Integrality over ideal semifiltrations

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Abstract. We study integrality over rings (all commutative in this paper) and over ideal semifiltrations (a generalization of integrality over ideals). We begin by reproving classical results, such as a version of the "faithful module" criterion for integrality over a ring, the transitivity of integrality, and the theorem that sums and products of integral elements are again integral. Then, we define the notion of integrality over an ideal semifiltration (a sequence $(I_0, I_1, I_2, \ldots)$ of ideals satisfying $I_0 = A$ and $I_a I_b \subseteq I_{a+b}$ for all $a, b \in \mathbb{N}$), which generalizes both integrality over a ring and integrality over an ideal (as considered, e.g., in Swanson/Huneke [5]). We prove a criterion that reduces this general notion to integrality over a ring using a variant of the Rees algebra. Using this criterion, we study this notion further and obtain transitivity and closedness under sums and products for it as well. Finally, we prove the curious fact that if u, x and y are three elements of a (commutative) A-algebra (for A a ring) such that u is both integral over $A[x]$ and integral over $A[y]$, then u is integral over $A[xy]$. We generalize this to integrality over ideal semifiltrations, too.

Contents

*updated and improved version of undergraduate work from 2010

Introduction

The purpose of this paper is to state (and prove) some theorems and proofs related to integrality in commutative algebra in somewhat greater generality than is common in the literature. I claim no novelty, at least not for the underlying ideas, but I hope that this paper will be useful as a reference (at least for myself).

Section 1 (Integrality over rings) mainly consists of known facts (Theorem 1.1, Theorem 1.5, Theorem 1.7) and a generalized exercise from [4] (Corollary 1.12) with a few minor variations (Theorem 1.11 and Corollary 1.13).

Section 2 (Integrality over ideal semifiltrations) merges the concept of integrality over rings (as considered in Section 1) and integrality over ideals (a less popular but still highly useful notion; the book [5] is devoted to it) into one general notion: that of integrality over ideal semifiltrations (Definition 2.3). This notion is very general, yet it can be reduced to the basic notion of integrality over rings by a suitable change of base ring (Theorem 2.11). This reduction allows to extend some standard properties of integrality over rings to the general case (Theorem 2.13, Theorem 2.14 and Theorem 2.16).

Section 3 (Generalizing to two ideal semifiltrations) continues Section 2, adding one more layer of generality. Its main results are a "relative" version of Theorem 2.11 (Theorem 3.2) and a known fact generalized once more (Theorem 3.4).

Section 4 (Accelerating ideal semifiltrations) generalizes Theorem 3.2 (and thus also Theorem 2.11) a bit further by considering accelerated ideal semifiltrations (a generalization of powers of an ideal).

Section 5 (On a lemma by Lombardi) is about an auxiliary result Henri Lombardi used in [6] to prove Kronecker's Theorem[1]. Here we show a variant of this result (generalized in one direction, less general in another).

This paper is supposed to be self-contained (only linear algebra and basic knowledge about rings, modules, ideals and polynomials is assumed).

All proofs given in this paper are constructive (with the exception of the proof of Lemma 5.1, which proceeds by contradiction out of sheer convenience; a constructive version of this proof can be found in [7]).

Note on the level of detail

This is the short version of this paper. It constitutes an attempt to balance clarity and brevity in the proofs. See [7] for a more detailed version.

Note on an old preprint

This is an updated and somewhat generalized version of my preprint "A few facts on integrality", which is still available in its old form as well:

- brief version:
  `https://www.cip.ifi.lmu.de/~grinberg/IntegralityBRIEF.pdf`
- long version:
  `https://www.cip.ifi.lmu.de/~grinberg/Integrality.pdf` .

Be warned that said preprint has been written in 2009–2010 when I was an undergraduate, and suffers from bad writing and formatting.

Acknowledgments

I thank Irena Swanson and Marco Fontana for enlightening conversations, and Irena Swanson in particular for making her book [5] freely available (which helped me discover the subject as an undergraduate).

[1] **Kronecker's Theorem.** Let B be a ring ("ring" always means "commutative ring with unity" in this paper). Let g and h be two elements of the polynomial ring $B[X]$. Let g_α be any coefficient of the polynomial g. Let h_β be any coefficient of the polynomial h. Let A be a subring of B which contains all coefficients of the polynomial gh. Then, the element $g_\alpha h_\beta$ of B is integral over the subring A.

0. Definitions and notations

We begin our study of integrality with some classical definitions and conventions from commutative algebra:

Definition 0.1. In the following, "ring" will always mean "commutative ring with unity". Furthermore, if A is a ring, then "A-algebra" shall always mean "commutative A-algebra with unity". The unity of a ring A will be denoted by 1_A or by 1 if no confusion can arise.

We denote the set $\{0, 1, 2, \ldots\}$ by $\mathbb{N}$, and the set $\{1, 2, 3, \ldots\}$ by $\mathbb{N}^+$.

Definition 0.2. Let A be a ring. Let M be an A-module.

If $n \in \mathbb{N}$, and if $m_1, m_2, \ldots, m_n$ are n elements of M, then we define an A-submodule $\langle m_1, m_2, \ldots, m_n \rangle_A$ of M by

$$\langle m_1, m_2, \ldots, m_n \rangle_A = \left\{ \sum_{i=1}^{n} a_i m_i \mid (a_1, a_2, \ldots, a_n) \in A^n \right\}.$$

This A-submodule $\langle m_1, m_2, \ldots, m_n \rangle_A$ is known as the A-submodule of M generated by $m_1, m_2, \ldots, m_n$ (or as the *A-linear span* of $m_1, m_2, \ldots, m_n$). It consists of all A-linear combinations of $m_1, m_2, \ldots, m_n$, and in particular contains all n elements $m_1, m_2, \ldots, m_n$. Thus, it satisfies $\{m_1, m_2, \ldots, m_n\} \subseteq \langle m_1, m_2, \ldots, m_n \rangle_A$.

Also, if S is a finite set, and m_s is an element of M for every $s \in S$, then we define an A-submodule $\langle m_s \mid s \in S \rangle_A$ of M by

$$\langle m_s \mid s \in S \rangle_A = \left\{ \sum_{s \in S} a_s m_s \mid (a_s)_{s \in S} \in A^S \right\}.$$

This A-submodule $\langle m_s \mid s \in S \rangle_A$ is known as the A-submodule of M generated by the family $(m_s)_{s \in S}$ (or as the *A-linear span* of $(m_s)_{s \in S}$). It consists of all A-linear combinations of the elements m_s with $s \in S$, and in particular contains all these elements themselves.

Of course, if $m_1, m_2, \ldots, m_n$ are n elements of M, then

$$\langle m_1, m_2, \ldots, m_n \rangle_A = \langle m_s \mid s \in \{1, 2, \ldots, n\} \rangle_A.$$

Let us observe a trivial fact that we shall use (often tacitly):

Lemma 0.3. Let A be a ring. Let M be an A-module. Let N be an A-submodule of M. Let S be a finite set; let m_s be an element of N for every $s \in S$. Then, $\langle m_s \mid s \in S \rangle_A \subseteq N$.

Definition 0.4. Let A be a ring, and let $n \in \mathbb{N}$. Let M be an A-module. We say that the A-module M is *n-generated* if there exist n elements $m_1, m_2, \ldots, m_n$ of M such that $M = \langle m_1, m_2, \ldots, m_n \rangle_A$. In other words, the A-module M is n-generated if and only if there exists a set S and an element m_s of M for every $s \in S$ such that $|S| = n$ and $M = \langle m_s \mid s \in S \rangle_A$.

Definition 0.5. Let A be a ring. Let B be an A-algebra. (Let us recall that both rings and algebras are always understood to be commutative and unital in this paper.)

If $u_1, u_2, \ldots, u_n$ are n elements of B, then we define an A-subalgebra $A[u_1, u_2, \ldots, u_n]$ of B by

$$A[u_1, u_2, \ldots, u_n] = \{P(u_1, u_2, \ldots, u_n) \mid P \in A[X_1, X_2, \ldots, X_n]\}$$

(where $A[X_1, X_2, \ldots, X_n]$ denotes the polynomial ring in n indeterminates $X_1, X_2, \ldots, X_n$ over A).

In particular, if u is an element of B, then the A-subalgebra $A[u]$ of B is defined by

$$A[u] = \{P(u) \mid P \in A[X]\}$$

(where $A[X]$ denotes the polynomial ring in a single indeterminate X over A). Since

$$A[X] = \left\{ \sum_{i=0}^{m} a_i X^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \ldots, a_m) \in A^{m+1} \right\},$$

this becomes

$$\begin{aligned} A[u] &= \left\{ \left(\sum_{i=0}^{m} a_i X^i \right)(u) \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \ldots, a_m) \in A^{m+1} \right\} \\ &\qquad \left(\begin{array}{c} \text{where } \left(\sum_{i=0}^{m} a_i X^i \right)(u) \text{ means the} \\ \text{polynomial } \sum_{i=0}^{m} a_i X^i \text{ evaluated at } X = u \end{array} \right) \\ &= \left\{ \sum_{i=0}^{m} a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \ldots, a_m) \in A^{m+1} \right\} \\ &\qquad \left(\text{because } \left(\sum_{i=0}^{m} a_i X^i \right)(u) = \sum_{i=0}^{m} a_i u^i \right). \end{aligned}$$

Obviously, $uA[u] \subseteq A[u]$ (since $A[u]$ is an A-algebra and $u \in A[u]$).

Definition 0.6. Let B be a ring, and let A be a subring of B. Then, B canonically becomes an A-algebra. The A-module structure of this A-algebra B is given by multiplication inside B.

Definition 0.6 shows that theorems about A-algebras (for a ring A) are always more general than theorems about rings that contain A as a subring. Hence, we shall study A-algebras in the following, even though most of the applications of the results we shall see are found at the level of rings containing A.

1. Integrality over rings

1.1. The fundamental equivalence

Most of the theory of integrality is based upon the following result:

Theorem 1.1. Let A be a ring. Let B be an A-algebra. Thus, B is canonically an A-module. Let $n \in \mathbb{N}$. Let $u \in B$. Then, the following four assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are equivalent:

- *Assertion $\mathcal{A}$:* There exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.
- *Assertion $\mathcal{B}$:* There exist a B-module C and an n-generated A-submodule U of C such that $uU \subseteq U$ and such that every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$. (Here, C is an A-module, since C is a B-module and B is an A-algebra.)
- *Assertion $\mathcal{C}$:* There exists an n-generated A-submodule U of B such that $1 \in U$ and $uU \subseteq U$. (Here and in the following, "1" means "1_B", that is, the unity of the ring B.)
- *Assertion $\mathcal{D}$:* We have $A[u] = \langle u^0, u^1, \ldots, u^{n-1} \rangle_A$.

We shall soon prove this theorem; first, let us explain what it is for:

Definition 1.2. Let A be a ring. Let B be an A-algebra. Let $n \in \mathbb{N}$. Let $u \in B$. We say that the element u of B is *n-integral over A* if it satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1.1.

Hence, in particular, the element u of B is n-integral over A if and only if it satisfies the assertion $\mathcal{A}$ of Theorem 1.1. In other words, u is n-integral over A if and only if there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.

The notion of "n-integral" elements that we have just defined is a refinement of the classical notion of integrality of elements over rings (see, e.g., [1, Definition

(10.21)] or [2, Chapter V, §1.1, Definition 1] or [3, Definition 8.1.1] for this classical notion, and [5, Definition 2.1.1] for its particular case when A is a subring of B). Indeed, the classical notion defines an element u of B to be *integral* over A if and only if (using the language of our Definition 1.2) there exists some $n \in \mathbb{N}$ such that u is n-integral over A. Since I believe the concrete value of n to be worth more than its mere existence, I prefer the specificity of the "n-integral" concept to the slickness of "integral".

Theorem 1.1 is one of several similar results providing equivalent criteria for the integrality of an element of an A-algebra. See [1, Proposition (10.23)], [2, Chapter V, Section 1.1, Theorem 1] or [3, Theorem 8.1.6] for other such results (some very close to Theorem 1.1, and all proven in similar ways).

Before we prove Theorem 1.1, let us recall a classical property of matrices:

Lemma 1.3. Let B be a ring. Let $n \in \mathbb{N}$. Let M be an $n \times n$-matrix over B. Then,

$$\det M \cdot I_n = \operatorname{adj} M \cdot M.$$

(Here, I_n means the $n \times n$ identity matrix and $\operatorname{adj} M$ denotes the adjugate of the matrix M. The expressions "$\det M \cdot I_n$" and "$\operatorname{adj} M \cdot M$" have to be understood as "$(\det M) \cdot I_n$" and "$(\operatorname{adj} M) \cdot M$", respectively.)

Lemma 1.3 is well-known (for example, it follows from [8, Theorem 6.100], applied to $\mathbb{K} = B$ and $A = M$).

Proof of Theorem 1.1. We will prove the implications $\mathcal{A} \Longrightarrow \mathcal{C}$, $\mathcal{C} \Longrightarrow \mathcal{B}$, $\mathcal{B} \Longrightarrow \mathcal{A}$, $\mathcal{A} \Longrightarrow \mathcal{D}$ and $\mathcal{D} \Longrightarrow \mathcal{C}$.

Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{C}$. Assume that Assertion $\mathcal{A}$ holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Consider this P. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements $a_0, a_1, \ldots, a_{n-1}$ of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Consider these $a_0, a_1, \ldots, a_{n-1}$. Substituting u for X in the equality $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$, we find $P(u) = u^n + \sum\limits_{k=0}^{n-1} a_k u^k$. Hence, the equality $P(u) = 0$ (which holds by definition of P) rewrites as $u^n + \sum\limits_{k=0}^{n-1} a_k u^k = 0$. Hence, $u^n = -\sum\limits_{k=0}^{n-1} a_k u^k$.

Let U be the A-submodule $\left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A$ of B. Then, $U = \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A$ and

$$u^n = -\sum_{k=0}^{n-1} a_k u^k \in \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A = U.$$

Moreover, the n elements $u^0, u^1, \ldots, u^{n-1}$ belong to U (since $U = \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A$). In other words,

$$u^i \in U \qquad \text{for each } i \in \{0, 1, \ldots, n-1\}. \tag{1}$$

This relation also holds for $i = n$ (since $u^n \in U$); thus, it holds for all $i \in \{0, 1, \ldots, n\}$. In other words, we have

$$u^i \in U \qquad \text{for each } i \in \{0, 1, \ldots, n\}. \tag{2}$$

Applying this to $i = 0$, we find $u^0 \in U$. Thus, $1 = u^0 \in U$.

Recall that $U = \langle u^0, u^1, \ldots, u^{n-1} \rangle_A$. Hence, U is an n-generated A-module (since $u^0, u^1, \ldots, u^{n-1}$ are n elements of U).

Now, for each $s \in \{0, 1, \ldots, n-1\}$, we have $s + 1 \in \{1, 2, \ldots, n\} \subseteq \{0, 1, \ldots, n\}$ and thus $u^{s+1} \in U$ (by (2), applied to $i = s + 1$). Hence, Lemma 0.3 (applied to $M = B$, $N = U$, $S = \{0, 1, \ldots, n-1\}$ and $m_s = u^{s+1}$) yields

$$\left\langle u^{s+1} \mid s \in \{0, 1, \ldots, n-1\} \right\rangle_A \subseteq U.$$

Now, from $U = \langle u^0, u^1, \ldots, u^{n-1} \rangle_A$, we obtain

$$\begin{aligned} uU &= u \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A = \left\langle u \cdot u^0, u \cdot u^1, \ldots, u \cdot u^{n-1} \right\rangle_A \\ &= \left\langle \underbrace{u \cdot u^s}_{=u^{s+1}} \mid s \in \{0, 1, \ldots, n-1\} \right\rangle_A = \left\langle u^{s+1} \mid s \in \{0, 1, \ldots, n-1\} \right\rangle_A \subseteq U. \end{aligned}$$

Thus, we have found an n-generated A-submodule U of B such that $1 \in U$ and $uU \subseteq U$. Hence, Assertion $\mathcal{C}$ holds. Hence, we have proved that $\mathcal{A} \Longrightarrow \mathcal{C}$.

Proof of the implication $\mathcal{C} \Longrightarrow \mathcal{B}$. Assume that Assertion $\mathcal{C}$ holds. Then, there exists an n-generated A-submodule U of B such that $1 \in U$ and $uU \subseteq U$. Consider this U. Every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$ (since $1 \in U$ and $vU = 0$ yield $v \cdot \underbrace{1}_{\in U} \in vU = 0$ and thus $v \cdot 1 = 0$, so that $v = 0$). Set $C = B$. Then, C is a B-module, and U is an n-generated A-submodule of C (since U is an n-generated A-submodule of B, and $C = B$) such that $uU \subseteq U$ and such that every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$. Thus, Assertion $\mathcal{B}$ holds. Hence, we have proved that $\mathcal{C} \Longrightarrow \mathcal{B}$.

