The Redei–Berge symmetric function of a directed graph

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joint work with Richard P. Stanley

KTH Stockholm, 2024-03-20;
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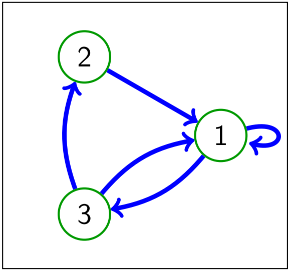
slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/kth2024.pdf

paper (draft): https://arxiv.org/abs/2307.05569

- **Definition.** A **digraph** (= directed graph) means a pair (V, A) of a finite set V and a subset $A \subseteq V \times V$.
  The elements $(u, v) \in A$ are called **arcs** of this digraph, and are drawn accordingly.
- **Example.**



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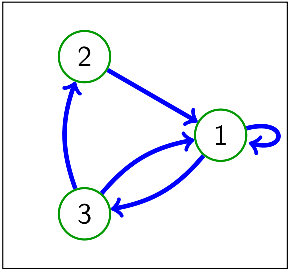

- Thus, we allow loops ($(u, u) \in A$) and antiparallel arcs ($(u, v) \in A$ and $(v, u) \in A$) but not parallel arcs (A is not a multiset).

The complement of a digraph

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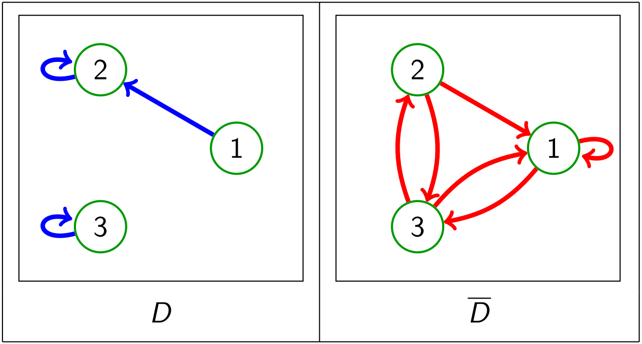

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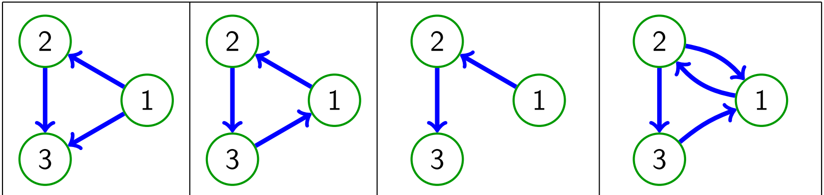
- **Examples.**



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- **Definition.** Let $D = (V, A)$ be a digraph. A **Hamiltonian path** (short: **hamp**) of D means a V-listing $(v_1, v_2, \ldots, v_n)$ such that

$$(v_i, v_{i+1}) \in A \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}.$$

  In other words (for $V \neq \varnothing$), it means a path of D that contains each vertex.

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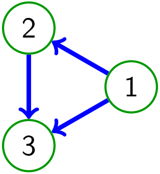
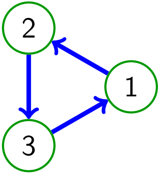
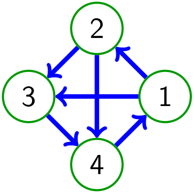
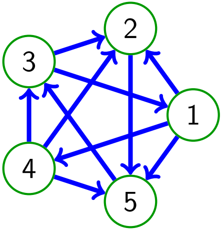
- **Example.** Here are some tournaments:

			
has 1 hamp: $(1,2,3)$	has 3 hamps: e.g., $(2,3,1)$	has 5 hamps: e.g., $(1,2,3,4)$	has 9 hamps: e.g., $(1,4,2,5,3)$

- Recall **Redei's Theorem:** Let D be a tournament. Then,

$$(\# \text{ of hamps of } D) \text{ is odd.}$$

- Rédei's proof is complicated and intransparent (see Moon, *Topics on Tournaments* for an English version).

