{\sum_{k=0}^{n-2} x_i x_{i+1} \cdots x_{i+k} \cdot u_{i+k+1} u_{i+k+2} \cdots u_{i+n-1}}_{\substack{= \sum_{k=1}^{n-1} x_i x_{i+1} \cdots x_{i+k-1} \cdot u_{i+k} u_{i+k+1} \cdots u_{i+n-1} \\ \text{(here, we have substituted } k-1 \text{ for } k \text{ in the sum)}}} \\
&= u_1 u_2 \cdots u_n + \sum_{k=1}^{n-1} x_i x_{i+1} \cdots x_{i+k-1} \cdot u_{i+k} u_{i+k+1} \cdots u_{i+n-1}. \qquad (17)
\end{aligned}$$

On the other hand, we have $u_{i-1} = u_{i+n-1}$ (since $i - 1 \equiv i + n - 1 \bmod n$) and

$$\begin{aligned}
t_{n-1,i-1} &= \sum_{k=0}^{n-1} \underbrace{x_{(i-1)+1} x_{(i-1)+2} \cdots x_{(i-1)+k}}_{= x_i x_{i+1} \cdots x_{i+k-1}} \cdot \underbrace{u_{(i-1)+k+1} u_{(i-1)+k+2} \cdots u_{(i-1)+n-1}}_{= u_{i+k} u_{i+k+1} \cdots u_{i+n-2}} \\
&\qquad (\text{by the definition of } t_{n-1,i-1}) \\
&= \sum_{k=0}^{n-1} x_i x_{i+1} \cdots x_{i+k-1} \cdot u_{i+k} u_{i+k+1} \cdots u_{i+n-2}.
\end{aligned}$$

Multiplying these two equalities, we obtain

$$\begin{aligned}
u_{i-1}t_{n-1,i-1} &= u_{i+n-1}\sum_{k=0}^{n-1} x_i x_{i+1}\cdots x_{i+k-1}\cdot u_{i+k}u_{i+k+1}\cdots u_{i+n-2} \\
&= \sum_{k=0}^{n-1} x_i x_{i+1}\cdots x_{i+k-1}\cdot \underbrace{u_{i+k}u_{i+k+1}\cdots u_{i+n-2}\cdot u_{i+n-1}}_{=u_{i+k}u_{i+k+1}\cdots u_{i+n-1}} \\
&= \sum_{k=0}^{n-1} x_i x_{i+1}\cdots x_{i+k-1}\cdot u_{i+k}u_{i+k+1}\cdots u_{i+n-1} \\
&= \underbrace{x_i x_{i+1}\cdots x_{i+0-1}}_{=(\text{empty product})=1}\cdot \underbrace{u_{i+0}u_{i+0+1}\cdots u_{i+n-1}}_{\substack{=u_iu_{i+1}\cdots u_{i+n-1} \\ =u_1u_2\cdots u_n \\ (\text{by Lemma 3.7})}} \\
&\qquad + \sum_{k=1}^{n-1} x_i x_{i+1}\cdots x_{i+k-1}\cdot u_{i+k}u_{i+k+1}\cdots u_{i+n-1} \\
&\qquad\qquad (\text{here, we have split off the addend for } k=0 \text{ from the sum}) \\
&= u_1u_2\cdots u_n + \sum_{k=1}^{n-1} x_i x_{i+1}\cdots x_{i+k-1}\cdot u_{i+k}u_{i+k+1}\cdots u_{i+n-1}.
\end{aligned}$$

Comparing this with (17), we obtain

$$u_{i-1}t_{n-1,i-1} = x_i t_{n-1,i}. \tag{18}$$

The same argument (applied to $i+1$ instead of i) yields

$$u_i t_{n-1,i} = x_{i+1}t_{n-1,i+1}. \tag{19}$$

Now, the definition of y yields

$$\begin{aligned}
y_i = u_i\cdot\frac{u_{i-1}t_{n-1,i-1}}{x_{i+1}t_{n-1,i+1}} &= u_i \underbrace{u_{i-1}t_{n-1,i-1}}_{\substack{=x_it_{n-1,i} \\ (\text{by (18)})}} / \underbrace{\left(x_{i+1}t_{n-1,i+1}\right)}_{\substack{=u_it_{n-1,i} \\ (\text{by (19)})}} \\
&= u_ix_it_{n-1,i}/\left(u_it_{n-1,i}\right) = x_i.
\end{aligned}$$

Now, forget that we fixed i. We thus have proved that $y_i = x_i$ for each $i\in\mathbb{Z}$. Thus, in particular, $y_i = x_i$ for each $i\in\{1,2,\ldots,n\}$. In other words, $y = x$. Hence, $\mathbf{f}_u(x) = y = x$. This proves Proposition 3.15. □

Remark 3.16. There is an alternative proof of Theorem 3.11 **(a)** that avoids the use of the more complicated parts of Lemma 3.12 (specifically, of parts **(g)**, **(i)**, **(j)** and **(k)**). Let us outline this proof:

The claim of Theorem 3.11 **(a)** can be restated as the equality $\mathbf{f}_u(\mathbf{f}_u(x)) = x$ for each $x\in\mathbb{K}^n$ and each $u\in\mathbb{K}^n$ (we are not regarding u as fixed here). This equality boils down to a set of identities between rational functions in the variables $u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n$ (since each entry of $\mathbf{f}_u(x)$ is a rational function in these variables, and each entry of $\mathbf{f}_u(\mathbf{f}_u(x))$ is a rational function in the former entries as well as $u_1, u_2, \ldots, u_n$). These rational functions are subtraction-free (i.e., no subtraction signs appear in them), and thus are defined over any semifield. But there is a general principle saying that if we need to prove an identity between two subtraction-free rational functions, it is sufficient to prove that it holds over the semifield $\mathbb{Q}_+$ from Example 3.2. (Indeed, this principle follows from the fact that any subtraction-free rational function can be rewritten as a ratio of two polynomials with nonnegative integer coefficients, and thus an identity between two subtraction-free rational functions can be rewritten as an identity between two such polynomials; but the latter kind of identity will necessarily be true if it has been checked on all positive rational numbers.)

As a consequence of this discussion, in order to prove Theorem 3.11 **(a)** in full generality, it suffices to prove Theorem 3.11 **(a)** in the case when $\mathbb{K} = \mathbb{Q}_+$. So let us restrict ourselves to this case. Let $x \in \mathbb{K}^n$. We must show that $\mathbf{f}_u(\mathbf{f}_u(x)) = x$.

Let $y = \mathbf{f}_u(x)$, and let $z = \mathbf{f}_u(y)$. We will show that $z = x$.

Assume the contrary. Thus, $z \neq x$. Hence, there exists some $i \in \{1, 2, \ldots, n\}$ such that $z_i \neq x_i$. Consider this i. Hence, either $z_i > x_i$ or $z_i < x_i$. We WLOG assume that $z_i > x_i$ (since the proof in the case of $z_i < x_i$ is identical, except that all inequality signs are reversed). But Theorem 3.11 **(c)** yields

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right).$$

Likewise, Theorem 3.11 **(c)** (applied to y and z instead of x and y) yields

$$(u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) = (u_i + z_i)\left(\frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}\right)$$

(since $z = \mathbf{f}_u(y)$). Hence,

$$\begin{aligned}(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) &= (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) \\ &= \left(u_i + \underbrace{z_i}_{>x_i}\right)\left(\frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}\right) \\ &> (u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}\right).\end{aligned}$$

Cancelling the positive number $u_i + x_i$ from this inequality, we obtain $\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}} > \frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}$. Hence, $\frac{1}{x_{i+1}} > \frac{1}{z_{i+1}}$, so that $z_{i+1} > x_{i+1}$. Thus, from $z_i > x_i$, we have obtained $z_{i+1} > x_{i+1}$. The same reasoning (but applied to $i + 1$ instead of i) now yields $z_{i+2} > x_{i+2}$ (since $z_{i+1} > x_{i+1}$). Proceeding in the same way, we successively obtain $z_{i+3} > x_{i+3}$ and $z_{i+4} > x_{i+4}$ and $z_{i+5} > x_{i+5}$ and so on. Hence,

$$\underbrace{z_i}_{>x_i} \underbrace{z_{i+1}}_{>x_{i+1}} \cdots \underbrace{z_{i+n-1}}_{>x_{i+n-1}} > x_i x_{i+1} \cdots x_{i+n-1}. \tag{20}$$

But Theorem 3.11 **(b)** yields

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 .$$

Also, Theorem 3.11 **(b)** (applied to y and z instead of x and y) yields

$$z_1 z_2 \cdots z_n \cdot y_1 y_2 \cdots y_n = (u_1 u_2 \cdots u_n)^2$$

(since $z = \mathbf{f}_u(y)$). Comparing these two equalities, we find $y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = z_1 z_2 \cdots z_n \cdot y_1 y_2 \cdots y_n$, so that

$$x_1 x_2 \cdots x_n = z_1 z_2 \cdots z_n. \tag{21}$$

But Lemma 3.7 yields $z_i z_{i+1} \cdots z_{i+n-1} = z_1 z_2 \cdots z_n$ and $x_i x_{i+1} \cdots x_{i+n-1} = x_1 x_2 \cdots x_n$. In light of these two equalities, we can rewrite (20) as $z_1 z_2 \cdots z_n > x_1 x_2 \cdots x_n$. This, however, contradicts (21). This contradiction shows that our assumption was false, thus concluding our proof of $z = x$.

Now, $\mathbf{f}_u\left(\underbrace{\mathbf{f}_u(x)}_{=y}\right) = \mathbf{f}_u(y) = z = x$, as we wanted to prove. Hence, Theorem 3.11 **(a)** is proved again.

We shall take up the study of the birational involution $\mathbf{f}_u$ again in Subsection 5.2, where we will pose several questions about its meaning and uniqueness properties.

4. Proof of the main theorem

We shall now slowly approach the proof of Theorem 2.3 through a long sequence of auxiliary results, some of them easy, some well-known.

4.1. From the life of snakes

Recall the conventions introduced in Section 1 and in Convention 2.1. Let us next introduce some further notations.

Definition 4.1.

(a) Let $\mathcal{L}$ denote the ring $\mathbf{k}\left[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ of Laurent polynomials in the n indeterminates $x_1, x_2, \ldots, x_n$ over $\mathbf{k}$. Clearly, the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$ is a subring of $\mathcal{L}$.

(b) We let x_Π denote the monomial $x_1 x_2 \cdots x_n \in \mathbf{k}[x_1, x_2, \ldots, x_n] \subseteq \mathcal{L}$.

If $f \in \Lambda$ is a symmetric function[4], and if $a_1, a_2, \ldots, a_n$ are n elements of a commutative $\mathbf{k}$-algebra A, then $f(a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots)$ means the result of substituting $a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots$ for $x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}, \ldots$ in f. This is a well-defined element of A (see [GriRei20, Exercise 2.1.2] for the proof), and is denoted by $f(a_1, a_2, \ldots, a_n)$. It is called the *evaluation* of f at $a_1, a_2, \ldots, a_n$.

For any symmetric function $f \in \Lambda$, the evaluation

$$f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$$

is a polynomial in $\mathbf{k}[x_1, x_2, \ldots, x_n]$ and thus a Laurent polynomial in $\mathcal{L}$. Moreover, for any symmetric function $f \in \Lambda$, the evaluation

$$f\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) = f\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}, 0, 0, 0, \ldots\right)$$

is a Laurent polynomial in $\mathcal{L}$ as well.

Convention 4.2. For the rest of Section 4, let us agree to the following notation: If γ is an n-tuple (of any objects), then we let γ_i denote the i-th entry of γ whenever $i \in \{1, 2, \ldots, n\}$. Thus, each n-tuple γ satisfies $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Definition 4.3.

(a) A *snake* means an n-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers (not necessarily nonnegative) such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(b) A snake λ is said to be *nonnegative* if it belongs to $\mathbb{N}^n$ (that is, if all its entries are nonnegative). Thus, a nonnegative snake is the same as a partition having length $\leq n$. In other words, a nonnegative snake is the same as a partition $\lambda \in \operatorname{Par}[n]$.

(c) If $\lambda \in \mathbb{Z}^n$ is an n-tuple, and d is an integer, then $\lambda + d$ denotes the n-tuple $(\lambda_1 + d, \lambda_2 + d, \ldots, \lambda_n + d) \in \mathbb{Z}^n$ (which is obtained from λ by adding d to each entry), whereas $\lambda - d$ denotes the n-tuple $(\lambda_1 - d, \lambda_2 - d, \ldots, \lambda_n - d) \in \mathbb{Z}^n$. (Thus, $\lambda - d = \lambda + (-d)$.)

(d) If $\lambda \in \mathbb{Z}^n$, then $\lambda^\vee$ denotes the n-tuple $(-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) \in \mathbb{Z}^n$.

[4] or, more generally, any formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that is of bounded degree

(e) We regard $\mathbb{Z}^n$ as a $\mathbb{Z}$-module in the obvious way. Thus, if $\lambda \in \mathbb{Z}^n$ and $\mu \in \mathbb{Z}^n$ are two n-tuples of integers, then

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_n + \mu_n),$$
$$\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_n - \mu_n).$$

(f) We let ρ denote the nonnegative snake $(n-1, n-2, \ldots, 2, 1, 0)$. Thus,

$$\rho_i = n - i \qquad \text{for each } i \in \{1, 2, \ldots, n\}. \tag{22}$$

Example 4.4. In this example, let $n = 3$.

(a) The four 3-tuples $(3,1,0)$, $(2,2,1)$, $(1,0,-1)$ and $(-1,-2,-5)$ are examples of snakes.

(b) The first two of these four snakes (but not the last two) are nonnegative.

(c) We have $(5,3,1) + 3 = (8,6,4)$ and $(5,3,1) - 3 = (2,0,-2)$.

(d) We have $(5,2,2)^\vee = (-2,-2,-5)$.

(e) We have $(2,1,2) + (3,4,5) = (5,5,7)$.

(f) We have $\rho = (2,1,0)$.

Note that what we call a "snake" here is called a "staircase of height n" in Stembridge's work [Stembr87], where he uses these snakes to index finite-dimensional polynomial representations of the group $\mathrm{GL}_n(\mathbb{C})$. We avoid calling them "staircases", as that word has since been used for other things (in particular, ρ is often called "the n-staircase" in the jargon of combinatorialists).

The notations introduced in Definition 4.3 have the following properties:

Proposition 4.5.

(a) If λ is a snake, and d is an integer, then $\lambda + d$ and $\lambda - d$ are snakes as well.

(b) If λ is a snake, then $\lambda^\vee$ is a snake as well.

(c) We have $(\lambda + \mu) + d = (\lambda + d) + \mu$ for any $\lambda \in \mathbb{Z}^n$, $\mu \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$.

(d) We have $\lambda + (d + e) = (\lambda + d) + e$ for any $\lambda \in \mathbb{Z}^n$, $d \in \mathbb{Z}$ and $e \in \mathbb{Z}$.

(e) We have $(\lambda + d) - d = (\lambda - d) + d = \lambda$ for any $\lambda \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$.

Proof of Proposition 4.5. Completely straightforward. □

Let us now assign a Laurent polynomial a_λ to each $\lambda \in \mathbb{Z}^n$:

Definition 4.6. Let $\lambda \in \mathbb{Z}^n$ be any n-tuple. Then, we define the Laurent polynomial

$$a_\lambda := \sum_{w \in \mathfrak{S}_n} (-1)^w \, x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n} \in \mathcal{L},$$

where $\mathfrak{S}_n$ is the symmetric group of the set $\{1, 2, \ldots, n\}$ (and where $(-1)^w$ denotes the sign of a permutation w). This Laurent polynomial a_λ is called the *alternant* corresponding to the n-tuple λ.

(The "a" in the notation "a_λ" has nothing to do with the a in Theorem 2.3.)

Example 4.7. We have

$$\begin{aligned} a_{(5,3,2)} &= \sum_{w \in \mathfrak{S}_3} (-1)^w \, x_{w(1)}^5 x_{w(2)}^3 x_{w(3)}^2 \\ &= x_1^5 x_2^3 x_3^2 + x_2^5 x_3^3 x_1^2 + x_3^5 x_1^3 x_2^2 - x_1^5 x_3^3 x_2^2 - x_2^5 x_1^3 x_3^2 - x_3^5 x_2^3 x_1^2. \end{aligned}$$

The sum in Definition 4.6 is the same kind of sum that appears in the definition of a determinant. Therefore, we can rewrite the alternant as follows:

Proposition 4.8. Let $\lambda \in \mathbb{Z}^n$ be an n-tuple. Then, the alternant $a_\lambda \in \mathcal{L}$ satisfies

$$a_\lambda = \det \left(\left(x_j^{\lambda_i} \right)_{1 \le i \le n, \ 1 \le j \le n} \right) = \det \left(\left(x_i^{\lambda_j} \right)_{1 \le i \le n, \ 1 \le j \le n} \right).$$

Thus, in particular, the alternant a_ρ corresponding to the snake

$$\rho = (n-1, n-2, \ldots, 2, 1, 0) = (n-1, n-2, \ldots, n-n)$$

satisfies

$$a_\rho = \det \left(\left(x_i^{n-j} \right)_{1 \le i \le n, \ 1 \le j \le n} \right) = \prod_{1 \le i < j \le n} (x_i - x_j)$$

(by the classical formula for the Vandermonde determinant).

We recall a standard concept from commutative algebra: An element a of a commutative ring A is said to be *regular* if it has the property that every $x \in A$ satisfying $ax = 0$ must satisfy $x = 0$. (Thus, regular elements are the same as elements that are not zero-divisors, if one does not require zero-divisors to be nonzero[5].)

[5]Unfortunately, there is no agreement in the literature on whether zero-divisors should be required to be nonzero. This is one of the reasons why we are avoiding this notion.

Lemma 4.9. The alternant a_ρ is a regular element of $\mathcal{L}$.

Proof of Lemma 4.9. Let $b \in \mathcal{L}$ be such that $a_\rho b = 0$. We want to show that $b = 0$.

We know that b is a Laurent polynomial, and thus has the form $b = \dfrac{c}{x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}}$ for some $u_1, u_2, \ldots, u_n \in \mathbb{Z}$ and some polynomial $c \in \mathbf{k}[x_1, x_2, \ldots, x_n]$. Consider these $u_1, u_2, \ldots, u_n$ and this c. From $b = \dfrac{c}{x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}}$, we obtain $c = b \cdot x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$. Multiplying this equality by a_ρ, we obtain

$$a_\rho c = \underbrace{a_\rho b}_{=0} \cdot x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} = 0.$$

But it is a well-known fact (see, e.g., [Grinbe19, Corollary 4.4]) that the polynomial $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ is a regular element of $\mathbf{k}[x_1, x_2, \ldots, x_n]$. In other words, a_ρ is a regular element of $\mathbf{k}[x_1, x_2, \ldots, x_n]$ (since $a_\rho = \prod_{1 \leq i < j \leq n} (x_i - x_j)$). Hence, from $a_\rho c = 0$, we obtain $c = 0$. Now, $b = \dfrac{c}{x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}} = 0$ (since $c = 0$).

Forget that we fixed b. We thus have shown that each $b \in \mathcal{L}$ satisfying $a_\rho b = 0$ satisfies $b = 0$. In other words, a_ρ is a regular element of $\mathcal{L}$. This proves Lemma 4.9. □

Lemma 4.9 shows that fractions of the form $\dfrac{u}{a_\rho}$ (where $u \in \mathcal{L}$) are well-defined if u is a multiple of a_ρ. (That is, there is never more than one $b \in \mathcal{L}$ that satisfies $a_\rho b = u$.)

We notice that the element $x_\Pi = x_1 x_2 \cdots x_n$ of $\mathcal{L}$ is invertible (since $x_1, x_2, \ldots, x_n$ are invertible in $\mathcal{L}$).

Lemma 4.10. Let $\lambda \in \mathbb{Z}^n$ be any n-tuple, and let $d \in \mathbb{Z}$. Then, $a_{\lambda + d} = x_\Pi^d a_\lambda$.