Proof of the implication $\mathcal{B} \Longrightarrow \mathcal{A}$. Assume that Assertion $\mathcal{B}$ holds. Then, there exist a B-module C and an n-generated A-submodule[2] U of C such that $uU \subseteq U$, and such that every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$. Consider these C and U.

The A-module U is n-generated. In other words, there exist n elements $m_1, m_2, \ldots, m_n$ of U such that $U = \langle m_1, m_2, \ldots, m_n \rangle_A$. Consider these $m_1, m_2, \ldots, m_n$. For any $k \in \{1, 2, \ldots, n\}$, we have $m_k \in U$ (since $U = \langle m_1, m_2, \ldots, m_n \rangle_A$) and thus

$$um_k \in uU \subseteq U = \langle m_1, m_2, \ldots, m_n \rangle_A,$$

so that there exist n elements $a_{k,1}, a_{k,2}, \ldots, a_{k,n}$ of A such that

$$um_k = \sum_{i=1}^{n} a_{k,i} m_i. \tag{3}$$

[2]where C is an A-module, since C is a B-module and B is an A-algebra

Consider these $a_{k,1}, a_{k,2}, \ldots, a_{k,n}$.

The A-algebra B gives rise to a canonical ring homomorphism $\iota : A \to B$ (sending each $a \in A$ to $a \cdot 1_B \in B$). This ring homomorphism, in turn, induces a ring homomorphism $\iota^{n\times n} : A^{n\times n} \to B^{n\times n}$ (which acts on an $n \times n$-matrix by applying ι to each entry of the matrix).

We are also going to work with matrices over U (that is, matrices whose entries lie in U). This might sound somewhat strange, because U is not a ring; however, we can still define matrices over U just as one defines matrices over any ring. While we cannot multiply two matrices over U (because U is not a ring), we can define the product of a matrix over A with a matrix over U as follows: If $P \in A^{\alpha\times\beta}$ is a matrix over A, and $Q \in U^{\beta\times\gamma}$ is a matrix over U (where $\alpha, \beta, \gamma \in \mathbb{N}$), then we define the product $PQ \in U^{\alpha\times\gamma}$ by setting

$$(PQ)_{x,y} = \sum_{z=1}^{\beta} P_{x,z} Q_{z,y} \qquad \text{for all } x \in \{1, 2, \ldots, \alpha\} \text{ and } y \in \{1, 2, \ldots, \gamma\}.$$

(Here, for any matrix T and any integers x and y, we denote by $T_{x,y}$ the entry of the matrix T in the x-th row and the y-th column.)

It is easy to see that whenever $P \in A^{\alpha\times\beta}$, $Q \in A^{\beta\times\gamma}$ and $R \in U^{\gamma\times\delta}$ are three matrices, then

$$(PQ)\,R = P\,(QR). \tag{4}$$

This is proven in the same way as the fact that the multiplication of matrices over a ring is associative.

Now define a matrix $V \in U^{n\times 1}$ by setting

$$V_{i,1} = m_i \qquad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Define another matrix $S \in A^{n\times n}$ by setting

$$S_{k,i} = a_{k,i} \qquad \text{for all } k \in \{1, 2, \ldots, n\} \text{ and } i \in \{1, 2, \ldots, n\}.$$

Then, for any $k \in \{1, 2, \ldots, n\}$, we have

$$(uV)_{k,1} = u \underbrace{V_{k,1}}_{\substack{=m_k \\ \text{(by the definition of } V)}} = um_k \qquad \text{and}$$

$$(SV)_{k,1} = \sum_{i=1}^{n} \underbrace{S_{k,i}}_{\substack{=a_{k,i} \\ \text{(by the definition of } S)}} \underbrace{V_{i,1}}_{\substack{=m_i \\ \text{(by the definition of } V)}} = \sum_{i=1}^{n} a_{k,i} m_i.$$

Hence, the equality (3) rewrites as $(uV)_{k,1} = (SV)_{k,1}$. Since this holds for every $k \in \{1, 2, \ldots, n\}$, we conclude that $uV = SV$. Thus,

$$0 = uV - SV = uI_nV - SV = (uI_n - S)\,V. \tag{5}$$

Here, the "S" in "$uI_n - S$" means not the matrix $S \in A^{n\times n}$ itself, but rather its image under the ring homomorphism $\iota^{n\times n} : A^{n\times n} \to B^{n\times n}$; thus, the matrix $uI_n - S$ is a well-defined matrix in $B^{n\times n}$.

Now, let $P \in A[X]$ be the characteristic polynomial of the matrix $S \in A^{n\times n}$. Then, P is monic, and $\deg P = n$. Besides, $P(X) = \det(XI_n - S)$, so that $P(u) = \det(uI_n - S)$. Thus,

$$\begin{aligned} P(u)\cdot V = \det(uI_n - S)\cdot V &= \underbrace{\det(uI_n - S)\,I_n}_{\substack{=\operatorname{adj}(uI_n-S)\cdot(uI_n-S) \\ \text{(by Lemma 1.3,} \\ \text{applied to } M=uI_n-S)}} \cdot V \\ &= (\operatorname{adj}(uI_n - S)\cdot(uI_n - S))\cdot V \\ &= \operatorname{adj}(uI_n - S)\cdot \underbrace{((uI_n - S)\,V)}_{\substack{=0 \\ \text{(by (5))}}} \qquad \text{(by (4))} \\ &= 0. \end{aligned}$$

Since the entries of the matrix V are $m_1, m_2, \ldots, m_n$, this yields $P(u)\cdot m_k = 0$ for every $k \in \{1, 2, \ldots, n\}$. Now, from $U = \langle m_1, m_2, \ldots, m_n\rangle_A$, we obtain

$$\begin{aligned} P(u)\cdot U &= P(u)\cdot\langle m_1, m_2, \ldots, m_n\rangle_A = \langle P(u)\cdot m_1, P(u)\cdot m_2, \ldots, P(u)\cdot m_n\rangle_A \\ &= \langle 0, 0, \ldots, 0\rangle_A \qquad (\text{since } P(u)\cdot m_k = 0 \text{ for any } k \in \{1, 2, \ldots, n\}) \\ &= 0. \end{aligned}$$

This implies $P(u) = 0$ (since every $v \in B$ satisfying $vU = 0$ satisfies $v = 0$). Thus, Assertion $\mathcal{A}$ holds. Hence, we have proved that $\mathcal{B} \Longrightarrow \mathcal{A}$.

Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{D}$. Assume that Assertion $\mathcal{A}$ holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Consider this P.

Let U be the A-submodule $\langle u^0, u^1, \ldots, u^{n-1}\rangle_A$ of B. As in the Proof of the implication $\mathcal{A} \Longrightarrow \mathcal{C}$, we can show that U is an n-generated A-module, and that $1 \in U$ and $uU \subseteq U$.

Now, it is easy to show that

$$u^i \in U \qquad \text{for any } i \in \mathbb{N}. \tag{6}$$

[Indeed, this follows easily from $uU \subseteq U$ and $1 \in U$ by induction on i.]

Hence, for any $m \in \mathbb{N}$ and any $(a_0, a_1, \ldots, a_m) \in A^{m+1}$, we have $\sum\limits_{i=0}^{m} a_i u^i \in U$ (since U is an A-module, and thus is closed under A-linear combination).

Now, the definition of $A[u]$ yields

$$\begin{aligned} A[u] &= \left\{ \sum_{i=0}^{m} a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \ldots, a_m) \in A^{m+1} \right\} \\ &\subseteq U \qquad \left(\text{since } \sum_{i=0}^{m} a_i u^i \in U \text{ for any } m \in \mathbb{N} \text{ and } (a_0, a_1, \ldots, a_m) \in A^{m+1} \right) \\ &= \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A . \end{aligned}$$

Combining this with the (obvious) relation $\langle u^0, u^1, \ldots, u^{n-1} \rangle_A \subseteq A[u]$, we find $A[u] = \langle u^0, u^1, \ldots, u^{n-1} \rangle_A$.

Thus, Assertion $\mathcal{D}$ holds. Hence, we have proved that $\mathcal{A} \Longrightarrow \mathcal{D}$.

Proof of the implication $\mathcal{D} \Longrightarrow \mathcal{C}$. Assume that Assertion $\mathcal{D}$ holds. Then, $A[u] = \langle u^0, u^1, \ldots, u^{n-1} \rangle_A$.

Let U be the A-submodule $A[u]$ of B. Then, U is an n-generated A-module (since $U = A[u] = \langle u^0, u^1, \ldots, u^{n-1} \rangle_A$). Besides, $1 = u^0 \in A[u] = U$. Finally, $U = A[u]$ yields $uU \subseteq U$. Thus, Assertion $\mathcal{C}$ holds. Hence, we have proved that $\mathcal{D} \Longrightarrow \mathcal{C}$.

Now, we have proved the implications $\mathcal{A} \Longrightarrow \mathcal{D}$, $\mathcal{D} \Longrightarrow \mathcal{C}$, $\mathcal{C} \Longrightarrow \mathcal{B}$ and $\mathcal{B} \Longrightarrow \mathcal{A}$ above. Thus, all four assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are equivalent, and Theorem 1.1 is proven. □

For the sake of completeness (and as a very easy exercise), let us state a basic property of integrality that we will not ever use:

Proposition 1.4. Let A be a ring. Let B be an A-algebra. Let $u \in B$. Let $q \in \mathbb{N}$ and $p \in \mathbb{N}$ be such that $p \geq q$. Assume that u is q-integral over A. Then, u is p-integral over A.

1.2. Transitivity of integrality

Let us now prove the first and probably most important consequence of Theorem 1.1:

Theorem 1.5. Let A be a ring. Let B be an A-algebra. Let $v \in B$ and $u \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that v is m-integral over A, and that u is n-integral over $A[v]$. Then, u is nm-integral over A.

(Here, we are using the fact that if A is a ring, and if v is an element of an A-algebra B, then $A[v]$ is a subring of B, and therefore B is an $A[v]$-algebra.)

Proof of Theorem 1.5. Since v is m-integral over A, we have $A[v] = \langle v^0, v^1, \ldots, v^{m-1} \rangle_A$ (this is the Assertion $\mathcal{D}$ of Theorem 1.1, stated for v and m in lieu of u and n).

Since u is n-integral over $A[v]$, we have $(A[v])[u] = \langle u^0, u^1, \ldots, u^{n-1} \rangle_{A[v]}$ (this is the Assertion $\mathcal{D}$ of Theorem 1.1, stated for $A[v]$ in lieu of A).

Let $S = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$. Then, S is a finite set with size $|S| = nm$.

Let $x \in (A[v])[u]$. Then, there exist n elements $b_0, b_1, \ldots, b_{n-1}$ of $A[v]$ such that $x = \sum_{i=0}^{n-1} b_i u^i$ (since $x \in (A[v])[u] = \langle u^0, u^1, \ldots, u^{n-1} \rangle_{A[v]}$). Consider these $b_0, b_1, \ldots, b_{n-1}$.

For each $i \in \{0, 1, \ldots, n-1\}$, there exist m elements $a_{i,0}, a_{i,1}, \ldots, a_{i,m-1}$ of A such that $b_i = \sum_{j=0}^{m-1} a_{i,j} v^j$ (because $b_i \in A[v] = \langle v^0, v^1, \ldots, v^{m-1} \rangle_A$). Consider these $a_{i,0}, a_{i,1}, \ldots, a_{i,m-1}$. Thus,

$$\begin{aligned} x &= \sum_{i=0}^{n-1} \underbrace{b_i}_{= \sum_{j=0}^{m-1} a_{i,j} v^j} u^i = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} a_{i,j} v^j u^i = \sum_{(i,j) \in \{0,1,\ldots,n-1\} \times \{0,1,\ldots,m-1\}} a_{i,j} v^j u^i \\ &= \sum_{(i,j) \in S} a_{i,j} v^j u^i \qquad (\text{since } \{0,1,\ldots,n-1\} \times \{0,1,\ldots,m-1\} = S) \\ &\in \left\langle v^j u^i \mid (i,j) \in S \right\rangle_A \qquad \left(\text{since } a_{i,j} \in A \text{ for every } (i,j) \in S\right). \end{aligned}$$

So we have proved that $x \in \langle v^j u^i \mid (i,j) \in S \rangle_A$ for every $x \in (A[v])[u]$. In other words, $(A[v])[u] \subseteq \langle v^j u^i \mid (i,j) \in S \rangle_A$. Conversely, $\langle v^j u^i \mid (i,j) \in S \rangle_A \subseteq (A[v])[u]$ (this is trivial). Combining these two relations, we find $(A[v])[u] = \langle v^j u^i \mid (i,j) \in S \rangle_A$. Thus, the A-module $(A[v])[u]$ is nm-generated (since $|S| = nm$).

Let $U = (A[v])[u]$. Then, the A-module U is nm-generated. Besides, U is an A-submodule of B, and we have $1 \in U$ and $uU \subseteq U$.

Thus, the element u of B satisfies the Assertion $\mathcal{C}$ of Theorem 1.1 with n replaced by nm. Hence, $u \in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1.1, all with n replaced by nm. Thus, u is nm-integral over A. This proves Theorem 1.5. □

1.3. Integrality of sums and products

Before the next significant consequence of Theorem 1.1, let us show an essentially trivial fact:

Theorem 1.6. Let A be a ring. Let B be an A-algebra. Let $a \in A$. Then, $a \cdot 1_B \in B$ is 1-integral over A.

Proof of Theorem 1.6. The polynomial $X - a \in A[X]$ is monic and satisfies $\deg(X - a) = 1$; moreover, evaluating this polynomial at $a \cdot 1_B \in B$ yields $a \cdot 1_B -$

$a \cdot 1_B = 0$. Hence, there exists a monic polynomial $P \in A[X]$ with $\deg P = 1$ and $P(a \cdot 1_B) = 0$ (namely, the polynomial $P \in A[X]$ defined by $P(X) = X - a$). Thus, $a \cdot 1_B$ is 1-integral over A. This proves Theorem 1.6. □

The following theorem is a standard result, generalizing (for example) the classical fact that sums and products of algebraic integers are again algebraic integers:

Theorem 1.7. Let A be a ring. Let B be an A-algebra. Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over A, and that y is n-integral over A.

(a) Then, $x + y$ is nm-integral over A.

(b) Furthermore, xy is nm-integral over A.

Our proof of this theorem will rely on a simple lemma:

Lemma 1.8. Let A be a ring. Let C be an A-algebra. Let $x \in C$.

Let $n \in \mathbb{N}$. Let $P \in A[X]$ be a monic polynomial with $\deg P = n$. Define a polynomial $Q \in C[X]$ by $Q(X) = P(X - x)$. Then, Q is a monic polynomial with $\deg Q = n$.

Proof of Lemma 1.8. Recall that P is a monic polynomial with $\deg P = n$; hence, we can write P in the form

$$P = X^n + \sum_{i=0}^{n-1} a_i X^i \tag{7}$$

for some $a_0, a_1, \ldots, a_{n-1} \in A$. Consider these $a_0, a_1, \ldots, a_{n-1}$.

Now,

$$\begin{aligned} Q(X) = P(X - x) &= \underbrace{(X - x)^n}_{= X^n + (\text{lower order terms})} + \underbrace{\sum_{i=0}^{n-1} a_i (X - x)^i}_{= (\text{lower order terms})} \\ &\qquad (\text{here, we have substituted } X - x \text{ for } X \text{ in (7)}) \\ &= X^n + (\text{lower order terms}), \end{aligned}$$

where "lower order terms" means a sum of terms of the form bX^i with $b \in C$ and $i < n$. Hence, Q is a monic polynomial with $\deg Q = n$. This proves Lemma 1.8. □

Proof of Theorem 1.7. Since y is n-integral over A, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(y) = 0$. Consider this P.

(a) Let C be the A-subalgebra $A[x]$ of B. Then, $C = A[x]$, so that $x \in A[x] = C$.

Now, define a polynomial $Q \in C[X]$ by $Q(X) = P(X - x)$. Then, Lemma 1.8 shows that Q is a monic polynomial with $\deg Q = n$. Also, substituting $x + y$ for X in the equality $Q(X) = P(X - x)$, we obtain $Q(x + y) = P\left(\underbrace{(x+y) - x}_{=y}\right) = P(y) = 0$.

Hence, there exists a monic polynomial $Q \in C[X]$ with $\deg Q = n$ and $Q(x + y) = 0$. Thus, $x + y$ is n-integral over C. In other words, $x + y$ is n-integral over $A[x]$ (since $C = A[x]$). Thus, Theorem 1.5 (applied to $v = x$ and $u = x + y$) yields that $x + y$ is nm-integral over A. This proves Theorem 1.7 **(a)**.