Berge's theorem

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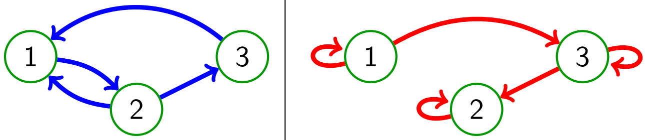
- Rédei's proof is complicated and intransparent (see Moon, *Topics on Tournaments* for an English version).
  To give a more conceptual proof, Berge discovered the following:
- **Theorem (Berge 1976):** Let D be a digraph. Then,

$$(\#\text{ of hamps of } \overline{D}) \equiv (\#\text{ of hamps of } D) \bmod 2.$$

- **Example.**



D	$\overline{D}$
has 3 hamps	has 1 hamp

- Berge proves his theorem (in his *Graphs* textbook) using an elegant inclusion-exclusion argument.
  Then he uses his theorem to prove Rédei's theorem via induction on the number of "inversions" (arcs directed the "wrong way").
  This proof is much cleaner than Rédei's, but still far from simple.
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- **Remark.** Can we improve on Rédei's theorem even further?
  MathOverflow question #232751 asks for the possible values of (# of hamps of D) for a tournament D.
  Among the numbers between 1 and 80555, the answer is "all odd numbers except for 7 and 21" (proved by bof and Gordon Royle).
  Question: Are these the only exceptions?

- Independently, Chow (*The Path-Cycle Symmetric Function of a Digraph*, 1996) introduced a symmetric function assigned to each digraph D.
  (This was inspired by Chung/Graham's cover polynomial in rook theory.)
- We only discuss a coarsening of his construction (Chow has two families of variables, and we set the second family to 0).
  Question: Which of the results below can be generalized to the full version?

- **Definition.** Let $n \in \mathbb{N}$, and let I be a subset of $\{1, 2, \ldots, n-1\}$. Then, we define the power series

$$L_{I,n} := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$$

(where the indices $i_1, i_2, \ldots, i_n$ range over $\{1, 2, 3, \ldots\}$).
Remark: This is a formal power series (but becomes a polynomial if you drop all but finitely many variables).
It is known as a **(Gessel's) fundamental quasisymmetric function**.

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- For instance,
$$L_{\{1\},3} = \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3};$$
$$L_{\{1\},4} = \sum_{i_1 < i_2 \leq i_3 \leq i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}.$$

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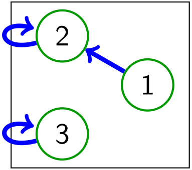
- **Definition.** Let $n \in \mathbb{N}$. Let $D = (V, A)$ be a digraph with n vertices. We define the **Redei–Berge symmetric function**

$$U_D := \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]],$$

where

$$\operatorname{Des}(w, D) := \{i \in \{1, 2, \ldots, n-1\} \mid (w_i, w_{i+1}) \in A\}$$
$$\text{for each } V\text{-listing } w = (w_1, w_2, \ldots, w_n).$$

- **Example:** Let

$D =$  .

Then,

$$U_D = \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ 3}$$
$$= L_{\operatorname{Des}((1,2,3),D),\ 3} + L_{\operatorname{Des}((1,3,2),D),\ 3} + L_{\operatorname{Des}((2,1,3),D),\ 3}$$
$$+ L_{\operatorname{Des}((2,3,1),D),\ 3} + L_{\operatorname{Des}((3,1,2),D),\ 3} + L_{\operatorname{Des}((3,2,1),D),\ 3}$$
$$= L_{\{1\},\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\{2\},\ 3} + L_{\varnothing,\ 3}$$
$$= 4 \cdot L_{\varnothing,\ 3} + L_{\{1\},\ 3} + L_{\{2\},\ 3}$$
$$= 4 \cdot \sum_{i_1 \leq i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}.$$

The Rédei–Berge symmetric function: restatement

- We can restate the definition of U_D directly as follows:
- **Proposition.** Let $D = (V, A)$ be a digraph. Then,

$$U_D = \sum_{f : V \to \{1,2,3,\ldots\}} a_{D,f} \prod_{v \in V} x_{f(v)},$$

where $a_{D,f}$ is the # of all V-listings $w = (w_1, w_2, \ldots, w_n)$ such that

  - we have $f(w_1) \leq f(w_2) \leq \cdots \leq f(w_n)$;
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- This is similar (though not directly related) to P-partition enumerators and chromatic symmetric functions.