Proof of Lemma 4.10. The definition of a_λ yields

$$a_\lambda = \sum_{w \in \mathfrak{S}_n} (-1)^w \, x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n}. \tag{23}$$

But the definition of $\lambda + d$ yields $\lambda + d = (\lambda_1 + d, \lambda_2 + d, \ldots, \lambda_n + d)$. Hence, the definition of $a_{\lambda+d}$ yields

$$\begin{aligned}
a_{\lambda+d} &= \sum_{w\in\mathfrak{S}_n} (-1)^w \underbrace{x_{w(1)}^{\lambda_1+d} x_{w(2)}^{\lambda_2+d} \cdots x_{w(n)}^{\lambda_n+d}}_{=\left(x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n}\right)\left(x_{w(1)}^{d} x_{w(2)}^{d} \cdots x_{w(n)}^{d}\right)} \\
&= \sum_{w\in\mathfrak{S}_n} (-1)^w \left(x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n}\right) \underbrace{\left(x_{w(1)}^{d} x_{w(2)}^{d} \cdots x_{w(n)}^{d}\right)}_{\substack{=x_1^d x_2^d \cdots x_n^d \\ \text{(since } w:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\} \\ \text{is a bijection)}}} \\
&= \sum_{w\in\mathfrak{S}_n} (-1)^w \left(x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n}\right) \underbrace{\left(x_1^d x_2^d \cdots x_n^d\right)}_{\substack{=(x_1x_2\cdots x_n)^d = x_\Pi^d \\ \text{(since } x_1x_2\cdots x_n = x_\Pi\text{)}}} \\
&= \sum_{w\in\mathfrak{S}_n} (-1)^w \left(x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n}\right) x_\Pi^d \\
&= x_\Pi^d \underbrace{\sum_{w\in\mathfrak{S}_n} (-1)^w x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n}}_{\substack{=a_\lambda \\ \text{(by (23))}}} = x_\Pi^d a_\lambda .
\end{aligned}$$

This proves Lemma 4.10. ☐

Lemma 4.11. Let λ be a snake. Then, $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$.

Proof of Lemma 4.11. Our proof will consist of two steps:

Step 1: We will prove Lemma 4.11 in the particular case when λ is nonnegative.

Step 2: We will use Lemma 4.10 to derive the general case of Lemma 4.11 from this particular case.

We will use this strategy again further on; we shall refer to it as the *right-shift strategy*.

Here are the details of the two steps:

Step 1: Let us prove that Lemma 4.11 holds in the particular case when λ is nonnegative.

Indeed, let us assume that λ is nonnegative. We must show that $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$.

We know that λ is a nonnegative snake, thus a partition of length $\leq n$. Hence, [GriRei20, Corollary 2.6.7] shows that $s_\lambda(x_1, x_2, \ldots, x_n) = \dfrac{a_{\lambda+\rho}}{a_\rho}$. Thus, $a_{\lambda+\rho} = a_\rho \cdot s_\lambda(x_1, x_2, \ldots, x_n)$. This shows that $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$ (since $s_\lambda(x_1, x_2, \ldots, x_n) \in \mathbf{k}[x_1, x_2, \ldots, x_n] \subseteq \mathcal{L}$). Thus, Lemma 4.11 is proved under the assumption that λ is nonnegative. This completes Step 1.

Step 2: Let us now prove Lemma 4.11 in the general case.

We know that λ is a snake. Thus, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Hence, each $i \in \{1, 2, \ldots, n\}$ satisfies $\lambda_i \geq \lambda_n$ and thus

$$\lambda_i - \lambda_n \geq 0. \tag{24}$$

The snake λ may or may not be nonnegative. However, there exists some integer d such that the snake $\lambda + d$ is nonnegative[6]. Consider this d. Proposition 4.5 **(c)** (applied to $\mu = \rho$) yields $(\lambda + \rho) + d = (\lambda + d) + \rho$.

The snake $\lambda + d$ is nonnegative; thus, we can apply Lemma 4.11 to $\lambda + d$ instead of λ (because in Step 1, we have proved that Lemma 4.11 holds in the particular case when λ is nonnegative). Thus we conclude that $a_{(\lambda+d)+\rho}$ is a multiple of a_ρ in $\mathcal{L}$. In other words, there exists some $u \in \mathcal{L}$ such that $a_{(\lambda+d)+\rho} = a_\rho u$. Consider this u. From $(\lambda + \rho) + d = (\lambda + d) + \rho$, we obtain $a_{(\lambda+\rho)+d} = a_{(\lambda+d)+\rho} = a_\rho u$.

Lemma 4.10 (applied to $\lambda + \rho$ instead of λ) yields $a_{(\lambda+\rho)+d} = x_\Pi^d a_{\lambda+\rho}$. Since the element x_Π of $\mathcal{L}$ is invertible, we thus obtain

$$a_{\lambda+\rho} = \underbrace{\left(x_\Pi^d\right)^{-1}}_{=x_\Pi^{-d}} \underbrace{a_{(\lambda+\rho)+d}}_{=a_\rho u} = x_\Pi^{-d} a_\rho u = a_\rho \cdot x_\Pi^{-d} u.$$

Hence, $a_{\lambda+\rho}$ is a multiple of a_ρ (since $x_\Pi^{-d} u \in \mathcal{L}$). This proves Lemma 4.11. Thus, Step 2 is complete, and Lemma 4.11 is proven. $\square$

Definition 4.12. Let λ be a snake. We define an element $\overline{s}_\lambda \in \mathcal{L}$ by $\overline{s}_\lambda = \dfrac{a_{\lambda+\rho}}{a_\rho}$. (This is well-defined, because Lemma 4.11 shows that $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$, and because Lemma 4.9 shows that the fraction $\dfrac{a_{\lambda+\rho}}{a_\rho}$ is uniquely defined.)

It makes sense to refer to the elements $\overline{s}_\lambda$ just defined as "*Schur Laurent polynomials*". In fact, as the following lemma shows, they are identical with the Schur polynomials $s_\lambda(x_1, x_2, \ldots, x_n)$ when the snake λ is nonnegative:

Lemma 4.13. Let $\lambda \in \operatorname{Par}[n]$. Then,

$$\overline{s}_\lambda = s_\lambda(x_1, x_2, \ldots, x_n).$$

Proof of Lemma 4.13. We know that λ is a partition of length $\leq n$ (since $\lambda \in \operatorname{Par}[n]$). Hence, λ is a nonnegative snake. Furthermore, since λ is a partition of length $\leq n$,

[6] Indeed, for example, we can take $d = -\lambda_n$. Then, all entries of $\lambda + d$ have the form $\lambda_i + \underbrace{d}_{=-\lambda_n} = \lambda_i - \lambda_n$ for some $i \in \{1, 2, \ldots, n\}$, and thus are nonnegative (because of (24)); this shows that the snake $\lambda + d$ is nonnegative.

we can apply [GriRei20, Corollary 2.6.7] and obtain $s_\lambda(x_1, x_2, \ldots, x_n) = \dfrac{a_{\lambda+\rho}}{a_\rho} = \overline{s}_\lambda$ (since $\overline{s}_\lambda$ was defined to be $\dfrac{a_{\lambda+\rho}}{a_\rho}$). This proves Lemma 4.13. $\square$

The Schur Laurent polynomials $\overline{s}_\lambda$ appear in Stembridge's [Stembr87], where they are named s_λ. (The equivalence of our definition with his follows from [Stembr87, Theorem 7.1].)

Furthermore, from Lemma 4.10, we can easily obtain an analogous property for Schur Laurent polynomials:

Lemma 4.14. Let $\lambda \in \mathbb{Z}^n$ be any snake, and let $d \in \mathbb{Z}$. Then, $\overline{s}_{\lambda+d} = x_\Pi^d \overline{s}_\lambda$.

Proof of Lemma 4.14. Proposition 4.5 **(c)** (applied to $\mu = \rho$) yields $(\lambda + \rho) + d = (\lambda + d) + \rho$. But Lemma 4.10 (applied to $\lambda + \rho$ instead of λ) yields $a_{(\lambda+\rho)+d} = x_\Pi^d a_{\lambda+\rho}$. This rewrites as $a_{(\lambda+d)+\rho} = x_\Pi^d a_{\lambda+\rho}$ (since $(\lambda + \rho) + d = (\lambda + d) + \rho$).

The definition of $\overline{s}_{\lambda+d}$ yields

$$\overline{s}_{\lambda+d} = \frac{a_{(\lambda+d)+\rho}}{a_\rho} = \frac{x_\Pi^d a_{\lambda+\rho}}{a_\rho} \qquad \left(\text{since } a_{(\lambda+d)+\rho} = x_\Pi^d a_{\lambda+\rho}\right).$$

Comparing this with

$$x_\Pi^d \underbrace{\overline{s}_\lambda}_{\substack{= \frac{a_{\lambda+\rho}}{a_\rho} \\ \text{(by the definition of } \overline{s}_\lambda)}} = x_\Pi^d \cdot \frac{a_{\lambda+\rho}}{a_\rho} = \frac{x_\Pi^d a_{\lambda+\rho}}{a_\rho},$$

we obtain $\overline{s}_{\lambda+d} = x_\Pi^d \overline{s}_\lambda$. This proves Lemma 4.14. $\square$

Lemma 4.15. Let $\mu, \nu \in \operatorname{Par}[n]$. Then,

$$\overline{s}_\mu \overline{s}_\nu = \sum_{\lambda \in \operatorname{Par}[n]} c_{\mu,\nu}^\lambda \overline{s}_\lambda.$$

Proof of Lemma 4.15. It is well-known (see, e.g., [GriRei20, Exercise 2.3.8(b)]) that if λ is a partition having length $> n$, then

$$s_\lambda(x_1, x_2, \ldots, x_n) = 0. \tag{25}$$

We know that $\mu \in \operatorname{Par}[n]$. Hence, Lemma 4.13 (applied to $\lambda = \mu$) yields $\overline{s}_\mu = s_\mu(x_1, x_2, \ldots, x_n)$. Likewise, $\overline{s}_\nu = s_\nu(x_1, x_2, \ldots, x_n)$. Multiplying these two equalities, we obtain

$$\begin{aligned}
\overline{s}_\mu \overline{s}_\nu &= s_\mu\left(x_1, x_2, \ldots, x_n\right) \cdot s_\nu\left(x_1, x_2, \ldots, x_n\right) \\
&= \underbrace{\left(s_\mu s_\nu\right)}_{\substack{=\sum\limits_{\lambda\in\operatorname{Par}} c^\lambda_{\mu,\nu} s_\lambda \\ \text{(by (1))}}} \left(x_1, x_2, \ldots, x_n\right) = \sum_{\lambda\in\operatorname{Par}} c^\lambda_{\mu,\nu} s_\lambda\left(x_1, x_2, \ldots, x_n\right) \\
&= \underbrace{\sum_{\substack{\lambda\in\operatorname{Par};\\ \lambda \text{ has length } \leq n}}}_{\substack{=\sum\limits_{\lambda\in\operatorname{Par}[n]} \\ \text{(by the definition of } \operatorname{Par}[n])}} c^\lambda_{\mu,\nu} s_\lambda\left(x_1, x_2, \ldots, x_n\right) + \sum_{\substack{\lambda\in\operatorname{Par};\\ \lambda \text{ has length } > n}} c^\lambda_{\mu,\nu} \underbrace{s_\lambda\left(x_1, x_2, \ldots, x_n\right)}_{\substack{=0\\ \text{(by (25))}}} \\
&= \sum_{\lambda\in\operatorname{Par}[n]} c^\lambda_{\mu,\nu} \underbrace{s_\lambda\left(x_1, x_2, \ldots, x_n\right)}_{\substack{=\overline{s}_\lambda \\ \text{(by Lemma 4.13)}}} + \underbrace{\sum_{\substack{\lambda\in\operatorname{Par};\\ \lambda \text{ has length } > n}} c^\lambda_{\mu,\nu} 0}_{=0} = \sum_{\lambda\in\operatorname{Par}[n]} c^\lambda_{\mu,\nu} \overline{s}_\lambda.
\end{aligned}$$

This proves Lemma 4.15. □

Lemma 4.16. The family $\left(\overline{s}_\lambda\right)_{\lambda\in\{\text{snakes}\}}$ of elements of $\mathcal{L}$ is $\mathbf{k}$-linearly independent.

Proof of Lemma 4.16. Let us define a *strict snake* to be an n-tuple $\alpha \in \mathbb{Z}^n$ of integers satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. It is easy to see that the map

$$\begin{aligned}
\{\text{snakes}\} &\to \{\text{strict snakes}\}, \\
\lambda &\mapsto \lambda + \rho
\end{aligned} \tag{26}$$

is a bijection.

It is also easy to see that if α and β are two strict snakes, then

$$\left(\text{the coefficient of } x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \text{ in } a_\alpha\right) = \delta_{\alpha,\beta}, \tag{27}$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta of α and β (that is, the integer $\begin{cases} 1, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha \neq \beta \end{cases}$).

Now, assume that $\left(u_\lambda\right)_{\lambda\in\{\text{snakes}\}} \in \mathbf{k}^{\{\text{snakes}\}}$ be a family of scalars with the property that (all but finitely many snakes λ satisfy $u_\lambda = 0$) and

$$\sum_{\lambda\in\{\text{snakes}\}} u_\lambda \overline{s}_\lambda = 0. \tag{28}$$

We shall show that $u_\lambda = 0$ for all snakes λ.

Indeed, fix a snake μ. Then, $\mu + \rho$ is a strict snake (since the map (26) is a bijection). Let us denote this strict snake by β. Thus, $\beta = \mu + \rho$.

If λ is any snake, then $\lambda + \rho$ is a strict snake (since the map (26) is a bijection), and thus satisfies

$$\begin{aligned}
&\left(\text{the coefficient of } x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \text{ in } a_{\lambda+\rho}\right) \\
&= \delta_{\lambda+\rho,\beta} \qquad \left(\text{by (27), applied to } \alpha = \lambda + \rho\right) \\
&= \delta_{\lambda+\rho,\mu+\rho} \qquad \left(\text{since } \beta = \mu + \rho\right) \\
&= \delta_{\lambda,\mu}
\end{aligned} \tag{29}$$

(since $\lambda + \rho = \mu + \rho$ holds if and only if $\lambda = \mu$ holds).

From (28), we obtain

$$0 = \sum_{\lambda \in \{\text{snakes}\}} u_\lambda \underbrace{\overline{s}_\lambda}_{\substack{= \dfrac{a_{\lambda+\rho}}{a_\rho} \\ \text{(by the definition of } \overline{s}_\lambda)}} = \sum_{\lambda \in \{\text{snakes}\}} u_\lambda \frac{a_{\lambda+\rho}}{a_\rho} = \frac{1}{a_\rho} \sum_{\lambda \in \{\text{snakes}\}} u_\lambda a_{\lambda+\rho}.$$

Multiplying both sides of this equality by a_ρ, we obtain

$$0 = \sum_{\lambda \in \{\text{snakes}\}} u_\lambda a_{\lambda+\rho}.$$

Hence,

$$\begin{aligned}
&\left(\text{the coefficient of } x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \text{ in } 0\right) \\
&= \left(\text{the coefficient of } x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \text{ in } \sum_{\lambda \in \{\text{snakes}\}} u_\lambda a_{\lambda+\rho}\right) \\
&= \sum_{\lambda \in \{\text{snakes}\}} u_\lambda \underbrace{\left(\text{the coefficient of } x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \text{ in } a_{\lambda+\rho}\right)}_{\substack{= \delta_{\lambda,\mu} \\ \text{(by (29))}}} \\
&= \sum_{\lambda \in \{\text{snakes}\}} u_\lambda \delta_{\lambda,\mu} = u_\mu,
\end{aligned}$$

so that

$$u_\mu = \left(\text{the coefficient of } x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \text{ in } 0\right) = 0.$$

Now, forget that we fixed μ. We thus have proved that $u_\mu = 0$ for all snakes μ. In other words, $u_\lambda = 0$ for all snakes λ.

Forget that we fixed $(u_\lambda)_{\lambda \in \{\text{snakes}\}}$. We thus have shown that if $(u_\lambda)_{\lambda \in \{\text{snakes}\}} \in \mathbf{k}^{\{\text{snakes}\}}$ is a family of scalars with the property that
(all but finitely many snakes λ satisfy $u_\lambda = 0$) and $\sum_{\lambda \in \{\text{snakes}\}} u_\lambda \overline{s}_\lambda = 0$, then $u_\lambda = 0$ for all snakes λ. In other words, the family $(\overline{s}_\lambda)_{\lambda \in \{\text{snakes}\}}$ of elements of $\mathcal{L}$ is $\mathbf{k}$-linearly independent. This proves Lemma 4.16. $\square$

Lemma 4.16 is actually part of a stronger claim: The family $(\overline{s}_\lambda)_{\lambda\in\{\text{snakes}\}}$ is a basis of the $\mathbf{k}$-module of symmetric Laurent polynomials in $x_1, x_2, \ldots, x_n$. We shall not need this, however, so we omit the proof[7].

Recall Definition 4.3 **(d)**. Our next lemma connects the Laurent polynomials $\overline{s}_\lambda$ and $\overline{s}_{\lambda^\vee}$ for every snake λ; it is folklore (see [GriRei20, Exercise 2.9.15(d)] for an equivalent version), but we shall prove it for the sake of completeness.

Lemma 4.17. Let λ be a snake. Then,

$$\overline{s}_{\lambda^\vee} = \overline{s}_\lambda\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right).$$

Here, of course, $\overline{s}_\lambda\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right)$ means the result of substituting $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$ for $x_1, x_2, \ldots, x_n$ in the Laurent polynomial $\overline{s}_\lambda \in \mathcal{L}$.

Proof of Lemma 4.17. Let $w_0 \in \mathfrak{S}_n$ be the permutation of $\{1, 2, \ldots, n\}$ that sends each $i \in \{1, 2, \ldots, n\}$ to $n+1-i$. The map $\mathfrak{S}_n \to \mathfrak{S}_n,\ w \mapsto w \circ w_0$ is a bijection (since $\mathfrak{S}_n$ is a group).

For each $i \in \{1, 2, \ldots, n\}$, we have

$$(\lambda+\rho)_i = \lambda_i + \underbrace{\rho_i}_{\substack{=n-i \\ \text{(by (22))}}} = \lambda_i + n - i. \tag{30}$$

Proposition 4.5 **(b)** shows that $\lambda^\vee$ is a snake. Thus, using Proposition 4.5 **(a)**, we conclude that $\lambda^\vee + (1-n)$ is a snake. Let us denote this snake by μ. Thus, $\mu = \lambda^\vee + (1-n)$. Hence, $\mu + \rho = (\lambda^\vee + (1-n)) + \rho = (\lambda^\vee + \rho) + (1-n)$ (this follows from Proposition 4.5 **(c)**, applied to $\lambda^\vee$, ρ and $1-n$ instead of λ, μ and d). Therefore,

$$a_{\mu+\rho} = a_{(\lambda^\vee+\rho)+(1-n)} = x_\Pi^{1-n} a_{\lambda^\vee+\rho} \tag{31}$$

(by Lemma 4.10, applied to $\lambda^\vee + \rho$ and $1-n$ instead of λ and d).

The definition of $\lambda^\vee$ yields $\lambda^\vee = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)$. Hence, for each $i \in \{1, 2, \ldots, n\}$, we have

$$\left(\lambda^\vee\right)_i = -\lambda_{n+1-i}. \tag{32}$$

Thus, for each $i \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}(\mu+\rho)_i &= \underbrace{\mu_i}_{\substack{=(\lambda^\vee+(1-n))_i \\ \text{(since } \mu=\lambda^\vee+(1-n))}} + \underbrace{\rho_i}_{\substack{=n-i \\ \text{(by (22))}}} = \underbrace{\left(\lambda^\vee + (1-n)\right)_i}_{\substack{=(\lambda^\vee)_i+(1-n) \\ \text{(by the definition of } \lambda^\vee+(1-n))}} + n - i \\ &= \underbrace{\left(\lambda^\vee\right)_i}_{\substack{=-\lambda_{n+1-i} \\ \text{(by (32))}}} + \underbrace{(1-n)+n-i}_{=1-i} \\ &= -\lambda_{n+1-i} + 1 - i. \end{aligned} \tag{33}$$

[7] Just in case: It follows easily from Lemma 4.14 and [GriRei20, Remark 2.3.9(d)].