(b) Recall that $P \in A[X]$ is a monic polynomial with $\deg P = n$. Thus, there exist elements $a_0, a_1, \ldots, a_{n-1}$ of A such that $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$. Consider these $a_0, a_1, \ldots, a_{n-1}$. Substituting y for X in $P(X) = X^n + \sum\limits_{k=0}^{n-1} a_k X^k$, we find $P(y) = y^n + \sum\limits_{k=0}^{n-1} a_k y^k$. Thus,

$$y^n + \sum_{k=0}^{n-1} a_k y^k = P(y) = 0. \tag{8}$$

Now, define a polynomial $Q \in (A[x])[X]$ by $Q(X) = X^n + \sum\limits_{k=0}^{n-1} x^{n-k} a_k X^k$. Then,

$$\begin{aligned} Q(xy) &= \underbrace{(xy)^n}_{=x^n y^n} + \sum_{k=0}^{n-1} x^{n-k} \underbrace{a_k (xy)^k}_{\substack{=a_k x^k y^k \\ =x^k a_k y^k}} = x^n y^n + \sum_{k=0}^{n-1} \underbrace{x^{n-k} x^k}_{=x^n} a_k y^k \\ &= x^n y^n + \sum_{k=0}^{n-1} x^n a_k y^k = x^n \underbrace{\left(y^n + \sum_{k=0}^{n-1} a_k y^k\right)}_{\substack{=0 \\ \text{(by (8))}}} = 0. \end{aligned}$$

Also, recall that $Q(X) = X^n + \sum\limits_{k=0}^{n-1} x^{n-k} a_k X^k$; hence, the polynomial $Q \in (A[x])[X]$ is monic and $\deg Q = n$. Thus, there exists a monic polynomial $Q \in (A[x])[X]$ with $\deg Q = n$ and $Q(xy) = 0$. Thus, xy is n-integral over $A[x]$. Hence, Theorem 1.5 (applied to $v = x$ and $u = xy$) yields that xy is nm-integral over A. This proves Theorem 1.7 **(b)**. $\square$

Corollary 1.9. Let A be a ring. Let B be an A-algebra. Let $x \in B$. Let $m \in \mathbb{N}$. Assume that x is m-integral over A. Then, $-x$ is m-integral over A.

Proof of Corollary 1.9. This is easy to prove directly (using Assertion $\mathcal{A}$ of Theorem 1.1), but the slickest proof is using Theorem 1.7 **(b)**: The element $(-1) \cdot 1_B \in B$ is 1-integral over A (by Theorem 1.6, applied to $a = -1$). Thus, $x \cdot ((-1) \cdot 1_B)$ is $1m$-integral over A (by Theorem 1.7 **(b)**, applied to $y = (-1) \cdot 1_B$ and $n = 1$). In other words, $-x$ is m-integral over A (since $x \cdot \underbrace{((-1) \cdot 1_B)}_{=-1_B} = x \cdot (-1_B) = -x \cdot 1_B = -x$ and $1m = m$). This proves Corollary 1.9. □

Corollary 1.10. Let A be a ring. Let B be an A-algebra. Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over A, and that y is n-integral over A. Then, $x - y$ is nm-integral over A.

Proof of Corollary 1.10. We know that y is n-integral over A. Hence, Corollary 1.9 (applied to y and n instead of x and m) shows that $-y$ is n-integral over A. Thus, Theorem 1.7 **(a)** (applied to $-y$ instead of y) shows that $x + (-y)$ is nm-integral over A. In other words, $x - y$ is nm-integral over A (since $x + (-y) = x - y$). This proves Corollary 1.10. □

1.4. Some further consequences

Theorem 1.11. Let A be a ring. Let B be an A-algebra. Let $n \in \mathbb{N}^+$. Let $v \in B$. Let $a_0, a_1, \ldots, a_n$ be $n+1$ elements of A such that $\sum\limits_{i=0}^{n} a_i v^i = 0$. Let $k \in \{0, 1, \ldots, n\}$. Then, $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over A.

Proof of Theorem 1.11. Let $u = \sum\limits_{i=0}^{n-k} a_{i+k} v^i$. Then,

$$\begin{aligned} 0 &= \sum_{i=0}^{n} a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=k}^{n} a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^{i+k}}_{=v^i v^k} \\ &\qquad \text{(here, we substituted } i+k \text{ for } i \text{ in the second sum)} \\ &= \sum_{i=0}^{k-1} a_i v^i + \underbrace{\sum_{i=0}^{n-k} a_{i+k} v^i v^k}_{=v^k \sum\limits_{i=0}^{n-k} a_{i+k} v^i} = \sum_{i=0}^{k-1} a_i v^i + v^k \underbrace{\sum_{i=0}^{n-k} a_{i+k} v^i}_{=u} = \sum_{i=0}^{k-1} a_i v^i + v^k u, \end{aligned}$$

so that

$$v^k u = -\sum_{i=0}^{k-1} a_i v^i.$$

Let U be the A-submodule $\left\langle v^0, v^1, \ldots, v^{n-1}\right\rangle_A$ of B. Thus, U is an n-generated A-module, and $1 = v^0 \in U$.

Now, we are going to show that

$$uv^s \in U \qquad \text{for any } s \in \{0, 1, \ldots, n-1\}. \tag{9}$$

[*Proof of* (9). Let $s \in \{0, 1, \ldots, n-1\}$. Thus, we have either $s < k$ or $s \geq k$. In the case $s < k$, the relation (9) follows from

$$\begin{aligned} uv^s &= \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^i \cdot v^s}_{=v^{i+s}} \qquad \left(\text{since } u = \sum_{i=0}^{n-k} a_{i+k} v^i\right) \\ &= \sum_{i=0}^{n-k} a_{i+k} v^{i+s} \in U \end{aligned}$$

(since every $i \in \{0, 1, \ldots, n-k\}$ satisfies $i + s \in \{0, 1, \ldots, n-1\}$ [3], and thus $\sum\limits_{i=0}^{n-k} a_{i+k} v^{i+s} \in \left\langle v^0, v^1, \ldots, v^{n-1}\right\rangle_A = U$). In the case $s \geq k$, the relation (9) follows from

$$\begin{aligned} uv^s &= u \underbrace{v^{k+(s-k)}}_{=v^k v^{s-k}} = v^k u \cdot v^{s-k} = -\sum_{i=0}^{k-1} a_i \underbrace{v^i \cdot v^{s-k}}_{=v^{i+(s-k)}} \qquad \left(\text{since } v^k u = -\sum_{i=0}^{k-1} a_i v^i\right) \\ &= -\sum_{i=0}^{k-1} a_i v^{i+(s-k)} \in U \end{aligned}$$

(since every $i \in \{0, 1, \ldots, k-1\}$ satisfies $i + (s-k) \in \{0, 1, \ldots, n-1\}$ [4], and thus $-\sum\limits_{i=0}^{k-1} a_i v^{i+(s-k)} \in \left\langle v^0, v^1, \ldots, v^{n-1}\right\rangle_A = U$). Hence, (9) is proven in both possible cases, and thus the proof of (9) is complete.]

Now, from $U = \left\langle v^0, v^1, \ldots, v^{n-1}\right\rangle_A$, we obtain

$$\begin{aligned} uU &= u \left\langle v^0, v^1, \ldots, v^{n-1}\right\rangle_A = \left\langle uv^0, uv^1, \ldots, uv^{n-1}\right\rangle_A \\ &= \langle uv^s \mid s \in \{0, 1, \ldots, n-1\}\rangle_A \subseteq U \qquad (\text{by (9) and Lemma 0.3}). \end{aligned}$$

Altogether, U is an n-generated A-submodule of B such that $1 \in U$ and $uU \subseteq U$. Thus, $u \in B$ satisfies Assertion $\mathcal{C}$ of Theorem 1.1. Hence, $u \in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1.1. Consequently, u is n-integral over A. Since $u = \sum\limits_{i=0}^{n-k} a_{i+k} v^i$, this means that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over A. This proves Theorem 1.11. $\square$

[3] Here we are using $s < k$.

[4] Here we are using $s \geq k$ and $s \in \{0, 1, \ldots, n-1\}$.

Corollary 1.12. Let A be a ring. Let B be an A-algebra. Let $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ be such that $\alpha + \beta \in \mathbb{N}^+$. Let $u \in B$ and $v \in B$. Let $s_0, s_1, \ldots, s_\alpha$ be $\alpha + 1$ elements of A such that $\sum_{i=0}^{\alpha} s_i v^i = u$. Let $t_0, t_1, \ldots, t_\beta$ be $\beta + 1$ elements of A such that $\sum_{i=0}^{\beta} t_i v^{\beta - i} = uv^\beta$. Then, u is $(\alpha + \beta)$-integral over A.

(This Corollary 1.12 generalizes [4, Exercise 2-5], which says that if v is an invertible element of an A-algebra B, then every element $u \in A[v] \cap A[v^{-1}]$ is integral over A. To see how this follows from Corollary 1.12, just pick $\alpha \in \mathbb{N}^+$ and $\beta \in \mathbb{N}^+$ and $s_0, s_1, \ldots, s_\alpha \in A$ and $t_0, t_1, \ldots, t_\beta \in A$ such that $\sum_{i=0}^{\alpha} s_i v^i = u$ and $\sum_{i=0}^{\beta} t_i (v^{-1})^i = u$.)

First proof of Corollary 1.12. Let $k = \beta$ and $n = \alpha + \beta$. Then, $k \in \{0, 1, \ldots, n\}$ (since $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$) and $n = \alpha + \beta \in \mathbb{N}^+$ and $n - \beta = \alpha$ (since $n = \alpha + \beta$). Define $n + 1$ elements $a_0, a_1, \ldots, a_n$ of A by

$$a_i = \begin{cases} t_{\beta - i}, & \text{if } i < \beta; \\ t_0 - s_0, & \text{if } i = \beta; \\ -s_{i-\beta}, & \text{if } i > \beta \end{cases} \qquad \text{for every } i \in \{0, 1, \ldots, n\}.$$

Then, from $n = \alpha + \beta$, we obtain

$$\begin{aligned}
\sum_{i=0}^{n} a_i v^i &= \sum_{i=0}^{\alpha+\beta} a_i v^i = \sum_{i=0}^{\beta-1} \underbrace{a_i}_{=t_{\beta-i}} v^i + \underbrace{a_\beta}_{=t_0 - s_0} v^\beta + \sum_{i=\beta+1}^{\alpha+\beta} \underbrace{a_i}_{=-s_{i-\beta}} v^i \\
&= \underbrace{\sum_{i=0}^{\beta-1} t_{\beta-i} v^i}_{\substack{= \sum_{i=1}^{\beta} t_i v^{\beta-i} \\ \text{(here, we have substituted } i \\ \text{for } \beta - i \text{ in the sum)}}} + \underbrace{(t_0 - s_0) v^\beta}_{= t_0 v^\beta - s_0 v^\beta} + \underbrace{\sum_{i=\beta+1}^{\alpha+\beta} \left(-s_{i-\beta}\right) v^i}_{\substack{= - \sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i = - \sum_{i=1}^{\alpha} s_i v^{i+\beta} \\ \text{(here, we have substituted } i \\ \text{for } i - \beta \text{ in the sum)}}} \\
&= \sum_{i=1}^{\beta} t_i v^{\beta - i} + t_0 v^\beta - s_0 v^\beta - \sum_{i=1}^{\alpha} s_i v^{i+\beta} \\
&= \underbrace{\sum_{i=1}^{\beta} t_i v^{\beta-i} + t_0 v^\beta}_{= \sum_{i=0}^{\beta} t_i v^{\beta - i} = uv^\beta} - \underbrace{\left(s_0 v^\beta + \sum_{i=1}^{\alpha} s_i v^{i+\beta} \right)}_{\substack{= \sum_{i=0}^{\alpha} s_i v^{i+\beta} = \left(\sum_{i=0}^{\alpha} s_i v^i \right) v^\beta = uv^\beta \\ \text{(since } \sum_{i=0}^{\alpha} s_i v^i = u)}} = uv^\beta - uv^\beta = 0.
\end{aligned}$$

Thus, Theorem 1.11 yields that $\sum\limits_{i=0}^{n-k} a_{i+k}v^i$ is n-integral over A.

But $k = \beta$ and thus

$$\begin{aligned}
\sum_{i=0}^{n-k} a_{i+k}v^i &= \sum_{i=0}^{n-\beta} a_{i+\beta}v^i = \sum_{i=\beta}^{n} a_i v^{i-\beta} \qquad \left(\begin{array}{c}\text{here, we have substituted } i-\beta \\ \text{for } i \text{ in the sum}\end{array}\right) \\
&= \sum_{i=\beta}^{\beta} \underbrace{a_i}_{\substack{=t_0-s_0 \\ \text{(by the} \\ \text{definition of } a_i, \\ \text{since } i=\beta)}} v^{i-\beta} + \sum_{i=\beta+1}^{n} \underbrace{a_i}_{\substack{=-s_{i-\beta} \\ \text{(by the} \\ \text{definition of } a_i, \\ \text{since } i>\beta)}} v^{i-\beta} \\
&= \underbrace{\sum_{i=\beta}^{\beta} (t_0 - s_0)\, v^{i-\beta}}_{\substack{=(t_0-s_0)v^{\beta-\beta} \\ =(t_0-s_0)v^0 \\ =t_0v^0-s_0v^0}} + \underbrace{\sum_{i=\beta+1}^{n} \left(-s_{i-\beta}\right) v^{i-\beta}}_{\substack{=-\sum\limits_{i=\beta+1}^{n} s_{i-\beta}v^{i-\beta}=-\sum\limits_{i=1}^{n-\beta} s_iv^i \\ \text{(here, we have substituted } i \\ \text{for } i-\beta \text{ in the sum)}}} \\
&= t_0v^0 - s_0v^0 - \sum_{i=1}^{n-\beta} s_iv^i = t_0v^0 - s_0v^0 - \sum_{i=1}^{\alpha} s_iv^i \qquad (\text{since } n-\beta=\alpha) \\
&= t_0 \underbrace{v^0}_{=1_B} - \underbrace{\left(s_0v^0 + \sum_{i=1}^{\alpha} s_iv^i\right)}_{=\sum\limits_{i=0}^{\alpha} s_iv^i=u} = t_0 \cdot 1_B - u.
\end{aligned}$$

Thus, $t_0 \cdot 1_B - u$ is n-integral over A (since $\sum\limits_{i=0}^{n-k} a_{i+k}v^i$ is n-integral over A). Thus, Corollary 1.9 (applied to $x = t_0 \cdot 1_B - u$ and $m = n$) shows that $-(t_0 \cdot 1_B - u)$ is n-integral over A. In other words, $u - t_0 \cdot 1_B$ is n-integral over A (since $-(t_0 \cdot 1_B - u) = u - t_0 \cdot 1_B$).

On the other hand, $t_0 \cdot 1_B$ is 1-integral over A (by Theorem 1.6, applied to $a = t_0$). Thus, $t_0 \cdot 1_B + (u - t_0 \cdot 1_B)$ is $n \cdot 1$-integral over A (by Theorem 1.7 **(a)**, applied to $x = t_0 \cdot 1_B$, $y = u - t_0 \cdot 1_B$ and $m = 1$). In other words, u is $(\alpha + \beta)$-integral over A (since $t_0 \cdot 1_B + (u - t_0 \cdot 1_B) = u$ and $n \cdot 1 = n = \alpha + \beta$). This proves Corollary 1.12. □

We will provide a second proof of Corollary 1.12 in Section 5.

Corollary 1.13. Let A be a ring. Let B be an A-algebra. Let $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$. Let $v \in B$. Let $b_0, b_1, \ldots, b_{n-1}$ be n elements of A, and let $u = \sum\limits_{i=0}^{n-1} b_iv^i$. Assume that vu is m-integral over A. Then, u is nm-integral over A.

Corollary 1.13 generalizes a folklore fact about integrality, which states that if B is an A-algebra, and if an invertible $v \in B$ satisfies $v^{-1} \in A[v]$, then v is integral over A. (Indeed, this latter fact follows from Corollary 1.13 by setting $u = v^{-1}$.)

Proof of Corollary 1.13. Define $n+1$ elements $a_0, a_1, \ldots, a_n$ of $A[vu]$ by

$$a_i = \begin{cases} -vu, & \text{if } i = 0; \\ b_{i-1} \cdot 1_B, & \text{if } i > 0 \end{cases} \qquad \text{for every } i \in \{0, 1, \ldots, n\}.$$

(These are well-defined, since every positive $i \in \{0, 1, \ldots, n\}$ satisfies $i \in \{1, 2, \ldots, n\}$ and thus $i - 1 \in \{0, 1, \ldots, n-1\}$ and thus $b_{i-1} \in A$ and therefore $b_{i-1} \cdot 1_B \in A \cdot 1_B \subseteq A[vu]$.)