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  - we have $f(w_i) < f(w_{i+1})$ if $(w_i, w_{i+1}) \in A$.
- **Remark.** We can restate the definition of $a_{D,f}$ in nicer terms. Namely, fix a digraph $D = (V, A)$ and a map $f : V \to \{1, 2, 3, \ldots\}$. For any $j \in f(V)$, let $\overline{D_j}$ denote the induced subdigraph of the complement $\overline{D}$ on the vertex set $f^{-1}(j) = \{v \in V \mid f(v) = j\}$. Then,

$$a_{D,f} = \prod_{j \in f(V)} \left(\# \text{ of hamps of } \overline{D_j}\right).$$

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- Formulas for U_D in some specific cases (D acyclic, D poset, D path) can be found in Additional Problem 120 to Chapter 7 of Stanley's EC2. Most prominently, if D is the "greater-than digraph" of a poset P, then $U_{\overline{D}}$ is the chromatic symmetric function of the incomparability graph of P.

- I called U_D the "Rédei–Berge symmetric function", but is it actually symmetric? Yes, and in fact something better holds:
- **Definition.** For each $k \geq 1$, let

$$p_k := x_1^k + x_2^k + x_3^k + \cdots$$

be the k**-th power-sum symmetric function**.

- **Theorem.** For any digraph D, we have

$$U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots].$$

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  That is, U_D can be written as a polynomial in $p_1, p_2, p_3, \ldots$ over $\mathbb{Z}$.
- Which polynomial, though?

- **Definition.** Fix a digraph $D = (V, A)$.
  Let $\mathfrak{S}_V$ be the symmetric group on the set V.
  For any $\sigma \in \mathfrak{S}_V$, we let

$$p_{\text{type}\,\sigma} := \prod_{\gamma \text{ is a cycle of } \sigma} p_{\text{length of } \gamma}.$$

  In other words, if σ has cycles of lengths $a, b, \ldots, k$ (including 1-cycles), then $p_{\text{type}\,\sigma} = p_a p_b \cdots p_k$.

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- **Main Theorem I.** Let $D = (V, A)$ be a digraph. Set

$$\varphi(\sigma) := \sum_{\substack{\gamma \text{ is a cycle of } \sigma; \\ \gamma \text{ is a } D\text{-cycle}}} ((\text{length of } \gamma) - 1) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

  Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} (-1)^{\varphi(\sigma)} p_{\text{type}\,\sigma}.$$

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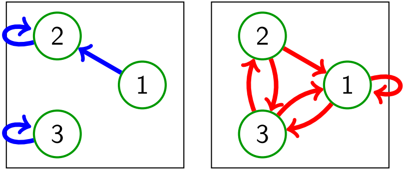
- This yields the $U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$ theorem, of course.

- **Example.** Recall our favorite example:



D $\qquad\qquad$ $\overline{D}$.

The cycles of D are $(2)_\sim$ and $(3)_\sim$, whereas the cycles of $\overline{D}$ are $(1)_\sim$, $(2,3)_\sim$, $(3,1)_\sim$ and $(1,3,2)_\sim$.
Thus, the $\sum\limits_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}}$ sum in Main Theorem I has four addends, corresponding to (σ written in one-line notation)

$\sigma =$	$[1,2,3]$	$[3,1,2]$	$[1,3,2]$	$[3,2,1]$
$(-1)^{\varphi(\sigma)} =$	1	1	1	1
$p_{\operatorname{type}\sigma} =$	p_1^3	p_3	$p_2 p_1$	$p_2 p_1$

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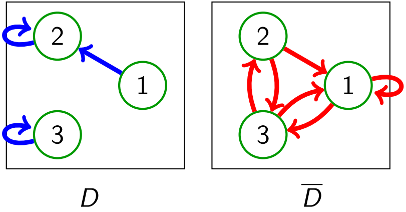

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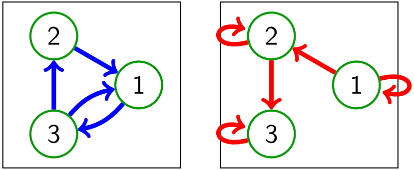
Hence, Main Theorem I yields

$$U_D = p_1^3 + p_3 + p_2 p_1 + p_2 p_1 = p_1^3 + 2p_1 p_2 + p_3.$$

- **Example.** Another example: Let



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Thus, the $\sum\limits_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}}$ sum in Main Theorem I has three addends, with

$\sigma =$	$[1, 2, 3]$	$[3, 1, 2]$	$[3, 2, 1]$
$(-1)^{\varphi(\sigma)} =$	1	1	-1
$p_{\operatorname{type} \sigma} =$	p_1^3	p_3	$p_2 p_1$

Hence, Main Theorem I yields $U_D = p_1^3 + p_3 - p_2 p_1$.