Hence, the definition of $a_{\mu+\rho}$ yields

$$\begin{aligned}
a_{\mu+\rho} &= \sum_{w\in\mathfrak{S}_n} (-1)^w \underbrace{x_{w(1)}^{(\mu+\rho)_1} x_{w(2)}^{(\mu+\rho)_2} \cdots x_{w(n)}^{(\mu+\rho)_n}}_{=\prod_{i\in\{1,2,\ldots,n\}} x_{w(i)}^{(\mu+\rho)_i}} \\
&= \sum_{w\in\mathfrak{S}_n} (-1)^w \prod_{i\in\{1,2,\ldots,n\}} \underbrace{x_{w(i)}^{(\mu+\rho)_i}}_{\substack{=x_{w(i)}^{-\lambda_{n+1-i}+1-i} \\ \text{(since (33) yields } (\mu+\rho)_i=-\lambda_{n+1-i}+1-i)}} \\
&= \sum_{w\in\mathfrak{S}_n} (-1)^w \prod_{i\in\{1,2,\ldots,n\}} x_{w(i)}^{-\lambda_{n+1-i}+1-i}. \qquad (34)
\end{aligned}$$

The definition of $a_{\lambda+\rho}$ yields

$$\begin{aligned}
a_{\lambda+\rho} &= \sum_{w\in\mathfrak{S}_n} (-1)^w \underbrace{x_{w(1)}^{(\lambda+\rho)_1} x_{w(2)}^{(\lambda+\rho)_2} \cdots x_{w(n)}^{(\lambda+\rho)_n}}_{=\prod_{i\in\{1,2,\ldots,n\}} x_{w(i)}^{(\lambda+\rho)_i}} \\
&= \sum_{w\in\mathfrak{S}_n} (-1)^w \prod_{i\in\{1,2,\ldots,n\}} \underbrace{x_{w(i)}^{(\lambda+\rho)_i}}_{\substack{=x_{w(i)}^{\lambda_i+n-i} \\ \text{(since (30) yields } (\lambda+\rho)_i=\lambda_i+n-i)}} \\
&= \sum_{w\in\mathfrak{S}_n} (-1)^w \prod_{i\in\{1,2,\ldots,n\}} x_{w(i)}^{\lambda_i+n-i}.
\end{aligned}$$

Substituting $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$ for $x_1, x_2, \ldots, x_n$ on both sides of this equality, we obtain

$$
\begin{aligned}
& a_{\lambda+\rho}\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \\
& = \sum_{w \in \mathfrak{S}_n} (-1)^w \prod_{i \in \{1,2,\ldots,n\}} \underbrace{\left(x_{w(i)}^{-1}\right)^{\lambda_i+n-i}}_{=x_{w(i)}^{-(\lambda_i+n-i)}} = \sum_{w \in \mathfrak{S}_n} (-1)^w \prod_{i \in \{1,2,\ldots,n\}} x_{w(i)}^{-(\lambda_i+n-i)} \\
& = \sum_{w \in \mathfrak{S}_n} \underbrace{(-1)^{w \circ w_0}}_{=(-1)^w (-1)^{w_0}} \prod_{i \in \{1,2,\ldots,n\}} \underbrace{x_{(w \circ w_0)(i)}^{-(\lambda_i+n-i)}}_{\substack{=x_{w(n+1-i)}^{-\lambda_i-n+i} \\ \text{(since } -(\lambda_i+n-i)=-\lambda_i-n+i \\ \text{and } (w \circ w_0)(i)=w(w_0(i))=w(n+1-i) \\ \text{(because } w_0(i)=n+1-i))}} \\
& \qquad \left(\begin{array}{c}\text{here, we have substituted } w \circ w_0 \text{ for } w \text{ in the sum,} \\ \text{since the map } \mathfrak{S}_n \to \mathfrak{S}_n,\ w \mapsto w \circ w_0 \text{ is a bijection}\end{array}\right) \\
& = \sum_{w \in \mathfrak{S}_n} (-1)^w (-1)^{w_0} \prod_{i \in \{1,2,\ldots,n\}} x_{w(n+1-i)}^{-\lambda_i-n+i} \\
& = (-1)^{w_0} \sum_{w \in \mathfrak{S}_n} (-1)^w \underbrace{\prod_{i \in \{1,2,\ldots,n\}} x_{w(n+1-i)}^{-\lambda_i-n+i}}_{\substack{=\prod_{i \in \{1,2,\ldots,n\}} x_{w(n+1-(n+1-i))}^{-\lambda_{n+1-i}-n+(n+1-i)} \\ \text{(here, we have substituted } n+1-i \text{ for } i \\ \text{in the product)}}} \\
& = (-1)^{w_0} \sum_{w \in \mathfrak{S}_n} (-1)^w \prod_{i \in \{1,2,\ldots,n\}} \underbrace{x_{w(n+1-(n+1-i))}^{-\lambda_{n+1-i}-n+(n+1-i)}}_{\substack{=x_{w(i)}^{-\lambda_{n+1-i}+1-i} \\ \text{(since } -\lambda_{n+1-i}-n+(n+1-i)=-\lambda_{n+1-i}+1-i \\ \text{and } n+1-(n+1-i)=i)}} \\
& = (-1)^{w_0} \underbrace{\sum_{w \in \mathfrak{S}_n} (-1)^w \prod_{i \in \{1,2,\ldots,n\}} x_{w(i)}^{-\lambda_{n+1-i}+1-i}}_{\substack{=a_{\mu+\rho} \\ \text{(by (34))}}} = (-1)^{w_0} \underbrace{a_{\mu+\rho}}_{\substack{=x_\Pi^{1-n} a_{\lambda^\vee+\rho} \\ \text{(by (31))}}} \\
& = (-1)^{w_0} x_\Pi^{1-n} a_{\lambda^\vee+\rho}.
\end{aligned} \tag{35}
$$

On the other hand, let us denote the snake $\left(\underbrace{0, 0, \ldots, 0}_{n \text{ times}}\right) \in \mathbb{Z}^n$ by $\varnothing$; note that it satisfies $\varnothing^\vee = \left(\underbrace{-0, -0, \ldots, -0}_{n \text{ times}}\right) = \left(\underbrace{0, 0, \ldots, 0}_{n \text{ times}}\right) = \varnothing$. We have proved (35) for any snake λ; thus, we can apply (35) to $\varnothing$ instead of λ. We thus obtain

$$
a_{\varnothing+\rho}\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) = (-1)^{w_0} x_\Pi^{1-n} a_{\varnothing^\vee+\rho}.
$$

In view of $\varnothing + \rho = \rho$ and $\underbrace{\varnothing^\vee}_{=\varnothing} + \rho = \varnothing + \rho = \rho$, we can rewrite this as

$$a_\rho \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right) = (-1)^{w_0} \, x_\Pi^{1-n} a_\rho. \tag{36}$$

The definition of $\overline{s}_\lambda$ yields $\overline{s}_\lambda = \dfrac{a_{\lambda+\rho}}{a_\rho}$. Hence, $a_{\lambda+\rho} = \overline{s}_\lambda a_\rho$. Substituting $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$ for $x_1, x_2, \ldots, x_n$ on both sides of this equality, we obtain

$$\begin{aligned}
a_{\lambda+\rho} \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right) &= \left(\overline{s}_\lambda a_\rho \right) \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right) \\
&= \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right) \cdot \underbrace{a_\rho \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right)}_{\substack{=(-1)^{w_0} x_\Pi^{1-n} a_\rho \\ \text{(by (36))}}} \\
&= \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right) \cdot (-1)^{w_0} \, x_\Pi^{1-n} a_\rho.
\end{aligned}$$

Comparing this with (35), we find

$$\overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right) \cdot (-1)^{w_0} \, x_\Pi^{1-n} a_\rho = (-1)^{w_0} \, x_\Pi^{1-n} a_{\lambda^\vee+\rho}. \tag{37}$$

The element $(-1)^{w_0} \, x_\Pi^{1-n}$ of $\mathcal{L}$ is invertible (since x_Π is invertible), and thus we can cancel it from the equality (37). As a result, we obtain

$$\overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right) \cdot a_\rho = a_{\lambda^\vee+\rho}. \tag{38}$$

But the definition of $\overline{s}_{\lambda^\vee}$ yields

$$\overline{s}_{\lambda^\vee} = \frac{a_{\lambda^\vee+\rho}}{a_\rho} = \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right)$$

(by (38)). This proves Lemma 4.17. □

4.2. h_k^+, h_k^- and the Pieri rule

Definition 4.18. Let $k \in \mathbb{Z}$. Then, we define two Laurent polynomials $h_k^+ \in \mathcal{L}$ and $h_k^- \in \mathcal{L}$ by

$$\begin{aligned}
h_k^+ &= h_k \left(x_1, x_2, \ldots, x_n \right) \qquad \text{and} \\
h_k^- &= h_k \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \right).
\end{aligned}$$

Note that if $k \in \mathbb{Z}$ is negative, then $h_k^+ = \underbrace{h_k}_{=0} \left(x_1, x_2, \ldots, x_n \right) = 0$ and $h_k^- = 0$ (similarly).

We begin by describing h_k^+ as a Schur Laurent polynomial:

Lemma 4.19. Let $k \in \mathbb{N}$. Then, the partition (k) is a nonnegative snake (when regarded as the n-tuple $(k, 0, 0, \ldots, 0)$), and satisfies

$$\overline{s}_{(k)} = h_k^+.$$

Proof of Lemma 4.19. The partition (k) has length $\leq n$ (since it has length ≤ 1, but we have $n \geq 1$). Thus, it is a nonnegative snake (since every partition having length $\leq n$ is a nonnegative snake) and belongs to $\operatorname{Par}[n]$. Hence, Lemma 4.13 (applied to $\lambda = (k)$) yields $\overline{s}_{(k)} = s_{(k)}(x_1, x_2, \ldots, x_n) = h_k(x_1, x_2, \ldots, x_n)$ (since $s_{(k)} = h_k$). Comparing this with $h_k^+ = h_k(x_1, x_2, \ldots, x_n)$, we obtain $\overline{s}_{(k)} = h_k^+$. This proves Lemma 4.19. $\square$

Next, we need to know what happens when a Schur Laurent polynomial $\overline{s}_\lambda$ is multiplied by some h_k^+. The answer to this question is classically given by the *first Pieri rule*; we shall state it in a form that will be most convenient to us. To do so, we introduce some more notation:

Definition 4.20. Let $\lambda \in \mathbb{Z}^n$. Then, we define the *size* $|\lambda|$ of λ to be the integer $\lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Definition 4.21. Let $\lambda, \mu \in \mathbb{Z}^n$. Then, we write that $\mu \rightharpoonup \lambda$ if and only if we have

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n. \tag{39}$$

In other words, we write that $\mu \rightharpoonup \lambda$ if and only if we have

$$\begin{aligned} &(\mu_i \geq \lambda_i \text{ for each } i \in \{1, 2, \ldots, n\}) \qquad \text{and} \\ &(\lambda_i \geq \mu_{i+1} \text{ for each } i \in \{1, 2, \ldots, n-1\}). \end{aligned}$$

The following properties of the sizes of n-tuples are obvious:

Proposition 4.22.

(a) If $\lambda, \mu \in \mathbb{Z}^n$, then $|\lambda + \mu| = |\lambda| + |\mu|$.

(b) If $\lambda \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$, then $|\lambda + d| = |\lambda| + nd$.

(c) If $\lambda \in \mathbb{Z}^n$, then $|\lambda^\vee| = -|\lambda|$.

The relation $\rightharpoonup$ defined in Definition 4.21 has the following simple properties:

Proposition 4.23. Let $\lambda, \mu \in \mathbb{Z}^n$.

(a) If $\mu \rightharpoonup \lambda$, then both λ and μ are snakes.

(b) We have $\mu \rightharpoonup \lambda$ if and only if $\lambda^\vee \rightharpoonup \mu^\vee$.

(c) Let $d \in \mathbb{Z}$. Then, we have $\mu \rightharpoonup \lambda$ if and only if $\mu + d \rightharpoonup \lambda + d$.

Proof of Proposition 4.23. **(a)** Assume that $\mu \rightharpoonup \lambda$. Thus, the chain of inequalities (39) holds (by Definition 4.21). But this chain of inequalities implies both $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Thus, μ and λ are snakes. This proves Proposition 4.23 **(a)**.

(b) The definition of $\mu^\vee$ yields $\mu^\vee = (-\mu_n, -\mu_{n-1}, \ldots, -\mu_1)$. Similarly, $\lambda^\vee = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)$. Hence, we have $\lambda^\vee \rightharpoonup \mu^\vee$ if and only if we have

$$-\lambda_n \geq -\mu_n \geq -\lambda_{n-1} \geq -\mu_{n-1} \geq \cdots \geq -\lambda_1 \geq -\mu_1$$

(because of Definition 4.21).

Thus, we have the following chain of equivalences:

$$\begin{aligned}
\left(\lambda^\vee \rightharpoonup \mu^\vee\right) &\iff \left(-\lambda_n \geq -\mu_n \geq -\lambda_{n-1} \geq -\mu_{n-1} \geq \cdots \geq -\lambda_1 \geq -\mu_1\right) \\
&\iff \left(\lambda_n \leq \mu_n \leq \lambda_{n-1} \leq \mu_{n-1} \leq \cdots \leq \lambda_1 \leq \mu_1\right) \\
&\iff \left(\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n\right) \\
&\iff \left(\mu \rightharpoonup \lambda\right) \qquad \text{(by Definition 4.21)}.
\end{aligned}$$

In other words, we have $\mu \rightharpoonup \lambda$ if and only if $\lambda^\vee \rightharpoonup \mu^\vee$. This proves Proposition 4.23 **(b)**.

(c) This follows easily from Definition 4.21. □

We can now state the Pieri rule in the form we need:

Proposition 4.24. Let λ be a snake. Let $k \in \mathbb{Z}$. Then,

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu| - |\lambda| = k}} \overline{s}_\mu. \tag{40}$$

This can be proven directly using alternants; but let us give a proof based on known theory:

Proof of Proposition 4.24. We follow the same right-shift strategy as we did in our proof of Lemma 4.11. Thus, our proof shall consist of two steps:

> *Step 1:* We will prove Proposition 4.24 in the particular case when λ is nonnegative.

Step 2: We will use Lemma 4.14 to derive the general case of Proposition 4.24 from this particular case.

Here are the details of the two steps:

Step 1: Let us prove that Proposition 4.24 holds in the particular case when λ is nonnegative.

Indeed, let us assume that λ is nonnegative. We must prove the equality (40).

If $k < 0$, then both sides of this equality are 0 [8]. Thus, the equality (40) holds if $k < 0$. Therefore, for the rest of Step 1, we WLOG assume that $k \geq 0$. In other words, $k \in \mathbb{N}$.

Note that λ is a partition of length $\leq n$ (since λ is a nonnegative snake). In other words, $\lambda \in \operatorname{Par}[n]$.

We will use some standard notations concerning partitions. Specifically:

- If $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ and $\beta = (\beta_1, \beta_2, \beta_3, \ldots)$ are two partitions, then we will write $\alpha \subseteq \beta$ if and only if each $i \in \{1, 2, 3, \ldots\}$ satisfies $\alpha_i \leq \beta_i$. (This is precisely the definition of $\alpha \subseteq \beta$ given in [GriRei20, Definition 2.3.1].)
- If $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ and $\beta = (\beta_1, \beta_2, \beta_3, \ldots)$ are two partitions, then we say that *α / β is a horizontal strip* if they satisfy
$$\beta \subseteq \alpha \text{ and } (\text{every } i \in \{1, 2, 3, \ldots\} \text{ satisfies } \beta_i \geq \alpha_{i+1}).$$
(This is not literally how a "horizontal strip" is defined in [GriRei20], but it is equivalent to that definition; the equivalence follows from [GriRei20, Exercise 2.7.5(a)].)
- If α and β are two partitions, and if $k \in \mathbb{N}$, then we say that *α / β is a horizontal k-strip* if α / β is a horizontal strip and we have $|\alpha| - |\beta| = k$. (This is equivalent to the definition of a "horizontal k-strip" in [GriRei20, §2.7]).

[8] *Proof.* Assume that $k < 0$. We must show that both sides of (40) are 0.

Indeed, from $k < 0$, we obtain $h_k = 0$, thus $h_k^+ = \underbrace{h_k}_{=0} (x_1, x_2, \ldots, x_n) = 0$. Hence, $\underbrace{h_k^+}_{=0} \cdot \overline{s}_\lambda = 0$. In other words, the left hand side of (40) is 0.

It remains to show that the right hand side of (40) is 0. This will follow if we can show that the sum on this right hand side is empty, i.e., that there exists no snake μ such that $\mu \rightharpoonup \lambda$ and $|\mu| - |\lambda| = k$. So let us show this.

Let μ be a snake such that $\mu \rightharpoonup \lambda$ and $|\mu| - |\lambda| = k$. We shall derive a contradiction.

From $\mu \rightharpoonup \lambda$, we obtain $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n$ (by the definition of "$\mu \rightharpoonup \lambda$"). Hence, $\mu_i \geq \lambda_i$ for each $i \in \{1, 2, \ldots, n\}$. Thus, $\sum_{i=1}^{n} \underbrace{\mu_i}_{\geq \lambda_i} \geq \sum_{i=1}^{n} \lambda_i$. In other words, $|\mu| \geq |\lambda|$ (since $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n = \sum_{i=1}^{n} \mu_i$ and similarly $|\lambda| = \sum_{i=1}^{n} \lambda_i$). Hence, $|\mu| - |\lambda| \geq 0$. But this contradicts $|\mu| - |\lambda| = k < 0$.

Forget that we fixed μ. We thus have found a contradiction whenever μ is a snake such that $\mu \rightharpoonup \lambda$ and $|\mu| - |\lambda| = k$. Hence, there exists no such snake μ. In other words, the sum on the right hand side of (40) is empty. Hence, the right hand side of (40) is 0. This completes our proof.

We note the following claim:

Claim 1: We have

$$\begin{aligned}&\{\text{partitions } \mu \in \operatorname{Par}[n] \text{ such that } \mu/\lambda \text{ is a horizontal } k\text{-strip}\}\\&= \{\text{snakes } \mu \text{ such that } \mu \rightharpoonup \lambda \text{ and } |\mu| - |\lambda| = k\}.\end{aligned}$$

[*Proof of Claim 1:* This is an exercise in unraveling definitions and recalling that partitions in Par $[n]$ are the same as nonnegative snakes. We leave the details to the reader (who can also find them expanded in the detailed version [Grinbe20] of this paper).]

From the first Pieri rule ([GriRei20, (2.7.1)][9], applied to k instead of n), we obtain

$$s_\lambda h_k = \sum_{\substack{\lambda^+ \in \operatorname{Par};\\ \lambda^+/\lambda \text{ is a horizontal } k\text{-strip}}} s_{\lambda^+} = \sum_{\substack{\mu \in \operatorname{Par};\\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} s_\mu$$

(here, we have renamed the summation index λ^+ as μ).