The definition of a_0 yields $a_0 = -vu$. Also,

$$\begin{aligned} \sum_{i=0}^{n} a_i v^i &= \underbrace{a_0}_{=-vu} \underbrace{v^0}_{=1} + \sum_{i=1}^{n} \underbrace{a_i}_{\substack{=b_{i-1} \cdot 1_B \\ \text{(by the definition} \\ \text{of } a_i, \text{ since } i>0)}} v^i = -vu + \sum_{i=1}^{n} b_{i-1} \cdot \underbrace{1_B v^i}_{=v^i = v^{i-1} v} \\ &= -vu + \sum_{i=1}^{n} b_{i-1} v^{i-1} v = -vu + \underbrace{\sum_{i=0}^{n-1} b_i v^i}_{=u} v \\ &\qquad \text{(here, we substituted } i \text{ for } i-1 \text{ in the sum)} \\ &= -vu + uv = 0. \end{aligned}$$

Let $k = 1$. Then, $k = 1 \in \{0, 1, \ldots, n\}$ (since $n \in \mathbb{N}^+$).

Now, $A[vu]$ is a subring of B; hence, B is an $A[vu]$-algebra. The $n+1$ elements $a_0, a_1, \ldots, a_n$ of $A[vu]$ satisfy $\sum\limits_{i=0}^{n} a_i v^i = 0$.

Hence, Theorem 1.11 (applied to the ring $A[vu]$ in lieu of A) yields that $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $A[vu]$. But from $k = 1$, we obtain

$$\sum_{i=0}^{n-k} a_{i+k} v^i = \sum_{i=0}^{n-1} \underbrace{a_{i+1}}_{\substack{=b_{(i+1)-1} \cdot 1_B \\ \text{(by the definition} \\ \text{of } a_{i+1}, \text{ since } i+1>0)}} v^i = \sum_{i=0}^{n-1} \underbrace{b_{(i+1)-1}}_{=b_i} \cdot \underbrace{1_B v^i}_{=v^i} = \sum_{i=0}^{n-1} b_i v^i = u.$$

Hence, u is n-integral over $A[vu]$ (since $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $A[vu]$). But vu is m-integral over A. Thus, Theorem 1.5 (applied to vu in lieu of v) yields that u is nm-integral over A. This proves Corollary 1.13. $\square$

2. Integrality over ideal semifiltrations

2.1. Definitions of ideal semifiltrations and integrality over them

We now set our sights at a more general notion of integrality.

Definition 2.1. Let A be a ring, and let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be a sequence of ideals of A. Then, $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ is called an *ideal semifiltration* of A if and only if it satisfies the two conditions

$$I_0 = A;$$
$$I_a I_b \subseteq I_{a+b} \qquad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}.$$

Two simple examples of ideal semifiltrations can easily be constructed from any ideal:

Example 2.2. Let A be a ring. Let I be an ideal of A. Then:

(a) The sequence $\left(I^\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A. (Here, I^ρ denotes the ρ-th power of I in the multiplicative monoid of ideals of A; in particular, $I^0 = A$.)

(b) The sequence $(A, I, I, I, \ldots) = \left(\begin{cases} A, & \text{if } \rho = 0; \\ I, & \text{if } \rho > 0 \end{cases}\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

Proof of Example 2.2. This is a straightforward exercise in checking axioms. □

Definition 2.3. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$.

We say that the element u of B is *n-integral over* $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$ if there exists some $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

This definition generalizes [5, Definition 1.1.1] in multiple ways. Indeed, if I is an ideal of a ring A, and if $u \in A$ and $n \in \mathbb{N}$, then u is n-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$ (here, $\left(I^\rho\right)_{\rho\in\mathbb{N}}$ is the ideal semifiltration from Example 2.2 **(a)**) if and only if there is an equation of integral dependence of u over I (in the sense of [5, Definition 1.1.1]).

We further notice that integrality over an ideal semifiltration of a ring A is a stronger claim than integrality over A:

Proposition 2.4. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$ be such that u is n-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$. Then, u is n-integral over A.

Proof of Proposition 2.4. We know that u is n-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$. Thus, by Definition 2.3, there exists some $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

Consider this $(a_0, a_1, \ldots, a_n)$.

Thus, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$ (namely, $P(X) = \sum\limits_{k=0}^{n} a_k X^k$).

In other words, u is n-integral over A. This proves Proposition 2.4. $\square$

We leave it to the reader to prove the following simple fact, which shows that nilpotency is an instance of integrality over ideal semifiltrations:

Proposition 2.5. Let A be a ring. Let $0A$ be the zero ideal of A. Let $n \in \mathbb{N}$. Let $u \in A$. Then, the element u of A is n-integral over $\left(A, \left((0A)^\rho\right)_{\rho\in\mathbb{N}}\right)$ if and only if $u^n = 0$.

2.2. Polynomial rings and Rees algebras

In order to study integrality over ideal semifiltrations, we shall now introduce the concept of a Rees algebra – a subalgebra of a polynomial ring that conveniently encodes an ideal semifiltration of the base ring. This, again, generalizes a classical notion for ideals (namely, the Rees algebra of an ideal – see [5, Definition 5.1.1]). First, we recall a basic fact:

Definition 2.6. Let A be a ring. Let B be an A-algebra. Then, there is a canonical ring homomorphism $\iota : A \to B$ that sends each $a \in A$ to $a \cdot 1_B \in B$. This ring homomorphism ι induces a canonical ring homomorphism $\iota[Y] : A[Y] \to B[Y]$ between the polynomial rings $A[Y]$ and $B[Y]$ that sends each polynomial $\sum\limits_{i=0}^{m} a_i Y^i \in A[Y]$ (with $m \in \mathbb{N}$ and $(a_0, a_1, \ldots, a_m) \in A^{m+1}$) to the polynomial $\sum\limits_{i=0}^{m} \iota(a_i) Y^i \in B[Y]$. Thus, the polynomial ring $B[Y]$ becomes an $A[Y]$-algebra.

Definition 2.7. Let A be a ring, and let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Thus, $I_0, I_1, I_2, \ldots$ are ideals of A, and we have $I_0 = A$.

Consider the polynomial ring $A[Y]$. For each $i \in \mathbb{N}$, the subset $I_i Y^i$ of $A[Y]$ is an A-submodule of the A-algebra $A[Y]$ (since I_i is an ideal of A). Hence, the sum $\sum\limits_{i\in\mathbb{N}} I_i Y^i$ of these A-submodules must also be an A-submodule of the A-algebra $A[Y]$.

Let $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ denote this A-submodule $\sum\limits_{i\in\mathbb{N}} I_i Y^i$ of the A-algebra $A[Y]$. Then,

$$\begin{aligned}
A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] &= \sum_{i\in\mathbb{N}} I_i Y^i \\
&= \left\{ \sum_{i\in\mathbb{N}} a_i Y^i \mid \left(a_i \in I_i \text{ for all } i \in \mathbb{N}\right), \right. \\
&\qquad\qquad \left. \text{and } \left(\text{only finitely many } i \in \mathbb{N} \text{ satisfy } a_i \neq 0\right) \right\} \\
&= \left\{ P \in A[Y] \mid \text{the } i\text{-th coefficient of the polynomial } P \right. \\
&\qquad\qquad \left. \text{lies in } I_i \text{ for every } i \in \mathbb{N} \right\}.
\end{aligned}$$

Clearly, $A \subseteq A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, since

$$A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] = \sum_{i\in\mathbb{N}} I_i Y^i \supseteq \underbrace{I_0}_{=A} \underbrace{Y^0}_{=1} = A \cdot 1 = A.$$

Hence, $1 \in A \subseteq A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Also, the A-submodule $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ of $A[Y]$ is an A-subalgebra of the A-algebra $A[Y]$ (by Lemma 2.8 below), and thus is a subring of $A[Y]$.

This A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is called the *Rees algebra* of the ideal semifiltration $\left(I_\rho\right)_{\rho\in\mathbb{N}}$.

Lemma 2.8. Let A be a ring, and let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Then, the A-submodule $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ of $A[Y]$ is an A-subalgebra of the A-algebra $A[Y]$.

Proof of Lemma 2.8. This is an easy exercise. (Use $I_a I_b \subseteq I_{a+b}$ to prove that $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is closed under multiplication.) $\square$

Remark 2.9. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A.

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ defined in Definition 2.7.

The polynomial ring $B[Y]$ is an $A[Y]$-algebra (as explained in Definition 2.6), and thus is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra (since $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A[Y]$). Hence, if $p \in B[Y]$ is a polynomial and $n \in \mathbb{N}$, then it makes sense to ask whether p is n-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Questions of this form will often appear in what follows.

We note in passing that the notion of a Rees algebra helps set up a bijection between ideal semifiltrations of a ring A and graded A-subalgebras of the polynomial ring $A[Y]$:

Proposition 2.10. Let A be a ring. Consider the polynomial ring $A[Y]$ as a graded A-algebra (with the usual degree of polynomials).

(a) If $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, then its Rees algebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a graded A-subalgebra of $A[Y]$.

(b) If B is any graded A-subalgebra of $A[Y]$, and if $\rho \in \mathbb{N}$, then we let $I_{B,\rho}$ denote the subset $\{a \in A \mid aY^\rho \in B\}$ of A. Then, $I_{B,\rho}$ is an ideal of A.

(c) The maps

$$\{\text{ideal semifiltrations of } A\} \to \{\text{graded } A\text{-subalgebras of } A[Y]\},$$
$$\left(I_\rho\right)_{\rho\in\mathbb{N}} \mapsto A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$$

and

$$\{\text{graded } A\text{-subalgebras of } A[Y]\} \to \{\text{ideal semifiltrations of } A\},$$
$$B \mapsto \left(I_{B,\rho}\right)_{\rho\in\mathbb{N}}$$

are mutually inverse bijections.

We shall not have any need for this proposition, so we omit its (straightforward and easy) proof.

2.3. Reduction to integrality over rings

We start with a theorem which reduces the question of n-integrality over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$ to that of n-integrality over a ring[5]:

[5] Theorem 2.11 is inspired by [5, Proposition 5.2.1].

Theorem 2.11. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$.

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ defined in Definition 2.7.

Then, the element u of B is n-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. (Here, $B[Y]$ is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra, as explained in Remark 2.9.)

Proof of Theorem 2.11. $\Longrightarrow$: Assume that u is n-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$. Thus, by Definition 2.3, there exists some $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

Consider this $(a_0, a_1, \ldots, a_n)$.

Hence, there exists a monic polynomial $P \in \left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ with $\deg P = n$ and $P(uY) = 0$ (viz., the polynomial $P(X) = \sum\limits_{k=0}^{n} a_k Y^{n-k} X^k$). Hence, uY is n-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. This proves the $\Longrightarrow$ direction of Theorem 2.11.

$\Longleftarrow$: Assume that uY is n-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Thus, there exists a monic polynomial $P \in \left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ with $\deg P = n$ and $P(uY) = 0$. Consider this P. Since $P \in \left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)[X]$ satisfies $\deg P = n$, there exists $(p_0, p_1, \ldots, p_n) \in \left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)^{n+1}$ such that $P(X) = \sum\limits_{k=0}^{n} p_k X^k$. Consider this $(p_0, p_1, \ldots, p_n)$. Note that $p_n = 1$ (since P is monic and $\deg P = n$).

For every $k \in \{0, 1, \ldots, n\}$, we have $p_k \in A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] = \sum\limits_{i\in\mathbb{N}} I_i Y^i$, and thus there exists a sequence $(p_{k,i})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^i$, such that ($p_{k,i} \in I_i$ for every $i \in \mathbb{N}$), and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Consider this sequence. Thus, $P(X) = \sum\limits_{k=0}^{n} p_k X^k$ rewrites as $P(X) = \sum\limits_{k=0}^{n} \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^i X^k$ (since $p_k = \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^i$). Hence,

$$P(uY) = \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} Y^i (uY)^k = \sum_{k=0}^{n} \sum_{i\in\mathbb{N}} p_{k,i} Y^{i+k} u^k.$$

Therefore, $P(uY) = 0$ rewrites as $\sum\limits_{k=0}^{n} \sum\limits_{i\in\mathbb{N}} p_{k,i} Y^{i+k} u^k = 0$. In other words, the polynomial $\sum\limits_{k=0}^{n}\sum\limits_{i\in\mathbb{N}} p_{k,i}Y^{i+k}u^k \in B[Y]$ equals 0. Hence, its coefficient before Y^n equals 0 as well. But its coefficient before Y^n is $\sum\limits_{k=0}^{n} p_{k,n-k}u^k$, so we get $\sum\limits_{k=0}^{n} p_{k,n-k}u^k = 0$.

Recall that $\sum\limits_{i\in\mathbb{N}} p_{k,i}Y^i = p_k$ for every $k \in \{0,1,\ldots,n\}$ (by the definition of the $p_{k,i}$). Thus, $\sum\limits_{i\in\mathbb{N}} p_{n,i}Y^i = p_n = 1$ in $A[Y]$, and thus $p_{n,0} = 1$ (by comparing coefficients before Y^0).

Define an $(n+1)$-tuple $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ by $a_k = p_{k,n-k}$ for every $k \in \{0,1,\ldots,n\}$. Then, $a_n = p_{n,0} = 1$. Besides, $\sum\limits_{k=0}^{n} \underbrace{a_k}_{=p_{k,n-k}} u^k = \sum\limits_{k=0}^{n} p_{k,n-k}u^k = 0$. Finally, $a_k = p_{k,n-k} \in I_{n-k}$ (since $p_{k,i} \in I_i$ for every $i \in \mathbb{N}$) for every $k \in \{0,1,\ldots,n\}$. In other words, $a_i \in I_{n-i}$ for every $i \in \{0,1,\ldots,n\}$.

Altogether, we now know that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} \text{ for every } i \in \{0,1,\ldots,n\}.$$

Thus, by Definition 2.3, the element u is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. This proves the $\Longleftarrow$ direction of Theorem 2.11. $\square$

2.4. Sums and products again

Let us next state an analogue of Theorem 1.6 for integrality over ideal semifiltrations:

Theorem 2.12. Let A be a ring. Let B be an A-algebra. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $u \in A$. Then, $u \cdot 1_B$ is 1-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$ if and only if $u \cdot 1_B \in I_1 \cdot 1_B$.

Proof of Theorem 2.12. Straightforward and left to the reader. $\square$

The next theorem is an analogue of Theorem 1.7 **(a)** for integrality over ideal semifiltrations:

Theorem 2.13. Let A be a ring. Let B be an A-algebra. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Then, $x + y$ is nm-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$.

Proof of Theorem 2.13. Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. The polynomial ring $B[Y]$ is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra (as explained in Remark 2.9).

Theorem 2.11 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$). Also, Theorem 2.11 (applied to y instead of u) yields that yY is n-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ (since y is n-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$).

Hence, Theorem 1.7 **(a)** (applied to $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$, xY and yY instead of A, B, x and y, respectively) yields that $xY + yY$ is nm-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Since $xY + yY = (x+y)Y$, this means that $(x+y)Y$ is nm-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Hence, Theorem 2.11 (applied to $x+y$ and nm instead of u and n) yields that $x+y$ is nm-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 2.13. □

Our next theorem is a somewhat asymmetric analogue of Theorem 1.7 **(b)** for integrality over ideal semifiltrations:

Theorem 2.14. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over A.

Then, xy is nm-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$.

Before we prove this theorem, we require a trivial observation:

Lemma 2.15. Let A be a ring. Let A' be an A-algebra. Let B' be an A'-algebra. Let $v \in B'$. Let $n \in \mathbb{N}$. Assume that v is n-integral over A. (Here, of course, we are using the fact that B' is an A-algebra, since B' is an A'-algebra while A' is an A-algebra.)

Then, v is n-integral over A'.

Proof of Theorem 2.14. Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. The polynomial ring $B[Y]$ is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra (as explained in Remark 2.9).

Theorem 2.11 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$). Also, we know that y is n-integral over A. Thus, Lemma 2.15 (applied to $A' = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, $B' = B[Y]$ and $v = y$) yields that y is n-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ (since $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is an A-algebra, and $B[Y]$ is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra).

On the other hand, we know that xY is m-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Hence, Theorem 1.7 **(b)** (applied to $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and xY instead of A, B and x, respectively) yields that $xY\cdot y$ is nm-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Since $xY\cdot y = xyY$, this means that xyY is nm-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Hence, Theorem 2.11 (applied to xy and nm instead of u and n) yields that xy is nm-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 2.14. $\square$

It is easy to state analogues of Corollary 1.9 and Corollary 1.10 for ideal semifiltrations. These analogues can be derived from Corollary 1.9 and Corollary 1.10 in the same way as how we derived Theorem 2.13 from Theorem 1.7 **(a)**.