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- Main Theorem I yields Berge's theorem, since the sum for D and the sum for $\overline{D}$ range over the same σ's, and the addends only differ in sign.

- Recall **Main Theorem I:** Let $D = (V, A)$ be a digraph. Set

$$\varphi(\sigma) := \sum_{\substack{\gamma \text{ is a cycle of } \sigma; \\ \gamma \text{ is a } D\text{-cycle}}} ((\text{length of } \gamma) - 1) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

  Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} (-1)^{\varphi(\sigma)} p_{\operatorname{type} \sigma}.$$

- Main Theorem I yields Berge's theorem, since the sum for D and the sum for $\overline{D}$ range over the same σ's, and the addends only differ in sign.
- **Corollary.** Let $D = (V, A)$ be a digraph. Assume that every D-cycle has odd length. Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} p_{\operatorname{type} \sigma} \in \mathbb{N}[p_1, p_2, p_3, \ldots].$$

- **Main Theorem II.** Let $D = (V, A)$ be a tournament. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. (A cycle is called **nontrivial** if it has length > 1.) Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is a } D\text{-cycle;} \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type} \sigma}$$

$$\in \mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots] = \mathbb{N}[p_1, \ 2p_i \ \mid \ i > 1 \text{ is odd}].$$

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- Main Theorem II easily yields Rédei's theorem, as the only addend with $2^{\psi(\sigma)}$ odd is the $\sigma = \mathrm{id}$ addend.

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- **Remark.** Not all p-positive U_D's are explained by this theorem.

p-expansions: Proof ideas, 1

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- **Pólya-style lemma.** Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Then,

$$\sum_{\substack{f:V\to\{1,2,3,\ldots\};\\ f\circ\sigma=f}} \prod_{v\in V} x_{f(v)} = p_{\operatorname{type}\sigma}.$$

  Proof. Easy exercise.

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- Using this lemma (and the above formula for $a_{D,f}$), we can easily reduce Main Theorem I to the following lemma:
- **Main combinatorial lemma.** Let $D = (V, A)$ be a digraph with n vertices. Let $f : V \to \{1, 2, 3, \ldots\}$ be any map. Then,

$$\prod_{j\in f(V)} \left(\#\text{ of hamps of } \overline{D_j}\right) = \sum_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is}\\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle;}\\ f\circ\sigma=f}} (-1)^{\varphi(\sigma)},$$

where $\overline{D_j}$ is the induced subdigraph of $\overline{D}$ on the vertex set $f^{-1}(j)$.

- So we need to prove the **Main combinatorial lemma:** Let $D = (V, A)$ be a digraph with n vertices. Let $f : V \to \{1, 2, 3, \ldots\}$ be any map. Then,

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- Work on each "level set" $f^{-1}(j)$ separately:
  Main combinatorial lemma (simplified). Let $D = (V, A)$ be a digraph. Then,

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- This can be proved using a nontrivial exclusion-inclusion.

- To prove Main Theorems II and III, start with the Main Theorem I sum, and combine σ's into equivalence classes by reversing certain cycles:
  - For Main Theorem II, call two permutations in $\mathfrak{S}_V$ equivalent if one can be obtained from the other by reversing (nontrivial) cycles. This turns D-cycles into $\overline{D}$-cycles and vice versa. The equivalence class of σ has $2^{\psi(\sigma)}$ elements if σ has no 2-cycles. Their addends in the sum cancel out if σ has an even-length cycle; otherwise they are all equal and sum up to $2^{\psi(\sigma)} p_{\operatorname{type} \sigma}$.

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  - For Main Theorem III, call a necklace $(v_1, v_2, \ldots, v_k)_\sim$ **risky** if its length k is even and either it or its inverse is a D-cycle. Call two permutations in $\mathfrak{S}_V$ equivalent if one can be obtained from the other by reversing risky cycles. The equivalence class of σ has $2^{r(\sigma)}$ elements, where $r(\sigma)$ is the number of risky cycles of σ. Their addends in the sum cancel out if σ has a risky cycle; otherwise there is only one of them.

- The proof of Main Theorem **I** is detailed in the preprint (https://arxiv.org/abs/2307.05569); the proofs of **II** and **III** are outlined.
  These would make a good project for formalization (Coq, Lean, etc.): only elementary combinatorics but some tricky reasoning with cycles and sums.