Evaluating both sides of this equality at $x_1, x_2, \ldots, x_n$, we find

$$\begin{aligned}&(s_\lambda h_k)(x_1, x_2, \ldots, x_n)\\&= \sum_{\substack{\mu \in \operatorname{Par};\\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} s_\mu(x_1, x_2, \ldots, x_n)\\&= \sum_{\substack{\mu \in \operatorname{Par};\\ \mu/\lambda \text{ is a horizontal } k\text{-strip};\\ \mu \text{ has length } \leq n}} s_\mu(x_1, x_2, \ldots, x_n) + \sum_{\substack{\mu \in \operatorname{Par};\\ \mu/\lambda \text{ is a horizontal } k\text{-strip};\\ \mu \text{ has length } > n}} \underbrace{s_\mu(x_1, x_2, \ldots, x_n)}_{\substack{=0\\ \text{(by (25),}\\ \text{applied to } \mu \text{ instead of } \lambda)}}\\&= \sum_{\substack{\mu \in \operatorname{Par};\\ \mu/\lambda \text{ is a horizontal } k\text{-strip};\\ \mu \text{ has length } \leq n}} s_\mu(x_1, x_2, \ldots, x_n) + \underbrace{\sum_{\substack{\mu \in \operatorname{Par};\\ \mu/\lambda \text{ is a horizontal } k\text{-strip};\\ \mu \text{ has length } > n}} 0}_{=0}\\&= \sum_{\substack{\mu \in \operatorname{Par};\\ \mu/\lambda \text{ is a horizontal } k\text{-strip};\\ \mu \text{ has length } \leq n}} s_\mu(x_1, x_2, \ldots, x_n).\end{aligned}$$

In view of

$$\begin{aligned}(s_\lambda h_k)(x_1, x_2, \ldots, x_n) &= \underbrace{s_\lambda(x_1, x_2, \ldots, x_n)}_{\substack{=\overline{s}_\lambda\\ \text{(by Lemma 4.13)}}} \cdot \underbrace{h_k(x_1, x_2, \ldots, x_n)}_{\substack{=h_k^+\\ \text{(since } h_k^+ = h_k(x_1, x_2, \ldots, x_n))}}\\&= \overline{s}_\lambda \cdot h_k^+ = h_k^+ \cdot \overline{s}_\lambda,\end{aligned}$$

[9]This also appears in [MenRem15, Theorem 5.3], in [Stanle01, Theorem 7.15.7] and in [Egge19, Theorem 9.3].

we can rewrite this as

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \operatorname{Par}; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}; \\ \mu \text{ has length } \leq n}} s_\mu \left(x_1, x_2, \ldots, x_n \right). \tag{41}$$

But we have the following equality of summation signs:

$$\sum_{\substack{\mu \in \operatorname{Par}; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}; \\ \mu \text{ has length } \leq n}} = \sum_{\substack{\mu \in \operatorname{Par}; \\ \mu \text{ has length } \leq n; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} = \sum_{\substack{\mu \in \operatorname{Par}[n]; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}}$$

$$\left(\begin{array}{c} \text{since the partitions } \mu \in \operatorname{Par} \text{ that have length } \leq n \\ \text{are precisely the partitions } \mu \in \operatorname{Par}[n] \end{array} \right).$$

Thus, we can rewrite (41) as

$$\begin{aligned} h_k^+ \cdot \overline{s}_\lambda &= \sum_{\substack{\mu \in \operatorname{Par}[n]; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} \underbrace{s_\mu \left(x_1, x_2, \ldots, x_n \right)}_{\substack{= \overline{s}_\mu \\ \text{(since Lemma 4.13} \\ \text{(applied to } \mu \text{ instead of } \lambda) \\ \text{yields } \overline{s}_\mu = s_\mu(x_1, x_2, \ldots, x_n))}} \\ &= \underbrace{\sum_{\substack{\mu \in \operatorname{Par}[n]; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}}}_{\substack{= \sum_{\substack{\mu \text{ is a snake}; \\ \mu \rightharpoonup \lambda;\ |\mu| - |\lambda| = k}} \\ \text{(by Claim 1)}}} \overline{s}_\mu = \sum_{\substack{\mu \text{ is a snake}; \\ \mu \rightharpoonup \lambda;\ |\mu| - |\lambda| = k}} \overline{s}_\mu. \end{aligned}$$

This proves (40). Thus, Proposition 4.24 is proved under the assumption that λ is nonnegative. This completes Step 1.

Step 2: Let us now prove Proposition 4.24 in the general case.

The snake λ may or may not be nonnegative. However, there exists some integer d such that the snake $\lambda + d$ is nonnegative[10]. Consider this d.

The map $\{\text{snakes}\} \to \{\text{snakes}\}$, $\mu \mapsto \mu + d$ is a bijection. (Indeed, its inverse is the map $\{\text{snakes}\} \to \{\text{snakes}\}$, $\mu \mapsto \mu - d$.) Moreover, every snake μ satisfies

$$\begin{aligned} \underbrace{|\mu + d|}_{\substack{= |\mu| + nd \\ \text{(by Proposition 4.22 } \textbf{(b)})}} - \underbrace{|\lambda + d|}_{\substack{= |\lambda| + nd \\ \text{(by Proposition 4.22 } \textbf{(b)})}} &= (|\mu| + nd) - (|\lambda| + nd) \\ &= |\mu| - |\lambda|. \end{aligned} \tag{42}$$

For any snake μ, we have the logical equivalence

$$(\mu \rightharpoonup \lambda) \iff (\mu + d \rightharpoonup \lambda + d)$$

[10] Indeed, this can be proved in the same way as it was proved during Step 2 of the proof of Lemma 4.11 above.

(by Proposition 4.23 **(c)**). In other words, for any snake μ, we have the logical equivalence

$$(\mu + d \rightharpoonup \lambda + d) \iff (\mu \rightharpoonup \lambda).$$

Hence, we have the following equality of summation signs:

$$\sum_{\substack{\mu \text{ is a snake;} \\ \mu+d\rightharpoonup\lambda+d;\ |\mu+d|-|\lambda+d|=k}} = \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda;\ |\mu+d|-|\lambda+d|=k}} = \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda;\ |\mu|-|\lambda|=k}} \tag{43}$$

(by (42)).

The snake $\lambda + d$ is nonnegative; thus, we can apply Proposition 4.24 to $\lambda + d$ instead of λ (because in Step 1, we have proved that Proposition 4.24 holds in the particular case when λ is nonnegative). Thus we conclude that

$$h_k^+ \cdot \overline{s}_{\lambda+d} = \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda+d;\ |\mu|-|\lambda+d|=k}} \overline{s}_\mu = \underbrace{\sum_{\substack{\mu \text{ is a snake;} \\ \mu+d\rightharpoonup\lambda+d;\ |\mu+d|-|\lambda+d|=k}}}_{\substack{= \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda;\ |\mu|-|\lambda|=k}} \\ \text{(by (43))}}} \underbrace{\overline{s}_{\mu+d}}_{\substack{=x_\Pi^d \overline{s}_\mu \\ \text{(by Lemma 4.14,} \\ \text{applied to } \mu \text{ instead of } \lambda)}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } \mu+d \text{ for } \mu \text{ in the sum,} \\ \text{since the map } \{\text{snakes}\} \to \{\text{snakes}\},\ \mu \mapsto \mu+d \\ \text{is a bijection}\end{array}\right)$$

$$= \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda;\ |\mu|-|\lambda|=k}} x_\Pi^d \overline{s}_\mu = x_\Pi^d \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu.$$

Comparing this with

$$h_k^+ \cdot \underbrace{\overline{s}_{\lambda+d}}_{\substack{=x_\Pi^d \overline{s}_\lambda \\ \text{(by Lemma 4.14)}}} = h_k^+ \cdot x_\Pi^d \overline{s}_\lambda,$$

we obtain

$$h_k^+ \cdot x_\Pi^d \overline{s}_\lambda = x_\Pi^d \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu.$$

We can divide both sides of this equality by x_Π^d (since $x_\Pi^d \in \mathcal{L}$ is invertible (because $x_\Pi \in \mathcal{L}$ is invertible)), and thus obtain

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \text{ is a snake;} \\ \mu\rightharpoonup\lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu.$$

This proves Proposition 4.24. Thus, Step 2 is complete, and Proposition 4.24 is proven. $\square$

Using Lemma 4.17, we can "turn Proposition 4.24 upside down", obtaining the following analogous result for h_k^- instead of h_k^+:

Proposition 4.25. Let λ be a snake. Let $k \in \mathbb{Z}$. Then,

$$h_k^- \cdot \overline{s}_\lambda = \sum_{\substack{\mu \text{ is a snake;} \\ \lambda \rightharpoonup \mu;\ |\lambda| - |\mu| = k}} \overline{s}_\mu . \tag{44}$$

Proof of Proposition 4.25. It is easy to see (from Definition 4.3 **(d)**) that $\left(\lambda^\vee\right)^\vee = \lambda$. Likewise, $\left(\mu^\vee\right)^\vee = \mu$ for any snake μ. Hence, the map $\{\text{snakes}\} \to \{\text{snakes}\}$, $\mu \mapsto \mu^\vee$ (which is well-defined because of Proposition 4.5 **(b)**) is inverse to itself. Thus, this map is a bijection.

If μ is a snake, then

$$\text{we have } \mu \rightharpoonup \lambda \text{ if and only if } \lambda^\vee \rightharpoonup \mu^\vee \tag{45}$$

(by Proposition 4.23 **(b)**). Moreover, if μ is a snake, then

$$\begin{aligned} &\underbrace{\left|\lambda^\vee\right|}_{\substack{= -|\lambda| \\ \text{(by Proposition 4.22 (c))}}} - \underbrace{\left|\mu^\vee\right|}_{\substack{= -|\mu| \\ \text{(by Proposition 4.22 (c))}}} \\ &= \left(-\left|\lambda\right|\right) - \left(-\left|\mu\right|\right) = \left|\mu\right| - \left|\lambda\right| . \end{aligned} \tag{46}$$

But comparing the definitions of h_k^- and h_k^+ easily yields

$$h_k^- = h_k^+ \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right).$$

Also, Lemma 4.17 yields $\overline{s}_{\lambda^\vee} = \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right)$. Multiplying these two equalities, we obtain

$$\begin{aligned} h_k^- \cdot \overline{s}_{\lambda^\vee} &= h_k^+ \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \cdot \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \\ &= \underbrace{\left(h_k^+ \cdot \overline{s}_\lambda\right)}_{\substack{= \sum\limits_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu| - |\lambda| = k}} \overline{s}_\mu \\ \text{(by Proposition 4.24)}}} \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \\ &= \underbrace{\sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu| - |\lambda| = k}}}_{\substack{= \sum\limits_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoonup \mu^\vee;\ |\mu| - |\lambda| = k}} \\ \text{(by (45))}}} \overline{s}_\mu \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \\ &= \sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoonup \mu^\vee;\ |\mu| - |\lambda| = k}} \overline{s}_\mu \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) . \end{aligned}$$

Comparing this with

$$\sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoondown \mu;\ |\lambda^\vee|-|\mu|=k}} \overline{s}_\mu = \underbrace{\sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoondown \mu^\vee;\ |\lambda^\vee|-|\mu^\vee|=k}}}_{\substack{= \sum \\ \mu \text{ is a snake;} \\ \lambda^\vee \rightharpoondown \mu^\vee;\ |\mu|-|\lambda|=k \\ \text{(by (46))}}} \underbrace{\overline{s}_{\mu^\vee}}_{\substack{=\overline{s}_\mu\left(x_1^{-1},x_2^{-1},\ldots,x_n^{-1}\right) \\ \text{(by Lemma 4.17,} \\ \text{applied to } \mu \text{ instead of } \lambda)}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } \mu^\vee \text{ for } \mu \text{ in the sum, since} \\ \text{the map } \{\text{snakes}\} \to \{\text{snakes}\},\ \mu \mapsto \mu^\vee \text{ is a bijection}\end{array}\right)$$

$$= \sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoondown \mu^\vee;\ |\mu|-|\lambda|=k}} \overline{s}_\mu\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right),$$

we obtain

$$h_k^- \cdot \overline{s}_{\lambda^\vee} = \sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoondown \mu;\ |\lambda^\vee|-|\mu|=k}} \overline{s}_\mu.$$

We have proved this equality for any snake λ. Thus, we can apply it to $\lambda^\vee$ instead of λ (since Proposition 4.5 **(b)** shows that $\lambda^\vee$ is a snake). We thus obtain

$$h_k^- \cdot \overline{s}_{(\lambda^\vee)^\vee} = \sum_{\substack{\mu \text{ is a snake;} \\ (\lambda^\vee)^\vee \rightharpoondown \mu;\ \left|(\lambda^\vee)^\vee\right|-|\mu|=k}} \overline{s}_\mu.$$

In view of $(\lambda^\vee)^\vee = \lambda$, this can be rewritten as follows:

$$h_k^- \cdot \overline{s}_\lambda = \sum_{\substack{\mu \text{ is a snake;} \\ \lambda \rightharpoondown \mu;\ |\lambda|-|\mu|=k}} \overline{s}_\mu.$$

This proves Proposition 4.25. □

4.3. Computing $\overline{s}_\alpha$

Convention 4.26. From now on, for the rest of Section 4, we assume that $n \geq 2$.

Our next goal is to obtain a simple formula for the Schur polynomial $\overline{s}_\alpha(x_1, x_2, \ldots, x_n)$, where α is as in Theorem 2.3. The first step is the following definition:

Definition 4.27. Let $a, b \in \mathbb{N}$. Then, $b \ominus a$ will denote the snake $\left(b, 0^{n-2}, -a\right)$. (This is indeed a well-defined snake, since $n \geq 2$ and since $b \geq 0 \geq -a$.)

Proposition 4.28. Let $a, b \in \mathbb{Z}$. Then,

$$h_a^- h_b^+ = \sum_{k=0}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)}. \tag{47}$$

Proof of Proposition 4.28. We must prove the equality (47). If (at least) one of the integers a and b is negative, then this equality boils down to $0 = 0$ [11]. Hence, for the rest of this proof, we WLOG assume that none of the integers a and b is negative. Hence, $a, b \in \mathbb{N}$.

Note that each $k \in \{0, 1, \ldots, \min\{a, b\}\}$ satisfies $b - k \in \mathbb{N}$ (since $k \leq \min\{a, b\} \leq b$) and $a - k \in \mathbb{N}$ (likewise). Hence, the snakes $(b - k) \ominus (a - k)$ on the right hand side of the equality (47) are well-defined.

Lemma 4.19 (applied to $k = b$) yields that the partition (b) is a nonnegative snake (when regarded as the n-tuple $(b, 0, 0, \ldots, 0)$), and satisfies $\overline{s}_{(b)} = h_b^+$.

Now, Proposition 4.25 (applied to $\lambda = (b)$ and $k = a$) yields

$$h_a^- \cdot \overline{s}_{(b)} = \sum_{\substack{\mu \text{ is a snake;} \\ (b) \rightharpoonup \mu;\ |(b)|-|\mu|=a}} \overline{s}_\mu .$$

In view of $\overline{s}_{(b)} = h_b^+$ and $|(b)| = b$, we can rewrite this as

$$h_a^- \cdot h_b^+ = \sum_{\substack{\mu \text{ is a snake;} \\ (b) \rightharpoonup \mu;\ b-|\mu|=a}} \overline{s}_\mu . \tag{48}$$

Now, we claim the following:

> *Claim 1:* The snakes μ satisfying $(b) \rightharpoonup \mu$ and $b - |\mu| = a$ are precisely the snakes of the form $(b - k) \ominus (a - k)$ for $k \in \{0, 1, \ldots, \min\{a, b\}\}$.

[*Proof of Claim 1:* The proof of this claim is straightforward (and can be found elaborated in the detailed version [Grinbe20] of this paper).]

Now, Claim 1 shows that

$$\sum_{\substack{\mu \text{ is a snake;} \\ (b) \rightharpoonup \mu;\ b-|\mu|=a}} \overline{s}_\mu = \sum_{k \in \{0,1,\ldots,\min\{a,b\}\}} \overline{s}_{(b-k)\ominus(a-k)}$$

(indeed, it is clear that the snakes $(b - k) \ominus (a - k)$ for $k \in \{0, 1, \ldots, \min\{a, b\}\}$ are all distinct). Hence, (48) becomes

$$h_a^- \cdot h_b^+ = \sum_{\substack{\mu \text{ is a snake;} \\ (b) \rightharpoonup \mu;\ b-|\mu|=a}} \overline{s}_\mu = \sum_{k \in \{0,1,\ldots,\min\{a,b\}\}} \overline{s}_{(b-k)\ominus(a-k)} = \sum_{k=0}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)}.$$

[11] Indeed, its left hand side is 0 in this case because every negative integer k satisfies $h_k^- = 0$ and $h_k^+ = 0$; but its right hand side is also 0 in this case, because the negativity of $\min\{a, b\}$ causes the sum to become empty.

This proves Proposition 4.28. $\square$

Proposition 4.29. Let $a, b \in \mathbb{N}$. Then,

$$\overline{s}_{b \ominus a} = h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+.$$

(Recall that every negative integer k satisfies $h_k^- = 0$ and $h_k^+ = 0$.)

Proof of Proposition 4.29. We have $\min\{a, b\} \in \mathbb{N}$ (since $a, b \in \mathbb{N}$), so that $\min\{a, b\} \geq 0$.

Proposition 4.28 yields

$$\begin{aligned}
h_a^- h_b^+ &= \sum_{k=0}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)} = \underbrace{\overline{s}_{(b-0)\ominus(a-0)}}_{=\overline{s}_{b\ominus a}} + \sum_{k=1}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend for } k=0 \text{ from the sum,} \\ \text{since } \min\{a,b\} \geq 0\end{array}\right) \\
&= \overline{s}_{b\ominus a} + \sum_{k=1}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)}.
\end{aligned} \tag{49}$$

Proposition 4.28 (applied to $a-1$ and $b-1$ instead of a and b) yields

$$\begin{aligned}
h_{a-1}^- h_{b-1}^+ &= \sum_{k=0}^{\min\{a-1,b-1\}} \overline{s}_{((b-1)-k)\ominus((a-1)-k)} = \sum_{k=0}^{\min\{a,b\}-1} \overline{s}_{((b-1)-k)\ominus((a-1)-k)} \\
&\qquad \left(\text{since } \min\{a-1, b-1\} = \min\{a,b\} - 1\right) \\
&= \sum_{k=1}^{\min\{a,b\}} \overline{s}_{((b-1)-(k-1))\ominus((a-1)-(k-1))} \\
&\qquad \left(\text{here, we have substituted } k-1 \text{ for } k \text{ in the sum}\right) \\
&= \sum_{k=1}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)} \\
&\qquad \left(\text{since } (b-1)-(k-1) = b-k \text{ and } (a-1)-(k-1) = a-k\right).
\end{aligned}$$

Subtracting this equality from (49), we obtain

$$h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+ = \left(\overline{s}_{b\ominus a} + \sum_{k=1}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)}\right) - \sum_{k=1}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)} = \overline{s}_{b\ominus a}.$$

This proves Proposition 4.29. $\square$

Corollary 4.30. Let $a, b \in \mathbb{N}$. Define the partition $\alpha = \left(a+b, a^{n-2}\right)$. Then, α is a nonnegative snake and satisfies

$$\overline{s}_\alpha = x_\Pi^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right).$$

Proof of Corollary 4.30. From $a, b \in \mathbb{N}$, we obtain $a + \underbrace{b}_{\geq 0} \geq a \geq 0$. Hence, α is a partition of length $\leq n$ (since $n-1 \leq n$). In other words, α is a nonnegative snake.

It is easy to see that $\alpha = (b \ominus a) + a$ (since $\alpha = \left(a+b, a^{n-2}\right) = \left(a+b, \underbrace{a, a, \ldots, a}_{n-2 \text{ times}}, 0\right)$ and $b \ominus a = \left(b, 0^{n-2}, -a\right) = \left(b, \underbrace{0, 0, \ldots, 0}_{n-2 \text{ times}}, -a\right)$). Hence,

$$\begin{aligned} \overline{s}_\alpha &= \overline{s}_{(b\ominus a)+a} = x_\Pi^a \overline{s}_{b\ominus a} && \text{(by Lemma 4.14, applied to } \lambda = b \ominus a \text{ and } d = a) \\ &= x_\Pi^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right) && \text{(by Proposition 4.29)}. \end{aligned}$$

This proves Corollary 4.30. □

We notice that the expression $h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+$ in Proposition 4.29 can be rewritten as $\det \begin{pmatrix} h_a^- & h_{a-1}^- \\ h_{b-1}^+ & h_b^+ \end{pmatrix}$. This suggests a way to generalize Proposition 4.29 (as well as the first Jacobi–Trudi formula [GriRei20, (2.4.16)]). Such a generalization indeed exists, and has been proved by Koike as well as by Hamel and King; see Proposition 5.1 below.

4.4. The sets $R_{\mu,a,b}(\gamma)$ and a formula for $h_a^- h_b^+ \overline{s}_\mu$

We shall next aim for a formula for $h_a^- h_b^+ \overline{s}_\mu$ (for a snake μ and integers $a, b \in \mathbb{Z}$), which will be obtained in a straightforward way by applying Propositions 4.24 and 4.25. We will need the following definition:

Definition 4.31. Let $\mu, \gamma \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$. Then, $R_{\mu,a,b}(\gamma)$ shall denote the set of all snakes ν satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = a \qquad \text{and} \qquad \gamma \rightharpoonup \nu \qquad \text{and} \qquad |\gamma| - |\nu| = b.$$

Lemma 4.32. Let $\mu, \gamma \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$. Assume that γ is not a snake. Then, $\left|R_{\mu,a,b}(\gamma)\right| = 0$.

Proof of Lemma 4.32. Let $\nu \in R_{\mu,a,b}(\gamma)$. We shall obtain a contradiction.

Indeed, we have $\nu \in R_{\mu,a,b}(\gamma)$. In other words, ν is a snake satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = a \qquad \text{and} \qquad \gamma \rightharpoonup \nu \qquad \text{and} \qquad |\gamma| - |\nu| = b$$

(by the definition of $R_{\mu,a,b}(\gamma)$). Thus, in particular, we have $\gamma \rightharpoonup \nu$. Hence, Proposition 4.23 **(a)** (applied to γ and ν instead of μ and λ) yields that both ν and γ are snakes. Hence, γ is a snake. This contradicts the fact that γ is not a snake.