2.5. Transitivity again

The next theorem imitates Theorem 1.5 for integrality over ideal semifiltrations:

Theorem 2.16. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A.

Let $v\in B$ and $u\in B$. Let $m\in\mathbb{N}$ and $n\in\mathbb{N}$.

(a) Then, $\left(I_\rho A[v]\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[v]$. (See Convention 2.17 below for the meaning of "$I_\rho A[v]$".)

(b) Assume that v is m-integral over A, and that u is n-integral over $\left(A[v], \left(I_\rho A[v]\right)_{\rho\in\mathbb{N}}\right)$. Then, u is nm-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$.

Here and in the following, we are using the following convention:

Convention 2.17. Let A be a ring. Let B be an A-algebra. Let $v\in B$, and let I be an ideal of A. Then, you should read the expression "$IA[v]$" as $I\cdot(A[v])$, not as $(IA)[v]$. For instance, you should read the term "$I_\rho A[v]$" (in Theorem 2.16 **(a)**) as $I_\rho\cdot(A[v])$, not as $\left(I_\rho A\right)[v]$.

Before we prove Theorem 2.16, let us state two lemmas. The first is a more general (but still obvious) version of Theorem 2.16 **(a)**:

Lemma 2.18. Let A be a ring. Let A' be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Then, $\left(I_\rho A'\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A'.

Proof of Lemma 2.18. This is a straightforward verification of axioms. $\square$

Lemma 2.19. Let A be a ring. Let A' be an A-algebra. Let B' be an A'-algebra. Let $v\in B'$. Then, $A'\cdot A[v] = A'[v]$ (an equality between A-submodules of B'). (Here, we are using the fact that B' is an A-algebra, because B' is an A'-algebra while A' is an A-algebra.)

Here, of course, the expression "$A' \cdot A[v]$" means "$A' \cdot (A[v])$", not "$(A' \cdot A)[v]$".

Proof of Lemma 2.19. Left to the reader (see [7]). $\square$

We are now ready to prove Theorem 2.16:

Proof of Theorem 2.16. **(a)** Lemma 2.18 (applied to $A' = A[v]$) yields that $(I_\rho A[v])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[v]$. This proves Theorem 2.16 **(a)**.

(b) Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$. Then, $(A[v])[Y]$ is an $A[Y]$-algebra (since $A[v]$ is an A-algebra) and therefore also an $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$-algebra (since $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A[Y]$). Hence, $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$ is an A-subalgebra of $(A[v])[Y]$ (since $v \in A[v] \subseteq (A[v])[Y]$). On the other hand, $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$ is an A-subalgebra of $(A[v])[Y]$ (by its definition).

Note that B is an $A[v]$-algebra (since $A[v]$ is a subring of B). Hence, (as explained in Definition 2.6) the polynomial ring $B[Y]$ is an $(A[v])[Y]$-algebra. Moreover, $B[Y]$ is an $A[Y]$-algebra (as explained in Definition 2.6) and also an $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$-algebra (as explained in Remark 2.9).

Now, we will show that $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v]$. (This is an equality between two subrings of $(A[v])[Y]$.)

In fact, Definition 2.7 yields $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right] = \sum_{i\in\mathbb{N}} I_i Y^i$. The same definition (but applied to $A[v]$ and $(I_\rho A[v])_{\rho\in\mathbb{N}}$ instead of A and $(I_\rho)_{\rho\in\mathbb{N}}$) yields

$$\begin{aligned}
(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right] &= \sum_{i\in\mathbb{N}} I_i \underbrace{A[v]\cdot Y^i}_{=Y^i\cdot A[v]} = \sum_{i\in\mathbb{N}} I_i Y^i \cdot A[v] \\
&= \underbrace{\left(\sum_{i\in\mathbb{N}} I_i Y^i\right)}_{=A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]} \cdot A[v] = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\cdot A[v] \\
&= \left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[v] \qquad (10)
\end{aligned}$$

(by Lemma 2.19, applied to $A' = A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ and $B' = (A[v])[Y]$).

Recall that $B[Y]$ is an $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$-algebra. Hence, Lemma 2.15 (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and m instead of A', B' and n) yields that v is m-integral over $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$ (since v is m-integral over A).

Now, Theorem 2.11 (applied to $A[v]$ and $(I_\rho A[v])_{\rho\in\mathbb{N}}$ instead of A and $(I_\rho)_{\rho\in\mathbb{N}}$) yields that the element uY is n-integral over $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$ (since u

is n-integral over $\left(A\left[v\right], \left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}\right)$. In view of (10), this rewrites as follows: The element uY is n-integral over $\left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)\left[v\right]$. Hence, Theorem 1.5 (applied to $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, $B\left[Y\right]$ and uY instead of A, B and u) yields that uY is nm-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ (since v is m-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$). Thus, Theorem 2.11 (applied to nm instead of n) yields that u is nm-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 2.16 **(b)**. $\square$

3. Generalizing to two ideal semifiltrations

Theorem 2.14 can be generalized: Instead of requiring y to be integral over the ring A, we can require y to be integral over a further ideal semifiltration $\left(J_\rho\right)_{\rho\in\mathbb{N}}$ of A. In that case, xy will be integral over the ideal semifiltration $\left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}$ (see Theorem 3.4 for the precise statement). To get a grip on this, let us study two ideal semifiltrations.

3.1. The product of two ideal semifiltrations

Theorem 3.1. Let A be a ring.

(a) Then, $\left(A\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

(b) Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ and $\left(J_\rho\right)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Then, $\left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

Proof of Theorem 3.1. The proof of this is just basic axiom checking (see [7] for details). $\square$

3.2. Half-reduction

Now let us generalize Theorem 2.11:

Theorem 3.2. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ and $\left(J_\rho\right)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Let $n \in \mathbb{N}$. Let $u \in B$.

We know that $\left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 3.1 **(b)**).

Consider the polynomial ring $A\left[Y\right]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$.

We will abbreviate this A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$.

(a) The sequence $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$.

(b) The element u of B is n-integral over $\left(A, \left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. (Here, we are using the fact that $B[Y]$ is an $A_{[I]}$-algebra, because $A_{[I]} = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A[Y]$ and because $B[Y]$ is an $A[Y]$-algebra as explained in Definition 2.6.)

Proof of Theorem 3.2. **(a)** We know that $(J_\tau)_{\tau\in\mathbb{N}} = \left(J_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A. Thus, by Lemma 2.18 (applied to $A_{[I]}$ and $(J_\tau)_{\tau\in\mathbb{N}}$ instead of A' and $\left(I_\rho\right)_{\rho\in\mathbb{N}}$), the sequence $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$. This proves Theorem 3.2 **(a)**.

(b) In order to verify Theorem 3.2 **(b)**, we have to prove the $\Longrightarrow$ and $\Longleftarrow$ statements.

$\Longrightarrow$: Assume that u is n-integral over $\left(A, \left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}\right)$. Thus, by Definition 2.3 (applied to $\left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}$ instead of $\left(I_\rho\right)_{\rho\in\mathbb{N}}$), there exists some $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} J_{n-i} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

Consider this $(a_0, a_1, \ldots, a_n)$.

For each $k \in \{0, 1, \ldots, n\}$, we have $a_k \in I_{n-k} J_{n-k} \subseteq I_{n-k}$ (since I_{n-k} is an ideal of A) and thus $a_k Y^{n-k} \in I_{n-k} Y^{n-k} \subseteq \sum_{i\in\mathbb{N}} I_i Y^i = A_{[I]}$. Thus, we can define an $(n+1)$-tuple $(b_0, b_1, \ldots, b_n) \in \left(A_{[I]}\right)^{n+1}$ by $b_k = a_k Y^{n-k}$ for every $k \in \{0, 1, \ldots, n\}$. This $(n+1)$-tuple satisfies

$$\sum_{k=0}^{n} b_k \cdot (uY)^k = 0, \qquad b_n = 1, \qquad \text{and} \qquad b_i \in J_{n-i} A_{[I]} \text{ for every } i \in \{0, 1, \ldots, n\}$$

(as can be easily checked). Hence, by Definition 2.3 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY and $(b_0, b_1, \ldots, b_n)$ instead of A, B, $\left(I_\rho\right)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, \ldots, a_n)$), the element uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. This proves the $\Longrightarrow$ direction of Theorem 3.2 **(b)**.

$\Longleftarrow$: Assume that uY is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. Thus, by Definition 2.3 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY and $(p_0, p_1, \ldots, p_n)$ instead of A, B, $\left(I_\rho\right)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, \ldots, a_n)$), there exists some $(p_0, p_1, \ldots, p_n) \in \left(A_{[I]}\right)^{n+1}$ such that

$$\sum_{k=0}^{n} p_k \cdot (uY)^k = 0, \qquad p_n = 1, \qquad \text{and} \qquad p_i \in J_{n-i} A_{[I]} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

Consider this $(p_0, p_1, \ldots, p_n)$. For every $k \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned} p_k &\in J_{n-k} A_{[I]} && \left(\text{since } p_i \in J_{n-i} A_{[I]} \text{ for every } i \in \{0, 1, \ldots, n\}\right) \\ &= J_{n-k} \sum_{i \in \mathbb{N}} I_i Y^i && \left(\text{since } A_{[I]} = A\left[\left(I_\rho\right)_{\rho \in \mathbb{N}} * Y\right] = \sum_{i \in \mathbb{N}} I_i Y^i\right) \\ &= \sum_{i \in \mathbb{N}} J_{n-k} I_i Y^i = \sum_{i \in \mathbb{N}} I_i J_{n-k} Y^i, \end{aligned}$$

and thus there exists a sequence $(p_{k,i})_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum\limits_{i \in \mathbb{N}} p_{k,i} Y^i$, such that ($p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$), and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Consider this sequence. Thus,

$$\sum_{k=0}^{n} \underbrace{p_k}_{= \sum\limits_{i \in \mathbb{N}} p_{k,i} Y^i} \cdot \underbrace{(uY)^k}_{\substack{= u^k Y^k \\ = Y^k u^k}} = \sum_{k=0}^{n} \sum_{i \in \mathbb{N}} p_{k,i} \underbrace{Y^i \cdot Y^k}_{= Y^{i+k}} u^k = \sum_{k=0}^{n} \sum_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k.$$

Hence, $\sum\limits_{k=0}^{n} p_k \cdot (uY)^k = 0$ rewrites as $\sum\limits_{k=0}^{n} \sum\limits_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k = 0$. In other words, the polynomial $\sum\limits_{k=0}^{n} \sum\limits_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k \in B[Y]$ equals 0. Hence, its coefficient before Y^n equals 0 as well. But its coefficient before Y^n is $\sum\limits_{k=0}^{n} p_{k,n-k} u^k$. Hence, we obtain $\sum\limits_{k=0}^{n} p_{k,n-k} u^k = 0$.

Recall that $\sum\limits_{i \in \mathbb{N}} p_{k,i} Y^i = p_k$ for every $k \in \{0, 1, \ldots, n\}$ (by the definition of the $p_{k,i}$). Thus, $\sum\limits_{i \in \mathbb{N}} p_{n,i} Y^i = p_n = 1$ in $A[Y]$, and thus $p_{n,0} = 1$ (by comparing coefficients before Y^0).

Define an $(n+1)$-tuple $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ by $a_k = p_{k,n-k}$ for every $k \in \{0, 1, \ldots, n\}$. Then, $a_n = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^{n} \underbrace{a_k}_{= p_{k,n-k}} u^k = \sum_{k=0}^{n} p_{k,n-k} u^k = 0.$$

Finally, for every $k \in \{0, 1, \ldots, n\}$, we have $n - k \in \mathbb{N}$ and thus $a_k = p_{k,n-k} \in I_{n-k} J_{n-k}$ (since $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$). Renaming the variable k as i in this statement, we obtain the following: For every $i \in \{0, 1, \ldots, n\}$, we have $a_i \in I_{n-i} J_{n-i}$.

Altogether, we now know that the $(n+1)$-tuple $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ satisfies

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{n-i} J_{n-i} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

Thus, by Definition 2.3 (applied to $(I_\rho J_\rho)_{\rho\in\mathbb{N}}$ instead of $(I_\rho)_{\rho\in\mathbb{N}}$), the element u is n-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$. This proves the $\Longleftarrow$ direction of Theorem 3.2 **(b)**, and thus Theorem 3.2 **(b)** is shown. $\square$

The reason why Theorem 3.2 **(b)** generalizes Theorem 2.11 (more precisely, Theorem 2.11 is the particular case of Theorem 3.2 **(b)** for $J_\rho = A$) is the following fact, which we mention here for the pure sake of completeness:

Theorem 3.3. Let A be a ring. Let B be an A-algebra. Let $n \in \mathbb{N}$. Let $u \in B$.

We know that $(A)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 3.1 **(a)**).

Then, the element u of B is n-integral over $\left(A, (A)_{\rho\in\mathbb{N}}\right)$ if and only if u is n-integral over A.

3.3. Integrality of products over the product semifiltration

Finally, let us generalize Theorem 2.14:

Theorem 3.4. Let A be a ring. Let B be an A-algebra. Let $(I_\rho)_{\rho\in\mathbb{N}}$ and $(J_\rho)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A.

Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$, and that y is n-integral over $\left(A, (J_\rho)_{\rho\in\mathbb{N}}\right)$. Then, xy is nm-integral over $\left(A, (I_\rho J_\rho)_{\rho\in\mathbb{N}}\right)$.

The proof of this theorem will require a generalization of Lemma 2.15:

Lemma 3.5. Let A be a ring. Let A' be an A-algebra. Let B' be an A'-algebra. Let $(I_\rho)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $v \in B'$. Let $n \in \mathbb{N}$. Assume that v is n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$. (Here, of course, we are using the fact that B' is an A-algebra, since B' is an A'-algebra while A' is an A-algebra.)

Then, v is n-integral over $\left(A', (I_\rho A')_{\rho\in\mathbb{N}}\right)$. (Note that $(I_\rho A')_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A', according to Lemma 2.18.)

Proof of Lemma 3.5. This becomes obvious upon unraveling the definitions of "n-integral over $\left(A, (I_\rho)_{\rho\in\mathbb{N}}\right)$" and of "$n$-integral over $\left(A', (I_\rho A')_{\rho\in\mathbb{N}}\right)$", and by realizing that every $\rho \in \mathbb{N}$ and every $a \in I_\rho$ satisfy $a \cdot 1_{A'} \in I_\rho A'$. (See [7] for details.) $\square$

Proof of Theorem 3.4. We have $(J_\rho)_{\rho\in\mathbb{N}} = (J_\tau)_{\tau\in\mathbb{N}}$. Hence, y is n-integral over $\left(A, (J_\tau)_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A, (J_\rho)_{\rho\in\mathbb{N}}\right)$). Also, $(J_\tau)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of A (since $\left(J_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, but we have $\left(J_\rho\right)_{\rho\in\mathbb{N}} = \left(J_\tau\right)_{\tau\in\mathbb{N}}$). Thus, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$ (by Lemma 2.18, applied to $A_{[I]}$ and $\left(J_\tau\right)_{\tau\in\mathbb{N}}$ instead of A' and $\left(I_\rho\right)_{\rho\in\mathbb{N}}$).

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. We will abbreviate this A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$. Thus, $A_{[I]} = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A[Y]$. Hence, $B[Y]$ is an $A_{[I]}$-algebra (since $B[Y]$ is an $A[Y]$-algebra as explained in Definition 2.6).

Theorem 2.11 (applied to x and m instead of u and n) yields that xY is m-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ (since x is m-integral over $\left(A, \left(I_\rho\right)_{\rho\in\mathbb{N}}\right)$). In other words, xY is m-integral over $A_{[I]}$ (since $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] = A_{[I]}$).

On the other hand, $A_{[I]}$ is an A-algebra, and $B[Y]$ is an $A_{[I]}$-algebra. Hence, Lemma 3.5 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau\right)_{\tau\in\mathbb{N}}$ and y instead of A', B', $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ and v) yields that y is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$ (since y is n-integral over $\left(A, \left(J_\tau\right)_{\tau\in\mathbb{N}}\right)$).

Hence, Theorem 2.14 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, y, xY, n and m instead of A, B, $\left(I_\rho\right)_{\rho\in\mathbb{N}}$, x, y, m and n, respectively) yields that $y \cdot xY$ is mn-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$ (since xY is m-integral over $A_{[I]}$).