- Rédei's theorem determines the # of hamps of a tournament D modulo 2. What about mod 4?

- Rédei's theorem determines the # of hamps of a tournament D modulo 2. What about $\mod 4$?
- **Theorem.** Let D be a tournament. Then,

$$\begin{aligned}&(\#\text{ of hamps of } D)\\ &\equiv 1 + 2\,(\#\text{ of nontrivial odd-length } D\text{-cycles}) \mod 4.\end{aligned}$$

  Here, "nontrivial" means "having length > 1".
- We can prove this using Main Theorem II. We have not seen this anywhere in the literature.

- Main Theorem I can be rewritten without speaking about digraphs:

- **Theorem.** Let $n \in \mathbb{N}$, and let V be an n-element set. Let $\mathbf{k}$ be a commutative ring.
  For any $a = (i, j) \in V \times V$, we fix an element $t_a = t_{(i,j)} \in \mathbf{k}$ and set $s_a := t_a + 1$.
  We define the **deformed Redei–Berge symmetric function**

$$\widetilde{U}_t := \sum_{\substack{w=(w_1,w_2,\ldots,w_n) \\ \text{is a } V\text{-listing}}} \ \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \left(\prod_{\substack{k \in [n-1]; \\ i_k = i_{k+1}}} s_{(w_k, w_{k+1})} \right) x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$\in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right].$$

  Then,

$$\widetilde{U}_t = \sum_{\sigma \in \mathfrak{S}_V} \left(\prod_{\gamma \text{ is a cycle of } \sigma} \left(\prod_{i \in \gamma} s_{(i, \sigma(i))} - \prod_{i \in \gamma} t_{(i, \sigma(i))} \right) \right) p_{\operatorname{type} \sigma}.$$

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- This generalizes Main Theorem I (set each t_a to 0 or -1), but also follows from it by multilinearity.

I thank

- **Richard P. Stanley** for the obvious reasons.
- **Hsin Chieh Liao, Anna Pun, Bruce Sagan, Mike Zabrocki** for helpful comments.
- **the organizers** for the opportunity to present this.
- **you** for your patience.

- In his paper *The Path-Cycle Symmetric Function of a Digraph* (1996), Timothy Y. Chow defined the **path-cycle symmetric function** of a digraph:

- **Definition.** Let $D = (V, A)$ be a digraph. The **path-cycle symmetric function** $\Xi_D = \Xi_D(\mathbf{x}, \mathbf{y})$ in two infinite families of indeterminates $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots)$ is the power series

$$\sum_{S \text{ is a PC-cover}} \ \sum_{\substack{f:V\to\{1,2,3,\ldots\} \\ \text{is an } S\text{-friendly} \\ \text{coloring}}} \ \prod_{\substack{v\in V \text{ is on} \\ \text{an } S\text{-path}}} x_{f(v)} \cdot \prod_{\substack{v\in V \text{ is on} \\ \text{an } S\text{-cycle}}} y_{f(v)}.$$

  Here:
  - A **PC-cover** means a set of paths and cycles of D (possibly trivial ones) such that each vertex in V belongs to exactly one of these.
  - If S is a PC-cover, then an S-**friendly coloring** means a map $f : V \to \{1, 2, 3, \ldots\}$ such that
    - $f(v) = f(w)$ whenever v and w lie on the same S-path;
    - $f(v) = f(w)$ whenever v and w lie on the same S-cycle;
    - $f(v) \neq f(w)$ whenever v and w lie on different S-paths;

    (but different S-cycles are unconstrained!).

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- Setting $\mathbf{y} = \mathbf{0}$ (that is, $y_i = 0$ for all i) amounts to forbidding cycles in PC-covers, thus turning them into P-covers. Thus, $\Xi_D(\mathbf{x}, \mathbf{0}) = U_{\overline{D}}$ (exercise!).

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- **Question.** Do any of our results extend to Ξ_D ?

- **Theorem (Gessel, Stanley, [EC2, Ch. 7, Additional problem 122 (h)]).** Let $D = (V, A)$ be the path digraph with $n > 0$ vertices. Then,

$$U_D = \sum_{i=1}^{n} f_i s_{(i,1^{n-i})}.$$

  Here, s_λ denotes the Schur function, whereas f_i means the # of permutations $\sigma \in S_i$ in which no entry j is followed by $j-1$ (OEIS sequence A000255).

- **Question.** When else is U_D Schur-positive?