Now, forget that we fixed ν. We thus have obtained a contradiction for each $\nu \in R_{\mu,a,b}(\gamma)$. Hence, there exists no $\nu \in R_{\mu,a,b}(\gamma)$. In other words, the set $R_{\mu,a,b}(\gamma)$ is empty. Thus, $\left|R_{\mu,a,b}(\gamma)\right| = 0$. This proves Lemma 4.32. □

Lemma 4.33. Let μ be a snake. Let $a, b \in \mathbb{Z}$. Then,

$$h_a^- h_b^+ \overline{s}_\mu = \sum_{\gamma \text{ is a snake}} \left|R_{\mu,a,b}(\gamma)\right| \overline{s}_\gamma.$$

Proof of Lemma 4.33. Proposition 4.25 (with the letters λ, k and μ renamed as μ, a and ν) says that

$$h_a^- \cdot \overline{s}_\mu = \sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoonup \nu;\ |\mu| - |\nu| = a}} \overline{s}_\nu. \tag{50}$$

Proposition 4.24 (with the letters λ, k and μ renamed as ν, b and γ) says that

$$h_b^+ \cdot \overline{s}_\nu = \sum_{\substack{\gamma \text{ is a snake;} \\ \gamma \rightharpoonup \nu;\ |\gamma| - |\nu| = b}} \overline{s}_\gamma \tag{51}$$

for each snake ν.

Now,

$$\begin{aligned}
h_a^- h_b^+ \overline{s}_\mu &= h_b^+ \cdot \underbrace{h_a^- \cdot \overline{s}_\mu}_{\substack{= \sum\limits_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoonup \nu;\ |\mu|-|\nu|=a}} \overline{s}_\nu \\ \text{(by (50))}}} = h_b^+ \cdot \sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoonup \nu;\ |\mu|-|\nu|=a}} \overline{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoonup \nu;\ |\mu|-|\nu|=a}} \underbrace{h_b^+ \cdot \overline{s}_\nu}_{\substack{= \sum\limits_{\substack{\gamma \text{ is a snake;} \\ \gamma \rightharpoonup \nu;\ |\gamma|-|\nu|=b}} \overline{s}_\gamma \\ \text{(by (51))}}} = \underbrace{\sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoonup \nu;\ |\mu|-|\nu|=a}} \sum_{\substack{\gamma \text{ is a snake;} \\ \gamma \rightharpoonup \nu;\ |\gamma|-|\nu|=b}}}_{= \sum\limits_{\gamma \text{ is a snake}} \sum\limits_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoonup \nu;\ |\mu|-|\nu|=a; \\ \gamma \rightharpoonup \nu;\ |\gamma|-|\nu|=b}}} \overline{s}_\gamma \\
&= \sum_{\gamma \text{ is a snake}} \underbrace{\sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoonup \nu;\ |\mu|-|\nu|=a; \\ \gamma \rightharpoonup \nu;\ |\gamma|-|\nu|=b}}}_{\substack{= \sum\limits_{\nu \in R_{\mu,a,b}(\gamma)} \\ \text{(by the definition of } R_{\mu,a,b}(\gamma))}} \overline{s}_\gamma = \sum_{\gamma \text{ is a snake}} \underbrace{\sum_{\nu \in R_{\mu,a,b}(\gamma)} \overline{s}_\gamma}_{= |R_{\mu,a,b}(\gamma)| \overline{s}_\gamma} \\
&= \sum_{\gamma \text{ is a snake}} \left| R_{\mu,a,b} \left(\gamma \right) \right| \overline{s}_\gamma .
\end{aligned}$$

This proves Lemma 4.33. □

Corollary 4.34. Let $\mu \in \operatorname{Par}[n]$. Let $a, b \in \mathbb{N}$. Define the partition $\alpha = \left(a+b, a^{n-2}\right)$. Then, every $\lambda \in \mathbb{Z}^n$ satisfies

$$c^{\lambda}_{\alpha,\mu} = \left| R_{\mu,a,b} \left(\lambda - a \right) \right| - \left| R_{\mu,a-1,b-1} \left(\lambda - a \right) \right| . \tag{52}$$

Here, we understand $c^{\lambda}_{\alpha,\mu}$ to mean 0 if λ is not a partition (i.e., if λ is not a nonnegative snake).

Proof of Corollary 4.34. Corollary 4.30 shows that α is a nonnegative snake and satisfies

$$\overline{s}_\alpha = x_\Pi^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+ \right) . \tag{53}$$

Every snake γ satisfies

$$\overline{s}_{\gamma+a} = x_\Pi^a \overline{s}_\gamma \tag{54}$$

(by Lemma 4.14, applied to γ and a instead of λ and d).

We have $\alpha \in \operatorname{Par}[n]$ (since α is a nonnegative snake). Hence, Lemma 4.15 (applied to α and μ instead of μ and ν) yields

$$\begin{aligned}
\overline{s}_\alpha \overline{s}_\mu &= \underbrace{\sum_{\lambda \in \operatorname{Par}[n]}}_{\substack{= \sum\limits_{\substack{\lambda \text{ is a snake;} \\ \lambda \text{ is nonnegative}}} \\ \text{(since the partitions } \lambda \in \operatorname{Par}[n] \text{ are} \\ \text{precisely the nonnegative snakes)}}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda = \sum_{\substack{\lambda \text{ is a snake;} \\ \lambda \text{ is nonnegative}}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda \\
&= \sum_{\lambda \text{ is a snake}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda - \sum_{\substack{\lambda \text{ is a snake;} \\ \lambda \text{ is not nonnegative}}} \underbrace{c^\lambda_{\alpha,\mu}}_{\substack{=0 \\ \text{(since we understand } c^\lambda_{\alpha,\mu} \\ \text{to mean 0 if } \lambda \text{ is not} \\ \text{a nonnegative snake)}}} \overline{s}_\lambda \\
&= \sum_{\lambda \text{ is a snake}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda - \underbrace{\sum_{\substack{\lambda \text{ is a snake;} \\ \lambda \text{ is not nonnegative}}} 0 \overline{s}_\lambda}_{=0} = \sum_{\lambda \text{ is a snake}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda .
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{\lambda \text{ is a snake}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda &= \overline{s}_\alpha \overline{s}_\mu = x^a_\Pi \cdot \left(h^-_a h^+_b - h^-_{a-1} h^+_{b-1} \right) \overline{s}_\mu \qquad \text{(by (53))} \\
&= x^a_\Pi \cdot \underbrace{h^-_a h^+_b \overline{s}_\mu}_{\substack{= \sum\limits_{\gamma \text{ is a snake}} \left| R_{\mu,a,b}(\gamma) \right| \overline{s}_\gamma \\ \text{(by Lemma 4.33)}}} - x^a_\Pi \cdot \underbrace{h^-_{a-1} h^+_{b-1} \overline{s}_\mu}_{\substack{= \sum\limits_{\gamma \text{ is a snake}} \left| R_{\mu,a-1,b-1}(\gamma) \right| \overline{s}_\gamma \\ \text{(by Lemma 4.33,} \\ \text{applied to } a-1 \text{ and } b-1 \\ \text{instead of } a \text{ and } b)}} \\
&= x^a_\Pi \cdot \sum_{\gamma \text{ is a snake}} \left| R_{\mu,a,b}(\gamma) \right| \overline{s}_\gamma - x^a_\Pi \cdot \sum_{\gamma \text{ is a snake}} \left| R_{\mu,a-1,b-1}(\gamma) \right| \overline{s}_\gamma \\
&= \sum_{\gamma \text{ is a snake}} \left(\left| R_{\mu,a,b}(\gamma) \right| - \left| R_{\mu,a-1,b-1}(\gamma) \right| \right) \underbrace{x^a_\Pi \overline{s}_\gamma}_{\substack{= \overline{s}_{\gamma + a} \\ \text{(by (54))}}} \\
&= \sum_{\gamma \text{ is a snake}} \left(\left| R_{\mu,a,b}(\gamma) \right| - \left| R_{\mu,a-1,b-1}(\gamma) \right| \right) \overline{s}_{\gamma + a} \\
&= \sum_{\lambda \text{ is a snake}} \left(\left| R_{\mu,a,b}(\lambda - a) \right| - \left| R_{\mu,a-1,b-1}(\lambda - a) \right| \right) \underbrace{\overline{s}_{(\lambda - a) + a}}_{= \overline{s}_\lambda} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } \lambda - a \text{ for } \gamma \text{ in the sum,} \\ \text{since the map } \{\text{snakes}\} \to \{\text{snakes}\},\ \lambda \mapsto \lambda - a \\ \text{is a bijection} \end{array} \right) \\
&= \sum_{\lambda \text{ is a snake}} \left(\left| R_{\mu,a,b}(\lambda - a) \right| - \left| R_{\mu,a-1,b-1}(\lambda - a) \right| \right) \overline{s}_\lambda .
\end{aligned}$$

We can compare coefficients on both sides of this equality (since Lemma 4.16 shows that the family $(\overline{s}_\lambda)_{\lambda \in \{\text{snakes}\}}$ of elements of $\mathcal{L}$ is $\mathbf{k}$-linearly independent), and thus conclude that

$$c_{\alpha,\mu}^{\lambda} = \left|R_{\mu,a,b}(\lambda - a)\right| - \left|R_{\mu,a-1,b-1}(\lambda - a)\right| \qquad \text{for every snake } \lambda.$$

This proves (52) in the case when λ is a snake.

However, it is easy to see that (52) also holds in the case when λ is not a snake[12]. Thus, (52) always holds. This proves Corollary 4.34. □

4.5. The map $\mathbf{f}_{\mu}$

Convention 4.35. For the whole Subsection 4.5, we shall use Convention 3.6 (not only for n-tuples $a \in \mathbb{K}^n$, but for any n-tuples a). This convention does not conflict with Convention 4.2, because both conventions define γ_i in the same way when γ is an n-tuple and $i \in \{1, 2, \ldots, n\}$ (whereas the latter convention does not define γ_i for any other values of i).

Convention 3.6 does conflict with our old convention (from Section 1) to identify partitions with finite tuples: Indeed, if we let γ be the n-tuple $\left(\underbrace{1, 1, \ldots, 1}_{n \text{ times}}\right)$, then Convention 3.6 yields $\gamma_{n+1} = \gamma_1 = 1$ when we regard γ as an n-tuple, but we get $\gamma_{n+1} = 0$ if we regard γ as a partition. We shall resolve this conflict by agreeing **not to identify partitions with finite tuples in Subsection 4.5**. (Thus, in particular, we will not identify a nonnegative snake $(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n$ with its corresponding partition $(\mu_1, \mu_2, \ldots, \mu_n, 0, 0, 0, \ldots) \in \operatorname{Par}[n]$.)

Let us now apply the results of Section 3. The abelian group $(\mathbb{Z}, +, 0)$ of integers is totally ordered (in the usual way). Thus, Example 3.3 (applied to $(\mathbb{A}, *, e) = (\mathbb{Z}, +, 0)$) shows that there is a semifield $(\mathbb{Z}, \min, +, 0)$ (that is, a semifield with ground set $\mathbb{Z}$, addition min, multiplication $+$ and unity 0), called the *min tropical semifield* of $(\mathbb{Z}, +, 0)$. We have the following little dictionary between various operations on this semifield $(\mathbb{Z}, \min, +, 0)$ and familiar operations on integers:

- The addition operation of the semifield $(\mathbb{Z}, \min, +, 0)$ is the binary operation min on $\mathbb{Z}$. That is, for any $a, b \in \mathbb{Z}$, the sum $a + b$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is precisely the integer $\min\{a, b\}$.

[12] *Proof.* Let $\lambda \in \mathbb{Z}^n$ be such that λ is not a snake. We must show that (52) holds for this λ.

We have assumed that λ is not a snake. Hence, $\lambda - a$ is not a snake (because it is easy to see from Definition 4.3 that λ is a snake if and only if $\lambda - a$ is a snake). Thus, Lemma 4.32 (applied to $\gamma = \lambda - a$) yields $\left|R_{\mu,a,b}(\lambda - a)\right| = 0$. Also, Lemma 4.32 (applied to $\lambda - a$, $a - 1$ and $b - 1$ instead of γ, a and b) yields $\left|R_{\mu,a-1,b-1}(\lambda - a)\right| = 0$. On the other hand, λ is not a snake, and thus not a nonnegative snake. Hence, $c_{\alpha,\mu}^{\lambda} = 0$ (since we have defined $c_{\alpha,\mu}^{\lambda}$ to be 0 if λ is not a nonnegative snake). Comparing this with $\underbrace{\left|R_{\mu,a,b}(\lambda - a)\right|}_{=0} - \underbrace{\left|R_{\mu,a-1,b-1}(\lambda - a)\right|}_{=0} = 0$, we obtain $c_{\alpha,\mu}^{\lambda} = \left|R_{\mu,a,b}(\lambda - a)\right| - \left|R_{\mu,a-1,b-1}(\lambda - a)\right|$. In other words, (52) holds.

Thus, we have shown that (52) holds in the case when λ is not a snake.

- Thus, (nonempty) finite sums in the semifield $(\mathbb{Z}, \min, +, 0)$ are minima of finite sets of integers. That is, if $r \in \mathbb{N}$, and if $a_0, a_1, \ldots, a_r$ are any $r+1$ integers, then the sum $\sum\limits_{k=0}^{r} a_k$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is $\min \{a_0, a_1, \ldots, a_r\} = \min \{a_k \mid k \in \{0, 1, \ldots, r\}\}$.

- Furthermore, the multiplication operation of the semifield $(\mathbb{Z}, \min, +, 0)$ is the addition $+$ of integers. That is, for any $a, b \in \mathbb{Z}$, the product ab understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is precisely the sum $a+b$ understood with respect to the integer ring $\mathbb{Z}$. Meanwhile, the unity of the semifield $(\mathbb{Z}, \min, +, 0)$ is the integer 0.

- Thus, the division operation of the semifield $(\mathbb{Z}, \min, +, 0)$ is the subtraction $-$ of integers. That is, for any $a, b \in \mathbb{Z}$, the quotient $\dfrac{a}{b}$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is precisely the difference $a - b$ understood with respect to the integer ring $\mathbb{Z}$.

- For the same reason, squaring an element of the semifield $(\mathbb{Z}, \min, +, 0)$ is tantamount to doubling it as an integer. That is, for any $a \in \mathbb{Z}$, the square a^2 understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the product $2a$ understood with respect to the integer ring $\mathbb{Z}$.

- For the same reason, taking reciprocals in the semifield $(\mathbb{Z}, \min, +, 0)$ is tantamount to negation of integers. That is, for any $a \in \mathbb{Z}$, the reciprocal $\dfrac{1}{a}$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the integer $-a$ understood with respect to the integer ring $\mathbb{Z}$.

- For the same reason, finite products in the semifield $(\mathbb{Z}, \min, +, 0)$ are sums of integers. That is, if $r \in \mathbb{N}$, and if $a_1, a_2, \ldots, a_r$ are any r integers, then the product $\prod\limits_{k=1}^{r} a_k$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the sum $\sum\limits_{k=1}^{r} a_k$ understood with respect to the integer ring $\mathbb{Z}$.

Thus, applying Definition 3.8 to $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ (and renaming everything[13]), we obtain the following:

---

[13]Namely, we are

- renaming the (fixed) n-tuple u as μ;
- renaming the (variable) n-tuple x as γ (in order to avoid a clash with the variables $x_1, x_2, \ldots, x_n$);
- renaming the elements $t_{r,j}$ as $\tau_{r,j}$;
- renaming the n-tuple y as η.

Definition 4.36. Fix any n-tuple $\mu \in \mathbb{Z}^n$.

We define a map $\mathbf{f}_\mu : \mathbb{Z}^n \to \mathbb{Z}^n$ as follows:

Let $\gamma \in \mathbb{Z}^n$ be an n-tuple. For each $j \in \mathbb{Z}$ and $r \in \mathbb{N}$, define an element $\tau_{r,j} \in \mathbb{Z}$ by

$$\tau_{r,j} = \min \left\{ \underbrace{\gamma_{j+1} + \gamma_{j+2} + \cdots + \gamma_{j+k}}_{= \sum_{i=1}^{k} \gamma_{j+i}} + \underbrace{\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+r}}_{= \sum_{i=k+1}^{r} \mu_{j+i}} \mid k \in \{0, 1, \ldots, r\} \right\}.$$

Define $\eta \in \mathbb{Z}^n$ by setting

$$\eta_i = \mu_i + (\mu_{i-1} + \tau_{n-1,i-1}) - (\gamma_{i+1} + \tau_{n-1,i+1}) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Set $\mathbf{f}_\mu(\gamma) = \eta$.

Applying Theorem 3.11 to $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ (and renaming everything[14], and using our above dictionary), we thus obtain the following:

Theorem 4.37. Fix any n-tuple $\mu \in \mathbb{Z}^n$.

(a) The map $\mathbf{f}_\mu$ is an involution (i.e., we have $\mathbf{f}_\mu \circ \mathbf{f}_\mu = \mathrm{id}$).

(b) Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then,

$$(\eta_1 + \eta_2 + \cdots + \eta_n) + (\gamma_1 + \gamma_2 + \cdots + \gamma_n) = 2(\mu_1 + \mu_2 + \cdots + \mu_n).$$

(c) Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then,

$$\min\{\mu_i, \gamma_i\} + \min\{-\mu_{i+1}, -\gamma_{i+1}\} = \min\{\mu_i, \eta_i\} + \min\{-\mu_{i+1}, -\eta_{i+1}\}$$

for each $i \in \mathbb{Z}$.

(d) Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then,

$$\sum_{i=1}^{n} (\min\{\mu_i, \gamma_i\} - \gamma_i) = \sum_{i=1}^{n} (\min\{\mu_i, \eta_i\} - \mu_i).$$

[14] Namely, we are

- renaming the (fixed) n-tuple u as μ;
- renaming the (variable) n-tuple x as γ;
- renaming the n-tuple y as η.

We obtain the following corollaries from Theorem 4.37:

Corollary 4.38. Fix any n-tuple $\mu \in \mathbb{Z}^n$. Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then:

(a) We have $|\eta| - |\mu| = |\mu| - |\gamma|$.

(b) We have

$$\min\{\mu_i, \eta_i\} - \min\{\mu_i, \gamma_i\} = \max\{\mu_{i+1}, \eta_{i+1}\} - \max\{\mu_{i+1}, \gamma_{i+1}\}$$

for each $i \in \{1, 2, \ldots, n-1\}$.

(c) We have

$$\sum_{i=1}^{n} (\mu_i - \min\{\mu_i, \eta_i\} + \min\{\mu_i, \gamma_i\}) = \sum_{i=1}^{n} \gamma_i.$$

(d) We have $\gamma = \mathbf{f}_\mu(\eta)$.

Proof of Corollary 4.38. **(a)** Theorem 4.37 **(b)** yields

$$(\eta_1 + \eta_2 + \cdots + \eta_n) + (\gamma_1 + \gamma_2 + \cdots + \gamma_n) = 2(\mu_1 + \mu_2 + \cdots + \mu_n).$$

In view of the equalities

$$|\eta| = \eta_1 + \eta_2 + \cdots + \eta_n, \qquad |\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$$
$$\text{and} \qquad |\mu| = \mu_1 + \mu_2 + \cdots + \mu_n,$$

we can rewrite this as $|\eta| + |\gamma| = 2|\mu|$. Equivalently, $|\eta| - |\mu| = |\mu| - |\gamma|$. This proves Corollary 4.38 **(a)**.

(b) Let $i \in \{1, 2, \ldots, n-1\}$. Then, Theorem 4.37 **(c)** yields

$$\min\{\mu_i, \gamma_i\} + \min\{-\mu_{i+1}, -\gamma_{i+1}\} = \min\{\mu_i, \eta_i\} + \min\{-\mu_{i+1}, -\eta_{i+1}\}.$$

In view of $\min\{-\mu_{i+1}, -\gamma_{i+1}\} = -\max\{\mu_{i+1}, \gamma_{i+1}\}$ and $\min\{-\mu_{i+1}, -\eta_{i+1}\} = -\max\{\mu_{i+1}, \eta_{i+1}\}$, we can rewrite this as

$$\min\{\mu_i, \gamma_i\} + (-\max\{\mu_{i+1}, \gamma_{i+1}\}) = \min\{\mu_i, \eta_i\} + (-\max\{\mu_{i+1}, \eta_{i+1}\}).$$

In other words,

$$\min\{\mu_i, \gamma_i\} - \max\{\mu_{i+1}, \gamma_{i+1}\} = \min\{\mu_i, \eta_i\} - \max\{\mu_{i+1}, \eta_{i+1}\}.$$

Equivalently,

$$\min\{\mu_i, \eta_i\} - \min\{\mu_i, \gamma_i\} = \max\{\mu_{i+1}, \eta_{i+1}\} - \max\{\mu_{i+1}, \gamma_{i+1}\}.$$

This proves Corollary 4.38 **(b)**.