Since $y \cdot xY = xyY$ and $mn = nm$, this means that xyY is nm-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. Hence, Theorem 3.2 **(b)** (applied to xy and nm instead of u and n) yields that xy is nm-integral over $\left(A, \left(I_\rho J_\rho\right)_{\rho\in\mathbb{N}}\right)$. This proves Theorem 3.4. $\square$

4. Accelerating ideal semifiltrations

4.1. Definition of λ-acceleration

We start this section with an obvious observation:

Theorem 4.1. Let A be a ring. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $\lambda \in \mathbb{N}$. Then, $\left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A.

I refer to the ideal semifiltration $\left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}}$ in Theorem 4.1 as the *λ-acceleration* of the ideal semifiltration $\left(I_\rho\right)_{\rho\in\mathbb{N}}$.

4.2. Half-reduction and reduction

Now, Theorem 3.2, itself a generalization of Theorem 2.11, can be generalized once more:

Theorem 4.2. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ and $\left(J_\rho\right)_{\rho\in\mathbb{N}}$ be two ideal semifiltrations of A. Let $n \in \mathbb{N}$. Let $u \in B$. Let $\lambda \in \mathbb{N}$.

We know that $\left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 4.1).

Hence, $\left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 3.1 **(b)**, applied to $\left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}}$ instead of $\left(I_\rho\right)_{\rho\in\mathbb{N}}$).

Consider the polynomial ring $A\left[Y\right]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$.

We will abbreviate this A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ by $A_{[I]}$.

(a) The sequence $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$.

(b) The element u of B is n-integral over $\left(A, \left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY^λ of the polynomial ring $B\left[Y\right]$ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. (Here, we are using the fact that $B\left[Y\right]$ is an $A_{[I]}$-algebra, because $A_{[I]} = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A\left[Y\right]$ and because $B\left[Y\right]$ is an $A\left[Y\right]$-algebra as explained in Definition 2.6.)

Proof of Theorem 4.2. **(a)** This is precisely Theorem 3.2 **(a)**.

(b) The definition of $A_{[I]}$ yields

$$\begin{aligned} A_{[I]} &= A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] = \sum_{i\in\mathbb{N}} I_i Y^i \qquad \text{(by Definition 2.7)} \\ &= \sum_{\ell\in\mathbb{N}} I_\ell Y^\ell \qquad \text{(here we renamed } i \text{ as } \ell \text{ in the sum)}. \end{aligned}$$

In order to verify Theorem 4.2 **(b)**, we have to prove the $\Longrightarrow$ and $\Longleftarrow$ statements.

$\Longrightarrow$: Assume that u is n-integral over $\left(A, \left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}\right)$. Thus, by Definition 2.3 (applied to $\left(I_{\lambda\rho}J_\rho\right)_{\rho\in\mathbb{N}}$ instead of $\left(I_\rho\right)_{\rho\in\mathbb{N}}$), there exists some $\left(a_0, a_1, \ldots, a_n\right) \in A^{n+1}$ such that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{\lambda(n-i)}J_{n-i} \text{ for every } i \in \left\{0, 1, \ldots, n\right\}.$$

Consider this $\left(a_0, a_1, \ldots, a_n\right)$.

For each $k \in \left\{0, 1, \ldots, n\right\}$, we have $a_k \in I_{\lambda(n-k)}J_{n-k} \subseteq I_{\lambda(n-k)}$ (since $I_{\lambda(n-k)}$ is an ideal of A) and thus $a_k Y^{\lambda(n-k)} \in I_{\lambda(n-k)} Y^{\lambda(n-k)} \subseteq \sum_{i\in\mathbb{N}} I_i Y^i = A_{[I]}$. Thus, we can find an $(n+1)$-tuple $(b_0, b_1, \ldots, b_n) \in \left(A_{[I]}\right)^{n+1}$ satisfying

$$\sum_{k=0}^{n} b_k \cdot \left(uY^\lambda\right)^k = 0, \qquad b_n = 1, \qquad \text{and} \qquad b_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

[6] Hence, by Definition 2.3 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY^λ and $(b_0, b_1, \ldots, b_n)$ instead of A, B, $\left(I_\rho\right)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, \ldots, a_n)$), the element uY^λ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. This proves the $\Longrightarrow$ direction of Theorem 4.2 **(b)**.

$\Longleftarrow$: Assume that uY^λ is n-integral over $\left(A_{[I]}, \left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}\right)$. Thus, by Definition 2.3 (applied to $A_{[I]}$, $B[Y]$, $\left(J_\tau A_{[I]}\right)_{\tau\in\mathbb{N}}$, uY^λ and $(p_0, p_1, \ldots, p_n)$ instead of A, B, $\left(I_\rho\right)_{\rho\in\mathbb{N}}$, u and $(a_0, a_1, \ldots, a_n)$), there exists some $(p_0, p_1, \ldots, p_n) \in \left(A_{[I]}\right)^{n+1}$ such that

$$\sum_{k=0}^{n} p_k \cdot \left(uY^\lambda\right)^k = 0, \qquad p_n = 1, \qquad \text{and} \qquad p_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

Consider this $(p_0, p_1, \ldots, p_n)$. For every $k \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned} p_k \in J_{n-k}A_{[I]} &= J_{n-k}\sum_{i\in\mathbb{N}} I_i Y^i \qquad \left(\text{since } A_{[I]} = \sum_{i\in\mathbb{N}} I_i Y^i\right) \\ &= \sum_{i\in\mathbb{N}} J_{n-k} I_i Y^i = \sum_{i\in\mathbb{N}} I_i J_{n-k} Y^i, \end{aligned}$$

and thus there exists a sequence $(p_{k,i})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum_{i\in\mathbb{N}} p_{k,i}Y^i$, such that ($p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$), and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Consider this sequence. Thus,

$$\sum_{k=0}^{n} \underbrace{p_k}_{=\sum_{i\in\mathbb{N}} p_{k,i}Y^i} \cdot \left(uY^\lambda\right)^k = \sum_{k=0}^{n}\sum_{i\in\mathbb{N}} p_{k,i} \underbrace{Y^i \cdot \left(uY^\lambda\right)^k}_{=u^k Y^{i+\lambda k}} = \sum_{k=0}^{n}\sum_{i\in\mathbb{N}} p_{k,i}u^k Y^{i+\lambda k}.$$

Compared with $\sum_{k=0}^{n} p_k \cdot \left(uY^\lambda\right)^k = 0$, this yields $\sum_{k=0}^{n}\sum_{i\in\mathbb{N}} p_{k,i}u^k Y^{i+\lambda k} = 0$. In other words, the polynomial $\sum_{k=0}^{n}\sum_{i\in\mathbb{N}} \underbrace{p_{k,i}u^k}_{\in B} Y^{i+\lambda k} \in B[Y]$ equals 0. Hence, its coefficient

[6]Namely, the $(n+1)$-tuple $(b_0, b_1, \ldots, b_n) \in \left(A_{[I]}\right)^{n+1}$ defined by $\left(b_k = a_k Y^{\lambda(n-k)} \text{ for every } k \in \{0, 1, \ldots, n\}\right)$ satisfies this. The proof is very easy (see [7] for details).

before $Y^{\lambda n}$ equals 0 as well. But its coefficient before $Y^{\lambda n}$ is $\sum\limits_{k=0}^{n} p_{k,\lambda(n-k)} u^k$ (since $i + \lambda k = \lambda n$ holds if and only if $i = \lambda(n-k)$). Hence, $\sum\limits_{k=0}^{n} p_{k,\lambda(n-k)} u^k = 0$.

Recall that $\sum\limits_{i \in \mathbb{N}} p_{k,i} Y^i = p_k$ for every $k \in \{0, 1, \ldots, n\}$ (by the definition of the $p_{k,i}$). Thus, $\sum\limits_{i \in \mathbb{N}} p_{n,i} Y^i = p_n = 1$ in $A[Y]$, and thus $p_{n,0} = 1$ (by comparing coefficients before Y^0).

Define an $(n+1)$-tuple $(a_0, a_1, \ldots, a_n) \in A^{n+1}$ by $a_k = p_{k,\lambda(n-k)}$ for every $k \in \{0, 1, \ldots, n\}$. Then, $a_n = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^{n} \underbrace{a_k}_{=p_{k,\lambda(n-k)}} u^k = \sum_{k=0}^{n} p_{k,\lambda(n-k)} u^k = 0.$$

Finally, for every $k \in \{0, 1, \ldots, n\}$, we have $a_k = p_{k,\lambda(n-k)} \in I_{\lambda(n-k)} J_{n-k}$ (since $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$). Renaming the variable k as i in this statement, we obtain the following: For every $i \in \{0, 1, \ldots, n\}$, we have $a_i \in I_{\lambda(n-i)} J_{n-i}$.

Altogether, we now know that

$$\sum_{k=0}^{n} a_k u^k = 0, \qquad a_n = 1, \qquad \text{and} \qquad a_i \in I_{\lambda(n-i)} J_{n-i} \text{ for every } i \in \{0, 1, \ldots, n\}.$$

Thus, by Definition 2.3 (applied to $(I_{\lambda\rho} J_\rho)_{\rho \in \mathbb{N}}$ instead of $(I_\rho)_{\rho \in \mathbb{N}}$), the element u is n-integral over $\left(A, (I_{\lambda\rho} J_\rho)_{\rho \in \mathbb{N}}\right)$. This proves the $\Longleftarrow$ direction of Theorem 4.2 **(b)**, and thus completes the proof. □

A particular case of Theorem 4.2 **(b)** is the following fact:

Theorem 4.3. Let A be a ring. Let B be an A-algebra. Let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}$. Let $u \in B$. Let $\lambda \in \mathbb{N}$.

We know that $(I_{\lambda\rho})_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 4.1).

Consider the polynomial ring $A[Y]$ and its A-subalgebra $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ defined in Definition 2.7.

Then, the element u of B is n-integral over $\left(A, (I_{\lambda\rho})_{\rho \in \mathbb{N}}\right)$ if and only if the element uY^λ of the polynomial ring $B[Y]$ is n-integral over the ring $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$. (Here, we are using the fact that $B[Y]$ is an $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$-algebra, because $A\left[(I_\rho)_{\rho \in \mathbb{N}} * Y\right]$ is a subring of $A[Y]$ and because $B[Y]$ is an $A[Y]$-algebra as explained in Definition 2.6.)

Proof of Theorem 4.3. Theorem 3.1 **(a)** states that $(A)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A.

Every $\rho \in \mathbb{N}$ satisfies $I_{\lambda\rho} = I_{\lambda\rho}A$ (since $I_{\lambda\rho}$ is an ideal of A). Thus, $\left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}} = \left(I_{\lambda\rho}A\right)_{\rho\in\mathbb{N}}$.

We will abbreviate the A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ of $A[Y]$ by $A_{[I]}$. Thus, $B[Y]$ is an $A_{[I]}$-algebra (since $B[Y]$ is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra).

It is easy to see that $AA_{[I]} = A_{[I]}$ (since $A_{[I]}$ is an A-algebra). Hence, $\left(AA_{[I]}\right)_{\tau\in\mathbb{N}} = \left(A_{[I]}\right)_{\tau\in\mathbb{N}} = \left(A_{[I]}\right)_{\rho\in\mathbb{N}}$.

Now, we have the following chain of equivalences:

$$\begin{aligned}
&\left(u \text{ is } n\text{-integral over } \left(A, \left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}}\right)\right) \\
\Longleftrightarrow\ &\left(u \text{ is } n\text{-integral over } \left(A, \left(I_{\lambda\rho}A\right)_{\rho\in\mathbb{N}}\right)\right) \\
&\left(\text{since } \left(I_{\lambda\rho}\right)_{\rho\in\mathbb{N}} = \left(I_{\lambda\rho}A\right)_{\rho\in\mathbb{N}}\right) \\
\Longleftrightarrow\ &\left(uY^\lambda \text{ is } n\text{-integral over } \left(A_{[I]}, \left(AA_{[I]}\right)_{\tau\in\mathbb{N}}\right)\right) \\
&\left(\text{by Theorem 4.2 \textbf{(b)}, applied to } \left(J_\rho\right)_{\rho\in\mathbb{N}} = \left(A\right)_{\rho\in\mathbb{N}}\right) \\
\Longleftrightarrow\ &\left(uY^\lambda \text{ is } n\text{-integral over } \left(A_{[I]}, \left(A_{[I]}\right)_{\rho\in\mathbb{N}}\right)\right) \\
&\left(\text{since } \left(AA_{[I]}\right)_{\tau\in\mathbb{N}} = \left(A_{[I]}\right)_{\rho\in\mathbb{N}}\right) \\
\Longleftrightarrow\ &\left(uY^\lambda \text{ is } n\text{-integral over } A_{[I]}\right) \\
&\left(\text{by Theorem 3.3, applied to } A_{[I]},\ B[Y] \text{ and } uY^\lambda \text{ instead of } A, B \text{ and } u\right) \\
\Longleftrightarrow\ &\left(uY^\lambda \text{ is } n\text{-integral over } A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right) \\
&\left(\text{since } A_{[I]} = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right).
\end{aligned}$$

This proves Theorem 4.3. □

Note that Theorem 2.11 is the particular case of Theorem 4.3 for $\lambda = 1$.

Finally we can generalize even Theorem 1.11:

Theorem 4.4. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $n \in \mathbb{N}^+$. Let $v \in B$. Let $a_0, a_1, \ldots, a_n$ be $n+1$ elements of A such that $\sum\limits_{i=0}^{n} a_i v^i = 0$. Assume further that $a_i \in I_{n-i}$ for every $i \in \{0, 1, \ldots, n\}$.

Let $k \in \{0, 1, \ldots, n\}$. We know that $\left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 4.1, applied to $\lambda = n - k$).

Then, $\sum\limits_{i=0}^{n-k} a_{i+k} v^i$ is n-integral over $\left(A, \left(I_{(n-k)\rho}\right)_{\rho \in \mathbb{N}}\right)$.

Proof of Theorem 4.4. Consider the polynomial ring $A\left[Y\right]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ defined in Definition 2.7. Note that $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A\left[Y\right]$; hence, $B\left[Y\right]$ is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra (because $B\left[Y\right]$ is an $A\left[Y\right]$-algebra as explained in Definition 2.6).

Definition 2.7 yields

$$A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right] = \sum_{i\in\mathbb{N}} I_i Y^i = \sum_{\ell\in\mathbb{N}} I_\ell Y^\ell \qquad \text{(here we renamed } i \text{ as } \ell \text{ in the sum)}.$$

Hence, $\sum\limits_{\ell\in\mathbb{N}} I_\ell Y^\ell = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$.

Define $u \in B$ by

$$u = \sum_{i=0}^{n-k} a_{i+k} v^i. \qquad (11)$$

In the ring $B\left[Y\right]$, we have

$$\sum_{i=0}^{n} a_i Y^{n-i} \underbrace{(vY)^i}_{=v^iY^i=Y^iv^i} = \sum_{i=0}^{n} a_i \underbrace{Y^{n-i}Y^i}_{=Y^n} v^i = Y^n \underbrace{\sum_{i=0}^{n} a_i v^i}_{=0} = 0.$$

Besides, every $i \in \{0, 1, \ldots, n\}$ satisfies

$$\underbrace{a_i}_{\substack{\in I_{n-i} \\ \text{(by assumption)}}} Y^{n-i} \in I_{n-i} Y^{n-i} \subseteq \sum_{\ell\in\mathbb{N}} I_\ell Y^\ell = A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right].$$

In other words, $a_0 Y^{n-0}, a_1 Y^{n-1}, \ldots, a_n Y^{n-n}$ are $n+1$ elements of $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Hence, Theorem 1.11 (applied to $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, $B\left[Y\right]$, vY and $a_i Y^{n-i}$ instead of A, B, v and a_i) yields that $\sum\limits_{i=0}^{n-k} a_{i+k} Y^{n-(i+k)} (vY)^i$ is n-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Since

$$\sum_{i=0}^{n-k} a_{i+k} Y^{n-(i+k)} \underbrace{(vY)^i}_{=v^iY^i=Y^iv^i} = \sum_{i=0}^{n-k} a_{i+k} \underbrace{Y^{n-(i+k)}Y^i}_{=Y^{(n-(i+k))+i}=Y^{n-k}} v^i = \underbrace{\sum_{i=0}^{n-k} a_{i+k} v^i}_{\substack{=u \\ \text{(by (11))}}} \cdot Y^{n-k} = uY^{n-k},$$

this means that uY^{n-k} is n-integral over $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$.