(c) Theorem 4.37 **(d)** yields

$$\sum_{i=1}^{n}\left(\min\left\{\mu_i,\gamma_i\right\}-\gamma_i\right)=\sum_{i=1}^{n}\left(\min\left\{\mu_i,\eta_i\right\}-\mu_i\right). \tag{55}$$

Now, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\underbrace{\left(\mu_i-\min\left\{\mu_i,\eta_i\right\}+\min\left\{\mu_i,\gamma_i\right\}\right)}_{=\min\left\{\mu_i,\gamma_i\right\}-\left(\min\left\{\mu_i,\eta_i\right\}-\mu_i\right)}\\
&=\sum_{i=1}^{n}\left(\min\left\{\mu_i,\gamma_i\right\}-\left(\min\left\{\mu_i,\eta_i\right\}-\mu_i\right)\right)\\
&=\sum_{i=1}^{n}\min\left\{\mu_i,\gamma_i\right\}-\underbrace{\sum_{i=1}^{n}\left(\min\left\{\mu_i,\eta_i\right\}-\mu_i\right)}_{\substack{=\sum_{i=1}^{n}\left(\min\left\{\mu_i,\gamma_i\right\}-\gamma_i\right)\\ \text{(by (55))}}}\\
&=\sum_{i=1}^{n}\min\left\{\mu_i,\gamma_i\right\}-\sum_{i=1}^{n}\left(\min\left\{\mu_i,\gamma_i\right\}-\gamma_i\right)\\
&=\sum_{i=1}^{n}\underbrace{\left(\min\left\{\mu_i,\gamma_i\right\}-\left(\min\left\{\mu_i,\gamma_i\right\}-\gamma_i\right)\right)}_{=\gamma_i}=\sum_{i=1}^{n}\gamma_i.
\end{aligned}$$

This proves Corollary 4.38 **(c)**.

(d) Theorem 4.37 **(a)** shows that $\mathbf{f}_\mu\circ\mathbf{f}_\mu=\mathrm{id}$. But recall that $\eta=\mathbf{f}_\mu\left(\gamma\right)$. Applying the map $\mathbf{f}_\mu$ to both sides of this equality, we obtain

$$\mathbf{f}_\mu\left(\eta\right)=\mathbf{f}_\mu\left(\mathbf{f}_\mu\left(\gamma\right)\right)=\underbrace{\left(\mathbf{f}_\mu\circ\mathbf{f}_\mu\right)}_{=\mathrm{id}}\left(\gamma\right)=\gamma.$$

This proves Corollary 4.38 **(d)**. □

We are now ready to prove the key lemma:

Lemma 4.39. Fix any n-tuple $\mu\in\mathbb{Z}^n$. Let $\gamma\in\mathbb{Z}^n$. Let $a,b\in\mathbb{Z}$. Then,

$$\left|R_{\mu,b,a}\left(\mathbf{f}_\mu\left(\gamma\right)\right)\right|=\left|R_{\mu,a,b}\left(\gamma\right)\right|.$$

Proof of Lemma 4.39. Define $\eta\in\mathbb{Z}^n$ by $\eta=\mathbf{f}_\mu\left(\gamma\right)$. We must then prove that $\left|R_{\mu,b,a}\left(\eta\right)\right|=\left|R_{\mu,a,b}\left(\gamma\right)\right|$.

We know that $R_{\mu,a,b}\left(\gamma\right)$ is the set of all snakes ν satisfying the four conditions

$$\mu\rightharpoonup\nu\qquad\text{and}\qquad\left|\mu\right|-\left|\nu\right|=a\qquad\text{and}\qquad\gamma\rightharpoonup\nu\qquad\text{and}\qquad\left|\gamma\right|-\left|\nu\right|=b.$$

Likewise, $R_{\mu,b,a}(\eta)$ is the set of all snakes ν satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = b \qquad \text{and} \qquad \eta \rightharpoonup \nu \qquad \text{and} \qquad |\eta| - |\nu| = a.$$

Now, fix $\nu \in R_{\mu,a,b}(\gamma)$. Thus, ν is a snake satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = a \qquad \text{and} \qquad \gamma \rightharpoonup \nu \qquad \text{and} \qquad |\gamma| - |\nu| = b$$

(by the definition of $R_{\mu,a,b}(\gamma)$). In particular, we have $\mu \rightharpoonup \nu$. In other words, we have

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \cdots \geq \mu_n \geq \nu_n \tag{56}$$

(by the definition of "$\mu \rightharpoonup \nu$"). Likewise, from $\gamma \rightharpoonup \nu$, we obtain

$$\gamma_1 \geq \nu_1 \geq \gamma_2 \geq \nu_2 \geq \cdots \geq \gamma_n \geq \nu_n. \tag{57}$$

We define an n-tuple $\zeta \in \mathbb{Z}^n$ by setting

$$\zeta_i = \min\{\mu_i, \eta_i\} - \min\{\mu_i, \gamma_i\} + \nu_i \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

We shall prove that $\zeta \in R_{\mu,b,a}(\eta)$.

We begin by proving several auxiliary claims:

Claim 1: We have $\min\{\mu_i, \eta_i\} \geq \zeta_i$ for each $i \in \{1, 2, \ldots, n\}$.

[*Proof of Claim 1:* Let $i \in \{1, 2, \ldots, n\}$. From (56), we obtain $\mu_i \geq \nu_i$. From (57), we obtain $\gamma_i \geq \nu_i$. Combining $\mu_i \geq \nu_i$ and $\gamma_i \geq \nu_i$, we obtain $\min\{\mu_i, \gamma_i\} \geq \nu_i$. Now, the definition of ζ yields

$$\zeta_i = \min\{\mu_i, \eta_i\} - \underbrace{\min\{\mu_i, \gamma_i\}}_{\geq \nu_i} + \nu_i \leq \min\{\mu_i, \eta_i\} - \nu_i + \nu_i = \min\{\mu_i, \eta_i\}.$$

In other words, $\min\{\mu_i, \eta_i\} \geq \zeta_i$. This proves Claim 1.]

Claim 2: We have $\zeta_i \geq \max\{\mu_{i+1}, \eta_{i+1}\}$ for each $i \in \{1, 2, \ldots, n-1\}$.

[*Proof of Claim 2:* Let $i \in \{1, 2, \ldots, n-1\}$. From (56), we obtain $\nu_i \geq \mu_{i+1}$. From (57), we obtain $\nu_i \geq \gamma_{i+1}$. Combining $\nu_i \geq \mu_{i+1}$ and $\nu_i \geq \gamma_{i+1}$, we obtain $\nu_i \geq \max\{\mu_{i+1}, \gamma_{i+1}\}$. Now, the definition of ζ yields

$$\begin{aligned} \zeta_i &= \underbrace{\min\{\mu_i, \eta_i\} - \min\{\mu_i, \gamma_i\}}_{\substack{=\max\{\mu_{i+1}, \eta_{i+1}\} - \max\{\mu_{i+1}, \gamma_{i+1}\} \\ \text{(by Corollary 4.38 (b))}}} + \underbrace{\nu_i}_{\geq \max\{\mu_{i+1}, \gamma_{i+1}\}} \\ &\geq \max\{\mu_{i+1}, \eta_{i+1}\} - \max\{\mu_{i+1}, \gamma_{i+1}\} + \max\{\mu_{i+1}, \gamma_{i+1}\} = \max\{\mu_{i+1}, \eta_{i+1}\}. \end{aligned}$$

This proves Claim 2.]

Claim 3: The n-tuple ζ is a snake and satisfies $\mu \rightharpoonup \zeta$ and $\eta \rightharpoonup \zeta$.

[*Proof of Claim 3:* For each $i \in \{1, 2, \ldots, n\}$, we have $\mu_i \geq \zeta_i$ (since $\mu_i \geq \min\{\mu_i, \eta_i\} \geq \zeta_i$ (by Claim 1)). For each $i \in \{1, 2, \ldots, n-1\}$, we have $\zeta_i \geq \mu_{i+1}$ (since Claim 2 yields $\zeta_i \geq \max\{\mu_{i+1}, \eta_{i+1}\} \geq \mu_{i+1}$). Combining the preceding two sentences, we obtain

$$\mu_1 \geq \zeta_1 \geq \mu_2 \geq \zeta_2 \geq \cdots \geq \mu_n \geq \zeta_n.$$

In other words, $\mu \rightharpoonup \zeta$ (by the definition of "$\mu \rightharpoonup \zeta$").

For each $i \in \{1, 2, \ldots, n\}$, we have $\eta_i \geq \zeta_i$ (since $\eta_i \geq \min\{\mu_i, \eta_i\} \geq \zeta_i$ (by Claim 1)). For each $i \in \{1, 2, \ldots, n-1\}$, we have $\zeta_i \geq \eta_{i+1}$ (since Claim 2 yields $\zeta_i \geq \max\{\mu_{i+1}, \eta_{i+1}\} \geq \eta_{i+1}$). Combining the preceding two sentences, we obtain

$$\eta_1 \geq \zeta_1 \geq \eta_2 \geq \zeta_2 \geq \cdots \geq \eta_n \geq \zeta_n.$$

In other words, $\eta \rightharpoonup \zeta$ (by the definition of "$\eta \rightharpoonup \zeta$"). Hence, Proposition 4.23 **(a)** (applied to η and ζ instead of μ and λ) yields that both ζ and η are snakes. Hence, ζ is a snake. This completes the proof of Claim 3.]

Claim 4: We have $|\mu| - |\zeta| = b$ and $|\eta| - |\zeta| = a$.

[*Proof of Claim 4:* We have

$$|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n = \sum_{i=1}^{n} \mu_i \qquad \text{and}$$
$$|\zeta| = \zeta_1 + \zeta_2 + \cdots + \zeta_n = \sum_{i=1}^{n} \zeta_i.$$

Subtracting these two equalities from one another, we find

$$\begin{aligned}
|\mu| - |\zeta| &= \sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \underbrace{\zeta_i}_{\substack{=\min\{\mu_i,\eta_i\}-\min\{\mu_i,\gamma_i\}+\nu_i \\ \text{(by the definition of } \zeta\text{)}}} \\
&= \sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \left(\min\left\{\mu_i, \eta_i\right\} - \min\left\{\mu_i, \gamma_i\right\} + \nu_i\right) \\
&= \sum_{i=1}^{n} \underbrace{\left(\mu_i - \left(\min\left\{\mu_i, \eta_i\right\} - \min\left\{\mu_i, \gamma_i\right\} + \nu_i\right)\right)}_{=(\mu_i - \min\{\mu_i,\eta_i\} + \min\{\mu_i,\gamma_i\}) - \nu_i} \\
&= \sum_{i=1}^{n} \left(\left(\mu_i - \min\left\{\mu_i, \eta_i\right\} + \min\left\{\mu_i, \gamma_i\right\}\right) - \nu_i\right) \\
&= \underbrace{\sum_{i=1}^{n} \left(\mu_i - \min\left\{\mu_i, \eta_i\right\} + \min\left\{\mu_i, \gamma_i\right\}\right)}_{\substack{=\sum_{i=1}^{n} \gamma_i \\ \text{(by Corollary 4.38 } \textbf{(c)}\text{)}}} - \sum_{i=1}^{n} \nu_i \\
&= \underbrace{\sum_{i=1}^{n} \gamma_i}_{=\gamma_1+\gamma_2+\cdots+\gamma_n=|\gamma|} - \underbrace{\sum_{i=1}^{n} \nu_i}_{=\nu_1+\nu_2+\cdots+\nu_n=|\nu|} = |\gamma| - |\nu| = b.
\end{aligned}$$

Furthermore,

$$|\mu| - |\gamma| = \underbrace{(|\mu| - |\nu|)}_{=a} - \underbrace{(|\gamma| - |\nu|)}_{=b} = a - b$$

and

$$|\eta| - |\zeta| = \underbrace{(|\eta| - |\mu|)}_{\substack{=|\mu|-|\gamma| \\ \text{(by Corollary 4.38 } \textbf{(a)}\text{)}}} + \underbrace{(|\mu| - |\zeta|)}_{=b} = \underbrace{|\mu| - |\gamma|}_{=a-b} + b = (a-b) + b = a.$$

Thus, Claim 4 is proven.]

We have now shown (in Claim 3 and Claim 4) that ζ is a snake satisfying the four conditions

$$\mu \rightharpoonup \zeta \qquad \text{and} \qquad |\mu| - |\zeta| = b \qquad \text{and} \qquad \eta \rightharpoonup \zeta \qquad \text{and} \qquad |\eta| - |\zeta| = a.$$

In other words, $\zeta \in R_{\mu,b,a}(\eta)$ (by the definition of $R_{\mu,b,a}(\eta)$).

Forget that we fixed ν. Thus, for each $\nu \in R_{\mu,a,b}(\gamma)$, we have constructed a $\zeta \in R_{\mu,b,a}(\eta)$. Let us denote this ζ by $\widetilde{\nu}$. We thus have defined a map

$$\begin{aligned}
R_{\mu,a,b}(\gamma) &\to R_{\mu,b,a}(\eta), \\
\nu &\mapsto \widetilde{\nu}.
\end{aligned}$$

Let us denote this map by $\mathbf{g}_{\gamma,a,b}$. Its definition shows that

$$\left(\underbrace{\mathbf{g}_{\gamma,a,b}\left(\nu\right)}_{=\widetilde{\nu}}\right)_i = \widetilde{\nu}_i = \min\left\{\mu_i,\eta_i\right\} - \min\left\{\mu_i,\gamma_i\right\} + \nu_i \tag{58}$$

$$\left(\begin{array}{c}\text{since } \widetilde{\nu} \text{ was defined as the } n\text{-tuple } \zeta\text{, whose entries} \\ \text{are given by } \zeta_i = \min\left\{\mu_i,\eta_i\right\} - \min\left\{\mu_i,\gamma_i\right\} + \nu_i\end{array}\right)$$

for each $\nu \in R_{\mu,a,b}\left(\gamma\right)$ and each $i \in \{1,2,\ldots,n\}$.

However, from $\eta = \mathbf{f}_\mu\left(\gamma\right)$, we obtain $\gamma = \mathbf{f}_\mu\left(\eta\right)$ (by Corollary 4.38 **(d)**). The relation between γ and η is thus symmetric. Hence, in the same way as we defined a map $\mathbf{g}_{\gamma,a,b} : R_{\mu,a,b}\left(\gamma\right) \to R_{\mu,b,a}\left(\eta\right)$, we can define a map $\mathbf{g}_{\eta,b,a} : R_{\mu,b,a}\left(\eta\right) \to R_{\mu,a,b}\left(\gamma\right)$ (by repeating the above construction of $\mathbf{g}_{\gamma,a,b}$ with b, a, η and γ taking the roles of a, b, γ and η, respectively). The resulting map $\mathbf{g}_{\eta,b,a}$ satisfies

$$\left(\mathbf{g}_{\eta,b,a}\left(\nu\right)\right)_i = \min\left\{\mu_i,\gamma_i\right\} - \min\left\{\mu_i,\eta_i\right\} + \nu_i \tag{59}$$

for each $\nu \in R_{\mu,b,a}\left(\eta\right)$ and each $i \in \{1,2,\ldots,n\}$. (Indeed, this can be proved just as we proved (58), but with b, a, η and γ taking the roles of a, b, γ and η.)

Now it is easy to see (using (58) and (59)) that the two maps $\mathbf{g}_{\gamma,a,b}$ and $\mathbf{g}_{\eta,b,a}$ are mutually inverse. Hence, these two maps are invertible, i.e., are bijections.

Thus, there exists a bijection from $R_{\mu,a,b}\left(\gamma\right)$ to $R_{\mu,b,a}\left(\eta\right)$ (namely, $\mathbf{g}_{\gamma,a,b}$). This yields $\left|R_{\mu,a,b}\left(\gamma\right)\right| = \left|R_{\mu,b,a}\left(\eta\right)\right| = \left|R_{\mu,b,a}\left(\mathbf{f}_\mu\left(\gamma\right)\right)\right|$ (since $\eta = \mathbf{f}_\mu\left(\gamma\right)$). This proves Lemma 4.39. □

Having learned a lot about the map $\mathbf{f}_\mu$, let us now connect it to the map φ defined in Theorem 2.3. For this, we shall use the following lemma:

Lemma 4.40. Fix any n-tuple $\mu \in \mathbb{Z}^n$.

Let $\nu \in \mathbb{Z}^n$ be an n-tuple. For each $j \in \mathbb{Z}$, let

$$\tau_j = \min\left\{\underbrace{\nu_{j+1} + \nu_{j+2} + \cdots + \nu_{j+k}}_{=\sum_{i=1}^{k}\nu_{j+i}} + \underbrace{\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+n-1}}_{=\sum_{i=k+1}^{n-1}\mu_{j+i}} \mid k \in \{0,1,\ldots,n-1\}\right\}.$$

Let $\eta \in \mathbb{Z}^n$ be such that

$$\eta_i = \mu_i + \left(\mu_{i-1} + \tau_{i-1}\right) - \left(\nu_{i+1} + \tau_{i+1}\right) \qquad \text{for each } i \in \{1,2,\ldots,n\}.$$

Then, $\mathbf{f}_\mu\left(\nu\right) = \eta$.

Proof of Lemma 4.40. Lemma 4.40 is obtained when we apply Lemma 3.13 to $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ (and rename everything[15], and use our above dictionary again). $\square$

We can now connect the map $\mathbf{f}_\mu$ with the map φ from Theorem 2.3:

Lemma 4.41. Let $a, b \in \mathbb{N}$. Fix any n-tuple $\mu \in \mathbb{Z}^n$. Define a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ as in Theorem 2.3. Then,

$$\varphi(\omega) = \mathbf{f}_\mu(\omega - a) + b \qquad \text{for each } \omega \in \mathbb{Z}^n.$$

Proof of Lemma 4.41. Let $\omega \in \mathbb{Z}^n$.

Define an n-tuple $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$ by

$$\nu_i = \omega_i - a \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Thus, $\nu = \omega - a$.

For each $i \in \mathbb{Z}$, we let $i\#$ denote the unique element of $\{1, 2, \ldots, n\}$ congruent to i modulo n.

For each $j \in \mathbb{Z}$, set

$$\begin{aligned} \tau_j = \min \Big\{ &\left(\nu_{(j+1)\#} + \nu_{(j+2)\#} + \cdots + \nu_{(j+k)\#}\right) \\ &+ \left(\mu_{(j+k+1)\#} + \mu_{(j+k+2)\#} + \cdots + \mu_{(j+n-1)\#}\right) \\ &\mid \ k \in \{0, 1, \ldots, n-1\} \Big\}. \end{aligned} \tag{60}$$

Define an n-tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{Z}^n$ by setting

$$\eta_i = \mu_{i\#} + \left(\mu_{(i-1)\#} + \tau_{(i-1)\#}\right) - \left(\nu_{(i+1)\#} + \tau_{(i+1)\#}\right) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

The definition of φ then yields

$$\varphi(\omega) = (\eta_1 + b, \eta_2 + b, \ldots, \eta_n + b) = \eta + b. \tag{61}$$

Our plan is now to show that $\mathbf{f}_\mu(\nu) = \eta$. We shall achieve this by applying Lemma 4.40; but in order to do so, we need to show that the assumptions of Lemma 4.40 are satisfied.

We shall do this piece by piece. First, we make the following two claims, which both follow from Convention 3.6:

[15] Namely, we are

- renaming the (fixed) n-tuple u as μ;
- renaming the n-tuple x as ν;
- renaming the elements q_j as τ_j;
- renaming the n-tuple z as η.

Claim 1: We have $\nu_{p\#} = \nu_p$ for each $p \in \mathbb{Z}$.

Claim 2: We have $\mu_{p\#} = \mu_p$ for each $p \in \mathbb{Z}$.

The next claim is an easy consequence of Claims 1 and 2:

Claim 3: For each $j \in \mathbb{Z}$, we have

$$\tau_j = \min \left\{ \underbrace{\nu_{j+1} + \nu_{j+2} + \cdots + \nu_{j+k}}_{= \sum_{i=1}^{k} \nu_{j+i}} + \underbrace{\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+n-1}}_{= \sum_{i=k+1}^{n-1} \mu_{j+i}} \right.$$
$$\left. \mid \ k \in \{0, 1, \ldots, n-1\} \right\}.$$

The next claim is an easy fact from elementary number theory:

Claim 4: We have $(p\# + q)\# = (p + q)\#$ for any $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$.

Using Claim 4, we easily obtain the following:

Claim 5: We have $\tau_{p\#} = \tau_p$ for each $p \in \mathbb{Z}$.

Now, let $i \in \{1, 2, \ldots, n\}$. Then, the definition of η yields

$$\eta_i = \underbrace{\mu_{i\#}}_{\substack{=\mu_i \\ \text{(by Claim 2)}}} + \left(\underbrace{\mu_{(i-1)\#}}_{\substack{=\mu_{i-1} \\ \text{(by Claim 2)}}} + \underbrace{\tau_{(i-1)\#}}_{\substack{=\tau_{i-1} \\ \text{(by Claim 5)}}} \right) - \left(\underbrace{\nu_{(i+1)\#}}_{\substack{=\nu_{i+1} \\ \text{(by Claim 1)}}} + \underbrace{\tau_{(i+1)\#}}_{\substack{=\tau_{i+1} \\ \text{(by Claim 5)}}} \right)$$
$$= \mu_i + (\mu_{i-1} + \tau_{i-1}) - (\nu_{i+1} + \tau_{i+1}).$$

Now, forget that we fixed i. We thus have proved that

$$\eta_i = \mu_i + (\mu_{i-1} + \tau_{i-1}) - (\nu_{i+1} + \tau_{i+1}) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Combining this with Claim 3, we conclude that the assumptions of Lemma 4.40 are satisfied. Hence, Lemma 4.40 yields $\mathbf{f}_\mu(\nu) = \eta$. In view of $\nu = \omega - a$, this rewrites as $\mathbf{f}_\mu(\omega - a) = \eta$. Hence, $\eta = \mathbf{f}_\mu(\omega - a)$, so that (61) becomes

$$\varphi(\omega) = \underbrace{\eta}_{=\mathbf{f}_\mu(\omega - a)} + b = \mathbf{f}_\mu(\omega - a) + b.$$

This proves Lemma 4.41. $\square$

4.6. The finale

Now, let us again use the convention (from Section 1) by which we identify partitions with finite tuples (and therefore identify partitions in $\operatorname{Par}[n]$ with nonnegative snakes). This is no longer problematic, since we are not using Convention 3.6 any more.

Lemma 4.42. Let $a, b \in \mathbb{N}$. Define the two partitions $\alpha = \left(a+b, a^{n-2}\right)$ and $\beta = \left(a+b, b^{n-2}\right)$.

Fix any partition $\mu \in \operatorname{Par}[n]$. Consider the map $\mathbf{f}_\mu : \mathbb{Z}^n \to \mathbb{Z}^n$ defined in Definition 4.36.

Then, for any $\lambda \in \mathbb{Z}^n$, we have

$$c_{\alpha,\mu}^{\lambda+a} = c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b}.$$

Here, we understand $c_{\alpha,\mu}^{\lambda+a}$ to mean 0 if $\lambda + a$ is not a partition, and likewise we understand $c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b}$ to mean 0 if $\mathbf{f}_\mu(\lambda) + b$ is not a partition.

Proof of Lemma 4.42. Let $\lambda \in \mathbb{Z}^n$. Corollary 4.34 (applied to $\lambda + a$ instead of λ) yields

$$\begin{aligned} c_{\alpha,\mu}^{\lambda+a} &= \left|R_{\mu,a,b}\left((\lambda+a)-a\right)\right| - \left|R_{\mu,a-1,b-1}\left((\lambda+a)-a\right)\right| \\ &= \left|R_{\mu,a,b}(\lambda)\right| - \left|R_{\mu,a-1,b-1}(\lambda)\right| \end{aligned} \tag{62}$$

(since $(\lambda+a) - a = \lambda$). On the other hand, $\beta = \left(\underbrace{a+b}_{=b+a}, b^{n-2}\right) = \left(b+a, b^{n-2}\right)$. Hence, Corollary 4.34 (applied to b, a, β and $\mathbf{f}_\mu(\lambda)+b$ instead of a, b, α and λ) yields

$$\begin{aligned} c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b} &= \left|R_{\mu,b,a}\left(\left(\mathbf{f}_\mu(\lambda)+b\right)-b\right)\right| - \left|R_{\mu,b-1,a-1}\left(\left(\mathbf{f}_\mu(\lambda)+b\right)-b\right)\right| \\ &= \underbrace{\left|R_{\mu,b,a}\left(\mathbf{f}_\mu(\lambda)\right)\right|}_{\substack{=\left|R_{\mu,a,b}(\lambda)\right| \\ \text{(by Lemma 4.39,} \\ \text{applied to } \gamma=\lambda)}} - \underbrace{\left|R_{\mu,b-1,a-1}\left(\mathbf{f}_\mu(\lambda)\right)\right|}_{\substack{=\left|R_{\mu,a-1,b-1}(\lambda)\right| \\ \text{(by Lemma 4.39,} \\ \text{applied to } \lambda,\ a-1 \text{ and } b-1 \\ \text{instead of } \gamma,\ a \text{ and } b)}} \\ &\qquad \left(\text{since } \left(\mathbf{f}_\mu(\lambda)+b\right)-b = \mathbf{f}_\mu(\lambda)\right) \\ &= \left|R_{\mu,a,b}(\lambda)\right| - \left|R_{\mu,a-1,b-1}(\lambda)\right|. \end{aligned}$$

Comparing this with (62), we find $c_{\alpha,\mu}^{\lambda+a} = c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b}$. This proves Lemma 4.42. □

We are now ready to prove Theorem 2.3:

Proof of Theorem 2.3. The map $\mathbf{f}_\mu$ is an involution (by Theorem 4.37 **(a)**), thus a bijection.

Let $\mathbf{a}^- : \mathbb{Z}^n \to \mathbb{Z}^n$ be the map that sends each $\omega \in \mathbb{Z}^n$ to $\omega - a$. This map $\mathbf{a}^-$ is clearly a bijection.

Let $\mathbf{b}^+ : \mathbb{Z}^n \to \mathbb{Z}^n$ be the map that sends each $\omega \in \mathbb{Z}^n$ to $\omega + b$. This map $\mathbf{b}^+$ is clearly a bijection.

From Lemma 4.41, we can easily see that

$$\varphi = \mathbf{b}^+ \circ \mathbf{f}_\mu \circ \mathbf{a}^-.$$

Recall that the maps $\mathbf{b}^+$, $\mathbf{f}_\mu$ and $\mathbf{a}^-$ are bijections. Hence, their composition $\mathbf{b}^+ \circ \mathbf{f}_\mu \circ \mathbf{a}^-$ is a bijection as well. In other words, φ is a bijection (since $\varphi = \mathbf{b}^+ \circ \mathbf{f}_\mu \circ \mathbf{a}^-$). This proves Theorem 2.3 **(a)**.

(b) Let $\omega \in \mathbb{Z}^n$. Then, Lemma 4.41 yields $\varphi(\omega) = \mathbf{f}_\mu(\omega - a) + b$. Hence, $\mathbf{f}_\mu(\omega - a) + b = \varphi(\omega)$. But Lemma 4.42 (applied to $\lambda = \omega - a$) yields

$$c_{\alpha,\mu}^{(\omega-a)+a} = c_{\beta,\mu}^{\mathbf{f}_\mu(\omega-a)+b} = c_{\beta,\mu}^{\varphi(\omega)} \qquad \left(\text{since } \mathbf{f}_\mu(\omega - a) + b = \varphi(\omega)\right).$$

In view of $(\omega - a) + a = \omega$, this rewrites as $c_{\alpha,\mu}^{\omega} = c_{\beta,\mu}^{\varphi(\omega)}$. This proves Theorem 2.3 **(b)**. $\square$

5. Final remarks

5.1. Aside: A Jacobi–Trudi formula for Schur Laurent polynomials

As mentioned above, Proposition 4.29 has the following generalization, which can be obtained from an identity of Koike [Koike89, Proposition 2.8] (via the map $\widetilde{\pi}_n$ from [Koike89] and the correspondence between Schur Laurent polynomials and rational representations of $\operatorname{GL}(n)$), which has later been extended by Hamel and King [HamKin11] (see [HamKin11, (6) and (10)] for the connection):[16]

Proposition 5.1. Let $p, q \in \mathbb{N}$ with $p + q \leq n$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_p)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_q)$ be two partitions. Let $\mathbf{b} \ominus \mathbf{a}$ denote the snake $(b_1, b_2, \ldots, b_q, 0^{n-p-q}, -a_p, -a_{p-1}, \ldots, -a_1)$. Let M be the $(p+q) \times (p+q)$-

[16] We recall Definition 4.1, Definition 4.3 **(a)**, Definition 4.6, Definition 4.12 and Definition 4.18, as well as the conventions made in Section 1, for the notations used in this proposition.

matrix

$$\left(\begin{cases} h^-_{a_{p-i+1}+i-j}, & \text{if } i \leq p; \\ h^+_{b_{i-p}-i+j}, & \text{if } i > p \end{cases}\right)_{i,j\in\{1,2,\ldots,p+q\}}$$

$$= \begin{pmatrix} h^-_{a_p} & h^-_{a_p-1} & \cdots & h^-_{a_p-p+1} & h^-_{a_p-p} & h^-_{a_p-p-1} & \cdots & h^-_{a_p-p-q+1} \\ h^-_{a_{p-1}+1} & h^-_{a_{p-1}} & \cdots & h^-_{a_{p-1}-p+2} & h^-_{a_{p-1}-p+1} & h^-_{a_{p-1}-p} & \cdots & h^-_{a_{p-1}-p-q+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h^-_{a_1+p-1} & h^-_{a_1+p-2} & \cdots & h^-_{a_1} & h^-_{a_1-1} & h^-_{a_1-2} & \cdots & h^-_{a_1-q} \\ h^+_{b_1-p} & h^+_{b_1-p+1} & \cdots & h^+_{b_1-1} & h^+_{b_1} & h^+_{b_1+1} & \cdots & h^+_{b_1+q-1} \\ h^+_{b_2-p-1} & h^+_{b_2-p} & \cdots & h^+_{b_2-2} & h^+_{b_2-1} & h^+_{b_2} & \cdots & h^+_{b_2+q-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h^+_{b_q-p-q+1} & h^+_{b_q-p-q+2} & \cdots & h^+_{b_q-q} & h^+_{b_q-q+1} & h^+_{b_q-q+2} & \cdots & h^+_{b_q} \end{pmatrix}.$$

Then,

$$\bar{s}_{\mathbf{b}\ominus\mathbf{a}} = \det M.$$

Remark 5.2. The analogous generalization of the second Jacobi–Trudi formula ([GriRei20, (2.4.17)]) can easily be proved (although we leave both stating and proving it to the reader). What makes it easy is the (fairly obvious) fact that the elementary symmetric functions e_k satisfy

$$e_k\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) = x_\Pi^{-1} e_{n-k}\left(x_1, x_2, \ldots, x_n\right)$$

for all $k \in \mathbb{Z}$. (See [GriRei20, Definition 2.2.1] for the definition of e_k.)

Proposition 5.1 generalizes Proposition 4.29 (which corresponds to the particular case when $p = 1$ and $q = 1$) as well as the first Jacobi–Trudi formula [GriRei20, (2.4.16)] (which corresponds to the particular case $p = 0$).

We notice that what we called $\mathbf{b} \ominus \mathbf{a}$ in Proposition 5.1 has been called $[\mathbf{b}, \mathbf{a}]$ in [Stembr87].

We thank Grigori Olshanski for informing us of the provenance of Proposition 5.1.

5.2. Questions on $\mathbf{f}_u$

We shall now pose several questions about the birational involution $\mathbf{f}_u$ studied in Section 3. Convention 3.4, Convention 3.5 and Convention 3.6 will be used throughout Subsection 5.2.

5.2.1. $\mathbf{f}_u$ as a composition?

Most of our questions are meant to attempt seeing the involution $\mathbf{f}_u$ from different directions. The first one is inspired by what is now known as the "*toggle approach*" to dynamical combinatorics (see, e.g., [Roby15]), but is really an application of the age-old "divide and conquer" paradigm to complicated maps:

Question 5.3. Is there an equivalent definition of $\mathbf{f}_u$ as a composition of toggles? (A *toggle* here means a birational map $\mathbb{K}^n \to \mathbb{K}^n$ that changes only one entry of the n-tuple. An example for a birational map that can be defined as a composition of toggles is *birational rowmotion* – see, e.g., [EinPro13]. Cluster mutations, as in the theory of cluster algebras, are another example of toggles.)

Another set of questions concern the *uniqueness* of $\mathbf{f}_u$. While we defined the map $\mathbf{f}_u$ explicitly, all we have then used are the properties listed in Theorem 3.11. Thus, it is a natural question to ask whether these properties characterize $\mathbf{f}_u$ uniquely. A pointwise version of this question can be asked as well: Given $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ satisfying some of the equalities in parts **(b)**, **(c)** and **(d)** of Theorem 3.11, does it follow that $y = \mathbf{f}_u(x)$? (Keep in mind that u is fixed.)

Depending on which equalities we require, we may of course get different answers. Let us first ask what happens if we require the equalities from Theorem 3.11 **(c)** only:

5.2.2. Characterizing $\mathbf{f}_u(x)$ via the cyclic equations

Question 5.4. Given $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ satisfying

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) \tag{63}$$

for all $i \in \mathbb{Z}$. Does it follow that $y = \mathbf{f}_u(x)$ or $y = x$?

Note that the "or $y = x$" part is needed here, since $y = x$ is obviously a solution to the equations (63).

The following example shows that the answer to Question 5.4 is "no" if $\mathbb{K}$ is the min tropical semifield $(\mathbb{Z}, \min, +, 0)$ of the totally ordered abelian group $\mathbb{Z}$.

Example 5.5. Let $k, g \in \mathbb{N}$ with $g \geq k$. Let $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ and $n = 3$ and $u = (0, 0, g)$ and $x = (1, 2, 0)$. Set $y = (k+1, 2, k)$ (where the "+" sign in "$k+1$" stands for addition of integers, not addition in $\mathbb{K}$). Then, the equations (63) hold in $\mathbb{K}$ for all $i \in \mathbb{Z}$. (Restated in terms of standard operations on integers, this is saying that

$$\min\{u_i, x_i\} + \min\{-u_{i+1}, -x_{i+1}\} = \min\{u_i, y_i\} + \min\{-u_{i+1}, -y_{i+1}\}$$

for all $i \in \mathbb{Z}$.) This is straightforward to verify, and shows that for a given x there can be an arbitrarily high (finite) number of $y \in \mathbb{K}^n$ satisfying the equations (63) for all $i \in \mathbb{Z}$.

It is not hard to show (see [Grinbe20] for details) that this number cannot be infinite.

Example 5.5 has shown that the answer to Question 5.4 is "no" when $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$. However, the answer to Question 5.4 is "yes" if $\mathbb{K} = \mathbb{Q}_+$ and, more generally, if the semifield $\mathbb{K}$ embeds into an integral domain:

Proposition 5.6. Assume that there is an integral domain $\mathbb{L}$ such that the semifield $\mathbb{K}$ is a subsemifield of $\mathbb{L}$ (in the sense that $\mathbb{K} \subseteq \mathbb{L}$ and that the operations $+$ and $\cdot$ of $\mathbb{K}$ are restrictions of those of $\mathbb{L}$, whereas the unity of $\mathbb{K}$ is the unity of $\mathbb{L}$). Let $x \in \mathbb{K}^n$. Then, the only n-tuples $y \in \mathbb{K}^n$ satisfying the equations (63) for all $i \in \mathbb{Z}$ are $y = \mathbf{f}_u(x)$ and $y = x$.

See the detailed version [Grinbe20] of this paper for a rough outline of the proof of Proposition 5.6.

5.2.3. Characterizing $\mathbf{f}_u(x)$ via the cyclic equations and the product equation

Another avatar of the uniqueness question is the following:

Question 5.7. Given $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ satisfying both (63) for all $i \in \mathbb{Z}$ and

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 . \tag{64}$$

Does it follow that $y = \mathbf{f}_u(x)$?

The answer to this question is definitely "yes" when $\mathbb{K} = \mathbb{Q}_+$, by essentially the same argument that was used in Remark 3.16. Again, however, the answer is "no" when $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$. For example, if $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ and $n = 4$ and $u = (2, 1, 1, 0)$ and $x = (1, 1, 1, 1)$, then the two n-tuples $(1, 1, 1, 1)$ and $(2, 2, 0, 0)$ both can be taken as y in Question 5.7, but clearly cannot both equal $\mathbf{f}_u(x)$. (On the other hand, if $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ and $n = 3$, then the answer is "yes" again; this can be shown by an unenlightening yet not particularly arduous case analysis.)

An even stronger version of Question 5.7 holds when $\mathbb{K} = \mathbb{Q}_+$:

Proposition 5.8. Assume that $\mathbb{K} = \mathbb{Q}_+$. Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$. Assume that (63) holds for all $i \in \{1, 2, \ldots, n-1\}$, and assume that (64) holds. Then, $y = \mathbf{f}_u(x)$.

The proof of Proposition 5.8 is sketched in the detailed version [Grinbe20] of this paper.

5.2.4. Understanding Lemma 3.12

Another question concerns Lemma 3.12:

Question 5.9. What is the "real meaning" of some of the more complicated parts of Lemma 3.12? In particular, Lemma 3.12 **(g)** reminds of the Plücker relation for minors of a $2 \times m$-matrix; can it be viewed that way? (Such a proof would not be superior to the one given above, as it wouldn't be subtraction-free and thus wouldn't work natively over arbitrary semifields. But it would shine more light on the lemma.)

5.3. On the genesis of φ (and $\mathbf{f}_u$)

As we mentioned in the introduction to this paper, Pelletier and Ressayre did not conjecture Theorem 2.3 in this exact form; instead, they conjectured the existence of a mysterious bijection φ that satisfies Theorem 2.3 **(b)**. Our definition of φ appears *ex caelis oblatus*; while we have seen that our φ duly plays its part, it is far from clear how we have found it in the first place. The following few paragraphs are meant to demystify this process.

We were looking for a bijection $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ satisfying Theorem 2.3 **(b)**. In other words, we were looking for a way to match[17] the nonzero coefficients in the product $s_\alpha(x_1, x_2, \ldots, x_n) \cdot s_\mu(x_1, x_2, \ldots, x_n)$ (when expanded in the basis $(s_\lambda(x_1, x_2, \ldots, x_n))_{\lambda \in \operatorname{Par}[n]}$ of the $\mathbf{k}$-module of symmetric polynomials in $x_1, x_2, \ldots, x_n$) with the nonzero coefficients in the product $s_\beta(x_1, x_2, \ldots, x_n) \cdot s_\mu(x_1, x_2, \ldots, x_n)$ in such a way that matching coefficients are equal.

The first step towards this goal was the discovery of the formula $s_\alpha(x_1, x_2, \ldots, x_n) = x_\Pi^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right)$: our Corollary 4.30. We originally proved this formula combinatorially, by analyzing the structure of semistandard tableaux of shape α. [18] The proof of Corollary 4.30 given above (using the Pieri rule) was an afterthought.

Corollary 4.30 was a visible step in the right direction, as it moved the problem from the world of Littlewood–Richardson coefficients into the simpler world of Pieri rules. Indeed, instead of expanding $s_\alpha(x_1, x_2, \ldots, x_n) \cdot s_\mu(x_1, x_2, \ldots, x_n)$, we now only had to expand $x_\Pi^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right) \cdot s_\mu(x_1, x_2, \ldots, x_n)$, which looked like an expansion that the Pieri rule could help with (to be fully honest, we only knew the Pieri rule for multiplying by h_k^+; but we soon would find one for multiplying by h_k^-). The x_Π^a factor was clearly a mere distraction, but in order to get rid of it, we had to extend our polynomial ring to the ring $\mathcal{L}$ of Laurent polynomials (since $\left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right) \cdot s_\mu(x_1, x_2, \ldots, x_n)$ is, in general, not a polynomial). This extension had already been done by Stembridge in [Stembr87], and all we had to do was rename "staircases" as "snakes", define Schur Laurent polynomials (by generalizing the alternant formula for Schur polynomials in the most obvious way), extend some basic properties of Schur polynomials to Schur Laurent poly-

[17] "Matching" means "perfect matching" here – i.e., every coefficient on either side should get a unique partner.