But Theorem 4.3 (applied to $\lambda = n - k$) yields that u is n-integral over $\left(A, \left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}\right)$ if and only if uY^{n-k} is n-integral over the ring $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. Since we know that uY^{n-k} is n-integral over the ring $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$, this yields that u is n-integral over $\left(A, \left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}\right)$. In other words, $\sum\limits_{i=0}^{n-k} a_{i+k}v^i$ is n-integral over $\left(A, \left(I_{(n-k)\rho}\right)_{\rho\in\mathbb{N}}\right)$ (since $u = \sum\limits_{i=0}^{n-k} a_{i+k}v^i$). This proves Theorem 4.4. $\square$

5. On a lemma by Lombardi

5.1. A lemma on products of powers

Now, we shall show a rather technical lemma:

Lemma 5.1. Let A be a ring. Let B be an A-algebra. Let $x \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Let $\mu \in \mathbb{N}$ and $\nu \in \mathbb{N}$ be such that $\mu + \nu \in \mathbb{N}^+$. Assume that

$$u^n \in \left\langle u^0, u^1, \ldots, u^{n-1}\right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\nu\right\rangle_A \tag{12}$$

and that

$$\begin{aligned} u^m x^\mu \in & \left\langle u^0, u^1, \ldots, u^{m-1}\right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\mu\right\rangle_A \\ & + \left\langle u^0, u^1, \ldots, u^m\right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{\mu-1}\right\rangle_A . \end{aligned} \tag{13}$$

Then, u is $(n\mu + m\nu)$-integral over A.

This lemma can be seen as a variant of [6, Theorem 2][7]. Indeed, the particular case of [6, Theorem 2] when $J = 0$ can easily be obtained from Lemma 5.1 (applied to x and α instead of u and x).

The proof of Lemma 5.1 is not difficult but rather elaborate. For a completely detailed writeup of this proof, see [7]. Here let me give the skeleton of the proof:

Proof of Lemma 5.1 (sketched). Define the set

$$\begin{aligned} S = & \left(\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, \mu - 1\}\right) \\ & \cup \left(\{0, 1, \ldots, m-1\} \times \{\mu, \mu+1, \ldots, \mu+\nu-1\}\right). \end{aligned}$$

Clearly, $|S| = n\mu + m\nu$ and

$$j < \mu + \nu \text{ for every } (i, j) \in S. \tag{14}$$

[7] *Caveat:* The notion "integral over (A, J)" defined in [6] has nothing to do with **our** notion "n-integral over $\left(A, (I_n)_{n\in\mathbb{N}}\right)$".

Let U be the A-submodule $\left\langle u^i x^j \mid (i,j) \in S \right\rangle_A$ of B. Then, U is an $(n\mu + m\nu)$-generated A-module (since $|S| = n\mu + m\nu$). Besides, clearly,

$$u^i x^j \in U \text{ for every } (i,j) \in S. \tag{15}$$

Now, we will show that

$$\text{every } i \in \mathbb{N} \text{ and } j \in \mathbb{N} \text{ satisfying } j < \mu + \nu \text{ satisfy } u^i x^j \in U. \tag{16}$$

[The *proof of (16)* can be done either by double induction (over i and over j) or by the well-ordering principle. The induction proof has the advantage that it is completely constructive, but it is clumsy (I give this induction proof in [7]). So, for the sake of brevity, the proof I am going to give here is by the well-ordering principle:

For the sake of contradiction, we assume that (16) is not true. That is, there exists some pair $(i,j) \in \mathbb{N}^2$ satisfying $j < \mu + \nu$ but **not** satisfying $u^i x^j \in U$. Let (I,J) be the lexicographically smallest[8] such pair[9]. Then, $J < \mu + \nu$ but $u^I x^J \notin U$, and since (I,J) is the lexicographically smallest such pair, we have

$$u^I x^j \in U \text{ for every } j \in \mathbb{N} \text{ such that } j < J \tag{17}$$

and

$$u^i x^j \in U \text{ for every } i \in \mathbb{N} \text{ and } j \in \mathbb{N} \text{ such that } i < I \text{ and } j < \mu + \nu. \tag{18}$$

Recall that U is an A-module. Hence, (17) entails

$$\left\langle u^I \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{J-1} \right\rangle_A \subseteq U, \tag{19}$$

and (18) entails

$$\left\langle u^0, u^1, \ldots, u^{I-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{\mu+\nu-1} \right\rangle_A \subseteq U. \tag{20}$$

Also, from $J < \mu + \nu$, we obtain $J \leq \mu + \nu - 1$ (since J and $\mu + \nu$ are integers).

We now want to prove that $u^I x^J \in U$.

We are in one of the following four cases:

Case 1: We have $I \geq m$ and $J \geq \mu$.

Case 2: We have $I < m$ and $J \geq \mu$.

Case 3: We have $I \geq n$ and $J < \mu$.

Case 4: We have $I < n$ and $J < \mu$.

[8]"Lexicographically smallest" means "smallest with respect to the lexicographic order". Here, the *lexicographic order* on $\mathbb{N}^2$ is defined to be the total order on $\mathbb{N}^2$ in which two pairs $(a_1, b_1) \in \mathbb{N}^2$ and $(a_2, b_2) \in \mathbb{N}^2$ satisfy $(a_1, b_1) < (a_2, b_2)$ if and only if either $a_1 < a_2$ or ($a_1 = a_2$ and $b_1 < b_2$). It is well-known that this total order is well-defined and turns $\mathbb{N}^2$ into a well-ordered set.

[9]This is well-defined, since the lexicographic order is a well-ordering on $\mathbb{N}^2$.

In Case 1, we have $I - m \geq 0$ (since $I \geq m$) and $J - \mu \geq 0$ (since $J \geq \mu$), thus

$$\begin{aligned}
&\underbrace{u^I}_{=u^{I-m}u^m} \underbrace{x^J}_{=x^\mu x^{J-\mu}} \\
&= u^{I-m} \underbrace{u^m x^\mu}_{\substack{\in \left\langle u^0,u^1,\ldots,u^{m-1}\right\rangle_A \cdot \left\langle x^0,x^1,\ldots,x^\mu\right\rangle_A + \left\langle u^0,u^1,\ldots,u^m\right\rangle_A \cdot \left\langle x^0,x^1,\ldots,x^{\mu-1}\right\rangle_A \\ \text{(by (13))}}} x^{J-\mu} \\
&\in u^{I-m} \left(\left\langle u^0, u^1, \ldots, u^{m-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\mu \right\rangle_A \right. \\
&\qquad\qquad \left. + \left\langle u^0, u^1, \ldots, u^m \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{\mu-1} \right\rangle_A \right) x^{J-\mu} \\
&= \underbrace{u^{I-m} \left\langle u^0, u^1, \ldots, u^{m-1} \right\rangle_A}_{\substack{= \left\langle u^{I-m},u^{I-m+1},\ldots,u^{I-1}\right\rangle_A \\ \subseteq \left\langle u^0,u^1,\ldots,u^{I-1}\right\rangle_A}} \cdot \underbrace{\left\langle x^0, x^1, \ldots, x^\mu \right\rangle_A x^{J-\mu}}_{\substack{= \left\langle x^{J-\mu},x^{J-\mu+1},\ldots,x^J\right\rangle_A \\ \subseteq \left\langle x^0,x^1,\ldots,x^{\mu+\nu-1}\right\rangle_A \\ \text{(since } J-\mu\geq 0 \text{ and } J\leq \mu+\nu-1\text{)}}} \\
&\qquad + \underbrace{u^{I-m} \left\langle u^0, u^1, \ldots, u^m \right\rangle_A}_{\substack{= \left\langle u^{I-m},u^{I-m+1},\ldots,u^I\right\rangle_A \\ \subseteq \left\langle u^0,u^1,\ldots,u^I\right\rangle_A}} \cdot \underbrace{\left\langle x^0, x^1, \ldots, x^{\mu-1} \right\rangle_A x^{J-\mu}}_{\substack{= \left\langle x^{J-\mu},x^{J-\mu+1},\ldots,x^{J-1}\right\rangle_A \\ \subseteq \left\langle x^0,x^1,\ldots,x^{J-1}\right\rangle_A}} \\
&\subseteq \underbrace{\left\langle u^0, u^1, \ldots, u^{I-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{\mu+\nu-1} \right\rangle_A}_{\substack{\subseteq U \\ \text{(by (20))}}} \\
&\qquad + \underbrace{\left\langle u^0, u^1, \ldots, u^I \right\rangle_A}_{= \left\langle u^0,u^1,\ldots,u^{I-1}\right\rangle_A + \left\langle u^I\right\rangle_A} \cdot \left\langle x^0, x^1, \ldots, x^{J-1} \right\rangle_A \\
&\subseteq U + \underbrace{\left(\left\langle u^0, u^1, \ldots, u^{I-1} \right\rangle_A + \left\langle u^I \right\rangle_A \right) \cdot \left\langle x^0, x^1, \ldots, x^{J-1} \right\rangle_A}_{= \left\langle u^0,u^1,\ldots,u^{I-1}\right\rangle_A \cdot \left\langle x^0,x^1,\ldots,x^{J-1}\right\rangle_A + \left\langle u^I\right\rangle_A \cdot \left\langle x^0,x^1,\ldots,x^{J-1}\right\rangle_A} \\
&= U + \left\langle u^0, u^1, \ldots, u^{I-1} \right\rangle_A \cdot \underbrace{\left\langle x^0, x^1, \ldots, x^{J-1} \right\rangle_A}_{\substack{\subseteq \left\langle x^0,x^1,\ldots,x^{\mu+\nu-1}\right\rangle_A \\ \text{(since } J-1\leq J\leq \mu+\nu-1\text{)}}} + \left\langle u^I \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{J-1} \right\rangle_A \\
&\subseteq U + \underbrace{\left\langle u^0, u^1, \ldots, u^{I-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{\mu+\nu-1} \right\rangle_A}_{\substack{\subseteq U \\ \text{(by (20))}}} + \underbrace{\left\langle u^I \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{J-1} \right\rangle_A}_{\substack{\subseteq U \\ \text{(by (19))}}} \\
&\subseteq U + U + U \subseteq U \qquad \text{(since } U \text{ is an } A\text{-module)}.
\end{aligned}$$

Thus, we have proved that $u^I x^J \in U$ holds in Case 1.

In Case 2, we have $(I, J) \in \{0, 1, \ldots, m-1\} \times \{\mu, \mu+1, \ldots, \mu+\nu-1\} \subseteq S$ and thus $u^I x^J \in U$ (by (15), applied to I and J instead of i and j). Thus, we have proved that $u^I x^J \in U$ holds in Case 2.

In Case 3, we have $I - n \geq 0$ (since $I \geq n$) and $J + \nu \leq \mu + \nu - 1$ (since $J < \mu$ yields $J + \nu < \mu + \nu$, and since $J + \nu$ and $\mu + \nu$ are integers), thus

$$\begin{aligned}
&\underbrace{u^I}_{=u^{I-n}u^n} x^J \\
&= u^{I-n} \underbrace{u^n}_{\substack{\in \left\langle u^0, u^1, \ldots, u^{n-1}\right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A \\ \text{(by (12))}}} x^J \\
&\in \underbrace{u^{I-n} \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A}_{\substack{= \left\langle u^{I-n}, u^{I-n+1}, \ldots, u^{I-1}\right\rangle_A \\ \subseteq \left\langle u^0, u^1, \ldots, u^{I-1}\right\rangle_A}} \cdot \underbrace{\left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A x^J}_{\substack{= \left\langle x^J, x^{J+1}, \ldots, x^{J+\nu}\right\rangle_A \\ \subseteq \left\langle x^0, x^1, \ldots, x^{\mu+\nu-1}\right\rangle_A \\ \text{(since } J+\nu \leq \mu+\nu-1)}} \\
&\subseteq \left\langle u^0, u^1, \ldots, u^{I-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{\mu+\nu-1} \right\rangle_A \subseteq U \qquad \text{(by (20))}.
\end{aligned}$$

Thus, we have proved that $u^I x^J \in U$ holds in Case 3.

In Case 4, we have $(I, J) \in \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, \mu - 1\} \subseteq S$ and thus $u^I x^J \in U$ (by (15), applied to I and J instead of i and j). Thus, we have proved that $u^I x^J \in U$ holds in Case 4.

By now, we have proved that $u^I x^J \in U$ holds in each of the four cases 1, 2, 3 and 4. Hence, $u^I x^J \in U$ always holds, contradicting $u^I x^J \notin U$. This contradiction completes the proof of (16).]

Now that (16) is proven, we can easily conclude that $uU \subseteq U$ (since every $(i, j) \in S$ satisfies $j < \mu + \nu$, and thus (16) shows that $u^{i+1} x^j \in U$) and $1 \in U$ (this follows by applying (16) to $i = 0$ and $j = 0$). Altogether, U is an $(n\mu + m\nu)$-generated A-submodule of B such that $1 \in U$ and $uU \subseteq U$. Thus, $u \in B$ satisfies Assertion $\mathcal{C}$ of Theorem 1.1 with n replaced by $n\mu + m\nu$. Hence, $u \in B$ satisfies the four equivalent assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Theorem 1.1 with n replaced by $n\mu + m\nu$. Consequently, u is $(n\mu + m\nu)$-integral over A. This proves Lemma 5.1. □

We record a weaker variant of Lemma 5.1:

Lemma 5.2. Let A be a ring. Let B be an A-algebra. Let $x \in B$ and $y \in B$ be such that $xy \in A$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Let $\mu \in \mathbb{N}$ and $\nu \in \mathbb{N}$ be such that $\mu + \nu \in \mathbb{N}^+$. Assume that

$$u^n \in \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A \tag{21}$$

and that

$$\begin{aligned} u^m \in &\left\langle u^0, u^1, \ldots, u^{m-1} \right\rangle_A \cdot \left\langle y^0, y^1, \ldots, y^\mu \right\rangle_A \\ &+ \left\langle u^0, u^1, \ldots, u^m \right\rangle_A \cdot \left\langle y^1, y^2, \ldots, y^\mu \right\rangle_A . \end{aligned} \tag{22}$$

Then, u is $(n\mu + m\nu)$-integral over A.

Proof of Lemma 5.2. (Again, this proof appears in greater detail in [7].) We have

$$\left\langle y^0, y^1, \ldots, y^\mu \right\rangle_A x^\mu \subseteq \left\langle x^0, x^1, \ldots, x^\mu \right\rangle_A, \tag{23}$$

since every $i \in \{0, 1, \ldots, \mu\}$ satisfies

$$\begin{aligned} y^i \underbrace{x^\mu}_{=x^{\mu-i}x^i} &= y^i x^{\mu-i} x^i = \underbrace{x^i y^i}_{\substack{=(xy)^i \in A \\ (\text{since } xy \in A)}} x^{\mu-i} \in A x^{\mu-i} = \left\langle x^{\mu-i} \right\rangle_A \\ &\subseteq \left\langle x^0, x^1, \ldots, x^\mu \right\rangle_A \qquad (\text{since } \mu - i \in \{0, 1, \ldots, \mu\}). \end{aligned} \tag{24}$$

Besides,

$$\left\langle y^1, y^2, \ldots, y^\mu \right\rangle_A x^\mu \subseteq \left\langle x^0, x^1, \ldots, x^{\mu-1} \right\rangle_A, \tag{25}$$

since every $i \in \{1, 2, \ldots, \mu\}$ satisfies

$$\begin{aligned} y^i x^\mu &\in \left\langle x^{\mu-i} \right\rangle_A \qquad (\text{by (24)}) \\ &\subseteq \left\langle x^0, x^1, \ldots, x^{\mu-1} \right\rangle_A \qquad (\text{since } \mu - i \in \{0, 1, \ldots, \mu - 1\}). \end{aligned}$$

Now, (22) yields

$$\begin{aligned} u^m x^\mu &\in \Big(\left\langle u^0, u^1, \ldots, u^{m-1} \right\rangle_A \cdot \left\langle y^0, y^1, \ldots, y^\mu \right\rangle_A \\ &\qquad + \left\langle u^0, u^1, \ldots, u^m \right\rangle_A \cdot \left\langle y^1, y^2, \ldots, y^\mu \right\rangle_A \Big) x^\mu \\ &= \left\langle u^0, u^1, \ldots, u^{m-1} \right\rangle_A \cdot \underbrace{\left\langle y^0, y^1, \ldots, y^\mu \right\rangle_A x^\mu}_{\substack{\subseteq \left\langle x^0, x^1, \ldots, x^\mu \right\rangle_A \\ (\text{by (23)})}} \\ &\qquad + \left\langle u^0, u^1, \ldots, u^m \right\rangle_A \cdot \underbrace{\left\langle y^1, y^2, \ldots, y^\mu \right\rangle_A x^\mu}_{\substack{\subseteq \left\langle x^0, x^1, \ldots, x^{\mu-1} \right\rangle_A \\ (\text{by (25)})}} \\ &\subseteq \left\langle u^0, u^1, \ldots, u^{m-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\mu \right\rangle_A \\ &\qquad + \left\langle u^0, u^1, \ldots, u^m \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^{\mu-1} \right\rangle_A. \end{aligned}$$

In other words, (13) holds. Also, (12) holds (because (21) holds, and because (12) is the same as (21)). Thus, Lemma 5.1 yields that u is $(n\mu + m\nu)$-integral over A. This proves Lemma 5.2. $\square$

We now come to something trivial:

Lemma 5.3. Let A be a ring. Let B be an A-algebra. Let $x \in B$. Let $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $A[x]$. Then, there exists some $\nu \in \mathbb{N}^+$ such that

$$u^n \in \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A .$$

Proof of Lemma 5.3. Again, see [7] for more details on this argument; here we only show a quick sketch: Since u is n-integral over $A[x]$, there exists a monic polynomial $P \in (A[x])[X]$ with $\deg P = n$ and $P(u) = 0$. Denoting the coefficients of this polynomial P by $\alpha_0, \alpha_1, \ldots, \alpha_n$ (where $\alpha_n = 1$), we can rewrite the equality $P(u) = 0$ as $u^n = -\sum\limits_{i=0}^{n-1} \alpha_i u^i$. Note that $\alpha_i \in A[x]$ for all i. Now, there exists some $\nu \in \mathbb{N}^+$ such that $\alpha_i \in \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A$ for every $i \in \{0, 1, \ldots, n-1\}$ (because for each $i \in \{0, 1, \ldots, n-1\}$, we have $\alpha_i \in A[x] = \bigcup\limits_{\nu=0}^{\infty} \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A$, so that $\alpha_i \in \left\langle x^0, x^1, \ldots, x^{\nu_i} \right\rangle_A$ for some $\nu_i \in \mathbb{N}$; now take $\nu = \max\{\nu_0, \nu_1, \ldots, \nu_{n-1}, 1\}$). This ν then satisfies

$$\begin{aligned} u^n &= -\sum_{i=0}^{n-1} \alpha_i u^i = -\sum_{i=0}^{n-1} \underbrace{u^i}_{\in \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A} \underbrace{\alpha_i}_{\in \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A} \\ &\in \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A , \end{aligned}$$

and Lemma 5.3 is proven. □

5.2. Integrality over $A[x]$ and over $A[y]$ implies integrality over $A[xy]$

A consequence of Lemma 5.2 and Lemma 5.3 is the following theorem:

Theorem 5.4. Let A be a ring. Let B be an A-algebra. Let $x \in B$ and $y \in B$ be such that $xy \in A$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $A[x]$, and that u is m-integral over $A[y]$. Then, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over A.