[18] Each of the first a columns of such a tableau would have the form $(1, 2, \ldots, i-1, i+1, \ldots, n)$ for some $i \in \{1, 2, \ldots, n\}$, and these numbers i would weakly increase as one moves right.

nomials, and find the "upside-down" Pieri rule (Proposition 4.25). None of this was difficult; in particular, the "upside-down" Pieri rule followed easily from the usual Pieri rule using Lemma 4.17 (which is our main device for turning things "upside down"). The Schur polynomial $s_\nu(x_1, x_2, \ldots, x_n)$ was generalized to the Schur Laurent polynomial $\overline{s}_\nu$.

Thus our problem was reduced to matching the nonzero coefficients in the product $\left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right) \cdot \overline{s}_\mu$ with the nonzero coefficients in the product $\left(h_b^- h_a^+ - h_{b-1}^- h_{a-1}^+\right) \cdot \overline{s}_\mu$. The products could both be expressed using the Pieri rules, but the differences were still a distraction. At this point, we made a fortunate guess: We hoped it would suffice to match the nonzero coefficients in the product $h_a^- h_b^+ \cdot \overline{s}_\mu$ with the nonzero coefficients in the product $h_b^- h_a^+ \cdot \overline{s}_\mu$. More precisely, we hoped to find such a matching that would not depend on a and b; then it would also provide a matching between the nonzero coefficients in the product $h_{a-1}^- h_{b-1}^+ \cdot \overline{s}_\mu$ and the nonzero coefficients in the product $h_{b-1}^- h_{a-1}^+ \cdot \overline{s}_\mu$, and therefore by taking differences we would obtain a matching between the nonzero coefficients in the product $\left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right) \cdot \overline{s}_\mu$ and the nonzero coefficients in the product $\left(h_b^- h_a^+ - h_{b-1}^- h_{a-1}^+\right) \cdot \overline{s}_\mu$.

Thus we needed to expand $h_a^- h_b^+ \cdot \overline{s}_\mu$. Using the Pieri rules, this was straightforward – the answer is in Lemma 4.33. Our problem was to connect this result with what we would similarly obtain from expanding $h_b^- h_a^+ \cdot \overline{s}_\mu$. In other words, we wanted to construct a bijection $\mathbf{f}_\mu : \mathbb{Z}^n \to \mathbb{Z}^n$ that would satisfy

$$\left|R_{\mu,b,a}\left(\mathbf{f}_\mu(\gamma)\right)\right| = \left|R_{\mu,a,b}(\gamma)\right| \qquad \text{for any } a, b \in \mathbb{Z} \text{ and } \gamma \in \mathbb{Z}^n.$$

Fixing $\gamma \in \mathbb{Z}^n$, we thus were looking for an n-tuple η (our $\mathbf{f}_\mu(\gamma)$-to-be) that would satisfy $\left|R_{\mu,b,a}(\eta)\right| = \left|R_{\mu,a,b}(\gamma)\right|$.

In the case when $b = 0$ (in which case this equality would be equivalent to saying "$\eta \rightharpoonup \mu$ if and only if $\mu \rightharpoonup \gamma$"), we found such an η directly, by setting

$$\eta_i = \mu_i + \mu_{i+1} - \gamma_{i+1} \qquad \text{for each } i \in \{1, 2, \ldots, n\},$$

where indices are cyclic modulo n (so that $\mu_0 = \mu_n$ and $\nu_0 = \nu_n$). This formula surprised us with its cyclic symmetry (which was not expected from the original problem, and which foreshadowed the usefulness of Convention 3.6, although we thought nothing of it at that point). Nevertheless, the formula failed in various examples for $b > 0$, and we could not easily fix it.

We tried to be more systematic. It was easy to rewrite the definition of $R_{\mu,a,b}(\gamma)$ as

$$\begin{aligned} &R_{\mu,a,b}(\gamma) \\ &= \{\nu \in \mathbb{Z}^n \mid (\min\{\mu_i, \gamma_i\} \geq \nu_i \geq \max\{\mu_{i+1}, \gamma_{i+1}\} \text{ for each } i \in \{1, 2, \ldots, n-1\}) \\ &\qquad\qquad \text{and } |\mu| - |\nu| = a \text{ and } |\gamma| - |\nu| = b\}. \end{aligned}$$

Thus, the size of this set would depend only

- on the differences $\min\{\mu_i, \gamma_i\} - \max\{\mu_{i+1}, \gamma_{i+1}\}$ for $i \in \{1, 2, \ldots, n-1\}$ (each of which differences would determine the "breathing space" for the corresponding ν_i),
- on the difference $|\mu| - |\gamma|$ (which would have to equal $a - b$ in order for the two conditions $|\mu| - |\nu| = a$ and $|\gamma| - |\nu| = b$ to be satisfiable simultaneously),
- as well as on something else we could not quite pinpoint (in order for $|\mu| - |\nu| = a$ and $|\gamma| - |\nu| = b$ to actually hold, as opposed to merely $|\mu| - |\gamma| = a - b$).

Analogous observations held for $R_{\mu,b,a}(\eta)$. With Occam's razor in hand, we suspected that $\left|R_{\mu,b,a}(\eta)\right| = \left|R_{\mu,a,b}(\gamma)\right|$ could best be achieved by requiring these differences to be the same for (μ, a, b, γ) as for (μ, b, a, η). Thus, in particular, we hoped to have

$$\min\{\mu_i, \gamma_i\} - \max\{\mu_{i+1}, \gamma_{i+1}\} = \min\{\mu_i, \eta_i\} - \max\{\mu_{i+1}, \eta_{i+1}\} \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}$$

and

$$|\mu| - |\gamma| = |\eta| - |\mu|.$$

(Due to the "mystery ingredient", this would likely neither be necessary nor sufficient for $\left|R_{\mu,b,a}(\eta)\right| = \left|R_{\mu,a,b}(\gamma)\right|$, but it looked like the right tree to bark up.) This is a system of equations that whose solution is neither unique nor straightforward. However, the system was a beacon rather than a destination to us, so we merely needed something like a good solution.

Systems of equations involving sums, differences, minima and maxima belong to *tropical geometry* – a discipline we were not expert in and could not hope to master quickly. However, we were aware of a surprisingly successful strategy for taming such systems: *detropicalization*. The mainstay of this strategy is the observation (made above in Example 3.3) that the binary operations min, max, $+$ and $-$ are the addition, the "harmonic addition"[19], the multiplication and the division of a certain semifield (the min tropical semifield of $(\mathbb{Z}, +, 0)$, or of whatever totally ordered abelian group our numbers belong to). Thus, even if we could not solve our system, we could generalize it to arbitrary semifields by replacing min, max, $+$ and $-$ by addition, "harmonic addition", multiplication and division, respectively. Thus our system would become

$$(\mu_i + \gamma_i) \,/\, \frac{1}{\dfrac{1}{\mu_{i+1}} + \dfrac{1}{\gamma_{i+1}}} = (\mu_i + \eta_i) \,/\, \frac{1}{\dfrac{1}{\mu_{i+1}} + \dfrac{1}{\eta_{i+1}}} \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}$$

[19] *Harmonic addition* is a binary operation defined on any semifield. It sends any pair (a, b) of elements of the semifield to $\dfrac{1}{\dfrac{1}{a} + \dfrac{1}{b}} = \dfrac{ab}{a+b}$.

and

$$\frac{\mu_1\mu_2\cdots\mu_n}{\gamma_1\gamma_2\cdots\gamma_n} = \frac{\eta_1\eta_2\cdots\eta_n}{\mu_1\mu_2\cdots\mu_n}.$$

Renaming μ, γ and η as u, x and y, and simplifying the fractions somewhat, we rewrote this as

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}$$

and

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2.$$

This new system was a system of polynomial equations (at least after clearing denominators), so we did the obvious thing: We left it to the computer for small values of n (specifically, $n = 2$, $n = 3$ and $n = 4$) and looked at the results. For $n = 3$, the computer (SageMath's `solve` function, to be precise) laid out the following two solutions:

- *Solution 1:*

$$\begin{aligned} y_1 &= \frac{u_1\left(u_1u_2u_3 + x_1u_2u_3 + x_1x_2u_3 + x_1x_2x_3\right)}{u_1x_2u_3 - x_1x_2x_3}, \\ y_2 &= \frac{-u_1u_2u_3}{x_1x_3}, \\ y_3 &= \frac{u_2u_3\left(x_1x_3 - u_1u_3\right)}{u_1u_2u_3 + x_1u_2u_3 + x_1x_2u_3 + x_1x_2x_3}. \end{aligned}$$

- *Solution 2:*

$$\begin{aligned} y_1 &= \frac{u_1u_3\left(u_1u_2 + x_1u_2 + x_1x_2\right)}{x_2\left(u_1u_3 + u_1x_3 + x_1x_3\right)}, \\ y_2 &= \frac{u_1u_2\left(u_2u_3 + x_2u_3 + x_2x_3\right)}{x_3\left(u_1u_2 + x_1u_2 + x_1x_2\right)}, \\ y_3 &= \frac{u_2u_3\left(u_1u_3 + u_1x_3 + x_1x_3\right)}{x_1\left(u_2u_3 + x_2u_3 + x_2x_3\right)}. \end{aligned}$$

The computer did not know that we were trying to work over a semifield (which had no subtraction), but we did, so we immediately discarded Solution 1 as useless due to the minus signs. The question was whether Solution 2 would be of any use. The omens were favorable: There were no minus signs; the (unexpected, but not unwelcome) cyclic symmetry reared its head again; finally, the nontrivial factors (such as $u_1u_2 + x_1u_2 + x_1x_2$) had a structure that appeared in the definition of the geometric crystal R-matrix (see, e.g., [Etingo03, (5)] or [NouYam02, (4.19)]) – a known successful case of detropicalization.

Solution 2 turned out to be generalizable indeed. Proving that the general formula indeed produced a solution to our system (parts **(b)** and **(c)** of Theorem 3.11) was not completely trivial, but not hard either. (The first few parts of Lemma 3.12 were discovered along the way.) Thus we had a candidate for the map $\mathbf{f}_\mu$ (and thus for the map φ, which was obtained from $\mathbf{f}_\mu$ by shifting by a and b, corresponding to the x_Π^a factor that we had dropped).

Why was this map $\mathbf{f}_\mu$ a bijection? Again, we believed that the easiest way lay through the birational realm (i.e., we had to detropicalize). Computer experiments suggested that $\mathbf{f}_\mu$ was not only a bijection but actually an involution (part **(a)** of Theorem 3.11). The first proof of this we found was the one sketched in Remark 3.16; the alternative, computational proof that we gave first was found afterwards.

Having found our bijection $\mathbf{f}_\mu$, we had to retrace our steps. Most of this was straightforward. The equality $\left|R_{\mu,b,a}\left(\mathbf{f}_\mu\left(\gamma\right)\right)\right| = \left|R_{\mu,a,b}\left(\gamma\right)\right|$ still had to be proved, but this turned out to be rather easy (part **(d)** of Theorem 3.11 was discovered along the way, as the missing ingredient from our previous analysis of the size of $R_{\mu,a,b}\left(\gamma\right)$). The way the proof was written up in the end was mostly decided by concerns of readability rather than authenticity; we believe that, had we followed the logic of its discovery in our writeup, we would have lost more in clarity than would be gained in motivation. We placed the study of the birational map $\mathbf{f}_u$ (Section 3) in front due to its self-contained nature and possible applicability to different problems; likewise, Section 4 begins with general properties of Schur Laurent polynomials and slowly progresses towards more technical lemmas tailored for the proof of Theorem 2.3. We would not be too surprised if some unnecessary detours were made along our way (Lemma 3.12 appears a particularly likely place for such), for which we apologize in advance (any simplifications are appreciated).

5.4. The birational R-matrix connection

In this section, we shall connect the map $\mathbf{f}_u$ from our Definition 3.8 with the *birational R-matrix* η defined in [LamPyl12, §6] and studied further (e.g.) in [CheLin20].

We fix a positive integer n and a semifield $\mathbb{K}$. We shall use Convention 3.4 and Convention 3.6. Let us recall the definition of the birational R-matrix η (no relation to the η in Theorem 2.3):

Definition 5.10. We define a map $\eta : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \times \mathbb{K}^n$ as follows:

Let $a \in \mathbb{K}^n$ and $b \in \mathbb{K}^n$ be two n-tuples. For any $i \in \mathbb{Z}$, define an element $\kappa_i\left(a,b\right) \in \mathbb{K}$ by

$$\kappa_i\left(a,b\right) = \sum_{j=i}^{i+n-1} \underbrace{b_{i+1}b_{i+2}\cdots b_j}_{=\prod_{p=i+1}^{j} b_p} \cdot \underbrace{a_{j+1}a_{j+2}\cdots a_{i+n-1}}_{=\prod_{p=j+1}^{i+n-1} a_p} .$$

Define $a' \in \mathbb{K}^n$ and $b' \in \mathbb{K}^n$ by setting

$$a'_i = \frac{a_{i-1}\kappa_{i-1}(a,b)}{\kappa_i(a,b)} \qquad \text{for each } i \in \{1,2,\ldots,n\}$$

and

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(a,b)}{\kappa_i(a,b)} \qquad \text{for each } i \in \{1,2,\ldots,n\}.$$

Set $\eta(a,b) = (a', b')$.

The map η we just defined is known as a *birational R-matrix*; related maps have previously appeared in the literature ([BraKaz00, Lemma 8.6], [Yamada01, Definition 2.1], [Etingo03, Proposition 3.1]). In particular, the map R from [Etingo03, Proposition 3.1] is equivalent to η (at least up to technical issues of where it is defined[20]). Indeed, it is not hard to see that the map η from Definition 5.10 becomes the map R from [Etingo03, Proposition 3.1] if we set $x_i = b_{i+1}$ and $y_i = a_i$ and $x'_i = b'_i$ and $y'_i = a'_{i+1}$ (that is, if we define x_i, y_i, x'_i, y'_i this way, then the equalities [Etingo03, (8), (9) and (10)] are satisfied, so that we have $R(x,y) = (x', y')$ where R is as defined in [Etingo03, Proposition 3.1]). This birational R-matrix R has its origins in the theory of geometric crystals and total positivity. A related map is the transformation $(x,a) \mapsto (y,b)$ in [NouYam02, §2.2] (see also [Zygour18]).

Now, we shall see that the map η is intimately related to our map $\mathbf{f}_u$ (even though $\mathbf{f}_u$ transforms a single n-tuple x into a single n-tuple y using the fixed n-tuple u, while η takes a pair of two n-tuples to another such pair). In order to state this relation, we define some more notation:

Definition 5.11. If $a \in \mathbb{K}^n$ and $b \in \mathbb{K}^n$ are two n-tuples, then we define two new n-tuples $ab \in \mathbb{K}^n$ and $\frac{a}{b} \in \mathbb{K}^n$ by setting

$$(ab)_i = a_i b_i \qquad \text{and} \qquad \left(\frac{a}{b}\right)_i = \frac{a_i}{b_i} \qquad \text{for each } i \in \{1,2,\ldots,n\}.$$

We can now express the map $\mathbf{f}_u$ from Definition 3.8 through the map η from Definition 5.10 as follows:

Theorem 5.12. Let $u \in \mathbb{K}^n$ and $x \in \mathbb{K}^n$ be two n-tuples. Let $(a', b') = \eta(u, x)$. Then,

$$\mathbf{f}_u(x) = u\frac{a'}{b'}.$$

[20]Namely: We have defined our map η as a literal map $\mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \times \mathbb{K}^n$ for any semifield $\mathbb{K}$, whereas [Etingo03, Proposition 3.1] defines R as a birational map $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$. Neither of these two settings generalizes the other, but it is not hard to transfer identities from one to the other (as long as they are *subtraction-free*, i.e., no minus signs appear in them).

Proof of Theorem 5.12. Set $a = u$ and $b = x$. We shall use the notations $\kappa_i(a,b)$ from Definition 5.10 and the notations $t_{r,j}$ and y from Definition 3.8. Then, $\mathbf{f}_u(x) = y$ (by Definition 3.8).

For each $i \in \mathbb{Z}$, we have

$$\begin{aligned}
\kappa_i(a,b) &= \sum_{j=i}^{i+n-1} b_{i+1}b_{i+2}\cdots b_j \cdot a_{j+1}a_{j+2}\cdots a_{i+n-1} && \text{(by the definition of } \kappa_i(a,b)) \\
&= \sum_{j=i}^{i+n-1} x_{i+1}x_{i+2}\cdots x_j \cdot u_{j+1}u_{j+2}\cdots u_{i+n-1} && \text{(since } a=u \text{ and } b=x) \\
&= \sum_{k=0}^{n-1} x_{i+1}x_{i+2}\cdots x_{i+k} \cdot u_{i+k+1}u_{i+k+2}\cdots u_{i+n-1} \\
&\qquad \text{(here, we have substituted } i+k \text{ for } j \text{ in the sum)} \\
&= t_{n-1,i}
\end{aligned} \tag{65}$$

(since the definition of $t_{n-1,i}$ yields $t_{n-1,i} = \sum_{k=0}^{n-1} x_{i+1}x_{i+2}\cdots x_{i+k}\cdot u_{i+k+1}u_{i+k+2}\cdots u_{i+n-1}$).

However, $(a',b') = \eta(u,x) = \eta(a,b)$ (since $u = a$ and $x = b$). Hence, Definition 5.10 yields that

$$a_i' = \frac{a_{i-1}\kappa_{i-1}(a,b)}{\kappa_i(a,b)} \qquad \text{for each } i \in \{1,2,\ldots,n\}$$

and

$$b_i' = \frac{b_{i+1}\kappa_{i+1}(a,b)}{\kappa_i(a,b)} \qquad \text{for each } i \in \{1,2,\ldots,n\}.$$

Hence, for each $i \in \{1,2,\ldots,n\}$, we have

$$\begin{aligned}
a_i'/b_i' &= \frac{a_{i-1}\kappa_{i-1}(a,b)}{\kappa_i(a,b)} \Big/ \frac{b_{i+1}\kappa_{i+1}(a,b)}{\kappa_i(a,b)} = \frac{a_{i-1}\kappa_{i-1}(a,b)}{b_{i+1}\kappa_{i+1}(a,b)} = \frac{a_{i-1}t_{n-1,i-1}}{b_{i+1}t_{n-1,i+1}} \\
&\qquad \left(\begin{array}{c}\text{since } \kappa_{i-1}(a,b) = t_{n-1,i-1} \text{ (by (65), applied to } i-1 \text{ instead of } i) \\ \text{and } \kappa_{i+1}(a,b) = t_{n-1,i+1} \text{ (by (65), applied to } i+1 \text{ instead of } i)\end{array}\right) \\
&= \frac{u_{i-1}t_{n-1,i-1}}{x_{i+1}t_{n-1,i+1}} \qquad \text{(since } a=u \text{ and } b=x) \\
&= y_i/u_i
\end{aligned} \tag{66}$$

(since the definition of y yields $y_i = u_i \cdot \dfrac{u_{i-1}t_{n-1,i-1}}{x_{i+1}t_{n-1,i+1}}$). Now, for each $i \in \{1,2,\ldots,n\}$, we have

$$\begin{aligned}
\left(u\frac{a'}{b'}\right)_i &= u_i\frac{a_i'}{b_i'} && \text{(by Definition 5.11)} \\
&= u_i \cdot \underbrace{a_i'/b_i'}_{\substack{=y_i/u_i \\ \text{(by (66))}}} = u_i \cdot y_i/u_i = y_i.
\end{aligned}$$

In other words, $u\dfrac{a'}{b'} = y$. Comparing this with $\mathbf{f}_u(x) = y$, we obtain $\mathbf{f}_u(x) = u\dfrac{a'}{b'}$. This proves Theorem 5.12. □

We finish by stating some "gauge-invariance" properties for $\mathbf{f}_u$ and η:

Proposition 5.13. Let $g, u, x \in \mathbb{K}^n$. Then, $\mathbf{f}_{gu}(gx) = g\mathbf{f}_u(x)$.

Proposition 5.14. Let $g, a, b \in \mathbb{K}^n$. Let $(a', b') = \eta(a, b)$. Then, $(ga', gb') = \eta(ga, gb)$.

Both of these propositions can be proved by fairly simple computations, which are left to the reader.

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