Proof of Theorem 5.4. Since u is n-integral over $A[x]$, Lemma 5.3 yields that there exists some $\nu \in \mathbb{N}^+$ such that

$$u^n \in \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_A \cdot \left\langle x^0, x^1, \ldots, x^\nu \right\rangle_A .$$

In other words, there exists some $\nu \in \mathbb{N}^+$ such that (21) holds. Consider this ν.

Since u is m-integral over $A[y]$, Lemma 5.3 (with x, n and ν replaced by y, m and μ) yields that there exists some $\mu \in \mathbb{N}^+$ such that

$$u^m \in \left\langle u^0, u^1, \ldots, u^{m-1} \right\rangle_A \cdot \left\langle y^0, y^1, \ldots, y^\mu \right\rangle_A . \tag{26}$$

Consider this μ. Hence, (22) holds as well (because (26) is even stronger than (22)).

Since both (21) and (22) hold, Lemma 5.2 yields that u is $(n\mu + m\nu)$-integral over A. Thus, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over A (namely, $\lambda = n\mu + m\nu$). This proves Theorem 5.4. $\square$

We record a generalization of Theorem 5.4 (which will turn out to be easily seen equivalent to Theorem 5.4):

Theorem 5.5. Let A be a ring. Let B be an A-algebra. Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $A[x]$, and that u is m-integral over $A[y]$. Then, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over $A[xy]$.

Proof of Theorem 5.5. Let C denote the A-subalgebra $A[xy]$ of A. Thus, $C = A[xy]$ is an A-subalgebra of B, hence a subring of B. Thus, $C[x]$ is a C-subalgebra of B, hence a subring of B. Note that $C = A[xy] = A[yx]$ (since $xy = yx$).

Furthermore, $A[x]$ is a subring of $C[x]$ [10]. Thus, $C[x]$ is an $A[x]$-algebra. Also, B is a $C[x]$-algebra (since $C[x]$ is a subring of B). Since u is n-integral over $A[x]$, Lemma 2.15 (applied to B, $C[x]$, $A[x]$ and u instead of B', A', A and v) yields that u is n-integral over $C[x]$. The same argument (but applied to y, x, n and m instead of x, y, m and n) shows that u is m-integral over $C[y]$ (since $C = A[yx]$).

Now, B is a C-algebra (since C is a subring of B) and we have $xy \in A[xy] = C$. Hence, Theorem 5.4 (applied to C instead of A) yields that there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over C (because u is n-integral over $C[x]$, and because u is m-integral over $C[y]$). In other words, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over $A[xy]$ (since $C = A[xy]$). This proves Theorem 5.5. $\square$

5.3. Generalization to ideal semifiltrations

Theorem 5.5 has a "relative version":

[10] *Proof.* Both $A[x]$ and $C[x]$ are subrings of B.

Now, let $\gamma \in A[x]$. Thus, there exist some $p \in \mathbb{N}$ and some elements $a_0, a_1, \ldots, a_p$ of A such that $\gamma = \sum\limits_{i=0}^{p} a_i x^i$. Consider this p and these $a_0, a_1, \ldots, a_p$. For each $i \in \{0, 1, \ldots, p\}$, we have $\underbrace{a_i}_{\in A} \cdot 1_B \in A \cdot 1_B \subseteq A[xy] = C$ (since $C = A[xy]$). Hence, $\sum\limits_{i=0}^{p} (a_i \cdot 1_B) x^i \in C[x]$. In view of

$$\sum_{i=0}^{p} (a_i \cdot 1_B) x^i = \sum_{i=0}^{p} a_i \cdot \underbrace{1_B x^i}_{=x^i} = \sum_{i=0}^{p} a_i x^i = \gamma \qquad \left(\text{since } \gamma = \sum_{i=0}^{p} a_i x^i\right),$$

this rewrites as $\gamma \in C[x]$.

Forget that we fixed γ. We thus have shown that $\gamma \in C[x]$ for each $\gamma \in A[x]$. In other words, $A[x] \subseteq C[x]$. Hence, $A[x]$ is a subring of $C[x]$ (since both $A[x]$ and $C[x]$ are subrings of B).

Theorem 5.6. Let A be a ring. Let B be an A-algebra. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Let $x \in B$ and $y \in B$.

(a) Then, $\left(I_\rho A\left[x\right]\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A\left[x\right]$. Besides, $\left(I_\rho A\left[y\right]\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A\left[y\right]$. Besides, $\left(I_\rho A\left[xy\right]\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A\left[xy\right]$.

(b) Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $u \in B$. Assume that u is n-integral over $\left(A\left[x\right], \left(I_\rho A\left[x\right]\right)_{\rho\in\mathbb{N}}\right)$, and that u is m-integral over $\left(A\left[y\right], \left(I_\rho A\left[y\right]\right)_{\rho\in\mathbb{N}}\right)$. Then, there exists some $\lambda \in \mathbb{N}$ such that u is λ-integral over $\left(A\left[xy\right], \left(I_\rho A\left[xy\right]\right)_{\rho\in\mathbb{N}}\right)$.

Our proof of this theorem will rely on a lemma:

Lemma 5.7. Let A be a ring. Let B be an A-algebra. Let $v \in B$. Let $\left(I_\rho\right)_{\rho\in\mathbb{N}}$ be an ideal semifiltration of A. Lemma 2.18 (applied to $A' = A\left[v\right]$) yields that $\left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A\left[v\right]$. Consider the polynomial ring $A\left[Y\right]$ and its A-subalgebra $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$. We know that $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A\left[Y\right]$, and (as explained in Definition 2.6) the polynomial ring $\left(A\left[v\right]\right)\left[Y\right]$ is an $A\left[Y\right]$-algebra (since $A\left[v\right]$ is an A-algebra). Hence, $\left(A\left[v\right]\right)\left[Y\right]$ is an $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$-algebra (since $A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $A\left[Y\right]$). On the other hand, $\left(A\left[v\right]\right)\left[\left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}} * Y\right] \subseteq \left(A\left[v\right]\right)\left[Y\right]$.

(a) We have

$$\left(A\left[v\right]\right)\left[\left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}} * Y\right] = \left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)\left[v\right]. \tag{27}$$

(b) Let $u \in B$. Let $n \in \mathbb{N}$. Then, the element u of B is n-integral over $\left(A\left[v\right], \left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B\left[Y\right]$ is n-integral over the ring $\left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)\left[v\right]$.

Proof of Lemma 5.7. **(a)** We have proven Lemma 5.7 **(a)** during the proof of Theorem 2.16 **(b)**.

(b) The ring B is an $A\left[v\right]$-algebra (since $A\left[v\right]$ is a subring of B). Hence, Theorem 2.11 (applied to $A\left[v\right]$ and $\left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}$ instead of A and $\left(I_\rho\right)_{\rho\in\mathbb{N}}$) yields that the element u of B is n-integral over $\left(A\left[v\right], \left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B\left[Y\right]$ is n-integral over the ring $\left(A\left[v\right]\right)\left[\left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}} * Y\right]$. In view of (27), this rewrites as follows: The element u of B is n-integral over $\left(A\left[v\right], \left(I_\rho A\left[v\right]\right)_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B\left[Y\right]$ is n-integral over the ring $\left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)\left[v\right]$. This proves Lemma 5.7 **(b)**. $\square$

Proof of Theorem 5.6. **(a)** Since $(I_\rho)_{\rho\in\mathbb{N}}$ is an ideal semifiltration of A, Lemma 2.18 (applied to $A' = A[x]$) yields that $(I_\rho A[x])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[x]$.

Similarly, the other two statements of Theorem 5.6 **(a)** are proven.

Thus, Theorem 5.6 **(a)** is proven.

(b) For every $v \in B$, the family $(I_\rho A[v])_{\rho\in\mathbb{N}}$ is an ideal semifiltration of $A[v]$ (by Lemma 2.18, applied to $A' = A[v]$), and thus we can consider the polynomial ring $(A[v])[Y]$ and its $A[v]$-subalgebra $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$. For every $v \in B$, the polynomial ring $B[Y]$ is an $(A[v])[Y]$-algebra (as explained in Definition 2.6), since B is an $A[v]$-algebra[11]. Hence, this ring $B[Y]$ is an $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$-algebra as well (because $(A[v])\left[(I_\rho A[v])_{\rho\in\mathbb{N}} * Y\right]$ is a subring of $(A[v])[Y]$). Similarly, the ring $B[Y]$ is an $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$-algebra.

Lemma 5.7 **(b)** (applied to $v = x$) yields that the element u of B is n-integral over $\left(A[x], (I_\rho A[x])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[x]$. But since the element u of B is n-integral over $\left(A[x], (I_\rho A[x])_{\rho\in\mathbb{N}}\right)$, this yields that the element uY of the polynomial ring $B[Y]$ is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[x]$.

Lemma 5.7 **(b)** (applied to y and m instead of v and n) yields that the element u of B is m-integral over $\left(A[y], (I_\rho A[y])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is m-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[y]$. But since the element u of B is m-integral over $\left(A[y], (I_\rho A[y])_{\rho\in\mathbb{N}}\right)$, this yields that the element uY of the polynomial ring $B[Y]$ is m-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[y]$.

Thus we know that uY is n-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[x]$, and that uY is m-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[y]$. Hence, Theorem 5.5 (applied to $A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]$, $B[Y]$ and uY instead of A, B and u) yields that there exists some $\lambda \in \mathbb{N}$ such that uY is λ-integral over $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[xy]$. Consider this λ.

Lemma 5.7 **(b)** (applied to xy and λ instead of v and n) yields that the element u of B is λ-integral over $\left(A[xy], (I_\rho A[xy])_{\rho\in\mathbb{N}}\right)$ if and only if the element uY of the polynomial ring $B[Y]$ is λ-integral over the ring $\left(A\left[(I_\rho)_{\rho\in\mathbb{N}} * Y\right]\right)[xy]$. But since the element uY of the polynomial ring $B[Y]$ is λ-integral over the

[11]because $A[v]$ is a subring of B

ring $\left(A\left[\left(I_\rho\right)_{\rho\in\mathbb{N}} * Y\right]\right)[xy]$, this yields that the element u of B is λ-integral over $\left(A[xy], \left(I_\rho A[xy]\right)_{\rho\in\mathbb{N}}\right)$. Thus, Theorem 5.6 **(b)** is proven. □

5.4. Second proof of Corollary 1.12

We notice that Corollary 1.12 can be derived from Lemma 5.1:

Second proof of Corollary 1.12. Let $n = 1$. Let $m = 1$. From $n = 1$, we obtain $\left\langle u^0, u^1, \ldots, u^{n-1}\right\rangle_A = \left\langle u^0\right\rangle_A = \left\langle 1_B\right\rangle_A$ (since $u^0 = 1_B$). Hence,

$$\begin{aligned}\underbrace{\left\langle u^0, u^1, \ldots, u^{n-1}\right\rangle_A}_{=\langle 1_B\rangle_A} \cdot \left\langle v^0, v^1, \ldots, v^\alpha\right\rangle_A &= \langle 1_B\rangle \cdot \left\langle v^0, v^1, \ldots, v^\alpha\right\rangle_A \\ &= \left\langle 1_B v^0, 1_B v^1, \ldots, 1_B v^\alpha\right\rangle_A \\ &= \left\langle v^0, v^1, \ldots, v^\alpha\right\rangle_A . \end{aligned} \tag{28}$$

The same argument (applied to m and β instead of n and α) yields

$$\left\langle u^0, u^1, \ldots, u^{m-1}\right\rangle_A \cdot \left\langle v^0, v^1, \ldots, v^\beta\right\rangle_A = \left\langle v^0, v^1, \ldots, v^\beta\right\rangle_A . \tag{29}$$

Now, we have

$$u^n \in \left\langle u^0, u^1, \ldots, u^{n-1}\right\rangle_A \cdot \left\langle v^0, v^1, \ldots, v^\alpha\right\rangle_A$$

[12] and

$$\begin{aligned} u^m v^\beta \in &\left\langle u^0, u^1, \ldots, u^{m-1}\right\rangle_A \cdot \left\langle v^0, v^1, \ldots, v^\beta\right\rangle_A \\ &+ \left\langle u^0, u^1, \ldots, u^m\right\rangle_A \cdot \left\langle v^0, v^1, \ldots, v^{\beta-1}\right\rangle_A \end{aligned}$$

[13]. Thus, Lemma 5.1 (applied to v, β and α instead of x, μ and ν) yields that u

[12] *Proof.* From $n = 1$, we obtain

$$\begin{aligned} u^n = u^1 = u = \sum_{i=0}^{\alpha} \underbrace{s_i}_{\in A} v^i \in \left\langle v^0, v^1, \ldots, v^\alpha\right\rangle_A &= \left\langle u^0, u^1, \ldots, u^{n-1}\right\rangle_A \cdot \left\langle v^0, v^1, \ldots, v^\alpha\right\rangle_A \\ = \left\langle u^0, u^1, \ldots, u^{n-1}\right\rangle_A \cdot \left\langle v^0, v^1, \ldots, v^\alpha\right\rangle_A \qquad &(\text{by (28)}) . \end{aligned}$$

[13] *Proof.* From $m = 1$, we obtain $u^m = u^1 = u$ and thus

$$\begin{aligned} \underbrace{u^m}_{=u} v^\beta &= u v^\beta = \sum_{i=0}^{\beta} t_i v^{\beta-i} = \sum_{i=0}^{\beta} t_{\beta-i} \underbrace{v^{\beta-(\beta-i)}}_{=v^i} \\ &\qquad \text{(here we substituted } \beta - i \text{ for } i \text{ in the sum)} \\ &= \sum_{i=0}^{\beta} \underbrace{t_{\beta-i}}_{\in A} v^i \in \left\langle v^0, v^1, \ldots, v^\beta\right\rangle_A = \left\langle u^0, u^1, \ldots, u^{m-1}\right\rangle_A \cdot \left\langle v^0, v^1, \ldots, v^\beta\right\rangle_A \qquad (\text{by (29)}) . \end{aligned}$$

is $(n\beta + m\alpha)$-integral over A (since $\beta + \alpha = \alpha + \beta \in \mathbb{N}^+$). This means that u is $(\alpha + \beta)$-integral over A (because $\underbrace{n}_{=1}\beta + \underbrace{m}_{=1}\alpha = \beta + \alpha = \alpha + \beta$). This proves Corollary 1.12 once again. $\square$

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The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .