

# Shuffle-compatible permutation statistics II: the exterior peak set (detailed version)

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This paper is a continuation of the work [GesZhu17] by Gessel and Zhuang (but can be read independently from the latter). It is devoted to the study of shuffle-compatibility of permutation statistics – a concept introduced in [GesZhu17], although various instances of it have appeared throughout the literature before.

In Section 1, we introduce the notations that we will need throughout this paper. In Section 2, we prove that the exterior peak set statistic  $\text{Epk}$  is shuffle-compatible (Theorem 2.56), as conjectured by Gessel and Zhuang in [GesZhu17]. In Section 3, we introduce the concept of an “LR-shuffle-compatible” statistic, which is stronger than shuffle-compatibility. We give a sufficient criterion for it and use it to show that  $\text{Epk}$  and some other statistics are LR-shuffle-compatible.

The last three sections relate all of this to quasisymmetric functions: In Section 4, we recall the concept of descent statistics introduced in [GesZhu17] and

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its connection to quasisymmetric functions. Motivated by this connection, in Section 5, we define the kernel of a descent statistic, and study this kernel for  $\text{Epk}$ , giving two explicit generating sets for this kernel. In Section 6, we extend the quasisymmetric functions connection to the concept of LR-shuffle-compatible statistics, and relate it to dendriform algebras.

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### 0.1. Remark on alternative versions

You are reading the detailed version of this paper. For the standard version (which is shorter by virtue of omitting some proofs and even some results), see [Grinbe18].

## 1. Notations and definitions

Let us first introduce the definitions and notations that we will use in the rest of this paper. Many of these definitions appear in [GesZhu17] already; we have tried to deviate from the notations of [GesZhu17] as little as possible.

### 1.1. Permutations and other basic concepts

**Definition 1.1.** We let  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$  and  $\mathbb{P} = \{1, 2, 3, \dots\}$ . Both of these sets are understood to be equipped with their standard total order. Elements of  $\mathbb{P}$  will be called *letters* (despite being numbers).

**Definition 1.2.** Let  $n \in \mathbb{Z}$ . We shall use the notation  $[n]$  for the totally ordered set  $\{1, 2, \dots, n\}$  (with the usual order relation inherited from  $\mathbb{Z}$ ). Note that  $[n] = \emptyset$  when  $n \leq 0$ .

**Definition 1.3.** Let  $n \in \mathbb{N}$ . An *n-permutation* shall mean a word with  $n$  letters, which are distinct and belong to  $\mathbb{P}$ . Equivalently, an *n-permutation* shall be regarded as an injective map  $[n] \rightarrow \mathbb{P}$  (the image of  $i$  under this map being the  $i$ -th letter of the word).

For example,  $(3, 6, 4)$  and  $(9, 1, 2)$  are 3-permutations, but  $(2, 1, 2)$  is not.

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**Definition 1.4.** A *permutation* is defined to be an  $n$ -permutation for some  $n \in \mathbb{N}$ . If  $\pi$  is an  $n$ -permutation for some  $n \in \mathbb{N}$ , then the number  $n$  is called the *size* of the permutation  $\pi$  and is denoted by  $|\pi|$ . A permutation is said to be *nonempty* if it is nonempty as a word (i.e., if its size is  $> 0$ ).

Note that the meaning of “permutation” we have just defined is unusual (most authors define a permutation to be a bijection from a set to itself); we are following [GesZhu17] in defining permutations this way.

**Definition 1.5.** Let  $n \in \mathbb{N}$ . Two  $n$ -permutations  $\alpha$  and  $\beta$  are said to be *order-isomorphic* if they have the following property: For every two integers  $i, j \in [n]$ , we have  $\alpha(i) < \alpha(j)$  if and only if  $\beta(i) < \beta(j)$ .

**Definition 1.6. (a)** A *permutation statistic* is a map  $st$  from the set of all permutations to an arbitrary set that has the following property: Whenever  $\alpha$  and  $\beta$  are two order-isomorphic permutations, we have  $st \alpha = st \beta$ .

**(b)** Let  $st$  be a permutation statistic. Two permutations  $\alpha$  and  $\beta$  are said to be *st-equivalent* if they satisfy  $|\alpha| = |\beta|$  and  $st \alpha = st \beta$ . The relation “st-equivalent” is an equivalence relation; its equivalence classes are called *st-equivalence classes*.

**Remark 1.7.** Let  $n \in \mathbb{N}$ . Let us call an  $n$ -permutation  $\pi$  *standard* if its letters are  $1, 2, \dots, n$  (in some order). The standard  $n$ -permutations are in bijection with the  $n!$  permutations of the set  $\{1, 2, \dots, n\}$  in the usual sense of this word (i.e., the bijections from this set to itself).

It is easy to see that for each  $n$ -permutation  $\sigma$ , there exists a **unique** standard  $n$ -permutation  $\pi$  order-isomorphic to  $\sigma$ . Thus, a permutation statistic is uniquely determined by its values on standard permutations. Consequently, we can view permutation statistics as statistics defined on standard permutations, i.e., on permutations in the usual sense of the word.

The word “permutation statistic” is often abbreviated as “statistic”.

## 1.2. Some examples of permutation statistics

**Definition 1.8.** Let  $n \in \mathbb{N}$ . Let  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$  be an  $n$ -permutation.

**(a)** The *descents* of  $\pi$  are the elements  $i \in [n - 1]$  satisfying  $\pi_i > \pi_{i+1}$ .

**(b)** The *descent set* of  $\pi$  is defined to be the set of all descents of  $\pi$ . This set is denoted by  $\text{Des } \pi$ , and is always a subset of  $[n - 1]$ .

**(c)** The *peaks* of  $\pi$  are the elements  $i \in \{2, 3, \dots, n - 1\}$  satisfying  $\pi_{i-1} < \pi_i > \pi_{i+1}$ .

**(d)** The *peak set* of  $\pi$  is defined to be the set of all peaks of  $\pi$ . This set is denoted by  $\text{Pk } \pi$ , and is always a subset of  $\{2, 3, \dots, n - 1\}$ .

(e) The *left peaks* of  $\pi$  are the elements  $i \in [n-1]$  satisfying  $\pi_{i-1} < \pi_i > \pi_{i+1}$ , where we set  $\pi_0 = 0$ .

(f) The *left peak set* of  $\pi$  is defined to be the set of all left peaks of  $\pi$ . This set is denoted by  $\text{Lpk } \pi$ , and is always a subset of  $[n-1]$ . It is easy to see that (for  $n \geq 2$ ) we have

$$\text{Lpk } \pi = \text{Pk } \pi \cup \{1 \mid \pi_1 > \pi_2\}.$$

(The strange notation “ $\{1 \mid \pi_1 > \pi_2\}$ ” means the set of all numbers 1 satisfying  $\pi_1 > \pi_2$ . In other words, it is the 1-element set  $\{1\}$  if  $\pi_1 > \pi_2$ , and the empty set  $\emptyset$  otherwise.)

(g) The *right peaks* of  $\pi$  are the elements  $i \in \{2, 3, \dots, n\}$  satisfying  $\pi_{i-1} < \pi_i > \pi_{i+1}$ , where we set  $\pi_{n+1} = 0$ .

(h) The *right peak set* of  $\pi$  is defined to be the set of all right peaks of  $\pi$ . This set is denoted by  $\text{Rpk } \pi$ , and is always a subset of  $\{2, 3, \dots, n\}$ . It is easy to see that (for  $n \geq 2$ ) we have

$$\text{Rpk } \pi = \text{Pk } \pi \cup \{n \mid \pi_{n-1} < \pi_n\}.$$

(i) The *exterior peaks* of  $\pi$  are the elements  $i \in [n]$  satisfying  $\pi_{i-1} < \pi_i > \pi_{i+1}$ , where we set  $\pi_0 = 0$  and  $\pi_{n+1} = 0$ .

(j) The *exterior peak set* of  $\pi$  is defined to be the set of all exterior peaks of  $\pi$ . This set is denoted by  $\text{Epk } \pi$ , and is always a subset of  $[n]$ . It is easy to see that (for  $n \geq 2$ ) we have

$$\begin{aligned} \text{Epk } \pi &= \text{Pk } \pi \cup \{1 \mid \pi_1 > \pi_2\} \cup \{n \mid \pi_{n-1} < \pi_n\} \\ &= \text{Lpk } \pi \cup \text{Rpk } \pi \end{aligned}$$

(where, again,  $\{n \mid \pi_{n-1} < \pi_n\}$  is the 1-element set  $\{n\}$  if  $\pi_{n-1} < \pi_n$ , and otherwise is the empty set).

(For  $n = 1$ , we have  $\text{Epk } \pi = \{1\}$ .)

For example, the 6-permutation  $\pi = (4, 1, 3, 9, 6, 8)$  has

$$\begin{aligned} \text{Des } \pi &= \{1, 4\}, & \text{Pk } \pi &= \{4\}, \\ \text{Lpk } \pi &= \{1, 4\}, & \text{Rpk } \pi &= \{4, 6\}, & \text{Epk } \pi &= \{1, 4, 6\}. \end{aligned}$$

For another example, the 6-permutation  $\pi = (1, 4, 3, 2, 9, 8)$  has

$$\begin{aligned} \text{Des } \pi &= \{2, 3, 5\}, & \text{Pk } \pi &= \{2, 5\}, \\ \text{Lpk } \pi &= \{2, 5\}, & \text{Rpk } \pi &= \{2, 5\}, & \text{Epk } \pi &= \{2, 5\}. \end{aligned}$$

Notice that Definition 1.8 actually defines several permutation statistics. For example, Definition 1.8 (b) defines the permutation statistic  $\text{Des}$ , whose codomain is the set of all subsets of  $\mathbb{P}$ . Likewise, Definition 1.8 (d) defines the permutation

statistic Pk, and Definition 1.8 (f) defines the permutation statistic Lpk, whereas Definition 1.8 (h) defines the permutation statistic Rpk. The main permutation statistic that we will study in this paper is Epk, which is defined in Definition 1.8 (j); its codomain is the set of all subsets of  $\mathbb{P}$ .

The following simple fact expresses the set  $\text{Epk } \pi$  corresponding to an  $n$ -permutation  $\pi$  in terms of  $\text{Des } \pi$ :

**Proposition 1.9.** Let  $n$  be a positive integer. Let  $\pi$  be an  $n$ -permutation. Then,

$$\text{Epk } \pi = (\text{Des } \pi \cup \{n\}) \setminus (\text{Des } \pi + 1),$$

where  $\text{Des } \pi + 1$  denotes the set  $\{i + 1 \mid i \in \text{Des } \pi\}$ .

*Proof of Proposition 1.9.* Write  $\pi$  in the form  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ . Set  $\pi_0 = 0$  and  $\pi_{n+1} = 0$ . Recall that  $\text{Des } \pi$  is defined as the set of all descents of  $\pi$ . In other words,

$$\text{Des } \pi = (\text{the set of all descents of } \pi) = \{i \in [n - 1] \mid \pi_i > \pi_{i+1}\}$$

(because the descents of  $\pi$  are defined to be the  $i \in [n - 1]$  satisfying  $\pi_i > \pi_{i+1}$ ).

But  $(\pi_1, \pi_2, \dots, \pi_n) = \pi$  is an  $n$ -permutation, and thus has no equal entries. Hence, for each  $i \in [n - 1]$ , we have  $\pi_i \neq \pi_{i+1}$ . Thus, for each  $i \in [n - 1]$ , we have the equivalence  $(\pi_i \geq \pi_{i+1}) \iff (\pi_i > \pi_{i+1})$ . Therefore,

$$\{i \in [n - 1] \mid \pi_i \geq \pi_{i+1}\} = \{i \in [n - 1] \mid \pi_i > \pi_{i+1}\} = \text{Des } \pi.$$

On the other hand,  $\pi_n \in [n]$ , so that  $\pi_n > 0 = \pi_{n+1}$ . Hence,  $n$  is an element of the set  $\{i \in \{n\} \mid \pi_i > \pi_{i+1}\}$ . Clearly, this set cannot have any other element (since it is a subset of  $\{n\}$ ); thus,  $\{i \in \{n\} \mid \pi_i > \pi_{i+1}\} = \{n\}$ .

But  $[n] = [n - 1] \cup \{n\}$ , so that

$$\begin{aligned} & \{i \in [n] \mid \pi_i > \pi_{i+1}\} \\ &= \{i \in [n - 1] \cup \{n\} \mid \pi_i > \pi_{i+1}\} \\ &= \underbrace{\{i \in [n - 1] \mid \pi_i > \pi_{i+1}\}}_{=\text{Des } \pi} \cup \underbrace{\{i \in \{n\} \mid \pi_i > \pi_{i+1}\}}_{=\{n\}} = \text{Des } \pi \cup \{n\}. \end{aligned} \quad (1)$$

On the other hand,  $\pi_1 \in [n]$ , so that  $\pi_1 > 0 = \pi_0$ . Hence, we do not have  $\pi_0 \geq \pi_1$ . Thus,  $0$  is not an element of the set  $\{i \in \{0\} \mid \pi_i \geq \pi_{i+1}\}$ . Clearly, this set cannot have any other element (since it is a subset of  $\{0\}$ ); thus,  $\{i \in \{0\} \mid \pi_i \geq \pi_{i+1}\} = \emptyset$ .

But  $\{0, 1, \dots, n - 1\} = \{0\} \cup [n - 1]$ , so that

$$\begin{aligned} & \{i \in \{0, 1, \dots, n - 1\} \mid \pi_i \geq \pi_{i+1}\} \\ &= \{i \in \{0\} \cup [n - 1] \mid \pi_i \geq \pi_{i+1}\} \\ &= \underbrace{\{i \in \{0\} \mid \pi_i \geq \pi_{i+1}\}}_{=\emptyset} \cup \underbrace{\{i \in [n - 1] \mid \pi_i \geq \pi_{i+1}\}}_{=\text{Des } \pi} = \emptyset \cup \text{Des } \pi = \text{Des } \pi. \end{aligned}$$

Hence,

$$\text{Des } \pi = \{i \in \{0, 1, \dots, n-1\} \mid \pi_i \geq \pi_{i+1}\},$$

so that

$$\begin{aligned} \text{Des } \pi + 1 &= \{i+1 \mid i \in \{0, 1, \dots, n-1\} \text{ satisfies } \pi_i \geq \pi_{i+1}\} \\ &= \{j \in [n] \mid \pi_{j-1} \geq \pi_j\} \\ &= \{i \in [n] \mid \pi_{i-1} \geq \pi_i\} \end{aligned} \tag{2}$$

(here, we have renamed the index  $j$  as  $i$ ).

But  $\text{Epk } \pi$  is the set of all exterior peaks of  $\pi$  (by the definition of  $\text{Epk } \pi$ ). Thus,

$$\begin{aligned} \text{Epk } \pi &= (\text{the set of all exterior peaks of } \pi) \\ &= \left\{ i \in [n] \mid \underbrace{\pi_{i-1} < \pi_i > \pi_{i+1}}_{\Leftrightarrow (\pi_i > \pi_{i+1} \text{ and } \pi_{i-1} < \pi_i)} \right\} \\ &\quad (\text{by the definition of an "exterior peak" of } \pi) \\ &= \left\{ i \in [n] \mid \pi_i > \pi_{i+1} \text{ and } \underbrace{\pi_{i-1} < \pi_i}_{\Leftrightarrow (\text{not } \pi_{i-1} \geq \pi_i)} \right\} \\ &= \{i \in [n] \mid \pi_i > \pi_{i+1} \text{ and not } \pi_{i-1} \geq \pi_i\} \\ &= \underbrace{\{i \in [n] \mid \pi_i > \pi_{i+1}\}}_{\substack{=\text{Des } \pi \cup \{n\} \\ \text{(by (1))}}} \setminus \underbrace{\{i \in [n] \mid \pi_{i-1} \geq \pi_i\}}_{\substack{=\text{Des } \pi + 1 \\ \text{(by (2))}}} \\ &= (\text{Des } \pi \cup \{n\}) \setminus (\text{Des } \pi + 1). \end{aligned}$$

This proves Proposition 1.9. □

### 1.3. Shuffles and shuffle-compatibility

**Definition 1.10.** Let  $\pi$  and  $\sigma$  be two permutations.

(a) We say that  $\pi$  and  $\sigma$  are *disjoint* if no letter appears in both  $\pi$  and  $\sigma$ .

(b) Assume that  $\pi$  and  $\sigma$  are disjoint. Set  $m = |\pi|$  and  $n = |\sigma|$ . Let  $\tau$  be an  $(m+n)$ -permutation. Then, we say that  $\tau$  is a *shuffle* of  $\pi$  and  $\sigma$  if both  $\pi$  and  $\sigma$  are subsequences of  $\tau$ .

(c) We let  $S(\pi, \sigma)$  be the set of all shuffles of  $\pi$  and  $\sigma$ .

For example, the permutations  $(3, 1)$  and  $(6, 2, 9)$  are disjoint, whereas the permutations  $(3, 1, 2)$  and  $(6, 2, 9)$  are not. The shuffles of the two disjoint permutations  $(3, 1)$  and  $(2, 6)$  are

$$\begin{array}{lll} (3, 1, 2, 6), & (3, 2, 1, 6), & (3, 2, 6, 1), \\ (2, 3, 1, 6), & (2, 3, 6, 1), & (2, 6, 3, 1). \end{array}$$

If  $\pi$  and  $\sigma$  are two disjoint permutations, and if  $\tau$  is a shuffle of  $\pi$  and  $\sigma$ , then each letter of  $\tau$  must be either a letter of  $\pi$  or a letter of  $\sigma$ .<sup>1</sup>

If  $\pi$  and  $\sigma$  are two disjoint permutations, then  $S(\pi, \sigma) = S(\sigma, \pi)$  is an  $\binom{m+n}{m}$ -element set, where  $m = |\pi|$  and  $n = |\sigma|$ .

Definition 1.10 (b) is used, e.g., in [Greene88]. From the point of view of combinatorics on words, it is somewhat naive, as it fails to properly generalize to the case when the words  $\pi$  and  $\sigma$  are no longer disjoint<sup>2</sup>. But we will not be considering this general case, since our results do not seem to straightforwardly extend to it (although we might have to look more closely); thus, Definition 1.10 will suffice for us.

**Definition 1.11. (a)** If  $a_1, a_2, \dots, a_k$  are finitely many arbitrary objects, then  $\{a_1, a_2, \dots, a_k\}_{\text{multi}}$  denotes the **multiset** whose elements are  $a_1, a_2, \dots, a_k$  (each appearing with the multiplicity with which it appears in the list  $(a_1, a_2, \dots, a_k)$ ).

**(b)** Let  $(a_i)_{i \in I}$  be a finite family of arbitrary objects. Then,  $\{a_i \mid i \in I\}_{\text{multi}}$  denotes the **multiset** whose elements are the elements of this family (each appearing with the multiplicity with which it appears in the family).

<sup>1</sup>*Proof.* Let  $\pi$  and  $\sigma$  be two disjoint permutations. Let  $\tau$  be a shuffle of  $\pi$  and  $\sigma$ . We must prove that each letter of  $\tau$  must be either a letter of  $\pi$  or a letter of  $\sigma$ .

Let  $P$  be the set of all letters of  $\pi$ . Let  $S$  be the set of all letters of  $\sigma$ . Let  $T$  be the set of all letters of  $\tau$ .

Let  $m = |\pi|$  and  $n = |\sigma|$ . Hence,  $\tau$  is an  $(m+n)$ -permutation (since  $\tau$  is a shuffle of  $\pi$  and  $\sigma$ ). In other words,  $\tau$  is a permutation with  $m+n$  letters.

The word  $\pi$  is a permutation with  $m$  letters (because  $m = |\pi|$ ). Thus, this word  $\pi$  has exactly  $m$  distinct letters. In other words,  $|P| = m$  (since  $P$  is the set of all letters of  $\pi$ ). Similarly,  $|S| = n$  and  $|T| = m+n$ . The sets  $P$  and  $S$  are disjoint (since the permutations  $\pi$  and  $\sigma$  are disjoint); thus,  $|P \cup S| = \underbrace{|P|}_{=m} + \underbrace{|S|}_{=n} = m+n$ .

But  $\pi$  is a subsequence of  $\tau$  (since  $\tau$  is a shuffle of  $\pi$  and  $\sigma$ ). Thus,  $P \subseteq T$ . Similarly,  $S \subseteq T$ . Combining  $P \subseteq T$  with  $S \subseteq T$ , we obtain  $P \cup S \subseteq T$ . Since  $|P \cup S| = m+n = |T|$ , we thus conclude that  $P \cup S$  is a subset of  $T$  but has the same size as  $T$ . By the pigeonhole principle, this entails that  $P \cup S = T$ .

Now, each letter of  $\tau$  must be an element of  $T$  (by the definition of  $T$ ), thus an element of  $P \cup S$  (since  $T = P \cup S$ ), and thus either an element of  $P$  or an element of  $S$ . In other words, each letter of  $\tau$  must be either a letter of  $\pi$  or a letter of  $\sigma$  (by the definitions of  $P$  and  $S$ ). Qed.

<sup>2</sup>In this general case, it is best to define a shuffle of two words  $\pi = (\pi_1, \pi_2, \dots, \pi_m)$  and  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$  as a word of the form  $(\gamma_{\eta(1)}, \gamma_{\eta(2)}, \dots, \gamma_{\eta(m+n)})$ , where  $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$  is the word  $(\pi_1, \pi_2, \dots, \pi_m, \sigma_1, \sigma_2, \dots, \sigma_n)$ , and where  $\eta$  is some permutation of the set  $\{1, 2, \dots, m+n\}$  (that is, a bijection from this set to itself) satisfying  $\eta^{-1}(1) < \eta^{-1}(2) < \dots < \eta^{-1}(m)$  (this causes the letters  $\pi_1, \pi_2, \dots, \pi_m$  to appear in the word  $(\gamma_{\eta(1)}, \gamma_{\eta(2)}, \dots, \gamma_{\eta(m+n)})$  in this order) and  $\eta^{-1}(m+1) < \eta^{-1}(m+2) < \dots < \eta^{-1}(m+n)$  (this causes the letters  $\sigma_1, \sigma_2, \dots, \sigma_n$  to appear in the word  $(\gamma_{\eta(1)}, \gamma_{\eta(2)}, \dots, \gamma_{\eta(m+n)})$  in this order). Furthermore, the proper generalization of  $S(\pi, \sigma)$  to this case would be a multiset, not a mere set.



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For example,  $\{k^2 \mid k \in \{-2, -1, 0, 1, 2\}\}_{\text{multi}}$  is the multiset that contains the element 4 twice, the element 1 twice, and the element 0 once (and no other elements). This multiset can also be written in the form  $\{4, 1, 0, 1, 4\}_{\text{multi}}$ , or in the form  $\{0, 1, 1, 4, 4\}_{\text{multi}}$ .

**Definition 1.12.** Let  $\text{st}$  be a permutation statistic. We say that  $\text{st}$  is *shuffle-compatible* if and only if it has the following property: For any two disjoint permutations  $\pi$  and  $\sigma$ , the multiset

$$\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$$

depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ .

In other words, a permutation statistic  $\text{st}$  is shuffle-compatible if and only if it has the following property:

- If  $\pi$  and  $\sigma$  are two disjoint permutations, and if  $\pi'$  and  $\sigma'$  are two disjoint permutations, and if these permutations satisfy

$$\begin{aligned} \text{st } \pi &= \text{st } (\pi'), & \text{st } \sigma &= \text{st } (\sigma'), \\ |\pi| &= |\pi'| & \text{and} & & |\sigma| &= |\sigma'|, \end{aligned}$$

then

$$\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}.$$

The notion of a shuffle-compatible permutation statistic was coined by Gesel and Zhuang in [GesZhu17], where various statistics were analyzed for their shuffle-compatibility. In particular, it was shown in [GesZhu17] that the statistics  $\text{Des}$ ,  $\text{Pk}$ ,  $\text{Lpk}$  and  $\text{Rpk}$  are shuffle-compatible. Our next goal is to prove the same for the statistic  $\text{Epk}$ .

## 2. Extending enriched $P$ -partitions and the exterior peak set

We are going to define  *$\mathcal{Z}$ -enriched  $P$ -partitions*, which are a straightforward generalization of the notions of “ $P$ -partitions” [Stanle72], “enriched  $P$ -partitions” [Stembr97, §2] and “left enriched  $P$ -partitions” [Peters05]. We will then consider a new particular case of this notion, which leads to a proof of the shuffle-compatibility of  $\text{Epk}$  conjectured in [GesZhu17] (Theorem 2.56 below).

We remark that Bruce Sagan and Duff Baker-Jarvis are currently working on an alternative, bijective approach to the shuffle-compatibility of permutation statistics, which may lead to a different proof of this fact.

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## 2.1. Lacunar sets

First, let us briefly study *lacunar sets*, a class of subsets of  $\mathbb{Z}$  that are closely connected to exterior peaks. We start with the definition:

**Definition 2.1.** A set  $S$  of integers is said to be *lacunar* if each  $s \in S$  satisfies  $s + 1 \notin S$ .

In other words, a set of integers is lacunar if and only if it contains no two consecutive integers. For example, the set  $\{2, 5, 7\}$  is lacunar, while the set  $\{2, 5, 6\}$  is not.

Lacunar sets of integers are also called *sparse* sets in some of the literature (though the latter word has several competing meanings).

**Definition 2.2.** Let  $n \in \mathbb{N}$ . We define a set  $\mathbf{L}_n$  of subsets of  $[n]$  as follows:

- If  $n$  is positive, then  $\mathbf{L}_n$  shall mean the set of all nonempty lacunar subsets of  $[n]$ .
- If  $n = 0$ , then  $\mathbf{L}_n$  shall mean the set  $\{\emptyset\}$ .

For example,

$$\begin{aligned} \mathbf{L}_0 &= \{\emptyset\}; & \mathbf{L}_1 &= \{\{1\}\}; & \mathbf{L}_2 &= \{\{1\}, \{2\}\}; \\ \mathbf{L}_3 &= \{\{1\}, \{2\}, \{3\}, \{1, 3\}\}. \end{aligned}$$

**Proposition 2.3.** Let  $(f_0, f_1, f_2, \dots)$  be the Fibonacci sequence (defined by  $f_0 = 0$  and  $f_1 = 1$  and the recursive relation  $f_m = f_{m-1} + f_{m-2}$  for all  $m \geq 2$ ). Let  $n$  be a positive integer. Then,  $|\mathbf{L}_n| = f_{n+2} - 1$ .

*Proof of Proposition 2.3.* Recall that  $\mathbf{L}_n$  is the set of all nonempty lacunar subsets of  $[n]$  (since  $n$  is positive). Thus,  $|\mathbf{L}_n|$  is the number of all lacunar subsets of  $[n]$  minus 1 (since the empty set  $\emptyset$ , which is clearly a lacunar subset of  $[n]$ , is withheld from the count). But a known fact (see, e.g., [Stanle11, Exercise 1.35 a.]) says that the number of lacunar subsets of  $[n]$  is  $f_{n+2}$ . Combining the preceding two sentences, we conclude that  $|\mathbf{L}_n| = f_{n+2} - 1$ . This proves Proposition 2.3.  $\square$

The following observation is easy:

**Proposition 2.4.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be an  $n$ -permutation. Then,  $\text{Epk } \pi \in \mathbf{L}_n$ .

*Proof of Proposition 2.4.* If  $n = 0$ , then the statement is obvious (since in this case, we have  $\text{Epk } \pi = \emptyset \in \mathbf{L}_0$ ). Thus, WLOG assume that  $n \neq 0$ . Hence,  $n$  is positive. Hence,  $\mathbf{L}_n$  is the set of all nonempty lacunar subsets of  $[n]$  (by the definition of  $\mathbf{L}_n$ ).

---

The set  $\text{Epk } \pi$  is lacunar (since two consecutive integers cannot both be exterior peaks of  $\pi$ ), and is also nonempty (since  $\pi^{-1}(n)$  is an exterior peak of  $\pi$ ).

[An alternative reason for the nonemptiness of  $\text{Epk } \pi$  is the following: If  $\text{Epk } \pi$  was empty, then  $\pi$  would have no peaks, so that the sequence  $(\pi(1), \pi(2), \dots, \pi(n))$  would be strictly decreasing up to a certain point and then strictly increasing from there on; but then, either  $n$  or  $1$  would be an exterior peak of  $\pi$ , which would contradict the emptiness of  $\text{Epk } \pi$ .]

Therefore,  $\text{Epk } \pi$  is a nonempty lacunar subset of  $[n]$ . In other words,  $\text{Epk } \pi \in \mathbf{L}_n$  (since  $\mathbf{L}_n$  is the set of all nonempty lacunar subsets of  $[n]$ ). This proves Proposition 2.4.  $\square$

Proposition 2.4 actually has a sort of converse:

**Proposition 2.5.** Let  $n \in \mathbb{N}$ . Let  $\Lambda$  be a subset of  $[n]$ . Then, there exists an  $n$ -permutation  $\pi$  satisfying  $\Lambda = \text{Epk } \pi$  if and only if  $\Lambda \in \mathbf{L}_n$ .

*Proof of Proposition 2.5.*  $\implies$ : We need to prove that for any  $n$ -permutation  $\pi$ , we have  $\text{Epk } \pi \in \mathbf{L}_n$ . But this follows immediately from Proposition 2.4. This proves the  $\implies$  direction of Proposition 2.5.

$\impliedby$ : Assume that  $\Lambda \in \mathbf{L}_n$ . We must prove that there exists an  $n$ -permutation  $\pi$  satisfying  $\Lambda = \text{Epk } \pi$ . Such an  $n$ -permutation  $\pi$  can be constructed as follows:

- If  $n = 0$ , then we simply set  $\pi = ()$ . Thus, for the rest of this construction, we WLOG assume that  $n \neq 0$ . Hence,  $n$  is positive. Thus, from  $\Lambda \in \mathbf{L}_n$ , we conclude that  $\Lambda$  is a nonempty lacunar subset of  $[n]$ .
- Write the set  $\Lambda$  in the form  $\Lambda = \{u_1 < u_2 < \dots < u_\ell\}$  (where  $\ell = |\Lambda|$ ). Thus,  $\ell \geq 1$  (since  $\Lambda$  is nonempty), and we can represent the set  $[n] \setminus \Lambda$  as a union of disjoint intervals as follows:

$$\begin{aligned} [n] \setminus \Lambda &= [1, u_1 - 1] \cup [u_1 + 1, u_2 - 1] \cup [u_2 + 1, u_3 - 1] \cup \dots \cup [u_{\ell-1} + 1, u_\ell - 1] \\ &\quad \cup [u_\ell + 1, n]. \end{aligned}$$

- Let  $\pi$  take the values  $n, n-1, \dots, n-\ell+1$  on the elements of  $\Lambda$ . (For example, this can be achieved by setting  $\pi(u_i) = n+1-i$  for each  $i \in [\ell]$ .)
- Let  $\pi$  take the values  $1, 2, \dots, n-\ell$  on the elements of  $[n] \setminus \Lambda$  in such a way that:
  - (A) on each of the intervals  $[1, u_1 - 1], [u_1 + 1, u_2 - 1], [u_2 + 1, u_3 - 1], \dots, [u_{\ell-1} + 1, u_\ell - 1], [u_\ell + 1, n]$ , the map  $\pi$  is either strictly increasing or strictly decreasing;
  - (B) if the interval  $[1, u_1 - 1]$  is nonempty, then the map  $\pi$  is strictly increasing on this interval;

(C) if the interval  $[u_\ell + 1, n]$  is nonempty, then the map  $\pi$  is strictly decreasing on this interval.

(This is indeed possible, because if the two intervals  $[1, u_1 - 1]$  and  $[u_\ell + 1, n]$  are both nonempty, then they are distinct (since  $\ell \geq 1$ )).

Any  $n$ -permutation  $\pi$  constructed in this way will satisfy  $\Lambda = \text{Epk } \pi$ . Indeed, it is clear that  $\pi$  satisfies

$$\pi(u) > \pi(v) \quad \text{for all } u \in \Lambda \text{ and } v \in [n] \setminus \Lambda.$$

Hence, any element of  $\Lambda$  is an exterior peak of  $\pi$ . Conversely, an element of  $[n] \setminus \Lambda$  cannot be an exterior peak of  $\pi$  (because our construction of  $\pi$  guarantees that any  $s \in [n] \setminus \Lambda$  satisfies either  $(s - 1 \in [n] \text{ and } \pi(s - 1) > \pi(s))$  or  $(s + 1 \in [n] \text{ and } \pi(s + 1) > \pi(s))$ ). Thus, the exterior peaks of  $\pi$  are precisely the elements of  $\Lambda$ ; in other words, we have  $\Lambda = \text{Epk } \pi$ . This proves the  $\Leftarrow$  direction of Proposition 2.5.  $\square$

Next, let us introduce a total order on the finite subsets of  $\mathbb{Z}$ :

**Definition 2.6. (a)** Let  $\mathbf{P}$  be the set of all finite subsets of  $\mathbb{Z}$ .

**(b)** If  $A$  and  $B$  are any two sets, then  $A \triangle B$  shall denote the *symmetric difference* of  $A$  and  $B$ . This is the set  $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ . It is well-known that the binary operation  $\triangle$  on sets is associative.

If  $A$  and  $B$  are two distinct sets, then the set  $A \triangle B$  is nonempty. Also, if  $A \in \mathbf{P}$  and  $B \in \mathbf{P}$ , then  $A \triangle B \in \mathbf{P}$ . Thus, if  $A$  and  $B$  are two distinct sets in  $\mathbf{P}$ , then  $\min(A \triangle B) \in \mathbb{Z}$  is well-defined.

**(c)** We define a binary relation  $<$  on  $\mathbf{P}$  as follows: For any  $A \in \mathbf{P}$  and  $B \in \mathbf{P}$ , we let  $A < B$  if and only if  $A \neq B$  and  $\min(A \triangle B) \in A$ . (This definition makes sense, because the condition  $A \neq B$  ensures that  $\min(A \triangle B)$  is well-defined.)

Note that this relation  $<$  is similar to the relation  $<$  in [AgBeNy03, Lemma 4.3].

**Proposition 2.7.** The relation  $<$  on  $\mathbf{P}$  is the smaller relation of a total order on  $\mathbf{P}$ .

*Proof of Proposition 2.7.* A quick proof of Proposition 2.7 can be obtained by imitating [AgBeNy03, proof of Lemma 4.3] (with the obvious changes made, such as replacing  $\max$  by  $\min$ , and reversing the relation). But let us instead give a completely pedestrian proof:

First, we claim the following fact: If  $X$  and  $Y$  are two finite nonempty subsets of  $\mathbb{Z}$  satisfying  $\min X \neq \min Y$ , then

$$X \neq Y \text{ and } \min(X \triangle Y) \in \{\min X, \min Y\}. \quad (3)$$

---

[*Proof of (3):* Let  $X$  and  $Y$  be two finite nonempty subsets of  $\mathbb{Z}$  satisfying  $\min X \neq \min Y$ . From  $\min X \neq \min Y$ , we obtain  $X \neq Y$ . Hence,  $\min(X \triangle Y) \in \mathbb{Z}$  is well-defined.

The definition of  $X \triangle Y$  yields  $X \triangle Y = (X \setminus Y) \cup (Y \setminus X) \supseteq X \setminus Y$ , so that  $X \setminus Y \subseteq X \triangle Y$ .

But our claim is symmetric with respect to  $X$  and  $Y$  (since  $Y \triangle X = X \triangle Y$  and  $\{\min Y, \min X\} = \{\min X, \min Y\}$ ). Hence, we can WLOG assume that  $\min X \leq \min Y$  (since otherwise, we can just swap  $X$  with  $Y$  to ensure this). Assume this. Combining  $\min X \leq \min Y$  with  $\min X \neq \min Y$ , we obtain  $\min X < \min Y$ . If we had  $\min X \in Y$ , then we would have  $\min X \geq \min Y$  (since every  $y \in Y$  satisfies  $y \geq \min Y$ ), which would contradict  $\min X < \min Y$ . Hence, we cannot have  $\min X \in Y$ . Thus,  $\min X \notin Y$ . Combining this with  $\min X \in X$ , we obtain  $\min X \in X \setminus Y \subseteq X \triangle Y$ . Hence,  $\min X \geq \min(X \triangle Y)$  (since every  $z \in X \triangle Y$  satisfies  $z \geq \min(X \triangle Y)$ ).

On the other hand, the definition of  $X \triangle Y$  yields  $X \triangle Y = (X \cup Y) \setminus (X \cap Y) \subseteq X \cup Y$  and thus

$$\min(X \triangle Y) \geq \min(X \cup Y) = \min\{\min X, \min Y\} = \min X$$

(since  $\min X < \min Y$ ). Combining this inequality with  $\min X \geq \min(X \triangle Y)$ , we obtain  $\min(X \triangle Y) = \min X \in \{\min X, \min Y\}$ . Thus, we have shown that  $X \neq Y$  and  $\min(X \triangle Y) \in \{\min X, \min Y\}$ . This proves (3).]

The binary relation  $<$  is transitive<sup>3</sup>, irreflexive<sup>4</sup> and asymmetric<sup>5</sup>. Hence,  $<$  is

<sup>3</sup>*Proof.* Let  $A \in \mathbf{P}$ ,  $B \in \mathbf{P}$  and  $C \in \mathbf{P}$  be such that  $A < B$  and  $B < C$ . We shall prove that  $A < C$ .

The sets  $A$ ,  $B$  and  $C$  are elements of  $\mathbf{P}$ , and thus are finite subsets of  $\mathbb{Z}$  (by the definition of  $\mathbf{P}$ ).

We have  $A < B$ . In other words,  $A \neq B$  and  $\min(A \triangle B) \in A$  (by the definition of the relation  $<$ ).

We have  $B < C$ . In other words,  $B \neq C$  and  $\min(B \triangle C) \in B$  (by the definition of the relation  $<$ ).

Let  $X = A \triangle B$ . Then, the set  $X$  is nonempty (since  $A \neq B$ ) and is a finite subset of  $\mathbb{Z}$  (since  $A$  and  $B$  are finite subsets of  $\mathbb{Z}$ ). Thus,  $\min X \in \mathbb{Z}$  is well-defined. From  $X = A \triangle B$ , we obtain  $\min X = \min(A \triangle B) \in A$ .

Let  $Y = B \triangle C$ . Then, the set  $Y$  is nonempty (since  $B \neq C$ ) and is a finite subset of  $\mathbb{Z}$  (since  $B$  and  $C$  are finite subsets of  $\mathbb{Z}$ ). Thus,  $\min Y \in \mathbb{Z}$  is well-defined. From  $Y = B \triangle C$ , we obtain  $\min Y = \min(B \triangle C) \in B$ .

From  $X = A \triangle B$  and  $Y = B \triangle C$ , we obtain

$$\begin{aligned} X \triangle Y &= (A \triangle B) \triangle (B \triangle C) = A \triangle \underbrace{B \triangle B}_{=\emptyset} \triangle C && \text{(since the operation } \triangle \text{ is associative)} \\ &= \underbrace{A \triangle \emptyset}_{=A} \triangle C = A \triangle C. \end{aligned}$$

We have  $\min X \in X = A \triangle B = (A \cup B) \setminus (A \cap B)$ , so that  $\min X \notin A \cap B$ . If we had  $\min X = \min Y$ , then we would have  $\min X \in A \cap B$  (since  $\min X \in A$  and  $\min X = \min Y \in B$ ), which would contradict  $\min X \notin A \cap B$ . Thus, we cannot have  $\min X = \min Y$ . Hence,  $\min X \neq \min Y$ . Thus, (3) yields  $X \neq Y$  and  $\min(X \triangle Y) \in \{\min X, \min Y\}$ .

If we had  $A = C$ , then we would have  $X = \underbrace{A}_{=C} \triangle B = C \triangle B = B \triangle C = Y$ , which would contradict  $X \neq Y$ . Thus, we cannot have  $A = C$ . Hence,  $A \neq C$ . Hence,  $\min(A \triangle C)$  is well-

defined. Let  $\mu = \min(A \triangle C)$ . Thus,  $\mu = \min\left(\underbrace{A \triangle C}_{=X \triangle Y}\right) = \min(X \triangle Y) \in \{\min X, \min Y\}$ .

Assume (for the sake of contradiction) that  $\mu \in C \setminus A$ . Thus,  $\mu \notin A$ . Hence,  $\mu \neq \min X$  (since  $\min X \in A$  but  $\mu \notin A$ ). Combining this with  $\mu \in \{\min X, \min Y\}$ , we obtain  $\mu \in \{\min X, \min Y\} \setminus \{\min X\} \subseteq \{\min Y\}$ . Hence,  $\mu = \min Y \in B$ . But also  $\mu \in C \setminus A \subseteq C$ . Combining  $\mu \in B$  with  $\mu \in C$ , we obtain  $\mu \in B \cap C$ . But  $\mu = \min Y \in Y = B \triangle C = (B \cup C) \setminus (B \cap C)$  (by the definition of  $B \triangle C$ ). Thus,  $\mu \notin B \cap C$ . This contradicts  $\mu \in B \cap C$ . This contradiction shows that our assumption (that  $\mu \in C \setminus A$ ) was wrong. Hence, we don't have  $\mu \in C \setminus A$ .

We have  $\mu = \min(A \triangle C) \in A \triangle C = (A \setminus C) \cup (C \setminus A)$ . Hence, either  $\mu \in A \setminus C$  or  $\mu \in C \setminus A$ . Thus,  $\mu \in A \setminus C$  (since we don't have  $\mu \in C \setminus A$ ). Therefore,  $\min(A \triangle C) = \mu \in A \setminus C \subseteq A$ . So we have shown that  $A \neq C$  and  $\min(A \triangle C) \in A$ . In other words,  $A < C$  (by the definition of the relation  $<$ ).

Now, forget that we fixed  $A$ ,  $B$  and  $C$ . We thus have shown that if  $A \in \mathbf{P}$ ,  $B \in \mathbf{P}$  and  $C \in \mathbf{P}$  are such that  $A < B$  and  $B < C$ , then  $A < C$ . In other words, the relation  $<$  is transitive.

<sup>4</sup>*Proof.* Let  $A \in \mathbf{P}$  be such that  $A < A$ . We shall derive a contradiction.

We have  $A < A$ . In other words,  $A \neq A$  and  $\min(A \triangle A) \in A$  (by the definition of the relation  $<$ ). But  $A \neq A$  is absurd. Hence, we have found a contradiction.

Now, forget that we fixed  $A$ . We thus have found a contradiction for each  $A \in \mathbf{P}$  satisfying

the smaller relation of a partial order on  $\mathbf{P}$ . It remains to prove that this partial order is a total order. In other words, we need to show that if  $A \in \mathbf{P}$  and  $B \in \mathbf{P}$  are distinct, then either  $A < B$  or  $B < A$ . But this is easy<sup>6</sup>. Thus, the proof of Proposition 2.7 is complete.  $\square$

In the following, we shall regard the set  $\mathbf{P}$  as a totally ordered set, equipped with the order from Proposition 2.7. Thus, for example, two sets  $A$  and  $B$  in  $\mathbf{P}$  satisfy  $A \geq B$  if and only if either  $A = B$  or  $B < A$ .

**Definition 2.8.** Let  $S$  be a subset of  $\mathbb{Z}$ . Then, we define a new subset  $S + 1$  of  $\mathbb{Z}$  by setting

$$S + 1 = \{i + 1 \mid i \in S\} = \{j \in \mathbb{Z} \mid j - 1 \in S\}.$$

Note that  $S + 1 \in \mathbf{P}$  if  $S \in \mathbf{P}$ .

For example,  $\{2, 5\} + 1 = \{3, 6\}$ . Note that a subset  $S$  of  $\mathbb{Z}$  is lacunar if and only if  $S \cap (S + 1) = \emptyset$ .

**Proposition 2.9.** Let  $\Lambda \in \mathbf{P}$  and  $R \in \mathbf{P}$  be such that the set  $R$  is lacunar and  $R \subseteq \Lambda \cup (\Lambda + 1)$ . Then,  $R \geq \Lambda$  (with respect to the total order on  $\mathbf{P}$ ).

---

$A < A$ . Thus, no  $A \in \mathbf{P}$  satisfies  $A < A$ . In other words, the relation  $<$  is irreflexive.

<sup>5</sup>*Proof.* We know that the binary relation  $<$  is transitive and irreflexive. Hence, this relation  $<$  is asymmetric (since every transitive irreflexive relation is asymmetric).

<sup>6</sup>*Proof.* Let  $A \in \mathbf{P}$  and  $B \in \mathbf{P}$  be distinct. We must show that either  $A < B$  or  $B < A$ .

The sets  $A$  and  $B$  are elements of  $\mathbf{P}$ , and thus are finite subsets of  $\mathbb{Z}$  (by the definition of  $\mathbf{P}$ ).

The set  $A \triangle B$  is nonempty (since  $A$  and  $B$  are distinct) and is a finite subset of  $\mathbb{Z}$  (since  $A$  and  $B$  are finite subsets of  $\mathbb{Z}$ ). Hence,  $\min(A \triangle B)$  is well-defined. Clearly,

$$\min(A \triangle B) \in A \triangle B = (A \cup B) \setminus (A \cap B) \subseteq A \cup B.$$

In other words, either  $\min(A \triangle B) \in A$  or  $\min(A \triangle B) \in B$ . In other words, we are in one of the following two cases:

*Case 1:* We have  $\min(A \triangle B) \in A$ .

*Case 2:* We have  $\min(A \triangle B) \in B$ .

Let us first consider Case 1. In this case, we have  $\min(A \triangle B) \in A$ . Thus,  $A \neq B$  (since  $A$  and  $B$  are distinct) and  $\min(A \triangle B) \in A$ . In other words,  $A < B$  (by the definition of the relation  $<$ ). Hence, either  $A < B$  or  $B < A$ . Thus, our claim (that either  $A < B$  or  $B < A$ ) is proven in Case 1.

Let us first consider Case 2. In this case, we have  $\min(A \triangle B) \in B$ . Hence,

$\min\left(\underbrace{B \triangle A}_{=A \triangle B}\right) = \min(A \triangle B) \in B$ . Thus,  $B \neq A$  (since  $A$  and  $B$  are distinct) and

$\min(B \triangle A) \in B$ . In other words,  $B < A$  (by the definition of the relation  $<$ ). Hence, either  $A < B$  or  $B < A$ . Thus, our claim (that either  $A < B$  or  $B < A$ ) is proven in Case 2.

Now, our claim (that either  $A < B$  or  $B < A$ ) has been proven in both Cases 1 and 2. Hence, this claim always holds. In other words, we always have either  $A < B$  or  $B < A$ . Qed.

---

*Proof of Proposition 2.9.* Assume the contrary. Thus,  $R < \Lambda$  (since  $\mathbf{P}$  is totally ordered). In other words,  $R \neq \Lambda$  and  $\min(R \triangle \Lambda) \in R$  (by the definition of the relation  $<$ ). Let  $\mu = \min(R \triangle \Lambda)$ . Thus,  $\mu = \min(R \triangle \Lambda) \in R \subseteq \Lambda \cup (\Lambda + 1)$ .

We have  $\mu = \min(R \triangle \Lambda) \in R \triangle \Lambda = (R \cup \Lambda) \setminus (R \cap \Lambda)$ . Hence,  $\mu \notin R \cap \Lambda$ . If we had  $\mu \in \Lambda$ , then we would have  $\mu \in R \cap \Lambda$  (since  $\mu \in R$  and  $\mu \in \Lambda$ ), which would contradict  $\mu \notin R \cap \Lambda$ . Thus, we cannot have  $\mu \in \Lambda$ . Hence,  $\mu \notin \Lambda$ . Combining  $\mu \in \Lambda \cup (\Lambda + 1)$  with  $\mu \notin \Lambda$ , we obtain  $\mu \in (\Lambda \cup (\Lambda + 1)) \setminus \Lambda \subseteq \Lambda + 1$ . In other words,  $\mu - 1 \in \Lambda$ .

Every  $x \in R \triangle \Lambda$  satisfies  $x \geq \min(R \triangle \Lambda)$ . Hence, if we had  $\mu - 1 \in R \triangle \Lambda$ , then we would have  $\mu - 1 \geq \min(R \triangle \Lambda) = \mu$ , which would contradict  $\mu - 1 < \mu$ . Thus, we cannot have  $\mu - 1 \in R \triangle \Lambda$ . Thus,  $\mu - 1 \notin R \triangle \Lambda$ . Combining this with  $\mu - 1 \in \Lambda$ , we obtain  $\mu - 1 \in \Lambda \setminus (R \triangle \Lambda) = R \cap \Lambda$  (since every two sets  $X$  and  $Y$  satisfy  $Y \setminus (X \triangle Y) = X \cap Y$ ). Thus,  $\mu - 1 \in R \cap \Lambda \subseteq R$ .

But the set  $R$  is lacunar. In other words, each  $s \in R$  satisfies  $s + 1 \notin R$  (by the definition of “lacunar”). Applying this to  $s = \mu - 1$ , we obtain  $(\mu - 1) + 1 \notin R$  (since  $\mu - 1 \in R$ ). This contradicts  $(\mu - 1) + 1 = \mu \in R$ . This contradiction shows that our assumption was wrong; hence, Proposition 2.9 is proven.  $\square$

**Corollary 2.10.** Let  $\Lambda_1$  and  $\Lambda_2$  be two finite lacunar subsets of  $\mathbb{Z}$  such that  $\Lambda_1 \cup (\Lambda_1 + 1) = \Lambda_2 \cup (\Lambda_2 + 1)$ . Then,  $\Lambda_1 = \Lambda_2$ .

*Proof of Corollary 2.10.* Both  $\Lambda_1$  and  $\Lambda_2$  are finite subsets of  $\mathbb{Z}$ , and thus belong to  $\mathbf{P}$  (by the definition of  $\mathbf{P}$ ). Also,

$$\Lambda_1 \subseteq \Lambda_1 \cup (\Lambda_1 + 1) = \Lambda_2 \cup (\Lambda_2 + 1).$$

Hence, Proposition 2.9 (applied to  $R = \Lambda_1$  and  $\Lambda = \Lambda_2$ ) yields  $\Lambda_1 \geq \Lambda_2$  (with respect to the total order on  $\mathbf{P}$ ). The same argument (with the roles of  $\Lambda_1$  and  $\Lambda_2$  interchanged) yields  $\Lambda_2 \geq \Lambda_1$ . Combining  $\Lambda_1 \geq \Lambda_2$  with  $\Lambda_2 \geq \Lambda_1$ , we obtain  $\Lambda_1 = \Lambda_2$ . This proves Corollary 2.10.  $\square$

## 2.2. $\mathcal{Z}$ -enriched $(P, \gamma)$ -partitions

**Convention 2.11.** By abuse of notation, we will often use the same notation for a poset  $P = (X, \leq)$  and its ground set  $X$  when there is no danger of confusion. In particular, if  $x$  is some object, then “ $x \in P$ ” shall mean “ $x \in X$ ”.

**Definition 2.12.** A *labeled poset* means a pair  $(P, \gamma)$  consisting of a finite poset  $P = (X, \leq)$  and an injective map  $\gamma : X \rightarrow A$  for some totally ordered set  $A$ . The injective map  $\gamma$  is called the *labeling* of the labeled poset  $(P, \gamma)$ . The poset  $P$  is called the *ground poset* of the labeled poset  $(P, \gamma)$ .



**Convention 2.13.** Let  $\mathcal{N}$  be a totally ordered set, whose (strict) order relation will be denoted by  $\prec$ . Let  $+$  and  $-$  be two distinct symbols. Let  $\mathcal{Z}$  be a subset of the set  $\mathcal{N} \times \{+, -\}$ . For each  $q = (n, s) \in \mathcal{Z}$ , we denote the element  $n \in \mathcal{N}$  by  $|q|$ , and we call the element  $s \in \{+, -\}$  the *sign* of  $q$ . If  $n \in \mathcal{N}$ , then we will denote the two elements  $(n, +)$  and  $(n, -)$  of  $\mathcal{N} \times \{+, -\}$  by  $+n$  and  $-n$ , respectively.

We equip the set  $\mathcal{Z}$  with a total order, whose (strict) order relation  $\prec$  is defined by

$$(n, s) \prec (n', s') \text{ if and only if either } n \prec n' \text{ or } (n = n' \text{ and } s = - \text{ and } s' = +).$$

Let  $\text{Pow } \mathcal{N}$  be the ring of all formal power series over  $\mathbb{Q}$  in the indeterminates  $x_n$  for  $n \in \mathcal{N}$ .

We fix  $\mathcal{N}$  and  $\mathcal{Z}$  throughout Subsection 2.2. That is, any result in this subsection is tacitly understood to begin with “Let  $\mathcal{N}$  be a totally ordered set, whose (strict) order relation will be denoted by  $\prec$ , and let  $\mathcal{Z}$  be a subset of the set  $\mathcal{N} \times \{+, -\}$ ”; and the notations of this convention shall always be in place throughout this Subsection.

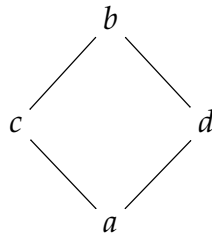
Whenever  $\prec$  denotes some strict order, the corresponding weak order will be denoted by  $\preceq$ . (Thus,  $a \preceq b$  means “ $a \prec b$  or  $a = b$ ”.)

**Definition 2.14.** Let  $(P, \gamma)$  be a labeled poset. A  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition means a map  $f : P \rightarrow \mathcal{Z}$  such that for all  $x < y$  in  $P$ , the following conditions hold:

- (i) We have  $f(x) \preceq f(y)$ .
- (ii) If  $f(x) = f(y) = +n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) < \gamma(y)$ .
- (iii) If  $f(x) = f(y) = -n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) > \gamma(y)$ .

(Of course, this concept depends on  $\mathcal{N}$  and  $\mathcal{Z}$ , but these will always be clear from the context.)

**Example 2.15.** Let  $P$  be the poset with the following Hasse diagram:



(that is, the ground set of  $P$  is  $\{a, b, c, d\}$ , and its order relation is given by  $a < c < b$  and  $a < d < b$ ). Let  $\gamma : P \rightarrow \mathbb{Z}$  be a map that satisfies  $\gamma(a) <$

$\gamma(b) < \gamma(c) < \gamma(d)$  (for example,  $\gamma$  could be the map that sends  $a, b, c, d$  to  $2, 3, 5, 7$ , respectively). Then,  $(P, \gamma)$  is a labeled poset. A  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition is a map  $f : P \rightarrow \mathcal{Z}$  satisfying the following conditions:

- (i) We have  $f(a) \preceq f(c) \preceq f(b)$  and  $f(a) \preceq f(d) \preceq f(b)$ .
- (ii) We cannot have  $f(c) = f(b) = +n$  with  $n \in \mathcal{N}$ .  
We cannot have  $f(d) = f(b) = +n$  with  $n \in \mathcal{N}$ .
- (iii) We cannot have  $f(a) = f(c) = -n$  with  $n \in \mathcal{N}$ .  
We cannot have  $f(a) = f(d) = -n$  with  $n \in \mathcal{N}$ .

For example, if  $\mathcal{N} = \mathbb{P}$  (the totally ordered set of positive integers, with its usual ordering) and  $\mathcal{Z} = \mathcal{N} \times \{+, -\}$ , then the map  $f : P \rightarrow \mathcal{Z}$  sending  $a, b, c, d$  to  $+2, -3, +2, -3$  (respectively) is a  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition. Notice that the total ordering on  $\mathcal{Z}$  in this case is given by

$$-1 \prec +1 \prec -2 \prec +2 \prec -3 \prec +3 \prec \dots,$$

rather than by the familiar total order on  $\mathbb{Z}$ .

The concept of a “ $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition” generalizes three notions in existing literature: that of a “ $(P, \gamma)$ -partition”, that of an “enriched  $(P, \gamma)$ -partition”, and that of a “left enriched  $(P, \gamma)$ -partition”<sup>7</sup>:

**Example 2.16. (a)** If  $\mathcal{N} = \mathbb{P}$  (the totally ordered set of positive integers) and  $\mathcal{Z} = \mathcal{N} \times \{+\} = \{+n \mid n \in \mathcal{N}\}$ , then the  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partitions are simply the  $(P, \gamma)$ -partitions into  $\mathcal{N}$ , composed with the canonical bijection  $\mathcal{N} \rightarrow \mathcal{Z}$ ,  $n \mapsto (+n)$ .

**(b)** If  $\mathcal{N} = \mathbb{P}$  (the totally ordered set of positive integers) and  $\mathcal{Z} = \mathcal{N} \times \{+, -\}$ , then the  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partitions are the enriched  $(P, \gamma)$ -partitions.

**(c)** If  $\mathcal{N} = \mathbb{N}$  (the totally ordered set of nonnegative integers) and  $\mathcal{Z} = (\mathcal{N} \times \{+, -\}) \setminus \{-0\}$ , then the  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partitions are the left en-

<sup>7</sup>The ideas behind these three concepts are due to Stanley [Stanle72], Stembridge [Stembr97, §2] and Petersen [Peters05], respectively, but the precise definitions are not standardized across the literature. We define a “ $(P, \gamma)$ -partition” as in [Stembr97, §1.1]; this definition differs noticeably from Stanley’s (in particular, Stanley requires  $f(x) \succeq f(y)$  instead of  $f(x) \preceq f(y)$ , but the differences do not end here). We define an “enriched  $(P, \gamma)$ -partition” as in [Stembr97, §2]. Finally, we define a “left enriched  $(P, \gamma)$ -partition” to be a  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition where  $\mathcal{N} = \mathbb{N}$  and  $\mathcal{Z} = (\mathcal{N} \times \{+, -\}) \setminus \{-0\}$ ; this definition is equivalent to Petersen’s [Peters06, Definition 3.4.1] up to some differences of notation (in particular, Petersen assumes that the ground set of  $P$  is already a subset of  $\mathbb{P}$ , and that the labeling  $\gamma$  is the canonical inclusion map  $P \rightarrow \mathbb{P}$ ; also, he identifies the elements  $+0, -1, +1, -2, +2, \dots$  of  $(\mathcal{N} \times \{+, -\}) \setminus \{-0\}$  with the integers  $0, -1, +1, -2, +2, \dots$ , respectively). Note that the definition Petersen gives in [Peters05, Definition 4.1] is incorrect, and the one in [Peters06, Definition 3.4.1] is probably his intent.

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riched  $(P, \gamma)$ -partitions. Note that  $+0$  and  $-0$  here stand for the pairs  $(0, +)$  and  $(0, -)$ ; thus, they are not equal.

**Definition 2.17.** If  $(P, \gamma)$  is a labeled poset, then  $\mathcal{E}(P, \gamma)$  shall denote the set of all  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partitions.

**Definition 2.18.** Let  $P$  be any finite poset. Then,  $\mathcal{L}(P)$  shall denote the set of all linear extensions of  $P$ . A linear extension of  $P$  shall be understood simultaneously as a totally ordered set extending  $P$  and as a list  $(w_1, w_2, \dots, w_n)$  of all elements of  $P$  such that no two integers  $i < j$  satisfy  $w_i \geq w_j$  in  $P$ .

Let us prove some basic facts about  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partitions, straightforwardly generalizing classical results proven by Stanley and Gessel (for the case of “plain”  $(P, \gamma)$ -partitions), Stembridge [Stembr97, Lemma 2.1] (for enriched  $(P, \gamma)$ -partitions) and Petersen [Peters06, Lemma 3.4.1] (for left enriched  $(P, \gamma)$ -partitions):

**Proposition 2.19.** For any labeled poset  $(P, \gamma)$ , we have

$$\mathcal{E}(P, \gamma) = \bigsqcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma).$$

*Proof of Proposition 2.19.* The following proof is a straightforward generalization of [Stembr97, proof of Lemma 2.1] (although we are rewriting it along the lines of [GriRei18, solution to Exercise 5.2.13]).

If  $z \in \mathcal{Z}$  is any element, then we say that  $z$  is *positive* if the sign of  $z$  is  $+$  (that is, if  $z = (n, +)$  for some  $n \in \mathcal{N}$ ), and we say that  $z$  is *negative* if the sign of  $z$  is  $-$  (that is, if  $z = (n, -)$  for some  $n \in \mathcal{N}$ ).

Let  $(P, \gamma)$  be a labeled poset.

Let  $f \in \mathcal{E}(P, \gamma)$  be arbitrary. Thus,  $f$  is a  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition (by the definition of  $\mathcal{E}(P, \gamma)$ ). We now define a binary relation  $<_f$  on the set  $P$  as follows: For any two elements  $x$  and  $y$  of  $P$ , we shall have  $x <_f y$  if and only if we have

- either**  $(f(x) \prec f(y))$
- or**  $(f(x) = f(y)$  is positive, and  $\gamma(x) < \gamma(y))$
- or**  $(f(x) = f(y)$  is negative, and  $\gamma(x) > \gamma(y))$ .

It is easy to see that this relation  $<_f$  is the smaller relation of a total order on the set  $P$  (indeed, it is transitive, irreflexive and asymmetric, and every two distinct elements  $x$  and  $y$  satisfy either  $x <_f y$  or  $y <_f x$ ). Denote this total order by  $\mathbf{w}_f$ . Thus,  $\mathbf{w}_f$  is a total order on the set  $P$ .

This total order  $\mathbf{w}_f$  is a linear extension of  $P$ .

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[*Proof:* Let  $x$  and  $y$  be two elements of  $P$  such that  $x < y$  in  $P$ . We shall show that  $x < y$  in  $\mathbf{w}_f$ .

Recall that  $f$  is a  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition. Thus, from  $x < y$  in  $P$ , we obtain the following three facts:

- (i) We have  $f(x) \preceq f(y)$ .
- (ii) If  $f(x) = f(y) = +n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) < \gamma(y)$ .
- (iii) If  $f(x) = f(y) = -n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) > \gamma(y)$ .

From these three facts, we conclude that **either**  $(f(x) \prec f(y))$  **or**  $(f(x) = f(y)$  is positive, and  $\gamma(x) < \gamma(y))$  **or**  $(f(x) = f(y)$  is negative, and  $\gamma(x) > \gamma(y))$ . In other words,  $x <_f y$  (by the definition of the relation  $<_f$ ). In other words,  $x < y$  in  $\mathbf{w}_f$  (since  $<_f$  is the smaller relation of  $\mathbf{w}_f$ ).

Now, forget that we fixed  $x$  and  $y$ . Thus, we have shown that if  $x$  and  $y$  are two elements of  $P$  such that  $x < y$  in  $P$ , then  $x < y$  in  $\mathbf{w}_f$ . Hence,  $\mathbf{w}_f$  is a linear extension of  $P$  (since  $\mathbf{w}_f$  is a total order).]

Furthermore, we have  $f \in \mathcal{E}(\mathbf{w}_f, \gamma)$ .

[*Proof:* Let  $x$  and  $y$  be two elements of  $\mathbf{w}_f$  such that  $x < y$  in  $\mathbf{w}_f$ . We shall prove the following three statements:

- (i) We have  $f(x) \preceq f(y)$ .
- (ii) If  $f(x) = f(y) = +n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) < \gamma(y)$ .
- (iii) If  $f(x) = f(y) = -n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) > \gamma(y)$ .

Indeed,  $x$  and  $y$  are elements of  $P$  (since  $\mathbf{w}_f = P$  as sets). Also,  $x < y$  in  $\mathbf{w}_f$ . In other words,  $x <_f y$  (since  $<_f$  is the smaller relation of  $\mathbf{w}_f$ ). In other words, we have

- either**  $(f(x) \prec f(y))$
- or**  $(f(x) = f(y)$  is positive, and  $\gamma(x) < \gamma(y))$
- or**  $(f(x) = f(y)$  is negative, and  $\gamma(x) > \gamma(y))$

(by the definition of the relation  $<_f$ ). In either of these three cases, the three statements (i), (ii) and (iii) above are true.

Now, forget that we fixed  $x$  and  $y$ . We thus have shown that if  $x$  and  $y$  are two elements of  $\mathbf{w}_f$  such that  $x < y$  in  $\mathbf{w}_f$ , then the three statements (i), (ii) and (iii) above are true. In other words,  $f$  is a  $\mathcal{Z}$ -enriched  $(\mathbf{w}_f, \gamma)$ -partition (by the definition of a  $\mathcal{Z}$ -enriched  $(\mathbf{w}_f, \gamma)$ -partition). In other words,  $f \in \mathcal{E}(\mathbf{w}_f, \gamma)$  (by the definition of  $\mathcal{E}(\mathbf{w}_f, \gamma)$ ).]

Furthermore, the following holds:

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*Observation 1:* If  $w \in \mathcal{L}(P)$  satisfies  $f \in \mathcal{E}(w, \gamma)$ , then  $w = \mathbf{w}_f$ .

[*Proof of Observation 1:* Let  $w \in \mathcal{L}(P)$  be such that  $f \in \mathcal{E}(w, \gamma)$ . Thus,  $f$  is a  $\mathcal{Z}$ -enriched  $(w, \gamma)$ -partition (by the definition of  $\mathcal{E}(w, \gamma)$ ).

From  $w \in \mathcal{L}(P)$ , we conclude that  $w = P$  as sets, and that  $w$  is a total order. Hence,  $w = P = \mathbf{w}_f$  as sets.

Now, let  $x$  and  $y$  be two elements of  $w$  such that  $x < y$  in  $w$ . Since  $f$  is a  $\mathcal{Z}$ -enriched  $(w, \gamma)$ -partition, we thus conclude that the following three statements hold:

- (i) We have  $f(x) \preceq f(y)$ .
- (ii) If  $f(x) = f(y) = +n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) < \gamma(y)$ .
- (iii) If  $f(x) = f(y) = -n$  for some  $n \in \mathcal{N}$ , then  $\gamma(x) > \gamma(y)$ .

From these three statements, we conclude that **either**  $(f(x) \prec f(y))$  **or**  $(f(x) = f(y)$  is positive, and  $\gamma(x) < \gamma(y))$  **or**  $(f(x) = f(y)$  is negative, and  $\gamma(x) > \gamma(y))$ . In other words,  $x <_f y$  (by the definition of the relation  $<_f$ ). In other words,  $x < y$  in  $\mathbf{w}_f$  (since  $<_f$  is the smaller relation of  $\mathbf{w}_f$ ).

Now, forget that we fixed  $x$  and  $y$ . We thus have shown that if  $x$  and  $y$  are two elements of  $w$  such that  $x < y$  in  $w$ , then  $x < y$  in  $\mathbf{w}_f$ . In other words,  $\mathbf{w}_f$  is a linear extension of  $w$  (since  $\mathbf{w}_f$  is a total order). But the only linear extension of  $w$  is  $w$  itself (since  $w$  is a total order). Thus, we conclude that  $\mathbf{w}_f = w$ . This proves Observation 1.]

Now, forget that we fixed  $f$ . Thus, for each  $f \in \mathcal{E}(P, \gamma)$ , we have constructed a linear extension  $\mathbf{w}_f$  of  $P$ , and we have shown that it satisfies  $f \in \mathcal{E}(\mathbf{w}_f, \gamma)$  and Observation 1.

Hence, for each  $f \in \mathcal{E}(P, \gamma)$ , we have

$$f \in \mathcal{E}(\mathbf{w}_f, \gamma) \subseteq \bigcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma)$$

(since  $\mathbf{w}_f \in \mathcal{L}(P)$  (because  $\mathbf{w}_f$  is a linear extension of  $P$ )). In other words,  $\mathcal{E}(P, \gamma) \subseteq \bigcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma)$ . On the other hand, every  $w \in \mathcal{L}(P)$  satisfies  $\mathcal{E}(w, \gamma) \subseteq \mathcal{E}(P, \gamma)$ <sup>8</sup>. Hence,  $\bigcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma) \subseteq \mathcal{E}(P, \gamma)$ . Combining this with  $\mathcal{E}(P, \gamma) \subseteq \bigcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma)$ , we obtain

$$\mathcal{E}(P, \gamma) = \bigcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma). \quad (4)$$

---

<sup>8</sup>*Proof.* Let  $w \in \mathcal{L}(P)$ . Thus,  $w$  is a linear extension of  $P$ . Hence,  $w = P$  as sets, and every two elements  $x$  and  $y$  of  $P$  satisfying  $x < y$  in  $P$  must also satisfy  $x < y$  in  $w$ . Thus, every  $\mathcal{Z}$ -enriched  $(w, \gamma)$ -partition is a  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition (since the axioms for a  $(w, \gamma)$ -partition are at least as strong as those for a  $(P, \gamma)$ -partition). In other words,  $\mathcal{E}(w, \gamma) \subseteq \mathcal{E}(P, \gamma)$ , qed.

Finally, the sets  $\mathcal{E}(w, \gamma)$  for distinct  $w \in \mathcal{L}(P)$  are disjoint.

[*Proof:* Let  $u$  and  $v$  be two elements of  $\mathcal{L}(P)$  such that  $\mathcal{E}(u, \gamma) \cap \mathcal{E}(v, \gamma) \neq \emptyset$ . We shall show that  $u = v$ .

We have  $\mathcal{E}(u, \gamma) \cap \mathcal{E}(v, \gamma) \neq \emptyset$ . Hence, there exists some  $f \in \mathcal{E}(u, \gamma) \cap \mathcal{E}(v, \gamma)$ . Consider this  $f$ .

We have

$$\begin{aligned} f \in \mathcal{E}(u, \gamma) \cap \mathcal{E}(v, \gamma) &\subseteq \mathcal{E}(u, \gamma) \subseteq \bigcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma) && \text{(since } u \in \mathcal{L}(P)) \\ &= \mathcal{E}(P, \gamma). \end{aligned}$$

Hence, Observation 1 (applied to  $w = u$ ) yields  $u = \mathbf{w}_f$ . The same argument (applied to  $v$  instead of  $u$ ) yields  $v = \mathbf{w}_f$ . Hence,  $u = \mathbf{w}_f = v$ .

Now, forget that we fixed  $u$  and  $v$ . We thus have shown that if  $u$  and  $v$  are two elements of  $\mathcal{L}(P)$  such that  $\mathcal{E}(u, \gamma) \cap \mathcal{E}(v, \gamma) \neq \emptyset$ , then  $u = v$ . In other words, the sets  $\mathcal{E}(w, \gamma)$  for distinct  $w \in \mathcal{L}(P)$  are disjoint.]

Thus, the union  $\bigcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma)$  is a disjoint union. Hence, (4) rewrites as  $\mathcal{E}(P, \gamma) = \bigsqcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma)$ . This proves Proposition 2.19.  $\square$

**Definition 2.20.** Let  $(P, \gamma)$  be a labeled poset. We define a power series  $\Gamma_{\mathcal{Z}}(P, \gamma) \in \text{Pow } \mathcal{N}$  by

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{f \in \mathcal{E}(P, \gamma)} \prod_{p \in P} x_{|f(p)|}.$$

This is easily seen to be convergent in the usual topology on  $\text{Pow } \mathcal{N}$ . (Indeed, for every monomial  $\mathbf{m}$  in  $\text{Pow } \mathcal{N}$ , there exist at most  $|P|! \cdot 2^{|P|}$  many  $f \in \mathcal{E}(P, \gamma)$  satisfying  $\prod_{p \in P} x_{|f(p)|} = \mathbf{m}$ .)

**Corollary 2.21.** For any labeled poset  $(P, \gamma)$ , we have

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{w \in \mathcal{L}(P)} \Gamma_{\mathcal{Z}}(w, \gamma).$$

*Proof of Corollary 2.21.* Follows straight from Proposition 2.19.  $\square$

**Definition 2.22.** Let  $P$  be any set. Let  $A$  be a totally ordered set. Let  $\gamma : P \rightarrow A$  and  $\delta : P \rightarrow A$  be two maps. We say that  $\gamma$  and  $\delta$  are *order-isomorphic* if the following holds: For every pair  $(p, q) \in P \times P$ , we have  $\gamma(p) \leq \gamma(q)$  if and only if  $\delta(p) \leq \delta(q)$ .

**Lemma 2.23.** Let  $(P, \alpha)$  and  $(P, \beta)$  be two labeled posets with the same ground poset  $P$ . Assume that the maps  $\alpha$  and  $\beta$  are order-isomorphic. Then:

- (a) We have  $\mathcal{E}(P, \alpha) = \mathcal{E}(P, \beta)$ .
- (b) We have  $\Gamma_{\mathcal{Z}}(P, \alpha) = \Gamma_{\mathcal{Z}}(P, \beta)$ .

*Proof of Lemma 2.23.* **(a)** If  $x$  and  $y$  are two elements of  $P$ , then we have the following equivalences:

$$\begin{aligned}(\alpha(x) \leq \alpha(y)) &\iff (\beta(x) \leq \beta(y)); \\(\alpha(x) > \alpha(y)) &\iff (\beta(x) > \beta(y)); \\(\alpha(x) < \alpha(y)) &\iff (\beta(x) < \beta(y)).\end{aligned}$$

(Indeed, the first of these equivalences holds because  $\alpha$  and  $\beta$  are order-isomorphic; the second is the contrapositive of the first; the third is obtained from the second by swapping  $x$  with  $y$ .)

Hence, the conditions “ $\alpha(x) > \alpha(y)$ ” and “ $\alpha(x) < \alpha(y)$ ” in the definition of a  $\mathcal{Z}$ -enriched  $(P, \alpha)$ -partition are equivalent to the conditions “ $\beta(x) > \beta(y)$ ” and “ $\beta(x) < \beta(y)$ ” in the definition of a  $\mathcal{Z}$ -enriched  $(P, \beta)$ -partition. Therefore, the  $\mathcal{Z}$ -enriched  $(P, \alpha)$ -partitions are precisely the  $\mathcal{Z}$ -enriched  $(P, \beta)$ -partitions. In other words,  $\mathcal{E}(P, \alpha) = \mathcal{E}(P, \beta)$ . This proves Lemma 2.23 **(a)**.

**(b)** Lemma 2.23 **(b)** follows from Lemma 2.23 **(a)**. □

Let us recall the notion of the disjoint union of two posets:

**Definition 2.24. (a)** Let  $P$  and  $Q$  be two sets. The *disjoint union* of  $P$  and  $Q$  is the set  $(\{0\} \times P) \cup (\{1\} \times Q)$ . This set is denoted by  $P \sqcup Q$ , and comes with two canonical injections

$$\begin{aligned}i_0 : P &\rightarrow P \sqcup Q, & p &\mapsto (0, p), & \text{and} \\i_1 : Q &\rightarrow P \sqcup Q, & q &\mapsto (1, q).\end{aligned}$$

The images of these two injections are disjoint, and their union is  $P \sqcup Q$ .

If  $f : P \sqcup Q \rightarrow X$  is any map, then the *restriction of  $f$  to  $P$*  is understood to be the map  $f \circ i_0 : P \rightarrow X$ , whereas the *restriction of  $f$  to  $Q$*  is understood to be the map  $f \circ i_1 : Q \rightarrow X$ . (Of course, this notation is ambiguous when  $P = Q$ .)

When the sets  $P$  and  $Q$  are already disjoint, it is common to identify their disjoint union  $P \sqcup Q$  with their union  $P \cup Q$  via the map

$$P \sqcup Q \rightarrow P \cup Q, \quad (i, r) \mapsto r.$$

Under this identification, the restriction of a map  $f : P \sqcup Q \rightarrow X$  to  $P$  becomes identical with the (literal) restriction  $f|_P$  of the map  $f : P \cup Q \rightarrow X$  (and similarly for the restrictions to  $Q$ ).

**(b)** Let  $P$  and  $Q$  be two posets. The *disjoint union* of the posets  $P$  and  $Q$  is the poset  $P \sqcup Q$  whose ground set is the disjoint union  $P \sqcup Q$ , and whose order relation is defined by the following rules:

- If  $p$  and  $p'$  are two elements of  $P$ , then  $(0, p) < (0, p')$  in  $P \sqcup Q$  if and only if  $p < p'$  in  $P$ .
- If  $q$  and  $q'$  are two elements of  $Q$ , then  $(1, q) < (1, q')$  in  $P \sqcup Q$  if and only if  $q < q'$  in  $Q$ .

- If  $p \in P$  and  $q \in Q$ , then the elements  $(0, p)$  and  $(1, q)$  of  $P \sqcup Q$  are incomparable.

**Proposition 2.25.** Let  $(P, \gamma)$  and  $(Q, \delta)$  be two labeled posets. Let  $(P \sqcup Q, \varepsilon)$  be a labeled poset whose ground poset  $P \sqcup Q$  is the disjoint union of  $P$  and  $Q$ , and whose labeling  $\varepsilon$  is such that the restriction of  $\varepsilon$  to  $P$  is order-isomorphic to  $\gamma$  and such that the restriction of  $\varepsilon$  to  $Q$  is order-isomorphic to  $\delta$ . Then,

$$\Gamma_{\mathcal{Z}}(P, \gamma) \Gamma_{\mathcal{Z}}(Q, \delta) = \Gamma_{\mathcal{Z}}(P \sqcup Q, \varepsilon).$$

*Proof of Proposition 2.25.* We WLOG assume that the ground sets  $P$  and  $Q$  are disjoint; thus, we can identify  $P \sqcup Q$  with the union  $P \cup Q$ . Let us make this identification.

The restriction  $\varepsilon|_P$  of  $\varepsilon$  to  $P$  is order-isomorphic to  $\gamma$ . Hence, Lemma 2.23 (a) (applied to  $\alpha = \varepsilon|_P$  and  $\beta = \gamma$ ) yields  $\mathcal{E}(P, \varepsilon|_P) = \mathcal{E}(P, \gamma)$ . Similarly,  $\mathcal{E}(Q, \varepsilon|_Q) = \mathcal{E}(Q, \delta)$ .

If  $x$  and  $y$  are two elements of  $P \sqcup Q$ , then  $x < y$  in  $P \sqcup Q$  holds if and only if

- **either**  $x$  and  $y$  both belong to  $P$  and satisfy  $x < y$  in  $P$ ,
- **or**  $x$  and  $y$  both belong to  $Q$  and satisfy  $x < y$  in  $Q$ .

Hence:

- If  $f$  is a  $\mathcal{Z}$ -enriched  $(P \sqcup Q, \varepsilon)$ -partition, then  $f|_P$  is a  $\mathcal{Z}$ -enriched  $(P, \varepsilon|_P)$ -partition and  $f|_Q$  is a  $\mathcal{Z}$ -enriched  $(Q, \varepsilon|_Q)$ -partition.
- Conversely, if  $g$  is a  $\mathcal{Z}$ -enriched  $(P, \varepsilon|_P)$ -partition, and if  $h$  is a  $\mathcal{Z}$ -enriched  $(Q, \varepsilon|_Q)$ -partition, then the unique map  $f : P \sqcup Q \rightarrow \mathcal{Z}$  satisfying  $(f|_P, f|_Q) = (g, h)$  (that is, the map  $P \sqcup Q \rightarrow \mathcal{Z}$  that sends each  $p \in P$  to  $g(p)$  and sends each  $q \in Q$  to  $h(q)$ ) is a  $\mathcal{Z}$ -enriched  $(P \sqcup Q, \varepsilon)$ -partition.

Therefore, the map

$$\begin{aligned} \mathcal{E}(P \sqcup Q, \varepsilon) &\rightarrow \mathcal{E}(P, \varepsilon|_P) \times \mathcal{E}(Q, \varepsilon|_Q), \\ f &\mapsto (f|_P, f|_Q) \end{aligned}$$

is a bijection (this is easy to see). In other words, the map

$$\begin{aligned} \mathcal{E}(P \sqcup Q, \varepsilon) &\rightarrow \mathcal{E}(P, \gamma) \times \mathcal{E}(Q, \delta), \\ f &\mapsto (f|_P, f|_Q) \end{aligned} \tag{5}$$



is a bijection (since  $\mathcal{E}(P, \varepsilon|_P) = \mathcal{E}(P, \gamma)$  and  $\mathcal{E}(Q, \varepsilon|_Q) = \mathcal{E}(Q, \delta)$ ). Now, the definition of  $\Gamma_{\mathcal{Z}}(P \sqcup Q, \varepsilon)$  yields

$$\begin{aligned}
\Gamma_{\mathcal{Z}}(P \sqcup Q, \varepsilon) &= \sum_{f \in \mathcal{E}(P \sqcup Q, \varepsilon)} \underbrace{\prod_{p \in P \sqcup Q} x_{|f(p)|}}_{=} \\
&= \left( \prod_{p \in P} x_{|f(p)|} \right) \left( \prod_{p \in Q} x_{|f(p)|} \right) \\
&= \sum_{f \in \mathcal{E}(P \sqcup Q, \varepsilon)} \underbrace{\left( \prod_{p \in P} x_{|f(p)|} \right)}_{= \prod_{p \in P} x_{|(f|_P)(p)|}} \underbrace{\left( \prod_{p \in Q} x_{|f(p)|} \right)}_{= \prod_{p \in Q} x_{|(f|_Q)(p)|}} \\
&= \sum_{f \in \mathcal{E}(P \sqcup Q, \varepsilon)} \left( \prod_{p \in P} x_{|(f|_P)(p)|} \right) \left( \prod_{p \in Q} x_{|(f|_Q)(p)|} \right) \\
&= \sum_{(g, h) \in \mathcal{E}(P, \gamma) \times \mathcal{E}(Q, \delta)} \left( \prod_{p \in P} x_{|g(p)|} \right) \left( \prod_{p \in Q} x_{|h(p)|} \right) \\
&\quad \left( \text{here, we have substituted } (g, h) \text{ for } (f|_P, f|_Q), \right. \\
&\quad \left. \text{since the map (5) is a bijection} \right) \\
&= \underbrace{\left( \sum_{g \in \mathcal{E}(P, \gamma)} \prod_{p \in P} x_{|g(p)|} \right)}_{= \sum_{f \in \mathcal{E}(P, \gamma)} \prod_{p \in P} x_{|f(p)|} = \Gamma_{\mathcal{Z}}(P, \gamma)} \cdot \underbrace{\left( \sum_{h \in \mathcal{E}(Q, \delta)} \prod_{p \in Q} x_{|h(p)|} \right)}_{= \sum_{f \in \mathcal{E}(Q, \delta)} \prod_{p \in Q} x_{|f(p)|} = \Gamma_{\mathcal{Z}}(Q, \delta)} \\
&= \Gamma_{\mathcal{Z}}(P, \gamma) \Gamma_{\mathcal{Z}}(Q, \delta).
\end{aligned}$$

This proves Proposition 2.25. □

**Definition 2.26.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be any  $n$ -permutation. (Recall that we have defined the concept of an “ $n$ -permutation” in Definition 1.3.) Then,  $([n], \pi)$  is a labeled poset (in fact,  $\pi$  is an injective map  $[n] \rightarrow \{1, 2, 3, \dots\}$ , and thus can be considered a labeling). We define  $\Gamma_{\mathcal{Z}}(\pi)$  to be the power series  $\Gamma_{\mathcal{Z}}([n], \pi)$ .

Let us recall the concept of a “poset homomorphism”:

**Definition 2.27.** Let  $P$  and  $Q$  be two posets. A map  $f : P \rightarrow Q$  is said to be a *poset homomorphism* if for any two elements  $x$  and  $y$  of  $P$  satisfying  $x \leq y$  in  $P$ , we have  $f(x) \leq f(y)$  in  $Q$ .

It is well-known that if  $U$  and  $V$  are any two finite totally ordered sets of the same size, then there is a unique poset isomorphism  $U \rightarrow V$ . Thus, if  $w$  is a finite totally ordered set with  $n$  elements, then there is a unique poset isomorphism  $w \rightarrow [n]$ . Now, we claim the following:

---

**Proposition 2.28.** Let  $w$  be a finite totally ordered set with ground set  $W$ . Let  $n = |W|$ . Let  $\bar{w}$  be the unique poset isomorphism  $w \rightarrow [n]$ . Let  $\gamma : W \rightarrow \{1, 2, 3, \dots\}$  be any injective map. Then,  $\Gamma_{\mathcal{Z}}(w, \gamma) = \Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1})$ .

*Proof of Proposition 2.28.* Clearly,  $(w, \gamma)$  is a labeled poset (since  $\gamma$  is injective). The map  $\gamma \circ \bar{w}^{-1} : [n] \rightarrow \{1, 2, 3, \dots\}$  is an injective map, thus an  $n$ -permutation. Hence,  $\Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1})$  is well-defined, and its definition yields  $\Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1}) = \Gamma_{\mathcal{Z}}([n], \gamma \circ \bar{w}^{-1})$ . But  $\bar{w}$  is a poset isomorphism  $w \rightarrow [n]$ , and thus is an isomorphism of labeled posets<sup>9</sup> from  $(w, \gamma)$  to  $([n], \gamma \circ \bar{w}^{-1})$ . Hence,

$$\begin{aligned} \mathcal{E}(w, \gamma) &\rightarrow \mathcal{E}([n], \gamma \circ \bar{w}^{-1}), \\ f &\mapsto f \circ \bar{w}^{-1} \end{aligned}$$

is a bijection (since any isomorphism of labeled posets induces a bijection between their  $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partitions). Every  $f \in \mathcal{E}(w, \gamma)$  satisfies

$$\begin{aligned} \prod_{p \in w} x_{|f(p)|} &= \prod_{p \in [n]} \underbrace{x_{|f(\bar{w}^{-1}(p))|}}_{=x_{|(f \circ \bar{w}^{-1})(p)|}} \\ &\left( \begin{array}{c} \text{here, we have substituted } \bar{w}^{-1}(p) \text{ for } p \text{ in the product,} \\ \text{since } \bar{w}^{-1} : [n] \rightarrow w \text{ is a bijection} \end{array} \right) \\ &= \prod_{p \in [n]} x_{|(f \circ \bar{w}^{-1})(p)|}. \end{aligned} \tag{6}$$

---

<sup>9</sup>We define the notion of an “isomorphism of labeled posets” in the obvious way: If  $(P, \alpha)$  and  $(Q, \beta)$  are two labeled posets, then a *homomorphism of labeled posets* from  $(P, \alpha)$  to  $(Q, \beta)$  means a poset homomorphism  $f : P \rightarrow Q$  satisfying  $\alpha = \beta \circ f$ . A *isomorphism of labeled posets* is an invertible homomorphism of labeled posets whose inverse also is a homomorphism of labeled posets. Note that this definition of an isomorphism is not equivalent to the definition given in [Stembr97, Section 1.1].

But the definition of  $\Gamma_{\mathcal{Z}}(w, \gamma)$  yields

$$\begin{aligned}
\Gamma_{\mathcal{Z}}(w, \gamma) &= \sum_{f \in \mathcal{E}(w, \gamma)} \underbrace{\prod_{p \in w} x_{|f(p)|}}_{= \prod_{p \in [n]} x_{|(f \circ \bar{w}^{-1})(p)|} \text{ (by (6))}} = \sum_{f \in \mathcal{E}(w, \gamma)} \prod_{p \in [n]} x_{|(f \circ \bar{w}^{-1})(p)|} \\
&= \sum_{f \in \mathcal{E}([n], \gamma \circ \bar{w}^{-1})} \prod_{p \in [n]} x_{|f(p)|} \\
&\quad \left( \begin{array}{l} \text{here, we have substituted } f \text{ for } f \circ \bar{w}^{-1} \text{ in the sum, since} \\ \text{the map } \mathcal{E}(w, \gamma) \rightarrow \mathcal{E}([n], \gamma \circ \bar{w}^{-1}), f \mapsto f \circ \bar{w}^{-1} \\ \text{is a bijection} \end{array} \right) \\
&= \Gamma_{\mathcal{Z}}([n], \gamma \circ \bar{w}^{-1}) \quad \left( \text{by the definition of } \Gamma_{\mathcal{Z}}([n], \gamma \circ \bar{w}^{-1}) \right) \\
&= \Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1}).
\end{aligned}$$

This proves Proposition 2.28. □

For the following corollary, let us recall that a bijective poset homomorphism is not necessarily an isomorphism of posets (since its inverse may and may not be a poset homomorphism).

**Corollary 2.29.** Let  $(P, \gamma)$  be a labeled poset. Let  $n = |P|$ . Then,

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{\substack{x: P \rightarrow [n] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}(\gamma \circ x^{-1}).$$

*Proof of Corollary 2.29.* For each totally ordered set  $w$  with ground set  $P$ , we let  $\bar{w}$  be the unique poset isomorphism  $w \rightarrow [n]$ . If  $w$  is a linear extension of  $P$ , then this map  $\bar{w}$  is also a bijective poset homomorphism  $P \rightarrow [n]$  (since every poset homomorphism  $w \rightarrow [n]$  is also a poset homomorphism  $P \rightarrow [n]$ ). Thus, for each  $w \in \mathcal{L}(P)$ , we have defined a bijective poset homomorphism  $\bar{w} : P \rightarrow [n]$ . We thus have defined a map

$$\begin{aligned}
\mathcal{L}(P) &\rightarrow \{\text{bijective poset homomorphisms } P \rightarrow [n]\}, \\
w &\mapsto \bar{w}.
\end{aligned} \tag{7}$$

This map is injective (indeed, a linear extension  $w \in \mathcal{L}(P)$  can be uniquely reconstructed from  $\bar{w}$ ) and surjective (because if  $x$  is a bijective poset homomorphism  $P \rightarrow [n]$ , then the linear extension  $w \in \mathcal{L}(P)$  defined (as a list) by  $w = (x^{-1}(1), x^{-1}(2), \dots, x^{-1}(n))$  satisfies  $x = \bar{w}$ ). Hence, this map is a bijection.

Corollary 2.21 yields

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{w \in \mathcal{L}(P)} \underbrace{\Gamma_{\mathcal{Z}}(w, \gamma)}_{=\Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1})} = \sum_{w \in \mathcal{L}(P)} \Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1}). \quad (8)$$

(by Proposition 2.28)

But recall that the map (7) is a bijection. Thus, we can substitute  $x$  for  $\bar{w}$  in the sum  $\sum_{w \in \mathcal{L}(P)} \Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1})$ , obtaining

$$\sum_{w \in \mathcal{L}(P)} \Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1}) = \sum_{\substack{x: P \rightarrow [n] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}(\gamma \circ x^{-1}).$$

Hence, (8) becomes

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{w \in \mathcal{L}(P)} \Gamma_{\mathcal{Z}}(\gamma \circ \bar{w}^{-1}) = \sum_{\substack{x: P \rightarrow [n] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}(\gamma \circ x^{-1}).$$

This proves Corollary 2.29. □

**Corollary 2.30.** Let  $n \in \mathbb{N}$  and  $m \in \mathbb{N}$ . Let  $\pi$  be an  $n$ -permutation and let  $\sigma$  be an  $m$ -permutation such that  $\pi$  and  $\sigma$  are disjoint. Then,

$$\Gamma_{\mathcal{Z}}(\pi) \Gamma_{\mathcal{Z}}(\sigma) = \sum_{\tau \in S(\pi, \sigma)} \Gamma_{\mathcal{Z}}(\tau).$$

*Proof of Corollary 2.30.* We first state a general fact about sets and maps:

*Observation 0:* Let  $U$ ,  $V$  and  $W$  be three sets. Let  $\alpha : U \rightarrow W$  and  $\beta : V \rightarrow W$  be two injective maps such that  $\alpha(U) = \beta(V)$ . Then, there exists a unique bijective map  $\lambda : U \rightarrow V$  satisfying  $\alpha = \beta \circ \lambda$ .

[*Proof of Observation 0:* This is a simple exercise. (Roughly speaking: If we replace the codomains of the maps  $\alpha$  and  $\beta$  by  $\alpha(U) = \beta(V)$ , then the injective maps  $\alpha$  and  $\beta$  become bijections, and the required map  $\lambda$  becomes  $\beta^{-1} \circ \alpha$ .)]

Consider the disjoint union  $[n] \sqcup [m]$  of the posets  $[n]$  and  $[m]$ . (Note that this disjoint union cannot be identified with the union  $[n] \cup [m]$ .) Let  $\varepsilon$  be the map  $[n] \sqcup [m] \rightarrow \{1, 2, 3, \dots\}$  whose restriction to  $[n]$  is  $\pi$  and whose restriction to  $[m]$  is  $\sigma$ . This map  $\varepsilon$  is injective, since  $\pi$  and  $\sigma$  are disjoint permutations. Thus,  $([n] \sqcup [m], \varepsilon)$  is a labeled poset.

Let us make some observations:

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*Observation 1:* If  $x$  is a bijective poset homomorphism  $[n] \sqcup [m] \rightarrow [n+m]$ , then  $\varepsilon \circ x^{-1} \in S(\pi, \sigma)$ .

[*Proof of Observation 1:* Let  $x$  be a bijective poset homomorphism  $[n] \sqcup [m] \rightarrow [n+m]$ . Then,  $x^{-1}$  is a well-defined bijective map  $[n+m] \rightarrow [n] \sqcup [m]$  (since  $x$  is bijective). Hence,  $\varepsilon \circ x^{-1}$  is an injective map  $[n+m] \rightarrow \{1, 2, 3, \dots\}$  (since  $\varepsilon$  is injective), therefore an  $(n+m)$ -permutation. Moreover,  $\sigma$  is a subsequence of  $\varepsilon \circ x^{-1}$ <sup>10</sup>. Similarly,  $\pi$  is a subsequence of  $\varepsilon \circ x^{-1}$ . Hence,  $\varepsilon \circ x^{-1}$  is an  $(n+m)$ -permutation having the property that both  $\pi$  and  $\sigma$  are subsequences of  $\varepsilon \circ x^{-1}$ . In other words,  $\varepsilon \circ x^{-1}$  is a shuffle of  $\pi$  and  $\sigma$  (by the definition of a shuffle). In other words,  $\varepsilon \circ x^{-1} \in S(\pi, \sigma)$  (by the definition of  $S(\pi, \sigma)$ ). This proves Observation 1.]

*Observation 2:* If  $\tau \in S(\pi, \sigma)$ , then there exists a unique bijective poset homomorphism  $x : [n] \sqcup [m] \rightarrow [n+m]$  satisfying  $\varepsilon \circ x^{-1} = \tau$ .

[*Proof of Observation 2:* Let  $\tau \in S(\pi, \sigma)$ . Thus,  $\tau$  is a shuffle of  $\pi$  and  $\sigma$  (by the definition of  $S(\pi, \sigma)$ ). In other words,  $\tau$  is an  $(n+m)$ -permutation having the property that both  $\pi$  and  $\sigma$  are subsequences of  $\tau$  (by the definition of a shuffle). Since  $\pi$  and  $\sigma$  are disjoint permutations, this entails that the letters of  $\tau$

---

<sup>10</sup>*Proof.* Let us denote the elements  $x((1,1)), x((1,2)), \dots, x((1,m))$  of  $[n+m]$  by  $i_1, i_2, \dots, i_m$ . Thus,  $i_q = x((1,q))$  for each  $q \in [m]$ . Hence, the elements  $i_1, i_2, \dots, i_m$  are the images of the distinct elements  $(1,1), (1,2), \dots, (1,m)$  under the map  $x$ . Therefore, these elements  $i_1, i_2, \dots, i_m$  are distinct themselves (since  $x$  is injective).

But  $x$  is a poset homomorphism  $[n] \sqcup [m] \rightarrow [n+m]$ . Thus,  $x((1,1)) \leq x((1,2)) \leq \dots \leq x((1,m))$  (since  $(1,1) \leq (1,2) \leq \dots \leq (1,m)$  in  $[n] \sqcup [m]$ ). In other words,  $i_1 \leq i_2 \leq \dots \leq i_m$  (since  $i_q = x((1,q))$  for each  $q \in [m]$ ). Hence,  $i_1 < i_2 < \dots < i_m$  (since the elements  $i_1, i_2, \dots, i_m$  are distinct). Hence,

$$\left( (\varepsilon \circ x^{-1})(i_1), (\varepsilon \circ x^{-1})(i_2), \dots, (\varepsilon \circ x^{-1})(i_m) \right)$$

is a subsequence of  $\varepsilon \circ x^{-1}$ . Since each  $q \in [m]$  satisfies

$$(\varepsilon \circ x^{-1}) \left( \underbrace{i_q}_{=x((1,q))} \right) = (\varepsilon \circ x^{-1})(x((1,q))) = \varepsilon((1,q)) = \sigma(q)$$

(since the restriction of  $\varepsilon$  to  $[m]$  is  $\sigma$ ),

we have

$$\left( (\varepsilon \circ x^{-1})(i_1), (\varepsilon \circ x^{-1})(i_2), \dots, (\varepsilon \circ x^{-1})(i_m) \right) = (\sigma(1), \sigma(2), \dots, \sigma(m)) = \sigma.$$

Thus,  $\sigma$  is a subsequence of  $\varepsilon \circ x^{-1}$  (since

$$\left( (\varepsilon \circ x^{-1})(i_1), (\varepsilon \circ x^{-1})(i_2), \dots, (\varepsilon \circ x^{-1})(i_m) \right)$$

is a subsequence of  $\varepsilon \circ x^{-1}$ ). Qed.

are the letters of  $\pi$  and the letters of  $\sigma$ . Hence, the image  $\tau([n+m])$  of the map  $\tau$  is the union  $\pi([n]) \cup \sigma([m])$ . But the image  $\varepsilon([n] \sqcup [m])$  is the same union  $\pi([n]) \cup \sigma([m])$  (by the definition of  $\varepsilon$ ). Hence,  $\varepsilon([n] \sqcup [m]) = \tau([n+m])$ . Also,  $\varepsilon : [n] \sqcup [m] \rightarrow \{1, 2, 3, \dots\}$  and  $\tau : [n+m] \rightarrow \{1, 2, 3, \dots\}$  are injective maps (indeed,  $\tau$  is injective since  $\tau$  is a permutation). Hence, Observation 0 (applied to  $U = [n] \sqcup [m]$ ,  $V = [n+m]$ ,  $W = \{1, 2, 3, \dots\}$ ,  $\alpha = \varepsilon$  and  $\beta = \tau$ ) shows that there exists a unique bijective map  $\lambda : [n] \sqcup [m] \rightarrow [n+m]$  satisfying  $\varepsilon = \tau \circ \lambda$ . In other words, there exists a unique bijective map  $\lambda : [n] \sqcup [m] \rightarrow [n+m]$  satisfying  $\varepsilon \circ \lambda^{-1} = \tau$  (since the condition  $\varepsilon = \tau \circ \lambda$  is equivalent to  $\varepsilon \circ \lambda^{-1} = \tau$ ). Consider this  $\lambda$ .

We have  $\lambda((0,1)) < \lambda((0,2)) < \dots < \lambda((0,n))$ <sup>11</sup> and  $\lambda((1,1)) < \lambda((1,2)) < \dots < \lambda((1,m))$ <sup>12</sup>. Thus,  $\lambda$  is a poset homomorphism  $[n] \sqcup [m] \rightarrow [n+m]$ . Hence, there exists at least one bijective poset homomorphism  $x : [n] \sqcup [m] \rightarrow [n+m]$  satisfying  $\varepsilon \circ x^{-1} = \tau$  (namely,  $x = \lambda$ ). Moreover, this  $x$  is unique (since  $\lambda$  is the **only** bijective map  $\lambda : [n] \sqcup [m] \rightarrow [n+m]$  satisfying  $\varepsilon \circ \lambda^{-1} = \tau$ ). Thus, there exists a unique bijective poset homomorphism  $x : [n] \sqcup [m] \rightarrow [n+m]$  satisfying  $\varepsilon \circ x^{-1} = \tau$ . This proves Observation 2.]

Now, the map

$$\{\text{bijective poset homomorphisms } x : [n] \sqcup [m] \rightarrow [n+m]\} \rightarrow S(\pi, \sigma), \\ x \mapsto \varepsilon \circ x^{-1}$$

is well-defined (by Observation 1) and is a bijection (by Observation 2). Hence, we can substitute  $\varepsilon \circ x^{-1}$  for  $\tau$  in the sum  $\sum_{\tau \in S(\pi, \sigma)} \Gamma_{\mathcal{Z}}(\tau)$ . We thus obtain

$$\sum_{\tau \in S(\pi, \sigma)} \Gamma_{\mathcal{Z}}(\tau) = \sum_{\substack{x: [n] \sqcup [m] \rightarrow [n+m] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}(\varepsilon \circ x^{-1}). \quad (9)$$

The definition of  $\Gamma_{\mathcal{Z}}(\pi)$  yields  $\Gamma_{\mathcal{Z}}(\pi) = \Gamma_{\mathcal{Z}}([n], \pi)$ . The definition of  $\Gamma_{\mathcal{Z}}(\sigma)$

<sup>11</sup>*Proof.* We know that  $\pi$  is a subsequence of  $\tau$ . In other words, there exist  $n$  elements  $i_1 < i_2 < \dots < i_n$  of  $[n+m]$  satisfying  $\pi = (\tau(i_1), \tau(i_2), \dots, \tau(i_n))$ . Consider these  $n$  elements.

We have  $\pi = (\tau(i_1), \tau(i_2), \dots, \tau(i_n))$ . In other words,  $\pi(p) = \tau(i_p)$  for each  $p \in [n]$ .

Now, let  $p \in [n]$ . Thus,  $\pi(p) = \tau(i_p)$  (as we have just seen). But recall that  $\varepsilon \circ \lambda^{-1} = \tau$ . In other words,  $\varepsilon = \tau \circ \lambda$ . On the other hand, the restriction of  $\varepsilon$  to  $[n]$  is  $\pi$  (by the definition of  $\varepsilon$ ). Hence,  $\varepsilon((0, p)) = \pi(p) = \tau(i_p)$ . Thus,

$$\tau(i_p) = \underbrace{\varepsilon}_{=\tau \circ \lambda}((0, p)) = (\tau \circ \lambda)((0, p)) = \tau(\lambda((0, p))).$$

Since the map  $\tau$  is injective (since  $\tau$  is a permutation), we thus obtain  $i_p = \lambda((0, p))$ .

Now, forget that we fixed  $p$ . Thus, we have shown that  $i_p = \lambda((0, p))$  for each  $p \in [n]$ . Hence, the chain of inequalities  $i_1 < i_2 < \dots < i_n$  rewrites as  $\lambda((0, 1)) < \lambda((0, 2)) < \dots < \lambda((0, n))$ . Qed.

<sup>12</sup>for similar reasons

yields  $\Gamma_{\mathcal{Z}}(\sigma) = \Gamma_{\mathcal{Z}}([m], \sigma)$ . Multiplying these two equalities, we obtain

$$\begin{aligned}
\Gamma_{\mathcal{Z}}(\pi) \Gamma_{\mathcal{Z}}(\sigma) &= \Gamma_{\mathcal{Z}}([n], \pi) \Gamma_{\mathcal{Z}}([m], \sigma) = \Gamma_{\mathcal{Z}}([n] \sqcup [m], \varepsilon) \\
&\quad \left( \begin{array}{c} \text{by Proposition 2.25, applied to } P = [n], \gamma = \pi, \\ Q = [m] \text{ and } \delta = \sigma \end{array} \right) \\
&= \sum_{\substack{x: [n] \sqcup [m] \rightarrow [n+m] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}(\varepsilon \circ x^{-1}) \\
&\quad \left( \begin{array}{c} \text{by Corollary 2.29, applied to } [n] \sqcup [m], \varepsilon \\ \text{and } n+m \text{ instead of } P, \gamma \text{ and } n \end{array} \right) \\
&= \sum_{\tau \in \mathcal{S}(\pi, \sigma)} \Gamma_{\mathcal{Z}}(\tau) \quad (\text{by (9)}).
\end{aligned}$$

This proves Corollary 2.30. □

### 2.3. Exterior peaks

So far we have been doing general nonsense. Let us now specialize to a situation that is connected to exterior peaks.

**Convention 2.31.** From now on, we set  $\mathcal{N} = \{0, 1, 2, \dots\} \cup \{\infty\}$ , with total order given by  $0 \prec 1 \prec 2 \prec \dots \prec \infty$ , and we set

$$\begin{aligned}
\mathcal{Z} &= (\mathcal{N} \times \{+, -\}) \setminus \{-0, +\infty\} \\
&= \{+0\} \cup \{+n \mid n \in \{1, 2, 3, \dots\}\} \cup \{-n \mid n \in \{1, 2, 3, \dots\}\} \cup \{-\infty\}.
\end{aligned}$$

Recall that the total order on  $\mathcal{Z}$  has

$$+0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \dots \prec -\infty.$$

**Definition 2.32.** Let  $S$  be a subset of  $\mathbb{Z}$ . A map  $\chi$  from  $S$  to a totally ordered set  $K$  is said to be *V-shaped* if there exists some  $t \in S$  such that the map  $\chi|_{\{s \in S \mid s \leq t\}}$  is strictly decreasing while the map  $\chi|_{\{s \in S \mid s \geq t\}}$  is strictly increasing. Notice that this  $t \in S$  is uniquely determined in this case; namely, it is the unique  $k \in S$  that minimizes  $\chi(k)$ .

Thus, roughly speaking, a map from a subset of  $\mathbb{Z}$  to a totally ordered set is *V-shaped* if and only if it is strictly decreasing up until a certain value of its argument, and then strictly increasing afterwards. For example, the 6-permutation  $(5, 1, 2, 3, 4)$  is V-shaped (keep in mind that we regard  $n$ -permutations as injective maps  $[n] \rightarrow \mathbb{P}$ ), whereas the 4-permutation  $(3, 1, 4, 2)$  is not.

For later use, let us crystallize a simple criterion for V-shapedness:

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**Lemma 2.33.** Let  $S$  be a finite nonempty subset of  $\mathbb{Z}$ . Let  $K$  be a totally ordered set. Let  $\chi : S \rightarrow K$  be an injective map. Let  $I$  and  $J$  be two subsets of  $S$  such that  $I \cup J = S$ . Assume that the map  $\chi|_I$  is strictly decreasing, whereas the map  $\chi|_J$  is strictly increasing. Assume further that each element of  $I$  is smaller than each element of  $J$ . Then, the map  $\chi$  is V-shaped.

*Proof of Lemma 2.33.* The set  $S$  is nonempty. Hence, there exists some  $k \in S$  that minimizes  $\chi(k)$ . Consider this  $k$ . Thus,

$$\chi(k) \leq \chi(s) \quad \text{for each } s \in S. \quad (10)$$

Now, we claim that if  $a \in S$  and  $b \in S$  satisfy  $a < b$  and  $\chi(a) < \chi(b)$ , then

$$b > k. \quad (11)$$

[*Proof of (11):* Let  $a \in S$  and  $b \in S$  be such that  $a < b$  and  $\chi(a) < \chi(b)$ . We must prove that  $b > k$ .

Assume the contrary. Thus,  $b \leq k$ . But (10) (applied to  $s = a$ ) yields  $\chi(k) \leq \chi(a)$ . Hence,  $\chi(k) \leq \chi(a) < \chi(b)$ , so that  $\chi(k) \neq \chi(b)$ . Thus,  $k \neq b$ . Combining this with  $b \leq k$ , we obtain  $b < k$ . Hence,  $a < b < k$ .

Assume (for the sake of contradiction) that  $b \notin I$ . Combining  $b \in S$  with  $b \notin I$ , we obtain  $b \in S \setminus I \subseteq J$  (since  $I \cup J = S$ ). Hence,  $k \in J$ <sup>13</sup>. Now, the map  $\chi|_J$  is strictly increasing. Thus, from  $b < k$ , we obtain  $(\chi|_J)(b) < (\chi|_J)(k)$  (since  $b \in J$  and  $k \in J$ ). Thus,  $\chi(b) = (\chi|_J)(b) < (\chi|_J)(k) = \chi(k) \leq \chi(b)$  (by (10), applied to  $s = b$ ). This is absurd. This contradiction shows that our assumption (that  $b \notin I$ ) was false. Hence, we have  $b \in I$ .

The same argument (but using  $a$  instead of  $b$ ) shows that  $a \in I$ . Now, recall that the map  $\chi|_I$  is strictly decreasing. Hence, from  $a < b$ , we obtain  $(\chi|_I)(a) > (\chi|_I)(b)$  (since  $a \in I$  and  $b \in I$ ). Thus,  $\chi(a) = (\chi|_I)(a) > (\chi|_I)(b) = \chi(b)$ . This contradicts  $\chi(a) < \chi(b)$ . This contradiction shows that our assumption was wrong. Hence,  $b > k$  is proven. Thus, (11) is proven.]

Next, we claim that if  $a \in S$  and  $b \in S$  satisfy  $a < b$  and  $\chi(a) > \chi(b)$ , then

$$a < k. \quad (12)$$

[*Proof of (12):* Let  $a \in S$  and  $b \in S$  be such that  $a < b$  and  $\chi(a) > \chi(b)$ . We must prove that  $a < k$ .

Assume the contrary. Thus,  $a \geq k$ . But (10) (applied to  $s = b$ ) yields  $\chi(k) \leq \chi(b)$ . Hence,  $\chi(k) \leq \chi(b) < \chi(a)$  (since  $\chi(a) > \chi(b)$ ), so that  $\chi(k) \neq \chi(a)$ . Thus,  $k \neq a$ . Combining this with  $a \geq k$ , we obtain  $a > k$ . Thus,  $k < a < b$ .

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<sup>13</sup>*Proof.* Assume the contrary. Thus,  $k \notin J$ . Combining  $k \in S$  with  $k \notin J$ , we find  $k \in S \setminus J \subseteq I$  (since  $I \cup J = S$ ). But recall that each element of  $I$  is smaller than each element of  $J$ . In other words,  $i < j$  for each  $i \in I$  and each  $j \in J$ . Applying this to  $i = k$  and  $j = b$ , we find  $k < b$ . This contradicts  $b < k$ . This contradiction shows that our assumption was false, qed.



Assume (for the sake of contradiction) that  $a \notin J$ . Combining  $a \in S$  with  $a \notin J$ , we obtain  $a \in S \setminus J \subseteq I$  (since  $I \cup J = S$ ). Hence,  $k \in I$ <sup>14</sup>. Now, the map  $\chi|_I$  is strictly decreasing. Thus, from  $k < a$ , we obtain  $(\chi|_I)(k) > (\chi|_I)(a)$  (since  $k \in I$  and  $a \in I$ ). Thus,  $\chi(k) = (\chi|_I)(k) > (\chi|_I)(a) = \chi(a) \geq \chi(k)$  (since (10) (applied to  $s = a$ ) yields  $\chi(k) \leq \chi(a)$ ). This is absurd. This contradiction shows that our assumption (that  $a \notin J$ ) was false. Hence, we have  $a \in J$ .

The same argument (but using  $b$  instead of  $a$ ) shows that  $b \in J$ . Now, recall that the map  $\chi|_J$  is strictly increasing. Hence, from  $a < b$ , we obtain  $(\chi|_J)(a) < (\chi|_J)(b)$  (since  $a \in J$  and  $b \in J$ ). Thus,  $\chi(a) = (\chi|_J)(a) < (\chi|_J)(b) = \chi(b)$ . This contradicts  $\chi(a) > \chi(b)$ . This contradiction shows that our assumption was wrong. Hence,  $a < k$  is proven. Thus, (12) is proven.]

Now, we claim that the map  $\chi|_{\{s \in S \mid s \leq k\}}$  is strictly decreasing.

[*Proof:* Let  $a$  and  $b$  be two elements of  $\{s \in S \mid s \leq k\}$  satisfying  $a < b$ . From  $a \in \{s \in S \mid s \leq k\}$ , it follows that  $a \in S$  and  $a \leq k$ . Similarly,  $b \in S$  and  $b \leq k$ .

We shall prove that  $\chi(a) > \chi(b)$ .

Indeed, assume the contrary. Thus,  $\chi(a) \leq \chi(b)$ . But  $a < b$ , so that  $a \neq b$  and thus  $\chi(a) \neq \chi(b)$  (since the map  $\chi$  is injective). Combining this with  $\chi(a) \leq \chi(b)$ , we obtain  $\chi(a) < \chi(b)$ . Hence, (11) yields  $b > k$ . This contradicts  $b \leq k$ . This contradiction shows that our assumption was false. Hence,  $\chi(a) > \chi(b)$  is proven.

Now, forget that we fixed  $a$  and  $b$ . We thus have shown that if  $a$  and  $b$  are two elements of  $\{s \in S \mid s \leq k\}$  satisfying  $a < b$ , then  $\chi(a) > \chi(b)$ . In other words, the map  $\chi|_{\{s \in S \mid s \leq k\}}$  is strictly decreasing.]

Next, we claim that the map  $\chi|_{\{s \in S \mid s \geq k\}}$  is strictly increasing.

[*Proof:* Let  $a$  and  $b$  be two elements of  $\{s \in S \mid s \geq k\}$  satisfying  $a < b$ . From  $a \in \{s \in S \mid s \geq k\}$ , it follows that  $a \in S$  and  $a \geq k$ . Similarly,  $b \in S$  and  $b \geq k$ .

We shall prove that  $\chi(a) < \chi(b)$ .

Indeed, assume the contrary. Thus,  $\chi(a) \geq \chi(b)$ . But  $a < b$ , so that  $a \neq b$  and thus  $\chi(a) \neq \chi(b)$  (since the map  $\chi$  is injective). Combining this with  $\chi(a) \geq \chi(b)$ , we obtain  $\chi(a) > \chi(b)$ . Hence, (12) yields  $a < k$ . This contradicts  $a \geq k$ . This contradiction shows that our assumption was false. Hence,  $\chi(a) < \chi(b)$  is proven.

Now, forget that we fixed  $a$  and  $b$ . We thus have shown that if  $a$  and  $b$  are two elements of  $\{s \in S \mid s \geq k\}$  satisfying  $a < b$ , then  $\chi(a) < \chi(b)$ . In other words, the map  $\chi|_{\{s \in S \mid s \geq k\}}$  is strictly increasing.]

Thus, we now know that the map  $\chi|_{\{s \in S \mid s \leq k\}}$  is strictly decreasing while the map  $\chi|_{\{s \in S \mid s \geq k\}}$  is strictly increasing. Hence, there exists some  $t \in S$  such that the map  $\chi|_{\{s \in S \mid s \leq t\}}$  is strictly decreasing while the map  $\chi|_{\{s \in S \mid s \geq t\}}$  is strictly increasing (namely,  $t = k$ ). In other words, the map  $\chi$  is V-shaped (by

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<sup>14</sup>*Proof.* Assume the contrary. Thus,  $k \notin I$ . Combining  $k \in S$  with  $k \notin I$ , we find  $k \in S \setminus I \subseteq J$  (since  $I \cup J = S$ ). But recall that each element of  $I$  is smaller than each element of  $J$ . In other words,  $i < j$  for each  $i \in I$  and each  $j \in J$ . Applying this to  $i = a$  and  $j = k$ , we find  $a < k$ . This contradicts  $k < a$ . This contradiction shows that our assumption was false, qed.

the definition of “V-shaped”). This proves Lemma 2.33.  $\square$

**Definition 2.34.** Let  $n \in \mathbb{N}$ .

- (a) Let  $f : [n] \rightarrow \mathcal{Z}$  be any map. Then,  $|f|$  shall denote the map  $[n] \rightarrow \mathcal{N}$ ,  $i \mapsto |f(i)|$ .
- (b) Let  $g : [n] \rightarrow \mathcal{N}$  be any map. Then, we define a monomial  $\mathbf{x}_g$  in  $\text{Pow } \mathcal{N}$  by  $\mathbf{x}_g = \prod_{i=1}^n x_{g(i)}$ .

Using this definition, we can rewrite the definition of  $\Gamma_{\mathcal{Z}}(\pi)$  as follows:

**Proposition 2.35.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be any  $n$ -permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = \sum_{f \in \mathcal{E}([n], \pi)} \prod_{p \in [n]} x_{|f(p)|} = \sum_{f \in \mathcal{E}([n], \pi)} \mathbf{x}_{|f|}. \quad (13)$$

*Proof of Proposition 2.35.* The definition of  $\Gamma_{\mathcal{Z}}(\pi)$  yields

$$\Gamma_{\mathcal{Z}}(\pi) = \Gamma_{\mathcal{Z}}([n], \pi) = \sum_{f \in \mathcal{E}([n], \pi)} \prod_{p \in [n]} x_{|f(p)|} \quad (\text{by the definition of } \Gamma_{\mathcal{Z}}([n], \pi)).$$

Also,

$$\begin{aligned} \sum_{f \in \mathcal{E}([n], \pi)} \underbrace{\mathbf{x}_{|f|}}_{\substack{= \prod_{i=1}^n x_{|f(i)|} \\ (\text{by the definition of } \mathbf{x}_{|f|})}} &= \sum_{f \in \mathcal{E}([n], \pi)} \underbrace{\prod_{i=1}^n x_{|f(i)|}}_{\substack{= \prod_{i \in [n]} x_{|f(i)|} \\ (\text{since } |f(i)| = |f(i)|)}} = \sum_{f \in \mathcal{E}([n], \pi)} \prod_{i \in [n]} x_{|f(i)|} \\ &= \sum_{f \in \mathcal{E}([n], \pi)} \prod_{p \in [n]} x_{|f(p)|} \end{aligned}$$

(here, we have renamed the index  $i$  as  $p$  in the product). Combining these two equalities, we obtain (13). Thus, Proposition 2.35 is proven.  $\square$

**Definition 2.36.** Let  $n \in \mathbb{N}$ . Let  $g : [n] \rightarrow \mathcal{N}$  be any map. Let  $\pi$  be an  $n$ -permutation. We shall say that  $g$  is  $\pi$ -amenable if it has the following properties:

- (i') The map  $\pi|_{g^{-1}(0)}$  is strictly increasing. (This allows the case when  $g^{-1}(0) = \emptyset$ .)
- (ii') For each  $h \in g([n]) \cap \{1, 2, 3, \dots\}$ , the map  $\pi|_{g^{-1}(h)}$  is V-shaped.

(iii') The map  $\pi \upharpoonright_{g^{-1}(\infty)}$  is strictly decreasing. (This allows the case when  $g^{-1}(\infty) = \emptyset$ .)

(iv') The map  $g$  is weakly increasing.

**Proposition 2.37.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be any  $n$ -permutation. Let  $f \in \mathcal{E}([n], \pi)$ . Then, the map  $|f| : [n] \rightarrow \mathcal{N}$  is  $\pi$ -amenable.

*Proof of Proposition 2.37.* We have  $f \in \mathcal{E}([n], \pi)$ . Thus,  $f$  is a  $\mathcal{Z}$ -enriched  $([n], \pi)$ -partition. In other words,  $f$  is a map  $[n] \rightarrow \mathcal{Z}$  such that for all  $x < y$  in  $[n]$ , the following conditions hold:

(i) We have  $f(x) \preceq f(y)$ .

(ii) If  $f(x) = f(y) = +h$  for some  $h \in \mathcal{N}$ , then  $\pi(x) < \pi(y)$ .

(iii) If  $f(x) = f(y) = -h$  for some  $h \in \mathcal{N}$ , then  $\pi(x) > \pi(y)$ .

(This follows from the definition of a  $\mathcal{Z}$ -enriched  $([n], \pi)$ -partition.)

Condition (i) shows that the map  $f$  is weakly increasing. Condition (ii) shows that for each  $h \in \mathcal{N}$ , the map  $\pi \upharpoonright_{f^{-1}(+h)}$  is strictly increasing. Condition (iii) shows that for each  $h \in \mathcal{N}$ , the map  $\pi \upharpoonright_{f^{-1}(-h)}$  is strictly decreasing.

Now, set  $g = |f|$ . Thus,  $g$  is a map  $[n] \rightarrow \mathcal{N}$ . We shall show that the map  $g$  is  $\pi$ -amenable. In order to prove this, we must show that the Properties (i'), (ii'), (iii') and (iv') in Definition 2.36 hold. We shall now do exactly this.

If  $x$  and  $y$  are two elements of  $[n]$  satisfying  $x < y$ , then  $g(x) \preceq g(y)$ <sup>15</sup>. Thus, the map  $g$  is weakly increasing. In other words, Property (iv') holds.

Recall that  $-0 \notin \mathcal{Z}$ . Thus,  $g^{-1}(0) = f^{-1}(+0)$ <sup>16</sup>. But the map  $\pi \upharpoonright_{f^{-1}(+0)}$  is strictly increasing<sup>17</sup>. In other words, the map  $\pi \upharpoonright_{g^{-1}(0)}$  is strictly increasing

<sup>15</sup>*Proof.* Let  $x$  and  $y$  be two elements of  $[n]$  satisfying  $x < y$ . We must prove that  $g(x) \preceq g(y)$ .

We have  $f(x) \preceq f(y)$  (by Condition (i)). Thus,  $|f(x)| \preceq |f(y)|$  (by the definition of the order on  $\mathcal{Z}$ ). But  $g = |f|$ . Hence,  $g(x) = |f|(x) = |f(x)|$  (by the definition of  $|f|$ ) and similarly  $g(y) = |f|(y) = |f(y)|$ . Thus,  $g(x) = |f(x)| \preceq |f(y)| = g(y)$ , qed.

<sup>16</sup>*Proof.* For each  $x \in [n]$ , we have the following chain of logical equivalences:

$$\begin{aligned} (x \in g^{-1}(0)) &\iff (g(x) = 0) \iff (|f(x)| = 0) && \left( \text{since } \underbrace{g(x)}_{=|f|} = |f|(x) = |f(x)| \right) \\ &\iff (f(x) = +0 \text{ or } f(x) = -0) \\ &\iff (f(x) = +0) && \left( \begin{array}{l} \text{since } f(x) = -0 \text{ cannot hold} \\ \text{(because } f(x) \in \mathcal{Z} \text{ but } -0 \notin \mathcal{Z}) \end{array} \right) \\ &\iff (x \in f^{-1}(+0)). \end{aligned}$$

Thus,  $g^{-1}(0) = f^{-1}(+0)$ .

<sup>17</sup>because for each  $h \in \mathcal{N}$ , the map  $\pi \upharpoonright_{f^{-1}(+h)}$  is strictly increasing

(since  $g^{-1}(0) = f^{-1}(+0)$ ). Hence, Property (i') holds. Similarly, Property (iii') also holds<sup>18</sup>.

Next, we shall show that Property (ii') holds. Indeed, fix  $h \in g([n]) \cap \{1, 2, 3, \dots\}$ . Thus,  $h \in g([n]) \cap \{1, 2, 3, \dots\} \subseteq g([n])$ . Hence, there exists some  $y \in [n]$  such that  $h = g(y)$ . In other words, the set  $g^{-1}(h)$  is nonempty.

This set  $g^{-1}(h)$  is an interval of  $[n]$  (since the map  $g$  is weakly increasing). The sets  $f^{-1}(-h)$  and  $f^{-1}(+h)$  are intervals of  $[n]$  (since the map  $f$  is weakly increasing), and their union is

$$f^{-1}(-h) \cup f^{-1}(+h) = g^{-1}(h)$$

<sup>19</sup> Moreover, each element of  $f^{-1}(-h)$  is smaller than each element of  $f^{-1}(+h)$  (since  $f$  is weakly increasing, and  $-h \prec +h$ ). Furthermore, recall that the map  $\pi|_{f^{-1}(-h)}$  is strictly decreasing while the map  $\pi|_{f^{-1}(+h)}$  is strictly increasing. In other words, the map  $(\pi|_{g^{-1}(h)})|_{f^{-1}(-h)}$  is strictly decreasing while the map  $(\pi|_{g^{-1}(h)})|_{f^{-1}(+h)}$  is strictly increasing (since  $\pi|_{f^{-1}(-h)} = (\pi|_{g^{-1}(h)})|_{f^{-1}(-h)}$  and  $\pi|_{f^{-1}(+h)} = (\pi|_{g^{-1}(h)})|_{f^{-1}(+h)}$ ). Hence, Lemma 2.33 (applied to  $S = g^{-1}(h)$ ,  $K = \mathbb{P}$ ,  $\chi = \pi|_{g^{-1}(h)}$ ,  $I = f^{-1}(-h)$  and  $J = f^{-1}(+h)$ ) shows that the map  $\pi|_{g^{-1}(h)}$  is V-shaped. Thus, Property (ii') holds. We have thus verified all four Properties (i'), (ii'), (iii') and (iv'); thus,  $g$  is  $\pi$ -amenable. In other words,  $|f|$  is  $\pi$ -amenable (since  $g = |f|$ ). This proves Proposition 2.37.  $\square$

**Proposition 2.38.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be any  $n$ -permutation. Let  $g : [n] \rightarrow \mathcal{N}$  be a  $\pi$ -amenable map. Let  $H$  be the set  $g([n]) \cap \{1, 2, 3, \dots\}$ . For each  $h \in H$ , we let  $m_h$  be the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum.

(a) The elements  $m_h$  for all  $h \in H$  are well-defined and distinct.

<sup>18</sup>To prove this, we first need to show that  $g^{-1}(\infty) = f^{-1}(-\infty)$  (this is similar to the proof of  $g^{-1}(0) = f^{-1}(+0)$ ), and that the map  $\pi|_{f^{-1}(-\infty)}$  is strictly decreasing (because for each  $h \in \mathcal{N}$ , the map  $\pi|_{f^{-1}(-h)}$  is strictly decreasing).

<sup>19</sup>*Proof.* For each  $x \in [n]$ , we have the following chain of logical equivalences:

$$\begin{aligned} (x \in g^{-1}(h)) &\iff (g(x) = h) \iff (|f(x)| = h) && \left( \text{since } \underbrace{g}_{=|f|}(x) = |f|(x) = |f(x)| \right) \\ &\iff \left( \underbrace{f(x) = +h}_{\iff (x \in f^{-1}(+h))} \text{ or } \underbrace{f(x) = -h}_{\iff (x \in f^{-1}(-h))} \right) \\ &\iff (x \in f^{-1}(+h) \text{ or } x \in f^{-1}(-h)) \\ &\iff (x \in f^{-1}(+h) \cup f^{-1}(-h)). \end{aligned}$$

Thus,  $g^{-1}(h) = f^{-1}(+h) \cup f^{-1}(-h) = f^{-1}(-h) \cup f^{-1}(+h)$ , qed.

**(b)** Let  $f : [n] \rightarrow \mathcal{Z}$  be any map. Then,  $(f \in \mathcal{E}([n], \pi)$  and  $|f| = g)$  if and only if the following five statements hold:

- (x<sub>1</sub>)** For each  $x \in g^{-1}(0)$ , we have  $f(x) = +0$ .
- (x<sub>2</sub>)** For each  $h \in H$  and each  $x \in g^{-1}(h)$  satisfying  $x < m_h$ , we have  $f(x) = -h$ .
- (x<sub>3</sub>)** For each  $h \in H$ , we have  $f(m_h) \in \{-h, +h\}$ .
- (x<sub>4</sub>)** For each  $h \in H$  and each  $x \in g^{-1}(h)$  satisfying  $x > m_h$ , we have  $f(x) = +h$ .
- (x<sub>5</sub>)** For each  $x \in g^{-1}(\infty)$ , we have  $f(x) = -\infty$ .

*Proof of Proposition 2.38.* The map  $\pi : [n] \rightarrow \mathbb{P}$  is injective (since  $\pi$  is an  $n$ -permutation). The map  $g$  is  $\pi$ -amenable, and thus satisfies the four Properties **(i')**, **(ii')**, **(iii')** and **(iv')** from Definition 2.36. In particular, this map  $g$  is weakly increasing (because of Property **(iv')**).

**(a)** The elements  $m_h$  for all  $h \in H$  are well-defined<sup>20</sup> and distinct<sup>21</sup>. This proves Proposition 2.38 **(a)**.

**(b)** We must prove the following two claims:

*Claim 1:* If  $(f \in \mathcal{E}([n], \pi)$  and  $|f| = g)$ , then the five statements **(x<sub>1</sub>)**, **(x<sub>2</sub>)**, **(x<sub>3</sub>)**, **(x<sub>4</sub>)** and **(x<sub>5</sub>)** hold.

*Claim 2:* If the five statements **(x<sub>1</sub>)**, **(x<sub>2</sub>)**, **(x<sub>3</sub>)**, **(x<sub>4</sub>)** and **(x<sub>5</sub>)** hold, then  $(f \in \mathcal{E}([n], \pi)$  and  $|f| = g)$ .

[*Proof of Claim 1:* Assume that  $(f \in \mathcal{E}([n], \pi)$  and  $|f| = g)$ . We must prove that the five statements **(x<sub>1</sub>)**, **(x<sub>2</sub>)**, **(x<sub>3</sub>)**, **(x<sub>4</sub>)** and **(x<sub>5</sub>)** hold.

We have  $f \in \mathcal{E}([n], \pi)$ . We have  $g = |f|$ . Hence, everything that we said in the proof of Proposition 2.37 is true in our current situation as well. In particular, the Properties **(i')**, **(ii')**, **(iii')** and **(iv')** in Definition 2.36 hold.

<sup>20</sup>*Proof.* Let  $h \in H$ . We must prove that the element  $m_h$  is well-defined.

We have  $h \in H = g([n]) \cap \{1, 2, 3, \dots\} \subseteq g([n])$ . Hence, there exists some  $y \in [n]$  such that  $h = g(y)$ . Therefore, the preimage  $g^{-1}(h)$  is nonempty. Therefore, there exists an element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum. Moreover, this element is unique (since  $\pi$  is injective). Thus, the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum is well-defined. In other words,  $m_h$  is well-defined. Qed.

<sup>21</sup>*Proof.* Let  $h_1 \in H$  and  $h_2 \in H$  be such that  $m_{h_1} = m_{h_2}$ . We must prove that  $h_1 = h_2$ .

We have  $m_h \in g^{-1}(h)$  for each  $h \in H$  (by the definition of  $m_h$ ). Hence,  $m_{h_1} \in g^{-1}(h_1)$ . In other words,  $g(m_{h_1}) = h_1$ . Similarly,  $g(m_{h_2}) = h_2$ . Hence,  $h_1 = g\left(\underbrace{m_{h_1}}_{=m_{h_2}}\right) = g(m_{h_2}) = h_2$ .

Qed.

Now, statement  $(x_1)$  holds<sup>22</sup>, and statement  $(x_5)$  holds<sup>23</sup>. Furthermore, statement  $(x_3)$  holds<sup>24</sup>. Also, statement  $(x_2)$  holds<sup>25</sup>, and statement  $(x_4)$  holds<sup>26</sup>. Thus, the five statements  $(x_1)$ ,  $(x_2)$ ,  $(x_3)$ ,  $(x_4)$  and  $(x_5)$  hold. This proves Claim 1.]

[*Proof of Claim 2:* Assume that five statements  $(x_1)$ ,  $(x_2)$ ,  $(x_3)$ ,  $(x_4)$  and  $(x_5)$  hold.

<sup>22</sup>*Proof.* Let  $x \in g^{-1}(0)$ . But  $g^{-1}(0) = f^{-1}(+0)$  (as we have shown in the proof of Proposition 2.37). Hence,  $x \in g^{-1}(0) = f^{-1}(+0)$ , so that  $f(x) = +0$ .

Now, forget that we fixed  $x$ . Thus, we have shown that for each  $x \in g^{-1}(0)$ , we have  $f(x) = +0$ . In other words, statement  $(x_1)$  holds.

<sup>23</sup>*Proof.* Let  $x \in g^{-1}(\infty)$ . But  $g^{-1}(\infty) = f^{-1}(-\infty)$  (as we have shown in the proof of Proposition 2.37). Hence,  $x \in g^{-1}(\infty) = f^{-1}(-\infty)$ , so that  $f(x) = -\infty$ .

Now, forget that we fixed  $x$ . Thus, we have shown that for each  $x \in g^{-1}(\infty)$ , we have  $f(x) = -\infty$ . In other words, statement  $(x_5)$  holds.

<sup>24</sup>*Proof.* Let  $h \in H$ . Then,  $m_h \in g^{-1}(h)$  (by the definition of  $m_h$ ), so that  $g(m_h) = h$ . Comparing this with  $\underbrace{g(m_h)}_{=|f|} = |f|(m_h) = |f(m_h)|$  (by the definition of  $|f|$ ), we obtain  $|f(m_h)| = h$ . In

other words, either  $f(m_h) = -h$  or  $f(m_h) = +h$ . In other words,  $f(m_h) \in \{-h, +h\}$ .

Now, forget that we fixed  $h$ . Thus, we have shown that for each  $h \in H$ , we have  $f(m_h) \in \{-h, +h\}$ . In other words, statement  $(x_3)$  holds.

<sup>25</sup>*Proof.* Let  $h \in H$  and  $x \in g^{-1}(h)$  be such that  $x < m_h$ . We shall show that  $f(x) = -h$ .

Indeed, assume the contrary. Thus,  $f(x) \neq -h$ .

We have  $x \in g^{-1}(h)$ , so that  $g(x) = h$ . Comparing this with  $\underbrace{g(x)}_{=|f|} = |f|(x) = |f(x)|$  (by

the definition of  $|f|$ ), we obtain  $|f(x)| = h$ . In other words, either  $f(x) = -h$  or  $f(x) = +h$ . Since  $f(x) \neq -h$ , we thus must have  $f(x) = +h$ . But the map  $f : [n] \rightarrow \mathcal{Z}$  is weakly increasing (as we have seen in the proof of Proposition 2.37). Hence, from  $x < m_h$ , we obtain  $f(x) \preceq f(m_h)$ . Hence, we cannot have  $f(m_h) = -h$  (because if we had  $f(m_h) = -h$ , then we would have  $+h = f(x) \preceq f(m_h) = -h$ , which would contradict the definition of the order on  $\mathcal{Z}$ ).

But statement  $(x_3)$  yields  $f(m_h) \in \{-h, +h\}$ . In other words, either  $f(m_h) = -h$  or  $f(m_h) = +h$ . Hence,  $f(m_h) = +h$  (since we cannot have  $f(m_h) = -h$ ). Combining  $f(x) = +h$  and  $f(m_h) = +h$ , we obtain  $f(x) = f(m_h) = +h$ .

But Condition (ii) from the proof of Proposition 2.37 holds. Applying this condition to  $y = m_h$ , we obtain  $\pi(x) < \pi(m_h)$ .

But  $m_h$  is the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum (by the definition of  $m_h$ ). Hence,  $\pi(m_h) \leq \pi(z)$  for each  $z \in g^{-1}(h)$ . Applying this to  $z = x$ , we obtain  $\pi(m_h) \leq \pi(x)$ . This contradicts  $\pi(x) < \pi(m_h)$ . This contradiction shows that our assumption was false. Hence,  $f(x) = -h$  is proven.

Now, forget that we fixed  $h$  and  $x$ . Thus, we have shown that for each  $h \in H$  and each  $x \in g^{-1}(h)$  satisfying  $x < m_h$ , we have  $f(x) = -h$ . In other words, statement  $(x_2)$  holds.

<sup>26</sup>*Proof.* Let  $h \in H$  and  $x \in g^{-1}(h)$  be such that  $x > m_h$ . Thus,  $m_h < x$ . We shall show that  $f(x) = +h$ .

Indeed, assume the contrary. Thus,  $f(x) \neq +h$ .

We have  $x \in g^{-1}(h)$ , so that  $g(x) = h$ . Comparing this with  $\underbrace{g(x)}_{=|f|} = |f|(x) = |f(x)|$  (by

the definition of  $|f|$ ), we obtain  $|f(x)| = h$ . In other words, either  $f(x) = -h$  or  $f(x) = +h$ . Since  $f(x) \neq +h$ , we thus must have  $f(x) = -h$ . But the map  $f : [n] \rightarrow \mathcal{Z}$  is weakly increasing (as we have seen in the proof of Proposition 2.37). Hence, from  $m_h < x$ , we obtain  $f(m_h) \preceq f(x)$ . Hence, we cannot have  $f(m_h) = +h$  (because if we had  $f(m_h) = +h$ , then we would have  $+h = f(m_h) \preceq f(x) = -h$ , which would contradict the definition of the order on  $\mathcal{Z}$ ).

We must prove that  $(f \in \mathcal{E}([n], \pi)$  and  $|f| = g)$ .

For each  $x \in [n]$ , we have

$$|f|(x) = g(x). \quad (14)$$

[*Proof of (14):* Let  $x \in [n]$ . We must prove (14). The definition of  $|f|$  yields  $|f|(x) = |f(x)|$ .

The equality (14) holds when  $g(x) = 0$ <sup>27</sup>. Hence, for the rest of this proof, we WLOG assume that  $g(x) \neq 0$ .

The equality (14) holds when  $g(x) = \infty$ <sup>28</sup>. Hence, for the rest of this proof, we WLOG assume that  $g(x) \neq \infty$ . Combining this with  $g(x) \neq 0$ , we obtain  $g(x) \notin \{0, \infty\}$ .

Set  $h = g(x)$ . We have

$$\begin{aligned} h = g(x) &\in \mathcal{N} \setminus \{0, \infty\} && (\text{since } g(x) \in \mathcal{N} \text{ and } g(x) \notin \{0, \infty\}) \\ &= \{1, 2, 3, \dots\}. \end{aligned}$$

Combining this with  $h = g\left(\underbrace{x}_{\in [n]}\right) \in g([n])$ , we obtain  $h \in g([n]) \cap \{1, 2, 3, \dots\} =$

$H$ . Also,  $x \in g^{-1}(h)$  (since  $g(x) = h$ ). Now, we are in one of the following three cases:

*Case 1:* We have  $x < m_h$ .

*Case 2:* We have  $x = m_h$ .

*Case 3:* We have  $x > m_h$ .

Let us first consider Case 1. In this case, we have  $x < m_h$ . Hence, statement  $(\mathbf{x}_2)$  yields  $f(x) = -h$  (since we know that statement  $(\mathbf{x}_2)$  holds). Hence,  $|f(x)| = |-h| = h$ . Hence,  $|f|(x) = |f(x)| = h = g(x)$ . Thus, the equality (14) holds in Case 1.

Let us next consider Case 2. In this case, we have  $x = m_h$ . But statement  $(\mathbf{x}_3)$  yields  $f(m_h) \in \{-h, +h\}$  (since we know that statement  $(\mathbf{x}_3)$  holds). Hence,

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But statement  $(\mathbf{x}_3)$  yields  $f(m_h) \in \{-h, +h\}$ . In other words, either  $f(m_h) = -h$  or  $f(m_h) = +h$ . Hence,  $f(m_h) = -h$  (since we cannot have  $f(m_h) = +h$ ). Combining  $f(x) = -h$  and  $f(m_h) = -h$ , we obtain  $f(m_h) = f(x) = -h$ .

But Condition **(iii)** from the proof of Proposition 2.37 holds. Applying this condition to  $m_h$  and  $x$  instead of  $x$  and  $y$ , we obtain  $\pi(m_h) > \pi(x)$ .

But  $m_h$  is the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum (by the definition of  $m_h$ ). Hence,  $\pi(m_h) \leq \pi(z)$  for each  $z \in g^{-1}(h)$ . Applying this to  $z = x$ , we obtain  $\pi(m_h) \leq \pi(x)$ . This contradicts  $\pi(m_h) > \pi(x)$ . This contradiction shows that our assumption was false. Hence,  $f(x) = +h$  is proven.

Now, forget that we fixed  $h$  and  $x$ . Thus, we have shown that for each  $h \in H$  and each  $x \in g^{-1}(h)$  satisfying  $x > m_h$ , we have  $f(x) = +h$ . In other words, statement  $(\mathbf{x}_4)$  holds.

<sup>27</sup>*Proof.* Assume that  $g(x) = 0$ . Thus,  $x \in g^{-1}(0)$ . Hence, statement  $(\mathbf{x}_1)$  yields  $f(x) = +0$  (since we know that statement  $(\mathbf{x}_1)$  holds). Thus,  $|f(x)| = |+0| = 0$ . Hence,  $|f|(x) = |f(x)| = 0 = g(x)$ . Thus, the equality (14) holds, qed.

<sup>28</sup>*Proof.* Assume that  $g(x) = \infty$ . Thus,  $x \in g^{-1}(\infty)$ . Hence, statement  $(\mathbf{x}_5)$  yields  $f(x) = -\infty$  (since we know that statement  $(\mathbf{x}_5)$  holds). Thus,  $|f(x)| = |-\infty| = \infty$ . Hence,  $|f|(x) = |f(x)| = \infty = g(x)$ . Thus, the equality (14) holds, qed.

$|f(m_h)| = h$ . Hence,  $|f|(x) = \left| f \left( \underbrace{x}_{=m_h} \right) \right| = |f(m_h)| = h = g(x)$ . Thus, the equality (14) holds in Case 2.

Let us finally consider Case 3. In this case, we have  $x > m_h$ . Hence, statement  $(\mathbf{x}_4)$  yields  $f(x) = +h$  (since we know that statement  $(\mathbf{x}_4)$  holds). Hence,  $|f(x)| = |+h| = h$ . Hence,  $|f|(x) = |f(x)| = h = g(x)$ . Thus, the equality (14) holds in Case 3.

We have now shown that the equality (14) holds in each of the three Cases 1, 2 and 3. Thus, (14) always holds. This completes the proof of (14).]

The equality (14) shows that  $|f| = g$ . It remains to prove that  $f \in \mathcal{E}([n], \pi)$ .

Consider the three Conditions **(i)**, **(ii)** and **(iii)** from the proof of Proposition 2.37. Now, let  $x$  and  $y$  be two elements of  $[n]$  satisfying  $x < y$ . We claim that the following three conditions hold:

- (i)** We have  $f(x) \preceq f(y)$ .
- (ii)** If  $f(x) = f(y) = +h$  for some  $h \in \mathcal{N}$ , then  $\pi(x) < \pi(y)$ .
- (iii)** If  $f(x) = f(y) = -h$  for some  $h \in \mathcal{N}$ , then  $\pi(x) > \pi(y)$ .

[*Proof of Condition (i):* Assume the contrary. Thus, we don't have  $f(x) \preceq f(y)$ .

But  $|f| = g$ , so that  $|f|(x) = g(x)$ . But the definition of  $|f|$  yields  $|f|(x) = |f(x)|$ . Hence,  $|f(x)| = |f|(x) = g(x)$ . Similarly,  $|f(y)| = g(y)$ . But the map  $g$  is weakly increasing; thus,  $g(x) \preceq g(y)$  (since  $x < y$ ). Hence,  $|f(x)| = g(x) \preceq g(y) = |f(y)|$ .

A look back at the definition of the total order on  $\mathcal{Z}$  reveals the following: If two elements  $\alpha$  and  $\beta$  of  $\mathcal{Z}$  satisfy  $|\alpha| \preceq |\beta|$  but not  $\alpha \preceq \beta$ , then there must exist some  $h \in \{1, 2, 3, \dots\}$  such that  $\alpha = +h$  and  $\beta = -h$ . Applying this to  $\alpha = f(x)$  and  $\beta = f(y)$ , we conclude that there must exist some  $h \in \{1, 2, 3, \dots\}$  such that  $f(x) = +h$  and  $f(y) = -h$  (because the elements  $f(x)$  and  $f(y)$  of  $\mathcal{Z}$  satisfy  $|f(x)| \preceq |f(y)|$  but not  $f(x) \preceq f(y)$ ). Consider this  $h$ . We have

$g(x) = \left| \underbrace{f(x)}_{=+h} \right| = |+h| = h$ , so that  $h = g \left( \underbrace{x}_{\in [n]} \right) \in g([n])$ . Combining this with

$h \in \{1, 2, 3, \dots\}$ , we obtain  $h \in g([n]) \cap \{1, 2, 3, \dots\} = H$ . Also,  $x \in g^{-1}(h)$  (since  $g(x) = h$ ). If we had  $x < m_h$ , then statement  $(\mathbf{x}_2)$  would yield  $f(x) = -h$ , which would contradict  $f(x) = +h \neq -h$ . Hence, we cannot have  $x < m_h$ . Thus, we

have  $x \geq m_h$ . But  $x < y$ , so that  $y > x \geq m_h$ . Also,  $g(y) = \left| \underbrace{f(y)}_{=-h} \right| = |-h| = h$ ,

so that  $y \in g^{-1}(h)$ . Hence, statement  $(\mathbf{x}_4)$  (applied to  $y$  instead of  $x$ ) yields  $f(y) = +h$ . This contradicts  $f(y) = -h \neq +h$ . This contradiction shows that our assumption was false. Hence, Condition **(i)** is proven.]



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[*Proof of Condition (ii)*]: Let  $f(x) = f(y) = +h$  for some  $h \in \mathcal{N}$ . We must show that  $\pi(x) < \pi(y)$ .

We have  $+h = f(x) \in \mathcal{Z}$ . Thus, we cannot have  $h = \infty$  (since  $h = \infty$  would lead to  $+h = +\infty \notin \mathcal{Z}$ , which would contradict  $+h \in \mathcal{Z}$ ). In other words,  $h \neq \infty$ . Combining this with  $h \in \mathcal{N}$ , we obtain  $h \in \mathcal{N} \setminus \{\infty\}$ .

The definition of  $|f|$  yields  $|f|(x) = |f(x)|$ . But  $g = |f|$ . Hence,  $g(x) = |f|(x) = \left| \underbrace{f(x)}_{=+h} \right| = |+h| = h$ , so that  $x \in g^{-1}(h)$ . Similarly,  $y \in g^{-1}(h)$ .

If  $h = 0$ , then it is easy to see that  $\pi(x) < \pi(y)$  <sup>29</sup>. Hence, for the rest of this proof of  $\pi(x) < \pi(y)$ , we WLOG assume that we don't have  $h = 0$ . Thus,  $h \neq 0$ . Combining this with  $h \in \mathcal{N} \setminus \{\infty\}$ , we obtain  $h \in (\mathcal{N} \setminus \{\infty\}) \setminus \{0\} = \mathcal{N} \setminus \{0, \infty\} = \{1, 2, 3, \dots\}$ .

Combining this with  $h = g\left(\underbrace{x}_{\in [n]}\right) \in g([n])$ , we obtain  $h \in g([n]) \cap \{1, 2, 3, \dots\} =$

$H$ . Also,  $x \in g^{-1}(h)$ . Hence, if we had  $x < m_h$ , then statement  $(\mathbf{x}_2)$  would yield  $f(x) = -h$ , which would contradict  $f(x) = +h \neq -h$ . Therefore, we cannot have  $x < m_h$ . Hence, we have  $x \geq m_h$ . In other words,  $x \in \{s \in g^{-1}(h) \mid s \geq m_h\}$  (since  $x \in g^{-1}(h)$ ). Similarly,  $y \in \{s \in g^{-1}(h) \mid s \geq m_h\}$ .

But  $h \in g([n]) \cap \{1, 2, 3, \dots\}$ . Thus, the map  $\pi|_{g^{-1}(h)}$  is V-shaped (since Condition (ii') from Definition 2.36 holds). In other words, there exists some  $t \in g^{-1}(h)$  such that the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \leq t\}}$  is strictly decreasing while the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \geq t\}}$  is strictly increasing. Clearly, this  $t$  must be the unique element  $\mu$  of  $g^{-1}(h)$  for which  $(\pi|_{g^{-1}(h)})(\mu)$  is minimum. In other words, this  $t$  must be the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum (since  $(\pi|_{g^{-1}(h)})(\mu) = \pi(\mu)$  for each  $\mu \in g^{-1}(h)$ ). In other words, this  $t$  must be  $m_h$  (because  $m_h$  was defined to be the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum). Thus, we conclude that the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}$  is strictly decreasing while the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \geq m_h\}}$  is strictly increasing.

In particular, the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \geq m_h\}}$  is strictly increasing. In other words, the map  $\pi|_{\{s \in g^{-1}(h) \mid s \geq m_h\}}$  is strictly increasing (since

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<sup>29</sup>*Proof.* Assume that  $h = 0$ . Now, recall that  $x \in g^{-1}(h)$ . In other words,  $x \in g^{-1}(0)$  (since  $h = 0$ ). Similarly,  $y \in g^{-1}(0)$ . But the map  $\pi|_{g^{-1}(0)}$  is strictly increasing (since Condition (i') from Definition 2.36 holds). Hence,  $(\pi|_{g^{-1}(0)})(x) < (\pi|_{g^{-1}(0)})(y)$  (because  $x < y$  and because both  $x$  and  $y$  belong to  $g^{-1}(0)$ ). Thus,  $\pi(x) = (\pi|_{g^{-1}(0)})(x) < (\pi|_{g^{-1}(0)})(y) = \pi(y)$ , qed.

$(\pi |_{g^{-1}(h)}) |_{\{s \in g^{-1}(h) \mid s \geq m_h\}} = \pi |_{\{s \in g^{-1}(h) \mid s \geq m_h\}}$ ). Since both  $x$  and  $y$  belong to the set  $\{s \in g^{-1}(h) \mid s \geq m_h\}$ , we thus conclude that

$$\left(\pi |_{\{s \in g^{-1}(h) \mid s \geq m_h\}}\right)(x) < \left(\pi |_{\{s \in g^{-1}(h) \mid s \geq m_h\}}\right)(y)$$

(since  $x < y$ ). Thus,

$$\pi(x) = \left(\pi |_{\{s \in g^{-1}(h) \mid s \geq m_h\}}\right)(x) < \left(\pi |_{\{s \in g^{-1}(h) \mid s \geq m_h\}}\right)(y) = \pi(y).$$

This proves Condition (ii).]

[*Proof of Condition (iii)*: Let  $f(x) = f(y) = -h$  for some  $h \in \mathcal{N}$ . We must show that  $\pi(x) > \pi(y)$ .

We have  $-h = f(x) \in \mathcal{Z}$ . Thus, we cannot have  $h = 0$  (since  $h = 0$  would lead to  $-h = -0 \notin \mathcal{Z}$ , which would contradict  $-h \in \mathcal{Z}$ ). In other words,  $h \neq 0$ . Combining this with  $h \in \mathcal{N}$ , we obtain  $h \in \mathcal{N} \setminus \{0\}$ .

The definition of  $|f|$  yields  $|f|(x) = |f(x)|$ . But  $g = |f|$ . Hence,  $g(x) = |f|(x) = \left| \underbrace{f(x)}_{=-h} \right| = |-h| = h$ , so that  $x \in g^{-1}(h)$ . Similarly,  $y \in g^{-1}(h)$ .

If  $h = \infty$ , then it is easy to see that  $\pi(x) > \pi(y)$ <sup>30</sup>. Hence, for the rest of this proof of  $\pi(x) > \pi(y)$ , we WLOG assume that we don't have  $h = \infty$ . Thus,  $h \neq \infty$ . Combining this with  $h \in \mathcal{N} \setminus \{0\}$ , we obtain  $h \in (\mathcal{N} \setminus \{0\}) \setminus \{\infty\} = \mathcal{N} \setminus \{0, \infty\} = \{1, 2, 3, \dots\}$ .

Combining this with  $h = g\left(\underbrace{x}_{\in [n]}\right) \in g([n])$ , we obtain  $h \in g([n]) \cap \{1, 2, 3, \dots\} =$

$H$ . Also,  $x \in g^{-1}(h)$ . Hence, if we had  $x > m_h$ , then statement (x<sub>4</sub>) would yield  $f(x) = +h$ , which would contradict  $f(x) = -h \neq +h$ . Therefore, we cannot have  $x > m_h$ . Hence, we have  $x \leq m_h$ . In other words,  $x \in \{s \in g^{-1}(h) \mid s \leq m_h\}$  (since  $x \in g^{-1}(h)$ ). Similarly,  $y \in \{s \in g^{-1}(h) \mid s \leq m_h\}$ .

But  $h \in g([n]) \cap \{1, 2, 3, \dots\}$ . Thus, the map  $\pi |_{g^{-1}(h)}$  is V-shaped (since Condition (ii') from Definition 2.36 holds). In other words, there exists some  $t \in g^{-1}(h)$  such that the map  $(\pi |_{g^{-1}(h)}) |_{\{s \in g^{-1}(h) \mid s \leq t\}}$  is strictly decreasing while the map  $(\pi |_{g^{-1}(h)}) |_{\{s \in g^{-1}(h) \mid s \geq t\}}$  is strictly increasing. Clearly, this  $t$  must be the unique element  $\mu$  of  $g^{-1}(h)$  for which  $(\pi |_{g^{-1}(h)})(\mu)$  is minimum. In other words, this  $t$  must be the unique element  $\mu$  of  $g^{-1}(h)$  for which

<sup>30</sup>*Proof.* Assume that  $h = \infty$ . Now, recall that  $x \in g^{-1}(h)$ . In other words,  $x \in g^{-1}(\infty)$  (since  $h = \infty$ ). Similarly,  $y \in g^{-1}(\infty)$ . But the map  $\pi |_{g^{-1}(\infty)}$  is strictly decreasing (since Condition (iii') from Definition 2.36 holds). Hence,  $(\pi |_{g^{-1}(\infty)})(x) > (\pi |_{g^{-1}(\infty)})(y)$  (because  $x < y$  and because both  $x$  and  $y$  belong to  $g^{-1}(\infty)$ ). Thus,  $\pi(x) = (\pi |_{g^{-1}(\infty)})(x) > (\pi |_{g^{-1}(\infty)})(y) = \pi(y)$ , qed.

$\pi(\mu)$  is minimum (since  $(\pi|_{g^{-1}(h)})(\mu) = \pi(\mu)$  for each  $\mu \in g^{-1}(h)$ ). In other words, this  $t$  must be  $m_h$  (because  $m_h$  was defined to be the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum). Thus, we conclude that the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}$  is strictly decreasing while the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \geq m_h\}}$  is strictly increasing.

In particular, the map  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}$  is strictly decreasing. In other words, the map  $\pi|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}$  is strictly decreasing (since  $(\pi|_{g^{-1}(h)})|_{\{s \in g^{-1}(h) \mid s \leq m_h\}} = \pi|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}$ ). Since both  $x$  and  $y$  belong to the set  $\{s \in g^{-1}(h) \mid s \leq m_h\}$ , we thus conclude that

$$\left(\pi|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}\right)(x) > \left(\pi|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}\right)(y)$$

(since  $x < y$ ). Thus,

$$\pi(x) = \left(\pi|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}\right)(x) > \left(\pi|_{\{s \in g^{-1}(h) \mid s \leq m_h\}}\right)(y) = \pi(y).$$

This proves Condition (iii).]

Now, forget that we fixed  $x$  and  $y$ . We thus have shown that for all  $x < y$  in  $[n]$ , the conditions (i), (ii) and (iii) stated above hold. Thus,  $f$  is a  $\mathcal{Z}$ -enriched  $([n], \pi)$ -partition (by the definition of a  $\mathcal{Z}$ -enriched  $([n], \pi)$ -partition). In other words,  $f \in \mathcal{E}([n], \pi)$ . Hence, we have shown that ( $f \in \mathcal{E}([n], \pi)$  and  $|f| = g$ ). This proves Claim 2.]

Combining Claim 1 with Claim 2, we conclude that ( $f \in \mathcal{E}([n], \pi)$  and  $|f| = g$ ) if and only if the five statements (x<sub>1</sub>), (x<sub>2</sub>), (x<sub>3</sub>), (x<sub>4</sub>) and (x<sub>5</sub>) hold. This proves Proposition 2.38 (b).  $\square$

**Proposition 2.39.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be any  $n$ -permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = \sum_{\substack{g: [n] \rightarrow \mathcal{N} \\ \text{is } \pi\text{-amenable}}} 2^{|g([n]) \cap \{1, 2, 3, \dots\}|} \mathbf{x}_g.$$

*Proof of Proposition 2.39.* We shall show the following observation:

*Observation 1:* If  $g: [n] \rightarrow \mathcal{N}$  is a  $\pi$ -amenable map, then there exist precisely  $2^{|g([n]) \cap \{1, 2, 3, \dots\}|}$  maps  $f \in \mathcal{E}([n], \pi)$  satisfying  $|f| = g$ .

[*Proof of Observation 1:* Let  $g: [n] \rightarrow \mathcal{N}$  be a  $\pi$ -amenable map. Let  $H$  be the set  $g([n]) \cap \{1, 2, 3, \dots\}$ . For each  $h \in H$ , we let  $m_h$  be the unique element  $\mu$  of  $g^{-1}(h)$  for which  $\pi(\mu)$  is minimum. Proposition 2.38 (a) shows that the elements  $m_h$  for all  $h \in H$  are well-defined and distinct.

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Proposition 2.38 (b) shows that a map  $f : [n] \rightarrow \mathcal{Z}$  satisfies ( $f \in \mathcal{E}([n], \pi)$  and  $|f| = g$ ) if and only if the five statements  $(\mathbf{x}_1)$ ,  $(\mathbf{x}_2)$ ,  $(\mathbf{x}_3)$ ,  $(\mathbf{x}_4)$  and  $(\mathbf{x}_5)$  from Proposition 2.38 (b) hold. Thus, we can construct every map  $f : [n] \rightarrow \mathcal{Z}$  that satisfies ( $f \in \mathcal{E}([n], \pi)$  and  $|f| = g$ ) by the following algorithm:

**Step 1:** For each  $x \in g^{-1}(0)$ , set  $f(x) = +0$ . (This is the only option, because we want statement  $(\mathbf{x}_1)$  to hold.)

**Step 2:** For each  $h \in H$  and each  $x \in g^{-1}(h)$ , set the value  $f(x)$  as follows:

- If  $x < m_h$ , then set  $f(x) = -h$ . (This is the only option, because we want statement  $(\mathbf{x}_2)$  to hold.)
- If  $x = m_h$ , then set  $f(x)$  to be either  $-h$  or  $+h$ . (These two options are the only options, because we want statement  $(\mathbf{x}_3)$  to hold. Notice that  $m_h \in g^{-1}(h)$  (by the definition of  $m_h$ ); therefore, this step ensures that  $f(m_h) \in \{-h, +h\}$  for each  $h \in H$ , and therefore statement  $(\mathbf{x}_3)$  holds indeed.)
- If  $x > m_h$ , then set  $f(x) = +h$ . (This is the only option, because we want statement  $(\mathbf{x}_4)$  to hold.)

**Step 3:** For each  $x \in g^{-1}(\infty)$ , set  $f(x) = -\infty$ . (This is the only option, because we want statement  $(\mathbf{x}_5)$  to hold.)

This algorithm indeed constructs a well-defined map  $f : [n] \rightarrow \mathcal{Z}$  (because for each  $x \in [n]$ , the value  $f(x)$  is set exactly once during the above algorithm<sup>31</sup>).

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<sup>31</sup>*Proof.* Let  $x \in [n]$  be arbitrary. Note that  $H = g([n]) \cap \{1, 2, 3, \dots\} \subseteq \{1, 2, 3, \dots\}$ .

We have  $g(x) \in \mathcal{N} = \{0\} \cup \{\infty\} \cup \{1, 2, 3, \dots\}$ . Hence, we have either  $g(x) = 0$  or  $g(x) = \infty$  or  $g(x) \in \{1, 2, 3, \dots\}$ . In other words, we are in one of the following cases:

*Case 1:* We have  $g(x) = 0$ .

*Case 2:* We have  $g(x) = \infty$ .

*Case 3:* We have  $g(x) \in \{1, 2, 3, \dots\}$ .

Let us first consider Case 1. In this case, we have  $g(x) = 0$ . Thus,  $x \in g^{-1}(0)$ . Hence,  $f(x)$  is set during Step 1 of the above algorithm. If we had  $0 \in H$ , then we would have  $0 \in H \subseteq \{1, 2, 3, \dots\}$ , which would contradict  $0 \notin \{1, 2, 3, \dots\}$ . Thus, we cannot have  $0 \in H$ . Hence,  $0 \notin H$ . Hence,  $x \notin g^{-1}(h)$  for all  $h \in H$  (because if we had  $x \in g^{-1}(h)$  for some  $h \in H$ , then this  $h$  would satisfy  $g(x) = h \in H$ , which would contradict  $g(x) = 0 \notin H$ ). Therefore, Step 2 of the above algorithm does not set  $f(x)$ . Finally,  $x \notin g^{-1}(\infty)$  (since  $g(x) = 0 \neq \infty$ ). Hence, Step 3 of the above algorithm does not set  $f(x)$ . Altogether, we have now shown that  $f(x)$  is set during Step 1 of the algorithm, but neither Step 2 nor Step 3 sets  $f(x)$ . Thus, the value  $f(x)$  is set exactly once during the above algorithm. So we have proven our claim (that the value  $f(x)$  is set exactly once during the above algorithm) in Case 1.

Let us next consider Case 2. In this case, we have  $g(x) = \infty$ . Thus,  $x \in g^{-1}(\infty)$ . Hence,  $f(x)$  is set during Step 3 of the above algorithm. If we had  $\infty \in H$ , then we would have  $\infty \in H \subseteq \{1, 2, 3, \dots\}$ , which would contradict  $\infty \notin \{1, 2, 3, \dots\}$ . Thus, we cannot have  $\infty \in H$ . Hence,  $\infty \notin H$ . Hence,  $x \notin g^{-1}(h)$  for all  $h \in H$  (because if we had  $x \in g^{-1}(h)$  for some  $h \in H$ , then this  $h$  would satisfy  $g(x) = h \in H$ , which would contradict  $g(x) = \infty \notin H$ ). Therefore, Step 2 of the above algorithm does not set  $f(x)$ . Finally,  $x \notin g^{-1}(0)$  (since

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Moreover, this map  $f$  that it constructs satisfies ( $f \in \mathcal{E}([n], \pi)$  and  $|f| = g$ ) (since the five statements  $(\mathbf{x}_1)$ ,  $(\mathbf{x}_2)$ ,  $(\mathbf{x}_3)$ ,  $(\mathbf{x}_4)$  and  $(\mathbf{x}_5)$  from Proposition 2.38 **(b)** hold for this map  $f$ ). Therefore, the number of all maps  $f : [n] \rightarrow \mathcal{Z}$  satisfying ( $f \in \mathcal{E}([n], \pi)$  and  $|f| = g$ ) equals the number of ways we can perform the above algorithm. But the latter number is easy to compute: The only choices we have in the algorithm are the choices we make during Step 2 when  $x = m_h$  (since in all the other decisions, we have only one option); there are altogether  $|H|$  many of these choices (one for each  $h \in H$ ), and each choice involves exactly 2 options. Hence, the number of ways we can perform the above algorithm is  $2^{|H|}$ . Thus, the number of all maps  $f : [n] \rightarrow \mathcal{Z}$  satisfying ( $f \in \mathcal{E}([n], \pi)$  and  $|f| = g$ ) is  $2^{|H|}$ . In other words,

$$\begin{aligned} & (\text{the number of all maps } f : [n] \rightarrow \mathcal{Z} \text{ satisfying } (f \in \mathcal{E}([n], \pi) \text{ and } |f| = g)) \\ &= 2^{|H|}. \end{aligned}$$

Now,

$$\begin{aligned} & (\text{the number of all maps } f \in \mathcal{E}([n], \pi) \text{ satisfying } |f| = g) \\ &= (\text{the number of all maps } f : [n] \rightarrow \mathcal{Z} \text{ satisfying } (f \in \mathcal{E}([n], \pi) \text{ and } |f| = g)) \\ & \quad (\text{since each } f \in \mathcal{E}([n], \pi) \text{ is automatically a map } [n] \rightarrow \mathcal{Z}) \\ &= 2^{|H|} = 2^{|g([n]) \cap \{1, 2, 3, \dots\}|} \quad (\text{since } H = g([n]) \cap \{1, 2, 3, \dots\}). \end{aligned}$$

In other words, there exist precisely  $2^{|g([n]) \cap \{1, 2, 3, \dots\}|}$  maps  $f \in \mathcal{E}([n], \pi)$  satisfying  $|f| = g$ . This proves Observation 1.]

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$g(x) = \infty \neq 0$ ). Hence, Step 1 of the above algorithm does not set  $f(x)$ . Altogether, we have now shown that  $f(x)$  is set during Step 3 of the algorithm, but neither Step 2 nor Step 1 sets  $f(x)$ . Thus, the value  $f(x)$  is set exactly once during the above algorithm. So we have proven our claim (that the value  $f(x)$  is set exactly once during the above algorithm) in Case 2.

Let us next consider Case 3. In this case, we have  $g(x) \in \{1, 2, 3, \dots\}$ . Combining this

with  $g\left(\underbrace{x}_{\in [n]}\right) \in g([n])$ , we obtain  $g(x) \in g([n]) \cap \{1, 2, 3, \dots\} = H$ . Hence, there exists an

$h \in H$  such that  $x \in g^{-1}(h)$  (namely,  $h = g(x)$ ). Of course, this  $h$  is unique (because the requirement  $x \in g^{-1}(h)$  entails  $g(x) = h$ , which uniquely determines  $h$ ). Therefore, the value  $f(x)$  is set exactly once during Step 2 of the algorithm. Moreover, from  $g(x) \in \{1, 2, 3, \dots\}$ , we obtain  $g(x) \neq 0$ , so that  $x \notin g^{-1}(0)$ . Hence, Step 1 of the above algorithm does not set  $f(x)$ . Similarly, Step 3 of the above algorithm does not set  $f(x)$ . Altogether, we have now shown that  $f(x)$  is set exactly once during Step 2 of the algorithm, but neither Step 1 nor Step 3 sets  $f(x)$ . Thus, the value  $f(x)$  is set exactly once during the above algorithm. So we have proven our claim (that the value  $f(x)$  is set exactly once during the above algorithm) in Case 3.

We have now proven our claim (that the value  $f(x)$  is set exactly once during the above algorithm) in each of the three Cases 1, 2 and 3. Hence, this claim always holds. Qed.

Now, (13) yields

$$\begin{aligned}
\Gamma_{\mathcal{Z}}(\pi) &= \sum_{f \in \mathcal{E}([n], \pi)} \mathbf{x}_{|f|} = \sum_{\substack{g: [n] \rightarrow \mathcal{N} \\ \text{is } \pi\text{-amenable}}} \sum_{\substack{f \in \mathcal{E}([n], \pi); \\ |f|=g}} \underbrace{\mathbf{x}_{|f|}}_{=\mathbf{x}_g} \quad (\text{since } |f|=g) \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \\ \text{is } \pi\text{-amenable}}} \sum_{\substack{f \in \mathcal{E}([n], \pi); \\ |f|=g}} \mathbf{x}_g \quad (\text{by Proposition 2.37}) \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \\ \text{is } \pi\text{-amenable}}} \underbrace{\sum_{\substack{f \in \mathcal{E}([n], \pi); \\ |f|=g}} \mathbf{x}_g}_{=2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g} \quad (\text{by Observation 1}) \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \\ \text{is } \pi\text{-amenable}}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g.
\end{aligned}$$

This proves Proposition 2.39.  $\square$

Now, let us observe that if  $g : [n] \rightarrow \mathcal{N}$  is a weakly increasing map (for some  $n \in \mathbb{N}$ ), then the fibers of  $g$  (that is, the subsets  $g^{-1}(h)$  of  $[n]$  for various  $h \in \mathcal{N}$ ) are intervals of  $[n]$  (possibly empty). Of course, when these fibers are nonempty, they have smallest elements and largest elements. We shall next study these elements more closely.

**Definition 2.40.** Let  $n \in \mathbb{N}$ . Let  $g : [n] \rightarrow \mathcal{N}$  be any map. We define a subset  $\text{FE}(g)$  of  $[n]$  as follows:

$$\begin{aligned}
\text{FE}(g) &= \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\
&\quad \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}.
\end{aligned}$$

In other words,  $\text{FE}(g)$  is the set comprising the smallest elements of all nonempty fibers of  $g$  except for  $g^{-1}(0)$  as well as the largest elements of all nonempty fibers of  $g$  except for  $g^{-1}(\infty)$ . We shall refer to the elements of  $\text{FE}(g)$  as the *fiber-ends* of  $g$ .

**Lemma 2.41.** Let  $n \in \mathbb{N}$ . Let  $\Lambda \in \mathbf{L}_n$ . Then, there exists a weakly increasing map  $g : [n] \rightarrow \mathcal{N}$  such that  $\text{FE}(g) = (\Lambda \cup (\Lambda + 1)) \cap [n]$ .

*Proof of Lemma 2.41.* If  $n = 0$ , then Lemma 2.41 holds<sup>32</sup>. Hence, for the rest of this proof, we WLOG assume that we don't have  $n = 0$ . Hence,  $n$  is a positive integer. Thus,  $\mathbf{L}_n$  is the set of all nonempty lacunar subsets of  $[n]$  (by the definition of  $\mathbf{L}_n$ ). Therefore, from  $\Lambda \in \mathbf{L}_n$ , we conclude that  $\Lambda$  is a nonempty lacunar subset

<sup>32</sup>*Proof.* Assume that  $n = 0$ . Thus,  $\Lambda \in \mathbf{L}_n = \mathbf{L}_0$  (since  $n = 0$ ), so that  $\Lambda \in \mathbf{L}_0 = \{\emptyset\}$ , so that  $\Lambda = \emptyset$ . Now, there is only one map  $g : [n] \rightarrow \mathcal{N}$  (since  $[n] = [0] = \emptyset$ ), and this map  $g$  is weakly increasing and satisfies  $\text{FE}(g) = (\Lambda \cup (\Lambda + 1)) \cap [n]$  (since both  $\text{FE}(g)$  and  $(\Lambda \cup (\Lambda + 1)) \cap [n]$  are the empty set). Hence, there exists a weakly increasing map  $g : [n] \rightarrow \mathcal{N}$  such that  $\text{FE}(g) = (\Lambda \cup (\Lambda + 1)) \cap [n]$ . In other words, Lemma 2.41 holds, qed.

of  $[n]$ . Write this subset  $\Lambda$  in the form  $\Lambda = \{j_1 < j_2 < \cdots < j_k\}$ . Thus,  $k \geq 1$  (since  $\Lambda$  is nonempty).

Hence,  $j_1 \in \{j_1 < j_2 < \cdots < j_k\} = \Lambda \subseteq [n]$ , so that  $j_1 > 0$ . In other words,  $0 < j_1$ . Combining this with  $j_1 < j_2 < \cdots < j_k$ , we obtain  $0 < j_1 < j_2 < \cdots < j_k$ .

From  $\Lambda = \{j_1 < j_2 < \cdots < j_k\}$ , we obtain  $\max \Lambda = j_k$ . Thus,  $j_k = \max \Lambda \leq n$  (since  $\Lambda \subseteq [n]$ ).

We have  $\Lambda = \{j_1 < j_2 < \cdots < j_k\} = \{j_1, j_2, \dots, j_k\}$  and thus

$$\Lambda + 1 = \{j_1 + 1, j_2 + 1, \dots, j_k + 1\}$$

(by the definition of  $\Lambda + 1$ ).

We are now in one of the following two cases:

*Case 1:* We have  $n \notin \Lambda$ .

*Case 2:* We have  $n \in \Lambda$ .

Let us consider Case 1. In this case, we have  $n \notin \Lambda$ . Hence,  $\Lambda \subseteq [n - 1]$  (since  $\Lambda \subseteq [n]$ ). Therefore,  $\Lambda + 1 \subseteq \{2, 3, \dots, n\} \subseteq [n]$ . Now,

$$\underbrace{\Lambda}_{\subseteq [n]} \cup \underbrace{(\Lambda + 1)}_{\subseteq [n]} \subseteq [n] \cup [n] = [n],$$

so that  $\Lambda \cup (\Lambda + 1) = (\Lambda \cup (\Lambda + 1)) \cap [n]$ .

But  $j_k = \max \Lambda \leq n - 1$  (since  $\Lambda \subseteq [n - 1]$ ). Thus,  $j_k \leq n - 1 < n$ . Combining this with  $0 < j_1 < j_2 < \cdots < j_k$ , we obtain  $0 < j_1 < j_2 < \cdots < j_k < n$ .

Now, consider the map  $g : [n] \rightarrow \mathcal{N}$  defined by

$$g(x) = \begin{cases} (\text{the number of all } \lambda \in \Lambda \text{ such that } \lambda < x), & \text{if } x \leq j_k; \\ \infty, & \text{if } x > j_k \end{cases}$$

for each  $x \in [n]$ .

Thus,

$$(g(1), g(2), \dots, g(n)) = \left( \underbrace{0, 0, \dots, 0}_{j_1 \text{ entries}} \underbrace{1, 1, \dots, 1}_{j_2 - j_1 \text{ entries}} \underbrace{2, 2, \dots, 2}_{j_3 - j_2 \text{ entries}}, \dots, \underbrace{k - 1, k - 1, \dots, k - 1}_{j_k - j_{k-1} \text{ entries}} \underbrace{\infty, \infty, \dots, \infty}_{n - j_k \text{ entries}} \right).$$

The  $n$ -tuple on the right hand side of this equality consists of a block of 0's, followed by a block of 1's, followed by a block of 2's, and so on, all the way up to a block of  $(k - 1)$ 's, which is then followed by a block of  $\infty$ 's. Each of these blocks is nonempty (since  $0 < j_1 < j_2 < \cdots < j_k < n$ ).

Hence, the map  $g$  is weakly increasing, and its nonempty fibers are

$$\begin{aligned}
g^{-1}(0) &= \{1, 2, \dots, j_1\}, \\
g^{-1}(1) &= \{j_1 + 1, j_1 + 2, \dots, j_2\}, \\
g^{-1}(2) &= \{j_2 + 1, j_2 + 2, \dots, j_3\}, \\
&\vdots \\
g^{-1}(k-1) &= \{j_{k-1} + 1, j_{k-1} + 2, \dots, j_k\}, \\
g^{-1}(\infty) &= \{j_k + 1, j_k + 2, \dots, n\}.
\end{aligned}$$

Thus, the definition of FE ( $g$ ) yields

$$\begin{aligned}
&\text{FE}(g) \\
&= \underbrace{\left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\}}_{= \{\min(g^{-1}(1)), \min(g^{-1}(2)), \dots, \min(g^{-1}(k-1)), \min(g^{-1}(\infty))\}} \\
&\quad \cup \underbrace{\left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}}_{= \{\max(g^{-1}(0)), \max(g^{-1}(1)), \max(g^{-1}(2)), \dots, \max(g^{-1}(k-1))\}} \\
&= \underbrace{\left\{ \min(g^{-1}(1)), \min(g^{-1}(2)), \dots, \min(g^{-1}(k-1)), \min(g^{-1}(\infty)) \right\}}_{\substack{= \{j_1+1, j_2+1, \dots, j_{k-1}+1, j_k+1\} \\ \text{(since } \min(g^{-1}(h)) = j_h+1 \text{ for each } h \in \{1, 2, \dots, k-1\}, \\ \text{and since } \min(g^{-1}(\infty)) = j_k+1)}} \\
&\quad \cup \underbrace{\left\{ \max(g^{-1}(0)), \max(g^{-1}(1)), \max(g^{-1}(2)), \dots, \max(g^{-1}(k-1)) \right\}}_{\substack{= \{j_1, j_2, \dots, j_k\} \\ \text{(since } \max(g^{-1}(h-1)) = j_h \text{ for each } h \in \{1, 2, \dots, k\})}} \\
&= \underbrace{\{j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1, j_k + 1\}}_{\substack{= \{j_1+1, j_2+1, \dots, j_k+1\} \\ = \Lambda+1}} \cup \underbrace{\{j_1, j_2, \dots, j_k\}}_{= \Lambda} = (\Lambda + 1) \cup \Lambda \\
&= \Lambda \cup (\Lambda + 1) = (\Lambda \cup (\Lambda + 1)) \cap [n].
\end{aligned}$$

Altogether, we have now shown that our map  $g : [n] \rightarrow \mathcal{N}$  is weakly increasing and satisfies  $\text{FE}(g) = (\Lambda \cup (\Lambda + 1)) \cap [n]$ . Hence, such a map  $g$  exists. Thus, Lemma 2.41 is proven in Case 1.

Let us consider Case 2. In this case, we have  $n \in \Lambda$ . Hence,  $n \leq \max \Lambda = j_k$ . Combined with  $j_k \leq n$ , this yields  $n = j_k$ .

We have  $\Lambda \subseteq [n]$  and thus  $\Lambda \cap [n] = \Lambda$ . Also,

$$(\Lambda + 1) \cap [n] = \{j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1\} \quad (15)$$

33.

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<sup>33</sup>*Proof.* We have  $j_1 < j_2 < \dots < j_k$  and thus  $j_1 + 1 < j_2 + 1 < \dots < j_k + 1$ . In other words,



Now, consider the map  $g : [n] \rightarrow \mathcal{N}$  defined by

$$g(x) = (\text{the number of all } \lambda \in \Lambda \text{ such that } \lambda < x) \\ \text{for each } x \in [n].$$

Thus,

$$(g(1), g(2), \dots, g(n)) \\ = \left( \underbrace{0, 0, \dots, 0}_{j_1 \text{ entries}}, \underbrace{1, 1, \dots, 1}_{j_2 - j_1 \text{ entries}}, \underbrace{2, 2, \dots, 2}_{j_3 - j_2 \text{ entries}}, \dots, \underbrace{k-1, k-1, \dots, k-1}_{j_k - j_{k-1} \text{ entries}} \right).$$

<sup>34</sup> The  $n$ -tuple on the right hand side of this equality consists of a block of 0's, followed by a block of 1's, followed by a block of 2's, and so on, all the way up to a block of  $(k-1)$ 's. Each of these blocks is nonempty (since  $0 < j_1 < j_2 < \dots < j_k$ ).

Hence, the map  $g$  is weakly increasing, and its nonempty fibers are

$$g^{-1}(0) = \{1, 2, \dots, j_1\}, \\ g^{-1}(1) = \{j_1 + 1, j_1 + 2, \dots, j_2\}, \\ g^{-1}(2) = \{j_2 + 1, j_2 + 2, \dots, j_3\}, \\ \vdots \\ g^{-1}(k-1) = \{j_{k-1} + 1, j_{k-1} + 2, \dots, j_k\}.$$

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$j_1 + 1 < j_2 + 1 < \dots < j_{k-1} + 1 < j_k + 1$ . In view of  $j_k = n$ , this rewrites as  $j_1 + 1 < j_2 + 1 < \dots < j_{k-1} + 1 < n + 1$ . In other words,  $j_1 + 1 < j_2 + 1 < \dots < j_{k-1} + 1 \leq n$  (since an integer is  $< n + 1$  if and only if it is  $\leq n$ ). Therefore, the  $k-1$  numbers  $j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1$  all belong to the set  $[n]$ . In other words,  $\{j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1\} \subseteq [n]$ .

On the other hand,  $\underbrace{j_k}_{=n} + 1 = n + 1 \notin [n]$  and thus  $\{j_k + 1\} \cap [n] = \emptyset$ . Now,

$$\begin{aligned} \underbrace{(\Lambda + 1)}_{=\{j_1+1, j_2+1, \dots, j_k+1\}} \cap [n] &= (\{j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1\} \cup \{j_k + 1\}) \cap [n] \\ &= \underbrace{(\{j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1\} \cap [n])}_{=\{j_1+1, j_2+1, \dots, j_{k-1}+1\} \text{ (since } \{j_1+1, j_2+1, \dots, j_{k-1}+1\} \subseteq [n])}} \cup \underbrace{(\{j_k + 1\} \cap [n])}_{=\emptyset} \\ &= \{j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1\} \cup \emptyset \\ &= \{j_1 + 1, j_2 + 1, \dots, j_{k-1} + 1\}. \end{aligned}$$

<sup>34</sup> There are no entries beyond the  $\underbrace{k-1, k-1, \dots, k-1}_{j_k - j_{k-1} \text{ entries}}$  block in this tuple, because  $j_k = n$ .

Thus, the definition of FE ( $g$ ) yields

$$\begin{aligned}
& \text{FE}(g) \\
&= \underbrace{\left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\}}_{=\{\min(g^{-1}(1)), \min(g^{-1}(2)), \dots, \min(g^{-1}(k-1))\}} \\
&\quad \cup \underbrace{\left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}}_{=\{\max(g^{-1}(0)), \max(g^{-1}(1)), \max(g^{-1}(2)), \dots, \max(g^{-1}(k-1))\}} \\
&= \underbrace{\left\{ \min(g^{-1}(1)), \min(g^{-1}(2)), \dots, \min(g^{-1}(k-1)) \right\}}_{=\{j_1+1, j_2+1, \dots, j_{k-1}+1\}} \\
&\quad \underbrace{\left\{ \max(g^{-1}(0)), \max(g^{-1}(1)), \max(g^{-1}(2)), \dots, \max(g^{-1}(k-1)) \right\}}_{=\{j_1, j_2, \dots, j_k\}} \\
&\quad \text{(since } \min(g^{-1}(h))=j_h+1 \text{ for each } h \in \{1, 2, \dots, k-1\}) \\
&\quad \text{(since } \max(g^{-1}(h-1))=j_h \text{ for each } h \in \{1, 2, \dots, k\}) \\
&= \underbrace{\{j_1+1, j_2+1, \dots, j_{k-1}+1\}}_{=\underbrace{(\Lambda+1) \cap [n]}_{\text{(by (15))}}} \cup \underbrace{\{j_1, j_2, \dots, j_k\}}_{=\underbrace{\Lambda \cap [n]}_{\text{(since } \Lambda \subseteq [n])}} = ((\Lambda+1) \cap [n]) \cup (\Lambda \cap [n]) \\
&= (\Lambda \cap [n]) \cup ((\Lambda+1) \cap [n]) = (\Lambda \cup (\Lambda+1)) \cap [n].
\end{aligned}$$

Altogether, we have now shown that our map  $g : [n] \rightarrow \mathcal{N}$  is weakly increasing and satisfies  $\text{FE}(g) = (\Lambda \cup (\Lambda+1)) \cap [n]$ . Hence, such a map  $g$  exists. Thus, Lemma 2.41 is proven in Case 2.

We have now proven Lemma 2.41 in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, we thus conclude that Lemma 2.41 always holds.  $\square$

**Lemma 2.42.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be an  $n$ -permutation. Let  $S$  be a nonempty interval of the totally ordered set  $[n]$  such that the map  $\pi|_S$  is V-shaped. Then,  $S \cap \text{Epk } \pi \subseteq \{\min S, \max S\}$ .

*Proof of Lemma 2.42.* Let  $j \in S \cap \text{Epk } \pi$ . We are going to show that  $j \in \{\min S, \max S\}$ .

Indeed, assume the contrary. Thus,  $j \notin \{\min S, \max S\}$ . Hence,  $j \neq \min S$  and  $j \neq \max S$ .

But  $j \in S \cap \text{Epk } \pi \subseteq S$ . Thus,  $j \geq \min S$  and  $j \leq \max S$ . Combining  $j \geq \min S$  with  $j \neq \min S$ , we obtain  $j > \min S$ . Combining  $j \leq \max S$  with  $j \neq \max S$ , we obtain  $j < \max S$ .

Recall the following basic fact: If  $T$  is a nonempty interval of  $[n]$ , and if  $t \in T$  satisfies  $t > \min T$ , then  $t-1 \in T$ . Applying this to  $T = S$  and  $t = j$ , we conclude that  $j-1 \in S$  (since  $j > \min S$ ).

Recall the following basic fact: If  $T$  is a nonempty interval of  $[n]$ , and if  $t \in T$  satisfies  $t < \max T$ , then  $t+1 \in T$ . Applying this to  $T = S$  and  $t = j$ , we conclude that  $j+1 \in S$  (since  $j < \max S$ ).

---

Recall that the map  $\pi|_S$  is V-shaped. According to the definition of “V-shaped”, this means the following: There exists some  $t \in S$  such that the map  $(\pi|_S)|_{\{s \in S \mid s \leq t\}}$  is strictly decreasing while the map  $(\pi|_S)|_{\{s \in S \mid s \geq t\}}$  is strictly increasing. Consider this  $t$ .

The map  $(\pi|_S)|_{\{s \in S \mid s \leq t\}}$  is strictly decreasing. In other words, the map  $\pi|_{\{s \in S \mid s \leq t\}}$  is strictly decreasing (since  $(\pi|_S)|_{\{s \in S \mid s \leq t\}} = \pi|_{\{s \in S \mid s \leq t\}}$ ).

The map  $(\pi|_S)|_{\{s \in S \mid s \geq t\}}$  is strictly increasing. In other words, the map  $\pi|_{\{s \in S \mid s \geq t\}}$  is strictly increasing (since  $(\pi|_S)|_{\{s \in S \mid s \geq t\}} = \pi|_{\{s \in S \mid s \geq t\}}$ ).

We have  $j \in S \cap \text{Epk } \pi \subseteq \text{Epk } \pi$ ; in other words,  $j$  is an exterior peak of  $\pi$  (by the definition of  $\text{Epk } \pi$ ). In other words,  $j$  is an element of  $[n]$  and satisfies  $\pi_{j-1} < \pi_j > \pi_{j+1}$  (by the definition of an exterior peak).

We are in one of the following two cases:

*Case 1:* We have  $j < t$ .

*Case 2:* We have  $j \geq t$ .

Let us first consider Case 1. In this case, we have  $j < t$ . Hence,  $j \leq t$  and  $j-1 \leq j \leq t$ . Thus,  $j-1 \in \{s \in S \mid s \leq t\}$  (since  $j-1 \in S$ ). Also,  $j \leq t$ . Hence,  $j \in \{s \in S \mid s \leq t\}$  (since  $j \in S$ ).

Now, we know that  $j-1$  and  $j$  are two elements of the set  $\{s \in S \mid s \leq t\}$ , and satisfy  $j-1 < j$ . Hence,

$$\left(\pi|_{\{s \in S \mid s \leq t\}}\right)(j-1) > \left(\pi|_{\{s \in S \mid s \leq t\}}\right)(j)$$

(since the map  $\pi|_{\{s \in S \mid s \leq t\}}$  is strictly decreasing). Thus,

$$\pi_{j-1} = \pi(j-1) = \left(\pi|_{\{s \in S \mid s \leq t\}}\right)(j-1) > \left(\pi|_{\{s \in S \mid s \leq t\}}\right)(j) = \pi(j) = \pi_j.$$

This contradicts  $\pi_{j-1} < \pi_j$ . Thus, we have obtained a contradiction in Case 1.

Let us now consider Case 2. In this case, we have  $j \geq t$ . Hence,  $j+1 \geq j \geq t$ . Thus,  $j+1 \in \{s \in S \mid s \geq t\}$  (since  $j+1 \in S$ ). Also,  $j \geq t$ . Hence,  $j \in \{s \in S \mid s \geq t\}$  (since  $j \in S$ ).

Now, we know that  $j$  and  $j+1$  are two elements of the set  $\{s \in S \mid s \geq t\}$ , and satisfy  $j < j+1$ . Hence,

$$\left(\pi|_{\{s \in S \mid s \geq t\}}\right)(j) < \left(\pi|_{\{s \in S \mid s \geq t\}}\right)(j+1)$$

(since the map  $\pi|_{\{s \in S \mid s \geq t\}}$  is strictly increasing). Thus,

$$\pi_j = \pi(j) = \left(\pi|_{\{s \in S \mid s \geq t\}}\right)(j) < \left(\pi|_{\{s \in S \mid s \geq t\}}\right)(j+1) = \pi(j+1) = \pi_{j+1}.$$

This contradicts  $\pi_j > \pi_{j+1}$ . Thus, we have obtained a contradiction in Case 2.

We have now obtained a contradiction in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, we thus conclude that we always have a contradiction. This contradiction shows that our assumption was wrong. Thus,  $j \in \{\min S, \max S\}$  is proven.

Now, forget that we fixed  $j$ . We thus have shown that  $j \in \{\min S, \max S\}$  for each  $j \in S \cap \text{Epk } \pi$ . In other words,  $S \cap \text{Epk } \pi \subseteq \{\min S, \max S\}$ . This proves Lemma 2.42.  $\square$

**Lemma 2.43.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be an  $n$ -permutation. Let  $S$  be a nonempty interval of the totally ordered set  $[n]$ .

(a) If  $1 \in S$  and  $S \cap \text{Epk } \pi \subseteq \{\max S\}$ , then the map  $\pi|_S$  is strictly increasing.

(b) If  $n \in S$  and  $S \cap \text{Epk } \pi \subseteq \{\min S\}$ , then the map  $\pi|_S$  is strictly decreasing.

(c) If  $S \cap \text{Epk } \pi \subseteq \{\min S, \max S\}$ , then the map  $\pi|_S$  is V-shaped.

*Proof of Lemma 2.43.* Set  $\pi_0 = 0$  and  $\pi_{n+1} = 0$ . Thus,  $\pi_i \in \mathbb{N}$  is well-defined for each  $i \in \{0, 1, \dots, n+1\}$ . The exterior peaks of  $\pi$  are the elements  $i \in [n]$  satisfying  $\pi_{i-1} < \pi_i > \pi_{i+1}$  (by the definition of an “exterior peak”).

Note that  $\pi$  is an  $n$ -permutation; thus, the entries of the word  $\pi$  are distinct. (But of course,  $\pi_0$  and  $\pi_{n+1}$  don’t count as entries of  $\pi$ .)

We now claim the following:

*Observation 1:* Let  $i$  and  $j$  be two elements of  $[n]$  satisfying  $i \leq j+1$  and  $\pi_{i-1} \leq \pi_i$  and  $\pi_j \geq \pi_{j+1}$ . Then,

$$\{i, i+1, \dots, j\} \cap \text{Epk } \pi \neq \emptyset.$$

[*Proof of Observation 1:* We have  $\pi_{i-1} \neq \pi_i$  <sup>35</sup>.

Combining  $\pi_{i-1} \leq \pi_i$  with  $\pi_{i-1} \neq \pi_i$ , we obtain  $\pi_{i-1} < \pi_i$ . If we had  $i = j+1$ , then we would have  $j = i-1$  and thus  $\pi_j = \pi_{i-1} < \pi_i = \pi_{j+1}$  (since  $i = j+1$ ); but this would contradict  $\pi_j \geq \pi_{j+1}$ . Hence, we cannot have  $i = j+1$ . Thus,  $i \neq j+1$ . Combining this with  $i \leq j+1$ , we obtain  $i < j+1$ . Thus,  $i \leq j$  (since  $i$  and  $j$  are integers).

Now,  $i \in \{i, i+1, \dots, j\}$  (since  $i \leq j$ ) and  $\pi_{i-1} < \pi_i$ . Hence,  $i$  is an element  $k \in \{i, i+1, \dots, j\}$  satisfying  $\pi_{k-1} < \pi_k$ . Hence, there exists at least one element  $k \in \{i, i+1, \dots, j\}$  satisfying  $\pi_{k-1} < \pi_k$  (namely,  $i$ ). Let  $p$  be the **largest** such element. Thus,  $p$  itself is an element of  $\{i, i+1, \dots, j\}$  and satisfies  $\pi_{p-1} < \pi_p$ , and furthermore, every element  $k \in \{i, i+1, \dots, j\}$  satisfying  $\pi_{k-1} < \pi_k$  must satisfy

$$k \leq p. \tag{16}$$

<sup>35</sup>*Proof.* Assume the contrary. Thus,  $\pi_{i-1} = \pi_i$ . If we have  $i = 1$ , then  $\pi_{i-1} = \pi_{1-1} = \pi_0 = 0 \notin \mathbb{P}$ , which contradicts  $\pi_{i-1} = \pi_i \in \mathbb{P}$  (since  $i \in [n]$ ). Thus, we cannot have  $i = 1$ . Hence, we have  $i \neq 1$  and therefore  $i > 1$  (since  $i \in [n]$ ). Thus,  $i-1 \in [n]$ . Hence, both  $i-1$  and  $i$  are elements of  $[n]$ .

Now,  $\pi_{i-1}$  and  $\pi_i$  are two distinct entries of the word  $\pi$  (since  $i-1$  and  $i$  are two distinct elements of  $[n]$ ). Thus,  $\pi_{i-1} \neq \pi_i$  (since the entries of  $\pi$  are distinct). This contradicts  $\pi_{i-1} = \pi_i$ . This contradiction shows that our assumption was false. Qed.

We have  $p \in \{i, i+1, \dots, j\} \subseteq [n]$  (since  $i$  and  $j$  are elements of  $[n]$ ), so that  $\pi_p \in \mathbb{P}$ . Thus,  $\pi_p > 0$ .

Next, we claim that  $\pi_p > \pi_{p+1}$ . Indeed, assume the contrary. Hence,  $\pi_p \leq \pi_{p+1}$ . But recall that  $\pi_p > 0$ . Hence,  $p \neq n$ <sup>36</sup>. Combining  $p \in [n]$  with  $p \neq n$ , we obtain  $p \in [n] \setminus \{n\} \subseteq [n-1]$ ; therefore,  $p+1 \in [n]$ . Now we know that  $p$  and  $p+1$  both are elements of  $[n]$ . Hence,  $\pi_p$  and  $\pi_{p+1}$  are two distinct entries of the word  $\pi$  (since  $p \neq p+1$ ). Hence,  $\pi_p \neq \pi_{p+1}$  (since the entries of the word  $\pi$  are distinct). Combining this with  $\pi_p \leq \pi_{p+1}$ , we obtain  $\pi_p < \pi_{p+1}$ . If we had  $p = j$ , then we would have  $\pi_p = \pi_j \geq \pi_{j+1} = \pi_{p+1}$  (since  $j = p$ ), which would contradict  $\pi_p < \pi_{p+1}$ . Thus, we cannot have  $p = j$ . In other words, we have  $p \neq j$ . Combining  $p \in \{i, i+1, \dots, j\}$  with  $p \neq j$ , we obtain  $p \in \{i, i+1, \dots, j\} \setminus \{j\} = \{i, i+1, \dots, j-1\}$ , so that

$$p+1 \in \{i+1, i+2, \dots, j\} \subseteq \{i, i+1, \dots, j\}.$$

Hence,  $p+1$  is an element  $k \in \{i, i+1, \dots, j\}$  satisfying  $\pi_{k-1} < \pi_k$  (since  $\pi_{(p+1)-1} = \pi_p < \pi_{p+1}$ ). Therefore, (16) (applied to  $k = p+1$ ) yields  $p+1 \leq p$ . But this contradicts  $p < p+1$ . This contradiction shows that our assumption was false. Hence,  $\pi_p > \pi_{p+1}$  is proven.

Now, we know that  $p \in [n]$  and  $\pi_{p-1} < \pi_p > \pi_{p+1}$ . In other words,  $p$  is an exterior peak of  $\pi$  (by the definition of an exterior peak). In other words,  $p \in \text{Epk } \pi$ . Combining this with  $p \in \{i, i+1, \dots, j\}$ , we obtain  $p \in \{i, i+1, \dots, j\} \cap \text{Epk } \pi$ . Hence, the set  $\{i, i+1, \dots, j\} \cap \text{Epk } \pi$  has at least one element (namely,  $p$ ). Therefore, this set is nonempty. In other words,  $\{i, i+1, \dots, j\} \cap \text{Epk } \pi \neq \emptyset$ . This proves Observation 1.]

(c) Assume that  $S \cap \text{Epk } \pi \subseteq \{\min S, \max S\}$ . We must prove that the map  $\pi|_S$  is V-shaped.

Write the interval  $S$  in the form  $S = \{a, a+1, \dots, b\}$  for some elements  $a$  and  $b$  of  $[n]$ . Thus,  $a \leq b$  (since the interval  $S$  is nonempty) and  $\min S = a$  and

$$\max S = b. \text{ Thus, } S \cap \text{Epk } \pi \subseteq \left\{ \underbrace{\min S}_{=a}, \underbrace{\max S}_{=b} \right\} = \{a, b\}.$$

The set  $S$  is nonempty. Hence, there exists some  $p \in S$  that minimizes  $\pi_p$ . Consider this  $p$ . Thus,

$$\pi_p \leq \pi_s \quad \text{for each } s \in S. \quad (17)$$

Note that  $p \in S = \{a, a+1, \dots, b\}$ , so that  $a \leq p \leq b$ . Also,  $p \in S \subseteq [n]$ .

From  $S = \{a, a+1, \dots, b\}$ , we obtain

$$\{s \in S \mid s \leq p\} = \{s \in \{a, a+1, \dots, b\} \mid s \leq p\} = \{a, a+1, \dots, p\}$$

(since  $p \leq b$ ). Also, from  $S = \{a, a+1, \dots, b\}$ , we obtain

$$\{s \in S \mid s \geq p\} = \{s \in \{a, a+1, \dots, b\} \mid s \geq p\} = \{p, p+1, \dots, b\}$$

<sup>36</sup>*Proof.* Assume the contrary. Thus,  $p = n$ , so that  $\pi_{p+1} = \pi_{n+1} = 0$ . Hence,  $\pi_p \leq \pi_{p+1} = 0$ . But this contradicts  $\pi_p > 0$ . This contradiction shows that our assumption was false. Qed.

(since  $p \geq a$ ).

Now, we claim that the map  $\pi |_{\{s \in S \mid s \leq p\}}$  is strictly decreasing.

[Proof: Let  $i \in \{a+1, a+2, \dots, p\}$ . Thus,

$$\begin{aligned} i-1 &\in \{a, a+1, \dots, p-1\} \subseteq \{a, a+1, \dots, b\} && (\text{since } p-1 < p \leq b) \\ &= S \subseteq [n]. \end{aligned}$$

Hence,  $\pi_{i-1}$  is well-defined. Also,  $i \in \{a+1, a+2, \dots, p\} \subseteq \{a, a+1, \dots, b\}$  (since  $a+1 > a \geq a$  and  $p \leq b$ ), so that  $i \in \{a, a+1, \dots, b\} = S \subseteq [n]$ . Thus,  $\pi_i$  is well-defined. Also, from  $i \in \{a, a+1, \dots, b\}$ , we obtain  $i \geq a$ .

Now, assume (for the sake of contradiction) that  $\pi_{i-1} \leq \pi_i$ .

But  $i \in \{a+1, a+2, \dots, p\}$  yields  $a+1 \leq i \leq p$ ; hence,  $a \leq p-1 \leq b$  (since  $p-1 < p \leq b$ ). Thus,  $p-1 \in \{a, a+1, \dots, b\} = S$ . Hence, (17) (applied to  $s = p-1$ ) yields  $\pi_p \leq \pi_{p-1}$ . Thus,  $\pi_{p-1} \geq \pi_p = \pi_{(p-1)+1}$ .

Also,  $i \in [n]$  and  $p-1 \in S \subseteq [n]$  and  $i \leq p = (p-1) + 1$ . Hence, Observation 1 (applied to  $j = p-1$ ) yields  $\{i, i+1, \dots, p-1\} \cap \text{Epk } \pi \neq \emptyset$ . In other words, the set  $\{i, i+1, \dots, p-1\} \cap \text{Epk } \pi$  has at least one element. Consider such an element, and denote it by  $z$ . Thus,

$$z \in \{i, i+1, \dots, p-1\} \cap \text{Epk } \pi.$$

Now,

$$\begin{aligned} z &\in \underbrace{\{i, i+1, \dots, p-1\}}_{\substack{\subseteq \{a, a+1, \dots, b\} \\ (\text{since } i \geq a \text{ and } p-1 \leq b)}} \cap \text{Epk } \pi \subseteq \underbrace{\{a, a+1, \dots, b\}}_{=S} \cap \text{Epk } \pi \\ &= S \cap \text{Epk } \pi \subseteq \{a, b\}. \end{aligned}$$

But  $z \in \{i, i+1, \dots, p-1\} \cap \text{Epk } \pi \subseteq \{i, i+1, \dots, p-1\}$ , so that  $z \leq p-1 < p \leq b$ . Hence,  $z \neq b$ . Combining  $z \in \{a, b\}$  with  $z \neq b$ , we obtain  $z \in \{a, b\} \setminus \{b\} \subseteq \{a\}$ . In other words,  $z = a$ . But from  $z \in \{i, i+1, \dots, p-1\}$ , we also obtain  $z \geq i \geq a+1$  (since  $i \in \{a+1, a+2, \dots, p\}$ ). Hence,  $z \geq a+1 > a$ ; this contradicts  $z = a$ .

This contradiction shows that our assumption (that  $\pi_{i-1} \leq \pi_i$ ) was wrong. Hence, we don't have  $\pi_{i-1} \leq \pi_i$ . In other words, we have  $\pi_{i-1} > \pi_i$ .

Now, forget that we fixed  $i$ . We thus have shown that  $\pi_{i-1} > \pi_i$  for each  $i \in \{a+1, a+2, \dots, p\}$ . In other words,  $\pi_a > \pi_{a+1} > \pi_{a+2} > \dots > \pi_p$ . In other words, the map  $\pi |_{\{a, a+1, \dots, p\}}$  is strictly decreasing. In other words, the map  $\pi |_{\{s \in S \mid s \leq p\}}$  is strictly decreasing (since  $\{s \in S \mid s \leq p\} = \{a, a+1, \dots, p\}$ ).

Next, we claim that the map  $\pi |_{\{s \in S \mid s \geq p\}}$  is strictly increasing.

[Proof: Let  $j \in \{p, p+1, \dots, b-1\}$ . Thus,

$$\begin{aligned} j+1 &\in \{p+1, p+2, \dots, b\} \subseteq \{a, a+1, \dots, b\} && (\text{since } p+1 > p \geq a) \\ &= S \subseteq [n]. \end{aligned}$$

Hence,  $\pi_{j+1}$  is well-defined. Also,  $j \in \{p, p+1, \dots, b-1\} \subseteq \{a, a+1, \dots, b\}$  (since  $p \geq a$  and  $b-1 < b$ ), so that  $j \in \{a, a+1, \dots, b\} = S \subseteq [n]$ . Thus,  $\pi_j$  is well-defined. Also, from  $j \in \{a, a+1, \dots, b\}$ , we obtain  $j \geq a$  and  $j \leq b$ .

Now, assume (for the sake of contradiction) that  $\pi_j \geq \pi_{j+1}$ .

But  $j \in \{p, p+1, \dots, b-1\}$  yields  $p \leq j \leq b-1$ ; hence,  $p+1 \leq b$ . Combined with  $a \leq p < p+1$ , this yields  $a \leq p+1 \leq b$ . Thus,  $p+1 \in \{a, a+1, \dots, b\} = S$ . Hence, (17) (applied to  $s = p+1$ ) yields  $\pi_p \leq \pi_{p+1}$ . In other words,  $\pi_{(p+1)-1} \leq \pi_{p+1}$  (since  $p = (p+1) - 1$ ).

Also,  $p+1 \in S \subseteq [n]$  and  $j \in [n]$  and  $p+1 \leq j+1$  (since  $p \leq j$ ). Hence, Observation 1 (applied to  $i = p+1$ ) yields  $\{p+1, p+2, \dots, j\} \cap \text{Epk } \pi \neq \emptyset$ . In other words, the set  $\{p+1, p+2, \dots, j\} \cap \text{Epk } \pi$  has at least one element. Consider such an element, and denote it by  $z$ . Thus,

$$z \in \{p+1, p+2, \dots, j\} \cap \text{Epk } \pi.$$

Now,

$$\begin{aligned} z \in \underbrace{\{p+1, p+2, \dots, j\}}_{\substack{\subseteq \{a, a+1, \dots, b\} \\ \text{(since } p+1 \geq a \text{ and } j \leq b)}} \cap \text{Epk } \pi &\subseteq \underbrace{\{a, a+1, \dots, b\}}_{=S} \cap \text{Epk } \pi \\ &= S \cap \text{Epk } \pi \subseteq \{a, b\}. \end{aligned}$$

But  $z \in \{p+1, p+2, \dots, j\} \cap \text{Epk } \pi \subseteq \{p+1, p+2, \dots, j\}$ , so that  $z \geq p+1 > p \geq a$ . Hence,  $z \neq a$ . Combining  $z \in \{a, b\}$  with  $z \neq a$ , we obtain  $z \in \{a, b\} \setminus \{a\} \subseteq \{b\}$ . In other words,  $z = b$ . But from  $z \in \{p+1, p+2, \dots, j\}$ , we also obtain  $z \leq j \leq b-1$  (since  $j \in \{p, p+1, \dots, b-1\}$ ). Hence,  $z \leq b-1 < b$ ; this contradicts  $z = b$ .

This contradiction shows that our assumption (that  $\pi_j \geq \pi_{j+1}$ ) was wrong. Hence, we don't have  $\pi_j \geq \pi_{j+1}$ . In other words, we have  $\pi_j < \pi_{j+1}$ .

Now, forget that we fixed  $j$ . We thus have shown that  $\pi_j < \pi_{j+1}$  for each  $j \in \{p, p+1, \dots, b-1\}$ . In other words,  $\pi_p < \pi_{p+1} < \pi_{p+2} < \dots < \pi_b$ . In other words, the map  $\pi|_{\{p, p+1, \dots, b\}}$  is strictly increasing. In other words, the map  $\pi|_{\{s \in S \mid s \geq p\}}$  is strictly increasing (since  $\{s \in S \mid s \geq p\} = \{p, p+1, \dots, b\}$ .)

Thus, we now know that the map  $\pi|_{\{s \in S \mid s \leq p\}}$  is strictly decreasing while the map  $\pi|_{\{s \in S \mid s \geq p\}}$  is strictly increasing. In view of  $\pi|_{\{s \in S \mid s \leq p\}} = (\pi|_S)|_{\{s \in S \mid s \leq p\}}$  and  $\pi|_{\{s \in S \mid s \geq p\}} = (\pi|_S)|_{\{s \in S \mid s \geq p\}}$ , this rewrites as follows: The map  $(\pi|_S)|_{\{s \in S \mid s \leq p\}}$  is strictly decreasing while the map  $(\pi|_S)|_{\{s \in S \mid s \geq p\}}$  is strictly increasing. Hence, there exists some  $t \in S$  such that the map  $(\pi|_S)|_{\{s \in S \mid s \leq t\}}$  is strictly decreasing while the map  $(\pi|_S)|_{\{s \in S \mid s \geq t\}}$  is strictly increasing (namely,  $t = p$ ). In other words, the map  $\pi|_S$  is V-shaped (by the definition of "V-shaped"). This proves Lemma 2.43 (c).

(a) Assume that  $1 \in S$  and  $S \cap \text{Epk } \pi \subseteq \{\max S\}$ . We must prove that the map  $\pi|_S$  is strictly increasing.

Write the interval  $S$  in the form  $S = \{a, a+1, \dots, b\}$  for some elements  $a$  and  $b$  of  $[n]$ . Thus,  $a \leq b$  (since the interval  $S$  is nonempty) and  $\min S = a$  and

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$\max S = b$ . But  $1 \in S$  and thus  $\min S \leq 1$ . Hence,  $a = \min S \leq 1$ . Combined with  $a \geq 1$  (since  $a \in [n]$ ), this yields  $a = 1$ . Now,

$$S = \{a, a + 1, \dots, b\} = \{1, 2, \dots, b\} \quad (\text{since } a = 1).$$

But  $S \cap \text{Epk } \pi \subseteq \{\max S\} = \{b\}$  (since  $\max S = b$ ).

Let  $j \in \{1, 2, \dots, b - 1\}$ . Thus,  $1 \leq j \leq b - 1$ . Hence,  $j \leq b - 1 < b \leq n$  (since  $b \in [n]$ ), so that  $j \in [n]$  (since  $1 \leq j \leq n$ ).

Assume (for the sake of contradiction) that  $\pi_j \geq \pi_{j+1}$ .

We have  $\pi_{1-1} = \pi_0 = 0 \leq \pi_1$  (since  $\pi_1 \in \mathbb{N}$ ) and  $1 \in S \subseteq [n]$  and  $j \in [n]$  and  $1 \leq j + 1$  (since  $j + 1 > j \geq 1$ ) and  $\pi_j \geq \pi_{j+1}$ . Thus, Observation 1 (applied to  $i = 1$ ) yields

$$\{1, 2, \dots, j\} \cap \text{Epk } \pi \neq \emptyset.$$

In other words, the set  $\{1, 2, \dots, j\} \cap \text{Epk } \pi$  has at least one element. Consider such an element, and denote it by  $z$ . Thus,

$$z \in \{1, 2, \dots, j\} \cap \text{Epk } \pi.$$

Now,

$$\begin{aligned} z \in \underbrace{\{1, 2, \dots, j\}}_{\substack{\subseteq \{a, a+1, \dots, b\} \\ (\text{since } 1=a \text{ and } j \leq b-1 \leq b)}} \cap \text{Epk } \pi &\subseteq \underbrace{\{a, a + 1, \dots, b\}}_{=S} \cap \text{Epk } \pi \\ &= S \cap \text{Epk } \pi \subseteq \{b\}. \end{aligned}$$

In other words,  $z = b$ .

But  $z \in \{1, 2, \dots, j\} \cap \text{Epk } \pi \subseteq \{1, 2, \dots, j\}$ , so that  $z \leq j \leq b - 1 < b$ . Hence,  $z \neq b$ . This contradicts  $z = b$ .

This contradiction shows that our assumption (that  $\pi_j \geq \pi_{j+1}$ ) was wrong. Hence, we don't have  $\pi_j \geq \pi_{j+1}$ . In other words, we have  $\pi_j < \pi_{j+1}$ .

Now, forget that we fixed  $j$ . We thus have shown that  $\pi_j < \pi_{j+1}$  for each  $j \in \{1, 2, \dots, b - 1\}$ . In other words,  $\pi_1 < \pi_2 < \dots < \pi_b$ . In other words, the map  $\pi|_{\{1, 2, \dots, b\}}$  is strictly increasing. In other words, the map  $\pi|_S$  is strictly increasing (since  $S = \{1, 2, \dots, b\}$ ). This proves Lemma 2.43 (a).

**(b)** Assume that  $n \in S$  and  $S \cap \text{Epk } \pi \subseteq \{\min S\}$ . We must prove that the map  $\pi|_S$  is strictly decreasing.

Write the interval  $S$  in the form  $S = \{a, a + 1, \dots, b\}$  for some elements  $a$  and  $b$  of  $[n]$ . Thus,  $a \leq b$  (since the interval  $S$  is nonempty) and  $\min S = a$  and  $\max S = b$ . But  $n \in S$  and thus  $\max S \geq n$ . Hence,  $b = \max S \geq n$ . Combined with  $b \leq n$  (since  $b \in [n]$ ), this yields  $b = n$ . Now,

$$S = \{a, a + 1, \dots, b\} = \{a, a + 1, \dots, n\} \quad (\text{since } b = n).$$

But  $S \cap \text{Epk } \pi \subseteq \{\min S\} = \{a\}$  (since  $\min S = a$ ).

Let  $i \in \{a + 1, a + 2, \dots, n\}$ . Thus,  $a + 1 \leq i \leq n$ . Hence,  $i \geq a + 1 > a \geq 1$  (since  $a \in [n]$ ), so that  $i \in [n]$  (since  $1 \leq i \leq n$ ).



Assume (for the sake of contradiction) that  $\pi_{i-1} \leq \pi_i$ .

We have  $\pi_{n+1} = 0 \leq \pi_n$  (since  $\pi_n \in \mathbb{N}$ ) and thus  $\pi_n \geq \pi_{n+1}$ . Also,  $n \in S \subseteq [n]$  and  $i \in [n]$  and  $i \leq n+1$  (since  $i \in [n] \subseteq [n+1]$ ) and  $\pi_{i-1} \leq \pi_i$ . Thus, Observation 1 (applied to  $j = n$ ) yields

$$\{i, i+1, \dots, n\} \cap \text{Epk } \pi \neq \emptyset.$$

In other words, the set  $\{i, i+1, \dots, n\} \cap \text{Epk } \pi$  has at least one element. Consider such an element, and denote it by  $z$ . Thus,

$$z \in \{i, i+1, \dots, n\} \cap \text{Epk } \pi.$$

Now,

$$\begin{aligned} z \in \underbrace{\{i, i+1, \dots, n\}}_{\substack{\subseteq \{a, a+1, \dots, b\} \\ \text{(since } i \geq a \text{ and } n = b)}} \cap \text{Epk } \pi &\subseteq \underbrace{\{a, a+1, \dots, b\}}_{=S} \cap \text{Epk } \pi \\ &= S \cap \text{Epk } \pi \subseteq \{a\}. \end{aligned}$$

In other words,  $z = a$ .

But  $z \in \{i, i+1, \dots, n\} \cap \text{Epk } \pi \subseteq \{i, i+1, \dots, n\}$ , so that  $z \geq i \geq a+1 > a$ . Hence,  $z \neq a$ . This contradicts  $z = a$ .

This contradiction shows that our assumption (that  $\pi_{i-1} \leq \pi_i$ ) was wrong. Hence, we don't have  $\pi_{i-1} \leq \pi_i$ . In other words, we have  $\pi_{i-1} > \pi_i$ .

Now, forget that we fixed  $i$ . We thus have shown that  $\pi_{i-1} > \pi_i$  for each  $i \in \{a+1, a+2, \dots, n\}$ . In other words,  $\pi_a > \pi_{a+1} > \pi_{a+2} > \dots > \pi_n$ . In other words, the map  $\pi|_{\{a, a+1, \dots, n\}}$  is strictly decreasing. In other words, the map  $\pi|_S$  is strictly decreasing (since  $S = \{a, a+1, \dots, n\}$ ). This proves Lemma 2.43 (b).  $\square$

**Lemma 2.44.** Let  $n \in \mathbb{N}$ . Let  $g : [n] \rightarrow \mathcal{N}$  be any weakly increasing map. Let  $u \in g([n])$ .

(a) If  $u = 0$ , then

$$\text{FE}(g) \cap g^{-1}(u) = \left\{ \max(g^{-1}(u)) \right\}.$$

(b) If  $u \in \{1, 2, 3, \dots\}$ , then

$$\text{FE}(g) \cap g^{-1}(u) = \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}.$$

(c) If  $u = \infty$ , then

$$\text{FE}(g) \cap g^{-1}(u) = \left\{ \min(g^{-1}(u)) \right\}.$$

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*Proof of Lemma 2.44.* From  $u \in g([n])$ , we conclude that there exists some  $j \in [n]$  such that  $u = g(j)$ . In other words, the fiber  $g^{-1}(u)$  is nonempty. In other words,  $g^{-1}(u) \neq \emptyset$ .

Recall that

$$\begin{aligned} \text{FE}(g) = & \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ & \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \end{aligned} \quad (18)$$

(by the definition of  $\text{FE}(g)$ ).

If  $u \in \{0, 1, 2, 3, \dots\}$ , then

$$\begin{aligned} \max(g^{-1}(u)) \in & \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ & \left( \begin{array}{l} \text{since } u \text{ is an } h \in \{0, 1, 2, 3, \dots\} \text{ satisfying } g^{-1}(h) \neq \emptyset \\ \text{(since } u \in \{0, 1, 2, 3, \dots\} \text{ and } g^{-1}(u) \neq \emptyset) \end{array} \right) \\ \subseteq & \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ & \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ = & \text{FE}(g) \quad (\text{by (18)}). \end{aligned} \quad (19)$$

If  $u \in \{1, 2, 3, \dots, \infty\}$ , then

$$\begin{aligned} \min(g^{-1}(u)) \in & \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ & \left( \begin{array}{l} \text{since } u \text{ is an } h \in \{1, 2, 3, \dots, \infty\} \text{ satisfying } g^{-1}(h) \neq \emptyset \\ \text{(since } u \in \{1, 2, 3, \dots, \infty\} \text{ and } g^{-1}(u) \neq \emptyset) \end{array} \right) \\ \subseteq & \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ & \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ = & \text{FE}(g) \quad (\text{by (18)}). \end{aligned} \quad (20)$$

**(a)** Assume that  $u = 0$ . We must prove that  $\text{FE}(g) \cap g^{-1}(u) = \{\max(g^{-1}(u))\}$ .

We have  $u = 0 \in \{0, 1, 2, 3, \dots\}$ . Thus, (19) shows that  $\max(g^{-1}(u)) \in \text{FE}(g)$ . Combining this with  $\max(g^{-1}(u)) \in g^{-1}(u)$  (which is obvious), we obtain  $\max(g^{-1}(u)) \in \text{FE}(g) \cap g^{-1}(u)$ . In other words,

$$\left\{ \max(g^{-1}(u)) \right\} \subseteq \text{FE}(g) \cap g^{-1}(u). \quad (21)$$

On the other hand, let  $j \in \text{FE}(g) \cap g^{-1}(u)$  be arbitrary. We shall show that  $j \in \{\max(g^{-1}(u))\}$ .

Indeed, we have  $j \in \text{FE}(g) \cap g^{-1}(u) \subseteq g^{-1}(u)$ , so that  $g(j) = u$ . On the other hand,

$$\begin{aligned} j &\in \text{FE}(g) \cap g^{-1}(u) \subseteq \text{FE}(g) \\ &= \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ &\quad \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ &\quad \text{(by (18)).} \end{aligned}$$

In other words, either  $j \in \{\min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset\}$  or  $j \in \{\max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset\}$ . Hence, we are in one of the following two cases:

*Case 1:* We have  $j \in \{\min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset\}$ .

*Case 2:* We have  $j \in \{\max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset\}$ .

Let us first consider Case 1. In this case, we have

$j \in \{\min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset\}$ . In other words,  $j = \min(g^{-1}(h))$  for some  $h \in \{1, 2, 3, \dots, \infty\}$  satisfying  $g^{-1}(h) \neq \emptyset$ . Consider this  $h$ . We have  $j = \min(g^{-1}(h)) \in g^{-1}(h)$ , so that  $g(j) = h$ . Hence,  $h = g(j) = u$ . Hence,  $u = h \in \{1, 2, 3, \dots, \infty\}$ . This contradicts  $u = 0 \notin \{1, 2, 3, \dots, \infty\}$ . Thus,  $j \in \{\max(g^{-1}(u))\}$  (because anything follows from a contradiction). Hence, we have proven  $j \in \{\max(g^{-1}(u))\}$  in Case 1.

Let us now consider Case 2. In this case, we have

$j \in \{\max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset\}$ . In other words,  $j = \max(g^{-1}(h))$  for some  $h \in \{0, 1, 2, 3, \dots\}$  satisfying  $g^{-1}(h) \neq \emptyset$ . Consider this  $h$ . We have  $j = \max(g^{-1}(h)) \in g^{-1}(h)$ , so that  $g(j) = h$ . Hence,  $h = g(j) = u$ .

Hence,  $j = \max\left(g^{-1}\left(\underbrace{h}_{=u}\right)\right) = \max(g^{-1}(u)) \in \{\max(g^{-1}(u))\}$ . Hence, we have proven  $j \in \{\max(g^{-1}(u))\}$  in Case 2.

We have now proven  $j \in \{\max(g^{-1}(u))\}$  in each of the two Cases 1 and 2. Thus,  $j \in \{\max(g^{-1}(u))\}$  always holds.

Now, forget that we fixed  $j$ . We thus have proven that  $j \in \{\max(g^{-1}(u))\}$  for each  $j \in \text{FE}(g) \cap g^{-1}(u)$ . In other words,

$$\text{FE}(g) \cap g^{-1}(u) \subseteq \{\max(g^{-1}(u))\}.$$

Combining this with (21), we obtain  $\text{FE}(g) \cap g^{-1}(u) = \{\max(g^{-1}(u))\}$ . This proves Lemma 2.44 (a).

(b) Assume that  $u \in \{1, 2, 3, \dots\}$ . We must prove that  $\text{FE}(g) \cap g^{-1}(u) = \{\min(g^{-1}(u)), \max(g^{-1}(u))\}$ .

We have  $u \in \{1, 2, 3, \dots\} \subseteq \{0, 1, 2, 3, \dots\}$ . Thus, (19) shows that  $\max(g^{-1}(u)) \in \text{FE}(g)$ . Combining this with  $\max(g^{-1}(u)) \in g^{-1}(u)$  (which is obvious), we obtain  $\max(g^{-1}(u)) \in \text{FE}(g) \cap g^{-1}(u)$ .

We have  $u \in \{1, 2, 3, \dots\} \subseteq \{1, 2, 3, \dots, \infty\}$ . Thus, (20) shows that  $\min(g^{-1}(u)) \in \text{FE}(g)$ . Combining this with  $\min(g^{-1}(u)) \in g^{-1}(u)$  (which is obvious), we obtain  $\min(g^{-1}(u)) \in \text{FE}(g) \cap g^{-1}(u)$ .

Now, we have shown that both numbers  $\min(g^{-1}(u))$  and  $\max(g^{-1}(u))$  belong to  $\text{FE}(g) \cap g^{-1}(u)$ . In other words,

$$\left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\} \subseteq \text{FE}(g) \cap g^{-1}(u). \quad (22)$$

On the other hand, let  $j \in \text{FE}(g) \cap g^{-1}(u)$  be arbitrary. We shall show that  $j \in \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}$ .

Indeed, we have  $j \in \text{FE}(g) \cap g^{-1}(u) \subseteq g^{-1}(u)$ , so that  $g(j) = u$ . On the other hand,

$$\begin{aligned} j &\in \text{FE}(g) \cap g^{-1}(u) \subseteq \text{FE}(g) \\ &= \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ &\quad \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ &\quad \text{(by (18))}. \end{aligned}$$

In other words, either  $j \in \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\}$  or  $j \in \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}$ . Hence, we are in one of the following two cases:

*Case 1:* We have  $j \in \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\}$ .

*Case 2:* We have  $j \in \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}$ .

Let us first consider Case 1. In this case, we have

$j \in \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\}$ . In other words,  $j = \min(g^{-1}(h))$  for some  $h \in \{1, 2, 3, \dots, \infty\}$  satisfying  $g^{-1}(h) \neq \emptyset$ . Consider this  $h$ . We have  $j = \min(g^{-1}(h)) \in g^{-1}(h)$ , so that  $g(j) = h$ . Hence,  $h = g(j) =$

$u$ . Hence,  $j = \min\left(g^{-1}\left(\underbrace{h}_{=u}\right)\right) = \min(g^{-1}(u)) \in \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}$ .

Hence, we have proven  $j \in \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}$  in Case 1.

Let us now consider Case 2. In this case, we have

$j \in \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}$ . In other words,  $j = \max(g^{-1}(h))$  for some  $h \in \{0, 1, 2, 3, \dots\}$  satisfying  $g^{-1}(h) \neq \emptyset$ . Consider this  $h$ . We have  $j = \max(g^{-1}(h)) \in g^{-1}(h)$ , so that  $g(j) = h$ . Hence,  $h = g(j) =$

Hence,  $j = \max\left(g^{-1}\left(\underbrace{h}_{=u}\right)\right) = \max(g^{-1}(u)) \in \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}$ .

Hence, we have proven  $j \in \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}$  in Case 2.

We have now proven  $j \in \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}$  in each of the two Cases 1 and 2. Thus,  $j \in \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}$  always holds.

Now, forget that we fixed  $j$ . We thus have proven that  $j \in \{\min(g^{-1}(u)), \max(g^{-1}(u))\}$  for each  $j \in \text{FE}(g) \cap g^{-1}(u)$ . In other words,

$$\text{FE}(g) \cap g^{-1}(u) \subseteq \{\min(g^{-1}(u)), \max(g^{-1}(u))\}.$$

Combining this with (22), we obtain  $\text{FE}(g) \cap g^{-1}(u) = \{\min(g^{-1}(u)), \max(g^{-1}(u))\}$ . This proves Lemma 2.44 **(b)**.

**(c)** Assume that  $u = \infty$ . We must prove that  $\text{FE}(g) \cap g^{-1}(u) = \{\min(g^{-1}(u))\}$ .

We have  $u = \infty \in \{1, 2, 3, \dots, \infty\}$ . Thus, (20) shows that  $\min(g^{-1}(u)) \in \text{FE}(g)$ . Combining this with  $\min(g^{-1}(u)) \in g^{-1}(u)$  (which is obvious), we obtain  $\min(g^{-1}(u)) \in \text{FE}(g) \cap g^{-1}(u)$ . In other words,

$$\{\min(g^{-1}(u))\} \subseteq \text{FE}(g) \cap g^{-1}(u). \quad (23)$$

On the other hand, let  $j \in \text{FE}(g) \cap g^{-1}(u)$  be arbitrary. We shall show that  $j \in \{\min(g^{-1}(u))\}$ .

Indeed, we have  $j \in \text{FE}(g) \cap g^{-1}(u) \subseteq g^{-1}(u)$ , so that  $g(j) = u$ . On the other hand,

$$\begin{aligned} j &\in \text{FE}(g) \cap g^{-1}(u) \subseteq \text{FE}(g) \\ &= \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ &\quad \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ &\quad \text{(by (18)).} \end{aligned}$$

In other words, either  $j \in \{\min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset\}$  or  $j \in \{\max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset\}$ . Hence, we are in one of the following two cases:

*Case 1:* We have  $j \in \{\min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset\}$ .

*Case 2:* We have  $j \in \{\max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset\}$ .

Let us first consider Case 1. In this case, we have

$j \in \{\min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset\}$ . In other words,  $j = \min(g^{-1}(h))$  for some  $h \in \{1, 2, 3, \dots, \infty\}$  satisfying  $g^{-1}(h) \neq \emptyset$ . Consider this  $h$ . We have  $j = \min(g^{-1}(h)) \in g^{-1}(h)$ , so that  $g(j) = h$ . Hence,  $h = g(j) = u$ . Hence,  $j = \min\left(g^{-1}\left(\underbrace{h}_{=u}\right)\right) = \min(g^{-1}(u)) \in \{\min(g^{-1}(u))\}$ . Hence,

we have proven  $j \in \{\min(g^{-1}(u))\}$  in Case 1.

Let us now consider Case 2. In this case, we have

$j \in \{\max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset\}$ . In other words,  $j = \max(g^{-1}(h))$  for some  $h \in \{0, 1, 2, 3, \dots\}$  satisfying  $g^{-1}(h) \neq \emptyset$ . Consider this  $h$ . We have  $j = \max(g^{-1}(h)) \in g^{-1}(h)$ , so that  $g(j) = h$ . Hence,  $h = g(j) = u$ . Hence,  $u = h \in \{0, 1, 2, 3, \dots\}$ . This contradicts  $u = \infty \notin \{0, 1, 2, 3, \dots\}$ . Thus,

$j \in \{\min(g^{-1}(u))\}$  (because anything follows from a contradiction). Hence, we have proven  $j \in \{\min(g^{-1}(u))\}$  in Case 2.

We have now proven  $j \in \{\min(g^{-1}(u))\}$  in each of the two Cases 1 and 2. Thus,  $j \in \{\min(g^{-1}(u))\}$  always holds.

Now, forget that we fixed  $j$ . We thus have proven that  $j \in \{\min(g^{-1}(u))\}$  for each  $j \in \text{FE}(g) \cap g^{-1}(u)$ . In other words,

$$\text{FE}(g) \cap g^{-1}(u) \subseteq \{\min(g^{-1}(u))\}.$$

Combining this with (23), we obtain  $\text{FE}(g) \cap g^{-1}(u) = \{\min(g^{-1}(u))\}$ . This proves Lemma 2.44 (c).  $\square$

**Proposition 2.45.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be an  $n$ -permutation. Let  $g : [n] \rightarrow \mathcal{N}$  be any weakly increasing map. Then, the map  $g$  is  $\pi$ -amenable if and only if  $\text{Epk } \pi \subseteq \text{FE}(g)$ .

*Proof of Proposition 2.45.* The map  $g$  is weakly increasing. Hence, each fiber  $g^{-1}(h)$  of  $g$  (with  $h \in \mathcal{N}$ ) is an interval of the totally ordered set  $[n]$ .

We shall prove the following two statements:

*Observation 1:* If the map  $g$  is  $\pi$ -amenable, then  $\text{Epk } \pi \subseteq \text{FE}(g)$ .

*Observation 2:* If  $\text{Epk } \pi \subseteq \text{FE}(g)$ , then the map  $g$  is  $\pi$ -amenable.

[*Proof of Observation 1:* Assume that the map  $g$  is  $\pi$ -amenable. We must show that  $\text{Epk } \pi \subseteq \text{FE}(g)$ .

We know that the map  $g$  is  $\pi$ -amenable. In other words, the four properties (i'), (ii'), (iii') and (iv') of Definition 2.36 hold (by the definition of " $\pi$ -amenable"). We shall use these four properties in the following.

Let  $j \in \text{Epk } \pi$ . We intend to show that  $j \in \text{FE}(g)$ .

We have  $j \in \text{Epk } \pi$ ; in other words,  $j$  is an exterior peak of  $\pi$  (by the definition of  $\text{Epk } \pi$ ). In other words,  $j$  is an element of  $[n]$  and satisfies  $\pi_{j-1} < \pi_j > \pi_{j+1}$  (by the definition of an exterior peak). Define  $u \in \mathcal{N}$  by  $u = g(j)$ . Thus,

$$j \in g^{-1}(u) \text{ and } u = g \left( \underbrace{j}_{\in [n]} \right) \in g([n]).$$

Recall that each fiber  $g^{-1}(h)$  of  $g$  (with  $h \in \mathcal{N}$ ) is an interval of the totally ordered set  $[n]$ . Applying this to  $h = u$ , we conclude that  $g^{-1}(u)$  is an interval of the totally ordered set  $[n]$ . This interval is furthermore nonempty (since  $j \in g^{-1}(u)$ ).

We have  $u \in \mathcal{N} = \{0\} \cup \{\infty\} \cup \{1, 2, 3, \dots\}$ . In other words, either  $u = 0$  or  $u = \infty$  or  $u \in \{1, 2, 3, \dots\}$ . Hence, we are in one of the following three cases:

*Case 1:* We have  $u = 0$ .

*Case 2:* We have  $u = \infty$ .

---

Case 3: We have  $u \in \{1, 2, 3, \dots\}$ .

Let us first consider Case 1. In this case, we have  $u = 0$ . Thus, Lemma 2.44 (a) yields

$$\text{FE}(g) \cap g^{-1}(u) = \left\{ \max \left( g^{-1}(u) \right) \right\}. \quad (24)$$

Recall that property (i') holds. In other words, the map  $\pi \big|_{g^{-1}(0)}$  is strictly increasing. Since  $0 = u$ , this rewrites as follows: The map  $\pi \big|_{g^{-1}(u)}$  is strictly increasing.

But  $j = \max(g^{-1}(u))$ <sup>37</sup>. Hence,

$$\begin{aligned} j = \max(g^{-1}(u)) &\in \left\{ \max(g^{-1}(u)) \right\} = \text{FE}(g) \cap g^{-1}(u) && \text{(by (24))} \\ &\subseteq \text{FE}(g). \end{aligned}$$

Thus, we have shown that  $j \in \text{FE}(g)$  in Case 1.

Let us next consider Case 2. In this case, we have  $u = \infty$ . Thus, Lemma 2.44 (c) yields

$$\text{FE}(g) \cap g^{-1}(u) = \left\{ \min(g^{-1}(u)) \right\}. \quad (25)$$

Recall that property (iii') holds. In other words, the map  $\pi \big|_{g^{-1}(\infty)}$  is strictly decreasing. Since  $\infty = u$ , this rewrites as follows: The map  $\pi \big|_{g^{-1}(u)}$  is strictly decreasing.

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<sup>37</sup>*Proof.* Assume the contrary. Thus,  $j \neq \max(g^{-1}(u))$ . Combining this with  $j \leq \max(g^{-1}(u))$  (which follows from  $j \in g^{-1}(u)$ ), we obtain  $j < \max(g^{-1}(u))$ .

But recall the following basic fact: If  $S$  is a nonempty interval of  $[n]$ , and if  $s \in S$  satisfies  $s < \max S$ , then  $s + 1 \in S$ . Applying this to  $S = g^{-1}(u)$  and  $s = j$ , we conclude that  $j + 1 \in g^{-1}(u)$  (because  $g^{-1}(u)$  is a nonempty interval of  $[n]$ , and because  $j \in g^{-1}(u)$  satisfies  $j < \max(g^{-1}(u))$ ). Hence, both  $j$  and  $j + 1$  are elements of  $g^{-1}(u)$ , and satisfy  $j < j + 1$ . Hence,  $(\pi \big|_{g^{-1}(u)})(j) < (\pi \big|_{g^{-1}(u)})(j + 1)$  (since the map  $\pi \big|_{g^{-1}(u)}$  is strictly increasing). Thus,

$$\pi_j = \pi(j) = (\pi \big|_{g^{-1}(u)})(j) < (\pi \big|_{g^{-1}(u)})(j + 1) = \pi(j + 1) = \pi_{j+1}.$$

This contradicts  $\pi_j > \pi_{j+1}$ . This contradiction shows that our assumption was false; qed.

But  $j = \min(g^{-1}(u))$  <sup>38</sup>. Hence,

$$\begin{aligned} j = \min(g^{-1}(u)) &\in \left\{ \min(g^{-1}(u)) \right\} = \text{FE}(g) \cap g^{-1}(u) && \text{(by (25))} \\ &\subseteq \text{FE}(g). \end{aligned}$$

Thus, we have shown that  $j \in \text{FE}(g)$  in Case 2.

Let us next consider Case 3. In this case, we have  $u \in \{1, 2, 3, \dots\}$ . Hence, Lemma 2.44 **(b)** yields

$$\text{FE}(g) \cap g^{-1}(u) = \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}. \quad (26)$$

Combining  $u \in g([n])$  with  $u \in \{1, 2, 3, \dots\}$ , we obtain  $u \in g([n]) \cap \{1, 2, 3, \dots\}$ .

Recall that property **(ii')** holds. In other words, for each  $h \in g([n]) \cap \{1, 2, 3, \dots\}$ , the map  $\pi|_{g^{-1}(h)}$  is V-shaped. Applying this to  $h = u$ , we conclude that the map  $\pi|_{g^{-1}(u)}$  is V-shaped. Hence, Lemma 2.42 (applied to  $S = g^{-1}(u)$ ) yields

$$g^{-1}(u) \cap \text{Epk } \pi \subseteq \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\}.$$

Combining  $j \in g^{-1}(u)$  with  $j \in \text{Epk } \pi$ , we obtain

$$\begin{aligned} j \in g^{-1}(u) \cap \text{Epk } \pi &\subseteq \left\{ \min(g^{-1}(u)), \max(g^{-1}(u)) \right\} \\ &= \text{FE}(g) \cap g^{-1}(u) && \text{(by (26))} \\ &\subseteq \text{FE}(g). \end{aligned}$$

Thus, we have shown that  $j \in \text{FE}(g)$  in Case 3.

We have now proven that  $j \in \text{FE}(g)$  in each of the three Cases 1, 2 and 3. Hence,  $j \in \text{FE}(g)$  always holds.

Now, forget that we fixed  $j$ . We thus have proven that  $j \in \text{FE}(g)$  for each  $j \in \text{Epk } \pi$ . In other words,  $\text{Epk } \pi \subseteq \text{FE}(g)$ . This proves Observation 1.]

[*Proof of Observation 2:* Assume that  $\text{Epk } \pi \subseteq \text{FE}(g)$ . We must prove that the map  $g$  is  $\pi$ -amenable.

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<sup>38</sup>*Proof.* Assume the contrary. Thus,  $j \neq \min(g^{-1}(u))$ . Combining this with  $j \geq \min(g^{-1}(u))$  (which follows from  $j \in g^{-1}(u)$ ), we obtain  $j > \min(g^{-1}(u))$ .

But recall the following basic fact: If  $S$  is a nonempty interval of  $[n]$ , and if  $s \in S$  satisfies  $s > \min S$ , then  $s - 1 \in S$ . Applying this to  $S = g^{-1}(u)$  and  $s = j$ , we conclude that  $j - 1 \in g^{-1}(u)$  (because  $g^{-1}(u)$  is a nonempty interval of  $[n]$ , and because  $j \in g^{-1}(u)$  satisfies  $j > \min(g^{-1}(u))$ ). Hence, both  $j - 1$  and  $j$  are elements of  $g^{-1}(u)$ , and satisfy  $j - 1 < j$ . Hence,  $(\pi|_{g^{-1}(u)})(j - 1) > (\pi|_{g^{-1}(u)})(j)$  (since the map  $\pi|_{g^{-1}(u)}$  is strictly decreasing). Thus,

$$\pi_{j-1} = \pi(j - 1) = (\pi|_{g^{-1}(u)})(j - 1) > (\pi|_{g^{-1}(u)})(j) = \pi(j) = \pi_j.$$

This contradicts  $\pi_{j-1} < \pi_j$ . This contradiction shows that our assumption was false; qed.



We are going to show that the four properties **(i')**, **(ii')**, **(iii')** and **(iv')** of Definition 2.36 hold. Clearly, property **(iv')** holds (since the map  $g$  is weakly increasing). Let us now prove the remaining three properties:

[*Proof of property (i')*: We must show that the map  $\pi|_{g^{-1}(0)}$  is strictly increasing. If  $g^{-1}(0) = \emptyset$ , then this is obvious. Thus, for the rest of this proof, we WLOG assume that we don't have  $g^{-1}(0) = \emptyset$ . Hence,  $g^{-1}(0) \neq \emptyset$ . In other words, there exists some  $j \in [n]$  such that  $g(j) = 0$ . Consider this  $j$ . Clearly,  $j \geq 1$  (since  $j \in [n]$ ), so that  $1 \leq j$ .

The map  $g$  is weakly increasing. Hence, from  $1 \leq j$ , we obtain  $g(1) \preceq g(j) = 0$ . Thus,  $g(1) = 0$  (since 0 is the smallest element of  $\mathcal{N}$ ). Thus,  $1 \in g^{-1}(0)$ .

Recall that each fiber  $g^{-1}(h)$  of  $g$  (with  $h \in \mathcal{N}$ ) is an interval of the totally ordered set  $[n]$ . Applying this to  $h = 0$ , we conclude that  $g^{-1}(0)$  is an interval of the totally ordered set  $[n]$ . This interval  $g^{-1}(0)$  is nonempty (since  $1 \in g^{-1}(0)$ ).

We have  $0 = g\left(\underbrace{j}_{\in [n]}\right) \in g([n])$ . Hence, Lemma 2.44 **(a)** (applied to  $u = 0$ ) yields  $\text{FE}(g) \cap g^{-1}(0) = \{\max(g^{-1}(0))\}$ . Now,

$$g^{-1}(0) \cap \underbrace{\text{Epk } \pi}_{\subseteq \text{FE}(g)} \subseteq g^{-1}(0) \cap \text{FE}(g) = \text{FE}(g) \cap g^{-1}(0) = \{\max(g^{-1}(0))\}.$$

Hence, Lemma 2.43 **(a)** (applied to  $S = g^{-1}(0)$ ) shows that the map  $\pi|_{g^{-1}(0)}$  is strictly increasing (since  $g^{-1}(0)$  is a nonempty interval of the totally ordered set  $[n]$  satisfying  $1 \in g^{-1}(0)$ ). This proves property **(i')**.]

[*Proof of property (ii')*: We must show that for each  $h \in g([n]) \cap \{1, 2, 3, \dots\}$ , the map  $\pi|_{g^{-1}(h)}$  is V-shaped. So let us fix  $h \in g([n]) \cap \{1, 2, 3, \dots\}$ .

Then,  $h \in g([n]) \cap \{1, 2, 3, \dots\} \subseteq g([n])$ . Thus, there exists some  $j \in [n]$  such that  $h = g(j)$ . In other words, the fiber  $g^{-1}(h)$  is nonempty. In other words,  $g^{-1}(h) \neq \emptyset$ . Also,  $g^{-1}(h)$  is an interval of the totally ordered set  $[n]$ <sup>39</sup>.

We have  $h \in g([n])$  and  $h \in g([n]) \cap \{1, 2, 3, \dots\} \subseteq \{1, 2, 3, \dots\}$ . Hence, Lemma 2.44 **(b)** (applied to  $u = h$ ) yields  $\text{FE}(g) \cap g^{-1}(h) = \{\min(g^{-1}(h)), \max(g^{-1}(h))\}$ . Now,

$$\begin{aligned} g^{-1}(h) \cap \underbrace{\text{Epk } \pi}_{\subseteq \text{FE}(g)} &\subseteq g^{-1}(h) \cap \text{FE}(g) = \text{FE}(g) \cap g^{-1}(h) \\ &= \{\min(g^{-1}(h)), \max(g^{-1}(h))\}. \end{aligned}$$

Hence, Lemma 2.43 **(c)** (applied to  $S = g^{-1}(h)$ ) shows that the map  $\pi|_{g^{-1}(h)}$  is V-shaped (since  $g^{-1}(h)$  is a nonempty interval of the totally ordered set  $[n]$ ). This proves property **(ii')**.]

<sup>39</sup>Indeed, we have previously shown that each fiber  $g^{-1}(h)$  of  $g$  (with  $h \in \mathcal{N}$ ) is an interval of the totally ordered set  $[n]$ .

[*Proof of property (iii')*]: We must show that the map  $\pi|_{g^{-1}(\infty)}$  is strictly decreasing. If  $g^{-1}(\infty) = \emptyset$ , then this is obvious. Thus, for the rest of this proof, we WLOG assume that we don't have  $g^{-1}(\infty) = \emptyset$ . Hence,  $g^{-1}(\infty) \neq \emptyset$ . In other words, there exists some  $j \in [n]$  such that  $g(j) = \infty$ . Consider this  $j$ . Clearly,  $j \leq n$  (since  $j \in [n]$ ).

The map  $g$  is weakly increasing. Hence, from  $j \leq n$ , we obtain  $g(j) \preceq g(n)$ . In view of  $g(j) = \infty$ , this rewrites as  $\infty \preceq g(n)$ . Thus,  $g(n) = \infty$  (since  $\infty$  is the largest element of  $\mathcal{N}$ ). Thus,  $n \in g^{-1}(\infty)$ .

Recall that each fiber  $g^{-1}(h)$  of  $g$  (with  $h \in \mathcal{N}$ ) is an interval of the totally ordered set  $[n]$ . Applying this to  $h = \infty$ , we conclude that  $g^{-1}(\infty)$  is an interval of the totally ordered set  $[n]$ . This interval  $g^{-1}(\infty)$  is nonempty (since  $n \in g^{-1}(\infty)$ ).

We have  $\infty = g\left(\underbrace{j}_{\in [n]}\right) \in g([n])$ . Hence, Lemma 2.44 (c) (applied to  $u = \infty$ ) yields  $\text{FE}(g) \cap g^{-1}(\infty) = \{\min(g^{-1}(\infty))\}$ . Now,

$$g^{-1}(\infty) \cap \underbrace{\text{Epk } \pi}_{\subseteq \text{FE}(g)} \subseteq g^{-1}(\infty) \cap \text{FE}(g) = \text{FE}(g) \cap g^{-1}(\infty) = \{\min(g^{-1}(\infty))\}.$$

Hence, Lemma 2.43 (b) (applied to  $S = g^{-1}(\infty)$ ) shows that the map  $\pi|_{g^{-1}(\infty)}$  is strictly decreasing (since  $g^{-1}(\infty)$  is a nonempty interval of the totally ordered set  $[n]$  satisfying  $n \in g^{-1}(\infty)$ ). This proves property (iii').]

We have now shown that the four properties (i'), (ii'), (iii') and (iv') of Definition 2.36 hold. In other words, the map  $g$  is  $\pi$ -amenable (by Definition 2.36). This proves Observation 2.]

Combining Observation 1 with Observation 2, we conclude that the map  $g$  is  $\pi$ -amenable if and only if  $\text{Epk } \pi \subseteq \text{FE}(g)$ . This proves Proposition 2.45.  $\square$

We can rewrite Proposition 2.39 as follows, exhibiting its analogy with [Stembr97, Proposition 2.2]:

**Proposition 2.46.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be any  $n$ -permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \text{Epk } \pi \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g.$$

*Proof of Proposition 2.46.* Each  $\pi$ -amenable map  $g: [n] \rightarrow \mathcal{N}$  is weakly increasing (because of Property (iv') in Definition 2.36). Thus, the  $\pi$ -amenable maps  $g$ :

$[n] \rightarrow \mathcal{N}$  are precisely the weakly increasing maps  $g : [n] \rightarrow \mathcal{N}$  that are  $\pi$ -amenable. Hence, we have the following equality of summation signs:

$$\sum_{\substack{g:[n] \rightarrow \mathcal{N} \\ \text{is } \pi\text{-amenable}}} = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ g \text{ is } \pi\text{-amenable}}} = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \text{Epk } \pi \subseteq \text{FE}(g)}}$$

(because for every weakly increasing map  $g : [n] \rightarrow \mathcal{N}$ , we have the logical equivalence

$$(g \text{ is } \pi\text{-amenable}) \iff (\text{Epk } \pi \subseteq \text{FE}(g))$$

(by Proposition 2.45)).

Now, Proposition 2.39 yields

$$\begin{aligned} \Gamma_{\mathcal{Z}}(\pi) &= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \\ \text{is } \pi\text{-amenable}}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \text{Epk } \pi \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g. \\ &= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \text{Epk } \pi \subseteq \text{FE}(g)}} \end{aligned}$$

This proves Proposition 2.46.  $\square$

**Definition 2.47.** Let  $n \in \mathbb{N}$ . If  $\Lambda$  is any subset of  $[n]$ , then we define a power series  $K_{n,\Lambda}^{\mathcal{Z}} \in \text{Pow } \mathcal{N}$  by

$$K_{n,\Lambda}^{\mathcal{Z}} = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g. \quad (27)$$

Thus, if  $\pi$  is an  $n$ -permutation, then Proposition 2.46 shows that

$$\Gamma_{\mathcal{Z}}(\pi) = K_{n,\text{Epk } \pi}^{\mathcal{Z}}. \quad (28)$$

(Indeed, (28) follows by comparing

$$\Gamma_{\mathcal{Z}}(\pi) = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \text{Epk } \pi \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g \quad (\text{by Proposition 2.46})$$

with

$$K_{n,\text{Epk } \pi}^{\mathcal{Z}} = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \text{Epk } \pi \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g. \quad (\text{by the definition of } K_{n,\text{Epk } \pi}^{\mathcal{Z}}).$$

)

**Remark 2.48.** Let  $n \in \mathbb{N}$ . Let  $\Lambda$  be any subset of  $[n]$ . It is easy to see that if  $g : [n] \rightarrow \mathcal{N}$  is a weakly increasing map, and if  $i \in [n]$ , then  $i \in \text{FE}(g)$  holds if and only if we don't have  $g(i-1) = g(i) = g(i+1)$ , where we use the convention that  $g(0) = 0$  and  $g(n+1) = \infty$ . Hence, a weakly increasing map  $g : [n] \rightarrow \mathcal{N}$  satisfies  $\Lambda \subseteq \text{FE}(g)$  if and only if no  $i \in \Lambda$  satisfies  $g(i-1) = g(i) = g(i+1)$ , where we use the convention that  $g(0) = 0$  and  $g(n+1) = \infty$ . Thus, (27) can be rewritten as follows:

$$\begin{aligned}
K_{n,\Lambda}^{\mathcal{Z}} &= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \text{no } i \in \Lambda \text{ satisfies } g(i-1)=g(i)=g(i+1) \\ \text{(where we set } g(0)=0 \text{ and } g(n+1)=\infty)}} 2^{|\{g([n]) \cap \{1,2,3,\dots\}|} x_g \\
&= \sum_{\substack{(g_1, g_2, \dots, g_n) \in \mathcal{N}^n; \\ 0 \leq g_1 \leq g_2 \leq \dots \leq g_n \leq \infty; \\ \text{no } i \in \Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ \text{(where we set } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1, g_2, \dots, g_n\} \cap \{1,2,3,\dots\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \quad (29)
\end{aligned}$$

(here, we have substituted  $(g_1, g_2, \dots, g_n)$  for  $(g(1), g(2), \dots, g(n))$  in the sum). For example,

$$\begin{aligned}
K_{3,\{1,3\}}^{\mathcal{Z}} &= \sum_{\substack{(g_1, g_2, g_3) \in \mathcal{N}^3; \\ 0 \leq g_1 \leq g_2 \leq g_3 \leq \infty; \\ \text{no } i \in \{1,3\} \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ \text{(where we set } g_0=0 \text{ and } g_4=\infty)}} 2^{|\{g_1, g_2, g_3\} \cap \{1,2,3,\dots\}|} x_{g_1} x_{g_2} x_{g_3} \\
&= \sum_{\substack{(g_1, g_2, g_3) \in \mathcal{N}^3; \\ 0 \leq g_1 \leq g_2 \leq g_3 \leq \infty; \\ \text{neither } 0=g_1=g_2 \text{ nor } g_2=g_3=\infty \text{ holds}}} 2^{|\{g_1, g_2, g_3\} \cap \{1,2,3,\dots\}|} x_{g_1} x_{g_2} x_{g_3}.
\end{aligned}$$

As a consequence of (29), we see that if we substitute 0 for  $x_0$  and for  $x_\infty$ , then  $K_{n,\Lambda}^{\mathcal{Z}}$  becomes the power series

$$\begin{aligned}
&\sum_{\substack{(g_1, g_2, \dots, g_n) \in \mathcal{N}^n; \\ 0 \leq g_1 \leq g_2 \leq \dots \leq g_n \leq \infty; \\ \text{no } i \in \Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ \text{(where we set } g_0=0 \text{ and } g_{n+1}=\infty); \\ \text{none of the } g_i \text{ equals 0 or } \infty}} 2^{|\{g_1, g_2, \dots, g_n\} \cap \{1,2,3,\dots\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \\
&= \sum_{\substack{(g_1, g_2, \dots, g_n) \in \{1,2,3,\dots\}^n; \\ g_1 \leq g_2 \leq \dots \leq g_n; \\ \text{no } i \in \Lambda \setminus \{1, n\} \text{ satisfies } g_{i-1}=g_i=g_{i+1}}} 2^{|\{g_1, g_2, \dots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}
\end{aligned}$$

in the indeterminates  $x_1, x_2, x_3, \dots$ . This is called the “shifted quasi-symmetric function  $\Theta_{\Lambda \setminus \{1, n\}}^n(X)$ ” in [BilHai95, (3.2)].

Corollary 2.30 now leads directly to the following multiplication rule (an analogue of [Stembr97, (3.1)]):

**Corollary 2.49.** Let  $n \in \mathbb{N}$  and  $m \in \mathbb{N}$ . Let  $\pi$  be an  $n$ -permutation. Let  $\sigma$  be an  $m$ -permutation such that  $\pi$  and  $\sigma$  are disjoint. Then,

$$K_{n,\text{Epk}\pi}^{\mathcal{Z}} \cdot K_{m,\text{Epk}\sigma}^{\mathcal{Z}} = \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\text{Epk}\tau}^{\mathcal{Z}}.$$

**Example 2.50.** Applying Corollary 2.49 to  $n = 2$ ,  $m = 1$ ,  $\pi = (1,2)$  and  $\sigma = (3)$ , we obtain

$$K_{2,\text{Epk}(1,2)}^{\mathcal{Z}} \cdot K_{1,\text{Epk}(3)}^{\mathcal{Z}} = K_{3,\text{Epk}(3,1,2)}^{\mathcal{Z}} + K_{3,\text{Epk}(1,3,2)}^{\mathcal{Z}} + K_{3,\text{Epk}(1,2,3)}^{\mathcal{Z}}.$$

In other words,

$$K_{2,\{2\}}^{\mathcal{Z}} \cdot K_{1,\{1\}}^{\mathcal{Z}} = K_{3,\{1,3\}}^{\mathcal{Z}} + K_{3,\{2\}}^{\mathcal{Z}} + K_{3,\{3\}}^{\mathcal{Z}}.$$

*Proof of Corollary 2.49.* From (28), we obtain  $\Gamma_{\mathcal{Z}}(\pi) = K_{n,\text{Epk}\pi}^{\mathcal{Z}}$ . Similarly,  $\Gamma_{\mathcal{Z}}(\sigma) = K_{m,\text{Epk}\sigma}^{\mathcal{Z}}$ . Multiplying these two equalities, we obtain  $\Gamma_{\mathcal{Z}}(\pi) \cdot \Gamma_{\mathcal{Z}}(\sigma) = K_{n+m,\text{Epk}\pi}^{\mathcal{Z}} \cdot K_{m,\text{Epk}\sigma}^{\mathcal{Z}}$ . Hence,

$$\begin{aligned} K_{n,\text{Epk}\pi}^{\mathcal{Z}} \cdot K_{m,\text{Epk}\sigma}^{\mathcal{Z}} &= \Gamma_{\mathcal{Z}}(\pi) \cdot \Gamma_{\mathcal{Z}}(\sigma) = \sum_{\tau \in S(\pi,\sigma)} \underbrace{\Gamma_{\mathcal{Z}}(\tau)}_{=K_{n+m,\text{Epk}\tau}^{\mathcal{Z}} \text{ (by (28))}} \quad (\text{by Corollary 2.30}) \\ &= \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\text{Epk}\tau}^{\mathcal{Z}}. \end{aligned}$$

This proves Corollary 2.49.  $\square$

The following lemma is a variant of the principle of inclusion and exclusion tailored to our setting:

**Lemma 2.51.** Let  $n \in \mathbb{N}$ . For each subset  $\Lambda$  of  $[n]$ , define a power series  $L_{n,\Lambda}^{\mathcal{Z}} \in \text{Pow } \mathcal{N}$  by

$$L_{n,\Lambda}^{\mathcal{Z}} = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \cap \text{FE}(g) = \emptyset}} 2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g. \quad (30)$$

Then:

(a) For each subset  $\Lambda$  of  $[n]$ , we have

$$K_{n,\Lambda}^{\mathcal{Z}} = \sum_{Q \subseteq \Lambda} (-1)^{|Q|} L_{n,Q}^{\mathcal{Z}}.$$

(b) For each subset  $\Lambda$  of  $[n]$ , we have

$$L_{n,\Lambda}^{\mathcal{Z}} = \sum_{Q \subseteq \Lambda} (-1)^{|Q|} K_{n,Q}^{\mathcal{Z}}.$$

*Proof of Lemma 2.51.* Recall the following known fact: If  $R$  is a finite set, then

$$\sum_{Q \subseteq R} (-1)^{|Q|} = \begin{cases} 1, & \text{if } R = \emptyset; \\ 0, & \text{if } R \neq \emptyset. \end{cases} \quad (31)$$

Let  $\Lambda$  be a subset of  $[n]$ . The subsets  $Q$  of  $\Lambda$  satisfying  $Q \cap \text{FE}(g) = \emptyset$  are precisely the subsets of  $\Lambda \setminus \text{FE}(g)$ . Therefore,

$$\begin{aligned} \sum_{\substack{Q \subseteq \Lambda; \\ Q \cap \text{FE}(g) = \emptyset}} (-1)^{|Q|} &= \sum_{Q \subseteq \Lambda \setminus \text{FE}(g)} (-1)^{|Q|} = \begin{cases} 1, & \text{if } \Lambda \setminus \text{FE}(g) = \emptyset; \\ 0, & \text{if } \Lambda \setminus \text{FE}(g) \neq \emptyset \end{cases} \quad (\text{by (31)}) \\ &= \begin{cases} 1, & \text{if } \Lambda \subseteq \text{FE}(g); \\ 0, & \text{if } \Lambda \not\subseteq \text{FE}(g). \end{cases} \quad (32) \end{aligned}$$

Also, the subsets  $Q$  of  $\Lambda$  satisfying  $Q \subseteq \text{FE}(g)$  are precisely the subsets of  $\Lambda \cap \text{FE}(g)$ . Thus,

$$\sum_{\substack{Q \subseteq \Lambda; \\ Q \subseteq \text{FE}(g)}} (-1)^{|Q|} = \sum_{Q \subseteq \Lambda \cap \text{FE}(g)} (-1)^{|Q|} = \begin{cases} 1, & \text{if } \Lambda \cap \text{FE}(g) = \emptyset; \\ 0, & \text{if } \Lambda \cap \text{FE}(g) \neq \emptyset \end{cases} \quad (33)$$

(by (31)).

Now, forget that we fixed  $\Lambda$ . We thus have proven (32) and (33) for each subset  $\Lambda$  of  $[n]$ .

(a) Let  $\Lambda$  be a subset of  $[n]$ . Then,

$$\begin{aligned}
& \sum_{Q \subseteq \Lambda} (-1)^{|Q|} \underbrace{L_{n,Q}^Z}_{=} \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ Q \cap \text{FE}(g) = \emptyset}} 2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g \\
&\quad \text{(by the definition of } L_{n,Q}^Z) \\
&= \sum_{Q \subseteq \Lambda} (-1)^{|Q|} \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ Q \cap \text{FE}(g) = \emptyset}} 2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing}}} \underbrace{\left( \sum_{\substack{Q \subseteq \Lambda; \\ Q \cap \text{FE}(g) = \emptyset}} (-1)^{|Q|} \right)}_{=} 2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g \\
&= \begin{cases} 1, & \text{if } \Lambda \subseteq \text{FE}(G); \\ 0, & \text{if } \Lambda \not\subseteq \text{FE}(G) \end{cases} \\
&\quad \text{(by (32))} \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing}}} \begin{cases} 1, & \text{if } \Lambda \subseteq \text{FE}(G); \\ 0, & \text{if } \Lambda \not\subseteq \text{FE}(G) \end{cases} 2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \subseteq \text{FE}(g)}} 2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g = K_{n,\Lambda}^Z
\end{aligned}$$

(by (27)). This proves Lemma 2.51 (a).

(b) Let  $\Lambda$  be a subset of  $[n]$ . Then,

$$\begin{aligned}
\sum_{Q \subseteq \Lambda} (-1)^{|Q|} &= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ Q \subseteq \text{FE}(g)}} \underbrace{K_{n,Q}^{\mathcal{Z}}}_{2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g} \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ Q \subseteq \text{FE}(g)}} (-1)^{|Q|} \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ Q \subseteq \text{FE}(g)}} 2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing}}} \underbrace{\left( \sum_{\substack{Q \subseteq \Lambda; \\ Q \subseteq \text{FE}(g)}} (-1)^{|Q|} \right)}_{\substack{= \begin{cases} 1, & \text{if } \Lambda \cap \text{FE}(G) = \emptyset; \\ 0, & \text{if } \Lambda \cap \text{FE}(G) \neq \emptyset \end{cases} \\ \text{(by (33))}}} 2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing}}} \begin{cases} 1, & \text{if } \Lambda \cap \text{FE}(G) = \emptyset; \\ 0, & \text{if } \Lambda \cap \text{FE}(G) \neq \emptyset \end{cases} 2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g \\
&= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \cap \text{FE}(g) = \emptyset}} 2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g = L_{n,\Lambda}^{\mathcal{Z}}
\end{aligned}$$

(by (30)). This proves Lemma 2.51 (b). □

Recall Definition 2.2.

**Proposition 2.52.** Let  $n \in \mathbb{N}$ . Then, the family

$$\left( K_{n,\Lambda}^{\mathcal{Z}} \right)_{\Lambda \in \mathbf{L}_n}$$

is  $\mathbb{Q}$ -linearly independent.

We shall give two proofs of Proposition 2.52. The first one relies on studying the coefficients of  $K_{n,\Lambda}^{\mathcal{Z}}$ ; it needs the following definition:

**Definition 2.53.** Let  $\mathfrak{m}$  be any monomial in  $\text{Pow } \mathcal{N}$  (that is, a formal commutative product of indeterminates  $x_h$  with  $h \in \mathcal{N}$ ). Let  $f \in \text{Pow } \mathcal{N}$ . Then,  $[\mathfrak{m}](f)$  shall mean the coefficient of  $\mathfrak{m}$  in the power series  $f$ . (For example,  $[x_0^2 x_3](3 + 5x_0^2 x_3 + 6x_0 + 9x_\infty) = 5$  and  $[x_0^2 x_3](x_1 - x_\infty) = 0$ .)



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**Lemma 2.54.** Let  $n \in \mathbb{N}$ .

(a) If  $g$  and  $h$  are two weakly increasing maps  $[n] \rightarrow \mathcal{N}$ , then

$$\begin{cases} 1, & \text{if } \mathbf{x}_g = \mathbf{x}_h; \\ 0, & \text{if } \mathbf{x}_g \neq \mathbf{x}_h \end{cases} = \begin{cases} 1, & \text{if } g = h; \\ 0, & \text{if } g \neq h \end{cases}.$$

(b) Let  $R \in \mathbf{L}_n$ . Let  $h : [n] \rightarrow \mathcal{N}$  be a weakly increasing map. Then,

$$[\mathbf{x}_h] \left( K_{n,R}^{\mathcal{Z}} \right) = \begin{cases} 2^{|h([n]) \cap \{1,2,3,\dots\}|}, & \text{if } R \subseteq \text{FE}(h); \\ 0, & \text{otherwise} \end{cases}.$$

*Proof of Lemma 2.54.* (a) A weakly increasing map  $g : [n] \rightarrow \mathcal{N}$  can be uniquely reconstructed from the **multiset**  $\{g(1), g(2), \dots, g(n)\}_{\text{multi}}$  of its values (because it is weakly increasing, so there is only one way in which these values can be ordered). Hence, a weakly increasing map  $g : [n] \rightarrow \mathcal{N}$  can be uniquely reconstructed from the monomial  $\mathbf{x}_g$  (since this monomial  $\mathbf{x}_g = x_{g(1)}x_{g(2)} \cdots x_{g(n)}$  encodes the multiset  $\{g(1), g(2), \dots, g(n)\}_{\text{multi}}$ ). In other words, if  $g$  and  $h$  are two weakly increasing maps  $[n] \rightarrow \mathcal{N}$ , then  $\mathbf{x}_g = \mathbf{x}_h$  holds if and only if  $g = h$ . Hence, if  $g$  and  $h$  are two weakly increasing maps  $[n] \rightarrow \mathcal{N}$ , then

$$\begin{cases} 1, & \text{if } \mathbf{x}_g = \mathbf{x}_h; \\ 0, & \text{if } \mathbf{x}_g \neq \mathbf{x}_h \end{cases} = \begin{cases} 1, & \text{if } g = h; \\ 0, & \text{if } g \neq h \end{cases}.$$

This proves Lemma 2.54 (a).

(b) The definition of  $K_{n,R}^{\mathcal{Z}}$  yields

$$K_{n,R}^{\mathcal{Z}} = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g.$$

Thus,

$$\begin{aligned}
[\mathbf{x}_h] \left( K_{n,R}^{\mathcal{Z}} \right) &= [\mathbf{x}_h] \left( \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g \right) \\
&= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \underbrace{[\mathbf{x}_h] (\mathbf{x}_g)}_{= \begin{cases} 1, & \text{if } \mathbf{x}_g = \mathbf{x}_h \\ 0, & \text{if } \mathbf{x}_g \neq \mathbf{x}_h \end{cases}} \\
&\hspace{15em} \text{(since } \mathbf{x}_h \text{ and } \mathbf{x}_g \text{ are two monomials)} \\
&= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \underbrace{\begin{cases} 1, & \text{if } \mathbf{x}_g = \mathbf{x}_h; \\ 0, & \text{if } \mathbf{x}_g \neq \mathbf{x}_h \end{cases}}_{= \begin{cases} 1, & \text{if } g = h; \\ 0, & \text{if } g \neq h \end{cases}} \\
&\hspace{15em} \text{(by Lemma 2.54 (a))} \\
&= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} \begin{cases} 1, & \text{if } g = h; \\ 0, & \text{if } g \neq h \end{cases} \\
&= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g); \\ g=h}} 2^{|g([n]) \cap \{1,2,3,\dots\}|}.
\end{aligned}$$

The sum on the right hand side of this equality has a unique addend (namely, its addend for  $g = h$ , which is  $2^{|h([n]) \cap \{1,2,3,\dots\}|}$ ) when  $R \subseteq \text{FE}(h)$ ; otherwise it is an empty sum. Hence, this sum simplifies as follows:

$$\sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g); \\ g=h}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} = \begin{cases} 2^{|h([n]) \cap \{1,2,3,\dots\}|}, & \text{if } R \subseteq \text{FE}(h); \\ 0, & \text{otherwise} \end{cases}.$$

Hence,

$$[\mathbf{x}_h] \left( K_{n,R}^{\mathcal{Z}} \right) = \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ R \subseteq \text{FE}(g); \\ g=h}} 2^{|g([n]) \cap \{1,2,3,\dots\}|} = \begin{cases} 2^{|h([n]) \cap \{1,2,3,\dots\}|}, & \text{if } R \subseteq \text{FE}(h); \\ 0, & \text{otherwise} \end{cases}.$$

This proves Lemma 2.54 (b). □

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*First proof of Proposition 2.52.* Recall Definition 2.6. In the following, we shall regard the set  $\mathbf{P}$  as a totally ordered set, equipped with the order from Proposition 2.7.

Clearly,  $\mathbf{L}_n$  is a set of subsets of  $[n]$ , and thus a set of finite subsets of  $\mathbb{Z}$ . In other words,  $\mathbf{L}_n$  is a subset of  $\mathbf{P}$ . Hence, we consider  $\mathbf{L}_n$  as a totally ordered set, whose total order is inherited from  $\mathbf{P}$ .

Let  $(a_R)_{R \in \mathbf{L}_n} \in \mathbb{Q}^{\mathbf{L}_n}$  be a family of scalars (in  $\mathbb{Q}$ ) such that  $\sum_{R \in \mathbf{L}_n} a_R K_{n,R}^{\mathbb{Z}} = 0$ . We are going to show that  $(a_R)_{R \in \mathbf{L}_n} = (0)_{R \in \mathbf{L}_n}$ .

Indeed, assume the contrary. Thus,  $(a_R)_{R \in \mathbf{L}_n} \neq (0)_{R \in \mathbf{L}_n}$ . Hence, there exists some  $R \in \mathbf{L}_n$  such that  $a_R \neq 0$ . Let  $\Lambda$  be the **largest** such  $R$  (with respect to the total order on  $\mathbf{L}_n$  we have introduced above). Hence,  $\Lambda$  is an element of  $\mathbf{L}_n$  and satisfies  $a_\Lambda \neq 0$ ; but every element  $R \in \mathbf{L}_n$  satisfying  $R > \Lambda$  must satisfy

$$a_R = 0. \quad (34)$$

We have  $\Lambda \in \mathbf{L}_n$ . Thus,  $\Lambda$  is a subset of  $[n]$  (since  $\mathbf{L}_n$  is a set of subsets of  $[n]$ ). In other words,  $\Lambda \subseteq [n]$ .

Lemma 2.41 shows that there exists a weakly increasing map  $g : [n] \rightarrow \mathcal{N}$  such that  $\text{FE}(g) = (\Lambda \cup (\Lambda + 1)) \cap [n]$ . Consider this  $g$ . Combining  $\Lambda \subseteq \Lambda \cup (\Lambda + 1)$  with  $\Lambda \subseteq [n]$ , we obtain

$$\Lambda \subseteq (\Lambda \cup (\Lambda + 1)) \cap [n] = \text{FE}(g).$$

For every  $R \in \mathbf{L}_n$  satisfying  $R \neq \Lambda$ , we have

$$[\mathbf{x}_g] \left( a_R K_{n,R}^{\mathbb{Z}} \right) = 0. \quad (35)$$

[*Proof of (35):* Let  $R \in \mathbf{L}_n$  be such that  $R \neq \Lambda$ . We must prove (35).

Assume the contrary. Thus,  $[\mathbf{x}_g] \left( a_R K_{n,R}^{\mathbb{Z}} \right) \neq 0$ . Therefore,  $a_R [\mathbf{x}_g] \left( K_{n,R}^{\mathbb{Z}} \right) = [\mathbf{x}_g] \left( a_R K_{n,R}^{\mathbb{Z}} \right) \neq 0$ . Hence,  $a_R \neq 0$  and  $[\mathbf{x}_g] \left( K_{n,R}^{\mathbb{Z}} \right) \neq 0$ . Thus, we don't have  $[\mathbf{x}_g] \left( K_{n,R}^{\mathbb{Z}} \right) = 0$  (since  $[\mathbf{x}_g] \left( K_{n,R}^{\mathbb{Z}} \right) \neq 0$ ).

Every element of  $\mathbf{L}_n$  is a lacunar subset of  $[n]$ <sup>40</sup>. Hence,  $R$  is a lacunar subset of  $[n]$  (since  $R \in \mathbf{L}_n$ ).

But Lemma 2.54 (b) (applied to  $h = g$ ) yields

$$[\mathbf{x}_g] \left( K_{n,R}^{\mathbb{Z}} \right) = \begin{cases} 2^{|\text{g}([n]) \cap \{1,2,3,\dots\}|}, & \text{if } R \subseteq \text{FE}(g); \\ 0, & \text{otherwise} \end{cases}.$$

Hence,  $[\mathbf{x}_g] \left( K_{n,R}^{\mathbb{Z}} \right) = 0$  if  $R \not\subseteq \text{FE}(g)$ . Thus, we cannot have  $R \not\subseteq \text{FE}(g)$  (since we don't have  $[\mathbf{x}_g] \left( K_{n,R}^{\mathbb{Z}} \right) = 0$ ). Therefore, we have

$$R \subseteq \text{FE}(g) = (\Lambda \cup (\Lambda + 1)) \cap [n] \subseteq \Lambda \cup (\Lambda + 1).$$

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<sup>40</sup>Indeed, this is obvious when  $n$  is positive (since  $\mathbf{L}_n$  is defined to be the set of all nonempty lacunar subsets of  $[n]$  in this case), but also obvious when  $n = 0$  (since  $\mathbf{L}_n = \{\emptyset\}$  in this case, but the set  $\emptyset$  is a lacunar subset of  $[n]$ ).

Thus, Proposition 2.9 yields that  $R \geq \Lambda$  (with respect to the total order on  $\mathbf{P}$ ). Combining this with  $R \neq \Lambda$ , we obtain  $R > \Lambda$ . Hence, (34) yields  $a_R = 0$ . This contradicts  $a_R \neq 0$ . This contradiction shows that our assumption was wrong. Hence, (35) is proven.]

On the other hand, Lemma 2.54 (b) (applied to  $h = g$  and  $R = \Lambda$ ) yields

$$[\mathbf{x}_g] \left( K_{n,\Lambda}^{\mathcal{Z}} \right) = \begin{cases} 2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|}, & \text{if } \Lambda \subseteq \text{FE}(g); \\ 0, & \text{otherwise} \end{cases} = 2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|}$$

(since  $\Lambda \subseteq \text{FE}(g)$ ).

Hence,

$$[\mathbf{x}_g] \left( a_{\Lambda} K_{n,\Lambda}^{\mathcal{Z}} \right) = a_{\Lambda} \underbrace{[\mathbf{x}_g] \left( K_{n,\Lambda}^{\mathcal{Z}} \right)}_{=2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|}} = \underbrace{a_{\Lambda}}_{\neq 0} \underbrace{2^{|\mathcal{g}([n]) \cap \{1,2,3,\dots\}|}}_{\neq 0} \neq 0.$$

Now, recall that  $\sum_{R \in \mathbf{L}_n} a_R K_{n,R}^{\mathcal{Z}} = 0$ . Hence,  $[\mathbf{x}_g] \left( \sum_{R \in \mathbf{L}_n} a_R K_{n,R}^{\mathcal{Z}} \right) = [\mathbf{x}_g] (0) = 0$ .

Therefore,

$$\begin{aligned} 0 &= [\mathbf{x}_g] \left( \sum_{R \in \mathbf{L}_n} a_R K_{n,R}^{\mathcal{Z}} \right) = \sum_{R \in \mathbf{L}_n} [\mathbf{x}_g] \left( a_R K_{n,R}^{\mathcal{Z}} \right) \\ &= [\mathbf{x}_g] \left( a_{\Lambda} K_{n,\Lambda}^{\mathcal{Z}} \right) + \sum_{\substack{R \in \mathbf{L}_n; \\ R \neq \Lambda}} \underbrace{[\mathbf{x}_g] \left( a_R K_{n,R}^{\mathcal{Z}} \right)}_{\substack{=0 \\ \text{(by (35))}}} \\ &\quad \left( \begin{array}{c} \text{here, we have split off the addend} \\ \text{for } R = \Lambda \text{ from the sum (since } \Lambda \in \mathbf{L}_n \text{)} \end{array} \right) \\ &= [\mathbf{x}_g] \left( a_{\Lambda} K_{n,\Lambda}^{\mathcal{Z}} \right) + \underbrace{\sum_{\substack{R \in \mathbf{L}_n; \\ R \neq \Lambda}} 0}_{=0} = [\mathbf{x}_g] \left( a_{\Lambda} K_{n,\Lambda}^{\mathcal{Z}} \right) \neq 0. \end{aligned}$$

This is absurd. This contradiction shows that our assumption was false. Hence,  $(a_R)_{R \in \mathbf{L}_n} = (0)_{R \in \mathbf{L}_n}$  is proven.

Now, forget that we fixed  $(a_R)_{R \in \mathbf{L}_n}$ . We thus have shown that if  $(a_R)_{R \in \mathbf{L}_n} \in \mathbf{Q}^{\mathbf{L}_n}$  is a family of scalars (in  $\mathbf{Q}$ ) such that  $\sum_{R \in \mathbf{L}_n} a_R K_{n,R}^{\mathcal{Z}} = 0$ , then  $(a_R)_{R \in \mathbf{L}_n} = (0)_{R \in \mathbf{L}_n}$ . In other words, the family  $\left( K_{n,R}^{\mathcal{Z}} \right)_{R \in \mathbf{L}_n}$  is  $\mathbf{Q}$ -linearly independent. In other words, the family  $\left( K_{n,\Lambda}^{\mathcal{Z}} \right)_{\Lambda \in \mathbf{L}_n}$  is  $\mathbf{Q}$ -linearly independent. This proves Proposition 2.52.  $\square$

*Second proof of Proposition 2.52 (sketched).* Let  $\Omega$  be the subset

$$\{1, 3, 5, \dots\} \cap [n] = \{i \in [n] \mid i \text{ is odd}\}$$

of  $[n]$ . This is clearly a lacunar subset of  $[n]$ .

We are going to prove the following claim:

*Claim 1: (a)* If  $n$  is odd, then the only syzygy<sup>41</sup> of the family  $\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is  $\sum_{\Lambda \subseteq \Omega} (-1)^{|\Lambda|} K_{n,\Lambda}^{\mathbb{Z}} = 0$ .

*(b)* If  $n$  is even, then the family  $\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is  $\mathbb{Q}$ -linearly independent.

Claim 1 (once proven) will clearly yield Proposition 2.52. Indeed, the family  $\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \in \mathbf{L}_n}$  is a subfamily of the family  $\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$ , which is obtained from the latter family by removing the element  $K_{n,\emptyset}^{\mathbb{Z}}$  when  $n > 0$  (by the definition of  $\mathbf{L}_n$ ). But Claim 1 shows that all nontrivial syzygies of the latter family (if there are any to begin with) involve the element  $K_{n,\emptyset}^{\mathbb{Z}}$ , and thus disappear when this element is removed. Hence, it lets us conclude that the former family is  $\mathbb{Q}$ -linearly independent. Thus, it remains to prove Claim 1.

For each subset  $\Lambda$  of  $[n]$ , define a power series  $L_{n,\Lambda}^{\mathbb{Z}} \in \text{Pow } \mathcal{N}$  by (30). Then, for each lacunar subset  $\Lambda$  of  $[n]$ , we have

$$K_{n,\Lambda}^{\mathbb{Z}} = \sum_{Q \subseteq \Lambda} (-1)^{|Q|} L_{n,Q}^{\mathbb{Z}} \quad (\text{by Lemma 2.51 (a)}) \text{ and}$$

$$L_{n,\Lambda}^{\mathbb{Z}} = \sum_{Q \subseteq \Lambda} (-1)^{|Q|} K_{n,Q}^{\mathbb{Z}} \quad (\text{by Lemma 2.51 (b)}).$$

Hence, the two families  $\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  and  $\left(L_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  can be obtained from each other by a unitriangular transition matrix (unitriangular with respect to inclusion<sup>42</sup>). Thus, the syzygies of these two families are in bijection with each other. Hence, in order to prove Claim 1, it suffices to prove the following claim:

*Claim 2: (a)* If  $n$  is odd, then the only syzygy of the family  $\left(L_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is  $L_{n,\Omega}^{\mathbb{Z}} = 0$ .

<sup>41</sup>If  $(v_h)_{h \in H}$  is a family of vectors in a vector space over a field  $\mathbb{F}$ , then a *syzygy* of this family  $(v_h)_{h \in H}$  means a family  $(\lambda_h)_{h \in H} \in \mathbb{F}^H$  of scalars in  $\mathbb{F}$  satisfying  $\sum_{h \in H} \lambda_h v_h = 0$ .

Thus, a syzygy is what is commonly called a “linear dependence relation” (at least when the scalars  $\lambda_h$  are not all 0). By abuse of notation, we shall speak of the “syzygy  $\sum_{h \in H} \lambda_h v_h = 0$ ” meaning not the equality  $\sum_{h \in H} \lambda_h v_h = 0$  but the family of coefficients  $(\lambda_h)_{h \in H}$ .

When we say “the only syzygy”, we mean “the only nonzero syzygy up to scalar multiples”.

<sup>42</sup>We are using the fact that a subset of a lacunar subset is lacunar.

(b) If  $n$  is even, then the family  $\left(L_{n,\Lambda}^{\mathcal{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is  $\mathbb{Q}$ -linearly independent.

Let  $G$  be the set of all weakly increasing maps  $g : [n] \rightarrow \mathcal{N}$ . Let  $R$  be the free  $\mathbb{Q}$ -vector space with basis  $G$ ; its standard basis will be denoted by  $([g])_{g \in G}$ . We define a  $\mathbb{Q}$ -linear map

$$\begin{aligned} \Phi : R &\rightarrow \text{Pow } \mathcal{N}, \\ [g] &\mapsto 2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g. \end{aligned}$$

This map  $\Phi$  is easily seen to be injective (since the maps  $g \in G$  are weakly increasing, and thus can be uniquely recovered from the monomials  $\mathbf{x}_g$ ).

For each subset  $\Lambda$  of  $[n]$ , we define an element  $\tilde{L}_\Lambda$  of  $R$  by

$$\tilde{L}_\Lambda = \sum_{\substack{g \in G; \\ \Lambda \cap \text{FE}(g) = \emptyset}} [g].$$

Then, each subset  $\Lambda$  of  $[n]$  satisfies

$$\begin{aligned} L_{n,\Lambda}^{\mathcal{Z}} &= \sum_{\substack{g: [n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \cap \text{FE}(g) = \emptyset}} \underbrace{2^{|g([n]) \cap \{1,2,3,\dots\}|} \mathbf{x}_g}_{\substack{= \Phi([g]) \\ \text{(by the definition of } \Phi)}} = \sum_{\substack{g \in G; \\ \Lambda \cap \text{FE}(g) = \emptyset}} \Phi([g]) \\ &= \sum_{\substack{g \in G; \\ \Lambda \cap \text{FE}(g) = \emptyset}} \underbrace{[g]}_{\substack{\text{(by the definition of } G)}} \\ &= \Phi \left( \underbrace{\sum_{\substack{g \in G; \\ \Lambda \cap \text{FE}(g) = \emptyset}} [g]}_{= \tilde{L}_\Lambda} \right) = \Phi(\tilde{L}_\Lambda). \end{aligned}$$

Hence, the family  $\left(L_{n,\Lambda}^{\mathcal{Z}}\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is the image of the family  $\left(\tilde{L}_\Lambda\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  under the map  $\Phi$ . Thus, the syzygies of the two families are in bijection (since the map  $\Phi$  is injective<sup>43</sup>). Hence, in order to prove Claim 2, it suffices to prove the following claim:

<sup>43</sup>We are here using the following obvious fact:

Let  $V$  and  $W$  be two vector spaces over a field  $\mathbb{F}$ . Let  $(v_h)_{h \in H} \in V^H$  be a family of vectors in  $V$ . Let  $\phi : V \rightarrow W$  be an injective  $\mathbb{F}$ -linear map. Then, the syzygies of the families  $(v_h)_{h \in H} \in V^H$  and  $(\phi(v_h))_{h \in H} \in W^H$  are in bijection. (Actually, these syzygies, when regarded as families of scalars, are literally the same.) In particular, the family  $(v_h)_{h \in H}$  is  $\mathbb{F}$ -linearly independent if and only if the family  $(\phi(v_h))_{h \in H}$  is  $\mathbb{F}$ -linearly independent.

*Claim 3: (a)* If  $n$  is odd, then the only syzygy of the family  $(\tilde{L}_\Lambda)_{\Lambda \subseteq [n]}$  is lacunar is  $\tilde{L}_\Omega = 0$ .

*(b)* If  $n$  is even, then the family  $(\tilde{L}_\Lambda)_{\Lambda \subseteq [n]}$  is  $\mathbb{Q}$ -linearly independent.

Let us agree that if  $g \in G$ , then we will set  $g(0) = 0$  and  $g(n+1) = \infty$ . Hence,  $g(i)$  will be a well-defined element of  $\mathcal{N}$  for each  $i \in \{0, 1, \dots, n+1\}$ .

If  $g \in G$ , then we let  $\text{Stag}(g)$  be the subset  $\{i \in [n+1] \mid g(i) = g(i-1)\}$  of  $[n+1]$ . It is easy to see that the family  $\left( \sum_{\substack{g \in G; \\ \text{Stag}(g)=T}} [g] \right)_{T \subseteq [n+1]; T \neq [n+1]}$  of elements

of  $R$  is  $\mathbb{Q}$ -linearly independent<sup>44</sup>. Hence, the family  $\left( \sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq T}} [g] \right)_{T \subseteq [n+1]; T \neq [n+1]}$  of elements of  $R$  is  $\mathbb{Q}$ -linearly independent, too (because this family is ob-

tained from the previous family  $\left( \sum_{\substack{g \in G; \\ \text{Stag}(g)=T}} [g] \right)_{T \subseteq [n+1]; T \neq [n+1]}$  via a unitriangular change-of-basis matrix<sup>45</sup>). Therefore, the only syzygy of the family

$\left( \sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq T}} [g] \right)_{T \subseteq [n+1]}$  is  $\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq [n+1]}} [g] = 0$  (since it is easy to see that no  $g \in G$  satisfies  $\text{Stag}(g) \supseteq [n+1]$ , which is why  $\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq [n+1]}} [g]$  is indeed 0).

But if  $\Lambda$  is a lacunar subset of  $[n]$ , and if  $g \in G$ , then we have the following

<sup>44</sup>*Proof.* Clearly, any two elements of this family are supported on different basis elements (i.e., any  $[g]$  appearing in one of them cannot appear in any other). It thus remains to show that these elements are  $\neq 0$ . In other words, it remains to show that for any proper subset  $T$  of  $[n+1]$ , we have  $\sum_{\substack{g \in G; \\ \text{Stag}(g)=T}} [g] \neq 0$ . But this is easy: Just construct some  $g \in G$  satisfying

$$\text{Stag}(g) = T.$$

<sup>45</sup>unitriangular with respect to the reverse inclusion order (notice that  $\sum_{\substack{g \in G; \\ \text{Stag}(g)=T}} [g] = 0$  for  $T = [n+1]$ , so the exclusion of  $[n+1]$  makes sense and does not mess up our computations)

logical equivalence:

$$\begin{aligned}
& (\Lambda \cap \text{FE}(g) = \emptyset) \\
& \iff (\text{no } i \in \Lambda \text{ satisfies } i \in \text{FE}(g)) \\
& \iff (\text{each } i \in \Lambda \text{ satisfies } i \notin \text{FE}(g)) \\
& \iff \left( \begin{array}{l} \text{each } i \in \Lambda \text{ satisfies } i \neq \underbrace{\min(g^{-1}(h))}_{\substack{\iff (g(i)=g(i-1)) \\ (\text{since } g \text{ is weakly increasing, and } g(0)=0)}} \text{ for all } h \in \{1, 2, 3, \dots, \infty\} \\ \\ \text{and } i \neq \underbrace{\max(g^{-1}(h))}_{\substack{\iff (g(i)=g(i+1)) \\ (\text{since } g \text{ is weakly increasing, and } g(n+1)=\infty)}} \text{ for all } h \in \{0, 1, 2, 3, \dots\}} \end{array} \right) \\
& \iff \left( \begin{array}{l} \text{each } i \in \Lambda \text{ satisfies } \underbrace{g(i) = g(i-1)}_{\iff (i \in \text{Stag}(g))} \text{ and } \underbrace{g(i) = g(i+1)}_{\iff (i+1 \in \text{Stag}(g))} \end{array} \right) \\
& \iff (\text{each } i \in \Lambda \text{ satisfies } i \in \text{Stag}(g) \text{ and } i+1 \in \text{Stag}(g)) \\
& \iff (\Lambda \cup (\Lambda + 1) \subseteq \text{Stag}(g)) \iff (\text{Stag}(g) \supseteq \Lambda \cup (\Lambda + 1)).
\end{aligned}$$

Hence, if  $\Lambda$  is a lacunar subset of  $[n]$ , then the condition  $\Lambda \cap \text{FE}(g) = \emptyset$  is equivalent to the condition  $\text{Stag}(g) \supseteq \Lambda \cup (\Lambda + 1)$ . Thus, for each lacunar subset  $\Lambda$  of  $[n]$ , the definition of  $\tilde{L}_\Lambda$  becomes

$$\tilde{L}_\Lambda = \sum_{\substack{g \in G; \\ \Lambda \cap \text{FE}(g) = \emptyset}} [g] = \sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq \Lambda \cup (\Lambda + 1)}} [g].$$

Hence, the family  $(\tilde{L}_\Lambda)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is a subfamily of the family  $\left( \sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq T}} [g] \right)_{T \subseteq [n+1]}$

(because if  $\Lambda$  is a lacunar subset of  $[n]$ , then  $\Lambda \cup (\Lambda + 1)$  is a well-defined subset of  $[n+1]$ , and moreover  $\Lambda$  can be uniquely recovered from  $\Lambda \cup (\Lambda + 1)$  <sup>46</sup>). The further argument depends on the parity of  $n$ :

- If  $n$  is odd, then the vanishing element  $\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq [n+1]}} [g]$  does appear in the

<sup>46</sup>This takes a bit of thought to check. You need to show that if  $\Lambda_1$  and  $\Lambda_2$  are two lacunar subsets of  $[n]$  satisfying  $\Lambda_1 \cup (\Lambda_1 + 1) = \Lambda_2 \cup (\Lambda_2 + 1)$ , then  $\Lambda_1 = \Lambda_2$ . This follows from Corollary 2.10.



family  $\left(\tilde{L}_\Lambda\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$ , because there exists a lacunar subset  $\Lambda$  of  $[n]$  satisfying  $\Lambda \cup (\Lambda + 1) = [n + 1]$ : Namely, this  $\Lambda$  is  $\Omega$ . Thus, the only syzygy of the family  $\left(\tilde{L}_\Lambda\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is  $\tilde{L}_\Omega = 0$  (since the only syzygy of

the family  $\left(\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq T}} [g]\right)_{T \subseteq [n+1]}$  is  $\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq [n+1]}} [g] = 0$ ).

- If  $n$  is even, then the vanishing element  $\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq [n+1]}} [g]$  does not appear

in the family  $\left(\tilde{L}_\Lambda\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$ , since no lacunar subset  $\Lambda$  of  $[n]$  satisfies  $\Lambda \cup (\Lambda + 1) = [n + 1]$ . Hence, the syzygy  $\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq [n+1]}} [g] = 0$  of the

family  $\left(\sum_{\substack{g \in G; \\ \text{Stag}(g) \supseteq T}} [g]\right)_{T \subseteq [n+1]}$  disappears when we pass to the subfamily

$\left(\tilde{L}_\Lambda\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$ . Consequently, the subfamily  $\left(\tilde{L}_\Lambda\right)_{\Lambda \subseteq [n] \text{ is lacunar}}$  is  $\mathbb{Q}$ -linearly independent.

This proves Claim 3. As explained above, this yields Claim 2, hence also Claim 1, and thus completes the proof of Proposition 2.52.  $\square$

**Corollary 2.55.** The family

$$\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{n \in \mathbb{N}; \Lambda \in \mathbf{L}_n}$$

is  $\mathbb{Q}$ -linearly independent.

*Proof of Corollary 2.55.* For each  $n \in \mathbb{N}$ , the family  $\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \in \mathbf{L}_n}$  is  $\mathbb{Q}$ -linearly independent (by Proposition 2.52). Furthermore, these families for varying  $n$  live in linearly disjoint subspaces of  $\text{Pow } \mathcal{N}$  (because for each  $n \in \mathbb{N}$  and  $\Lambda \subseteq [n]$ , the power series  $K_{n,\Lambda}^{\mathbb{Z}}$  is homogeneous of degree  $n$ ). Thus, the union  $\left(K_{n,\Lambda}^{\mathbb{Z}}\right)_{n \in \mathbb{N}; \Lambda \in \mathbf{L}_n}$  of all these families must also be  $\mathbb{Q}$ -linearly independent. This proves Corollary 2.55.  $\square$

We can now finally prove what we came here for:

■ **Theorem 2.56.** The permutation statistic  $\text{Epk}$  is shuffle-compatible.

*Proof of Theorem 2.56.* We must prove that  $\text{Epk}$  is shuffle-compatible. In other words, we must prove that for any two disjoint permutations  $\pi$  and  $\sigma$ , the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{Epk } \pi$ ,  $\text{Epk } \sigma$ ,  $|\pi|$  and  $|\sigma|$ . In other words, we must prove that if  $\pi$  and  $\sigma$  are two disjoint permutations, and if  $\pi'$  and  $\sigma'$  are two disjoint permutations satisfying  $\text{Epk } \pi = \text{Epk } (\pi')$ ,  $\text{Epk } \sigma = \text{Epk } (\sigma')$ ,  $|\pi| = |\pi'|$  and  $|\sigma| = |\sigma'|$ , then the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  equals the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}$ .

So let  $\pi$  and  $\sigma$  be two disjoint permutations, and let  $\pi'$  and  $\sigma'$  be two disjoint permutations satisfying  $\text{Epk } \pi = \text{Epk } (\pi')$ ,  $\text{Epk } \sigma = \text{Epk } (\sigma')$ ,  $|\pi| = |\pi'|$  and  $|\sigma| = |\sigma'|$ .

Define  $n \in \mathbb{N}$  by  $n = |\pi| = |\pi'|$  (this is well-defined, since  $|\pi| = |\pi'|$ ). Likewise, define  $m \in \mathbb{N}$  by  $m = |\sigma| = |\sigma'|$ . Thus,  $\pi$  is an  $n$ -permutation, while  $\sigma$  is an  $m$ -permutation. Hence, each  $\tau \in S(\pi, \sigma)$  is an  $(n+m)$ -permutation, and therefore satisfies  $\text{Epk } \tau \in \mathbf{L}_{n+m}$  (by Proposition 2.4, applied to  $n+m$  and  $\tau$  instead of  $n$  and  $\pi$ ). Thus, the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  consists of elements of  $\mathbf{L}_{n+m}$ . The same holds for the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}$  (for similar reasons).

Corollary 2.49 yields

$$\begin{aligned} & K_{n, \text{Epk } \pi}^{\mathbb{Z}} \cdot K_{m, \text{Epk } \sigma}^{\mathbb{Z}} \\ &= \sum_{\tau \in S(\pi, \sigma)} K_{n+m, \text{Epk } \tau}^{\mathbb{Z}} = \sum_{\Lambda \in \mathbf{L}_{n+m}} \underbrace{\sum_{\substack{\tau \in S(\pi, \sigma); \\ \text{Epk } \tau = \Lambda}} K_{n+m, \Lambda}^{\mathbb{Z}}}_{=|\{\tau \in S(\pi, \sigma) \mid \text{Epk } \tau = \Lambda\}| K_{n+m, \Lambda}^{\mathbb{Z}}} \\ & \quad \text{(because each } \tau \in S(\pi, \sigma) \text{ satisfies } \text{Epk } \tau \in \mathbf{L}_{n+m}) \\ &= \sum_{\Lambda \in \mathbf{L}_{n+m}} |\{\tau \in S(\pi, \sigma) \mid \text{Epk } \tau = \Lambda\}| K_{n+m, \Lambda}^{\mathbb{Z}}. \end{aligned}$$

The same argument (but using  $\pi'$  and  $\sigma'$  instead of  $\pi$  and  $\sigma$ ) yields

$$K_{n, \text{Epk } (\pi')}^{\mathbb{Z}} \cdot K_{m, \text{Epk } (\sigma')}^{\mathbb{Z}} = \sum_{\Lambda \in \mathbf{L}_{n+m}} |\{\tau \in S(\pi', \sigma') \mid \text{Epk } \tau = \Lambda\}| K_{n+m, \Lambda}^{\mathbb{Z}}.$$

The left-hand sides of these two equalities are identical (since  $\text{Epk } \pi = \text{Epk } (\pi')$  and  $\text{Epk } \sigma = \text{Epk } (\sigma')$ ). Thus, their right-hand sides must also be identical. In other words, we have

$$\begin{aligned} & \sum_{\Lambda \in \mathbf{L}_{n+m}} |\{\tau \in S(\pi, \sigma) \mid \text{Epk } \tau = \Lambda\}| K_{n+m, \Lambda}^{\mathbb{Z}} \\ &= \sum_{\Lambda \in \mathbf{L}_{n+m}} |\{\tau \in S(\pi', \sigma') \mid \text{Epk } \tau = \Lambda\}| K_{n+m, \Lambda}^{\mathbb{Z}}. \end{aligned}$$

---

Since the family  $\left(K_{n+m,\Lambda}^{\mathbb{Z}}\right)_{\Lambda \in \mathbf{L}_{n+m}}$  is  $\mathbb{Q}$ -linearly independent (by Proposition 2.52), this shows that

$$|\{\tau \in S(\pi, \sigma) \mid \text{Epk } \tau = \Lambda\}| = |\{\tau \in S(\pi', \sigma') \mid \text{Epk } \tau = \Lambda\}|$$

for each  $\Lambda \in \mathbf{L}_{n+m}$ . In other words, for each  $\Lambda \in \mathbf{L}_{n+m}$ , the multiplicity with which  $\Lambda$  appears in the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  equals the multiplicity with which  $\Lambda$  appears in the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}$ . In other words, the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  equals the multiset  $\{\text{Epk } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}$  (because both of these multisets consist of elements of  $\mathbf{L}_{n+m}$ , and the previous sentence shows that each of these elements appears with equal multiplicities in them). This completes our proof of Theorem 2.56.  $\square$

We end this section with a tangential remark for readers of [GesZhu17]:

**Remark 2.57.** Let us use the notations of [GesZhu17] (specifically, the concept of “equivalent” statistics defined in [GesZhu17, Section 3.1]; and various specific statistics defined in [GesZhu17, Section 2.2]). The permutation statistics  $(\text{Lpk}, \text{val})$ ,  $(\text{Lpk}, \text{udr})$  and  $(\text{Pk}, \text{udr})$  are equivalent to  $\text{Epk}$ , and therefore are shuffle-compatible.

*Proof of Remark 2.57 (sketched).* If  $\text{st}_1$  and  $\text{st}_2$  are two permutation statistics, then we shall write  $\text{st}_1 \sim \text{st}_2$  to mean “ $\text{st}_1$  is equivalent to  $\text{st}_2$ ”.

The permutation statistic  $\text{val}$  is equivalent to  $\text{epk}$ , because of [GesZhu17, Lemma 2.1 (e)]. In other words,  $\text{val} \sim \text{epk}$ . Hence,  $(\text{Lpk}, \text{val}) \sim (\text{Lpk}, \text{epk})$ . But if  $\pi$  is an  $n$ -permutation, then  $\text{Epk } \pi$  can be computed from the knowledge of  $\text{Lpk } \pi$  and  $\text{epk } \pi$  (indeed,  $\text{Epk } \pi$  differs from  $\text{Lpk } \pi$  only in the possible element  $n$ , so that

$$\text{Epk } \pi = \begin{cases} \text{Lpk } \pi, & \text{if } \text{epk } \pi = |\text{Lpk } \pi|; \\ \text{Lpk } \pi \cup \{n\}, & \text{if } \text{epk } \pi \neq |\text{Lpk } \pi| \end{cases}$$

) and vice versa (since  $\text{Lpk } \pi = (\text{Epk } \pi) \setminus \{n\}$  and  $\text{epk } \pi = |\text{Epk } \pi|$ ). Thus,  $(\text{Lpk}, \text{epk}) \sim \text{Epk}$ . Hence, altogether, we obtain  $(\text{Lpk}, \text{val}) \sim (\text{Lpk}, \text{epk}) \sim \text{Epk}$ . In other words,  $(\text{Lpk}, \text{val})$  is equivalent to  $\text{Epk}$ .

Moreover, [GesZhu17, Lemma 2.2 (a)] shows that for any permutation  $\pi$ , the knowledge of  $\text{Lpk } \pi$  allows us to compute  $\text{udr } \pi$  from  $\text{val } \pi$  and vice versa. Hence,  $(\text{Lpk}, \text{udr}) \sim (\text{Lpk}, \text{val}) \sim \text{Epk}$ . In other words,  $(\text{Lpk}, \text{udr})$  is equivalent to  $\text{Epk}$ .

On the other hand,  $(\text{Pk}, \text{lpk}) \sim \text{Lpk}$ . (This is proven similarly to our proof of  $(\text{Lpk}, \text{epk}) \sim \text{Epk}$ .)

Also,  $\text{udr} \sim (\text{lpk}, \text{val})$  (indeed, [GesZhu17, Lemma 2.2 (b) and (c)] show how the value  $(\text{lpk}, \text{val})(\pi)$  can be computed from  $\text{udr } \pi$ , whereas [GesZhu17, Lemma 2.2 (a)] shows the opposite direction). Hence,  $(\text{Pk}, \text{udr}) \sim (\text{Pk}, \text{lpk}, \text{val}) \sim$

$(\text{Lpk}, \text{val})$  (since  $(\text{Pk}, \text{lpk}) \sim \text{Lpk}$ ). Therefore,  $(\text{Pk}, \text{udr}) \sim (\text{Lpk}, \text{val}) \sim \text{Epk}$ . In other words,  $(\text{Pk}, \text{udr})$  is equivalent to  $\text{Epk}$ .

We have now shown that the statistics  $(\text{Lpk}, \text{val})$ ,  $(\text{Lpk}, \text{udr})$  and  $(\text{Pk}, \text{udr})$  are equivalent to  $\text{Epk}$ . Thus, [GesZhu17, Theorem 3.2] shows that they are shuffle-compatible (since  $\text{Epk}$  is shuffle-compatible). This proves Remark 2.57.  $\square$

**Question 2.58.** Our concept of a “ $\mathcal{Z}$ -enriched  $(P, \gamma)$ -partition” generalizes the concept of an “enriched  $(P, \gamma)$ -partition” by restricting ourselves to a subset  $\mathcal{Z}$  of  $\mathcal{N} \times \{+, -\}$ . (This does not sound like much of a generalization when stated like this, but as we have seen the behavior of the power series  $\Gamma_{\mathcal{Z}}(P, \gamma)$  depends strongly on what  $\mathcal{Z}$  is, and is not all anticipated by the  $\mathcal{Z} = \mathcal{N} \times \{+, -\}$  case.) A different generalization of enriched  $(P, \gamma)$ -partitions (introduced by Hsiao and Petersen in [HsiPet10]) are the *colored  $(P, \gamma)$ -partitions*, where the two-element set  $\{+, -\}$  is replaced by the set  $\{1, \omega, \dots, \omega^{m-1}\}$  of all  $m$ -th roots of unity (where  $m$  is a chosen positive integer, and  $\omega$  is a fixed primitive  $m$ -th root of unity). We can play various games with this concept. The most natural thing to do seems to be to consider  $m$  arbitrary total orders  $<_0, <_1, \dots, <_{m-1}$  on the codomain  $A$  of the labeling  $\gamma$  (perhaps with some nice properties such as all intervals being finite) and an arbitrary subset  $\mathcal{Z}$  of  $\mathcal{N} \times \{1, \omega, \dots, \omega^{m-1}\}$ , and define a  $\mathcal{Z}$ -enriched colored  $(P, \gamma)$ -partition to be a map  $f : P \rightarrow \mathcal{Z}$  such that every  $x < y$  in  $P$  satisfy the following conditions:

- (i) We have  $f(x) \preceq f(y)$ . (Here, the total order on  $\mathcal{N} \times \{1, \omega, \dots, \omega^{m-1}\}$  is defined by

$$(n, \omega^i) \prec (n', \omega^{i'}) \text{ if and only if either } n \prec n' \text{ or } (n = n' \text{ and } i < i')$$

(for  $i, i' \in \{0, 1, \dots, m-1\}$ .)

- (ii) If  $f(x) = f(y) = (n, \omega^i)$  for some  $n \in \mathcal{N}$  and  $i \in \{0, 1, \dots, m-1\}$ , then  $\gamma(x) <_i \gamma(y)$ .

Is this a useful concept, and can it be used to study permutation statistics?

**Question 2.59.** Corollary 2.49 provides a formula for rewriting a product of the form  $K_{n, \Lambda}^{\mathcal{Z}} \cdot K_{m, \Omega}^{\mathcal{Z}}$  as a  $\mathbb{Q}$ -linear combination of  $K_{n+m, \Xi}^{\mathcal{Z}}$ 's when  $\Lambda \in \mathbf{L}_n$  and  $\Omega \in \mathbf{L}_m$  (because any such  $\Lambda$  and  $\Omega$  can be written as  $\Lambda = \text{Epk } \pi$  and  $\Omega = \text{Epk } \sigma$  for appropriate permutations  $\pi$  and  $\sigma$ ). Thus, in particular, any such product belongs to the  $\mathbb{Q}$ -linear span of the  $K_{n+m, \Xi}^{\mathcal{Z}}$ 's. Is this still true if  $\Lambda$  and  $\Omega$  are arbitrary subsets of  $[n]$  and  $[m]$  rather than having to belong to  $\mathbf{L}_n$  and to  $\mathbf{L}_m$ ? Computations with SageMath suggest that the answer is “yes”.

For example,

$$\begin{aligned} K_{2,\{1,2\}}^{\mathbb{Z}} \cdot K_{1,\{1\}}^{\mathbb{Z}} &= K_{3,\{2\}}^{\mathbb{Z}} + 2 \cdot K_{3,\{1,3\}}^{\mathbb{Z}} && \text{and} \\ K_{2,\emptyset}^{\mathbb{Z}} \cdot K_{1,\{1\}}^{\mathbb{Z}} &= K_{3,\emptyset}^{\mathbb{Z}} + K_{3,\{2\}}^{\mathbb{Z}} + K_{3,\{1,3\}}^{\mathbb{Z}} = K_{3,\{1\}}^{\mathbb{Z}} + K_{3,\{2\}}^{\mathbb{Z}} + K_{3,\{3\}}^{\mathbb{Z}}. \end{aligned}$$

Note that the  $\mathbb{Q}$ -linear span of the  $K_{n+m,\Xi}^{\mathbb{Z}}$ 's for all  $\Xi \subseteq [n+m]$  is (generally) larger than that of the  $K_{n+m,\Xi}^{\mathbb{Z}}$ 's with  $\Xi \in \mathbf{L}_{n+m}$ .

### 3. LR-shuffle-compatibility

In this section, we shall introduce the concept of “LR-shuffle-compatibility” (short for “left-and-right-shuffle-compatibility”), which is stronger than usual shuffle-compatibility. We shall prove that  $\text{Epk}$  still is LR-shuffle-compatible, and study some other statistics that are and some that are not.

#### 3.1. Left and right shuffles

We begin by introducing “left shuffles” and “right shuffles”. There is a well-known notion of left and right shuffles of words (see, e.g., the operations  $\prec$  and  $\succ$  in [EbMaPa07, Example 1]). Specialized to permutations, it can be defined in the following simple way:

**Definition 3.1.** Let  $\pi$  and  $\sigma$  be two disjoint permutations. Then:

- A *left shuffle* of  $\pi$  and  $\sigma$  means a shuffle  $\tau$  of  $\pi$  and  $\sigma$  such that the first letter of  $\tau$  is the first letter of  $\pi$ . (This makes sense only when  $\pi$  is nonempty. Otherwise, there are no left shuffles of  $\pi$  and  $\sigma$ .)
- A *right shuffle* of  $\pi$  and  $\sigma$  means a shuffle  $\tau$  of  $\pi$  and  $\sigma$  such that the first letter of  $\tau$  is the first letter of  $\sigma$ . (This makes sense only when  $\sigma$  is nonempty. Otherwise, there are no right shuffles of  $\pi$  and  $\sigma$ .)
- We let  $S_{\prec}(\pi, \sigma)$  denote the set of all left shuffles of  $\pi$  and  $\sigma$ .
- We let  $S_{\succ}(\pi, \sigma)$  denote the set of all right shuffles of  $\pi$  and  $\sigma$ .

For example, the left shuffles of the two disjoint permutations  $(3, 1)$  and  $(2, 6)$  are

$$(3, 1, 2, 6), \quad (3, 2, 1, 6), \quad (3, 2, 6, 1),$$

whereas their right shuffles are

$$(2, 3, 1, 6), \quad (2, 3, 6, 1), \quad (2, 6, 3, 1).$$

The permutations  $()$  and  $(1,3)$  have only one right shuffle, which is  $(1,3)$ , and they have no left shuffles.

Clearly, if  $\pi$  and  $\sigma$  are two disjoint permutations such that at least one of  $\pi$  and  $\sigma$  is nonempty, then the two sets  $S_{\prec}(\pi, \sigma)$  and  $S_{\succ}(\pi, \sigma)$  are disjoint and their union is  $S(\pi, \sigma)$  (because every shuffle of  $\pi$  and  $\sigma$  is either a left shuffle or a right shuffle, but not both).

Left and right shuffles have a recursive structure that makes them amenable to inductive arguments. To state it, we need one more definition:

**Definition 3.2.** Let  $n \in \mathbb{N}$ . Let  $\pi$  be an  $n$ -permutation.

(a) For each  $i \in \{1, 2, \dots, n\}$ , we let  $\pi_i$  denote the  $i$ -th entry of  $\pi$ . Thus,  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ .

(b) If  $a$  is a positive integer that does not appear in  $\pi$ , then  $a : \pi$  denotes the  $(n+1)$ -permutation  $(a, \pi_1, \pi_2, \dots, \pi_n)$ .

(c) If  $n > 0$ , then  $\pi_{\sim 1}$  denotes the  $(n-1)$ -permutation  $(\pi_2, \pi_3, \dots, \pi_n)$ .

**Proposition 3.3.** Let  $\pi$  and  $\sigma$  be two disjoint permutations.

(a) We have  $S_{\prec}(\pi, \sigma) = S_{\succ}(\sigma, \pi)$ .

(b) If  $\pi$  is nonempty, then the permutations  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  are well-defined and disjoint, and satisfy  $S_{\prec}(\pi, \sigma) = S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ .

(c) If  $\sigma$  is nonempty, then the permutations  $\sigma_{\sim 1}$  and  $\sigma_1 : \pi$  are well-defined and disjoint, and satisfy  $S_{\succ}(\pi, \sigma) = S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1})$ .

*Proof of Proposition 3.3.* (a) The definition of left shuffles shows that the left shuffles of  $\pi$  and  $\sigma$  are the shuffles  $\tau$  of  $\pi$  and  $\sigma$  such that the first letter of  $\tau$  is the first letter of  $\pi$ . Meanwhile, the definition of right shuffles shows that the right shuffles of  $\sigma$  and  $\pi$  are the shuffles  $\tau$  of  $\sigma$  and  $\pi$  such that the first letter of  $\tau$  is the first letter of  $\pi$ . Comparing these two descriptions, we conclude that the left shuffles of  $\pi$  and  $\sigma$  are the same as the right shuffles of  $\sigma$  and  $\pi$  (since the shuffles of  $\pi$  and  $\sigma$  are the same as the shuffles of  $\sigma$  and  $\pi$ ). In other words,  $S_{\prec}(\pi, \sigma) = S_{\succ}(\sigma, \pi)$ . This proves Proposition 3.3 (a).

(b) We first make a simple observation:

*Claim 1:* Let  $\alpha$  and  $\beta$  be two permutations such that  $\beta$  is nonempty. Assume that  $\alpha$  is a subsequence of  $\beta$ , but does not contain the letter  $\beta_1$ . Then, the permutation  $\beta_1 : \alpha$  also is a subsequence of  $\beta$ .

[*Proof of Claim 1:* The letter  $\beta_1$  does not appear in  $\alpha$  (since  $\alpha$  does not contain  $\beta_1$ ). Thus,  $\beta_1 : \alpha$  is a well-defined permutation.

We have assumed that  $\alpha$  is a subsequence of  $\beta$ . In other words,  $\alpha = (\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k})$  for some  $k \in \mathbb{N}$  and some positive integers  $i_1, i_2, \dots, i_k$  satisfying  $i_1 < i_2 < \dots < i_k$ . Consider these  $i_1, i_2, \dots, i_k$ . From  $\alpha = (\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k})$ , we obtain  $\beta_1 : \alpha = (\beta_1, \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k})$ .

Let  $g \in \{1, 2, \dots, k\}$ . Then,  $\alpha = (\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k})$  clearly contains the letter  $\beta_{i_g}$ . If we had  $i_g = 1$ , then this would yield that  $\alpha$  contains the letter  $\beta_1$  (since  $i_g = 1$ );

but this would contradict the assumption that  $\alpha$  does not contain the letter  $\beta_1$ . Hence, we cannot have  $i_g = 1$ . Thus,  $i_g > 1$ , so that  $1 < i_g$ .

Now, forget that we fixed  $g$ . We thus have shown that  $1 < i_g$  for each  $g \in \{1, 2, \dots, k\}$ . Combining this with  $i_1 < i_2 < \dots < i_k$ , we obtain  $1 < i_1 < i_2 < \dots < i_k$ . Hence,  $(\beta_1, \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k})$  is a subsequence of  $\beta$ . In view of  $\beta_1 : \alpha = (\beta_1, \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k})$ , this rewrites as follows:  $\beta_1 : \alpha$  is a subsequence of  $\beta$ . This proves Claim 1.]

Assume that  $\pi$  is nonempty. The first letter of  $\pi$  does not appear in  $\sigma$  (since  $\pi$  and  $\sigma$  are disjoint). In other words, the letter  $\pi_1$  does not appear in  $\sigma$ . Thus, the permutation  $\pi_1 : \sigma$  is well-defined. The permutation  $\pi_{\sim 1}$  is clearly well-defined. Furthermore, the permutations  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  are disjoint<sup>47</sup>. It thus remains to show that  $S_{\prec}(\pi, \sigma) = S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ .

Set  $m = |\pi|$  and  $n = |\sigma|$ . Thus,  $\pi_{\sim 1}$  is an  $(m - 1)$ -permutation, whereas  $\pi_1 : \sigma$  is an  $(n + 1)$ -permutation.

Now, we shall prove that

$$S_{\prec}(\pi, \sigma) \subseteq S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma) \quad (36)$$

and

$$S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma) \subseteq S_{\prec}(\pi, \sigma). \quad (37)$$

[*Proof of (36):* Let  $\tau \in S_{\prec}(\pi, \sigma)$ . We shall show that  $\tau \in S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ .

We have  $\tau \in S_{\prec}(\pi, \sigma)$ . In other words,  $\tau$  is a left shuffle of  $\pi$  and  $\sigma$  (by the definition of  $S_{\prec}(\pi, \sigma)$ ). In other words,  $\tau$  is a shuffle of  $\pi$  and  $\sigma$  such that the first letter of  $\tau$  is the first letter of  $\pi$  (by the definition of a left shuffle).

So the first letter of  $\tau$  is the first letter of  $\pi$ . In other words,  $\tau_1 = \pi_1$ . Thus, the permutation  $\tau$  is nonempty.

Also,  $\tau$  is a shuffle of  $\pi$  and  $\sigma$ . In other words,  $\tau$  is an  $(m + n)$ -permutation such that both  $\pi$  and  $\sigma$  are subsequences of  $\tau$  (by the definition of a shuffle).

So  $\pi$  is a subsequence of  $\tau$ . Thus,  $\pi_{\sim 1}$  is a subsequence of  $\tau$  as well (since  $\pi_{\sim 1}$  is a subsequence of  $\pi$ ).

Also,  $\sigma$  is a subsequence of  $\tau$ , but does not contain the letter  $\tau_1$  (because  $\tau_1 = \pi_1$ , which does not appear in  $\sigma$ ). Therefore, Claim 1 (applied to  $\alpha = \sigma$  and

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<sup>47</sup>*Proof.* Let  $\ell$  be a letter that appears in both  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$ . We shall derive a contradiction.

The permutations  $\pi$  and  $\sigma$  are disjoint; thus, no letter appears in both  $\pi$  and  $\sigma$ . In other words, a letter that appears in  $\pi$  cannot appear in  $\sigma$ . But the letter  $\ell$  appears in  $\pi_{\sim 1}$ , thus in  $\pi$  (since  $\pi_{\sim 1}$  is a subsequence of  $\pi$ ). Hence,  $\ell$  does not appear in  $\sigma$  (since a letter that appears in  $\pi$  cannot appear in  $\sigma$ ). But  $\ell$  appears in  $\pi_1 : \sigma$ . But the only letter that appears in  $\pi_1 : \sigma$  but not in  $\sigma$  is the letter  $\pi_1$  (due to the construction of  $\pi_1 : \sigma$ ). Thus, the letter  $\ell$  must be  $\pi_1$  (since  $\ell$  is a letter that appears in  $\pi_1 : \sigma$  but not in  $\sigma$ ). In other words,  $\ell = \pi_1$ . Hence, the letter  $\pi_1$  appears in  $\pi_{\sim 1}$  (since the letter  $\ell$  appears in  $\pi_{\sim 1}$ ).

But  $\pi$  is a permutation, and thus the letters of  $\pi$  are distinct. Hence, the letter  $\pi_1$  does not appear in  $\pi_{\sim 1}$ . This contradicts the fact that the letter  $\pi_1$  appears in  $\pi_{\sim 1}$ .

Now, forget that we fixed  $\ell$ . We thus have found a contradiction for each letter  $\ell$  that appears in both  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$ . Hence, no letter appears in both  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$ . In other words, the permutations  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  are disjoint.

$\beta = \tau$ ) yields that  $\tau_1 : \sigma$  also is a subsequence of  $\tau$ . In other words,  $\pi_1 : \sigma$  is a subsequence of  $\tau$  (since  $\tau_1 = \pi_1$ ).

Finally, recall that  $\pi_{\sim 1}$  is an  $(m - 1)$ -permutation, whereas  $\pi_1 : \sigma$  is an  $(n + 1)$ -permutation. But  $\tau$  is an  $(m + n)$ -permutation, hence an  $((m - 1) + (n + 1))$ -permutation (since  $m + n = (m - 1) + (n + 1)$ ). So we know that  $\tau$  is an  $((m - 1) + (n + 1))$ -permutation such that both  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  are subsequences of  $\tau$ . In other words,  $\tau$  is a shuffle of  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  (by the definition of a shuffle). Since the first letter of  $\tau$  is the first letter of  $\pi_1 : \sigma$  (because the first letter of  $\tau$  is  $\tau_1 = \pi_1$ , whereas the first letter of  $\pi_1 : \sigma$  is  $\pi_1$  as well), we thus conclude that  $\tau$  is a right shuffle of  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  (by the definition of a right shuffle). In other words,  $\tau \in S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$  (by the definition of  $S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ ).

Since we have proven this for all  $\tau \in S_{\prec}(\pi, \sigma)$ , we thus conclude that  $S_{\prec}(\pi, \sigma) \subseteq S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ . This proves (36).]

[Proof of (37): Let  $\tau \in S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ . We shall show that  $\tau \in S_{\prec}(\pi, \sigma)$ .

We have  $\tau \in S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ . In other words,  $\tau$  is a right shuffle of  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  (by the definition of  $S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ ). In other words,  $\tau$  is a shuffle of  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  such that the first letter of  $\tau$  is the first letter of  $\pi_1 : \sigma$  (by the definition of a right shuffle).

So the first letter of  $\tau$  is the first letter of  $\pi_1 : \sigma$ . In other words, the first letter of  $\tau$  is  $\pi_1$  (since the first letter of  $\pi_1 : \sigma$  is  $\pi_1$ ). In other words,  $\tau_1 = \pi_1$ . Note that the entries of  $\pi$  are distinct (since  $\pi$  is a permutation); thus, the letter  $\pi_1$  does not appear in  $\pi_{\sim 1}$ . In other words, the letter  $\tau_1$  does not appear in  $\pi_{\sim 1}$  (since  $\tau_1 = \pi_1$ ). Also, the permutation  $\tau$  is nonempty (since  $\tau_1$  exists).

The definitions of  $\pi_1$ ,  $\pi_{\sim 1}$  and  $\pi_1 : \pi_{\sim 1}$  yield  $\pi_1 : \pi_{\sim 1} = \pi$ . In view of  $\tau_1 = \pi_1$ , this rewrites as  $\tau_1 : \pi_{\sim 1} = \pi$ .

Also,  $\tau$  is a shuffle of  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$ . In other words,  $\tau$  is an  $((m - 1) + (n + 1))$ -permutation such that both  $\pi_{\sim 1}$  and  $\pi_1 : \sigma$  are subsequences of  $\tau$  (by the definition of a shuffle).

So  $\pi_1 : \sigma$  is a subsequence of  $\tau$ . Thus,  $\sigma$  is a subsequence of  $\tau$  as well (since  $\sigma$  is a subsequence of  $\pi_1 : \sigma$ ).

Also,  $\pi_{\sim 1}$  is a subsequence of  $\tau$ , but does not contain the letter  $\tau_1$  (since the letter  $\tau_1$  does not appear in  $\pi_{\sim 1}$ ). Therefore, Claim 1 (applied to  $\alpha = \pi_{\sim 1}$  and  $\beta = \tau$ ) yields that  $\tau_1 : \pi_{\sim 1}$  also is a subsequence of  $\tau$ . In other words,  $\pi$  also is a subsequence of  $\tau$  (since  $\tau_1 : \pi_{\sim 1} = \pi$ ).

Finally, recall that  $\tau$  is an  $((m - 1) + (n + 1))$ -permutation, hence an  $(m + n)$ -permutation (since  $(m - 1) + (n + 1) = m + n$ ). So we know that  $\tau$  is an  $(m + n)$ -permutation such that both  $\pi$  and  $\sigma$  are subsequences of  $\tau$ . In other words,  $\tau$  is a shuffle of  $\pi$  and  $\sigma$  (by the definition of a shuffle). Since the first letter of  $\tau$  is the first letter of  $\pi$  (because the first letter of  $\tau$  is  $\tau_1 = \pi_1$ ), we thus conclude that  $\tau$  is a left shuffle of  $\pi$  and  $\sigma$  (by the definition of a left shuffle). In other words,  $\tau \in S_{\prec}(\pi, \sigma)$  (by the definition of  $S_{\prec}(\pi, \sigma)$ ).

Since we have proven this for all  $\tau \in S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ , we thus conclude that  $S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma) \subseteq S_{\prec}(\pi, \sigma)$ . This proves (37).]

Combining (36) with (37), we obtain  $S_{\prec}(\pi, \sigma) = S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma)$ . This com-



pletes the proof of Proposition 3.3 **(b)**.

**(c)** Assume that  $\sigma$  is nonempty. Proposition 3.3 **(a)** (applied to  $\sigma$  and  $\pi$  instead of  $\pi$  and  $\sigma$ ) yields  $S_{\prec}(\sigma, \pi) = S_{\succ}(\pi, \sigma)$ .

Proposition 3.3 **(b)** (applied to  $\sigma$  and  $\pi$  instead of  $\pi$  and  $\sigma$ ) yields that the permutations  $\sigma_{\sim 1}$  and  $\sigma_1 : \pi$  are well-defined and disjoint, and satisfy  $S_{\prec}(\sigma, \pi) = S_{\succ}(\sigma_{\sim 1}, \sigma_1 : \pi)$ . Thus, Proposition 3.3 **(a)** (applied to  $\sigma_1 : \pi$  and  $\sigma_{\sim 1}$  instead of  $\pi$  and  $\sigma$ ) yields  $S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1}) = S_{\succ}(\sigma_{\sim 1}, \sigma_1 : \pi)$ . Combining all the equalities we have now proven, we obtain

$$S_{\succ}(\pi, \sigma) = S_{\prec}(\sigma, \pi) = S_{\succ}(\sigma_{\sim 1}, \sigma_1 : \pi) = S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1}).$$

This completes the proof of Proposition 3.3 **(c)**. □

## 3.2. LR-shuffle-compatibility

We shall use the so-called *Iverson bracket notation* for truth values:

**Definition 3.4.** If  $\mathcal{A}$  is any logical statement, then we define an integer  $[\mathcal{A}] \in \{0, 1\}$  by

$$[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$$

This integer  $[\mathcal{A}]$  is known as the *truth value* of  $\mathcal{A}$ .

Thus, for example,  $[4 > 2] = 1$  whereas  $[2 > 4] = 0$ .

We can now define a notion similar to shuffle-compatibility:

**Definition 3.5.** Let  $st$  be a permutation statistic. We say that  $st$  is *LR-shuffle-compatible* if and only if it has the following property: For any two disjoint nonempty permutations  $\pi$  and  $\sigma$ , the multisets

$$\{st \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \quad \text{and} \quad \{st \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$$

depend only on  $st \pi$ ,  $st \sigma$ ,  $|\pi|$ ,  $|\sigma|$  and  $[\pi_1 > \sigma_1]$ .

In other words, a permutation statistic  $st$  is LR-shuffle-compatible if and only if every two disjoint nonempty permutations  $\pi$  and  $\sigma$  and every two disjoint nonempty permutations  $\pi'$  and  $\sigma'$  satisfying

$$\begin{aligned} st \pi &= st(\pi'), & st \sigma &= st(\sigma'), \\ |\pi| &= |\pi'|, & |\sigma| &= |\sigma'| \quad \text{and} \quad [\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1] \end{aligned}$$

satisfy

$$\begin{aligned} \{st \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} &= \{st \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}} & \text{and} \\ \{st \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} &= \{st \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}. \end{aligned}$$

For example, the permutation statistic  $\text{Pk}$  is not LR-shuffle-compatible. Indeed, if we take  $\pi = (4, 2, 3)$ ,  $\sigma = (1)$ ,  $\pi' = (2, 3, 4)$  and  $\sigma' = (1)$ , then the equalities

$$\begin{aligned} \text{Pk } \pi &= \text{Pk } (\pi'), & \text{Pk } \sigma &= \text{Pk } (\sigma'), \\ |\pi| &= |\pi'|, & |\sigma| &= |\sigma'| \quad \text{and} \quad [\pi_1 > \sigma_1] &= [\pi'_1 > \sigma'_1] \end{aligned}$$

are all satisfied, but

$$\{\text{Pk } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \left\{ \underbrace{\text{Pk } (1, 4, 2, 3)}_{=\{2\}} \right\}_{\text{multi}} = \{\{2\}\}_{\text{multi}}$$

is not the same as

$$\{\text{Pk } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}} = \left\{ \underbrace{\text{Pk } (1, 2, 3, 4)}_{=\emptyset} \right\}_{\text{multi}} = \{\emptyset\}_{\text{multi}}.$$

Similarly, the permutation statistic  $\text{Rpk}$  is not LR-shuffle-compatible. As we will see in Theorem 3.12 further below, the three statistics  $\text{Des}$ ,  $\text{Lpk}$  and  $\text{Epk}$  are LR-shuffle-compatible.

### 3.3. Head-graft-compatibility

We shall now define another compatibility concept for a permutation statistic, which will later prove a useful stepping stone for checking the LR-shuffle-compatibility of this statistic.

**Definition 3.6.** Let  $\text{st}$  be a permutation statistic. We say that  $\text{st}$  is *head-graft-compatible* if and only if it has the following property: For any nonempty permutation  $\pi$  and any letter  $a$  that does not appear in  $\pi$ , the element  $\text{st}(a : \pi)$  depends only on  $\text{st } \pi$ ,  $|\pi|$  and  $[a > \pi_1]$ .

In other words, a permutation statistic  $\text{st}$  is head-graft-compatible if and only if every nonempty permutation  $\pi$ , every letter  $a$  that does not appear in  $\pi$ , every nonempty permutation  $\pi'$  and every letter  $a'$  that does not appear in  $\pi'$  satisfying

$$\text{st } \pi = \text{st } (\pi'), \quad |\pi| = |\pi'| \quad \text{and} \quad [a > \pi_1] = [a' > \pi'_1]$$

satisfy  $\text{st}(a : \pi) = \text{st}(a' : \pi')$ .

For example, the permutation statistic  $\text{Pk}$  is not head-graft-compatible, because if we take  $\pi = (3, 1)$ ,  $a = 2$ ,  $\pi' = (3, 4)$  and  $a' = 2$ , then we do have

$$\text{Pk } \pi = \text{Pk } (\pi'), \quad |\pi| = |\pi'| \quad \text{and} \quad [a > \pi_1] = [a' > \pi'_1]$$

---

but we don't have  $\text{Pk}(a : \pi) = \text{Pk}(a' : \pi')$  (in fact,  $\text{Pk}(a : \pi) = \text{Pk}(2, 3, 1) = \{2\}$  whereas  $\text{Pk}(a' : \pi') = \text{Pk}(2, 3, 4) = \emptyset$ ). Similarly, it can be shown that  $\text{Rpk}$  is not head-graft-compatible. As we will see below (in Proposition 3.8), the permutation statistics  $\text{Des}$ ,  $\text{Lpk}$  and  $\text{Epk}$  are head-graft-compatible; we will analyze a few other statistics in Subsection 3.5.

**Remark 3.7.** Let  $\text{st}$  be a head-graft-compatible permutation statistic. Then, it is easy to see that

$$\text{st}(3, 1, 2) = \text{st}(2, 1, 3) \quad \text{and} \quad \text{st}(2, 3, 1) = \text{st}(1, 3, 2).$$

Moreover, these are the only restrictions that head-graft-compatibility places on the values of  $\text{st}$  at 3-permutations. The restrictions placed on the values of  $\text{st}$  at permutations of length  $n > 3$  are more complicated, and depend on its values on shorter permutations.

It is usually easy to check if a given permutation statistic is head-graft-compatible. For example:

- Proposition 3.8.** (a) The permutation statistic  $\text{Des}$  is head-graft-compatible.  
 (b) The permutation statistic  $\text{Lpk}$  is head-graft-compatible.  
 (c) The permutation statistic  $\text{Epk}$  is head-graft-compatible.

*Proof of Proposition 3.8.* In this proof, we shall use the following notation: If  $S$  is a set of integers, and  $p$  is an integer, then  $S + p$  shall denote the set  $\{s + p \mid s \in S\}$ .

(a) Let  $\pi$  be a nonempty permutation. Let  $a$  be a letter that does not appear in  $\pi$ . We shall express the element  $\text{Des}(a : \pi)$  in terms of  $\text{Des} \pi$ ,  $|\pi|$  and  $[a > \pi_1]$ .

Let  $n = |\pi|$ . Thus,  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ . Therefore,  $a : \pi = (a, \pi_1, \pi_2, \dots, \pi_n)$ . Hence, the descents of  $a : \pi$  are obtained as follows:

- The number 1 is a descent of  $a : \pi$  if and only if  $a > \pi_1$ .
- Adding 1 to each descent of  $\pi$  yields a descent of  $a : \pi$ . (That is, if  $i$  is a descent of  $\pi$ , then  $i + 1$  is a descent of  $a : \pi$ .)

These are all the descents of  $a : \pi$ . Thus,

$$\text{Des}(a : \pi) = \{1 \mid a > \pi_1\} \cup (\text{Des} \pi + 1). \quad (38)$$

(The strange notation “ $\{1 \mid a > \pi_1\}$ ” means exactly what it says: It is the set of all numbers 1 satisfying  $a > \pi_1$ . In other words, it is  $\{1\}$  if  $a > \pi_1$ , and  $\emptyset$  otherwise.)

The equality (38) shows that  $\text{Des}(a : \pi)$  depends only on  $\text{Des} \pi$ ,  $|\pi|$  and  $[a > \pi_1]$  (indeed, the truth value  $[a > \pi_1]$  determines whether  $a > \pi_1$  is true). In other words,  $\text{Des}$  is head-graft-compatible (by the definition of “head-graft-compatible”). This proves Proposition 3.8 (a).

(b) Let  $\pi$  be a nonempty permutation. Let  $a$  be a letter that does not appear in  $\pi$ . We shall express the element  $\text{Lpk}(a : \pi)$  in terms of  $\text{Lpk } \pi$ ,  $|\pi|$  and  $[a > \pi_1]$ .

Notice first that  $a \neq \pi_1$  (since  $a$  does not appear in  $\pi$ ). Thus,  $a < \pi_1$  is true if and only if  $a > \pi_1$  is false.

Let  $n = |\pi|$ . Therefore,  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ . Thus,  $a : \pi = (a, \pi_1, \pi_2, \dots, \pi_n)$ . Hence, the left peaks of  $a : \pi$  are obtained as follows:

- The number 1 is a left peak of  $a : \pi$  if and only if  $a > \pi_1$ .
- Adding 1 to each left peak  $i$  of  $\pi$  yields a left peak  $i + 1$  of  $a : \pi$ , except for the case when  $i = 1$  (in which case  $i + 1 = 2$  is a left peak of  $a : \pi$  only if  $a < \pi_1$ ).

These are all the left peaks of  $a : \pi$ . Thus,

$$\text{Lpk}(a : \pi) = \{1 \mid a > \pi_1\} \cup \begin{cases} \text{Lpk } \pi + 1, & \text{if } a < \pi_1; \\ (\text{Lpk } \pi + 1) \setminus \{2\}, & \text{if not } a < \pi_1. \end{cases} \quad (39)$$

This equality shows that  $\text{Lpk}(a : \pi)$  depends only on  $\text{Lpk } \pi$ ,  $|\pi|$  and  $[a > \pi_1]$  (indeed, the truth value  $[a > \pi_1]$  determines whether  $a > \pi_1$  is true and also determines whether  $a < \pi_1$  is true<sup>48</sup>). In other words,  $\text{Lpk}$  is head-graft-compatible (by the definition of “head-graft-compatible”). This proves Proposition 3.8 (b).

(c) To obtain a proof of Proposition 3.8 (c), it suffices to take our above proof of Proposition 3.8 (b) and replace every appearance of “left peak” and “Lpk” by “exterior peak” and “Epk”.  $\square$

### 3.4. Proving LR-shuffle-compatibility

Let us now state a sufficient criterion for the LR-shuffle-compatibility of a statistic:

**Theorem 3.9.** Let  $st$  be a permutation statistic that is both shuffle-compatible and head-graft-compatible. Then,  $st$  is LR-shuffle-compatible.

Before we prove this theorem, let us introduce some terminology and state an almost-trivial fact:

**Definition 3.10. (a)** If  $A$  is a finite multiset, and if  $g$  is any object, then  $|A|_g$  means the multiplicity of  $g$  in  $A$ .

(b) If  $A$  and  $B$  are two finite multisets, then we say that  $B \subseteq A$  if and only if each object  $g$  satisfies  $|B|_g \leq |A|_g$ .

(c) If  $A$  and  $B$  are two finite multisets satisfying  $B \subseteq A$ , then  $A - B$  shall denote the “multiset difference” of  $A$  and  $B$ ; this is the finite multiset  $C$  such that each object  $g$  satisfies  $|C|_g = |A|_g - |B|_g$ .

<sup>48</sup>Indeed,  $a < \pi_1$  is true if and only if  $a > \pi_1$  is false.

For example,  $\{2, 3, 3\}_{\text{multi}} \subseteq \{1, 2, 2, 3, 3\}_{\text{multi}}$  and  $\{1, 2, 2, 3, 3\}_{\text{multi}} - \{2, 3, 3\}_{\text{multi}} = \{1, 2\}_{\text{multi}}$ .

**Lemma 3.11.** Let  $\pi$  and  $\sigma$  be two disjoint permutations such that at least one of  $\pi$  and  $\sigma$  is nonempty. Let  $\text{st}$  be any permutation statistic. Then:

(a) We have

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} - \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}. \end{aligned}$$

(b) We have

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} - \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}. \end{aligned}$$

*Proof of Lemma 3.11.* Recall that the two sets  $S_{\prec}(\pi, \sigma)$  and  $S_{\succ}(\pi, \sigma)$  are disjoint and their union is  $S(\pi, \sigma)$ . Thus,  $S_{\succ}(\pi, \sigma) \subseteq S(\pi, \sigma)$  and  $S_{\prec}(\pi, \sigma) = S(\pi, \sigma) \setminus S_{\succ}(\pi, \sigma)$ . Hence,

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} - \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}. \end{aligned}$$

This proves Lemma 3.11 (a). The proof of Lemma 3.11 (b) is analogous.  $\square$

*Proof of Theorem 3.9.* We shall first show the following:

*Claim 1:* Let  $\pi, \pi'$  and  $\sigma$  be three nonempty permutations. Assume that  $\pi$  and  $\sigma$  are disjoint. Assume that  $\pi'$  and  $\sigma$  are disjoint. Assume furthermore that

$$\text{st } \pi = \text{st } (\pi'), \quad |\pi| = |\pi'| \quad \text{and} \quad [\pi_1 > \sigma_1] = [\pi'_1 > \sigma_1].$$

Then,

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma)\}_{\text{multi}} \end{aligned} \tag{40}$$

and

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma)\}_{\text{multi}}. \end{aligned} \tag{41}$$

[*Proof of Claim 1:* We shall prove Claim 1 by induction on  $|\sigma|$ :

*Induction base:* The case  $|\sigma| = 0$  cannot happen (because  $\sigma$  is assumed to be nonempty). Thus, Claim 1 is true in the case  $|\sigma| = 0$ . This completes the induction base.

*Induction step:* Let  $N$  be a positive integer. Assume (as the induction hypothesis) that Claim 1 holds when  $|\sigma| = N - 1$ . We must now prove that Claim 1 holds when  $|\sigma| = N$ .

Indeed, let  $\pi, \pi'$  and  $\sigma$  be as in Claim 1, and assume that  $|\sigma| = N$ . We must prove (40) and (41).

Proposition 3.3 (c) yields that the permutations  $\sigma_{\sim 1}$  and  $\sigma_1 : \pi$  are well-defined and disjoint, and satisfy

$$S_{\succ}(\pi, \sigma) = S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1}). \quad (42)$$

Furthermore,  $|\sigma_{\sim 1}| = |\sigma| - 1 = N - 1$  (since  $|\sigma| = N$ ).

Proposition 3.3 (c) (applied to  $\pi'$  instead of  $\pi$ ) yields that the permutations  $\sigma_{\sim 1}$  and  $\sigma_1 : \pi'$  are well-defined and disjoint, and satisfy

$$S_{\succ}(\pi', \sigma) = S_{\prec}(\sigma_1 : \pi', \sigma_{\sim 1}). \quad (43)$$

The letter  $\sigma_1$  does not appear in the permutation  $\pi$  (since  $\pi$  and  $\sigma$  are disjoint). Similarly, the letter  $\sigma_1$  does not appear in the permutation  $\pi'$ . Also,  $|\sigma_1 : \pi| = \underbrace{|\pi|}_{=|\pi'|} + 1 = |\pi'| + 1 = |\sigma_1 : \pi'|$ .

We have  $\sigma_1 \neq \pi_1$  (since  $\pi$  and  $\sigma$  are disjoint). Thus, the statement  $(\sigma_1 > \pi_1)$  is equivalent to  $(\text{not } \pi_1 > \sigma_1)$ . Hence,  $[\sigma_1 > \pi_1] = [\text{not } \pi_1 > \sigma_1] = 1 - [\pi_1 > \sigma_1]$ . Similarly,  $[\sigma_1 > \pi'_1] = 1 - [\pi'_1 > \sigma_1]$ . Hence,

$$[\sigma_1 > \pi_1] = 1 - \underbrace{[\pi_1 > \sigma_1]}_{=[\pi'_1 > \sigma_1]} = 1 - [\pi'_1 > \sigma_1] = [\sigma_1 > \pi'_1].$$

Both permutations  $\sigma_1 : \pi$  and  $\sigma_1 : \pi'$  begin with the letter  $\sigma_1$ . Thus, both  $(\sigma_1 : \pi)_1$  and  $(\sigma_1 : \pi')_1$  equal  $\sigma_1$ . Hence,  $(\sigma_1 : \pi)_1 = (\sigma_1 : \pi')_1$ .

The statistic  $\text{st}$  is head-graft-compatible. In other words, for any nonempty permutation  $\varphi$  and any letter  $a$  that does not appear in  $\varphi$ , the element  $\text{st}(a : \varphi)$  depends only on  $\text{st}(\varphi)$ ,  $|\varphi|$  and  $[a > \varphi_1]$  (by the definition of ‘‘head-graft-compatible’’). Hence, if  $\varphi$  and  $\varphi'$  are two nonempty permutations, and if  $a$  is any letter that does not appear in  $\varphi$  and does not appear in  $\varphi'$ , and if we have  $\text{st } \varphi = \text{st } (\varphi')$  and  $|\varphi| = |\varphi'|$  and  $[a > \varphi_1] = [a > \varphi'_1]$ , then  $\text{st}(a : \varphi) = \text{st}(a : \varphi')$ . Applying this to  $a = \sigma_1$ ,  $\varphi = \pi$  and  $\varphi' = \pi'$ , we obtain

$$\text{st}(\sigma_1 : \pi) = \text{st}(\sigma_1 : \pi')$$

(since  $\text{st } \pi = \text{st } (\pi')$  and  $|\pi| = |\pi'|$  and  $[\sigma_1 > \pi_1] = [\sigma_1 > \pi'_1]$ ).

Next, we claim that

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1})\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\prec}(\sigma_1 : \pi', \sigma_{\sim 1})\}_{\text{multi}}. \end{aligned} \quad (44)$$

[Proof of (44): The permutations  $\sigma_1 : \pi$  and  $\sigma_1 : \pi'$  are clearly nonempty. Hence, if  $\sigma_{\sim 1}$  is the 0-permutation  $()$ , then  $S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1}) = \{\sigma_1 : \pi\}$  and  $S_{\prec}(\sigma_1 : \pi', \sigma_{\sim 1}) = \{\sigma_1 : \pi'\}$ . Thus, if  $\sigma_{\sim 1}$  is the 0-permutation  $()$ , then (44) follows from

$$\begin{aligned}
& \left\{ \text{st } \tau \mid \tau \in \underbrace{S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1})}_{=\{\sigma_1 : \pi\}} \right\}_{\text{multi}} \\
&= \{ \text{st } \tau \mid \tau \in \{\sigma_1 : \pi\} \}_{\text{multi}} = \left\{ \underbrace{\text{st}(\sigma_1 : \pi)}_{=\text{st}(\sigma_1 : \pi')} \right\}_{\text{multi}} \\
&= \{ \text{st}(\sigma_1 : \pi') \}_{\text{multi}} = \left\{ \text{st } \tau \mid \tau \in \underbrace{\{\sigma_1 : \pi'\}}_{=S_{\prec}(\sigma_1 : \pi', \sigma_{\sim 1})} \right\}_{\text{multi}} \\
&= \{ \text{st } \tau \mid \tau \in S_{\prec}(\sigma_1 : \pi', \sigma_{\sim 1}) \}_{\text{multi}}.
\end{aligned}$$

Thus, for the rest of our proof of (44), we WLOG assume that  $\sigma_{\sim 1}$  is not the 0-permutation  $()$ . Thus,  $\sigma_{\sim 1}$  is nonempty.

But recall that  $|\sigma_{\sim 1}| = N - 1$ . Hence, the induction hypothesis allows us to apply Claim 1 to  $\sigma_1 : \pi$ ,  $\sigma_1 : \pi'$  and  $\sigma_{\sim 1}$  instead of  $\pi$ ,  $\pi'$  and  $\sigma$  (because we know that the permutations  $\sigma_{\sim 1}$  and  $\sigma_1 : \pi$  are disjoint; that the permutations  $\sigma_{\sim 1}$  and  $\sigma_1 : \pi'$  are disjoint; that  $\text{st}(\sigma_1 : \pi) = \text{st}(\sigma_1 : \pi')$  and  $|\sigma_1 : \pi| = |\sigma_1 : \pi'|$ ; and that

$$\left[ \underbrace{(\sigma_1 : \pi)_1}_{=(\sigma_1 : \pi')_1} > (\sigma_{\sim 1})_1 \right] = [(\sigma_1 : \pi')_1 > (\sigma_{\sim 1})_1].$$

We therefore obtain

$$\{ \text{st } \tau \mid \tau \in S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1}) \}_{\text{multi}} = \{ \text{st } \tau \mid \tau \in S_{\prec}(\sigma_1 : \pi', \sigma_{\sim 1}) \}_{\text{multi}}$$

and

$$\{ \text{st } \tau \mid \tau \in S_{\succ}(\sigma_1 : \pi, \sigma_{\sim 1}) \}_{\text{multi}} = \{ \text{st } \tau \mid \tau \in S_{\succ}(\sigma_1 : \pi', \sigma_{\sim 1}) \}_{\text{multi}}.$$

The first of these two equalities is precisely (44). Thus, (44) is proven.]

Now,

$$\begin{aligned}
& \left\{ \text{st } \tau \mid \tau \in \underbrace{S_{\succ}(\pi, \sigma)}_{\substack{=S_{\prec}(\sigma_1: \pi, \sigma_{\sim 1}) \\ \text{(by (42))}}} \right\}_{\text{multi}} \\
&= \left\{ \text{st } \tau \mid \tau \in S_{\prec}(\sigma_1: \pi, \sigma_{\sim 1}) \right\}_{\text{multi}} \\
&= \left\{ \text{st } \tau \mid \tau \in \underbrace{S_{\prec}(\sigma_1: \pi', \sigma_{\sim 1})}_{\substack{=S_{\succ}(\pi', \sigma) \\ \text{(by (43))}}} \right\}_{\text{multi}} \quad \text{(by (44))} \\
&= \left\{ \text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma) \right\}_{\text{multi}}. \tag{45}
\end{aligned}$$

This proves (41). It remains to prove (40).

Lemma 3.11 (a) yields

$$\begin{aligned}
& \left\{ \text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma) \right\}_{\text{multi}} \\
&= \left\{ \text{st } \tau \mid \tau \in S(\pi, \sigma) \right\}_{\text{multi}} - \left\{ \text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma) \right\}_{\text{multi}}. \tag{46}
\end{aligned}$$

Lemma 3.11 (a) (applied to  $\pi'$  instead of  $\pi$ ) yields

$$\begin{aligned}
& \left\{ \text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma) \right\}_{\text{multi}} \\
&= \left\{ \text{st } \tau \mid \tau \in S(\pi', \sigma) \right\}_{\text{multi}} - \left\{ \text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma) \right\}_{\text{multi}}. \tag{47}
\end{aligned}$$

But recall that the statistic  $\text{st}$  is shuffle-compatible. In other words, for any two disjoint permutations  $\alpha$  and  $\beta$ , the multiset

$$\left\{ \text{st } \tau \mid \tau \in S(\alpha, \beta) \right\}_{\text{multi}}$$

depends only on  $\text{st } \alpha$ ,  $\text{st } \beta$ ,  $|\alpha|$  and  $|\beta|$  (by the definition of shuffle-compatibility). In other words, if  $\alpha$  and  $\beta$  are two disjoint permutations, and if  $\alpha'$  and  $\beta'$  are two disjoint permutations, and if we have

$$\text{st } \alpha = \text{st } (\alpha'), \quad \text{st } \beta = \text{st } (\beta'), \quad |\alpha| = |\alpha'| \quad \text{and} \quad |\beta| = |\beta'|,$$

then

$$\left\{ \text{st } \tau \mid \tau \in S(\alpha, \beta) \right\}_{\text{multi}} = \left\{ \text{st } \tau \mid \tau \in S(\alpha', \beta') \right\}_{\text{multi}}.$$

Applying this to  $\alpha = \pi$ ,  $\beta = \sigma$ ,  $\alpha' = \pi'$  and  $\beta' = \sigma$ , we obtain

$$\left\{ \text{st } \tau \mid \tau \in S(\pi, \sigma) \right\}_{\text{multi}} = \left\{ \text{st } \tau \mid \tau \in S(\pi', \sigma) \right\}_{\text{multi}} \tag{48}$$



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(since  $\text{st } \pi = \text{st } (\pi')$ ,  $\text{st } \sigma = \text{st } \sigma$ ,  $|\pi| = |\pi'|$  and  $|\sigma| = |\sigma|$ ). Now, (46) becomes

$$\begin{aligned}
& \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \\
&= \underbrace{\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}}_{=\{\text{st } \tau \mid \tau \in S(\pi', \sigma)\}_{\text{multi}} \text{ (by (48))}} - \underbrace{\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}}_{=\{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma)\}_{\text{multi}} \text{ (by (45))}} \\
&= \{\text{st } \tau \mid \tau \in S(\pi', \sigma)\}_{\text{multi}} - \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma)\}_{\text{multi}} \\
&= \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma)\}_{\text{multi}} \quad \text{(by (47))}.
\end{aligned}$$

Thus, (40) is proven. Hence, we have proven both (40) and (41). This shows that Claim 1 holds for our  $\pi$ ,  $\pi'$  and  $\sigma$ . This completes the induction step. Thus, Claim 1 is proven by induction.]

We shall next derive a ‘‘mirror version’’ of Claim 1:

*Claim 2:* Let  $\pi$ ,  $\sigma$  and  $\sigma'$  be three nonempty permutations. Assume that  $\pi$  and  $\sigma$  are disjoint. Assume that  $\pi$  and  $\sigma'$  are disjoint. Assume furthermore that

$$\text{st } \sigma = \text{st } (\sigma'), \quad |\sigma| = |\sigma'| \quad \text{and} \quad [\pi_1 > \sigma_1] = [\pi_1 > \sigma'_1].$$

Then,

$$\begin{aligned}
& \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \\
&= \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma')\}_{\text{multi}}
\end{aligned}$$

and

$$\begin{aligned}
& \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} \\
&= \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma')\}_{\text{multi}}.
\end{aligned}$$

[*Proof of Claim 2:* We have  $\sigma_1 \neq \pi_1$  (since  $\pi$  and  $\sigma$  are disjoint). Thus, the statement  $(\sigma_1 > \pi_1)$  is equivalent to (not  $\pi_1 > \sigma_1$ ). Hence,  $[\sigma_1 > \pi_1] = [\text{not } \pi_1 > \sigma_1] = 1 - [\pi_1 > \sigma_1]$ . Similarly,  $[\sigma'_1 > \pi_1] = 1 - [\pi_1 > \sigma'_1]$ . Hence,

$$\begin{aligned}
[\sigma_1 > \pi_1] &= 1 - \underbrace{[\pi_1 > \sigma_1]}_{=[\pi_1 > \sigma'_1]} = 1 - [\pi_1 > \sigma'_1] = [\sigma'_1 > \pi_1].
\end{aligned}$$

Hence, Claim 1 (applied to  $\sigma$ ,  $\sigma'$  and  $\pi$  instead of  $\pi$ ,  $\pi'$  and  $\sigma$ ) shows that

$$\begin{aligned}
& \{\text{st } \tau \mid \tau \in S_{\prec}(\sigma, \pi)\}_{\text{multi}} \\
&= \{\text{st } \tau \mid \tau \in S_{\prec}(\sigma', \pi)\}_{\text{multi}}
\end{aligned}$$

and

$$\begin{aligned}
& \{\text{st } \tau \mid \tau \in S_{\succ}(\sigma, \pi)\}_{\text{multi}} \\
&= \{\text{st } \tau \mid \tau \in S_{\succ}(\sigma', \pi)\}_{\text{multi}}.
\end{aligned}$$

But Proposition 3.3 **(a)** yields  $S_{\prec}(\pi, \sigma) = S_{\succ}(\sigma, \pi)$ . Similarly,  $S_{\prec}(\pi, \sigma') = S_{\succ}(\sigma', \pi)$ . Also, Proposition 3.3 **(a)** (applied to  $\sigma$  and  $\pi$  instead of  $\pi$  and  $\sigma$ ) yields  $S_{\prec}(\sigma, \pi) = S_{\succ}(\pi, \sigma)$ . Similarly,  $S_{\prec}(\sigma', \pi) = S_{\succ}(\pi, \sigma')$ . Using all these equalities, we find

$$\begin{aligned} & \left\{ \text{st } \tau \mid \tau \in \underbrace{S_{\prec}(\pi, \sigma)}_{=S_{\succ}(\sigma, \pi)} \right\}_{\text{multi}} \\ &= \{ \text{st } \tau \mid \tau \in S_{\succ}(\sigma, \pi) \}_{\text{multi}} = \left\{ \text{st } \tau \mid \tau \in \underbrace{S_{\succ}(\sigma', \pi)}_{=S_{\prec}(\pi, \sigma')} \right\}_{\text{multi}} \\ &= \{ \text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma') \}_{\text{multi}} \end{aligned}$$

and

$$\begin{aligned} & \left\{ \text{st } \tau \mid \tau \in \underbrace{S_{\succ}(\pi, \sigma)}_{=S_{\prec}(\sigma, \pi)} \right\}_{\text{multi}} \\ &= \{ \text{st } \tau \mid \tau \in S_{\prec}(\sigma, \pi) \}_{\text{multi}} = \left\{ \text{st } \tau \mid \tau \in \underbrace{S_{\prec}(\sigma', \pi)}_{=S_{\succ}(\pi, \sigma')} \right\}_{\text{multi}} \\ &= \{ \text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma') \}_{\text{multi}}. \end{aligned}$$

Thus, Claim 2 is proven.]

Finally, we are ready to take on the LR-shuffle-compatibility of st:

*Claim 3:* Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations. Let  $\pi'$  and  $\sigma'$  be two disjoint nonempty permutations. Assume that

$$\begin{aligned} \text{st } \pi = \text{st } (\pi'), \quad \text{st } \sigma = \text{st } (\sigma'), \\ |\pi| = |\pi'|, \quad |\sigma| = |\sigma'| \quad \text{and} \quad [\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1]. \end{aligned}$$

Then,

$$\{ \text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma) \}_{\text{multi}} = \{ \text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma') \}_{\text{multi}}$$

and

$$\{ \text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma) \}_{\text{multi}} = \{ \text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma') \}_{\text{multi}}.$$

[*Proof of Claim 3:* We have  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1]$ . Since  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1]$  is either 1 or 0, we must therefore be in one of the following two cases:

*Case 1:* We have  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1] = 1$ .

Case 2: We have  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1] = 0$ .

Let us first consider Case 1. In this case, we have  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1] = 1$ .

There clearly exists a positive integer  $N$  that is larger than all entries of  $\sigma$  and larger than all entries of  $\sigma'$ . Consider such an  $N$ . Let  $n = |\pi|$ ; thus,  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ . Let  $\gamma$  be the permutation  $(\pi_1 + N, \pi_2 + N, \dots, \pi_n + N)$ . This permutation  $\gamma$  is order-isomorphic to  $\pi$ , but is disjoint from  $\sigma$  (since all its entries are  $> N$ , while all the entries of  $\sigma$  are  $< N$ ) and disjoint from  $\sigma'$  (for similar reasons). Also,  $\gamma_1 = \underbrace{\pi_1}_{>0} + N > N > \sigma_1$  (since  $N$  is larger than all entries of  $\sigma$ ),

so that  $[\gamma_1 > \sigma_1] = 1$ . Similarly,  $[\gamma_1 > \sigma'_1] = 1$ .

The permutation  $\gamma$  is order-isomorphic to  $\pi$ . Thus,  $\text{st } \gamma = \text{st } \pi$  (since  $\text{st}$  is a permutation statistic) and  $|\gamma| = |\pi|$ . The permutation  $\gamma$  is furthermore nonempty (since it is order-isomorphic to the nonempty permutation  $\pi$ ). Also,  $\text{st } \gamma = \text{st } \pi = \text{st } (\pi')$  and  $|\gamma| = |\pi| = |\pi'|$ . Moreover,  $[\pi_1 > \sigma_1] = 1 = [\gamma_1 > \sigma_1]$  and  $[\gamma_1 > \sigma_1] = 1 = [\gamma_1 > \sigma'_1]$  and  $[\gamma_1 > \sigma'_1] = 1 = [\pi'_1 > \sigma'_1]$ . Hence, Claim 1 (applied to  $\gamma$  instead of  $\pi'$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\gamma, \sigma)\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\gamma, \sigma)\}_{\text{multi}}.$$

Furthermore, Claim 2 (applied to  $\gamma$  instead of  $\pi$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\gamma, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\gamma, \sigma')\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\gamma, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\gamma, \sigma')\}_{\text{multi}}.$$

Finally, Claim 1 (applied to  $\gamma$  and  $\sigma'$  instead of  $\pi$  and  $\sigma$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\gamma, \sigma')\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\gamma, \sigma')\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

Combining the equalities we have found, we obtain

$$\begin{aligned} \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} &= \{\text{st } \tau \mid \tau \in S_{\prec}(\gamma, \sigma)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\prec}(\gamma, \sigma')\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}. \end{aligned}$$

The same argument (but with the symbols “ $S_{\prec}$ ” and “ $S_{\succ}$ ” interchanged) yields

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

Thus, Claim 3 is proven in Case 1.

Let us now consider Case 2. In this case, we have  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1] = 0$ .

There clearly exists a positive integer  $N$  that is larger than all entries of  $\pi$  and larger than all entries of  $\pi'$ . Consider such an  $N$ . Set  $m = |\sigma|$ . Thus,  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$ . Let  $\delta$  be the permutation  $(\sigma_1 + N, \sigma_2 + N, \dots, \sigma_m + N)$ . This permutation  $\delta$  is order-isomorphic to  $\sigma$ , but is disjoint from  $\pi$  (since all its entries are  $> N$ , while all the entries of  $\pi$  are  $< N$ ) and disjoint from  $\pi'$  (for similar reasons). Also,  $\delta_1 = \underbrace{\sigma_1}_{>0} + N > N > \pi_1$  (since  $N$  is larger than all entries of  $\pi$ ),

so that we don't have  $\pi_1 > \delta_1$ . Thus,  $[\pi_1 > \delta_1] = 0$ . Similarly,  $[\pi'_1 > \delta_1] = 0$ .

The permutation  $\delta$  is order-isomorphic to  $\sigma$ . Thus,  $\text{st } \delta = \text{st } \sigma$  (since  $\text{st}$  is a permutation statistic) and  $|\delta| = |\sigma|$ . The permutation  $\delta$  is furthermore nonempty (since it is order-isomorphic to the nonempty permutation  $\sigma$ ). Also,  $\text{st } \delta = \text{st } \sigma = \text{st } (\sigma')$  and  $|\delta| = |\sigma| = |\sigma'|$ . Moreover,  $[\pi_1 > \sigma_1] = 0 = [\pi_1 > \delta_1]$  and  $[\pi_1 > \delta_1] = 0 = [\pi'_1 > \delta_1]$  and  $[\pi'_1 > \delta_1] = 0 = [\pi'_1 > \sigma'_1]$ . Hence, Claim 2 (applied to  $\delta$  instead of  $\sigma'$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \delta)\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \delta)\}_{\text{multi}}.$$

Furthermore, Claim 1 (applied to  $\delta$  instead of  $\sigma$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \delta)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \delta)\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \delta)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \delta)\}_{\text{multi}}.$$

Finally, Claim 2 (applied to  $\pi'$  and  $\delta$  instead of  $\pi$  and  $\sigma$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \delta)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \delta)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

Combining the equalities we have found, we obtain

$$\begin{aligned} \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} &= \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \delta)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \delta)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}. \end{aligned}$$

The same argument (but with the symbols “ $S_{\prec}$ ” and “ $S_{\succ}$ ” interchanged) yields

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

Thus, Claim 3 is proven in Case 2.

We have now proven Claim 3 in each of the two Cases 1 and 2. Hence, Claim 3 always holds.]

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Claim 3 says that for any two disjoint nonempty permutations  $\pi$  and  $\sigma$ , the multisets

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \quad \text{and} \quad \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$$

depend only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$ ,  $|\sigma|$  and  $[\pi_1 > \sigma_1]$ . In other words, the statistic  $\text{st}$  is LR-shuffle-compatible (by the definition of “LR-shuffle-compatible”). This proves Theorem 3.9.  $\square$

Combining Theorem 3.9 with Proposition 3.8, we obtain the following:

- Theorem 3.12.** (a) The permutation statistic  $\text{Des}$  is LR-shuffle-compatible.  
 (b) The permutation statistic  $\text{Lpk}$  is LR-shuffle-compatible.  
 (c) The permutation statistic  $\text{Epk}$  is LR-shuffle-compatible.

*Proof of Theorem 3.12.* (a) The permutation statistic  $\text{Des}$  is shuffle-compatible (by [GesZhu17, §2.4]) and head-graft-compatible (by Proposition 3.8 (a)). Thus, Theorem 3.9 (applied to  $\text{st} = \text{Des}$ ) shows that the permutation statistic  $\text{Des}$  is LR-shuffle-compatible. This proves Theorem 3.12 (a).

(b) The permutation statistic  $\text{Lpk}$  is shuffle-compatible (by [GesZhu17, Theorem 4.9 (a)]) and head-graft-compatible (by Proposition 3.8 (b)). Thus, Theorem 3.9 (applied to  $\text{st} = \text{Lpk}$ ) shows that the permutation statistic  $\text{Lpk}$  is LR-shuffle-compatible. This proves Theorem 3.12 (b).

(c) The permutation statistic  $\text{Epk}$  is shuffle-compatible (by Theorem 2.56) and head-graft-compatible (by Proposition 3.8 (c)). Thus, Theorem 3.9 (applied to  $\text{st} = \text{Epk}$ ) shows that the permutation statistic  $\text{Epk}$  is LR-shuffle-compatible. This proves Theorem 3.12 (c).  $\square$

### 3.5. Some other statistics

The question of LR-shuffle-compatibility can be asked about any statistic. We have so far answered it for  $\text{Des}$ ,  $\text{Pk}$ ,  $\text{Lpk}$ ,  $\text{Rpk}$  and  $\text{Epk}$ . In this section, we shall analyze it for some further statistics.

#### 3.5.1. The descent number $\text{des}$

The permutation statistic  $\text{des}$  (called the *descent number*) is defined as follows: For each permutation  $\pi$ , we set  $\text{des } \pi = |\text{Des } \pi|$  (that is,  $\text{des } \pi$  is the number of all descents of  $\pi$ ). It was proven in [GesZhu17, Theorem 4.6 (a)] that this statistic  $\text{des}$  is shuffle-compatible. We now claim the following:

- Proposition 3.13.** The permutation statistic  $\text{des}$  is head-graft-compatible and LR-shuffle-compatible.

---

*Proof of Proposition 3.13.* From (38), we easily obtain the following: If  $\pi$  is a nonempty permutation, and if  $a$  is a letter that does not appear in  $\pi$ , then

$$\text{des}(a : \pi) = \text{des } \pi + [a > \pi_1].$$

Thus,  $\text{des}(a : \pi)$  depends only on  $\text{des } \pi$ ,  $|\pi|$  and  $[a > \pi_1]$ . In other words,  $\text{des}$  is head-graft-compatible (by the definition of “head-graft-compatible”). Hence, Theorem 3.9 (applied to  $\text{st} = \text{des}$ ) shows that the permutation statistic  $\text{des}$  is LR-shuffle-compatible. This proves Proposition 3.13.  $\square$

### 3.5.2. The major index $\text{maj}$

The permutation statistic  $\text{maj}$  (called the *major index*) is defined as follows: For each permutation  $\pi$ , we set  $\text{maj } \pi = \sum_{i \in \text{Des } \pi} i$  (that is,  $\text{maj } \pi$  is the sum of all descents of  $\pi$ ). It was proven in [GesZhu17, Theorem 3.1 (a)] that this statistic  $\text{maj}$  is shuffle-compatible.

However,  $\text{maj}$  is neither head-graft-compatible nor LR-shuffle-compatible. For example, if we take  $\pi = (5, 4, 2, 3)$ ,  $a = 1$ ,  $\pi' = (3, 4, 5, 2)$  and  $a' = 1$ , then we do have

$$\text{maj } \pi = \text{maj } (\pi'), \quad |\pi| = |\pi'| \quad \text{and} \quad [a > \pi_1] = [a' > \pi'_1]$$

but we don't have  $\text{maj}(a : \pi) = \text{maj}(a' : \pi')$ . Thus,  $\text{maj}$  is not head-graft-compatible. Using Proposition 3.18 below, this entails that  $\text{maj}$  is not LR-shuffle-compatible.

### 3.5.3. The joint statistic $(\text{des}, \text{maj})$

The next permutation statistic we shall study is the so-called joint statistic  $(\text{des}, \text{maj})$ . This statistic is defined as the permutation statistic that sends each permutation  $\pi$  to the ordered pair  $(\text{des } \pi, \text{maj } \pi)$ . (Calling it  $(\text{des}, \text{maj})$  is thus a slight abuse of notation.) It was proven in [GesZhu17, Theorem 4.5 (a)] that this statistic  $(\text{des}, \text{maj})$  is shuffle-compatible. We now claim the following:

**Proposition 3.14.** The permutation statistic  $(\text{des}, \text{maj})$  is head-graft-compatible and LR-shuffle-compatible.

*Proof of Proposition 3.14.* From (38), we easily obtain the following: If  $\pi$  is a nonempty permutation, and if  $a$  is a letter that does not appear in  $\pi$ , then

$$\begin{aligned} \text{des}(a : \pi) &= \text{des } \pi + [a > \pi_1] & \text{and} \\ \text{maj}(a : \pi) &= \text{des } \pi + \text{maj } \pi + [a > \pi_1]. \end{aligned}$$

Thus,  $(\text{des}, \text{maj})(a : \pi)$  depends only on  $(\text{des}, \text{maj})(\pi)$ ,  $|\pi|$  and  $[a > \pi_1]$ . In other words,  $(\text{des}, \text{maj})$  is head-graft-compatible (by the definition of “head-graft-compatible”). Hence, Theorem 3.9 (applied to  $\text{st} = (\text{des}, \text{maj})$ ) shows that the permutation statistic  $(\text{des}, \text{maj})$  is LR-shuffle-compatible. This proves Proposition 3.14.  $\square$

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### 3.5.4. The comajor index comaj

The permutation statistic comaj (called the *comajor index*) is defined as follows: For each permutation  $\pi$ , we set  $\text{comaj } \pi = \sum_{k \in \text{Des } \pi} (n - k)$ , where  $n = |\pi|$ . It was proven in [GesZhu17, §3.2] that this statistic comaj is shuffle-compatible. We now claim the following:

**Proposition 3.15.** The permutation statistic comaj is head-graft-compatible and LR-shuffle-compatible.

*Proof of Proposition 3.15.* From (38), we easily obtain the following: If  $\pi$  is a nonempty permutation, and if  $a$  is a letter that does not appear in  $\pi$ , then

$$\text{comaj}(a : \pi) = \text{comaj } \pi + [a > \pi_1] \cdot |\pi|.$$

Thus,  $\text{comaj}(a : \pi)$  depends only on  $\text{comaj } \pi$ ,  $|\pi|$  and  $[a > \pi_1]$ . In other words, comaj is head-graft-compatible (by the definition of “head-graft-compatible”). Hence, Theorem 3.9 (applied to  $\text{st} = \text{comaj}$ ) shows that the permutation statistic comaj is LR-shuffle-compatible. This proves Proposition 3.15.  $\square$

### 3.6. Left- and right-shuffle-compatibility

In this section, we shall study two notions closely related to LR-shuffle-compatibility:

**Definition 3.16.** Let  $\text{st}$  be a permutation statistic.

(a) We say that  $\text{st}$  is *left-shuffle-compatible* if for any two disjoint nonempty permutations  $\pi$  and  $\sigma$  having the property that  $\pi_1 > \sigma_1$ , the multiset  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ .

(b) We say that  $\text{st}$  is *right-shuffle-compatible* if for any two disjoint nonempty permutations  $\pi$  and  $\sigma$  having the property that  $\pi_1 > \sigma_1$ , the multiset  $\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ .

For a shuffle-compatible permutation statistic, these two notions are equivalent to the notions of LR-shuffle-compatibility and head-graft-compatibility, as the following proposition reveals:

**Proposition 3.17.** Let  $\text{st}$  be a shuffle-compatible permutation statistic. Then, the following assertions are equivalent:

- *Assertion  $\mathcal{A}_1$ :* The statistic  $\text{st}$  is LR-shuffle-compatible.
- *Assertion  $\mathcal{A}_2$ :* The statistic  $\text{st}$  is left-shuffle-compatible.
- *Assertion  $\mathcal{A}_3$ :* The statistic  $\text{st}$  is right-shuffle-compatible.
- *Assertion  $\mathcal{A}_4$ :* The statistic  $\text{st}$  is head-graft-compatible.

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*Proof of Proposition 3.17 (sketched).* We shall prove the implications  $\mathcal{A}_1 \implies \mathcal{A}_2$ ,  $\mathcal{A}_2 \implies \mathcal{A}_3$ ,  $\mathcal{A}_3 \implies \mathcal{A}_2$ ,  $\mathcal{A}_3 \implies \mathcal{A}_4$  and  $\mathcal{A}_4 \implies \mathcal{A}_1$ .

The implication  $\mathcal{A}_4 \implies \mathcal{A}_1$  follows from Theorem 3.9.

*Proof of the implication  $\mathcal{A}_1 \implies \mathcal{A}_2$ :* Assume that Assertion  $\mathcal{A}_1$  holds. In other words, the statistic  $\text{st}$  is LR-shuffle-compatible. We shall show that Assertion  $\mathcal{A}_2$  holds.

The statistic  $\text{st}$  is LR-shuffle-compatible. In other words, for any two disjoint nonempty permutations  $\pi$  and  $\sigma$ , the multisets

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \quad \text{and} \quad \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$$

depend only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$ ,  $|\sigma|$  and  $[\pi_1 > \sigma_1]$ . Hence, for any two disjoint nonempty permutations  $\pi$  and  $\sigma$ , the multiset  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$ ,  $|\sigma|$  and  $[\pi_1 > \sigma_1]$ . Hence, for any two disjoint nonempty permutations  $\pi$  and  $\sigma$  having the property that  $\pi_1 > \sigma_1$ , the multiset  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$  (indeed, it no longer depends on  $[\pi_1 > \sigma_1]$ , because our condition  $\pi_1 > \sigma_1$  ensures that  $[\pi_1 > \sigma_1] = 1$ ). In other words, the statistic  $\text{st}$  is left-shuffle-compatible (by the definition of “left-shuffle-compatible”). In other words, Assertion  $\mathcal{A}_2$  holds. This proves the implication  $\mathcal{A}_1 \implies \mathcal{A}_2$ .

*Proof of the implication  $\mathcal{A}_2 \implies \mathcal{A}_3$ :* Assume that Assertion  $\mathcal{A}_2$  holds. In other words, the statistic  $\text{st}$  is left-shuffle-compatible. We shall show that Assertion  $\mathcal{A}_3$  holds.

If  $\pi$  and  $\sigma$  are two disjoint nonempty permutations, then

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} - \{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} \end{aligned}$$

(by Lemma 3.11 **(b)**). Hence, if  $\pi$  and  $\sigma$  are two disjoint nonempty permutations, then the multiset  $\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$  is uniquely determined by  $\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  and  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$ .

Thus, altogether, we know that for any two disjoint nonempty permutations  $\pi$  and  $\sigma$  having the property that  $\pi_1 > \sigma_1$ , the following holds:

- The multiset  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$  (since the statistic  $\text{st}$  is left-shuffle-compatible);
- The multiset  $\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$  (since the statistic  $\text{st}$  is shuffle-compatible);
- The multiset  $\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$  is uniquely determined by  $\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  and  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$ .

Hence, the multiset  $\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$  also depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ . In other words, the statistic  $\text{st}$  is right-shuffle-compatible (by the



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definition of “right-shuffle-compatible”). In other words, Assertion  $\mathcal{A}_3$  holds. This proves the implication  $\mathcal{A}_2 \implies \mathcal{A}_3$ .

*Proof of the implication  $\mathcal{A}_3 \implies \mathcal{A}_2$ :* The proof of the implication  $\mathcal{A}_3 \implies \mathcal{A}_2$  is entirely analogous to the above proof of the implication  $\mathcal{A}_2 \implies \mathcal{A}_3$  (except that we need to interchange “left” and “right” and likewise interchange “ $S_{\prec}$ ” and “ $S_{\succ}$ ”).

*Proof of the implication  $\mathcal{A}_3 \implies \mathcal{A}_4$ :* Assume that Assertion  $\mathcal{A}_3$  holds. We shall show that Assertion  $\mathcal{A}_4$  holds.

We have already proven the implication  $\mathcal{A}_3 \implies \mathcal{A}_2$ . Thus, Assertion  $\mathcal{A}_2$  holds (since  $\mathcal{A}_3$  holds). In other words, the statistic  $\text{st}$  is left-shuffle-compatible. In other words, the following claim holds:

*Claim 1:* Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations having the property that  $\pi_1 > \sigma_1$ . Let  $\pi'$  and  $\sigma'$  be two disjoint nonempty permutations having the property that  $\pi'_1 > \sigma'_1$ . Assume that

$$\begin{aligned} \text{st } \pi &= \text{st } (\pi'), & \text{st } \sigma &= \text{st } (\sigma'), \\ |\pi| &= |\pi'| & \text{and} & & |\sigma| &= |\sigma'|. \end{aligned}$$

Then,

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}.$$

Also, Assertion  $\mathcal{A}_3$  holds. In other words, the statistic  $\text{st}$  is right-shuffle-compatible. In other words, the following claim holds:

*Claim 2:* Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations having the property that  $\pi_1 > \sigma_1$ . Let  $\pi'$  and  $\sigma'$  be two disjoint nonempty permutations having the property that  $\pi'_1 > \sigma'_1$ . Assume that

$$\begin{aligned} \text{st } \pi &= \text{st } (\pi'), & \text{st } \sigma &= \text{st } (\sigma'), \\ |\pi| &= |\pi'| & \text{and} & & |\sigma| &= |\sigma'|. \end{aligned}$$

Then,

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

We are now going to prove the following claim:

*Claim 3:* Let  $\pi$  be a nonempty permutation, and let  $a$  be a letter that does not appear in  $\pi$ . Let  $\pi'$  be a nonempty permutation, and let  $a'$  be a letter that does not appear in  $\pi'$ . Assume that

$$\text{st } \pi = \text{st } (\pi'), \quad |\pi| = |\pi'| \quad \text{and} \quad [a > \pi_1] = [a' > \pi'_1].$$

Then,  $\text{st}(a : \pi) = \text{st}(a' : \pi')$ .

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[*Proof of Claim 3:* The 1-permutations  $(a)$  and  $(a')$  are order-isomorphic. Thus,  $\text{st}((a)) = \text{st}((a'))$  (since  $\text{st}$  is a permutation statistic). Also,  $|(a)| = 1 = |(a')|$ .

We are in one of the following two cases:

*Case 1:* We have  $a > \pi_1$ .

*Case 2:* We have  $a \leq \pi_1$ .

Let us first consider Case 1. In this case, we have  $a > \pi_1$ . Thus,  $[a > \pi_1] = 1$ . Comparing this with  $[a > \pi_1] = [a' > \pi'_1]$ , we obtain  $[a' > \pi'_1] = 1$ , so that  $a' > \pi'_1$ .

Consider the 1-permutations  $(a)$  and  $(a')$ . Then, the first entry of  $(a)$  is  $(a)_1 = a > \pi_1$ . Also,  $(a')_1 = a' > \pi'_1$ . Also, the permutations  $(a)$  and  $\pi$  are disjoint (since  $a$  does not appear in  $\pi$ ). Similarly, the permutations  $(a')$  and  $\pi'$  are disjoint. Furthermore,  $\text{st}((a)) = \text{st}((a'))$ ,  $\text{st} \pi = \text{st}(\pi')$ ,  $|(a)| = |(a')|$  and  $|\pi| = |\pi'|$ . Hence, Claim 1 (applied to  $(a)$ ,  $\pi$ ,  $(a')$  and  $\pi'$  instead of  $\pi$ ,  $\sigma$ ,  $\pi'$  and  $\sigma'$ ) yields

$$\{\text{st} \tau \mid \tau \in S_{\prec}((a), \pi)\}_{\text{multi}} = \{\text{st} \tau \mid \tau \in S_{\prec}((a'), \pi')\}_{\text{multi}}. \quad (49)$$

But  $S_{\prec}((a), \pi) = \{a : \pi\}$ . Hence,

$$\{\text{st} \tau \mid \tau \in S_{\prec}((a), \pi)\}_{\text{multi}} = \{\text{st} \tau \mid \tau \in \{a : \pi\}\}_{\text{multi}} = \{\text{st}(a : \pi)\}_{\text{multi}}.$$

Similarly,

$$\{\text{st} \tau \mid \tau \in S_{\prec}((a'), \pi')\}_{\text{multi}} = \{\text{st}(a' : \pi')\}_{\text{multi}}.$$

In light of the last two equalities, the equality (49) rewrites as  $\{\text{st}(a : \pi)\}_{\text{multi}} = \{\text{st}(a' : \pi')\}_{\text{multi}}$ . In other words,  $\text{st}(a : \pi) = \text{st}(a' : \pi')$ . Thus, we have proven  $\text{st}(a : \pi) = \text{st}(a' : \pi')$  in Case 1.

Let us now consider Case 2. In this case, we have  $a \leq \pi_1$ . But  $a$  does not appear in  $\pi$ ; therefore,  $a \neq \pi_1$ . Combined with  $a \leq \pi_1$ , this yields  $a < \pi_1$ . In other words,  $\pi_1 > a$ . Also, from  $a \leq \pi_1$ , we obtain  $[a > \pi_1] = 0$ . Comparing this with  $[a > \pi_1] = [a' > \pi'_1]$ , we obtain  $[a' > \pi'_1] = 0$ , so that  $a' \leq \pi'_1$ . But  $a'$  does not appear in  $\pi'$ ; thus,  $a' \neq \pi'_1$ . Combined with  $a' \leq \pi'_1$ , this yields  $a' < \pi'_1$ . In other words,  $\pi'_1 > a'$ .

Consider the 1-permutations  $(a)$  and  $(a')$ . Then, the first entry of  $(a)$  is  $(a)_1 = a$ ; similarly,  $(a')_1 = a'$ . Now,  $\pi_1 > a = (a)_1$  and  $\pi'_1 > a' = (a')_1$ . Also, the permutations  $\pi$  and  $(a)$  are disjoint (since  $a$  does not appear in  $\pi$ ). Similarly, the permutations  $\pi'$  and  $(a')$  are disjoint. Furthermore,  $\text{st}((a)) = \text{st}((a'))$ ,  $\text{st} \pi = \text{st}(\pi')$ ,  $|(a)| = |(a')|$  and  $|\pi| = |\pi'|$ . Hence, Claim 2 (applied to  $\pi$ ,  $(a)$ ,  $\pi'$  and  $(a')$  instead of  $\pi$ ,  $\sigma$ ,  $\pi'$  and  $\sigma'$ ) yields

$$\{\text{st} \tau \mid \tau \in S_{\succ}(\pi, (a))\}_{\text{multi}} = \{\text{st} \tau \mid \tau \in S_{\succ}(\pi', (a'))\}_{\text{multi}}. \quad (50)$$

But  $S_{\succ}(\pi, (a)) = \{a : \pi\}$ . Hence,

$$\{\text{st} \tau \mid \tau \in S_{\succ}(\pi, (a))\}_{\text{multi}} = \{\text{st} \tau \mid \tau \in \{a : \pi\}\}_{\text{multi}} = \{\text{st}(a : \pi)\}_{\text{multi}}.$$

Similarly,

$$\{\text{st} \tau \mid \tau \in S_{\succ}(\pi', (a'))\}_{\text{multi}} = \{\text{st}(a' : \pi')\}_{\text{multi}}.$$

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In light of the last two equalities, the equality (50) rewrites as  $\{\text{st}(a : \pi)\}_{\text{multi}} = \{\text{st}(a' : \pi')\}_{\text{multi}}$ . In other words,  $\text{st}(a : \pi) = \text{st}(a' : \pi')$ . Thus, we have proven  $\text{st}(a : \pi) = \text{st}(a' : \pi')$  in Case 2.

We have now proven  $\text{st}(a : \pi) = \text{st}(a' : \pi')$  in both Cases 1 and 2. Therefore,  $\text{st}(a : \pi) = \text{st}(a' : \pi')$  always holds. This proves Claim 3.]

Claim 3 shows that the statistic  $\text{st}$  is head-graft-compatible. In other words, Assertion  $\mathcal{A}_4$  holds. This proves the implication  $\mathcal{A}_3 \implies \mathcal{A}_4$ .

We have now proven the implications  $\mathcal{A}_1 \implies \mathcal{A}_2$ ,  $\mathcal{A}_2 \implies \mathcal{A}_3$ ,  $\mathcal{A}_3 \implies \mathcal{A}_4$  and  $\mathcal{A}_4 \implies \mathcal{A}_1$ . Thus, all four statements  $\mathcal{A}_1$ ,  $\mathcal{A}_2$ ,  $\mathcal{A}_3$  and  $\mathcal{A}_4$  are equivalent. This proves Proposition 3.17.  $\square$

Note that on their own, the properties of left-shuffle-compatibility and right-shuffle-compatibility are not equivalent. For example, the permutation statistic that sends each nonempty permutation  $\pi$  to the truth value  $[\pi_1 > \pi_i \text{ for all } i > 1]$  (and the 0-permutation  $()$  to 0) is right-shuffle-compatible (because in the definition of right-shuffle-compatibility, all the  $\text{st } \tau$  will be 0), but not left-shuffle-compatible.

### 3.7. Properties of compatible statistics

The following converse to Theorem 3.9 holds:

**Proposition 3.18.** Let  $\text{st}$  be a permutation statistic that is LR-shuffle-compatible. Then,  $\text{st}$  is head-graft-compatible and shuffle-compatible.

Before we can prove this, we need three lemmas:

**Lemma 3.19.** Let  $\text{st}$  be a head-graft-compatible permutation statistic. If  $\alpha$  and  $\beta$  are two permutations satisfying  $\text{Des } \alpha = \text{Des } \beta$  and  $|\alpha| = |\beta|$ , then  $\text{st } \alpha = \text{st } \beta$ .

*Proof of Lemma 3.19 (sketched).* We must prove the following claim:

*Claim 1:* Let  $\alpha$  and  $\beta$  be two permutations satisfying  $\text{Des } \alpha = \text{Des } \beta$  and  $|\alpha| = |\beta|$ . Then,  $\text{st } \alpha = \text{st } \beta$ .

[*Proof of Claim 1:* We shall prove Claim 1 by induction on  $|\alpha|$ :

*Induction base:* If  $|\alpha| = 0$ , then Claim 1 is true (because if  $|\alpha| = 0$ , then both  $\alpha$  and  $\beta$  equal the 0-permutation  $()$ , and therefore satisfy  $\alpha = \beta$  and thus  $\text{st } \alpha = \text{st } \beta$ ). This completes the induction base.

*Induction step:* Let  $N$  be a positive integer. Assume (as the induction hypothesis) that Claim 1 holds for  $|\alpha| = N - 1$ . We must now prove that Claim 1 holds for  $|\alpha| = N$ .

Let  $\alpha$  and  $\beta$  be as in Claim 1, and assume that  $|\alpha| = N$ . We must prove that  $\text{st } \alpha = \text{st } \beta$ .

If  $N = 1$ , then this holds<sup>49</sup>. Hence, for the rest of this proof, we WLOG assume that  $N \neq 1$ . Hence,  $N \geq 2$  (since  $N$  is a positive integer).

The permutation  $\alpha$  is nonempty (since  $|\alpha| = N > 0$ ). Thus,  $\alpha_{\sim 1}$  and  $\alpha_1$  are well-defined. Similarly,  $\beta_{\sim 1}$  and  $\beta_1$  are well-defined (since  $|\beta| = |\alpha| = N > 0$ ). Clearly, the letter  $\alpha_1$  does not appear in  $\alpha_{\sim 1}$  (since the letters of the permutation  $\alpha$  are distinct). Similarly, the letter  $\beta_1$  does not appear in  $\beta_{\sim 1}$ .

We have  $|\alpha_{\sim 1}| = \underbrace{|\alpha|}_{=N} - 1 = N - 1$  and similarly  $|\beta_{\sim 1}| = N - 1$ . Thus,  $|\alpha_{\sim 1}| = N - 1 = |\beta_{\sim 1}|$ . Also,  $|\alpha_{\sim 1}| = N - 1 \geq 1$  (since  $N \geq 2$ ); thus, the permutation  $\alpha_{\sim 1}$  is nonempty. Similarly,  $\beta_{\sim 1}$  is nonempty.

It is easy to see that

$$\begin{aligned} \text{Des}(\alpha_{\sim 1}) &= \{i - 1 \mid i \in (\text{Des } \alpha) \setminus \{1\}\} & \text{and} \\ \text{Des}(\beta_{\sim 1}) &= \{i - 1 \mid i \in (\text{Des } \beta) \setminus \{1\}\}. \end{aligned}$$

The right hand sides of these two equalities are equal (since  $\text{Des } \alpha = \text{Des } \beta$ ). Thus, their left hand sides are equal as well. In other words,  $\text{Des}(\alpha_{\sim 1}) = \text{Des}(\beta_{\sim 1})$ .

Moreover,  $|\alpha_{\sim 1}| = N - 1$ . Hence, the induction hypothesis reveals that we can apply Claim 1 to  $\alpha_{\sim 1}$  and  $\beta_{\sim 1}$  instead of  $\alpha$  and  $\beta$ . We thus obtain  $\text{st}(\alpha_{\sim 1}) = \text{st}(\beta_{\sim 1})$ .

Furthermore,  $(\alpha_{\sim 1})_1 = \alpha_2$ , so that

$$[\alpha_1 > (\alpha_{\sim 1})_1] = [\alpha_1 > \alpha_2] = [1 \in \text{Des } \alpha].$$

Similarly,

$$[\beta_1 > (\beta_{\sim 1})_1] = [1 \in \text{Des } \beta].$$

The right hand sides of these two equalities are equal (since  $\text{Des } \alpha = \text{Des } \beta$ ). Thus, their left hand sides are equal as well. In other words,  $[\alpha_1 > (\alpha_{\sim 1})_1] = [\beta_1 > (\beta_{\sim 1})_1]$ .

But recall that  $\text{st}$  is head-graft-compatible. In other words, every nonempty permutation  $\pi$ , every letter  $a$  that does not appear in  $\pi$ , every nonempty permutation  $\pi'$  and every letter  $a'$  that does not appear in  $\pi'$  satisfying

$$\text{st } \pi = \text{st}(\pi'), \quad |\pi| = |\pi'| \quad \text{and} \quad [a > \pi_1] = [a' > \pi'_1]$$

satisfy  $\text{st}(a : \pi) = \text{st}(a' : \pi')$ . Applying this to  $\pi = \alpha_{\sim 1}$ ,  $a = \alpha_1$ ,  $\pi' = \beta_{\sim 1}$  and  $a' = \beta_1$ , we obtain  $\text{st}(\alpha_1 : \alpha_{\sim 1}) = \text{st}(\beta_1 : \beta_{\sim 1})$ . In view of  $\alpha_1 : \alpha_{\sim 1} = \alpha$  and  $\beta_1 : \beta_{\sim 1} = \beta$ , this rewrites as  $\text{st } \alpha = \text{st } \beta$ .

Thus, we have proven that  $\text{st } \alpha = \text{st } \beta$ . Hence, Claim 1 holds for  $|\alpha| = N$ . This completes the induction step. Thus, Claim 1 is proven.]

Lemma 3.19 follows immediately from Claim 1.  $\square$

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<sup>49</sup>*Proof.* Assume that  $N = 1$ . Thus,  $\alpha$  and  $\beta$  are 1-permutations (since  $|\alpha| = N = 1$  and  $|\beta| = |\alpha| = N = 1$ ) and thus are order-isomorphic. Hence,  $\text{st } \alpha = \text{st } \beta$  (since  $\text{st}$  is a permutation statistic).

---

**Lemma 3.20.** Let  $\text{st}$  be a head-graft-compatible permutation statistic. Let  $X$  be the codomain of  $\text{st}$ . Let  $n \in \mathbb{N}$ . Then, there exists a map

$$F_n : \{\text{subsets of } [n-1]\} \rightarrow X$$

such that every  $n$ -permutation  $\tau$  satisfies  $\text{st } \tau = F_n(\text{Des } \tau)$ .

*Proof of Lemma 3.20.* We define the map  $F_n$  as follows:

Let  $Z$  be any subset of  $[n-1]$ . Then, it is well-known that there exists some  $n$ -permutation  $\tau$  satisfying  $Z = \text{Des } \tau$ . Pick any such  $\tau$ . Then,  $\text{st } \tau$  does not depend on the choice of  $\tau$  (because if  $\alpha$  and  $\beta$  are two different  $n$ -permutations  $\tau$  satisfying  $Z = \text{Des } \tau$ , then Lemma 3.19 yields  $\text{st } \alpha = \text{st } \beta$ ). Hence, we can set  $F_n(Z)$  to be  $\text{st } \tau$ .

Thus, we have defined  $F_n(Z)$  for each subset  $Z$  of  $[n-1]$ . This completes the definition of  $F_n$ . This definition shows that every  $n$ -permutation  $\tau$  satisfies  $\text{st } \tau = F_n(\text{Des } \tau)$ . Thus, Lemma 3.20 is proven.  $\square$

**Lemma 3.21.** Let  $\pi$  and  $\sigma$  be two disjoint permutations. Let  $\pi'$  and  $\sigma'$  be two disjoint permutations. Assume that

$$\begin{aligned} \text{Des } \pi = \text{Des } (\pi'), & \quad \text{Des } \sigma = \text{Des } (\sigma'), \\ |\pi| = |\pi'| & \quad \text{and} \quad |\sigma| = |\sigma'|. \end{aligned}$$

Then,

$$\{\text{Des } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{Des } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}.$$

*Proof of Lemma 3.21.* Lemma 3.21 is simply the statement that the permutation statistic  $\text{Des}$  is shuffle-compatible. But this has been proven in [GesZhu17, §2.4].  $\square$

*Proof of Proposition 3.18 (sketched).* We know that  $\text{st}$  is LR-shuffle-compatible. In other words, the following holds:

*Claim 1:* Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations. Let  $\pi'$  and  $\sigma'$  be two disjoint nonempty permutations. Assume that

$$\begin{aligned} \text{st } \pi = \text{st } (\pi'), & \quad \text{st } \sigma = \text{st } (\sigma'), \\ |\pi| = |\pi'|, & \quad |\sigma| = |\sigma'| \quad \text{and} \quad [\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1]. \end{aligned}$$

Then,

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

---

Next, we want to show that  $\text{st}$  is head-graft-compatible. In other words, we want to show that the following holds:

*Claim 2:* Let  $\pi$  be a nonempty permutation, and let  $a$  be a letter that does not appear in  $\pi$ . Let  $\pi'$  be a nonempty permutation, and let  $a'$  be a letter that does not appear in  $\pi'$ . Assume that

$$\text{st } \pi = \text{st } (\pi'), \quad |\pi| = |\pi'| \quad \text{and} \quad [a > \pi_1] = [a' > \pi'_1].$$

Then,  $\text{st } (a : \pi) = \text{st } (a' : \pi')$ .

[*Proof of Claim 2:* The 1-permutations  $(a)$  and  $(a')$  are order-isomorphic. Thus,  $\text{st } ((a)) = \text{st } ((a'))$  (since  $\text{st}$  is a permutation statistic).

The 1-permutations  $(a)$  and  $(a')$  satisfy  $(a)_1 = a$  and  $(a')_1 = a'$ . Thus, the inequality  $[a > \pi_1] = [a' > \pi'_1]$  (which is true by assumption) rewrites as  $[(a)_1 > \pi_1] = [(a')_1 > \pi'_1]$ . Also,  $\text{st } ((a)) = \text{st } ((a'))$  and  $|(a)| = 1 = |(a')|$ . Also, the 1-permutation  $(a)$  is disjoint from  $\pi$  (since  $a$  does not appear in  $\pi$ ). Similarly, the 1-permutation  $(a')$  is disjoint from  $\pi'$ . Thus, Claim 1 (applied to  $\sigma = (a)$  and  $\sigma' = (a')$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, (a))\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', (a'))\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, (a))\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', (a'))\}_{\text{multi}}.$$

But it is easily seen that  $S_{\succ}(\pi, (a)) = \{a : \pi\}$ . Hence,

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, (a))\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in \{a : \pi\}\}_{\text{multi}} = \{\text{st } (a : \pi)\}_{\text{multi}}.$$

Similarly,

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi', (a'))\}_{\text{multi}} = \{\text{st } (a' : \pi')\}_{\text{multi}}.$$

Thus,

$$\begin{aligned} \{\text{st } (a : \pi)\}_{\text{multi}} &= \{\text{st } \tau \mid \tau \in S_{\succ}(\pi, (a))\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', (a'))\}_{\text{multi}} = \{\text{st } (a' : \pi')\}_{\text{multi}}, \end{aligned}$$

so that  $\text{st } (a : \pi) = \text{st } (a' : \pi')$ . This proves Claim 2.]

Now, Claim 2 shows that  $\text{st}$  is head-graft-compatible. It remains to show that  $\text{st}$  is shuffle-compatible. First, we show an auxiliary statement:

*Claim 3:* Let  $\pi, \pi'$  and  $\sigma$  be three permutations. Assume that  $\pi$  and  $\pi'$  are order-isomorphic. Assume that  $\pi$  and  $\sigma$  are disjoint. Assume that  $\pi'$  and  $\sigma$  are disjoint. Then,

$$\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\pi', \sigma)\}_{\text{multi}}.$$

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[*Proof of Claim 3:* We have  $|\pi| = |\pi'|$  (since  $\pi$  and  $\pi'$  are order-isomorphic). Define  $n \in \mathbb{N}$  by  $n = |\pi| = |\pi'|$ . Let  $X$  be the codomain of  $\text{st}$ . Lemma 3.20 shows that there exists a map

$$F_n : \{\text{subsets of } [n-1]\} \rightarrow X$$

such that every  $n$ -permutation  $\tau$  satisfies

$$\text{st } \tau = F_n (\text{Des } \tau). \quad (51)$$

Consider this map  $F_n$ .

We know that  $\text{Des}$  is a permutation statistic. Thus,  $\text{Des } \pi = \text{Des } (\pi')$  (since  $\pi$  and  $\pi'$  are order-isomorphic). Also,  $|\pi| = |\pi'|$ ,  $\text{Des } \sigma = \text{Des } \sigma$  and  $|\sigma| = |\sigma|$ . Hence, Lemma 3.21 (applied to  $\sigma' = \sigma$ ) yields

$$\{\text{Des } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{Des } \tau \mid \tau \in S(\pi', \sigma)\}_{\text{multi}}. \quad (52)$$

Now,

$$\begin{aligned} & \left\{ \underbrace{\text{st } \tau}_{=F_n(\text{Des } \tau)} \mid \tau \in S(\pi, \sigma) \right\}_{\text{multi}} \\ & \quad \text{(by (51))} \\ &= \{F_n(\text{Des } \tau) \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = F_n(\{\text{Des } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}) \\ &= F_n(\{\text{Des } \tau \mid \tau \in S(\pi', \sigma)\}_{\text{multi}}) \quad \text{(by (52))} \\ &= \left\{ \underbrace{F_n(\text{Des } \tau)}_{=\text{st } \tau} \mid \tau \in S(\pi', \sigma) \right\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\pi', \sigma)\}_{\text{multi}}. \end{aligned}$$

This proves Claim 3.]

Let us finally prove that  $\text{st}$  is shuffle-compatible. In other words, let us prove the following claim:

*Claim 4:* Let  $\pi$  and  $\sigma$  be two disjoint permutations. Let  $\pi'$  and  $\sigma'$  be two disjoint permutations. Assume that

$$\begin{aligned} \text{st } \pi &= \text{st } (\pi'), & \text{st } \sigma &= \text{st } (\sigma'), \\ |\pi| &= |\pi'| & \text{and} & & |\sigma| &= |\sigma'|. \end{aligned}$$

Then,

$$\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}.$$

[*Proof of Claim 4:* Recall that  $S(\pi, \sigma) = S(\sigma, \pi)$  and  $S(\pi', \sigma') = S(\sigma', \pi')$ . Hence, Claim 4 does not change if we swap  $\pi$  with  $\sigma$  while simultaneously swapping  $\pi'$  with  $\sigma'$ . Thus, we WLOG assume that  $\pi_1 \geq \sigma_1$  (since otherwise, we can ensure this by performing these swaps). But  $\pi_1 \neq \sigma_1$  (since  $\pi$  and  $\sigma$  are disjoint). Combining this with  $\pi_1 \geq \sigma_1$ , we obtain  $\pi_1 > \sigma_1$ .

Define  $n \in \mathbb{N}$  by  $n = |\pi| = |\pi'|$ . Define  $m \in \mathbb{N}$  by  $m = |\sigma| = |\sigma'|$ .

If  $n = 0$ , then both  $\pi$  and  $\pi'$  equal the 0-permutation  $()$  (since  $n = |\pi| = |\pi'|$ ). Hence, if  $n = 0$ , then Claim 4 is true (because in this case, we have

$$\begin{aligned} \left\{ \text{st } \tau \mid \tau \in S \left( \underbrace{\pi}_{=()} , \sigma \right) \right\}_{\text{multi}} &= \left\{ \text{st } \tau \mid \tau \in S \left( \underbrace{()}_{=\{\sigma\}} , \sigma \right) \right\}_{\text{multi}} \\ &= \{ \text{st } \tau \mid \tau \in \{\sigma\} \}_{\text{multi}} = \{ \text{st } \sigma \}_{\text{multi}} \end{aligned}$$

and similarly  $\{ \text{st } \tau \mid \tau \in S(\pi', \sigma') \}_{\text{multi}} = \{ \text{st } (\sigma') \}_{\text{multi}}$ , and therefore the assertion of Claim 4 reduces to  $\{ \text{st } \sigma \}_{\text{multi}} = \{ \text{st } (\sigma') \}_{\text{multi}}$ , which follows immediately from the assumption  $\text{st } \sigma = \text{st } (\sigma')$ . Thus, for the rest of this proof, we WLOG assume that  $n \neq 0$ . For similar reasons, we WLOG assume that  $m \neq 0$ .

The permutations  $\pi$  and  $\pi'$  are nonempty (since  $|\pi| = |\pi'| = n \neq 0$ ). Similarly, the permutations  $\sigma$  and  $\sigma'$  are nonempty.

There clearly exists a positive integer  $N$  that is larger than all entries of  $\sigma$  and larger than all entries of  $\sigma'$ . Consider such an  $N$ . From  $n = |\pi'|$ , we obtain  $\pi' = (\pi'_1, \pi'_2, \dots, \pi'_n)$ . Let  $\gamma$  be the permutation  $(\pi'_1 + N, \pi'_2 + N, \dots, \pi'_n + N)$ . This permutation  $\gamma$  is order-isomorphic to  $\pi'$ , but is disjoint from  $\sigma$  (since all its entries are  $> N$ , while all the entries of  $\sigma$  are  $< N$ ) and disjoint from  $\sigma'$  (for similar reasons). Also,  $\gamma_1 = \underbrace{\pi'_1}_{>0} + N > N > \sigma_1$ , so that  $[\gamma_1 > \sigma_1] = 1$ . Similarly,

$$[\gamma_1 > \sigma'_1] = 1.$$

The permutation  $\gamma$  is order-isomorphic to  $\pi'$ . Thus,  $\text{st } \gamma = \text{st } (\pi')$  (since  $\text{st}$  is a permutation statistic) and  $|\gamma| = |\pi'|$ . The permutation  $\gamma$  is furthermore nonempty (since it is order-isomorphic to the nonempty permutation  $\pi'$ ). Also,  $\text{st } \gamma = \text{st } (\pi') = \text{st } \pi$  and  $|\gamma| = |\pi'| = |\pi|$ . Moreover, from  $\pi_1 > \sigma_1$ , we obtain  $[\pi_1 > \sigma_1] = 1 = [\gamma_1 > \sigma'_1]$ . Hence, Claim 1 (applied to  $\gamma$  instead of  $\pi'$ ) yields

$$\{ \text{st } \tau \mid \tau \in S_{<}(\pi, \sigma) \}_{\text{multi}} = \{ \text{st } \tau \mid \tau \in S_{<}(\gamma, \sigma') \}_{\text{multi}} \quad (53)$$

and

$$\{ \text{st } \tau \mid \tau \in S_{>}(\pi, \sigma) \}_{\text{multi}} = \{ \text{st } \tau \mid \tau \in S_{>}(\gamma, \sigma') \}_{\text{multi}}. \quad (54)$$

Now, if  $A$  and  $B$  are two finite multisets, then  $A + B$  shall denote the multiset union of  $A$  and  $B$ ; this is the finite multiset  $C$  such that each object  $g$  satisfies  $|C|_g = |A|_g + |B|_g$ . Then, it is easy to see that

$$\begin{aligned} &\{ \text{st } \tau \mid \tau \in S(\pi, \sigma) \}_{\text{multi}} \\ &= \{ \text{st } \tau \mid \tau \in S_{<}(\pi, \sigma) \}_{\text{multi}} + \{ \text{st } \tau \mid \tau \in S_{>}(\pi, \sigma) \}_{\text{multi}} \end{aligned}$$



and

$$\begin{aligned} & \{\text{st } \tau \mid \tau \in S(\gamma, \sigma')\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S_{\prec}(\gamma, \sigma')\}_{\text{multi}} + \{\text{st } \tau \mid \tau \in S_{\succ}(\gamma, \sigma')\}_{\text{multi}}. \end{aligned}$$

Hence, by adding together the equalities (53) and (54) (using the operation + that we have just defined), we obtain the equality

$$\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\gamma, \sigma')\}_{\text{multi}}.$$

On the other hand, Claim 3 (applied to  $\gamma$  and  $\sigma'$  instead of  $\pi$  and  $\sigma$ ) yields

$$\{\text{st } \tau \mid \tau \in S(\gamma, \sigma')\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}.$$

Hence,

$$\begin{aligned} \{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} &= \{\text{st } \tau \mid \tau \in S(\gamma, \sigma')\}_{\text{multi}} \\ &= \{\text{st } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}. \end{aligned}$$

This proves Claim 4.]

Claim 4 shows that  $\text{st}$  is shuffle-compatible. This completes the proof of Proposition 3.18.  $\square$

**Corollary 3.22.** Let  $\text{st}$  be a LR-shuffle-compatible permutation statistic. Then,  $\text{st}$  is shuffle-compatible, left-shuffle-compatible, right-shuffle-compatible and head-graft-compatible.

*Proof of Corollary 3.22.* Proposition 3.18 yields that  $\text{st}$  is head-graft-compatible and shuffle-compatible. Thus, Proposition 3.17 yields that  $\text{st}$  is left-shuffle-compatible and right-shuffle-compatible as well. This proves Corollary 3.22.  $\square$

**Corollary 3.23.** Let  $\text{st}$  be a permutation statistic that is left-shuffle-compatible and right-shuffle-compatible. Then,  $\text{st}$  is LR-shuffle-compatible.

*Proof of Corollary 3.23.* We have assumed that  $\text{st}$  is left-shuffle-compatible. In other words, the following claim holds:

*Claim 1:* Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations having the property that  $\pi_1 > \sigma_1$ . Let  $\pi'$  and  $\sigma'$  be two disjoint nonempty permutations having the property that  $\pi'_1 > \sigma'_1$ . Assume that

$$\begin{aligned} \text{st } \pi &= \text{st } (\pi'), & \text{st } \sigma &= \text{st } (\sigma'), \\ |\pi| &= |\pi'| & \text{and} & & |\sigma| &= |\sigma'|. \end{aligned}$$

Then,

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}.$$

---

Also, the statistic  $\text{st}$  is right-shuffle-compatible. In other words, the following claim holds:

*Claim 2:* Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations having the property that  $\pi_1 > \sigma_1$ . Let  $\pi'$  and  $\sigma'$  be two disjoint nonempty permutations having the property that  $\pi'_1 > \sigma'_1$ . Assume that

$$\begin{aligned} \text{st } \pi &= \text{st } (\pi'), & \text{st } \sigma &= \text{st } (\sigma'), \\ |\pi| &= |\pi'| & \text{and} & & |\sigma| &= |\sigma'|. \end{aligned}$$

Then,

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

On the other hand, we want to prove that  $\text{st}$  is LR-shuffle-compatible. In other words, we want to prove the following claim:

*Claim 3:* Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations. Let  $\pi'$  and  $\sigma'$  be two disjoint nonempty permutations. Assume that

$$\begin{aligned} \text{st } \pi &= \text{st } (\pi'), & \text{st } \sigma &= \text{st } (\sigma'), \\ |\pi| &= |\pi'|, & |\sigma| &= |\sigma'| & \text{and} & & [\pi_1 > \sigma_1] &= [\pi'_1 > \sigma'_1]. \end{aligned}$$

Then,

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}$$

and

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

[*Proof of Claim 3:* We are in one of the following two cases:

*Case 1:* We have  $\pi_1 > \sigma_1$ .

*Case 2:* We have  $\pi_1 \leq \sigma_1$ .

Let us first consider Case 1. In this case, we have  $\pi_1 > \sigma_1$ . Hence,  $[\pi_1 > \sigma_1] = 1$ . Comparing this with  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1]$ , we find  $[\pi'_1 > \sigma'_1] = 1$ . Hence,  $\pi'_1 > \sigma'_1$ . Recall also that  $\pi_1 > \sigma_1$ . Hence, Claim 1 yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}.$$

But Claim 2 yields

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

Thus, Claim 3 is proven in Case 1.

Let us next consider Case 2. In this case, we have  $\pi_1 \leq \sigma_1$ .

Applying Proposition 3.3 (a) to various pairs of disjoint permutations, we obtain  $S_{\prec}(\pi, \sigma) = S_{\succ}(\sigma, \pi)$  and  $S_{\prec}(\pi', \sigma') = S_{\succ}(\sigma', \pi')$  and  $S_{\prec}(\sigma, \pi) = S_{\succ}(\pi, \sigma)$  and  $S_{\prec}(\sigma', \pi') = S_{\succ}(\pi', \sigma')$ .

But  $\pi_1 \neq \sigma_1$  (since  $\pi$  and  $\sigma$  are disjoint). Combined with  $\pi_1 \leq \sigma_1$ , this yields  $\pi_1 < \sigma_1$ ; thus,  $\sigma_1 > \pi_1$ . Also,  $[\pi_1 > \sigma_1] = 0$  (since  $\pi_1 \leq \sigma_1$ ). Comparing this with  $[\pi_1 > \sigma_1] = [\pi'_1 > \sigma'_1]$ , we find  $[\pi'_1 > \sigma'_1] = 0$ . Hence,  $\pi'_1 \leq \sigma'_1$ . But  $\pi'_1 \neq \sigma'_1$  (since  $\pi'$  and  $\sigma'$  are disjoint). Combined with  $\pi'_1 \leq \sigma'_1$ , this yields  $\pi'_1 < \sigma'_1$ ; thus,  $\sigma'_1 > \pi'_1$ . Hence, Claim 2 (applied to  $\sigma, \pi, \sigma'$  and  $\pi'$  instead of  $\pi, \sigma, \pi'$  and  $\sigma'$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\sigma, \pi)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\sigma', \pi')\}_{\text{multi}}.$$

In light of  $S_{\succ}(\sigma, \pi) = S_{\prec}(\pi, \sigma)$  and  $S_{\succ}(\sigma', \pi') = S_{\prec}(\pi', \sigma')$ , this rewrites as

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}.$$

Also, Claim 1 (applied to  $\sigma, \pi, \sigma'$  and  $\pi'$  instead of  $\pi, \sigma, \pi'$  and  $\sigma'$ ) yields

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\sigma, \pi)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\sigma', \pi')\}_{\text{multi}}.$$

In light of  $S_{\prec}(\sigma, \pi) = S_{\succ}(\pi, \sigma)$  and  $S_{\prec}(\sigma', \pi') = S_{\succ}(\pi', \sigma')$ , this rewrites as

$$\{\text{st } \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\succ}(\pi', \sigma')\}_{\text{multi}}.$$

Thus, Claim 3 is proven in Case 2.

We have now proven Claim 3 in both Cases 1 and 2. Thus, Claim 3 is always proven.]

Claim 3 shows that  $\text{st}$  is LR-shuffle-compatible. This proves Corollary 3.23.  $\square$

## 4. Descent statistics and quasisymmetric functions

In this section, we shall recall the concepts of descent statistics and their shuffle algebras (introduced in [GesZhu17]), and apply them to  $\text{Epk}$ .

### 4.1. Compositions

**Definition 4.1.** A *composition* is a finite list of positive integers. If  $I = (i_1, i_2, \dots, i_n)$  is a composition, then the nonnegative integer  $i_1 + i_2 + \dots + i_n$  is called the *size* of  $I$  and is denoted by  $|I|$ ; we furthermore say that  $I$  is a *composition of*  $|I|$ .

**Definition 4.2.** Let  $n \in \mathbb{N}$ . For each composition  $I = (i_1, i_2, \dots, i_k)$  of  $n$ , we define a subset  $\text{Des } I$  of  $[n - 1]$  by

$$\begin{aligned} \text{Des } I &= \{i_1, i_1 + i_2, i_1 + i_2 + i_3, \dots, i_1 + i_2 + \dots + i_{k-1}\} \\ &= \{i_1 + i_2 + \dots + i_s \mid s \in [k - 1]\}. \end{aligned}$$

---

On the other hand, for each subset  $A = \{a_1 < a_2 < \cdots < a_k\}$  of  $[n - 1]$ , we define a composition  $\text{Comp } A$  of  $n$  by

$$\text{Comp } A = (a_1, a_2 - a_1, a_3 - a_2, \dots, a_k - a_{k-1}, n - a_k).$$

(The definition of  $\text{Comp } A$  should be understood to give  $\text{Comp } A = (n)$  if  $A = \emptyset$  and  $n > 0$ , and to give  $\text{Comp } A = ()$  if  $A = \emptyset$  and  $n = 0$ . Note that  $\text{Comp } A$  depends not only on the set  $A$  itself, but also on  $n$ . We hope that  $n$  will always be clear from the context when we use this notation.)

We thus have defined a map  $\text{Des}$  (from the set of all compositions of  $n$  to the set of all subsets of  $[n - 1]$ ) and a map  $\text{Comp}$  (in the opposite direction). These two maps are mutually inverse bijections.

**Definition 4.3.** Let  $n \in \mathbb{N}$ . Let  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$  be an  $n$ -permutation. The *descent composition* of  $\pi$  is defined to be the composition  $\text{Comp}(\text{Des } \pi)$  of  $n$ . This composition is denoted by  $\text{Comp } \pi$ .

For example, the 6-permutation  $\pi = (4, 1, 3, 9, 6, 8)$  has  $\text{Comp } \pi = (1, 3, 2)$ . For another example, the 6-permutation  $\pi = (1, 4, 3, 2, 9, 8)$  has  $\text{Comp } \pi = (2, 1, 2, 1)$ .

Definition 4.3 defines the permutation statistic  $\text{Comp}$ , whose codomain is the set of all compositions.

## 4.2. Descent statistics

**Definition 4.4.** Let  $\text{st}$  be a permutation statistic. We say that  $\text{st}$  is a *descent statistic* if and only if  $\text{st } \pi$  (for  $\pi$  a permutation) depends only on the descent composition  $\text{Comp } \pi$  of  $\pi$ . In other words,  $\text{st}$  is a descent statistic if and only if every two permutations  $\pi$  and  $\sigma$  satisfying  $\text{Comp } \pi = \text{Comp } \sigma$  satisfy  $\text{st } \pi = \text{st } \sigma$ .

Equivalently, a permutation statistic  $\text{st}$  is a descent statistic if and only if every two permutations  $\pi$  and  $\sigma$  satisfying  $|\pi| = |\sigma|$  and  $\text{Des } \pi = \text{Des } \sigma$  satisfy  $\text{st } \pi = \text{st } \sigma$ . (This is indeed equivalent, because for two permutations  $\pi$  and  $\sigma$ , the condition  $(|\pi| = |\sigma| \text{ and } \text{Des } \pi = \text{Des } \sigma)$  is equivalent to  $(\text{Comp } \pi = \text{Comp } \sigma)$ .)

For example, the permutation statistic  $\text{Des}$  is a descent statistic, because each permutation  $\pi$  satisfies  $\text{Des } \pi = \text{Des}(\text{Comp } \pi)$ . Also,  $\text{Pk}$  is a descent statistic, since each permutation  $\pi$  satisfies

$$\text{Pk } \pi = (\text{Des } \pi) \setminus (\{1\} \cup (\text{Des } \pi + 1)),$$

where  $\text{Des } \pi + 1$  denotes the set  $\{i + 1 \mid i \in \text{Des } \pi\}$  (and, as we have just said,  $\text{Des } \pi$  can be recovered from  $\text{Comp } \pi$ ). Furthermore,  $\text{Epk}$  is a descent statistic, since each  $n$ -permutation  $\pi$  (for a positive integer  $n$ ) satisfies

$$\text{Epk } \pi = (\text{Des } \pi \cup \{n\}) \setminus (\text{Des } \pi + 1)$$

---

(and both  $\text{Des } \pi$  and  $n$  can be recovered from  $\text{Comp } \pi$ ). The permutation statistics  $\text{Lpk}$  and  $\text{Rpk}$  (and, of course,  $\text{Comp}$ ) are descent statistics as well, as one can easily check.

In [Oguz18, Corollary 1.6], Ezgi Kantarcı Oğuz has demonstrated that not every shuffle-compatible permutation statistic is a descent statistic. However, this changes if we require LR-shuffle-compatibility, because of Corollary 3.22 and of the following fact:

**Proposition 4.5.** Every head-graft-compatible permutation statistic is a descent statistic.

*Proof of Proposition 4.5.* Let  $st$  be a head-graft-compatible permutation statistic. Then, we must show that  $st$  is a descent statistic. But this follows directly from Lemma 3.19.  $\square$

Combining Proposition 4.5 with Corollary 3.22, we conclude that every LR-shuffle-compatible permutation statistic is a descent statistic.

**Definition 4.6.** Let  $st$  be a descent statistic. Then, we can regard  $st$  as a map from the set of all compositions (rather than from the set of all permutations). Namely, for any composition  $I$ , we define  $st I$  (an element of the codomain of  $st$ ) by setting

$$st I = st \pi \quad \text{for any permutation } \pi \text{ satisfying } \text{Comp } \pi = I.$$

This is well-defined (because for every composition  $I$ , there exists at least one permutation  $\pi$  satisfying  $\text{Comp } \pi = I$ , and all such permutations  $\pi$  have the same value of  $st \pi$ ). In the following, we shall regard every descent statistic  $st$  simultaneously as a map from the set of all permutations and as a map from the set of all compositions.

Note that this definition leads to a new interpretation of  $\text{Des } I$  for a composition  $I$ : It is now defined as  $\text{Des } \pi$  for any permutation  $\pi$  satisfying  $\text{Comp } \pi = I$ . This could clash with the old meaning of  $\text{Des } I$  introduced in Definition 4.2. Fortunately, these two meanings of  $\text{Des } I$  are exactly the same, so there is no conflict of notation.

However, Definition 4.6 causes an ambiguity for expressions like “ $\text{Des } (i_1, i_2, \dots, i_n)$ ”: Here, the “ $(i_1, i_2, \dots, i_n)$ ” might be understood either as a permutation, or as a composition, and the resulting descent sets  $\text{Des } (i_1, i_2, \dots, i_n)$  are not the same. A similar ambiguity occurs for any descent statistic  $st$  instead of  $\text{Des}$ . We hope that this ambiguity will not arise in this paper due to our explicit typecasting of permutations and compositions; but the reader should be warned that it can arise if one takes the notation too literally.

**Definition 4.7.** Let  $st$  be a descent statistic.

(a) Two compositions  $J$  and  $K$  are said to be *st-equivalent* if and only if they have the same size and satisfy  $st J = st K$ . Equivalently, two compositions  $J$  and  $K$  are *st-equivalent* if and only if there exist two *st-equivalent* permutations  $\pi$  and  $\sigma$  satisfying  $J = \text{Comp } \pi$  and  $K = \text{Comp } \sigma$ .

(b) The relation “*st-equivalent*” is an equivalence relation on compositions; its equivalence classes are called *st-equivalence classes of compositions*.

### 4.3. Quasisymmetric functions

We now recall the definition of quasisymmetric functions; see [GriRei18, Chapter 5] (and various other modern textbooks) for more details about this:

**Definition 4.8. (a)** Consider the ring of power series  $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$  in infinitely many commuting indeterminates over  $\mathbb{Q}$ . A power series  $f \in \mathbb{Q}[[x_1, x_2, x_3, \dots]]$  is said to be *quasisymmetric* if it has the following property:

- For any positive integers  $a_1, a_2, \dots, a_k$  and any two strictly increasing sequences  $(i_1 < i_2 < \dots < i_k)$  and  $(j_1 < j_2 < \dots < j_k)$  of positive integers, the coefficient of  $x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_k}^{a_k}$  in  $f$  equals the coefficient of  $x_{j_1}^{a_1} x_{j_2}^{a_2} \dots x_{j_k}^{a_k}$  in  $f$ .

(b) A *quasisymmetric function* is a quasisymmetric power series  $f \in \mathbb{Q}[[x_1, x_2, x_3, \dots]]$  that has bounded degree (i.e., there exists an  $N \in \mathbb{N}$  such that each monomial appearing in  $f$  has degree  $\leq N$ ).

(c) The quasisymmetric functions form a  $\mathbb{Q}$ -subalgebra of  $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ ; this  $\mathbb{Q}$ -subalgebra is denoted by  $\text{QSym}$  and called the *ring of quasisymmetric functions* over  $\mathbb{Q}$ . This  $\mathbb{Q}$ -algebra  $\text{QSym}$  is graded (in the obvious way, i.e., by the degree of a monomial).

The  $\mathbb{Q}$ -algebra  $\text{QSym}$  has much interesting structure (e.g., it is a Hopf algebra), some of which we will introduce later when we need it. One simple yet crucial feature of  $\text{QSym}$  that we will immediately use is the *fundamental basis* of  $\text{QSym}$ :

**Definition 4.9.** For any composition  $\alpha$ , we define the *fundamental quasisymmetric function*  $F_\alpha$  to be the power series

$$\sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ i_j < i_{j+1} \text{ for each } j \in \text{Des } \alpha}} x_{i_1} x_{i_2} \dots x_{i_n} \in \text{QSym},$$

where  $n = |\alpha|$  is the size of  $\alpha$ . The family  $(F_\alpha)_\alpha$  is a composition is a basis of the  $\mathbb{Q}$ -vector space  $\text{QSym}$ ; it is known as the *fundamental basis* of  $\text{QSym}$ .

The fundamental quasisymmetric function  $F_\alpha$  is denoted by  $L_\alpha$  in [GriRei18, §5.2].

---

The multiplication of fundamental quasisymmetric functions is intimately related to shuffles of permutations:

**Theorem 4.10.** Let  $\pi$  and  $\sigma$  be two disjoint permutations. Let  $J = \text{Comp } \pi$  and  $K = \text{Comp } \sigma$ . For any composition  $L$ , let  $c_{J,K}^L$  be the number of permutations with descent composition  $L$  among the shuffles of  $\pi$  and  $\sigma$ . Then,

$$F_J F_K = \sum_L c_{J,K}^L F_L$$

(where the sum is over all compositions  $L$ ).

Theorem 4.10 is [GesZhu17, Theorem 4.1]; it can also be written in the following form:

**Proposition 4.11.** Let  $\pi$  and  $\sigma$  be two disjoint permutations. Then,

$$F_{\text{Comp } \pi} F_{\text{Comp } \sigma} = \sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp } \chi}.$$

For a proof of Proposition 4.11 (and therefore also of the equivalent Theorem 4.10), we refer to [GriRei18, (5.2.6)] (which makes the additional requirement that the letters of  $\pi$  are  $1, 2, \dots, |\pi|$  and the letters of  $\sigma$  are  $|\pi| + 1, |\pi| + 2, \dots, |\pi| + |\sigma|$ ; but this requirement is not used in the proof and thus can be dropped).

## 4.4. Shuffle algebras

Any shuffle-compatible permutation statistic  $\text{st}$  gives rise to a *shuffle algebra*  $\mathcal{A}_{\text{st}}$ , defined as follows:

**Definition 4.12.** Let  $\text{st}$  be a shuffle-compatible permutation statistic. For each permutation  $\pi$ , let  $[\pi]_{\text{st}}$  denote the  $\text{st}$ -equivalence class of  $\pi$ .

Let  $\mathcal{A}_{\text{st}}$  be the free  $\mathbb{Q}$ -vector space whose basis is the set of all  $\text{st}$ -equivalence classes of permutations. We define a multiplication on  $\mathcal{A}_{\text{st}}$  by setting

$$[\pi]_{\text{st}} [\sigma]_{\text{st}} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{\text{st}}$$

for any two disjoint permutations  $\pi$  and  $\sigma$ . It is easy to see that this multiplication is well-defined and associative, and turns  $\mathcal{A}_{\text{st}}$  into a  $\mathbb{Q}$ -algebra whose unity is the  $\text{st}$ -equivalence class of the 0-permutation  $()$ . (In the particular case when  $\text{st}$  is a descent statistic, this shall be proven again in Proposition 4.13 (a) below.) This  $\mathbb{Q}$ -algebra is denoted by  $\mathcal{A}_{\text{st}}$ , and is called the *shuffle algebra* of  $\text{st}$ . It is a graded  $\mathbb{Q}$ -algebra; its  $n$ -th graded component (for each  $n \in \mathbb{N}$ ) is spanned by the  $\text{st}$ -equivalence classes of all  $n$ -permutations.

---

This definition originates in [GesZhu17, §3.1]. The following fact is implicit in [GesZhu17]:

**Proposition 4.13.** Let  $st$  be a shuffle-compatible descent statistic.

(a) The multiplication on  $\mathcal{A}_{st}$  defined in Definition 4.12 is well-defined and associative, and turns  $\mathcal{A}_{st}$  into a  $\mathbb{Q}$ -algebra whose unity is the  $st$ -equivalence class of the 0-permutation  $()$ .

(b) There is a surjective  $\mathbb{Q}$ -algebra homomorphism  $p_{st} : \text{QSym} \rightarrow \mathcal{A}_{st}$  that satisfies

$$p_{st}(F_{\text{Comp } \pi}) = [\pi]_{st} \quad \text{for every permutation } \pi.$$

A central result, connecting shuffle-compatibility of a descent statistic with  $\text{QSym}$ , is [GesZhu17, Theorem 4.3], which we restate as follows:

**Theorem 4.14.** Let  $st$  be a descent statistic.

(a) The descent statistic  $st$  is shuffle-compatible if and only if there exist a  $\mathbb{Q}$ -algebra  $A$  with basis  $(u_\alpha)$  (indexed by  $st$ -equivalence classes  $\alpha$  of compositions) and a  $\mathbb{Q}$ -algebra homomorphism  $\phi_{st} : \text{QSym} \rightarrow A$  with the property that whenever  $\alpha$  is an  $st$ -equivalence class of compositions, we have

$$\phi_{st}(F_L) = u_\alpha \quad \text{for each } L \in \alpha.$$

(b) In this case, the  $\mathbb{Q}$ -linear map

$$\mathcal{A}_{st} \rightarrow A, \quad [\pi]_{st} \mapsto u_\alpha,$$

where  $\alpha$  is the  $st$ -equivalence class of the composition  $\text{Comp } \pi$ , is a  $\mathbb{Q}$ -algebra isomorphism  $\mathcal{A}_{st} \rightarrow A$ .

As we said, Theorem 4.14 is precisely [GesZhu17, Theorem 4.3]. For the sake of completeness, we shall give proofs of Proposition 4.13 and Theorem 4.14 (independent of [GesZhu17]) in Subsection 5.4.

## 4.5. The shuffle algebra of $\text{Epk}$

Theorem 2.56 yields that the permutation statistic  $\text{Epk}$  is shuffle-compatible. Hence, the shuffle algebra  $\mathcal{A}_{\text{Epk}}$  is well-defined. We have little to say about it:

**Theorem 4.15.** (a) The shuffle algebra  $\mathcal{A}_{\text{Epk}}$  is a graded quotient algebra of  $\text{QSym}$ .

(b) Define the Fibonacci sequence  $(f_0, f_1, f_2, \dots)$  as in Proposition 2.3. Let  $n$  be a positive integer. The  $n$ -th graded component of  $\mathcal{A}_{\text{Epk}}$  has dimension  $f_{n+2} - 1$ .



*Proof of Theorem 4.15 (sketched).* **(a)** Proposition 4.13 **(b)** (applied to  $\text{st} = \text{Epk}$ ) yields a surjective  $\mathbb{Q}$ -algebra homomorphism  $p_{\text{Epk}} : \text{QSym} \rightarrow \mathcal{A}_{\text{Epk}}$ . It is easy to see that this  $p_{\text{Epk}}$  is furthermore graded (i.e., degree-preserving). Thus,  $\mathcal{A}_{\text{Epk}}$  is isomorphic to a quotient of  $\text{QSym}$  as a graded algebra. This proves Theorem 4.15 **(a)**.

**(b)** The  $n$ -th graded component of  $\mathcal{A}_{\text{Epk}}$  has a basis indexed by  $\text{Epk}$ -equivalence classes of compositions of  $n$ . These latter classes are in bijection with  $\text{Epk}$ -equivalence classes of  $n$ -permutations. In turn, the latter classes are in bijection with the images of  $n$ -permutations under the map  $\text{Epk}$ . Finally, the latter images are the elements of  $\mathbf{L}_n$  (according to Proposition 2.5). Hence, the number of these images is  $|\mathbf{L}_n| = f_{n+2} - 1$  (by Proposition 2.3). Combining all of the preceding sentences, we conclude that the dimension of the  $n$ -th graded component of  $\mathcal{A}_{\text{Epk}}$  is  $f_{n+2} - 1$ . This proves Theorem 4.15 **(b)**.  $\square$

We can describe  $\mathcal{A}_{\text{Epk}}$  using the notations of Section 2:

**Definition 4.16.** Let  $\Pi_{\mathcal{Z}}$  be the  $\mathbb{Q}$ -vector subspace of  $\text{Pow } \mathcal{N}$  spanned by the family  $(K_{n,\Lambda}^{\mathcal{Z}})_{n \in \mathbb{N}; \Lambda \in \mathbf{L}_n}$ . Then,  $\Pi_{\mathcal{Z}}$  is also the  $\mathbb{Q}$ -vector subspace of  $\text{Pow } \mathcal{N}$  spanned by the family  $(K_{n,\text{Epk } \pi}^{\mathcal{Z}})_{n \in \mathbb{N}; \pi \text{ is an } n\text{-permutation}}$  (by Proposition 2.5). In other words,  $\Pi_{\mathcal{Z}}$  is also the  $\mathbb{Q}$ -vector subspace of  $\text{Pow } \mathcal{N}$  spanned by the family  $(\Gamma_{\mathcal{Z}}(\pi))_{n \in \mathbb{N}; \pi \text{ is an } n\text{-permutation}}$  (because of (28)). Hence, Corollary 2.30 shows that  $\Pi_{\mathcal{Z}}$  is closed under multiplication. Since furthermore  $\Gamma_{\mathcal{Z}}(( )) = 1$  (for the 0-permutation  $( )$ ), we can thus conclude that  $\Pi_{\mathcal{Z}}$  is a  $\mathbb{Q}$ -subalgebra of  $\text{Pow } \mathcal{N}$ .

**Theorem 4.17.** The  $\mathbb{Q}$ -linear map

$$\mathcal{A}_{\text{Epk}} \rightarrow \Pi_{\mathcal{Z}}, \quad [\pi]_{\text{Epk}} \mapsto K_{n,\text{Epk } \pi}^{\mathcal{Z}}$$

is a  $\mathbb{Q}$ -algebra isomorphism.

In the process of proving Theorem 4.17, we will also prove Theorem 2.56 again.

*Proof of Theorem 4.17 (sketched).* For each positive integer  $n$  and each  $n$ -permutation  $\pi$ , we have

$$\text{Epk } \pi = (\text{Des } \pi \cup \{n\}) \setminus (\text{Des } \pi + 1)$$

(by Proposition 1.9). Thus,  $\text{Epk } \pi$  is uniquely determined by  $\text{Des } \pi$  and  $n$ . (Of course, this holds for  $n = 0$  as well, because in this case only one  $\pi$  exists.) Hence,  $\text{Epk}$  is a descent statistic. Thus, for every composition  $L$ , a set  $\text{Epk } L$  is defined (according to Definition 4.6); explicitly,  $\text{Epk } L = \text{Epk } \pi$  whenever  $\pi$  is a permutation satisfying  $\text{Comp } \pi = L$ .

Recall that  $(F_L)_{L \text{ is a composition}}$  is a basis of the  $\mathbb{Q}$ -vector space  $\text{QSym}$ . Let  $\phi_{\text{Epk}} : \text{QSym} \rightarrow \Pi_{\mathcal{Z}}$  be the  $\mathbb{Q}$ -linear map that sends each  $F_L$  (for each  $n \in \mathbb{N}$  and

each composition  $L$  of  $n$ ) to  $K_{n, \text{Epk} L}^{\mathcal{Z}} \in \Pi_{\mathcal{Z}}$ . This  $\mathbb{Q}$ -linear map  $\phi_{\text{Epk}}$  respects multiplication<sup>50</sup> and sends  $1 \in \text{QSym}$  to  $1 \in \Pi_{\mathcal{Z}}$ <sup>51</sup>. Thus,  $\phi_{\text{Epk}}$  is a  $\mathbb{Q}$ -algebra homomorphism.

The family  $\left(K_{n, \Lambda}^{\mathcal{Z}}\right)_{n \in \mathbb{N}; \Lambda \in \mathbf{L}_n}$  spans  $\Pi_{\mathcal{Z}}$  (by the definition of  $\Pi_{\mathcal{Z}}$ ) and is  $\mathbb{Q}$ -linearly independent (by Corollary 2.55). Thus, it is a basis of  $\Pi_{\mathcal{Z}}$ .

For each positive integer  $n$ , there is a canonical bijection between the  $\text{Epk}$ -equivalence classes of  $n$ -permutations and the  $\Lambda \in \mathbf{L}_n$  (indeed, the bijection sends any equivalence class  $[\pi]_{\text{Epk}}$  to  $\text{Epk} \pi$ )<sup>52</sup>. Hence, Theorem 4.14 (a) (applied to  $A = \Pi_{\mathcal{Z}}$ ,  $\text{st} = \text{Epk}$  and  $u_{\alpha} = K_{n, \Lambda}^{\mathcal{Z}}$ , where  $\alpha$  is an  $\text{Epk}$ -equivalence class of  $n$ -permutations and where  $\Lambda$  is the corresponding element of  $\mathbf{L}_n$ ) shows that the descent statistic  $\text{Epk}$  is shuffle-compatible. This proves Theorem 2.56 again. Theorem 4.14 (b) then yields that the  $\mathbb{Q}$ -linear map

$$\mathcal{A}_{\text{Epk}} \rightarrow \Pi_{\mathcal{Z}}, \quad [\pi]_{\text{Epk}} \mapsto K_{n, \text{Epk} \pi}^{\mathcal{Z}}$$

is a  $\mathbb{Q}$ -algebra isomorphism from  $\mathcal{A}_{\text{Epk}}$  to  $\Pi_{\mathcal{Z}}$ . Hence, Theorem 4.17 is proven.  $\square$

<sup>50</sup>*Proof.* Let  $n \in \mathbb{N}$  and  $m \in \mathbb{N}$ . Let  $J$  be a composition of  $n$ , and let  $K$  be a composition of  $m$ . Fix an  $n$ -permutation  $\pi$  satisfying  $\text{Comp} \pi = J$ , and fix an  $m$ -permutation  $\sigma$  satisfying  $\text{Comp} \sigma = K$  such that  $\pi$  and  $\sigma$  are disjoint. Now,

$$\begin{aligned} & \underbrace{\phi_{\text{Epk}}(F_J)}_{=K_{n, \text{Epk} J}^{\mathcal{Z}}=K_{n, \text{Epk} \pi}^{\mathcal{Z}} \text{ (since } \text{Epk} J = \text{Epk} \pi)} \cdot \underbrace{\phi_{\text{Epk}}(F_K)}_{=K_{m, \text{Epk} K}^{\mathcal{Z}}=K_{m, \text{Epk} \sigma}^{\mathcal{Z}} \text{ (since } \text{Epk} K = \text{Epk} \sigma)} \\ &= K_{n, \text{Epk} \pi}^{\mathcal{Z}} \cdot K_{m, \text{Epk} \sigma}^{\mathcal{Z}} = \sum_{\tau \in \mathcal{S}(\pi, \sigma)} K_{n+m, \text{Epk} \tau}^{\mathcal{Z}} \quad \text{(by Corollary 2.49)}. \end{aligned}$$

Comparing this with

$$\begin{aligned} \phi_{\text{Epk}} \left( \underbrace{F_J F_K}_{= \sum_{\tau \in \mathcal{S}(\pi, \sigma)} F_{\text{Comp} \tau} \text{ (by Proposition 4.11)}} \right) &= \phi_{\text{Epk}} \left( \sum_{\tau \in \mathcal{S}(\pi, \sigma)} F_{\text{Comp} \tau} \right) = \sum_{\tau \in \mathcal{S}(\pi, \sigma)} \underbrace{\phi_{\text{Epk}}(F_{\text{Comp} \tau})}_{=K_{n+m, \text{Epk}(\text{Comp} \tau)}^{\mathcal{Z}}=K_{n+m, \text{Epk} \tau}^{\mathcal{Z}} \text{ (since } \text{Epk}(\text{Comp} \tau) = \text{Epk} \tau)} \\ &= \sum_{\tau \in \mathcal{S}(\pi, \sigma)} K_{n+m, \text{Epk} \tau}^{\mathcal{Z}} \end{aligned}$$

we obtain  $\phi_{\text{Epk}}(F_J) \cdot \phi_{\text{Epk}}(F_K) = \phi_{\text{Epk}}(F_J F_K)$ . Since the map  $\phi_{\text{Epk}}$  is  $\mathbb{Q}$ -linear, this yields that  $\phi_{\text{Epk}}$  respects multiplication (since  $(F_L)_L$  is a composition is a basis of the  $\mathbb{Q}$ -vector space  $\text{QSym}$ ).

<sup>51</sup>This is easy, since  $1 = F_{()}.$

<sup>52</sup>Proposition 2.5 shows that this is indeed a bijection.

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## 5. The kernel of the map $\text{QSym} \rightarrow \mathcal{A}_{\text{Epk}}$

### 5.1. The kernel of a descent statistic

Now, we shall focus on a feature of shuffle-compatible descent statistics that seems to have been overlooked so far: their kernels.

**Definition 5.1.** Let  $\text{st}$  be a descent statistic. Then,  $\mathcal{K}_{\text{st}}$  shall mean the  $\mathbb{Q}$ -vector subspace of  $\text{QSym}$  spanned by all elements of the form  $F_J - F_K$ , where  $J$  and  $K$  are two  $\text{st}$ -equivalent compositions. (See Definition 4.7 (a) for the definition of “ $\text{st}$ -equivalent compositions”.) We shall refer to  $\mathcal{K}_{\text{st}}$  as the *kernel* of  $\text{st}$ .

The following basic linear-algebraic lemma will be useful:

**Lemma 5.2.** Let  $\text{st}$  be a descent statistic. Let  $A$  be a  $\mathbb{Q}$ -vector space with basis  $(u_\alpha)$  indexed by  $\text{st}$ -equivalence classes  $\alpha$  of compositions. Let  $\phi_{\text{st}} : \text{QSym} \rightarrow A$  be a  $\mathbb{Q}$ -linear map with the property that whenever  $\alpha$  is an  $\text{st}$ -equivalence class of compositions, we have

$$\phi_{\text{st}}(F_L) = u_\alpha \quad \text{for each } L \in \alpha. \quad (55)$$

Then,  $\text{Ker}(\phi_{\text{st}}) = \mathcal{K}_{\text{st}}$ .

*Proof of Lemma 5.2.* Let us first show that  $\text{Ker}(\phi_{\text{st}}) \subseteq \mathcal{K}_{\text{st}}$ .

Indeed, let  $x \in \text{Ker}(\phi_{\text{st}})$  be arbitrary. Write  $x \in \text{QSym}$  in the form  $x = \sum_L x_L F_L$ , where the sum ranges over all compositions  $L$ , and where the  $x_L$  are elements of  $\mathbb{Q}$  (all but finitely many of which are zero). (This can be done, since  $(F_L)_L$  is a composition is a basis of the  $\mathbb{Q}$ -vector space  $\text{QSym}$ .) Now,  $x \in \text{Ker}(\phi_{\text{st}})$ , so that  $\phi_{\text{st}}(x) = 0$ . Thus,

$$\begin{aligned} 0 &= \phi_{\text{st}}(x) = \sum_L x_L \phi_{\text{st}}(F_L) && \left( \text{since } x = \sum_L x_L F_L \right) \\ &= \sum_\alpha \sum_{L \in \alpha} x_L \underbrace{\phi_{\text{st}}(F_L)}_{\substack{= u_\alpha \\ \text{(by (55))}}} && \left( \begin{array}{c} \text{where the first sum is over} \\ \text{all st-equivalence classes } \alpha \text{ of compositions} \end{array} \right) \\ &= \sum_\alpha \sum_{L \in \alpha} x_L u_\alpha = \sum_\alpha \left( \sum_{L \in \alpha} x_L \right) u_\alpha. \end{aligned}$$

Since the family  $(u_\alpha)$  is linearly independent (because it is a basis of  $A$ ), we thus conclude that

$$\sum_{L \in \alpha} x_L = 0 \quad (56)$$

for each  $\text{st}$ -equivalence class  $\alpha$  of compositions.

Now, for each st-equivalence class  $\alpha$  of compositions, we fix an element  $L_\alpha$  of  $\alpha$ . Then, for each st-equivalence class  $\alpha$  of compositions and each composition  $L \in \alpha$ , we have

$$F_L - F_{L_\alpha} \in \mathcal{K}_{\text{st}} \quad (57)$$

(since the compositions  $L$  and  $L_\alpha$  are st-equivalent<sup>53</sup>).

Now,

$$\begin{aligned} x &= \sum_L x_L F_L \\ &= \sum_\alpha \sum_{L \in \alpha} x_L \underbrace{F_L}_{=(F_L - F_{L_\alpha}) + F_{L_\alpha}} \quad \left( \begin{array}{l} \text{where the first sum is over} \\ \text{all st-equivalence classes } \alpha \text{ of compositions} \end{array} \right) \\ &= \sum_\alpha \sum_{L \in \alpha} x_L ((F_L - F_{L_\alpha}) + F_{L_\alpha}) \\ &= \sum_\alpha \sum_{L \in \alpha} x_L \underbrace{(F_L - F_{L_\alpha})}_{\substack{\in \mathcal{K}_{\text{st}} \\ \text{(by (57))}}} + \sum_\alpha \sum_{L \in \alpha} x_L \underbrace{F_{L_\alpha}}_{=0 \text{ (by (56))}} \\ &\in \underbrace{\sum_\alpha \sum_{L \in \alpha} x_L \mathcal{K}_{\text{st}}}_{\subseteq \mathcal{K}_{\text{st}}} + \underbrace{\sum_\alpha 0 F_{L_\alpha}}_{=0} \subseteq \mathcal{K}_{\text{st}}. \end{aligned}$$

Now, forget that we fixed  $x$ . We thus have proven that  $x \in \mathcal{K}_{\text{st}}$  for each  $x \in \text{Ker}(\phi_{\text{st}})$ . In other words,  $\text{Ker}(\phi_{\text{st}}) \subseteq \mathcal{K}_{\text{st}}$ .

Conversely, it is easy to see that  $\mathcal{K}_{\text{st}} \subseteq \text{Ker}(\phi_{\text{st}})$ <sup>54</sup>. Combining this with  $\text{Ker}(\phi_{\text{st}}) \subseteq \mathcal{K}_{\text{st}}$ , we obtain  $\mathcal{K}_{\text{st}} = \text{Ker}(\phi_{\text{st}})$ . This proves Lemma 5.2.  $\square$

Theorem 4.14 easily yields the following fact:

**Proposition 5.3.** Let st be a descent statistic. Then, st is shuffle-compatible if and only if  $\mathcal{K}_{\text{st}}$  is an ideal of  $\text{QSym}$ . Furthermore, in this case,  $\mathcal{A}_{\text{st}} \cong \text{QSym} / \mathcal{K}_{\text{st}}$  as  $\mathbb{Q}$ -algebras.

*Proof of Proposition 5.3.*  $\implies$ : Assume that st is shuffle-compatible. Thus, Theorem 4.14 (a) shows that there exist a  $\mathbb{Q}$ -algebra  $A$  with basis  $(u_\alpha)$  indexed

<sup>53</sup>since both compositions  $L$  and  $L_\alpha$  lie in the same st-equivalence class  $\alpha$

<sup>54</sup>*Proof.* Recall that  $\mathcal{K}_{\text{st}}$  is the  $\mathbb{Q}$ -vector subspace of  $\text{QSym}$  spanned by all elements of the form  $F_J - F_K$ , where  $J$  and  $K$  are two st-equivalent compositions. Hence, in order to prove that  $\mathcal{K}_{\text{st}} \subseteq \text{Ker}(\phi_{\text{st}})$ , it suffices to show that  $F_J - F_K \in \text{Ker}(\phi_{\text{st}})$ , whenever  $J$  and  $K$  are two st-equivalent compositions. So let  $J$  and  $K$  be two st-equivalent compositions. We must show that  $F_J - F_K \in \text{Ker}(\phi_{\text{st}})$ .

The two compositions  $J$  and  $K$  are st-equivalent. Hence, they lie in one and the same st-equivalence class. Let  $\alpha$  be this st-equivalence class. Then,  $J \in \alpha$  and therefore  $\phi_{\text{st}}(F_J) = u_\alpha$  (by (55), applied to  $L = J$ ). Similarly,  $\phi_{\text{st}}(F_K) = u_\alpha$ . Now,  $\phi_{\text{st}}(F_J - F_K) = \underbrace{\phi_{\text{st}}(F_J)}_{=u_\alpha} - \underbrace{\phi_{\text{st}}(F_K)}_{=u_\alpha} = u_\alpha - u_\alpha = 0$ . In other words,  $F_J - F_K \in \text{Ker}(\phi_{\text{st}})$ . This completes our proof.

by st-equivalence classes  $\alpha$  of compositions, and a  $\mathbb{Q}$ -algebra homomorphism  $\phi_{\text{st}} : \text{QSym} \rightarrow A$  with the property that whenever  $\alpha$  is an st-equivalence class of compositions, we have

$$\phi_{\text{st}}(F_L) = u_\alpha \quad \text{for each } L \in \alpha. \quad (58)$$

Consider this  $A$  and this  $\phi_{\text{st}}$ .

Lemma 5.2 yields that  $\mathcal{K}_{\text{st}} = \text{Ker}(\phi_{\text{st}})$ . But the map  $\phi_{\text{st}}$  is a  $\mathbb{Q}$ -algebra homomorphism. Thus, its kernel  $\text{Ker}(\phi_{\text{st}})$  is an ideal of  $\text{QSym}$ . In other words,  $\mathcal{K}_{\text{st}}$  is an ideal of  $\text{QSym}$  (since  $\mathcal{K}_{\text{st}} = \text{Ker}(\phi_{\text{st}})$ ).

It remains to show that  $\mathcal{A}_{\text{st}} \cong \text{QSym} / \mathcal{K}_{\text{st}}$  as  $\mathbb{Q}$ -algebras. This is easy: Each element of the basis  $(u_\alpha)$  of the  $\mathbb{Q}$ -vector space  $A$  is contained in the image of  $\phi_{\text{st}}$  (because of (58)). Therefore, the homomorphism  $\phi_{\text{st}}$  is surjective. Thus,  $\phi_{\text{st}}(\text{QSym}) = A$ . Hence,  $A = \phi_{\text{st}}(\text{QSym}) \cong \text{QSym} / \text{Ker}(\phi_{\text{st}})$  (by the homomorphism theorem). But Theorem 4.14 (b) shows that  $\mathcal{A}_{\text{st}} \cong A$ . Thus,  $\mathcal{A}_{\text{st}} \cong A \cong \text{QSym} / \underbrace{\text{Ker}(\phi_{\text{st}})}_{=\mathcal{K}_{\text{st}}} = \text{QSym} / \mathcal{K}_{\text{st}}$ . This finishes the proof of the  $\implies$

direction of Proposition 5.3.

$\Leftarrow$ : Assume that  $\mathcal{K}_{\text{st}}$  is an ideal of  $\text{QSym}$ . We must prove that st is shuffle-compatible.

We shall not use this direction of Proposition 5.3, so let us merely sketch the proof. Let  $A$  be the  $\mathbb{Q}$ -algebra  $\text{QSym} / \mathcal{K}_{\text{st}}$ . Let  $\phi_{\text{st}}$  be the canonical projection  $\text{QSym} \rightarrow A$ ; this is clearly a  $\mathbb{Q}$ -algebra homomorphism.

For each st-equivalence class  $\alpha$  of compositions, we define an element  $u_\alpha$  of  $A$  by requiring that

$$u_\alpha = \phi_{\text{st}}(F_L) \quad \text{whenever } L \in \alpha.$$

This is easily seen to be well-defined, because the image  $\phi_{\text{st}}(F_L)$  depends only on  $\alpha$  but not on  $L$  (indeed, if  $J$  and  $K$  are two elements of  $\alpha$ , then  $J$  and  $K$  are st-equivalent, whence  $F_J - F_K \in \mathcal{K}_{\text{st}}$ , whence  $F_J \equiv F_K \pmod{\mathcal{K}_{\text{st}}}$  and therefore  $\phi_{\text{st}}(F_J) = \phi_{\text{st}}(F_K)$ ).

It is not hard to see that the family  $(u_\alpha)$  (where  $\alpha$  ranges over all st-equivalence classes of compositions) is a basis of the  $\mathbb{Q}$ -algebra  $A$ . Hence, Theorem 4.14 (a) yields that st is shuffle-compatible. This proves the  $\Leftarrow$  direction of Proposition 5.3.  $\square$

**Corollary 5.4.** The kernel  $\mathcal{K}_{\text{Epk}}$  of the descent statistic  $\text{Epk}$  is an ideal of  $\text{QSym}$ .

*Proof of Corollary 5.4.* This follows from Proposition 5.3 (applied to  $\text{st} = \text{Epk}$ ), because of Theorem 2.56.  $\square$

We can study the kernel of any descent statistic; in particular, the case of shuffle-compatible descent statistics appears interesting. Since  $\text{QSym}$  is isomorphic to a polynomial ring (as an algebra), it has many ideals, which are rather hopeless to classify or tame; but the ones obtained as kernels of shuffle-compatible descent statistics might be worth discussing.

## 5.2. An F-generating set of $\mathcal{K}_{\text{Epk}}$

Let us now focus on  $\mathcal{K}_{\text{Epk}}$ , the kernel of  $\text{Epk}$ .

**Proposition 5.5.** If  $J = (j_1, j_2, \dots, j_m)$  and  $K$  are two compositions, then we shall write  $J \rightarrow K$  if there exists an  $\ell \in \{2, 3, \dots, m\}$  such that  $j_\ell > 2$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$ . (In other words, we write  $J \rightarrow K$  if  $K$  can be obtained from  $J$  by “splitting” some entry  $j_\ell > 2$  into two consecutive entries<sup>55</sup> 1 and  $j_\ell - 1$ , provided that this entry was not the first entry – i.e., we had  $\ell > 1$  – and that this entry was greater than 2.)

The ideal  $\mathcal{K}_{\text{Epk}}$  of  $\text{QSym}$  is spanned (as a  $\mathbb{Q}$ -vector space) by all differences of the form  $F_J - F_K$ , where  $J$  and  $K$  are two compositions satisfying  $J \rightarrow K$ .

**Example 5.6.** We have  $(2, 1, 4, 4) \rightarrow (2, 1, 1, 3, 4)$ , since the composition  $(2, 1, 1, 3, 4)$  is obtained from  $(2, 1, 4, 4)$  by splitting the third entry (which is  $4 > 2$ ) into two consecutive entries 1 and 3.

Similarly,  $(2, 1, 4, 4) \rightarrow (2, 1, 4, 1, 3)$ .

But we do not have  $(3, 1) \rightarrow (1, 2, 1)$ , because splitting the first entry of the composition is not allowed in the definition of the relation  $\rightarrow$ . Also, we do not have  $(1, 2, 1) \rightarrow (1, 1, 1, 1)$ , because the entry we are splitting must be  $> 2$ .

Two compositions  $J$  and  $K$  satisfying  $J \rightarrow K$  must necessarily satisfy  $|J| = |K|$ .

Here are all relations  $\rightarrow$  between compositions of size 4:

$$(1, 3) \rightarrow (1, 1, 2).$$

Here are all relations  $\rightarrow$  between compositions of size 5:

$$\begin{aligned} (1, 4) &\rightarrow (1, 1, 3), \\ (1, 3, 1) &\rightarrow (1, 1, 2, 1), \\ (1, 1, 3) &\rightarrow (1, 1, 1, 2), \\ (2, 3) &\rightarrow (2, 1, 2). \end{aligned}$$

There are no relations  $\rightarrow$  between compositions of size  $\leq 3$ .

*Proof of Proposition 5.5.* We begin by proving some simple claims.

*Claim 1:* Let  $n \in \mathbb{N}$ . Let  $J$  and  $K$  be two compositions of size  $n$ . Then,  $J \rightarrow K$  if and only if there exists some  $k \in [n - 1]$  such that

$$\begin{aligned} \text{Des } K &= \text{Des } J \cup \{k\}, & k &\notin \text{Des } J, \\ k - 1 &\in \text{Des } J & \text{and} & & k + 1 &\notin \text{Des } J \cup \{k\}. \end{aligned}$$

<sup>55</sup>The word “consecutive” here means “in consecutive positions of  $J$ ”, not “consecutive integers”. So two consecutive entries of  $J$  are two entries of the form  $j_p$  and  $j_{p+1}$  for some  $p \in \{1, 2, \dots, m - 1\}$ .

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[*Proof of Claim 1:* This is straightforward to check: “Splitting” an entry of a composition  $C$  into two consecutive entries (summing up to the original entry) is always tantamount to adding a new element to  $\text{Des } C$ . The rest is translating conditions.]

If  $n$  is a positive integer, and  $L$  is any composition of  $n$ , then

$$\text{Epk } L = (\text{Des } L \cup \{n\}) \setminus (\text{Des } L + 1). \quad (59)$$

(This is a consequence of Proposition 1.9, applied to any  $n$ -permutation  $\pi$  satisfying  $L = \text{Comp } \pi$ .)

*Claim 2:* Let  $J$  and  $K$  be two compositions satisfying  $J \rightarrow K$ . Then,  $\text{Epk } J = \text{Epk } K$ .

[*Proof of Claim 2:* Easy consequence of Claim 1 and (59).]

For any two integers  $a$  and  $b$ , we set  $[a, b] = \{a, a + 1, \dots, b\}$ . (This is an empty set if  $a > b$ .)

It is easy to see that every composition  $J$  of size  $n > 0$  satisfies

$$[\max(\text{Epk } J), n - 1] \subseteq \text{Des } J. \quad (60)$$

[*Proof of (60):* Let  $J$  be a composition of size  $n > 0$ . We shall show that  $[\max(\text{Epk } J), n - 1] \subseteq \text{Des } J$ .

Indeed, assume the contrary. Thus,  $[\max(\text{Epk } J), n - 1] \not\subseteq \text{Des } J$ . Hence, there exists some  $q \in [\max(\text{Epk } J), n - 1]$  satisfying  $q \notin \text{Des } J$ . Let  $r$  be the **largest** such  $q$ .

Thus,  $r \in [\max(\text{Epk } J), n - 1]$  but  $r \notin \text{Des } J$ . From  $r \in [\max(\text{Epk } J), n - 1] \subseteq [n - 1]$ , we obtain  $r + 1 \in [n]$ . Also, from  $r \notin \text{Des } J$ , we obtain  $r + 1 \notin \text{Des } J + 1$ .

From  $r \in [\max(\text{Epk } J), n - 1]$ , we obtain  $r \geq \max(\text{Epk } J)$ , so that  $r + 1 > r \geq \max(\text{Epk } J)$  and therefore  $r + 1 \notin \text{Epk } J$  (since a number that is higher than  $\max(\text{Epk } J)$  cannot belong to  $\text{Epk } J$ ).

From (59), we obtain  $\text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1)$ .

If we had  $r + 1 \in \text{Des } J \cup \{n\}$ , then we would have  $r + 1 \in (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1)$  (since  $r + 1 \notin \text{Des } J + 1$ ). This would contradict  $r + 1 \notin \text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1)$ . Thus, we cannot have  $r + 1 \in \text{Des } J \cup \{n\}$ . Therefore,  $r + 1 \notin \text{Des } J \cup \{n\}$ .

Hence,  $r + 1 \neq n$  (since  $r + 1 \notin \text{Des } J \cup \{n\}$  but  $n \in \{n\} \subseteq \text{Des } J \cup \{n\}$ ). Combined with  $r + 1 \in [n]$ , this yields  $r + 1 \in [n] \setminus \{n\} = [n - 1]$ . Combined with  $r + 1 > \max(\text{Epk } J)$ , this yields  $r + 1 \in [\max(\text{Epk } J), n - 1]$ . Also,  $r + 1 \notin \text{Des } J$  (since  $r + 1 \notin \text{Des } J \cup \{n\}$ ).

Thus,  $r + 1$  is a  $q \in [\max(\text{Epk } J), n - 1]$  satisfying  $q \notin \text{Des } J$ . This contradicts the fact that  $r$  is the **largest** such  $q$  (since  $r + 1$  is clearly larger than  $r$ ). This contradiction proves that our assumption was wrong; thus, (60) is proven.]

For each  $n \in \mathbb{N}$  and each subset  $S$  of  $[n - 1]$ , we define a subset  $S^\circ$  of  $[n - 1]$  as follows:

$$S_n^\circ = \{s \in S \mid s - 1 \notin S \text{ or } [s, n - 1] \subseteq S\}.$$

Also, for each  $n \in \mathbb{N}$  and each nonempty subset  $T$  of  $[n]$ , we define a subset  $\rho_n(T)$  of  $[n-1]$  as follows:

$$\rho_n(T) = \begin{cases} T \setminus \{n\}, & \text{if } n \in T; \\ T \cup [\max T, n-1], & \text{if } n \notin T. \end{cases}$$

*Claim 3:* Let  $n \in \mathbb{N}$ . Let  $J$  be a composition of size  $n$ . Then,  $(\text{Des } J)_n^\circ = \rho_n(\text{Epk } J)$ .

[*Proof of Claim 3:* Let  $g \in (\text{Des } J)_n^\circ$ . We shall show that  $g \in \rho_n(\text{Epk } J)$ .

We have  $g \in (\text{Des } J)_n^\circ \subseteq \text{Des } J$  (since  $S_n^\circ \subseteq S$  for each subset  $S$  of  $[n-1]$ ) and therefore  $\text{Des } J \neq \emptyset$ . Hence,  $J$  is not the empty composition. In other words,  $n > 0$ .

From (59), we obtain  $\text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1)$ . Thus, the set  $\text{Epk } J$  is disjoint from  $\text{Des } J + 1$ . Furthermore, the set  $\text{Epk } J$  is nonempty<sup>56</sup>.

We have  $g \in (\text{Des } J)_n^\circ$ . Thus,  $g$  is an element of  $\text{Des } J$  satisfying  $g-1 \notin \text{Des } J$  or  $[g, n-1] \subseteq \text{Des } J$  (by the definition of  $(\text{Des } J)_n^\circ$ ). We are thus in one of the following two cases:

*Case 1:* We have  $g-1 \notin \text{Des } J$ .

*Case 2:* We have  $[g, n-1] \subseteq \text{Des } J$ .

Let us first consider Case 1. In this case, we have  $g-1 \notin \text{Des } J$ . In other words,  $g \notin \text{Des } J + 1$ . Combined with  $g \in \text{Des } J \subseteq \text{Des } J \cup \{n\}$ , this yields  $g \in (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1) = \text{Epk } J$ . Moreover,  $g \neq n$  (since  $g \in \text{Des } J \subseteq [n-1]$ ) and thus  $g \in (\text{Epk } J) \setminus \{n\}$  (since  $g \in \text{Epk } J$ ). But each nonempty subset  $T$  of  $[n]$  satisfies  $T \setminus \{n\} \subseteq \rho_n(T)$  (by the definition of  $\rho_n(T)$ ). Applying this to  $T = \text{Epk } J$ , we obtain  $(\text{Epk } J) \setminus \{n\} \subseteq \rho_n(\text{Epk } J)$ . Hence,  $g \in (\text{Epk } J) \setminus \{n\} \subseteq \rho_n(\text{Epk } J)$ . Thus,  $g \in \rho_n(\text{Epk } J)$  is proven in Case 1.

Let us now consider Case 2. In this case, we have  $[g, n-1] \subseteq \text{Des } J$ . Hence, each of the elements  $g, g+1, \dots, n-1$  belongs to  $\text{Des } J$ . In other words, each of the elements  $g+1, g+2, \dots, n$  belongs to  $\text{Des } J + 1$ . Hence, none of the elements  $g+1, g+2, \dots, n$  belongs to  $\text{Epk } J$  (since the set  $\text{Epk } J$  is disjoint from  $\text{Des } J + 1$ ). Thus,  $\max(\text{Epk } J) \leq g$ . Therefore,  $g \in [\max(\text{Epk } J), n-1]$  (since  $g \in \text{Des } J \subseteq [n-1]$ ).

Also,  $n \notin \text{Epk } J$ <sup>57</sup>. Hence, the definition of  $\rho_n(\text{Epk } J)$  yields  $\rho_n(\text{Epk } J) = \text{Epk } J \cup [\max(\text{Epk } J), n-1]$ . Now,

$$g \in [\max(\text{Epk } J), n-1] \subseteq \text{Epk } J \cup [\max(\text{Epk } J), n-1] = \rho_n(\text{Epk } J).$$

Hence,  $g \in \rho_n(\text{Epk } J)$  is proven in Case 2.

<sup>56</sup>Indeed, it contains at least the smallest element of the set  $\text{Des } J \cup \{n\}$  (since  $\text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1)$ ).

<sup>57</sup>*Proof.* Assume the contrary. Thus,  $n \in \text{Epk } J$ . But none of the elements  $g+1, g+2, \dots, n$  belongs to  $\text{Epk } J$ . Hence,  $n$  is not among the elements  $g+1, g+2, \dots, n$ . Therefore,  $g \geq n$ , so that  $g = n$ . This contradicts  $g \in \text{Des } J \subseteq [n-1]$ . This contradiction shows that our assumption was wrong, qed.



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Thus,  $g \in \rho_n(\text{Epk } J)$  is proven in both Cases 1 and 2. This shows that  $g \in \rho_n(\text{Epk } J)$  always holds.

Forget that we fixed  $g$ . We thus have proven that  $g \in \rho_n(\text{Epk } J)$  for each  $g \in (\text{Des } J)_n^\circ$ . In other words,  $(\text{Des } J)_n^\circ \subseteq \rho_n(\text{Epk } J)$ .

Now, let  $h \in \rho_n(\text{Epk } J)$  be arbitrary. We shall prove that  $h \in (\text{Des } J)_n^\circ$ .

We are in one of the following two cases:

*Case 1:* We have  $n \in \text{Epk } J$ .

*Case 2:* We have  $n \notin \text{Epk } J$ .

Let us first consider Case 1. In this case, we have  $n \in \text{Epk } J$ , and thus  $\rho_n(\text{Epk } J) = \text{Epk } J \setminus \{n\}$  (by the definition of  $\rho_n(\text{Epk } J)$ ). Hence,

$$h \in \rho_n(\text{Epk } J) = \text{Epk } J \setminus \{n\} \subseteq \text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1).$$

In other words,  $h \in \text{Des } J \cup \{n\}$  and  $h \notin \text{Des } J + 1$ . Since  $h \in \text{Des } J \cup \{n\}$  and  $h \neq n$  (because  $h \in \rho_n(\text{Epk } J) \subseteq [n-1]$ ), we obtain  $h \in (\text{Des } J \cup \{n\}) \setminus \{n\} \subseteq \text{Des } J$ . From  $h \notin \text{Des } J + 1$ , we obtain  $h-1 \notin \text{Des } J$ . Thus,  $h$  is an element of  $\text{Des } J$  satisfying  $h-1 \notin \text{Des } J$  or  $[h, n-1] \subseteq \text{Des } J$  (in fact,  $h-1 \notin \text{Des } J$  holds). Thus,  $h \in (\text{Des } J)_n^\circ$  (by the definition of  $(\text{Des } J)_n^\circ$ ). Thus,  $h \in (\text{Des } J)_n^\circ$  is proven in Case 1.

Let us now consider Case 2. In this case, we have  $n \notin \text{Epk } J$ . Hence, the definition of  $\rho_n(\text{Epk } J)$  yields  $\rho_n(\text{Epk } J) = (\text{Epk } J) \cup [\max(\text{Epk } J), n-1]$ . Thus,  $h \in \rho_n(\text{Epk } J) = (\text{Epk } J) \cup [\max(\text{Epk } J), n-1]$ .

If  $h \in \text{Epk } J$ , then we can prove  $h \in (\text{Des } J)_n^\circ$  just as in Case 1. Hence, let us WLOG assume that we don't have  $h \in \text{Epk } J$ . Thus,  $h \notin \text{Epk } J$ . Combined with  $h \in (\text{Epk } J) \cup [\max(\text{Epk } J), n-1]$ , this yields

$$\begin{aligned} h \in ((\text{Epk } J) \cup [\max(\text{Epk } J), n-1]) \setminus (\text{Epk } J) &= [\max(\text{Epk } J), n-1] \setminus (\text{Epk } J) \\ &\subseteq [\max(\text{Epk } J), n-1] \subseteq \text{Des } J \quad (\text{by (60)}). \end{aligned}$$

Moreover, from  $h \in [\max(\text{Epk } J), n-1]$ , we obtain  $h \geq \max(\text{Epk } J)$ , so that

$$[h, n-1] \subseteq [\max(\text{Epk } J), n-1] \subseteq \text{Des } J \quad (\text{by (60)}).$$

Hence,  $h$  is an element of  $\text{Des } J$  satisfying  $h-1 \notin \text{Des } J$  or  $[h, n-1] \subseteq \text{Des } J$  (namely,  $[h, n-1] \subseteq \text{Des } J$ ). In other words,  $h \in (\text{Des } J)_n^\circ$  (by the definition of  $(\text{Des } J)_n^\circ$ ). Thus,  $h \in (\text{Des } J)_n^\circ$  is proven in Case 2.

We have now proven  $h \in (\text{Des } J)_n^\circ$  in both Cases 1 and 2. Hence,  $h \in (\text{Des } J)_n^\circ$  always holds.

Forget that we fixed  $h$ . We thus have shown that  $h \in (\text{Des } J)_n^\circ$  for each  $h \in \rho_n(\text{Epk } J)$ . In other words,  $\rho_n(\text{Epk } J) \subseteq (\text{Des } J)_n^\circ$ . Combining this with  $(\text{Des } J)_n^\circ \subseteq \rho_n(\text{Epk } J)$ , we obtain  $(\text{Des } J)_n^\circ = \rho_n(\text{Epk } J)$ . This proves Claim 3.]

*Claim 4:* Let  $n \in \mathbb{N}$ . Let  $J$  and  $K$  be two compositions of size  $n$  satisfying  $\text{Epk } J = \text{Epk } K$ . Then,  $(\text{Des } J)_n^\circ = (\text{Des } K)_n^\circ$ .

[*Proof of Claim 4:* Claim 3 yields  $(\text{Des } J)_n^\circ = \rho_n(\text{Epk } J)$  and similarly  $(\text{Des } K)_n^\circ = \rho_n(\text{Epk } K)$ . Hence,

$$(\text{Des } J)_n^\circ = \rho_n \left( \underbrace{\text{Epk } J}_{=\text{Epk } K} \right) = \rho_n(\text{Epk } K) = (\text{Des } K)_n^\circ.$$

This proves Claim 4.]

We let  $\xrightarrow{*}$  be the transitive-and-reflexive closure of the relation  $\rightarrow$ . Thus, two compositions  $J$  and  $K$  satisfy  $J \xrightarrow{*} K$  if and only if there exists a sequence  $(L_0, L_1, \dots, L_\ell)$  of compositions satisfying  $L_0 = J$  and  $L_\ell = K$  and  $L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_\ell$ .

*Claim 5:* Let  $n \in \mathbb{N}$ . Let  $K$  be a composition of size  $n$ . Then,  
 $\text{Comp}((\text{Des } K)_n^\circ) \xrightarrow{*} K$ .

[*Proof of Claim 5:* We shall prove Claim 5 by strong induction on  $|(\text{Des } K) \setminus (\text{Des } K)_n^\circ|$ . Thus, we fix an  $n \in \mathbb{N}$  and a composition  $K$  of size  $n$ , and we assume (as the induction hypothesis) that each composition  $J$  of size  $n$  satisfying  $|(\text{Des } J) \setminus (\text{Des } J)_n^\circ| < |(\text{Des } K) \setminus (\text{Des } K)_n^\circ|$  satisfies  $\text{Comp}((\text{Des } J)_n^\circ) \xrightarrow{*} J$ . Our goal is to prove that  $\text{Comp}((\text{Des } K)_n^\circ) \xrightarrow{*} K$ .

Let  $A = \text{Des } K$ . Thus,  $K = \text{Comp } A$  and  $A \subseteq [n-1]$ .

Applying (59) to  $L = K$ , we obtain  $\text{Epk } K = (\text{Des } K \cup \{n\}) \setminus (\text{Des } K + 1) = (A \cup \{n\}) \setminus (A + 1)$  (since  $\text{Des } K = A$ ).

Also,  $A_n^\circ \subseteq A$  (since  $S_n^\circ \subseteq S$  for any subset  $S$  of  $[n-1]$ ). If  $A_n^\circ = A$ , then we are done (because if  $A_n^\circ = A$ , then  $\text{Comp} \left( \left( \underbrace{\text{Des } K}_{=A} \right)_n^\circ \right) = \text{Comp} \left( \underbrace{A_n^\circ}_{=A} \right) = \text{Comp } A = K$ , and therefore the reflexivity of  $\xrightarrow{*}$  shows that  $\text{Comp}((\text{Des } K)_n^\circ) \xrightarrow{*} K$ ). Hence, we WLOG assume that  $A_n^\circ \neq A$ . Thus,  $A_n^\circ$  is a proper subset of  $A$  (since  $A_n^\circ \subseteq A$ ). Therefore, there exists a  $q \in A$  satisfying  $q \notin A_n^\circ$ . Let  $k$  be the **largest** such  $q$ . Thus,  $k \in A$  and  $k \notin A_n^\circ$ . Hence,  $k \in A \setminus A_n^\circ$ . Also,  $k \in A \subseteq [n-1]$ .

Let  $J = \text{Comp}(A \setminus \{k\})$ . Thus,  $\text{Des } J = A \setminus \{k\}$ , so that  $A = \text{Des } J \cup \{k\}$  (since  $k \in A$ ). Hence,  $\text{Des } K = A = \text{Des } J \cup \{k\}$ . Also,  $k \notin A \setminus \{k\} = \text{Des } J$ .

Furthermore,  $k-1 \in \text{Des } J$  <sup>58</sup> and  $k+1 \notin \text{Des } J \cup \{n\}$  <sup>59</sup>. Hence, we have

<sup>58</sup>*Proof.* Assume the contrary. Thus,  $k-1 \notin \text{Des } J$ . Hence,  $k-1 \notin \text{Des } J \cup \{k\}$  as well (since  $k-1 \neq k$ ). In other words,  $k-1 \notin A$  (since  $\text{Des } J \cup \{k\} = A$ ). Therefore, the element  $k$  of  $A$  satisfies  $k-1 \notin A$  or  $[k, n-1] \subseteq A$ . Thus, the definition of  $A_n^\circ$  yields  $k \in A_n^\circ$ . This contradicts  $k \notin A_n^\circ$ . This contradiction shows that our assumption was false; qed.

<sup>59</sup>*Proof.* Assume the contrary. Thus,  $k+1 \in \text{Des } J \cup \{n\}$ . In other words, we have  $k+1 \in \text{Des } J$  or  $k+1 = n$ . In other words, we are in one of the following two cases:

*Case 1:* We have  $k+1 \in \text{Des } J$ .

found a  $k \in [n - 1]$  satisfying

$$\begin{aligned} \text{Des } K &= \text{Des } J \cup \{k\}, & k &\notin \text{Des } J, \\ k - 1 &\in \text{Des } J & \text{and} & & k + 1 &\notin \text{Des } J \cup \{n\}. \end{aligned}$$

Therefore, Claim 1 yields  $J \rightarrow K$ . Thus, Claim 2 yields  $\text{Epk } J = \text{Epk } K$ . Claim 4 therefore yields  $(\text{Des } J)_n^\circ = (\text{Des } K)_n^\circ = A_n^\circ$  (since  $\text{Des } K = A$ ). Thus,

$$\begin{aligned} \left| \underbrace{(\text{Des } J) \setminus (\text{Des } J)_n^\circ}_{=A \setminus \{k\}} \right| &= \left| \underbrace{(A \setminus \{k\}) \setminus A_n^\circ}_{=(A \setminus A_n^\circ) \setminus \{k\}} \right| = |(A \setminus A_n^\circ) \setminus \{k\}| \\ &= |A \setminus A_n^\circ| - 1 && \text{(since } k \in A \setminus A_n^\circ) \\ &< |A \setminus A_n^\circ| = |(\text{Des } K) \setminus (\text{Des } K)_n^\circ| \end{aligned}$$

(since  $A = \text{Des } K$ ). Thus, the induction hypothesis shows that  $\text{Comp}((\text{Des } J)_n^\circ) \xrightarrow{*} J$ . Combining this with  $J \rightarrow K$ , we obtain  $\text{Comp}((\text{Des } J)_n^\circ) \xrightarrow{*} K$  (since  $\xrightarrow{*}$  is the transitive-and-reflexive closure of the relation  $\rightarrow$ ). In light of  $(\text{Des } J)_n^\circ = (\text{Des } K)_n^\circ$ , this rewrites as  $\text{Comp}((\text{Des } K)_n^\circ) \xrightarrow{*} K$ . Thus, Claim 5 is proven by induction.]

Now, let  $\mathcal{K}'$  be the  $\mathbb{Q}$ -vector subspace of  $\text{QSym}$  spanned by all differences of the form  $F_J - F_K$ , where  $J$  and  $K$  are two compositions satisfying  $J \rightarrow K$ .

*Claim 6:* Let  $J$  and  $K$  be two compositions such that  $J \xrightarrow{*} K$ . Then,  $F_J - F_K \in \mathcal{K}'$ .

[*Proof of Claim 6:* We have  $J \xrightarrow{*} K$ . By the definition of the relation  $\xrightarrow{*}$ , this means that there exists a sequence  $(L_0, L_1, \dots, L_\ell)$  of compositions satisfying

*Case 2:* We have  $k + 1 = n$ .

Let us first consider Case 1. In this case, we have  $k + 1 \in \text{Des } J$ . Hence,  $k + 1 \in \text{Des } J \subseteq \text{Des } J \cup \{k\} = A$ . If we had  $k + 1 \notin A_n^\circ$ , then  $k + 1$  would be a  $q \in A$  satisfying  $q \notin A_n^\circ$ ; this would contradict the fact that  $k$  is the **largest** such  $q$  (since  $k + 1$  is larger than  $k$ ). Hence, we cannot have  $k + 1 \notin A_n^\circ$ . Thus, we must have  $k + 1 \in A_n^\circ$ . In other words,  $k + 1$  is an element of  $A$  satisfying  $(k + 1) - 1 \notin A$  or  $[k + 1, n - 1] \subseteq A$  (by the definition of  $A_n^\circ$ ). Since  $(k + 1) - 1 \notin A$  is impossible (because  $(k + 1) - 1 = k \in A$ ), we thus have  $[k + 1, n - 1] \subseteq A$ . Now,  $[k, n - 1] = \underbrace{\{k\}}_{\subseteq A \text{ (since } k \in A)} \cup \underbrace{[k + 1, n - 1]}_{\subseteq A} \subseteq A \cup A = A$ . Thus, the element  $k$  of  $A$  satisfies

$k - 1 \notin A$  or  $[k, n - 1] \subseteq A$ . In other words,  $k \in A_n^\circ$  (by the definition of  $A_n^\circ$ ). This contradicts  $k \notin A_n^\circ$ . Thus, we have found a contradiction in Case 1.

Let us now consider Case 2. In this case, we have  $k + 1 = n$ . Hence,  $k = n - 1$ , so that  $[k, n - 1] = \{k\} \subseteq A$  (since  $k \in A$ ). Thus, the element  $k$  of  $A$  satisfies  $k - 1 \notin A$  or  $[k, n - 1] \subseteq A$ . In other words,  $k \in A_n^\circ$  (by the definition of  $A_n^\circ$ ). This contradicts  $k \notin A_n^\circ$ . Thus, we have found a contradiction in Case 2.

We have therefore found a contradiction in each of the two Cases 1 and 2. Thus, we always get a contradiction, so our assumption must have been wrong. Qed.

$L_0 = J$  and  $L_\ell = K$  and  $L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_\ell$ . Consider this sequence. For each  $i \in \{0, 1, \dots, \ell - 1\}$ , we have  $L_i \rightarrow L_{i+1}$  and thus  $F_{L_i} - F_{L_{i+1}} \in \mathcal{K}'$  (by the definition of  $\mathcal{K}'$ ). Therefore,  $\sum_{i=0}^{\ell-1} (F_{L_i} - F_{L_{i+1}}) \in \mathcal{K}'$ . In light of

$$\begin{aligned} \sum_{i=0}^{\ell-1} (F_{L_i} - F_{L_{i+1}}) &= F_{L_0} - F_{L_\ell} && \text{(by the telescope principle)} \\ &= F_J - F_K && \text{(since } L_0 = J \text{ and } L_\ell = K \text{),} \end{aligned}$$

this rewrites as  $F_J - F_K \in \mathcal{K}'$ . This proves Claim 6.]

*Claim 7:* We have  $\mathcal{K}_{\text{Epk}} \subseteq \mathcal{K}'$ .

[*Proof of Claim 7:* Recall that  $\mathcal{K}_{\text{Epk}}$  is the  $\mathbb{Q}$ -vector subspace of  $\text{QSym}$  spanned by all elements of the form  $F_J - F_K$ , where  $J$  and  $K$  are two Epk-equivalent compositions. Thus, it suffices to show that if  $J$  and  $K$  are two Epk-equivalent compositions, then  $F_J - F_K \in \mathcal{K}'$ .

So let  $J$  and  $K$  be two Epk-equivalent compositions. We must prove that  $F_J - F_K \in \mathcal{K}'$ .

The compositions  $J$  and  $K$  are Epk-equivalent; in other words, they have the same size and satisfy  $\text{Epk } J = \text{Epk } K$ . Let  $n = |J| = |K|$ . (This is well-defined, since the compositions  $J$  and  $K$  have the same size.)

Claim 4 yields  $(\text{Des } J)_n^\circ = (\text{Des } K)_n^\circ$ . But Claim 5 yields  $\text{Comp}((\text{Des } K)_n^\circ) \xrightarrow{*} K$ . Hence, Claim 6 (applied to  $\text{Comp}((\text{Des } K)_n^\circ)$  instead of  $J$ ) shows that  $F_{\text{Comp}((\text{Des } K)_n^\circ)} - F_K \in \mathcal{K}'$ . The same argument (applied to  $J$  instead of  $K$ ) shows that  $F_{\text{Comp}((\text{Des } J)_n^\circ)} - F_J \in \mathcal{K}'$ . Now,

$$\begin{aligned} & \left( F_{\text{Comp}((\text{Des } K)_n^\circ)} - F_K \right) - \left( F_{\text{Comp}((\text{Des } J)_n^\circ)} - F_J \right) \\ &= \left( F_{\text{Comp}((\text{Des } K)_n^\circ)} - \underbrace{F_{\text{Comp}((\text{Des } J)_n^\circ)}}_{\substack{= F_{\text{Comp}((\text{Des } K)_n^\circ)} \\ \text{(since } (\text{Des } J)_n^\circ = (\text{Des } K)_n^\circ)}} \right) + F_J - F_K \\ &= \underbrace{\left( F_{\text{Comp}((\text{Des } K)_n^\circ)} - F_{\text{Comp}((\text{Des } K)_n^\circ)} \right)}_{=0} + F_J - F_K = F_J - F_K, \end{aligned}$$

so that

$$F_J - F_K = \underbrace{\left( F_{\text{Comp}((\text{Des } K)_n^\circ)} - F_K \right)}_{\in \mathcal{K}'} - \underbrace{\left( F_{\text{Comp}((\text{Des } J)_n^\circ)} - F_J \right)}_{\in \mathcal{K}'} \in \mathcal{K}' - \mathcal{K}' \subseteq \mathcal{K}'.$$

This proves Claim 7.]

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*Claim 8:* We have  $\mathcal{K}' \subseteq \mathcal{K}_{\text{Epk}}$ .

[*Proof of Claim 8:* Recall that  $\mathcal{K}'$  is the  $\mathbb{Q}$ -vector subspace of  $\text{QSym}$  spanned by all differences of the form  $F_J - F_K$ , where  $J$  and  $K$  are two compositions satisfying  $J \rightarrow K$ . Thus, it suffices to show that if  $J$  and  $K$  are two compositions satisfying  $J \rightarrow K$ , then  $F_J - F_K \in \mathcal{K}_{\text{Epk}}$ .

So let  $J$  and  $K$  be two compositions satisfying  $J \rightarrow K$ . We must prove that  $F_J - F_K \in \mathcal{K}_{\text{Epk}}$ .

We have  $J \rightarrow K$ . Therefore,  $|J| = |K|$  and  $\text{Epk } J = \text{Epk } K$  (by Claim 2). Hence, the compositions  $J$  and  $K$  are  $\text{Epk}$ -equivalent. Thus, the definition of  $\mathcal{K}_{\text{Epk}}$  shows that  $F_J - F_K \in \mathcal{K}_{\text{Epk}}$ . This proves Claim 8.]

Combining Claim 7 and Claim 8, we obtain  $\mathcal{K}_{\text{Epk}} = \mathcal{K}'$ . Recalling the definition of  $\mathcal{K}'$ , we can rewrite this as follows:  $\mathcal{K}_{\text{Epk}}$  is the  $\mathbb{Q}$ -vector subspace of  $\text{QSym}$  spanned by all differences of the form  $F_J - F_K$ , where  $J$  and  $K$  are two compositions satisfying  $J \rightarrow K$ . This proves Proposition 5.5.  $\square$

### 5.3. An $M$ -generating set of $\mathcal{K}_{\text{Epk}}$

Another characterization of the ideal  $\mathcal{K}_{\text{Epk}}$  of  $\text{QSym}$  can be obtained using the monomial basis of  $\text{QSym}$ . Let us first recall how said basis is defined:

For any composition  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ , we let

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_\ell}^{\alpha_\ell}$$

(where the sum is over all strictly increasing  $\ell$ -tuples  $(i_1, i_2, \dots, i_\ell)$  of positive integers). This power series  $M_\alpha$  belongs to  $\text{QSym}$ . The family  $(M_\alpha)_\alpha$  is a composition is a basis of the  $\mathbb{Q}$ -vector space  $\text{QSym}$ ; it is called the *monomial basis* of  $\text{QSym}$ .

**Proposition 5.7.** If  $J = (j_1, j_2, \dots, j_m)$  and  $K$  are two compositions, then we shall write  $J \xrightarrow{M} K$  if there exists an  $\ell \in \{2, 3, \dots, m\}$  such that  $j_\ell > 2$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 2, j_\ell - 2, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$ . (In other words, we write  $J \xrightarrow{M} K$  if  $K$  can be obtained from  $J$  by “splitting” some entry  $j_\ell > 2$  into two consecutive entries 2 and  $j_\ell - 2$ , provided that this entry was not the first entry – i.e., we had  $\ell > 1$  – and that this entry was greater than 2.)

The ideal  $\mathcal{K}_{\text{Epk}}$  of  $\text{QSym}$  is spanned (as a  $\mathbb{Q}$ -vector space) by all sums of the form  $M_J + M_K$ , where  $J$  and  $K$  are two compositions satisfying  $J \xrightarrow{M} K$ .

**Example 5.8.** We have  $(2, 1, 4, 4) \xrightarrow{M} (2, 1, 2, 2, 4)$ , since the composition  $(2, 1, 2, 2, 4)$  is obtained from  $(2, 1, 4, 4)$  by splitting the third entry (which is  $4 > 2$ ) into two consecutive entries 2 and 2.

Similarly,  $(2, 1, 4, 4) \xrightarrow{M} (2, 1, 4, 2, 2)$  and  $(2, 1, 5, 4) \xrightarrow{M} (2, 1, 2, 3, 4)$ .

But we do not have  $(3, 1) \xrightarrow{M} (2, 1, 1)$ , because splitting the first entry of the composition is not allowed in the definition of the relation  $\xrightarrow{M}$ .

Two compositions  $J$  and  $K$  satisfying  $J \xrightarrow{M} K$  must necessarily satisfy  $|J| = |K|$ .

Here are all relations  $\xrightarrow{M}$  between compositions of size 4:

$$(1, 3) \xrightarrow{M} (1, 2, 1).$$

Here are all relations  $\xrightarrow{M}$  between compositions of size 5:

$$\begin{aligned} (1, 4) &\xrightarrow{M} (1, 2, 2), \\ (1, 3, 1) &\xrightarrow{M} (1, 2, 1, 1), \\ (1, 1, 3) &\xrightarrow{M} (1, 1, 2, 1), \\ (2, 3) &\xrightarrow{M} (2, 2, 1). \end{aligned}$$

There are no relations  $\xrightarrow{M}$  between compositions of size  $\leq 3$ .

Before we start proving Proposition 5.7, let us recall a basic formula ([GriRei18, (5.2.2)]) that connects the monomial quasisymmetric functions with the fundamental quasisymmetric functions:

**Proposition 5.9.** Let  $n \in \mathbb{N}$ . Let  $\alpha$  be any composition of  $n$ . Then,

$$M_\alpha = \sum_{\substack{\beta \text{ is a composition} \\ \text{of } n \text{ that refines } \alpha}} (-1)^{\ell(\beta) - \ell(\alpha)} F_\beta.$$

Here, if  $\gamma$  is any composition, then  $\ell(\gamma)$  denotes the *length* of  $\gamma$  (that is, the number of entries of  $\gamma$ ).

*Proof of Proposition 5.9.* This is precisely [GriRei18, (5.2.2)]. □

**Proposition 5.10.** Let  $n$  be a positive integer. Let  $C$  be a subset of  $[n - 1]$ .

(a) Then,

$$M_{\text{Comp } C} = \sum_{B \supseteq C} (-1)^{|B \setminus C|} F_{\text{Comp } B}.$$

(The bound variable  $B$  in this sum and any similar sums is supposed to be a subset of  $[n - 1]$ ; thus, the above sum ranges over all subsets  $B$  of  $[n - 1]$  satisfying  $B \supseteq C$ .)

(b) Let  $k \in [n - 1]$  be such that  $k \notin C$ . Then,

$$M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{k\})} = \sum_{\substack{B \supseteq C; \\ k \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B}.$$

(c) Let  $k \in [n-1]$  be such that  $k \notin C$  and  $k-1 \notin C \cup \{0\}$ . Then,

$$M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{k\})} = \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|B \setminus C|} \left( F_{\text{Comp } B} - F_{\text{Comp}(B \cup \{k-1\})} \right).$$

*Proof of Proposition 5.10. (a)* If  $U$  is any subset of  $[n-1]$ , then  $\ell(\text{Comp } U) = \begin{cases} |U| + 1, & \text{if } n > 0; \\ 0, & \text{if } n = 0. \end{cases}$  This easy fact yields that every subset  $B$  of  $[n-1]$  satisfies

$$\begin{aligned} \ell(\text{Comp } B) - \ell(\text{Comp } C) &= \begin{cases} |B| + 1, & \text{if } n > 0; \\ 0, & \text{if } n = 0 \end{cases} - \begin{cases} |C| + 1, & \text{if } n > 0; \\ 0, & \text{if } n = 0 \end{cases} \\ &= \begin{cases} |B| - |C|, & \text{if } n > 0; \\ 0, & \text{if } n = 0 \end{cases} = |B| - |C| \end{aligned} \quad (61)$$

(indeed, if  $n = 0$ , then both  $B$  and  $C$  must be the empty set, and thus  $0 = |B| - |C|$  holds in this case).

Proposition 5.9 (applied to  $\alpha = \text{Comp } C$ ) yields

$$\begin{aligned} &M_{\text{Comp } C} \\ &= \sum_{\substack{\beta \text{ is a composition} \\ \text{of } n \text{ that refines } \text{Comp } C}} (-1)^{\ell(\beta) - \ell(\text{Comp } C)} F_{\beta} \\ &= \underbrace{\sum_{\substack{B \subseteq [n-1]; \\ \text{Comp } B \text{ refines } \text{Comp } C}}}_{= \sum_{\substack{B \subseteq [n-1]; \\ B \supseteq C}}}} \underbrace{(-1)^{\ell(\text{Comp } B) - \ell(\text{Comp } C)}}_{= (-1)^{|B| - |C|} \text{ (by (61))}} F_{\text{Comp } B} \\ &\quad \text{(because for a subset } B \text{ of } [n-1], \\ &\quad \text{we have } (\text{Comp } B \text{ refines } \text{Comp } C) \\ &\quad \text{if and only if } B \supseteq C) \\ &\quad \left( \begin{array}{c} \text{here, we have substituted } \text{Comp } B \text{ for } \beta, \text{ since} \\ \text{the map } \text{Comp} : \{\text{subsets of } [n-1]\} \rightarrow \{\text{compositions of } n\} \\ \text{is a bijection} \end{array} \right) \\ &= \underbrace{\sum_{\substack{B \subseteq [n-1]; \\ B \supseteq C}}}_{= \sum_{B \supseteq C}} \underbrace{(-1)^{|B| - |C|}}_{= (-1)^{|B \setminus C|} \text{ (since } B \supseteq C)} F_{\text{Comp } B} = \sum_{B \supseteq C} (-1)^{|B \setminus C|} F_{\text{Comp } B}. \end{aligned}$$

This proves Proposition 5.10 (a).

(b) Proposition 5.10 (a) (applied to  $C \cup \{k\}$  instead of  $C$ ) yields

$$\begin{aligned}
M_{\text{Comp}(C \cup \{k\})} &= \sum_{\substack{B \supseteq C \cup \{k\} \\ B \supseteq C; \\ k \in B}} \underbrace{(-1)^{|B \setminus (C \cup \{k\})|}}_{\substack{= (-1)^{|B \setminus C| - 1} \\ \text{(since } |B \setminus (C \cup \{k\})| = |B \setminus C| - 1 \\ \text{(since } k \notin C))}} F_{\text{Comp } B} = \sum_{\substack{B \supseteq C; \\ k \in B}} \underbrace{(-1)^{|B \setminus C| - 1}}_{= -(-1)^{|B \setminus C|}} F_{\text{Comp } B} \\
&= - \sum_{\substack{B \supseteq C; \\ k \in B}} (-1)^{|B \setminus C|} F_{\text{Comp } B}. \tag{62}
\end{aligned}$$

But Proposition 5.10 (a) yields

$$\begin{aligned}
M_{\text{Comp } C} &= \sum_{B \supseteq C} (-1)^{|B \setminus C|} F_{\text{Comp } B} \\
&= \sum_{\substack{B \supseteq C; \\ k \in B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \sum_{\substack{B \supseteq C; \\ k \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B}.
\end{aligned}$$

Adding (62) to this equality, we obtain

$$\begin{aligned}
&M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{k\})} \\
&= \sum_{\substack{B \supseteq C; \\ k \in B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \sum_{\substack{B \supseteq C; \\ k \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \left( - \sum_{\substack{B \supseteq C; \\ k \in B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} \right) \\
&= \sum_{\substack{B \supseteq C; \\ k \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B}.
\end{aligned}$$

This proves Proposition 5.10 (b).

(c) We have  $k - 1 \notin C \cup \{0\}$ . Thus,  $k - 1 \notin C$  and  $k - 1 \neq 0$ . From  $k - 1 \neq 0$ , we obtain  $k - 1 \in [n - 1]$ .

The map

$$\begin{aligned}
&\{B \subseteq [n - 1] \mid B \supseteq C \text{ and } k \notin B \text{ and } k - 1 \notin B\} \\
&\rightarrow \{B \subseteq [n - 1] \mid B \supseteq C \text{ and } k \notin B \text{ and } k - 1 \in B\}
\end{aligned}$$

sending each  $B$  to  $B \cup \{k - 1\}$  is a bijection (this is easy to check using the facts that  $k - 1 \notin C$  and  $k - 1 \in [n - 1]$ ). We shall denote this map by  $\Phi$ .



Proposition 5.10 (b) yields

$$\begin{aligned}
& M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{k\})} \\
&= \sum_{\substack{B \supseteq C; \\ k \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} \\
&= \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \underbrace{\sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \in B}} (-1)^{|B \setminus C|} F_{\text{Comp } B}}_{= \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|(B \cup \{k-1\}) \setminus C|} F_{\text{Comp}(B \cup \{k-1\})}} \\
&\quad \text{(here, we have substituted } B \cup \{k-1\} \text{ for } B \text{ in the sum} \\
&\quad \text{(since the map } \Phi \text{ is a bijection))} \\
&= \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} \underbrace{(-1)^{|(B \cup \{k-1\}) \setminus C|}}_{= (-1)^{|B \setminus C|+1}} F_{\text{Comp}(B \cup \{k-1\})} \\
&\quad \text{(since } |(B \cup \{k-1\}) \setminus C| = |B \setminus C| + 1 \\
&\quad \text{(since } k-1 \notin B \text{ and } k-1 \notin C))} \\
&= \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} \underbrace{(-1)^{|B \setminus C|+1}}_{= -(-1)^{|B \setminus C|}} F_{\text{Comp}(B \cup \{k-1\})} \\
&= \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B} - \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp}(B \cup \{k-1\})} \\
&= \sum_{\substack{B \supseteq C; \\ k \notin B; \\ k-1 \notin B}} (-1)^{|B \setminus C|} \left( F_{\text{Comp } B} - F_{\text{Comp}(B \cup \{k-1\})} \right).
\end{aligned}$$

This proves Proposition 5.10 (c).  $\square$

*Proof of Proposition 5.7.* We shall use the notation  $\langle f_i \mid i \in I \rangle$  for the  $\mathbb{Q}$ -linear span of a family  $(f_i)_{i \in I}$  of elements of a  $\mathbb{Q}$ -vector space.

Define a  $\mathbb{Q}$ -vector subspace  $\mathcal{M}$  of  $\text{QSym}$  by

$$\mathcal{M} = \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions satisfying } J \xrightarrow{M} K \right\rangle.$$

Then, our goal is to prove that  $\mathcal{K}_{\text{Epk}} = \mathcal{M}$ .

We have

$$\begin{aligned}
\mathcal{M} &= \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions satisfying } J \xrightarrow{M} K \right\rangle \\
&= \sum_{n \in \mathbb{N}} \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \xrightarrow{M} K \right\rangle
\end{aligned}$$

---

(because if  $J$  and  $K$  are two compositions satisfying  $J \xrightarrow{M} K$ , then  $J$  and  $K$  have the same size).

Consider the binary relation  $\rightarrow$  defined in Proposition 5.5. Then, Proposition 5.5 yields

$$\begin{aligned} \mathcal{K}_{\text{Epk}} &= \langle F_J - F_K \mid J \text{ and } K \text{ are compositions satisfying } J \rightarrow K \rangle \\ &= \sum_{n \in \mathbb{N}} \langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \rangle \end{aligned}$$

(because if  $J$  and  $K$  are two compositions satisfying  $J \rightarrow K$ , then  $J$  and  $K$  have the same size).

Now, fix  $n \in \mathbb{N}$ . Let  $\Omega$  be the set of all pairs  $(C, k)$  in which  $C$  is a subset of  $[n-1]$  and  $k$  is an element of  $[n-1]$  satisfying  $k \notin C$ ,  $k-1 \in C$  and  $k+1 \notin C \cup \{n\}$ .

For every  $(C, k) \in \Omega$ , we define two elements  $\mathbf{m}_{C,k}$  and  $\mathbf{f}_{C,k}$  of  $\text{QSym}$  by

$$\mathbf{m}_{C,k} = M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{k+1\})} \quad \text{and} \quad (63)$$

$$\mathbf{f}_{C,k} = F_{\text{Comp } C} - F_{\text{Comp}(C \cup \{k\})}. \quad (64)$$

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We have the following:

*Claim 1:* We have

$$\begin{aligned} &\left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \xrightarrow{M} K \right\rangle \\ &= \langle \mathbf{m}_{C,k} \mid (C, k) \in \Omega \rangle. \end{aligned}$$

[*Proof of Claim 1:* It is easy to see that two subsets  $C$  and  $D$  of  $[n-1]$  satisfy  $\text{Comp } C \xrightarrow{M} \text{Comp } D$  if and only if there exists some  $k \in [n-1]$  satisfying  $D =$

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<sup>60</sup>These two elements are well-defined, because both  $C \cup \{k\}$  and  $C \cup \{k+1\}$  are subsets of  $[n-1]$  (since  $k+1 \notin C \cup \{n\}$  shows that  $k+1 \neq n$ ).

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$C \cup \{k+1\}$ ,  $k \notin C$ ,  $k-1 \in C$  and  $k+1 \notin C \cup \{n\}$ .<sup>61</sup> Thus,

$$\begin{aligned}
& \langle M_{\text{Comp } C} + M_{\text{Comp } D} \mid C \text{ and } D \text{ are subsets of } [n-1] \\
& \quad \text{satisfying } \text{Comp } C \xrightarrow{M} \text{Comp } D \rangle \\
&= \langle M_{\text{Comp } C} + M_{\text{Comp } D} \mid C \text{ and } D \text{ are subsets of } [n-1] \\
& \quad \text{such that there exists some } k \in [n-1] \\
& \quad \text{satisfying } D = C \cup \{k+1\}, k \notin C, k-1 \in C \text{ and } k+1 \notin C \cup \{n\} \rangle \\
&= \langle M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{k+1\})} \mid C \subseteq [n-1] \text{ and } k \in [n-1] \\
& \quad \text{are such that } k \notin C, k-1 \in C \text{ and } k+1 \notin C \cup \{n\} \rangle \\
&= \langle \underbrace{M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{k+1\})}}_{\substack{= \mathbf{m}_{C,k} \\ \text{(by (63))}}} \mid (C, k) \in \Omega \rangle \\
& \quad \text{(by the definition of } \Omega) \\
&= \langle \mathbf{m}_{C,k} \mid (C, k) \in \Omega \rangle.
\end{aligned}$$

Now, recall that  $\text{Comp}$  is a bijection between the subsets of  $[n-1]$  and the compositions of  $n$ . Hence,

$$\begin{aligned}
& \langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \xrightarrow{M} K \rangle \\
&= \langle M_{\text{Comp } C} + M_{\text{Comp } D} \mid C \text{ and } D \text{ are subsets of } [n-1] \\
& \quad \text{satisfying } \text{Comp } C \xrightarrow{M} \text{Comp } D \rangle \\
&= \langle \mathbf{m}_{C,k} \mid (C, k) \in \Omega \rangle.
\end{aligned}$$

This proves Claim 1.]

*Claim 2:* We have

$$\begin{aligned}
& \langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \rangle \\
&= \langle \mathbf{f}_{C,k} \mid (C, k) \in \Omega \rangle.
\end{aligned}$$

[*Proof of Claim 2:* It is easy to see that two subsets  $C$  and  $D$  of  $[n-1]$  satisfy  $\text{Comp } C \rightarrow \text{Comp } D$  if and only if there exists some  $k \in [n-1]$  satisfying  $D =$

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<sup>61</sup>To prove this, recall that “splitting” an entry of a composition  $J$  into two consecutive entries (summing up to the original entry) is always tantamount to adding a new element to  $\text{Des } J$ . It suffices to show that the conditions under which an entry of a composition  $J$  can be split in the definition of the relation  $\xrightarrow{M}$  are precisely the conditions  $k \notin C$ ,  $k-1 \in C$  and  $k+1 \notin C \cup \{n\}$  on  $C = \text{Des } J$ , where  $k+1$  denotes the new element that we are adding to  $\text{Des } J$ . This is straightforward.

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$C \cup \{k\}, k \notin C, k-1 \in C$  and  $k+1 \notin C \cup \{n\}$ .<sup>62</sup> Thus,

$$\begin{aligned}
& \langle F_{\text{Comp } C} - F_{\text{Comp } D} \mid C \text{ and } D \text{ are subsets of } [n-1] \\
& \quad \text{satisfying } \text{Comp } C \rightarrow \text{Comp } D \rangle \\
&= \langle F_{\text{Comp } C} - F_{\text{Comp } D} \mid C \text{ and } D \text{ are subsets of } [n-1] \\
& \quad \text{such that there exists some } k \in [n-1] \\
& \quad \text{satisfying } D = C \cup \{k\}, k \notin C, k-1 \in C \text{ and } k+1 \notin C \cup \{n\} \rangle \\
&= \langle F_{\text{Comp } C} - F_{\text{Comp}(C \cup \{k\})} \mid C \subseteq [n-1] \text{ and } k \in [n-1] \\
& \quad \text{are such that } k \notin C, k-1 \in C \text{ and } k+1 \notin C \cup \{n\} \rangle \\
&= \left\langle \underbrace{F_{\text{Comp } C} - F_{\text{Comp}(C \cup \{k\})}}_{\substack{= \mathbf{f}_{C,k} \\ \text{(by (64))}}} \mid (C, k) \in \Omega \right\rangle \\
& \quad \text{(by the definition of } \Omega \text{)} \\
&= \langle \mathbf{f}_{C,k} \mid (C, k) \in \Omega \rangle.
\end{aligned}$$

Now, recall that  $\text{Comp}$  is a bijection between the subsets of  $[n-1]$  and the compositions of  $n$ . Hence,

$$\begin{aligned}
& \langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \rangle \\
&= \langle F_{\text{Comp } C} - F_{\text{Comp } D} \mid C \text{ and } D \text{ are subsets of } [n-1] \\
& \quad \text{satisfying } \text{Comp } C \rightarrow \text{Comp } D \rangle \\
&= \langle \mathbf{f}_{C,k} \mid (C, k) \in \Omega \rangle.
\end{aligned}$$

This proves Claim 2.]

We define a partial order on the set  $\Omega$  by setting

$$(B, k) \geq (C, \ell) \quad \text{if and only if} \quad (k = \ell \text{ and } B \supseteq C).$$

Thus,  $\Omega$  is a finite poset.

*Claim 3:* For every  $(C, \ell) \in \Omega$ , we have

$$\mathbf{m}_{C,\ell} = \sum_{\substack{(B,k) \in \Omega; \\ (B,k) \geq (C,\ell)}} (-1)^{|B \setminus C|} \mathbf{f}_{B,k}.$$

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<sup>62</sup>To prove this, recall that “splitting” an entry of a composition  $J$  into two consecutive entries (summing up to the original entry) is always tantamount to adding a new element to  $\text{Des } J$ . It suffices to show that the conditions under which an entry of a composition  $J$  can be split in the definition of the relation  $\rightarrow$  are precisely the conditions  $k \notin C, k-1 \in C$  and  $k+1 \notin C \cup \{n\}$  on  $C = \text{Des } J$ , where  $k$  denotes the new element that we are adding to  $\text{Des } J$ . This is straightforward.

[Proof of Claim 3: Let  $(C, \ell) \in \Omega$ . Thus,  $C$  is a subset of  $[n-1]$  and  $\ell$  is an element of  $[n-1]$  satisfying  $\ell \notin C$ ,  $\ell-1 \in C$  and  $\ell+1 \notin C \cup \{n\}$ . From  $\ell+1 \notin C \cup \{n\}$ , we obtain  $\ell+1 \notin C$  and  $\ell+1 \neq n$ . From  $\ell+1 \neq n$ , we obtain  $\ell+1 \in [n-1]$ . Also,  $(\ell+1)-1 = \ell \notin C \cup \{0\}$  (since  $\ell \notin C$  and  $\ell \neq 0$ ). Thus, Proposition 5.10 (c) (applied to  $k = \ell+1$ ) yields<sup>63</sup>

$$M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{\ell+1\})} = \sum_{\substack{B \supseteq C; \\ \ell+1 \notin B; \\ \ell \notin B}} (-1)^{|B \setminus C|} \left( F_{\text{Comp } B} - F_{\text{Comp}(B \cup \{\ell\})} \right).$$

But every  $B \subseteq [n-1]$  satisfying  $B \supseteq C$  must satisfy  $\ell-1 \in B$  (since  $\ell-1 \in C \subseteq B$ ). Hence, we can manipulate summation signs as follows:

$$\begin{aligned} \sum_{\substack{B \supseteq C; \\ \ell+1 \notin B; \\ \ell \notin B}} &= \sum_{\substack{B \supseteq C; \\ \ell+1 \notin B; \\ \ell \notin B; \\ \ell-1 \in B}} = \sum_{\substack{B \supseteq C; \\ \ell+1 \notin B \cup \{n\}; \\ \ell \notin B; \\ \ell-1 \in B}} &\quad \left( \begin{array}{l} \text{since } \ell+1 \notin B \text{ is equivalent to } \ell+1 \notin B \cup \{n\} \\ \text{(because } \ell+1 \neq n) \end{array} \right) \\ &= \sum_{\substack{B \supseteq C; \\ \ell \notin B; \\ \ell-1 \in B; \\ \ell+1 \notin B \cup \{n\}}} \\ &= \sum_{\substack{B \supseteq C; \\ (B, \ell) \in \Omega}} \left( \begin{array}{l} \text{since the} \\ \text{condition } (\ell \notin B, \ell-1 \in B \text{ and } \ell+1 \notin B \cup \{n\}) \\ \text{on a subset } B \text{ of } [n-1] \text{ is equivalent to } (B, \ell) \in \Omega \\ \text{(by the definition of } \Omega) \end{array} \right) \\ &= \sum_{\substack{(B, \ell) \in \Omega; \\ B \supseteq C}} = \sum_{\substack{(B, k) \in \Omega; \\ k = \ell; \\ B \supseteq C}} = \sum_{\substack{(B, k) \in \Omega; \\ (B, k) \geq (C, \ell)}} &\quad (65) \end{aligned}$$

(since the condition  $(k = \ell \text{ and } B \supseteq C)$  on a  $(B, k) \in \Omega$  is equivalent to  $(B, k) \geq (C, \ell)$  (by the definition of the partial order on  $\Omega$ )).

<sup>63</sup>Here and in the following, the bound variable  $B$  in a sum always is understood to be a subset of  $[n-1]$ .

Now, the definition of  $\mathbf{m}_{C,\ell}$  yields

$$\begin{aligned}
\mathbf{m}_{C,\ell} &= M_{\text{Comp } C} + M_{\text{Comp}(C \cup \{\ell+1\})} \\
&= \sum_{\substack{B \supseteq C; \\ \ell+1 \notin B; \\ \ell \notin B}} (-1)^{|B \setminus C|} \left( F_{\text{Comp } B} - F_{\text{Comp}(B \cup \{\ell\})} \right) \\
&= \sum_{\substack{(B,k) \in \Omega; \\ (B,k) \geq (C,\ell) \\ \text{(by (65))}}} \\
&= \sum_{\substack{(B,k) \in \Omega; \\ (B,k) \geq (C,\ell)}} (-1)^{|B \setminus C|} \left( F_{\text{Comp } B} - \underbrace{F_{\text{Comp}(B \cup \{\ell\})}}_{\substack{= F_{\text{Comp}(B \cup \{k\})} \\ \text{(since } \ell=k \\ \text{(since } (B,k) \geq (C,\ell) \text{ and thus } k=\ell))}} \right) \\
&= \sum_{\substack{(B,k) \in \Omega; \\ (B,k) \geq (C,\ell)}} (-1)^{|B \setminus C|} \underbrace{\left( F_{\text{Comp } B} - F_{\text{Comp}(B \cup \{k\})} \right)}_{\substack{= \mathbf{f}_{B,k} \\ \text{(since (64) yields} \\ \mathbf{f}_{B,k} = F_{\text{Comp } B} - F_{\text{Comp}(B \cup \{k\})})}} \\
&= \sum_{\substack{(B,k) \in \Omega; \\ (B,k) \geq (C,\ell)}} (-1)^{|B \setminus C|} \mathbf{f}_{B,k}.
\end{aligned}$$

This proves Claim 3.]

Now, Claim 3 shows that the family  $(\mathbf{m}_{C,k})_{(C,k) \in \Omega}$  expands triangularly with respect to the family  $(\mathbf{f}_{C,k})_{(C,k) \in \Omega}$  with respect to the poset structure on  $\Omega$ . Moreover, the expansion is **unitriangular** (because if  $(B,k) = (C,\ell)$ , then  $B = C$  and thus  $(-1)^{|B \setminus C|} = (-1)^{|C \setminus C|} = (-1)^0 = 1$ ) and thus invertibly triangular (this means that the diagonal entries are invertible). Therefore, by a standard fact from linear algebra (see, e.g., [GriRei18, Corollary 11.1.19 **(b)**]), we conclude that the span of the family  $(\mathbf{m}_{C,k})_{(C,k) \in \Omega}$  equals the span of the family  $(\mathbf{f}_{C,k})_{(C,k) \in \Omega}$ . In other words,

$$\langle \mathbf{m}_{C,k} \mid (C,k) \in \Omega \rangle = \langle \mathbf{f}_{C,k} \mid (C,k) \in \Omega \rangle.$$

Now, Claim 1 yields

$$\begin{aligned}
&\left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \xrightarrow{M} K \right\rangle \\
&= \langle \mathbf{m}_{C,k} \mid (C,k) \in \Omega \rangle = \langle \mathbf{f}_{C,k} \mid (C,k) \in \Omega \rangle \\
&= \langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \rangle
\end{aligned}$$

(by Claim 2).

Now, forget that we fixed  $n$ . We thus have proven that

$$\begin{aligned} & \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \xrightarrow{M} K \right\rangle \\ &= \left\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \right\rangle \end{aligned}$$

for each  $n \in \mathbb{N}$ . Thus,

$$\begin{aligned} & \sum_{n \in \mathbb{N}} \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \xrightarrow{M} K \right\rangle \\ &= \sum_{n \in \mathbb{N}} \left\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \right\rangle. \end{aligned}$$

In light of

$$\mathcal{K}_{\text{Epk}} = \sum_{n \in \mathbb{N}} \left\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \right\rangle$$

and

$$\mathcal{M} = \sum_{n \in \mathbb{N}} \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \xrightarrow{M} K \right\rangle,$$

this rewrites as  $\mathcal{M} = \mathcal{K}_{\text{Epk}}$ . In other words,  $\mathcal{K}_{\text{Epk}} = \mathcal{M}$ . This proves Proposition 5.7.  $\square$

**Question 5.11.** It is worth analyzing the kernels of other known descent statistics (shuffle-compatible or not). Let us say that a descent statistic  $\text{st}$  is *M-binomial* if its kernel  $\mathcal{K}_{\text{st}}$  can be spanned by elements of the form  $\lambda M_J + \mu M_K$  with  $\lambda, \mu \in \mathbb{Q}$  and compositions  $J, K$ . Then, Proposition 5.7 yields that  $\text{Epk}$  is *M-binomial*. It is easy to see that the statistics  $\text{Des}$  and  $\text{des}$  are *M-binomial* as well. Computations using SageMath suggest that the statistics  $\text{Lpk}$ ,  $\text{Rpk}$ ,  $\text{Pk}$ ,  $\text{Val}$ ,  $\text{pk}$ ,  $\text{lpk}$ ,  $\text{rpk}$  and  $\text{val}$  (see [GesZhu17] for some of their definitions) are *M-binomial*, too (at least for compositions of size  $\leq 9$ ); this would be nice to prove. On the other hand, the statistics  $\text{maj}$ ,  $(\text{des}, \text{maj})$  and  $(\text{val}, \text{des})$  (again, see [GesZhu17] for definitions) are not *M-binomial*.

#### 5.4. Appendix: Proof of Proposition 4.13 and Theorem 4.14

Let us now give proofs of Proposition 4.13 and Theorem 4.14, which we have promised above. We will mostly rely on Lemma 5.2 and on Proposition 4.11.

For the rest of Subsection 5.4, we shall make the following conventions:

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**Convention 5.12.** Let  $st$  be a permutation statistic. For each permutation  $\pi$ , let  $[\pi]_{st}$  denote the  $st$ -equivalence class of  $\pi$ . Let  $\mathcal{A}_{st}$  be the free  $\mathbb{Q}$ -vector space whose basis is the set of all  $st$ -equivalence classes of permutations. (This is well-defined whether or not  $st$  is shuffle-compatible.)

*Proof of Proposition 4.13.* A *magmatic algebra* shall mean a  $\mathbb{Q}$ -vector space equipped with a binary operation which is written as multiplication (i.e., we write  $ab$  for the image of a pair  $(a, b)$  under this operation), but is not required to be associative (or have a unity). An (actual, i.e., associative unital) algebra is thus a magmatic algebra whose multiplication is associative and has a unity. In particular, any actual algebra is a magmatic algebra. A *magmatic algebra homomorphism* is a  $\mathbb{Q}$ -linear map between two magmatic algebras that preserves the multiplication.

We make  $\mathcal{A}_{st}$  into a magmatic algebra by setting

$$[\pi]_{st} [\sigma]_{st} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{st} \quad (66)$$

for any two disjoint permutations  $\pi$  and  $\sigma$ . This is well-defined, because the right-hand side of (66) depends only on the  $st$ -equivalence classes  $[\pi]_{st}$  and  $[\sigma]_{st}$  rather than on the permutations  $\pi$  and  $\sigma$  themselves (this is because  $st$  is shuffle-compatible).

Define a  $\mathbb{Q}$ -linear map  $p : \text{QSym} \rightarrow \mathcal{A}_{st}$  by requiring that

$$p(F_L) = [\pi]_{st} \quad \text{for every composition } L \text{ and every permutation } \pi \text{ with } \text{Comp } \pi = L.$$

This is well-defined, because for any given composition  $L$ , any two permutations  $\pi$  with  $\text{Comp } \pi = L$  will have the same  $st$ -equivalence class  $[\pi]_{st}$  (since  $st$  is a descent statistic).

Thus, each permutation  $\pi$  satisfies

$$p(F_{\text{Comp } \pi}) = [\pi]_{st} \quad (67)$$

and therefore  $[\pi]_{st} = p(F_{\text{Comp } \pi}) \in p(\text{QSym})$ . Hence,  $\mathcal{A}_{st} \subseteq p(\text{QSym})$  (since the  $st$ -equivalence classes  $[\pi]_{st}$  form a basis of  $\mathcal{A}_{st}$ ). Consequently, the map  $p$  is surjective.

Moreover, we have

$$p(ab) = p(a)p(b) \quad \text{for all } a, b \in \text{QSym}. \quad (68)$$

[*Proof of (68):* Let  $a, b \in \text{QSym}$ . We must prove the equality (68). Since this equality is  $\mathbb{Q}$ -linear in each of  $a$  and  $b$ , we WLOG assume that  $a$  and  $b$  belong to the fundamental basis of  $\text{QSym}$ . That is,  $a = F_J$  and  $b = F_K$  for two compositions  $J$  and  $K$ . Consider these  $J$  and  $K$ . Fix any two disjoint permutations  $\pi$  and  $\sigma$



such that  $\text{Comp } \pi = J$  and  $\text{Comp } \sigma = K$ . (Such  $\pi$  and  $\sigma$  are easy to find.) The definition of  $p$  thus yields  $p(F_J) = [\pi]_{\text{st}}$  and  $p(F_K) = [\sigma]_{\text{st}}$ . Hence,

$$\begin{aligned} p\left(\underbrace{a}_{=F_J}\right) p\left(\underbrace{b}_{=F_K}\right) &= \underbrace{p(F_J)}_{=[\pi]_{\text{st}}} \underbrace{p(F_K)}_{=[\sigma]_{\text{st}}} = [\pi]_{\text{st}} [\sigma]_{\text{st}} \\ &= \sum_{\tau \in S(\pi, \sigma)} [\tau]_{\text{st}} \quad (\text{by (66)}) \\ &= \sum_{\chi \in S(\pi, \sigma)} [\chi]_{\text{st}} \quad (69) \end{aligned}$$

(here, we have renamed the summation index  $\tau$  as  $\chi$ ). On the other hand,  $a = F_J = F_{\text{Comp } \pi}$  (since  $J = \text{Comp } \pi$ ) and  $b = F_{\text{Comp } \sigma}$  (similarly); multiplying these equalities, we get

$$ab = F_{\text{Comp } \pi} F_{\text{Comp } \sigma} = \sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp } \chi}$$

(by Proposition 4.11). Applying the map  $p$  to this equality, we find

$$\begin{aligned} p(ab) &= p\left(\sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp } \chi}\right) = \sum_{\chi \in S(\pi, \sigma)} \underbrace{p(F_{\text{Comp } \chi})}_{\substack{=[\chi]_{\text{st}} \\ (\text{by (67))}}} = \sum_{\chi \in S(\pi, \sigma)} [\chi]_{\text{st}} \\ &= p(a) p(b) \quad (\text{by (69)}). \end{aligned}$$

This proves (68).]

The equality (68) shows that  $p$  is a magmatic algebra homomorphism (since  $p$  is  $\mathbb{Q}$ -linear). Thus, using the surjectivity of  $p$ , we can easily see that the magmatic algebra  $\mathcal{A}_{\text{st}}$  is associative<sup>64</sup>. In other words, the multiplication on  $\mathcal{A}_{\text{st}}$  defined in Definition 4.12 is associative. Moreover, it is clear that the st-equivalence class of the 0-permutation  $()$  serves as a neutral element for this multiplication (because if  $\emptyset$  denotes the 0-permutation  $()$ , then  $S(\emptyset, \sigma) = S(\sigma, \emptyset) = \{\sigma\}$  for every

<sup>64</sup>*Proof.* Let  $u, v, w \in \mathcal{A}_{\text{st}}$ . We must show that  $(uv)w = u(vw)$ .

There exist  $a, b, c \in \text{QSym}$  such that  $u = p(a)$ ,  $v = p(b)$  and  $w = p(c)$  (since  $p$  is surjective).

Fix such  $a, b, c$ . Since  $\text{QSym}$  is an actual (i.e., associative unital) algebra, we have

$$\begin{aligned} p(abc) &= p((ab)c) = \underbrace{p(ab)}_{\substack{=p(a)p(b) \\ (\text{since } p \text{ is a magmatic} \\ \text{algebra homomorphism})}} p(c) \quad (\text{since } p \text{ is a magmatic algebra homomorphism}) \\ &= \left(\underbrace{p(a)}_{=u} \underbrace{p(b)}_{=v}\right) \underbrace{p(c)}_{=w} = (uv)w. \end{aligned}$$

A similar argument shows that  $p(abc) = u(vw)$ . Thus,  $(uv)w = p(abc) = u(vw)$ , qed.

permutation  $\sigma$ ). Thus, the multiplication on  $\mathcal{A}_{\text{st}}$  defined in Definition 4.12 is well-defined and associative, and turns  $\mathcal{A}_{\text{st}}$  into a  $\mathbb{Q}$ -algebra whose unity is the st-equivalence class of the 0-permutation  $()$ . This proves Proposition 4.13 (a).

(b) The map  $p : \text{QSym} \rightarrow \mathcal{A}_{\text{st}}$  is  $\mathbb{Q}$ -linear and respects multiplication (by (68)). Moreover, it sends the unity of  $\text{QSym}$  to the unity of the algebra  $\mathcal{A}_{\text{st}}$ <sup>65</sup>. Thus,  $p$  is a  $\mathbb{Q}$ -algebra homomorphism. Moreover, recall that  $p$  is surjective and satisfies  $p(F_{\text{Comp } \pi}) = [\pi]_{\text{st}}$  for every permutation  $\pi$ . Hence, there is a surjective  $\mathbb{Q}$ -algebra homomorphism  $p_{\text{st}} : \text{QSym} \rightarrow \mathcal{A}_{\text{st}}$  that satisfies

$$p_{\text{st}}(F_{\text{Comp } \pi}) = [\pi]_{\text{st}} \quad \text{for every permutation } \pi$$

(namely,  $p_{\text{st}} = p$ ). This proves Proposition 4.13 (b).  $\square$

*Proof of Theorem 4.14.* (a)  $\implies$ : Assume that st is shuffle-compatible. Proposition 4.13 (b) shows that there is a surjective  $\mathbb{Q}$ -algebra homomorphism  $p_{\text{st}} : \text{QSym} \rightarrow \mathcal{A}_{\text{st}}$  that satisfies

$$p_{\text{st}}(F_{\text{Comp } \pi}) = [\pi]_{\text{st}} \quad \text{for every permutation } \pi. \quad (70)$$

Consider this  $p_{\text{st}}$ .

If  $\alpha$  is an st-equivalence class of compositions, then we let  $u_\alpha$  denote the st-equivalence class  $[\pi]_{\text{st}}$  of all permutations  $\pi$  whose descent composition  $\text{Comp } \pi$  belongs to  $\alpha$ . (This is indeed a well-defined st-equivalence class, because st is a descent statistic.) This establishes a bijection between the st-equivalence classes of compositions and the st-equivalence classes of permutations. Thus, the family  $(u_\alpha)$  (indexed by st-equivalence classes  $\alpha$  of compositions) is just a reindexing of the basis of  $\mathcal{A}_{\text{st}}$  consisting of the st-equivalence classes  $[\pi]_{\text{st}}$  of permutations. Consequently, this family is a basis of the  $\mathbb{Q}$ -vector space  $\mathcal{A}_{\text{st}}$ . Moreover,  $p_{\text{st}}$  is a  $\mathbb{Q}$ -algebra homomorphism  $\text{QSym} \rightarrow \mathcal{A}_{\text{st}}$  with the property that whenever  $\alpha$  is an st-equivalence class of compositions, we have

$$p_{\text{st}}(F_L) = u_\alpha \quad \text{for each } L \in \alpha.$$

(Indeed, this follows from applying (70) to any permutation  $\pi$  satisfying  $\text{Comp } \pi = L$ .)

Thus, there exist a  $\mathbb{Q}$ -algebra  $A$  (namely,  $\mathcal{A} = \mathcal{A}_{\text{st}}$ ) with basis  $(u_\alpha)$  (indexed by st-equivalence classes  $\alpha$  of compositions) and a  $\mathbb{Q}$ -algebra homomorphism  $\phi_{\text{st}} : \text{QSym} \rightarrow A$  (namely,  $\phi_{\text{st}} = p_{\text{st}}$ ) with the property that whenever  $\alpha$  is an st-equivalence class of compositions, we have

$$\phi_{\text{st}}(F_L) = u_\alpha \quad \text{for each } L \in \alpha.$$

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<sup>65</sup>*Proof.* The unity of  $\text{QSym}$  is  $1 = F_{()}$ , where  $()$  denotes the empty composition. Now, let  $\emptyset$  denote the 0-permutation  $()$ . Then, the st-equivalence class  $[\emptyset]_{\text{st}}$  is the unity of the algebra  $\mathcal{A}_{\text{st}}$ . But the 0-permutation  $\emptyset = ()$  has descent composition  $\text{Comp } \emptyset = ()$ . Hence, the definition of  $p$  yields  $p(F_{()}) = [\emptyset]_{\text{st}}$ . In view of what we just said, this equality says that  $p$  sends the unity of  $\text{QSym}$  to the unity of the algebra  $\mathcal{A}_{\text{st}}$ .

This proves the  $\implies$  direction of Theorem 4.14 (a).

$\Leftarrow$ : Assume that there exist a  $\mathbb{Q}$ -algebra  $A$  with basis  $(u_\alpha)$  (indexed by st-equivalence classes  $\alpha$  of compositions) and a  $\mathbb{Q}$ -algebra homomorphism  $\phi_{\text{st}} : \text{QSym} \rightarrow A$  with the property that whenever  $\alpha$  is an st-equivalence class of compositions, we have

$$\phi_{\text{st}}(F_L) = u_\alpha \quad \text{for each } L \in \alpha.$$

Consider this  $A$ , this  $(u_\alpha)$  and this  $\phi_{\text{st}}$ . Lemma 5.2 shows that  $\text{Ker}(\phi_{\text{st}}) = \mathcal{K}_{\text{st}}$ . But  $\text{Ker}(\phi_{\text{st}})$  is an ideal of  $\text{QSym}$  (since  $\phi_{\text{st}}$  is a  $\mathbb{Q}$ -algebra homomorphism). In other words,  $\mathcal{K}_{\text{st}}$  is an ideal of  $\text{QSym}$  (since  $\text{Ker}(\phi_{\text{st}}) = \mathcal{K}_{\text{st}}$ ).

Now, consider any two disjoint permutations  $\pi$  and  $\sigma$ . Also, consider two further disjoint permutations  $\pi'$  and  $\sigma'$  satisfying  $\text{st } \pi = \text{st } (\pi')$ ,  $\text{st } \sigma = \text{st } (\sigma')$ ,  $|\pi| = |\pi'|$  and  $|\sigma| = |\sigma'|$ . We shall show that  $\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}$  as multisets. This will show that the multiset  $\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ .

From  $\text{st } \pi = \text{st } (\pi')$  and  $|\pi| = |\pi'|$ , we conclude that  $\pi$  and  $\pi'$  are st-equivalent. In other words,  $\text{Comp } \pi$  and  $\text{Comp } (\pi')$  are st-equivalent. Hence,  $F_{\text{Comp } \pi} - F_{\text{Comp } (\pi')} \in \mathcal{K}_{\text{st}}$  (by the definition of  $\mathcal{K}_{\text{st}}$ ), so that  $F_{\text{Comp } \pi} \equiv F_{\text{Comp } (\pi')} \pmod{\mathcal{K}_{\text{st}}}$ . Similarly,  $F_{\text{Comp } \sigma} \equiv F_{\text{Comp } (\sigma')} \pmod{\mathcal{K}_{\text{st}}}$ . These two congruences, combined, yield  $F_{\text{Comp } \pi} F_{\text{Comp } \sigma} \equiv F_{\text{Comp } (\pi')} F_{\text{Comp } (\sigma')} \pmod{\mathcal{K}_{\text{st}}}$ , because  $\mathcal{K}_{\text{st}}$  is an ideal of  $\text{QSym}$ .

Let  $X$  be the codomain of the map  $\text{st}$ . Let  $\mathbb{Q}[X]$  be the free  $\mathbb{Q}$ -vector space with basis  $([x])_{x \in X}$ . Then, we can define a  $\mathbb{Q}$ -linear map  $\mathbf{st} : \text{QSym} \rightarrow \mathbb{Q}[X]$ ,  $F_j \mapsto [\text{st } j]$ . This map  $\mathbf{st}$  sends each of the generators of  $\mathcal{K}_{\text{st}}$  to 0 (by the definition of  $\mathcal{K}_{\text{st}}$ ), and therefore sends the whole  $\mathcal{K}_{\text{st}}$  to 0. In other words,  $\mathbf{st}(\mathcal{K}_{\text{st}}) = 0$ .

We have  $F_{\text{Comp } \pi} F_{\text{Comp } \sigma} \equiv F_{\text{Comp } (\pi')} F_{\text{Comp } (\sigma')} \pmod{\mathcal{K}_{\text{st}}}$  and thus

$$\mathbf{st}(F_{\text{Comp } \pi} F_{\text{Comp } \sigma}) = \mathbf{st}(F_{\text{Comp } (\pi')} F_{\text{Comp } (\sigma')}) \quad (71)$$

(since  $\mathbf{st}(\mathcal{K}_{\text{st}}) = 0$ ). But Proposition 4.11 yields

$$F_{\text{Comp } \pi} F_{\text{Comp } \sigma} = \sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp } \chi}.$$

Applying the map  $\mathbf{st}$  to both sides of this equality, we find

$$\begin{aligned} \mathbf{st}(F_{\text{Comp } \pi} F_{\text{Comp } \sigma}) &= \mathbf{st}\left(\sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp } \chi}\right) \\ &= \sum_{\chi \in S(\pi, \sigma)} \underbrace{\mathbf{st}(F_{\text{Comp } \chi})}_{=[\text{st}(\text{Comp } \chi)]=[\text{st } \chi]} = \sum_{\chi \in S(\pi, \sigma)} [\text{st } \chi]. \end{aligned}$$

Similarly,

$$\mathbf{st}(F_{\text{Comp } (\pi')} F_{\text{Comp } (\sigma')}) = \sum_{\chi \in S(\pi', \sigma')} [\text{st } \chi].$$

But the left-hand sides of the last two equalities are equal (because of (71)); therefore, the right-hand sides must be equal as well. In other words,

$$\sum_{\chi \in S(\pi, \sigma)} [\text{st } \chi] = \sum_{\chi \in S(\pi', \sigma')} [\text{st } \chi].$$

This shows exactly that  $\{\text{st } \chi \mid \chi \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \chi \mid \chi \in S(\pi', \sigma')\}_{\text{multi}}$ . In other words,  $\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S(\pi', \sigma')\}_{\text{multi}}$ . Thus, we have proven that the multiset  $\{\text{st } \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ . Hence, the statistic  $\text{st}$  is shuffle-compatible. This proves the  $\Leftarrow$  direction of Theorem 4.14 (a).

(b) Proposition 4.13 (b) shows that there is a surjective  $\mathbb{Q}$ -algebra homomorphism  $p_{\text{st}} : \text{QSym} \rightarrow \mathcal{A}_{\text{st}}$  that satisfies

$$p_{\text{st}}(F_{\text{Comp}} \pi) = [\pi]_{\text{st}} \quad \text{for every permutation } \pi. \quad (72)$$

Consider this  $p_{\text{st}}$ .

Let  $\gamma$  be the  $\mathbb{Q}$ -linear map

$$\mathcal{A}_{\text{st}} \rightarrow A, \quad [\pi]_{\text{st}} \mapsto u_{\alpha},$$

where  $\alpha$  is the  $\text{st}$ -equivalence class of the composition  $\text{Comp } \pi$ . This map  $\gamma$  is clearly well-defined (since the  $\text{st}$ -equivalence classes  $[\pi]_{\text{st}}$  form a basis of  $\mathcal{A}_{\text{st}}$ , and since the  $\text{st}$ -equivalence class of the composition  $\text{Comp } \pi$  depends only on the  $\text{st}$ -equivalence class  $[\pi]_{\text{st}}$  and not on the permutation  $\pi$  itself). Moreover,  $\gamma$  sends a basis of  $\mathcal{A}_{\text{st}}$  (the basis formed by the  $\text{st}$ -equivalence classes  $[\pi]_{\text{st}}$  of permutations) to a basis of  $A$  (namely, to the basis  $(u_{\alpha})$ ) bijectively; thus,  $\gamma$  is an isomorphism of  $\mathbb{Q}$ -vector spaces.

The diagram

$$\begin{array}{ccc} \text{QSym} & \xrightarrow{p_{\text{st}}} & \mathcal{A}_{\text{st}} \\ & \searrow \phi_{\text{st}} & \downarrow \gamma \\ & & A \end{array}$$

is commutative (as one can easily check by tracing an arbitrary basis element  $F_L$  of  $\text{QSym}$  through the diagram). Since the maps  $p_{\text{st}}$  and  $\phi_{\text{st}}$  in this diagram are  $\mathbb{Q}$ -algebra homomorphisms, and since  $p_{\text{st}}$  is surjective, we thus conclude that  $\gamma$  is also a  $\mathbb{Q}$ -algebra homomorphism<sup>66</sup>. Since  $\gamma$  is an isomorphism of  $\mathbb{Q}$ -vector spaces, we thus conclude that  $\gamma$  is a  $\mathbb{Q}$ -algebra isomorphism  $\mathcal{A}_{\text{st}} \rightarrow A$ . This proves Theorem 4.14 (b).  $\square$

<sup>66</sup>*Proof.* Let  $a, b \in \mathcal{A}_{\text{st}}$ . We shall show that  $\gamma(ab) = \gamma(a)\gamma(b)$ .

There exist  $a', b' \in \text{QSym}$  such that  $a = p_{\text{st}}(a')$  and  $b = p_{\text{st}}(b')$  (since  $p_{\text{st}}$  is surjective).

Consider these  $a', b'$ . Then,  $\gamma\left(\underbrace{a}_{=p_{\text{st}}(a')}\right) = \gamma(p_{\text{st}}(a')) = \phi_{\text{st}}(a')$  (since the diagram is commutative) and  $\gamma(b) = \phi_{\text{st}}(b')$  (similarly). But from  $a = p_{\text{st}}(a')$  and  $b = p_{\text{st}}(b')$ , we obtain

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## 6. Dendriform structures

Next, we shall study how the ideal  $\mathcal{K}_{\text{Epk}}$  interacts with some additional structure on  $\text{QSym}$ , viz. the *dendriform operations*  $\prec$  and  $\succeq$  and the “runic” operations  $\Phi$  and  $\star$ . These operations were introduced in [Grinbe16]. Our study shall lead us back to the notions of left-shuffle-compatibility and right-shuffle-compatibility from Section 3. We shall reprove that  $\text{Epk}$  is left-shuffle-compatible and right-shuffle-compatible; similar studies can probably be made for other descent statistics.

### 6.1. Four operations on $\text{QSym}$

We begin with some definitions. We will use some notations from [Grinbe16], but we set  $\mathbf{k} = \mathbb{Q}$  because we are working over the ring  $\mathbb{Q}$  in this paper. Monomials always mean formal expressions of the form  $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$  with  $a_1 + a_2 + a_3 + \cdots < \infty$  (see [Grinbe16, Section 2] for details). If  $\mathfrak{m}$  is a monomial, then  $\text{Supp } \mathfrak{m}$  will denote the finite subset

$$\{i \in \{1, 2, 3, \dots\} \mid \text{the exponent with which } x_i \text{ occurs in } \mathfrak{m} \text{ is } > 0\}$$

of  $\{1, 2, 3, \dots\}$ . Next, we define four binary operations

- $\prec$  (called “dendriform less-than”; but it’s an operation, not a relation),
- $\succeq$  (called “dendriform greater-or-equal”; but it’s an operation, not a relation),
- $\Phi$  (called “belgthor”),
- $\star$  (called “tvimadur”)

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$ab = p_{\text{st}}(a') p_{\text{st}}(b') = p_{\text{st}}(a'b')$  (since  $p_{\text{st}}$  is a  $\mathbb{Q}$ -algebra homomorphism), so that

$$\begin{aligned} \gamma(ab) &= \gamma(p_{\text{st}}(a'b')) = \phi_{\text{st}}(a'b') && \text{(since the diagram is commutative)} \\ &= \underbrace{\phi_{\text{st}}(a')}_{=\gamma(a)} \underbrace{\phi_{\text{st}}(b')}_{=\gamma(b)} && \text{(since } \phi_{\text{st}} \text{ is a } \mathbb{Q}\text{-algebra homomorphism)} \\ &= \gamma(a) \gamma(b). \end{aligned}$$

Now, forget that we fixed  $a, b$ . We thus have proven that  $\gamma(ab) = \gamma(a) \gamma(b)$  for all  $a, b \in \mathcal{A}_{\text{st}}$ . Similarly,  $\gamma(1) = 1$ . Hence,  $\gamma$  is a  $\mathbb{Q}$ -algebra homomorphism (since  $\gamma$  is  $\mathbb{Q}$ -linear).

on the ring  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  of power series by first defining how they act on monomials:

$$\begin{aligned} m \prec n &= \begin{cases} m \cdot n, & \text{if } \min(\text{Supp } m) < \min(\text{Supp } n); \\ 0, & \text{if } \min(\text{Supp } m) \geq \min(\text{Supp } n) \end{cases}; \\ m \succeq n &= \begin{cases} m \cdot n, & \text{if } \min(\text{Supp } m) \geq \min(\text{Supp } n); \\ 0, & \text{if } \min(\text{Supp } m) < \min(\text{Supp } n) \end{cases}; \\ m \phi n &= \begin{cases} m \cdot n, & \text{if } \max(\text{Supp } m) \leq \min(\text{Supp } n); \\ 0, & \text{if } \max(\text{Supp } m) > \min(\text{Supp } n) \end{cases}; \\ m \ast n &= \begin{cases} m \cdot n, & \text{if } \max(\text{Supp } m) < \min(\text{Supp } n); \\ 0, & \text{if } \max(\text{Supp } m) \geq \min(\text{Supp } n) \end{cases}; \end{aligned}$$

and then requiring that they all be  $\mathbf{k}$ -bilinear and continuous (so their action on pairs of arbitrary power series can be computed by “opening the parentheses”). These operations  $\prec$ ,  $\succeq$ ,  $\phi$  and  $\ast$  all restrict to the subset  $\text{QSym}$  of  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  (this is proven in [Grinbe16, detailed version, Section 3]). They furthermore satisfy numerous relations<sup>67</sup>:

- The dendriform operations satisfy the four rules

$$\begin{aligned} a \prec b + a \succeq b &= ab; & (73) \\ (a \prec b) \prec c &= a \prec (bc); \\ (a \succeq b) \prec c &= a \succeq (b \prec c); \\ a \succeq (b \succeq c) &= (ab) \succeq c \end{aligned}$$

for all  $a, b, c \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$ . (In other words, they turn  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  into what is called a dendriform algebra.)

- For any  $a \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$ , we have

$$1 \prec a = 0; \tag{74}$$

$$a \prec 1 = a - \varepsilon(a); \tag{75}$$

$$1 \succeq a = a; \tag{76}$$

$$a \succeq 1 = \varepsilon(a), \tag{77}$$

where  $\varepsilon(a)$  denotes the constant term of the power series  $a$ .

- The binary operation  $\phi$  is associative and unital (with 1 serving as the unity).
- The binary operation  $\ast$  is associative and unital (with 1 serving as the unity).

<sup>67</sup>These relations are all easy to prove (by linearity, it suffices to verify them on monomials only, and this verification is straightforward). A proof of the associativity of  $\phi$  was given in [Grinbe16, detailed version, Proposition 3.4].

Recall that we are using the notations  $M_\alpha$  for the monomial quasisymmetric functions and  $F_\alpha$  for the fundamental quasisymmetric functions.

- For any two nonempty compositions  $\alpha$  and  $\beta$ , we have  $M_\alpha \diamond M_\beta = M_{[\alpha, \beta]} + M_{\alpha \odot \beta}$ , where  $[\alpha, \beta]$  and  $\alpha \odot \beta$  are two compositions defined by

$$\begin{aligned} [(\alpha_1, \alpha_2, \dots, \alpha_\ell), (\beta_1, \beta_2, \dots, \beta_m)] &= (\alpha_1, \alpha_2, \dots, \alpha_\ell, \beta_1, \beta_2, \dots, \beta_m); \\ (\alpha_1, \alpha_2, \dots, \alpha_\ell) \odot (\beta_1, \beta_2, \dots, \beta_m) &= (\alpha_1, \alpha_2, \dots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \dots, \beta_m). \end{aligned}$$

- For any two compositions  $\alpha$  and  $\beta$ , we have  $M_\alpha \ast M_\beta = M_{[\alpha, \beta]}$ .
- For any two compositions  $\alpha$  and  $\beta$ , we have  $F_\alpha \diamond F_\beta = F_{\alpha \odot \beta}$ . (Here,  $\alpha \odot \beta$  is defined to be  $\alpha$  if  $\beta$  is the empty composition, and is defined to be  $\beta$  if  $\alpha$  is the empty composition.)
- For any two compositions  $\alpha$  and  $\beta$ , we have  $F_\alpha \ast F_\beta = F_{[\alpha, \beta]}$ .

Furthermore, we shall use two theorems from [Grinbe16, detailed version, Section 3]:

**Theorem 6.1.** Let  $S$  denote the antipode of the Hopf algebra  $\text{QSym}$ . Let us use Sweedler's notation  $\sum_{(b)} b_{(1)} \otimes b_{(2)}$  for  $\Delta(b)$ , where  $b$  is any element of  $\text{QSym}$ .

Then,

$$\sum_{(b)} \left( S(b_{(1)}) \diamond a \right) b_{(2)} = a \prec b$$

for any  $a \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$  and  $b \in \text{QSym}$ .

**Theorem 6.2.** Let  $S$  denote the antipode of the Hopf algebra  $\text{QSym}$ . Let us use Sweedler's notation  $\sum_{(b)} b_{(1)} \otimes b_{(2)}$  for  $\Delta(b)$ , where  $b$  is any element of  $\text{QSym}$ .

Then,

$$\sum_{(b)} \left( S(b_{(1)}) \ast a \right) b_{(2)} = b \succeq a$$

for any  $a \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$  and  $b \in \text{QSym}$ .

(Notice that Theorem 6.2 differs from [Grinbe16, detailed version, Theorem 3.15] in that we are writing  $b \succeq a$  instead of  $a \preceq b$ . But this is the same thing, since  $a \preceq b = b \succeq a$  for all  $a, b \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$ .)

## 6.2. The dendriform operations on the fundamental basis

Recall Definition 3.1 and Definition 3.2. The following theorem is analogous to Theorem 4.10:

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**Theorem 6.3.** Let  $\pi$  be a nonempty permutation with descent composition  $J$ . Let  $\sigma$  be a nonempty permutation with descent composition  $K$ . Assume that the permutations  $\pi$  and  $\sigma$  are disjoint, and that  $\pi_1 > \sigma_1$ . For any composition  $L$ , let  $c_{J,K}^{L,\prec}$  be the number of permutations with descent composition  $L$  among the left shuffles of  $\pi$  and  $\sigma$ , and let  $c_{J,K}^{L,\succ}$  be the number of permutations with descent composition  $L$  among the right shuffles of  $\pi$  and  $\sigma$ . Then,

$$F_J \prec F_K = \sum_L c_{J,K}^{L,\prec} F_L$$

and

$$F_J \succeq F_K = \sum_L c_{J,K}^{L,\succ} F_L.$$

Note the condition  $\pi_1 > \sigma_1$ , which is not present in Theorem 4.10, and which makes Theorem 6.3 somewhat harder to apply.

Theorem 6.3 can be proven similarly to [GriRei18, (5.2.6)], but it relies on some variants of the disjoint union of two posets. We shall show this proof after first establishing some auxiliary facts.

Throughout this section, the notion of a “labelled poset” will be understood in the sense of [GriRei18, Definition 5.2.1]: Namely, a *labelled poset* simply means a poset whose underlying set is a finite subset of  $\mathbb{Z}$  (but whose order is not necessarily inherited from  $\mathbb{Z}$ ). Thus, a labelled poset is not equipped with a map that serves as its labelling, but instead its underlying set must be a finite set of integers.

Let us first recall a well-known fact about posets:

**Lemma 6.4.** Let  $R$  be a poset. Let  $u$  and  $v$  be two distinct elements of  $R$  such that we don’t have  $u > v$  in  $R$ . Then, there exists a unique poset  $R'$  such that

- we have  $R' = R$  as sets, and
- we have the logical equivalence

$$(x < y \text{ in } R') \iff ((x < y \text{ in } R) \text{ or } (x \leq u \text{ and } v \leq y \text{ in } R))$$

for every two elements  $x$  and  $y$  of  $R'$ .

We say that this poset  $R'$  is obtained by *adding the relation*  $u < v$  to  $R$ . Clearly,  $u < v$  holds in  $R'$ .

If  $R$  is a labelled poset in the sense of [GriRei18, Section 5.2], then  $R'$  is a labelled poset in the sense of [GriRei18, Section 5.2] as well (since  $R' = R$  as sets).

If we add a relation to a labelled poset  $R$ , obtaining a new labelled poset  $R'$ , then how do the  $R'$ -partitions differ from the  $R$ -partitions? The following two



lemmas answer this question:<sup>68</sup>

**Lemma 6.5.** Let  $R$  be a labelled poset. Let  $u$  and  $v$  be two elements of  $R$  such that  $u >_{\mathbb{Z}} v$ . Assume that we don't have  $u > v$  in  $R$ . Let  $f$  be an  $R$ -partition. Let  $R'$  be the labelled poset obtained from  $R$  by adding the relation  $u < v$  to  $R$ . Then,  $f$  is an  $R'$ -partition if and only if  $f(u) < f(v)$ .

*Proof of Lemma 6.5.* The “only if” direction is obvious (since  $u < v$  in  $R'$  and  $u >_{\mathbb{Z}} v$ ). It thus remains to prove the “if” direction. So let us assume that  $f(u) < f(v)$ . We must then show that  $f$  is an  $R'$ -partition.

The poset  $R'$  is obtained from  $R$  by adding the relation  $u < v$  to  $R$ . Thus,  $R' = R$  as sets, and we have the logical equivalence

$$(x < y \text{ in } R') \iff ((x < y \text{ in } R) \text{ or } (x \leq u \text{ and } v \leq y \text{ in } R))$$

for every two elements  $x$  and  $y$  of  $R'$ . Thus, any pair  $(x, y)$  of elements of  $R'$  that satisfies  $x < y$  in  $R'$  must either already satisfy  $x < y$  in  $R$ , or satisfy  $x \leq u$  and  $v \leq y$  in  $R$ .

The map  $f$  is an  $R$ -partition, and thus is weakly increasing as a map from  $R$  to  $\mathbb{Z}$ .

Now, we have

$$(\text{if } i \in R' \text{ and } j \in R' \text{ satisfy } i < j \text{ in } R' \text{ and } i <_{\mathbb{Z}} j, \text{ then } f(i) \leq f(j))$$

<sup>69</sup> and

$$(\text{if } i \in R' \text{ and } j \in R' \text{ satisfy } i < j \text{ in } R' \text{ and } i >_{\mathbb{Z}} j, \text{ then } f(i) < f(j))$$

<sup>70</sup> These two statements entail that  $f$  is an  $R'$ -partition (by the definition of an  $R'$ -partition). This concludes the proof of the “if” direction of Lemma 6.5. Thus, Lemma 6.5 is proven.  $\square$

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<sup>68</sup>We are going to use the following notation: If  $P$  is a poset, then we let  $\leq_P$ ,  $<_P$ ,  $\geq_P$  and  $>_P$  denote the smaller-or-equal relation of  $P$ , the smaller relation of  $P$ , the greater-or-equal relation of  $P$ , and the greater relation of  $P$ , respectively. Thus, in particular,  $\leq_{\mathbb{Z}}$ ,  $<_{\mathbb{Z}}$ ,  $\geq_{\mathbb{Z}}$  and  $>_{\mathbb{Z}}$  denote the usual smaller-or-equal, smaller, greater-or-equal and greater relations of the totally ordered set  $\mathbb{Z}$  of integers. (For example,  $a \leq_{\mathbb{Z}} b$  if and only if  $b - a \in \mathbb{N}$ .)

<sup>69</sup>*Proof.* Let  $i \in R'$  and  $j \in R'$  be such that  $i < j$  in  $R$  and  $i <_{\mathbb{Z}} j$ . We must prove that  $f(i) \leq f(j)$ .

We have  $i \in R' = R$  and  $j \in R' = R$ .

Recall that any pair  $(x, y)$  of elements of  $R'$  that satisfies  $x < y$  in  $R'$  must either already satisfy  $x < y$  in  $R$ , or satisfy  $x \leq u$  and  $v \leq y$  in  $R$ . Applying this to  $(x, y) = (i, j)$ , we conclude that the pair  $(i, j)$  must either already satisfy  $i < j$  in  $R$ , or satisfy  $i \leq u$  and  $v \leq j$  in  $R$ . In the first of these two cases, we immediately obtain  $f(i) \leq f(j)$ , because  $f$  is an  $R$ -partition (and because  $i < j$  in  $R$  and  $i <_{\mathbb{Z}} j$ ). Hence, we can WLOG assume that we are in the second case. In other words, we have  $i \leq u$  and  $v \leq j$  in  $R$ . In particular,  $i \leq u$  in  $R$  and therefore  $f(i) \leq f(u)$  (since the map  $f$  is weakly increasing). Similarly,  $f(v) \leq f(j)$ . Thus,  $f(i) \leq f(u) < f(v) \leq f(j)$ , so that  $f(i) \leq f(j)$ , qed.

<sup>70</sup>*Proof.* Let  $i \in R'$  and  $j \in R'$  be such that  $i < j$  in  $R$  and  $i >_{\mathbb{Z}} j$ . We must prove that  $f(i) < f(j)$ .

We have  $i \in R' = R$  and  $j \in R' = R$ .

Recall that any pair  $(x, y)$  of elements of  $R'$  that satisfies  $x < y$  in  $R'$  must either already

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**Lemma 6.6.** Let  $R$  be a labelled poset. Let  $u$  and  $v$  be two elements of  $R$  such that  $u <_{\mathbb{Z}} v$ . Assume that we don't have  $u > v$  in  $R$ . Let  $f$  be an  $R$ -partition. Let  $R'$  be the labelled poset obtained from  $R$  by adding the relation  $u < v$  to  $R$ . Then,  $f$  is an  $R'$ -partition if and only if  $f(u) \leq f(v)$ .

*Proof of Lemma 6.6.* The “only if” direction is obvious (since  $u < v$  in  $R'$  and  $u <_{\mathbb{Z}} v$ ). It thus remains to prove the “if” direction. So let us assume that  $f(u) \leq f(v)$ . We must then show that  $f$  is an  $R'$ -partition.

The poset  $R'$  is obtained from  $R$  by adding the relation  $u < v$  to  $R$ . Thus,  $R' = R$  as sets, and we have the logical equivalence

$$(x < y \text{ in } R') \iff ((x < y \text{ in } R) \text{ or } (x \leq u \text{ and } v \leq y \text{ in } R))$$

for every two elements  $x$  and  $y$  of  $R'$ . Thus, any pair  $(x, y)$  of elements of  $R'$  that satisfies  $x < y$  in  $R'$  must either already satisfy  $x < y$  in  $R$ , or satisfy  $x \leq u$  and  $v \leq y$  in  $R$ .

The map  $f$  is an  $R$ -partition, and thus is weakly increasing as a map from  $R$  to  $\mathbb{Z}$ .

Now, we have

$$(\text{if } i \in R' \text{ and } j \in R' \text{ satisfy } i < j \text{ in } R' \text{ and } i <_{\mathbb{Z}} j, \text{ then } f(i) \leq f(j))$$

<sup>71</sup> and

$$(\text{if } i \in R' \text{ and } j \in R' \text{ satisfy } i < j \text{ in } R' \text{ and } i >_{\mathbb{Z}} j, \text{ then } f(i) < f(j))$$

<sup>72</sup>. These two statements entail that  $f$  is an  $R'$ -partition (by the definition of an  $R'$ -partition). This concludes the proof of the “if” direction of Lemma 6.6. Thus, Lemma 6.6 is proven.  $\square$

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satisfy  $x < y$  in  $R$ , or satisfy  $x \leq u$  and  $v \leq y$  in  $R$ . Applying this to  $(x, y) = (i, j)$ , we conclude that the pair  $(i, j)$  must either already satisfy  $i < j$  in  $R$ , or satisfy  $i \leq u$  and  $v \leq j$  in  $R$ . In the first of these two cases, we immediately obtain  $f(i) < f(j)$ , because  $f$  is an  $R$ -partition (and because  $i < j$  in  $R$  and  $i >_{\mathbb{Z}} j$ ). Hence, we can WLOG assume that we are in the second case. In other words, we have  $i \leq u$  and  $v \leq j$  in  $R$ . In particular,  $i \leq u$  in  $R$  and therefore  $f(i) \leq f(u)$  (since the map  $f$  is weakly increasing). Similarly,  $f(v) \leq f(j)$ . Thus,  $f(i) \leq f(u) < f(v) \leq f(j)$ , so that  $f(i) < f(j)$ , qed.

<sup>71</sup>*Proof.* Let  $i \in R'$  and  $j \in R'$  be such that  $i < j$  in  $R$  and  $i <_{\mathbb{Z}} j$ . We must prove that  $f(i) \leq f(j)$ .

We have  $i \in R' = R$  and  $j \in R' = R$ .

Recall that any pair  $(x, y)$  of elements of  $R'$  that satisfies  $x < y$  in  $R'$  must either already satisfy  $x < y$  in  $R$ , or satisfy  $x \leq u$  and  $v \leq y$  in  $R$ . Applying this to  $(x, y) = (i, j)$ , we conclude that the pair  $(i, j)$  must either already satisfy  $i < j$  in  $R$ , or satisfy  $i \leq u$  and  $v \leq j$  in  $R$ . In the first of these two cases, we immediately obtain  $f(i) \leq f(j)$ , because  $f$  is an  $R$ -partition (and because  $i < j$  in  $R$  and  $i <_{\mathbb{Z}} j$ ). Hence, we can WLOG assume that we are in the second case. In other words, we have  $i \leq u$  and  $v \leq j$  in  $R$ . In particular,  $i \leq u$  in  $R$  and therefore  $f(i) \leq f(u)$  (since the map  $f$  is weakly increasing). Similarly,  $f(v) \leq f(j)$ . Thus,  $f(i) \leq f(u) \leq f(v) \leq f(j)$ , so that  $f(i) \leq f(j)$ , qed.

<sup>72</sup>*Proof.* Let  $i \in R'$  and  $j \in R'$  be such that  $i < j$  in  $R$  and  $i >_{\mathbb{Z}} j$ . We must prove that  $f(i) < f(j)$ .

Assume the contrary. Thus,  $f(i) \geq f(j)$ , so that  $f(j) \leq f(i)$ .

We have  $i \in R' = R$  and  $j \in R' = R$ .

For our next results, we need the following notation:

**Definition 6.7.** A *minimum element* of a poset  $P$  means an element  $m \in P$  such that every  $p \in P$  satisfies  $m \leq p$ . This is not the same as a minimal element. A minimum element of a poset  $P$  is unique if it exists; it is denoted by  $\min P$ .

This notation overrides the notation  $\min S$  for a nonempty finite subset  $S$  of  $\mathbb{Z}$ . Thus, if  $P$  is a labelled poset that has a minimum element, then  $\min P$  shall denote this minimum element, not the smallest element of  $P$  as a subset of  $\mathbb{Z}$ . (For example, if  $P$  is the labelled poset  $\{6 < 2 < 3 < 4\}$ , then  $\min P$  shall mean 6, not 2.)

The following fact is an analogue of [GriRei18, Lemma 5.2.17]:

**Proposition 6.8.** We shall use the notations of [GriRei18, Section 5.2]. Let  $P$  and  $Q$  be two disjoint labelled posets, each of which has a minimum element. Assume that  $\min P >_{\mathbb{Z}} \min Q$ . Consider the disjoint union  $P \sqcup Q$  of  $P$  and  $Q$ .

(a) Add a further relation  $\min P < \min Q$  to  $P \sqcup Q$ ; denote the resulting labelled poset by  $P \prec Q$ . Then,  $F_P(\mathbf{x}) \prec F_Q(\mathbf{x}) = F_{P \prec Q}(\mathbf{x})$ .

(b) Add a further relation  $\min P > \min Q$  to  $P \sqcup Q$ ; denote the resulting labelled poset by  $P \succeq Q$ . Then,  $F_P(\mathbf{x}) \succeq F_Q(\mathbf{x}) = F_{P \succeq Q}(\mathbf{x})$ .

Let us recall that (as agreed in Definition 6.7) we are using the notation  $\min P$  for the minimum element of the **poset**  $P$ ; not the smallest element of  $P$  as a subset of  $\mathbb{Z}$ . The same applies to the notation  $\min Q$ .

*Proof of Proposition 6.8 (sketched).* We imitate the proof of [GriRei18, Lemma 5.2.17]:

(a) If  $f : P \sqcup Q \rightarrow \{1, 2, 3, \dots\}$  is a  $P \prec Q$ -partition, then its restrictions  $f|_P$  and  $f|_Q$  are a  $P$ -partition and a  $Q$ -partition, respectively, and have the property

Recall that any pair  $(x, y)$  of elements of  $R'$  that satisfies  $x < y$  in  $R'$  must either already satisfy  $x < y$  in  $R$ , or satisfy  $x \leq u$  and  $v \leq y$  in  $R$ . Applying this to  $(x, y) = (i, j)$ , we conclude that the pair  $(i, j)$  must either already satisfy  $i < j$  in  $R$ , or satisfy  $i \leq u$  and  $v \leq j$  in  $R$ . In the first of these two cases, we immediately obtain  $f(i) < f(j)$ , because  $f$  is an  $R$ -partition (and because  $i < j$  in  $R$  and  $i >_{\mathbb{Z}} j$ ). Hence, we can WLOG assume that we are in the second case. In other words, we have  $i \leq u$  and  $v \leq j$  in  $R$ . In particular,  $i \leq u$  in  $R$  and therefore  $f(i) \leq f(u)$  (since the map  $f$  is weakly increasing). Similarly,  $f(v) \leq f(j)$ . Thus,  $f(i) \leq f(u) \leq f(v) \leq f(j) \leq f(i)$ . All inequality signs in this chain of inequalities must be equalities (since its left and right hand sides are equal). In other words, we have  $f(i) = f(u) = f(v) = f(j) = f(i)$ .

If we had  $i >_{\mathbb{Z}} u$ , then we would have  $i \neq u$  and therefore  $i < u$  in  $R$  (since  $i \leq u$  in  $R$ ). Therefore, if we had  $i >_{\mathbb{Z}} u$ , then we would have  $f(i) < f(u)$  (since  $i < u$  in  $R$ , but  $f$  is an  $R$ -partition), which would contradict  $f(i) = f(u)$ . Hence, we cannot have  $i >_{\mathbb{Z}} u$ . Thus,  $i \leq_{\mathbb{Z}} u$ .

If we had  $v >_{\mathbb{Z}} j$ , then we would have  $v \neq j$  and therefore  $v < j$  in  $R$  (since  $v \leq j$  in  $R$ ). Therefore, if we had  $v >_{\mathbb{Z}} j$ , then we would have  $f(v) < f(j)$  (since  $v < j$  in  $R$ , but  $f$  is an  $R$ -partition), which would contradict  $f(v) = f(j)$ . Hence, we cannot have  $v >_{\mathbb{Z}} j$ . Thus,  $v \leq_{\mathbb{Z}} j$ .

Now,  $i \leq_{\mathbb{Z}} u <_{\mathbb{Z}} v \leq_{\mathbb{Z}} j$ . This contradicts  $i >_{\mathbb{Z}} j$ . This contradiction shows that our assumption was wrong, qed.

that  $(f|_P)(\min P) < (f|_Q)(\min Q)$  (indeed, this property must hold because  $\min P < \min Q$  in  $P \prec Q$  but  $\min P >_{\mathbb{Z}} \min Q$ ). Conversely, any pair of a  $P$ -partition  $g$  and a  $Q$ -partition  $h$  having the property that  $g(\min P) < h(\min Q)$  can be combined to form a  $P \prec Q$ -partition<sup>73</sup>.

Using the notations of [GriRei18, Definition 5.2.1], we have

$$\sum_{f \text{ is a } P \prec Q\text{-partition}} \mathbf{x}_f = \sum_{\substack{g \text{ is a } P\text{-partition;} \\ h \text{ is a } Q\text{-partition;} \\ g(\min P) < h(\min Q)}} \mathbf{x}_g \mathbf{x}_h \quad (78)$$

(since the previous two sentences establish a bijection between the addends on the left hand side of this equality and the addends on its right hand side).

But the definitions of  $F_P(\mathbf{x})$  and  $F_Q(\mathbf{x})$  yield  $F_P(\mathbf{x}) = \sum_{g \text{ is a } P\text{-partition}} \mathbf{x}_g$  and  $F_Q(\mathbf{x}) = \sum_{h \text{ is a } Q\text{-partition}} \mathbf{x}_h$ . Thus,

$$\begin{aligned} & F_P(\mathbf{x}) \prec F_Q(\mathbf{x}) \\ &= \left( \sum_{g \text{ is a } P\text{-partition}} \mathbf{x}_g \right) \prec \left( \sum_{h \text{ is a } Q\text{-partition}} \mathbf{x}_h \right) \\ &= \sum_{\substack{g \text{ is a } P\text{-partition;} \\ h \text{ is a } Q\text{-partition}}} \underbrace{\mathbf{x}_g \prec \mathbf{x}_h} \\ &= \sum_{\substack{g \text{ is a } P\text{-partition;} \\ h \text{ is a } Q\text{-partition}}} \begin{cases} \mathbf{x}_g \mathbf{x}_h, & \text{if } \min(\text{Supp}(\mathbf{x}_g)) < \min(\text{Supp}(\mathbf{x}_h)); \\ 0, & \text{if } \min(\text{Supp}(\mathbf{x}_g)) \geq \min(\text{Supp}(\mathbf{x}_h)) \end{cases} \\ & \quad \text{(by the definition of } \mathbf{x}_g \prec \mathbf{x}_h) \\ &= \sum_{\substack{g \text{ is a } P\text{-partition;} \\ h \text{ is a } Q\text{-partition}}} \underbrace{\begin{cases} \mathbf{x}_g \mathbf{x}_h, & \text{if } \min(\text{Supp}(\mathbf{x}_g)) < \min(\text{Supp}(\mathbf{x}_h)); \\ 0, & \text{if } \min(\text{Supp}(\mathbf{x}_g)) \geq \min(\text{Supp}(\mathbf{x}_h)) \end{cases}} \\ & \quad = \begin{cases} \mathbf{x}_g \mathbf{x}_h, & \text{if } g(\min P) < h(\min Q); \\ 0, & \text{if } g(\min P) \geq h(\min Q) \end{cases} \\ & \quad \text{(since } \min(\text{Supp}(\mathbf{x}_g)) = g(\min P) \text{ (because the map } g \text{ is a } P\text{-partition, thus} \\ & \quad \text{weakly increasing on } P, \text{ and therefore } g(\min P) = \min(g(P)) = \min(\text{Supp}(\mathbf{x}_g))) \\ & \quad \text{and similarly } \min(\text{Supp}(\mathbf{x}_h)) = h(\min Q)) \\ &= \sum_{\substack{g \text{ is a } P\text{-partition;} \\ h \text{ is a } Q\text{-partition}}} \begin{cases} \mathbf{x}_g \mathbf{x}_h, & \text{if } g(\min P) < h(\min Q); \\ 0, & \text{if } g(\min P) \geq h(\min Q) \end{cases} \\ &= \sum_{\substack{g \text{ is a } P\text{-partition;} \\ h \text{ is a } Q\text{-partition;} \\ g(\min P) < h(\min Q)}} \mathbf{x}_g \mathbf{x}_h = \sum_{f \text{ is a } P \prec Q\text{-partition}} \mathbf{x}_f \quad \text{(by (78))} \\ &= F_{P \prec Q}(\mathbf{x}) \end{aligned}$$

(by the definition of  $F_{P \prec Q}(\mathbf{x})$ ). This proves Proposition 6.8 (a).

<sup>73</sup>To see this, we need to apply Lemma 6.5 to  $R = P \sqcup Q$ ,  $u = \min P$ ,  $v = \min Q$  and  $R' = P \prec Q$ .

(b) Proposition 6.8 (b) is proven similarly to Proposition 6.8 (a) (but this time we need to use Lemma 6.6 instead of Lemma 6.5).  $\square$

Also, the following simple fact is used:

**Lemma 6.9.** We shall use the notations of [GriRei18, Section 5.2]. Let  $P$  and  $Q$  be two disjoint posets, each of which has a minimum element. Consider the disjoint union  $P \sqcup Q$  of  $P$  and  $Q$  as the set-theoretic union  $P \cup Q$ . Assume that  $P$  and  $Q$  are subsets of  $\mathbb{P}$  (the set of positive integers); thus, any linear extension of  $P$  or of  $Q$  or of  $P \sqcup Q$  is a permutation (a word with letters in  $\mathbb{P}$ ).

(a) Add a further relation  $\min P < \min Q$  to  $P \sqcup Q$ ; denote the resulting poset by  $P \prec Q$ . Then,

$$\mathcal{L}(P \prec Q) = \bigsqcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma).$$

(b) Add a further relation  $\min P > \min Q$  to  $P \sqcup Q$ ; denote the resulting poset by  $P \succeq Q$ . Then,

$$\mathcal{L}(P \succeq Q) = \bigsqcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\succ}(\pi, \sigma).$$

*Proof of Lemma 6.9 (sketched).* Recall that we regard linear extensions of a finite poset  $R$  as lists of elements of  $R$ . Thus, if  $R$  is a finite poset, if  $u$  and  $v$  are two elements of  $R$ , and if  $w$  is a linear extension of  $R$ , then we have  $u < v$  in  $w$  if and only if  $u$  appears before  $v$  in the list  $w$ .

We shall also use the following notation: If  $w$  is a list of elements of some set  $U$ , and if  $V$  is a subset of  $U$ , then  $w \upharpoonright_V$  means the result of removing all entries from  $w$  that don't belong to  $V$ . For example,  $(2, 7, 1, 6, 3, 4) \upharpoonright_{\{1, 2, 4, 5, 6\}} = (2, 1, 6, 4)$ .

Now, consider our two posets  $P$  and  $Q$ . Recall that a linear extension of  $P \sqcup Q$  is simply a list  $(w_1, w_2, \dots, w_n)$  of all elements of  $P \sqcup Q$  such that no two integers  $i < j$  satisfy  $w_i \geq w_j$  in  $P \sqcup Q$ . In other words, a linear extension of  $P \sqcup Q$  is a list  $w$  of all elements of  $P \sqcup Q$  such that if  $x$  and  $y$  are two elements of  $P \sqcup Q$  satisfying  $x < y$  in  $P \sqcup Q$ , then  $x$  appears before<sup>74</sup>  $y$  in the list  $w$ . By the definition of  $P \sqcup Q$ , this rewrites as follows: A linear extension of  $P \sqcup Q$  is a list  $w$  of all elements of  $P \sqcup Q$  with the following two properties:

- If  $x$  and  $y$  are two elements of  $P$  satisfying  $x < y$  in  $P$ , then  $x$  appears before  $y$  in the list  $w$ .
- If  $x$  and  $y$  are two elements of  $Q$  satisfying  $x < y$  in  $Q$ , then  $x$  appears before  $y$  in the list  $w$ .

<sup>74</sup>“Before” doesn't imply “immediately before”. For example, 2 appears before 4 in the list  $(1, 2, 3, 4)$ .

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In other words, a linear extension  $P \sqcup Q$  is a list  $w$  of all elements of  $P \sqcup Q$  such that  $w \upharpoonright_P$  is a linear extension of  $P$  and  $w \upharpoonright_Q$  is a linear extension of  $Q$ .

This yields the following:

- If  $w$  is a linear extension of  $P \sqcup Q$ , then  $w \upharpoonright_P$  is a linear extension of  $P$ , and  $w \upharpoonright_Q$  is a linear extension of  $Q$ , and we have  $w \in S(w \upharpoonright_P, w \upharpoonright_Q)$  (since both  $w \upharpoonright_P$  and  $w \upharpoonright_Q$  are subsequences of  $w$ , and their sizes add up to the size of  $w$ ). Therefore, if  $w$  is a linear extension of  $P \sqcup Q$ , then  $w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S(\pi, \sigma)$  (because  $w \in S(\pi, \sigma)$  for  $\pi = w \upharpoonright_P \in \mathcal{L}(P)$  and  $\sigma = w \upharpoonright_Q \in \mathcal{L}(Q)$ ).
- Conversely, any  $w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S(\pi, \sigma)$  is a linear extension of  $P \sqcup Q$ .

Combining these two facts, we conclude the following:

*Observation 1:* The linear extensions of  $P \sqcup Q$  are precisely the elements of  $\bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S(\pi, \sigma)$ .

Next, we notice the following:

*Observation 2:* The union  $\bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S(\pi, \sigma)$  is a disjoint union (i.e., the sets  $S(\pi, \sigma)$  for distinct pairs  $(\pi, \sigma)$  are disjoint).

[*Proof of Observation 2:* If we are given an element  $w \in S(\pi, \sigma)$  for some  $\pi \in \mathcal{L}(P)$  and  $\sigma \in \mathcal{L}(Q)$ , then we can uniquely reconstruct  $(\pi, \sigma)$  from  $w$  (namely,  $(\pi, \sigma)$  is given by  $\pi = w \upharpoonright_P$  and  $\sigma = w \upharpoonright_Q$ ). Thus, the sets  $S(\pi, \sigma)$  for distinct pairs  $(\pi, \sigma)$  are disjoint. This proves Observation 2.]

We will furthermore need a simple auxiliary claim:

*Observation 3:* Let  $R$  be a finite poset. Let  $u$  and  $v$  be two elements of  $R$ . Assume that we don't have  $u > v$  in  $R$ . Let  $w$  be a linear extension of  $R$ . Let  $R'$  be the poset obtained from  $R$  by adding the relation  $u < v$  to  $R$ . Then,  $w$  is a linear extension of  $R'$  if and only if we have  $u < v$  in  $w$ .

[*Proof of Observation 3:* The "only if" direction is obvious. The "if" direction is easily derived from the definition of a linear extension, once you recall that any pair  $(x, y)$  of elements of  $R'$  that satisfies  $x < y$  in  $R'$  must either already satisfy  $x < y$  in  $R$ , or satisfy  $x \leq u$  and  $v \leq y$  in  $R$ . The details are left to the reader.]

(a) Observation 2 shows that the union  $\bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S(\pi, \sigma)$  is a disjoint union. Hence, the union  $\bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma)$  is a disjoint union as well (since  $S_{\prec}(\pi, \sigma)$  is a subset of  $S(\pi, \sigma)$  for all  $\pi$  and  $\sigma$ ).

We don't have  $\min P > \min Q$  in  $P \sqcup Q$ . Thus, applying Observation 3 to  $R = P \sqcup Q$ ,  $u = \min P$ ,  $v = \min Q$  and  $R' = P \prec Q$ , we obtain the following:

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*Observation 4:* Let  $w$  be a linear extension of  $P \sqcup Q$ . Then,  $w$  is a linear extension of  $P \prec Q$  if and only if we have  $\min P < \min Q$  in  $w$ .

Next, we claim that

$$\mathcal{L}(P \prec Q) \subseteq \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma). \quad (79)$$

[*Proof:* Let  $w \in \mathcal{L}(P \prec Q)$ . Thus,  $w$  is a linear extension of  $P \prec Q$ . Hence,  $w$  is a linear extension of  $P \sqcup Q$ , and we have  $\min P < \min Q$  in  $w$  (by Observation 4). Since  $w$  is a linear extension of  $P \sqcup Q$ , we have  $w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S(\pi, \sigma)$  (by Observation 1). In other words,  $w \in S(\pi, \sigma)$  for some  $\pi \in \mathcal{L}(P)$  and  $\sigma \in \mathcal{L}(Q)$ . Consider these  $\pi$  and  $\sigma$ .

The element  $\min P$  is the minimum element of the poset  $P$ , and thus must be the first letter of  $\pi$  (since  $\pi$  is a linear extension of  $P$ ). Similarly,  $\min Q$  must be the first letter of  $\sigma$ . But  $w \in S(\pi, \sigma)$ . Hence, the first letter of  $w$  is either the first letter of  $\pi$  or the first letter of  $\sigma$ .

We have  $\min P < \min Q$  in  $w$ . In other words,  $\min P$  appears before  $\min Q$  in the list  $w$ . Hence,  $\min Q$  cannot be the first letter of  $w$ . In other words, the first letter of  $w$  cannot be  $\min Q$ . In other words, the first letter of  $w$  cannot be the first letter of  $\sigma$  (since  $\min Q$  is the first letter of  $\sigma$ ). Thus, the first letter of  $w$  is the first letter of  $\pi$  (since the first letter of  $w$  is either the first letter of  $\pi$  or the first letter of  $\sigma$ ). In other words,  $w$  is a left shuffle of  $\pi$  and  $\sigma$  (since  $w \in S(\pi, \sigma)$ ). In other words,  $w \in S_{\prec}(\pi, \sigma)$  (since  $S_{\prec}(\pi, \sigma)$  is the set of all left shuffles of  $\pi$  and  $\sigma$ ).

Now, forget that we have defined  $\pi$  and  $\sigma$ . We thus have shown that  $w \in S_{\prec}(\pi, \sigma)$  for some  $\pi \in \mathcal{L}(P)$  and  $\sigma \in \mathcal{L}(Q)$ . Thus,  $w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma)$ . Since we have proven this for any  $w \in \mathcal{L}(P \prec Q)$ , we thus have proven (79).]

On the other hand, we claim that

$$\bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma) \subseteq \mathcal{L}(P \prec Q). \quad (80)$$

[*Proof:* Let  $w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma)$ .

We have

$$w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} \underbrace{S_{\prec}(\pi, \sigma)}_{\subseteq S(\pi, \sigma)} \subseteq \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S(\pi, \sigma).$$

Thus,  $w$  is a linear extension of  $P \sqcup Q$  (by Observation 1).

Also,  $w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma)$ . Thus,  $w \in S_{\prec}(\pi, \sigma)$  for some  $\pi \in \mathcal{L}(P)$  and  $\sigma \in \mathcal{L}(Q)$ . Consider these  $\pi$  and  $\sigma$ . From  $w \in S_{\prec}(\pi, \sigma)$ , we conclude that  $w$  is a left shuffle of  $\pi$  and  $\sigma$ . In other words,  $w$  is a shuffle of  $\pi$  and  $\sigma$  such that the first letter of  $w$  is the first letter of  $\pi$ .

The element  $\min P$  is the minimum element of the poset  $P$ , and thus must be the first letter of  $\pi$  (since  $\pi$  is a linear extension of  $P$ ). In other words,  $\min P$

is the first letter of  $w$  (since the first letter of  $w$  is the first letter of  $\pi$ ). Hence,  $\min P$  appears before  $\min Q$  in the list  $w$  (since  $\min P$  and  $\min Q$  are two distinct letters of  $w$ ). In other words, we have  $\min P < \min Q$  in  $w$ . Hence, Observation 4 yields that  $w$  is a linear extension of  $P \prec Q$  (since  $w$  is a linear extension of  $P \sqcup Q$ ). In other words,  $w \in \mathcal{L}(P \prec Q)$ . Since we have proven this for any  $w \in \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma)$ , we thus have proven (80).]

Combining (79) and (80), we obtain

$$\mathcal{L}(P \prec Q) = \bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma) = \bigsqcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma)$$

(since the union  $\bigcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma)$  is a disjoint union). This proves Lemma 6.9 (a).

(b) Lemma 6.9 (b) is proven similarly to Lemma 6.9 (a).  $\square$

We can now prove Theorem 6.3. First, let us rewrite Theorem 6.3 as follows:<sup>75</sup>

**Corollary 6.10.** Let  $\pi$  and  $\sigma$  be two disjoint nonempty permutations. Assume that  $\pi_1 > \sigma_1$ . Then,

$$F_{\text{Comp } \pi} \prec F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\prec}(\pi, \sigma)} F_{\text{Comp } \chi}$$

and

$$F_{\text{Comp } \pi} \succeq F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\succ}(\pi, \sigma)} F_{\text{Comp } \chi}.$$

*Proof of Corollary 6.10 (sketched).* Let  $n = |\pi|$  and  $m = |\sigma|$ . We shall use the notations of [GriRei18, Section 5.2]; in particular, “labelled poset” will be defined as in [GriRei18, Definition 5.2.1].

If  $R$  is a labelled poset, and if  $w$  is a linear extension of  $R$ , then  $w$  can be regarded as a labelled poset itself, but also as a permutation (since  $w$  is a list of distinct elements of  $R$ , and thus a word over the alphabet  $\mathbb{P}$  with no two equal letters). The first interpretation (as a labelled poset) gives rise to a quasisymmetric function  $F_w(\mathbf{x})$  (defined as in [GriRei18, Definition 5.2.1]). The second interpretation (as a permutation) leads to a composition  $\text{Comp } w$ . These two objects are connected by the equality

$$F_w(\mathbf{x}) = F_{\text{Comp } w}. \quad (81)$$

(This follows from [GriRei18, Proposition 5.2.10]; but keep in mind that [GriRei18, Proposition 5.2.10] denotes  $F_{\text{Comp } \pi}$  by  $L_\alpha$  in this context.)

Let  $P$  be the labelled poset whose elements are  $\pi_1, \pi_2, \dots, \pi_n$  and whose order is the total order given by  $\pi_1 < \pi_2 < \dots < \pi_n$ . Thus,  $P$  is totally ordered, and

<sup>75</sup>Recall that for any permutation  $\varphi$ , we have let  $\text{Comp } \varphi$  denote the descent composition of  $\varphi$ .



its minimum element is  $\min P = \pi_1$ . Also, [GriRei18, Proposition 5.2.10] yields  $F_P(\mathbf{x}) = F_{\text{Comp } \pi}$ . (Keep in mind that [GriRei18, Proposition 5.2.10] denotes  $F_{\text{Comp } \pi}$  by  $L_\alpha$  in this context.)

Let  $Q$  be the labelled poset whose elements are  $\sigma_1, \sigma_2, \dots, \sigma_m$  and whose order is the total order given by  $\sigma_1 < \sigma_2 < \dots < \sigma_m$ . Thus,  $Q$  is totally ordered, and its minimum element is  $\min Q = \sigma_1$ . Also, [GriRei18, Proposition 5.2.10] yields  $F_Q(\mathbf{x}) = F_{\text{Comp } \sigma}$ .

The posets  $P$  and  $Q$  are disjoint (since the permutations  $\pi$  and  $\sigma$  are disjoint). Define three labelled posets  $P \sqcup Q$ ,  $P \prec Q$  and  $P \succ Q$  as in Proposition 6.8.

Lemma 6.9 (a) yields  $\mathcal{L}(P \prec Q) = \bigsqcup_{\pi' \in \mathcal{L}(P); \sigma' \in \mathcal{L}(Q)} S_{\prec}(\pi', \sigma')$  (where we are using the letters  $\pi'$  and  $\sigma'$  for our subscripts, since the letters  $\pi$  and  $\sigma$  are already taken). But  $P$  is totally ordered; thus, there exists only one linear extension  $\pi' \in \mathcal{L}(P)$ , namely,  $\pi' = \pi$ . In other words,  $\mathcal{L}(P) = \{\pi\}$ . Similarly,  $\mathcal{L}(Q) = \{\sigma\}$ . Hence,

$$\begin{aligned} \mathcal{L}(P \prec Q) &= \bigsqcup_{\pi' \in \mathcal{L}(P); \sigma' \in \mathcal{L}(Q)} S_{\prec}(\pi', \sigma') \\ &= \bigsqcup_{\pi' \in \{\pi\}; \sigma' \in \{\sigma\}} S_{\prec}(\pi', \sigma') \quad \left( \begin{array}{l} \text{since } \mathcal{L}(P) = \{\pi\} \\ \text{and } \mathcal{L}(Q) = \{\sigma\} \end{array} \right) \\ &= S_{\prec}(\pi, \sigma). \end{aligned} \tag{82}$$

Recall that every labelled poset  $R$  satisfies

$$F_R(\mathbf{x}) = \sum_{w \in \mathcal{L}(R)} F_w(\mathbf{x}) \tag{83}$$

(by [GriRei18, Theorem 5.2.11]).

Now,  $\pi_1 > \sigma_1$  in  $\mathbb{Z}$ . In other words,  $\pi_1 >_{\mathbb{Z}} \sigma_1$ . In other words,  $\min P >_{\mathbb{Z}} \min Q$  (since  $\min P = \pi_1$  and  $\min Q = \sigma_1$ ). Thus,

$$\begin{aligned} &\underbrace{F_{\text{Comp } \pi}}_{=F_P(\mathbf{x})} \prec \underbrace{F_{\text{Comp } \sigma}}_{=F_Q(\mathbf{x})} \\ &= F_P(\mathbf{x}) \prec F_Q(\mathbf{x}) = F_{P \prec Q}(\mathbf{x}) \quad (\text{by Proposition 6.8 (a)}) \\ &= \sum_{w \in \mathcal{L}(P \prec Q)} F_w(\mathbf{x}) \quad (\text{by (83)}) \\ &= \sum_{w \in S_{\prec}(\pi, \sigma)} \underbrace{F_w(\mathbf{x})}_{=F_{\text{Comp } w}} \quad (\text{by (82)}) \\ &\quad (\text{by (81)}) \\ &= \sum_{w \in S_{\prec}(\pi, \sigma)} F_{\text{Comp } w} = \sum_{\chi \in S_{\prec}(\pi, \sigma)} F_{\text{Comp } \chi}. \end{aligned}$$

A similar argument (using Proposition 6.8 (b) and Lemma 6.9 (b) instead of Proposition 6.8 (a) and Lemma 6.9 (a)) shows that

$$F_{\text{Comp } \pi} \succeq F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\succ}(\pi, \sigma)} F_{\text{Comp } \chi}.$$

Thus, Corollary 6.10 is proven.  $\square$

Hence, Theorem 6.3 is proven as well (since Corollary 6.10 was just a restatement of Theorem 6.3).

### 6.3. Ideals

**Definition 6.11.** Let  $A$  be a  $\mathbf{k}$ -module equipped with some binary operation  $*$  (written infix).

(a) If  $B$  and  $C$  are two  $\mathbf{k}$ -submodules of  $A$ , then  $B * C$  shall mean the  $\mathbf{k}$ -submodule of  $A$  spanned by all elements of the form  $b * c$  with  $b \in B$  and  $c \in C$ .

(b) A  $\mathbf{k}$ -submodule  $M$  of  $A$  is said to be a *left  $*$ -ideal* if and only if it satisfies  $A * M \subseteq M$ .

(c) A  $\mathbf{k}$ -submodule  $M$  of  $A$  is said to be a *right  $*$ -ideal* if and only if it satisfies  $M * A \subseteq M$ .

(d) A  $\mathbf{k}$ -submodule  $M$  of  $A$  is said to be a  *$*$ -ideal* if and only if it is both a left  $*$ -ideal and a right  $*$ -ideal.

**Theorem 6.12.** Let  $M$  be an ideal of  $\text{QSym}$ . Let  $A = \text{QSym}$ .

(a) If  $A \phi M \subseteq M$ , then  $M \prec A \subseteq M$ .

(b) If  $A \varkappa M \subseteq M$ , then  $A \succeq M \subseteq M$ .

(c) If  $A \varkappa M \subseteq M$  and  $A \phi M \subseteq M$ , then  $M$  is a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

*Proof of Theorem 6.12.* (a) Assume that  $A \phi M \subseteq M$ . If  $a \in M$  and  $b \in A$ , then

$$\begin{aligned} a \prec b &= \sum_{(b)} \left( \underbrace{S(b_{(1)})}_{\in A} \phi \underbrace{a}_{\in M} \right) \underbrace{b_{(2)}}_{\in A} && \text{(by Theorem 6.1)} \\ &\in \underbrace{(A \phi M)}_{\subseteq M} A \subseteq MA \subseteq M && \text{(since } M \text{ is an ideal of } A). \end{aligned}$$

Thus,  $M \prec A \subseteq M$ . This proves Theorem 6.12 (a).

(b) Assume that  $A \varkappa M \subseteq M$ . If  $a \in M$  and  $b \in A$ , then

$$\begin{aligned} b \succeq a &= \sum_{(b)} \left( \underbrace{S(b_{(1)})}_{\in A} \varkappa \underbrace{a}_{\in M} \right) \underbrace{b_{(2)}}_{\in A} && \text{(by Theorem 6.2)} \\ &\in \underbrace{(A \varkappa M)}_{\subseteq M} A \subseteq MA \subseteq M && \text{(since } M \text{ is an ideal of } A). \end{aligned}$$

Thus,  $A \succeq M \subseteq M$ . This proves Theorem 6.12 (b).

(c) Assume that  $A \bowtie M \subseteq M$  and  $A \phi M \subseteq M$ . Then, Theorem 6.12 (b) yields  $A \succeq M \subseteq M$ . Thus,  $M$  is a left  $\succeq$ -ideal.

Now, any  $b \in M$  and  $a \in A$  satisfy

$$\begin{aligned} a \prec b &= \underbrace{a}_{\in A} \underbrace{b}_{\in M} - \underbrace{a}_{\in A} \succeq \underbrace{b}_{\in M} && \text{(by (73))} \\ &\in \underbrace{AM}_{\subseteq M} - \underbrace{A \succeq M}_{\subseteq M} \subseteq M - M \subseteq M. \\ &\text{(since } M \text{ is an ideal of } A) \end{aligned}$$

In other words,  $A \prec M \subseteq M$ . In other words,  $M$  is a left  $\prec$ -ideal.

But Theorem 6.12 (a) yields  $M \prec A \subseteq M$ . In other words,  $M$  is a right  $\prec$ -ideal.

Any  $a \in M$  and  $b \in A$  satisfy

$$\begin{aligned} a \succeq b &= \underbrace{a}_{\in M} \underbrace{b}_{\in A} - \underbrace{a}_{\in M} \prec \underbrace{b}_{\in A} && \text{(by (73))} \\ &\in \underbrace{MA}_{\subseteq M} - \underbrace{M \prec A}_{\subseteq M} \subseteq M - M \subseteq M. \\ &\text{(since } M \text{ is an ideal of } A) \end{aligned}$$

In other words,  $M \succeq A \subseteq M$ . In other words,  $M$  is a right  $\succeq$ -ideal.

Hence,  $M$  is a  $\prec$ -ideal (since  $M$  is a left  $\prec$ -ideal and a right  $\prec$ -ideal) and a  $\succeq$ -ideal (since  $M$  is a left  $\succeq$ -ideal and a right  $\succeq$ -ideal). This proves Theorem 6.12 (c).  $\square$

Another simple fact is the following:

**Proposition 6.13.** Let  $M$  be simultaneously a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ . Then,  $M$  is an ideal of  $\text{QSym}$ .

*Proof of Proposition 6.13.* Any  $a \in M$  and  $b \in \text{QSym}$  satisfy

$$\begin{aligned} ab &= \underbrace{a}_{\in M} \prec \underbrace{b}_{\in \text{QSym}} + \underbrace{a}_{\in M} \succeq \underbrace{b}_{\in \text{QSym}} && \text{(by (73))} \\ &\in \underbrace{M \prec \text{QSym}}_{\subseteq M} + \underbrace{M \succeq \text{QSym}}_{\subseteq M} \subseteq M + M \subseteq M. \\ &\text{(since } M \text{ is a } \prec\text{-ideal of } \text{QSym}) \quad \text{(since } M \text{ is a } \succeq\text{-ideal of } \text{QSym}) \end{aligned}$$

In other words,  $M$  is an ideal of  $\text{QSym}$ . This proves Proposition 6.13.  $\square$

**Question 6.14.** Proposition 6.13 says that if a  $\mathbb{Q}$ -vector subspace  $M$  of  $\text{QSym}$  is simultaneously a  $\prec$ -ideal and a  $\succeq$ -ideal, then it is also an ideal. Similarly, if  $M$  is an ideal and a  $\prec$ -ideal, then it is a  $\succeq$ -ideal. Can we state any other such criteria?

## 6.4. Application to $\mathcal{K}_{\text{Epk}}$

We now claim the following:

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**Theorem 6.15.** The ideal  $\mathcal{K}_{\text{Epk}}$  of  $\text{QSym}$  is a  $\ast$ -ideal, a  $\phi$ -ideal, a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

*Proof of Theorem 6.15.* Let  $A = \text{QSym}$ . Corollary 5.4 shows that  $\mathcal{K}_{\text{Epk}}$  is an ideal of  $\text{QSym}$ .

Let us recall the binary relation  $\rightarrow$  on the set of compositions defined in Proposition 5.5.

*Claim 1:* Let  $J$  and  $K$  be two compositions satisfying  $J \rightarrow K$ . Let  $G$  be a further composition. Then,  $[G, J] \rightarrow [G, K]$ .

[*Proof of Claim 1:* Write the composition  $J$  in the form  $J = (j_1, j_2, \dots, j_m)$ . Write the composition  $G$  in the form  $G = (g_1, g_2, \dots, g_p)$ .

We have  $J \rightarrow K$ . In other words, there exists an  $\ell \in \{2, 3, \dots, m\}$  such that  $j_\ell > 2$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$  (by the definition of the relation  $\rightarrow$ ). Consider this  $\ell$ . Clearly,  $\ell > 1$  (since  $\ell \in \{2, 3, \dots, m\}$ ), so that  $p + \ell > \underbrace{p}_{\geq 0} + 1 \geq 1$ .

From  $G = (g_1, g_2, \dots, g_p)$  and  $J = (j_1, j_2, \dots, j_m)$ , we obtain

$$[G, J] = (g_1, g_2, \dots, g_p, j_1, j_2, \dots, j_m). \quad (84)$$

From  $G = (g_1, g_2, \dots, g_p)$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$ , we obtain

$$[G, K] = (g_1, g_2, \dots, g_p, j_1, j_2, \dots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m). \quad (85)$$

From looking at (84) and (85), we conclude immediately that the composition  $[G, K]$  is obtained from  $[G, J]$  by “splitting” the entry  $j_\ell > 2$  into two consecutive entries 1 and  $j_\ell - 1$ , and that this entry  $j_\ell$  was not the first entry (indeed, this entry is the  $(p + \ell)$ -th entry, but  $p + \ell > 1$ ). Hence,  $[G, J] \rightarrow [G, K]$  (by the definition of the relation  $\rightarrow$ ). This proves Claim 1.]

*Claim 2:* We have  $A \ast \mathcal{K}_{\text{Epk}} \subseteq \mathcal{K}_{\text{Epk}}$ .

[*Proof of Claim 2:* We must show that  $a \ast m \in \mathcal{K}_{\text{Epk}}$  for every  $a \in A$  and  $m \in \mathcal{K}_{\text{Epk}}$ . So let us fix  $a \in A$  and  $m \in \mathcal{K}_{\text{Epk}}$ .

Proposition 5.5 shows that the  $\mathbb{Q}$ -vector space  $\mathcal{K}_{\text{Epk}}$  is spanned by all differences of the form  $F_J - F_K$ , where  $J$  and  $K$  are two compositions satisfying  $J \rightarrow K$ . Hence, we can WLOG assume that  $m$  is such a difference (because the relation  $a \ast m \in \mathcal{K}_{\text{Epk}}$ , which we must prove, is  $\mathbb{Q}$ -linear in  $m$ ). Assume this. Thus,  $m = F_J - F_K$  for some two compositions  $J$  and  $K$  satisfying  $J \rightarrow K$ . Consider these  $J$  and  $K$ .

From  $J \rightarrow K$ , we easily conclude that the composition  $J$  is nonempty. Thus,  $|J| \neq 0$ . But from  $J \rightarrow K$ , we also obtain  $|J| = |K|$ . Hence,  $|K| = |J| \neq 0$ . Thus, the composition  $K$  is nonempty.

Recall that the family  $(F_L)_{L \text{ is a composition}}$  is a basis of the  $\mathbb{Q}$ -vector space  $\text{QSym} = A$ . Hence, we can WLOG assume that  $a$  belongs to this family (since the relation  $a \ast m \in \mathcal{K}_{\text{Epk}}$ , which we must prove, is  $\mathbb{Q}$ -linear in  $a$ ). Assume this. Thus,  $a = F_G$  for some composition  $G$ . Consider this  $G$ .

If  $G$  is the empty composition, then  $a = F_G = 1$ , and therefore  $\underbrace{a}_{=1} \ast m = 1 \ast m = m \in \mathcal{K}_{\text{Epk}}$  holds. Thus, for the rest of this proof, we WLOG assume that  $G$  is not the empty composition. Thus,  $G$  is nonempty.

Recall that for any two compositions  $\alpha$  and  $\beta$ , we have  $F_\alpha \ast F_\beta = F_{[\alpha, \beta]}$ . Applying this to  $\alpha = G$  and  $\beta = J$ , we obtain  $F_G \ast F_J = F_{[G, J]}$ . Similarly,  $F_G \ast F_K = F_{[G, K]}$ .

But Claim 1 yields  $[G, J] \rightarrow [G, K]$ . Hence, the difference  $F_{[G, J]} - F_{[G, K]}$  is one of the differences which span the ideal  $\mathcal{K}_{\text{Epk}}$  according to Proposition 5.5. Thus, in particular, this difference lies in  $\mathcal{K}_{\text{Epk}}$ . In other words,  $F_{[G, J]} - F_{[G, K]} \in \mathcal{K}_{\text{Epk}}$ .

Now,

$$\begin{aligned} \underbrace{a}_{=F_G} \ast \underbrace{m}_{=F_J - F_K} &= F_G \ast (F_J - F_K) = \underbrace{F_G \ast F_J}_{=F_{[G, J]}} - \underbrace{F_G \ast F_K}_{=F_{[G, K]}} \\ &= F_{[G, J]} - F_{[G, K]} \in \mathcal{K}_{\text{Epk}}. \end{aligned}$$

This proves Claim 2.]

*Claim 3:* Let  $J$  and  $K$  be two compositions satisfying  $J \rightarrow K$ . Let  $G$  be a further composition. Then,  $[J, G] \rightarrow [K, G]$ .

[*Proof of Claim 3:* This is proven in the same way as we proved Claim 1, with the only difference that  $j_\ell$  is now the  $\ell$ -th entry of  $[J, G]$  and not the  $(p + \ell)$ -th entry (but this is still sufficient, since  $\ell > 1$ ).]

*Claim 4:* We have  $\mathcal{K}_{\text{Epk}} \ast A \subseteq \mathcal{K}_{\text{Epk}}$ .

[*Proof of Claim 4:* This is proven in the same way as we proved Claim 2, with the only difference that now we need to use Claim 3 instead of Claim 1.]

Combining Claim 2 and Claim 4, we conclude that  $\mathcal{K}_{\text{Epk}}$  is a  $\ast$ -ideal of  $A = \text{QSym}$ .

*Claim 5:* Let  $J$  and  $K$  be two nonempty compositions satisfying  $J \rightarrow K$ . Let  $G$  be a further nonempty composition. Then,  $G \odot J \rightarrow G \odot K$ .

[*Proof of Claim 5:* Write the composition  $J$  in the form  $J = (j_1, j_2, \dots, j_m)$ . Write the composition  $G$  in the form  $G = (g_1, g_2, \dots, g_p)$ . Thus,  $p > 0$  (since the composition  $G$  is nonempty).

We have  $J \rightarrow K$ . In other words, there exists an  $\ell \in \{2, 3, \dots, m\}$  such that  $j_\ell > 2$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$  (by the definition of the

relation  $\rightarrow$ ). Consider this  $\ell$ . Clearly,  $\ell \geq 2$  (since  $\ell \in \{2, 3, \dots, m\}$ ), so that

$$\underbrace{p}_{>0} + \underbrace{\ell}_{\geq 2} - 1 > 0 + 2 - 1 = 1.$$

From  $G = (g_1, g_2, \dots, g_p)$  and  $J = (j_1, j_2, \dots, j_m)$ , we obtain

$$G \odot J = (g_1, g_2, \dots, g_{p-1}, g_p + j_1, j_2, j_3, \dots, j_m). \quad (86)$$

From  $G = (g_1, g_2, \dots, g_p)$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_{\ell} - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$ , we obtain

$$G \odot K = (g_1, g_2, \dots, g_{p-1}, g_p + j_1, j_2, j_3, \dots, j_{\ell-1}, 1, j_{\ell} - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m) \quad (87)$$

(notice that the  $g_p + j_1$  term is **not** a  $g_p + 1$  term, because  $\ell \geq 2$ ).

From looking at (86) and (87), we conclude immediately that the composition  $G \odot K$  is obtained from  $G \odot J$  by “splitting” the entry  $j_{\ell} > 2$  into two consecutive entries 1 and  $j_{\ell} - 1$ , and that this entry  $j_{\ell}$  was not the first entry (indeed, this entry is the  $(p + \ell - 1)$ -th entry, but  $p + \ell - 1 > 1$ ). Hence,  $G \odot J \rightarrow G \odot K$  (by the definition of the relation  $\rightarrow$ ). This proves Claim 5.]

*Claim 6:* We have  $A \diamond \mathcal{K}_{\text{Epk}} \subseteq \mathcal{K}_{\text{Epk}}$ .

[*Proof of Claim 6:* This is proven in the same way as we proved Claim 2, with the only difference that now we need to use Claim 5 instead of Claim 1 and that we need to use the formula  $F_{\alpha} \diamond F_{\beta} = F_{\alpha \odot \beta}$  instead of  $F_{\alpha} \ast F_{\beta} = F_{[\alpha, \beta]}$ .]

*Claim 7:* Let  $J$  and  $K$  be two nonempty compositions satisfying  $J \rightarrow K$ . Let  $G$  be a further nonempty composition. Then,  $J \odot G \rightarrow K \odot G$ .

[*Proof of Claim 7:* Write the composition  $J$  in the form  $J = (j_1, j_2, \dots, j_m)$ . Write the composition  $G$  in the form  $G = (g_1, g_2, \dots, g_p)$ . Thus,  $p > 0$  (since the composition  $G$  is nonempty).

We have  $J \rightarrow K$ . In other words, there exists an  $\ell \in \{2, 3, \dots, m\}$  such that  $j_{\ell} > 2$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_{\ell} - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$  (by the definition of the relation  $\rightarrow$ ). Consider this  $\ell$ . Clearly,  $\ell \geq 2$  (since  $\ell \in \{2, 3, \dots, m\}$ ), so that  $\ell > 1$ .

From  $G = (g_1, g_2, \dots, g_p)$  and  $J = (j_1, j_2, \dots, j_m)$ , we obtain

$$J \odot G = (j_1, j_2, \dots, j_{m-1}, j_m + g_1, g_2, g_3, \dots, g_p). \quad (88)$$

Now, we distinguish between the following two cases:

*Case 1:* We have  $\ell = m$ .

*Case 2:* We have  $\ell \neq m$ .

Let us first consider Case 1. In this case, we have  $\ell = m$ . Thus,  $m = \ell \geq 2 > 1$  and  $j_m + g_1 = \underbrace{j_{\ell}}_{>2} + \underbrace{g_1}_{\geq 0} > 2$ .

From  $G = (g_1, g_2, \dots, g_p)$  and

$$\begin{aligned} K &= (j_1, j_2, \dots, j_{\ell-1}, 1, j_{\ell} - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m) \\ &= (j_1, j_2, \dots, j_{m-1}, 1, j_m - 1) \quad (\text{since } \ell = m), \end{aligned}$$

we obtain

$$\begin{aligned} K \odot G &= (j_1, j_2, \dots, j_{m-1}, 1, (j_m - 1) + g_1, g_2, g_3, \dots, g_p) \\ &= (j_1, j_2, \dots, j_{m-1}, 1, j_m + g_1 - 1, g_2, g_3, \dots, g_p). \end{aligned} \quad (89)$$

From looking at (88) and (89), we conclude immediately that the composition  $K \odot G$  is obtained from  $J \odot G$  by “splitting” the entry  $j_m + g_1 > 2$  into two consecutive entries 1 and  $j_m + g_1 - 1$ , and that this entry  $j_m + g_1$  was not the first entry (indeed, this entry is the  $m$ -th entry, but  $m > 1$ ). Hence,  $J \odot G \rightarrow K \odot G$  (by the definition of the relation  $\rightarrow$ ). This proves Claim 7 in Case 1.

Let us next consider Case 2. In this case, we have  $\ell \neq m$ . Hence,  $\ell \in \{2, 3, \dots, m-1\}$  (since  $\ell \in \{2, 3, \dots, m\}$ ).

From  $G = (g_1, g_2, \dots, g_p)$  and  $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_{\ell} - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$ , we obtain

$$\begin{aligned} K \odot G &= (j_1, j_2, \dots, j_{\ell-1}, 1, j_{\ell} - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_{m-1}, j_m + g_1, g_2, g_3, \dots, g_p) \end{aligned} \quad (90)$$

(notice that the  $j_m + g_1$  term is **not** a  $(j_{\ell} - 1) + g_1$  term, because  $\ell \neq m$ ).

From looking at (88) and (90), we conclude immediately that the composition  $K \odot G$  is obtained from  $J \odot G$  by “splitting” the entry  $j_{\ell} > 2$  into two consecutive entries 1 and  $j_{\ell} - 1$ , and that this entry  $j_{\ell}$  was not the first entry (indeed, this entry is the  $\ell$ -th entry, but  $\ell > 1$ ). Hence,  $J \odot G \rightarrow K \odot G$  (by the definition of the relation  $\rightarrow$ ). This proves Claim 7 in Case 2.

We have now proven Claim 7 in both Cases 1 and 2. Thus, Claim 7 always holds.]

*Claim 8:* We have  $\mathcal{K}_{\text{Epk}} \phi A \subseteq \mathcal{K}_{\text{Epk}}$ .

[*Proof of Claim 8:* This is proven in the same way as we proved Claim 6, with the only difference that now we need to use Claim 7 instead of Claim 5.]

Combining Claim 6 and Claim 8, we conclude that  $\mathcal{K}_{\text{Epk}}$  is a  $\phi$ -ideal of  $A = \text{QSym}$ .

Finally, Theorem 6.12 (c) (applied to  $M = \mathcal{K}_{\text{Epk}}$ ) shows that  $\mathcal{K}_{\text{Epk}}$  is a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

Thus, altogether, we have proven that  $\mathcal{K}_{\text{Epk}}$  is a  $\ast$ -ideal, a  $\phi$ -ideal, a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ . This proves Theorem 6.15.  $\square$

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**Question 6.16.** What other descent statistics  $st$  have the property that  $\mathcal{K}_{st}$  is an  $\ast$ -ideal,  $\phi$ -ideal,  $\prec$ -ideal and/or  $\succeq$ -ideal? We will see some answers in Subsection 6.7, but a more systematic study would be interesting.

## 6.5. Dendriform shuffle-compatibility

We have seen (in Proposition 5.3) that the kernel  $\mathcal{K}_{st}$  of a descent statistic  $st$  is an ideal of  $\text{QSym}$  if and only if  $st$  is shuffle-compatible. It is natural to ask whether similar combinatorial interpretations exist for when the kernel  $\mathcal{K}_{st}$  of a descent statistic  $st$  is a  $\ast$ -ideal, a  $\phi$ -ideal, a  $\prec$ -ideals or a  $\succeq$ -ideal. In this section, we shall prove such interpretations.

Now, let us define two further variants of LR-shuffle-compatibility (to be compared with those introduced in Definition 3.16):

**Definition 6.17.** Let  $st$  be a permutation statistic.

(a) We say that  $st$  is *weakly left-shuffle-compatible* if for any two disjoint nonempty permutations  $\pi$  and  $\sigma$  having the property that

$$\text{each entry of } \pi \text{ is greater than each entry of } \sigma, \quad (91)$$

the multiset  $\{st \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $st \pi$ ,  $st \sigma$ ,  $|\pi|$  and  $|\sigma|$ .

(b) We say that  $st$  is *weakly right-shuffle-compatible* if for any two disjoint nonempty permutations  $\pi$  and  $\sigma$  having the property that

$$\text{each entry of } \pi \text{ is greater than each entry of } \sigma,$$

the multiset  $\{st \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $st \pi$ ,  $st \sigma$ ,  $|\pi|$  and  $|\sigma|$ .

Then, the following analogues to the first part of Proposition 5.3 hold:

**Theorem 6.18.** Let  $st$  be a descent statistic. Then, the following three statements are equivalent:

- *Statement A:* The statistic  $st$  is left-shuffle-compatible.
- *Statement B:* The statistic  $st$  is weakly left-shuffle-compatible.
- *Statement C:* The set  $\mathcal{K}_{st}$  is an  $\prec$ -ideal of  $\text{QSym}$ .

**Theorem 6.19.** Let  $st$  be a descent statistic. Then, the following three statements are equivalent:

- *Statement A:* The statistic  $st$  is right-shuffle-compatible.
- *Statement B:* The statistic  $st$  is weakly right-shuffle-compatible.



- *Statement C:* The set  $\mathcal{K}_{\text{st}}$  is an  $\succeq$ -ideal of  $\text{QSym}$ .

Let us prove Theorem 6.18 directly, without using shuffle algebras:

*Proof of Theorem 6.18 (sketched).* The implication  $A \implies B$  is obvious.

*Proof of the implication  $B \implies C$ :* Assume that Statement B holds. Thus, the statistic  $\text{st}$  is weakly left-shuffle-compatible.

Let us show that the set  $\mathcal{K}_{\text{st}}$  is a  $\prec$ -ideal of  $\text{QSym}$ . Indeed, it suffices to show that every two  $\text{st}$ -equivalent compositions  $J$  and  $K$  and every further composition  $L$  satisfy

$$(F_J - F_K) \prec F_L \in \mathcal{K}_{\text{st}} \quad \text{and} \quad F_L \prec (F_J - F_K) \in \mathcal{K}_{\text{st}} \quad (92)$$

(because of the definition of  $\mathcal{K}_{\text{st}}$ ). So let  $J$  and  $K$  be two  $\text{st}$ -equivalent compositions, and let  $L$  be a further composition. If  $J = K$ , then (92) follows immediately from realizing that  $F_J - F_K = 0$ ; thus, we WLOG assume that  $J \neq K$ . But  $|J| = |K|$  (since  $J$  and  $K$  are  $\text{st}$ -equivalent). Hence,  $|J| = |K| > 0$  (since otherwise, we would have  $|J| = |K| = 0$ , which would imply that both  $J$  and  $K$  would be the empty composition, contradicting  $J \neq K$ ). Thus, the power series  $F_J$  and  $F_K$  are homogeneous of degree  $|J| = |K| > 0$ ; consequently,  $\varepsilon(F_J) = 0$  and  $\varepsilon(F_K) = 0$ . Hence,  $\varepsilon(F_J - F_K) = \underbrace{\varepsilon(F_J)}_{=0} - \underbrace{\varepsilon(F_K)}_{=0} = 0$ .

The compositions  $J$  and  $K$  are nonempty (since  $|J| = |K| > 0$ ). If  $L$  is empty, then (92) holds for easy reasons (indeed, we have  $F_L = 1$  in this case, and therefore (75) yields

$$(F_J - F_K) \prec F_L = (F_J - F_K) - \underbrace{\varepsilon(F_J - F_K)}_{=0} = F_J - F_K \in \mathcal{K}_{\text{st}},$$

and similarly (74) leads to  $F_L \prec (F_J - F_K) \in \mathcal{K}_{\text{st}}$ ). Hence, we WLOG assume that  $L$  is nonempty.

Pick three disjoint permutations  $\varphi$ ,  $\psi$  and  $\sigma$  having descent compositions  $J$ ,  $K$  and  $L$ , respectively, and having the property that

each entry of  $\varphi$  is greater than each entry of  $\sigma$

and

each entry of  $\psi$  is greater than each entry of  $\sigma$ .

(Such permutations  $\varphi$ ,  $\psi$  and  $\sigma$  exist, since the set  $\mathbb{P}$  is infinite.)

The permutations  $\varphi$  and  $\psi$  are  $\text{st}$ -equivalent (since their descent compositions  $J$  and  $K$  are  $\text{st}$ -equivalent). In other words,  $|\varphi| = |\psi|$  and  $\text{st } \varphi = \text{st } \psi$ .

The statistic  $\text{st}$  is weakly left-shuffle-compatible. Thus, the multiset  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  (where  $\pi$  is a nonempty permutation disjoint from  $\sigma$  and having the property that each entry of  $\pi$  is greater than each entry

of  $\sigma$ ) depends only on  $\text{st } \pi$  and  $|\pi|$  (by the definition of “weakly left-shuffle-compatible”)<sup>76</sup>. Therefore, the multisets  $\{\text{st } \tau \mid \tau \in S_{\prec}(\varphi, \sigma)\}_{\text{multi}}$  and  $\{\text{st } \tau \mid \tau \in S_{\prec}(\psi, \sigma)\}_{\text{multi}}$  are equal (since  $|\varphi| = |\psi|$  and  $\text{st } \varphi = \text{st } \psi$ ). Hence, there exists a bijection  $\alpha : S_{\prec}(\varphi, \sigma) \rightarrow S_{\prec}(\psi, \sigma)$  such that each  $\chi \in S_{\prec}(\varphi, \sigma)$  satisfies

$$\text{st}(\alpha(\chi)) = \text{st } \chi. \quad (93)$$

Consider this  $\alpha$ . Clearly, each  $\chi \in S_{\prec}(\varphi, \sigma)$  satisfies

$$(\chi \text{ and } \alpha(\chi) \text{ are st-equivalent})$$

(because of (93) and since  $|\chi| = \underbrace{|\varphi|}_{=|\psi|} + |\sigma| = |\psi| + |\sigma| = |\alpha(\chi)|$ ) and therefore

$$(\text{Comp } \chi \text{ and } \text{Comp}(\alpha(\chi)) \text{ are st-equivalent})$$

(since  $\text{st}$  is a descent statistic) and thus  $F_{\text{Comp } \chi} - F_{\text{Comp}(\alpha(\chi))} \in \mathcal{K}_{\text{st}}$  (by the definition of  $\mathcal{K}_{\text{st}}$ ) and therefore

$$F_{\text{Comp } \chi} \equiv F_{\text{Comp}(\alpha(\chi))} \pmod{\mathcal{K}_{\text{st}}}. \quad (94)$$

The first claim of Corollary 6.10 yields

$$\begin{aligned} F_{\text{Comp } \varphi} \prec F_{\text{Comp } \sigma} &= \sum_{\chi \in S_{\prec}(\varphi, \sigma)} F_{\text{Comp } \chi} \quad \text{and} \\ F_{\text{Comp } \psi} \prec F_{\text{Comp } \sigma} &= \sum_{\chi \in S_{\prec}(\psi, \sigma)} F_{\text{Comp } \chi}. \end{aligned}$$

Hence,

$$\begin{aligned} F_{\text{Comp } \varphi} \prec F_{\text{Comp } \sigma} &= \sum_{\chi \in S_{\prec}(\varphi, \sigma)} \underbrace{F_{\text{Comp } \chi}}_{\equiv F_{\text{Comp}(\alpha(\chi))} \pmod{\mathcal{K}_{\text{st}}} \text{ (by (94))}} \equiv \sum_{\chi \in S_{\prec}(\varphi, \sigma)} F_{\text{Comp}(\alpha(\chi))} \\ &= \sum_{\chi \in S_{\prec}(\psi, \sigma)} F_{\text{Comp } \chi} \\ &\quad \left( \begin{array}{l} \text{here, we have substituted } \chi \text{ for } \alpha(\chi) \text{ in the sum,} \\ \text{since the map } \alpha : S_{\prec}(\varphi, \sigma) \rightarrow S_{\prec}(\psi, \sigma) \text{ is a bijection} \end{array} \right) \\ &= F_{\text{Comp } \psi} \prec F_{\text{Comp } \sigma} \pmod{\mathcal{K}_{\text{st}}}. \end{aligned}$$

Since  $\text{Comp } \varphi = J$ ,  $\text{Comp } \psi = K$  and  $\text{Comp } \sigma = L$  (by the definition of  $\varphi$ ,  $\psi$  and  $\sigma$ ), this rewrites as  $F_J \prec F_L \equiv F_K \prec F_L \pmod{\mathcal{K}_{\text{st}}}$ . In other words,  $F_J \prec F_L - F_K \prec F_L \in \mathcal{K}_{\text{st}}$ . In other words,  $(F_J - F_K) \prec F_L \in \mathcal{K}_{\text{st}}$ . This proves the first claim of (92). The second is proven similarly. Altogether, we thus conclude that  $\mathcal{K}_{\text{st}}$

<sup>76</sup>Recall that  $\sigma$  is fixed here, which is why we don't have to say that it depends on  $\text{st } \sigma$  and  $|\sigma|$  as well.

is a  $\prec$ -ideal of  $\mathbf{QSym}$ . In other words, Statement C holds. This proves the implication  $B \implies C$ .

*Proof of the implication  $C \implies A$ :* Assume that Statement C holds. Thus, the set  $\mathcal{K}_{\text{st}}$  is an  $\prec$ -ideal of  $\mathbf{QSym}$ .

Let  $X$  be the codomain of the map  $\text{st}$ . Let  $\mathbf{Q}[X]$  be the free  $\mathbf{Q}$ -vector space with basis  $([x])_{x \in X}$ . Then, we can define a  $\mathbf{Q}$ -linear map  $\mathbf{st} : \mathbf{QSym} \rightarrow \mathbf{Q}[X]$ ,  $F_J \mapsto [\text{st } J]$ . This map  $\mathbf{st}$  sends each of the generators of  $\mathcal{K}_{\text{st}}$  to 0 (by the definition of  $\mathcal{K}_{\text{st}}$ ), and therefore sends the whole  $\mathcal{K}_{\text{st}}$  to 0. In other words,  $\mathbf{st}(\mathcal{K}_{\text{st}}) = 0$ .

Now, consider any two disjoint nonempty permutations  $\pi$  and  $\sigma$  having the property that  $\pi_1 > \sigma_1$ . Also, consider two further disjoint nonempty permutations  $\pi'$  and  $\sigma'$  having the property that  $\pi'_1 > \sigma'_1$  and satisfying  $\text{st } \pi = \text{st}(\pi')$ ,  $\text{st } \sigma = \text{st}(\sigma')$ ,  $|\pi| = |\pi'|$  and  $|\sigma| = |\sigma'|$ . We shall show that

$$\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}.$$

This will show that the multiset  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ .

From  $\text{st } \pi = \text{st}(\pi')$  and  $|\pi| = |\pi'|$ , we conclude that  $\pi$  and  $\pi'$  are  $\text{st}$ -equivalent. In other words,  $\text{Comp } \pi$  and  $\text{Comp}(\pi')$  are  $\text{st}$ -equivalent. Hence,  $F_{\text{Comp } \pi} - F_{\text{Comp}(\pi')} \in \mathcal{K}_{\text{st}}$  (by the definition of  $\mathcal{K}_{\text{st}}$ ), so that  $F_{\text{Comp } \pi} \equiv F_{\text{Comp}(\pi')} \pmod{\mathcal{K}_{\text{st}}}$ . Similarly,  $F_{\text{Comp } \sigma} \equiv F_{\text{Comp}(\sigma')} \pmod{\mathcal{K}_{\text{st}}}$ . These two congruences, combined, yield  $F_{\text{Comp } \pi} \prec F_{\text{Comp } \sigma} \equiv F_{\text{Comp}(\pi')} \prec F_{\text{Comp}(\sigma')} \pmod{\mathcal{K}_{\text{st}}}$ . (Indeed, we can conclude  $a \prec c \equiv b \prec d \pmod{\mathcal{K}_{\text{st}}}$  whenever we have  $a \equiv b \pmod{\mathcal{K}_{\text{st}}}$  and  $c \equiv d \pmod{\mathcal{K}_{\text{st}}}$ ; this is because we know that  $\mathcal{K}_{\text{st}}$  is a  $\prec$ -ideal of  $\mathbf{QSym}$ .)

From  $F_{\text{Comp } \pi} \prec F_{\text{Comp } \sigma} \equiv F_{\text{Comp}(\pi')} \prec F_{\text{Comp}(\sigma')} \pmod{\mathcal{K}_{\text{st}}}$ , we obtain

$$\mathbf{st}(F_{\text{Comp } \pi} \prec F_{\text{Comp } \sigma}) = \mathbf{st}(F_{\text{Comp}(\pi')} \prec F_{\text{Comp}(\sigma')}) \quad (95)$$

(since  $\mathbf{st}(\mathcal{K}_{\text{st}}) = 0$ ).

The first claim of Corollary 6.10 yields

$$F_{\text{Comp } \pi} \prec F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\prec}(\pi, \sigma)} F_{\text{Comp } \chi}.$$

Applying the map  $\mathbf{st}$  to both sides of this equality, we find

$$\begin{aligned} \mathbf{st}(F_{\text{Comp } \pi} \prec F_{\text{Comp } \sigma}) &= \mathbf{st}\left(\sum_{\chi \in S_{\prec}(\pi, \sigma)} F_{\text{Comp } \chi}\right) \\ &= \sum_{\chi \in S_{\prec}(\pi, \sigma)} \underbrace{\mathbf{st}(F_{\text{Comp } \chi})}_{=[\text{st}(\text{Comp } \chi)]=[\text{st } \chi]} = \sum_{\chi \in S_{\prec}(\pi, \sigma)} [\text{st } \chi]. \end{aligned}$$

Similarly,

$$\mathbf{st}(F_{\text{Comp}(\pi')} \prec F_{\text{Comp}(\sigma')}) = \sum_{\chi \in S_{\prec}(\pi', \sigma')} [\text{st } \chi].$$

But the left-hand sides of the last two equalities are equal (because of (95)); therefore, the right-hand sides must be equal as well. In other words,

$$\sum_{\chi \in S_{\prec}(\pi, \sigma)} [\text{st } \chi] = \sum_{\chi \in S_{\prec}(\pi', \sigma')} [\text{st } \chi].$$

This shows exactly that  $\{\text{st } \chi \mid \chi \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \chi \mid \chi \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}$ . In other words,  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}} = \{\text{st } \tau \mid \tau \in S_{\prec}(\pi', \sigma')\}_{\text{multi}}$ . Thus, we have proven that the multiset  $\{\text{st } \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multi}}$  depends only on  $\text{st } \pi$ ,  $\text{st } \sigma$ ,  $|\pi|$  and  $|\sigma|$ . Hence, the statistic  $\text{st}$  is left-shuffle-compatible. In other words, Statement A holds. This proves the implication  $C \implies A$ .

Now that we have proven all three implications  $A \implies B$ ,  $B \implies C$  and  $C \implies A$ , the proof of Theorem 6.18 is complete.  $\square$

*Proof of Theorem 6.19.* The proof of Theorem 6.19 is analogous to the above proof of Theorem 6.18.  $\square$

**Corollary 6.20.** Let  $\text{st}$  be a permutation statistic that is LR-shuffle-compatible. Then,  $\text{st}$  is a shuffle-compatible descent statistic, and the set  $\mathcal{K}_{\text{st}}$  is an ideal and a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

*Proof of Corollary 6.20 (sketched).* Proposition 3.18 yields that  $\text{st}$  is head-graft-compatible and shuffle-compatible. Proposition 3.17 shows that  $\text{st}$  is left-shuffle-compatible, right-shuffle-compatible and head-graft-compatible (since  $\text{st}$  is LR-shuffle-compatible). Hence, Proposition 4.5 shows that  $\text{st}$  is a descent statistic. Thus, Theorem 6.18 yields that  $\mathcal{K}_{\text{st}}$  is a  $\prec$ -ideal of  $\text{QSym}$  (since  $\text{st}$  is left-shuffle-compatible). Likewise, Theorem 6.19 yields that  $\mathcal{K}_{\text{st}}$  is a  $\succeq$ -ideal of  $\text{QSym}$  (since  $\text{st}$  is right-shuffle-compatible). Finally, Proposition 5.3 yields that  $\mathcal{K}_{\text{st}}$  is an ideal of  $\text{QSym}$  (since  $\text{st}$  is a shuffle-compatible descent statistic). This proves Corollary 6.20.  $\square$

A converse of Corollary 6.20 also holds:

**Corollary 6.21.** Let  $\text{st}$  be a descent statistic such that  $\mathcal{K}_{\text{st}}$  is a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ . Then,  $\text{st}$  is LR-shuffle-compatible and shuffle-compatible.

*Proof of Corollary 6.21 (sketched).* Theorem 6.18 yields that  $\text{st}$  is left-shuffle-compatible (since  $\mathcal{K}_{\text{st}}$  is a  $\prec$ -ideal of  $\text{QSym}$ ). Likewise, Theorem 6.19 yields that  $\text{st}$  is right-shuffle-compatible (since  $\mathcal{K}_{\text{st}}$  is a  $\succeq$ -ideal of  $\text{QSym}$ ). Hence, Corollary 3.23 shows that  $\text{st}$  is LR-shuffle-compatible. Thus, Proposition 3.18 yields that  $\text{st}$  is head-graft-compatible and shuffle-compatible. This proves Corollary 6.21.  $\square$

As a consequence of Theorem 6.18 and Theorem 6.19, we can see that any descent statistic that is weakly left-shuffle-compatible and weakly right-shuffle-compatible must automatically be shuffle-compatible<sup>77</sup>. Note that this is only

<sup>77</sup>*Proof.* Let  $\text{st}$  be a descent statistic that is weakly left-shuffle-compatible and weakly right-shuffle-compatible. We must prove that  $\text{st}$  is shuffle-compatible.

The implication  $B \implies C$  in Theorem 6.18 shows that the set  $\mathcal{K}_{\text{st}}$  is a  $\prec$ -ideal of  $\text{QSym}$ . Similarly, the set  $\mathcal{K}_{\text{st}}$  is a  $\succeq$ -ideal of  $\text{QSym}$ . Hence, Proposition 6.13 (applied to  $M = \mathcal{K}_{\text{st}}$ ) yields that  $\mathcal{K}_{\text{st}}$  is an ideal of  $\text{QSym}$ . By Proposition 5.3, this shows that  $\text{st}$  is shuffle-compatible.

true for descent statistics! As far as arbitrary permutation statistics are concerned, this is false; for example, the number of inversions is weakly left-shuffle-compatible and weakly right-shuffle-compatible but not shuffle-compatible.

Recall that every permutation statistic that is left-shuffle-compatible and right-shuffle-compatible must automatically be LR-shuffle-compatible (by Corollary 3.23) and therefore also shuffle-compatible (by Corollary 3.22) and head-graft-compatible (again by Corollary 3.22) and therefore a descent statistic (by Proposition 4.5).

**Corollary 6.22.** The descent statistic  $\text{Epk}$  is left-shuffle-compatible and right-shuffle-compatible.

Corollary 6.22 follows by combining Theorem 3.12 (c) with Theorem 3.17. But we can also give a proof using Theorem 6.18:

*Proof of Corollary 6.22.* To prove that  $\text{Epk}$  is left-shuffle-compatible, combine Theorem 6.18 with Theorem 6.15. Similarly for right-shuffle-compatibility.  $\square$

Using Theorem 6.3, we can state an analogue of Theorem 4.14. Let us first define the notion of dendriform algebras:

**Definition 6.23. (a)** A *dendriform algebra* over a field  $\mathbf{k}$  means a  $\mathbf{k}$ -algebra  $A$  equipped with two further  $\mathbf{k}$ -bilinear binary operations  $\prec$  and  $\succeq$  (these are operations, not relations, despite the symbols) from  $A \times A$  to  $A$  that satisfy the four rules

$$\begin{aligned} a \prec b + a \succeq b &= ab; \\ (a \prec b) \prec c &= a \prec (bc); \\ (a \succeq b) \prec c &= a \succeq (b \prec c); \\ a \succeq (b \succeq c) &= (ab) \succeq c \end{aligned}$$

for all  $a, b, c \in A$ . (Depending on the situation, it is useful to also impose a few axioms that relate the unity 1 of the  $\mathbf{k}$ -algebra  $A$  with the operations  $\prec$  and  $\succeq$ . For example, we could require  $1 \prec a = 0$  for each  $a \in A$ . For what we are going to do in the following, it does not matter whether we make this requirement.)

**(b)** If  $A$  and  $B$  are two dendriform algebras over  $\mathbf{k}$ , then a *dendriform algebra homomorphism* from  $A$  to  $B$  means a  $\mathbf{k}$ -algebra homomorphism  $\phi : A \rightarrow B$  preserving the operations  $\prec$  and  $\succeq$  (that is, satisfying  $\phi(a \prec b) = \phi(a) \prec \phi(b)$  and  $\phi(a \succeq b) = \phi(a) \succeq \phi(b)$  for all  $a, b \in A$ ). (Some authors only require it to be a  $\mathbf{k}$ -linear map instead of being a  $\mathbf{k}$ -algebra homomorphism; this boils down to the question whether  $\phi(1)$  must be 1 or not. This does not make a difference for us here.)

Thus,  $\text{QSym}$  (with its two operations  $\prec$  and  $\succeq$ ) becomes a dendriform algebra over  $\mathbb{Q}$ .

Notice that if  $A$  and  $B$  are two dendriform algebras over  $\mathbf{k}$ , then the kernel of any dendriform algebra homomorphism  $A \rightarrow B$  is an  $\prec$ -ideal and a  $\succeq$ -ideal of  $A$ . Conversely, if  $A$  is a dendriform algebra over  $\mathbf{k}$ , and  $I$  is simultaneously a  $\prec$ -ideal and a  $\succeq$ -ideal of  $A$ , then  $A/I$  canonically becomes a dendriform algebra, and the canonical projection  $A \rightarrow A/I$  becomes a dendriform algebra homomorphism.

Therefore, Theorem 6.18 and Theorem 6.19 (and the  $\mathcal{A}_{\text{st}} \cong \text{QSym} / \mathcal{K}_{\text{st}}$  isomorphism from Proposition 5.3) yield the following:

**Corollary 6.24.** If a descent statistic  $\text{st}$  is left-shuffle-compatible and right-shuffle-compatible, then its shuffle algebra  $\mathcal{A}_{\text{st}}$  canonically becomes a dendriform algebra.

We furthermore have the following analogue of Theorem 4.14, which easily follows from Theorem 6.18 and Theorem 6.19:

**Theorem 6.25.** Let  $\text{st}$  be a descent statistic.

(a) The descent statistic  $\text{st}$  is left-shuffle-compatible and right-shuffle-compatible if and only if there exist a dendriform algebra  $A$  with basis  $(u_\alpha)$  (indexed by  $\text{st}$ -equivalence classes  $\alpha$  of compositions) and a dendriform algebra homomorphism  $\phi_{\text{st}} : \text{QSym} \rightarrow A$  with the property that whenever  $\alpha$  is an  $\text{st}$ -equivalence class of compositions, we have

$$\phi_{\text{st}}(F_L) = u_\alpha \quad \text{for each } L \in \alpha.$$

(b) In this case, the  $\mathbb{Q}$ -linear map

$$\mathcal{A}_{\text{st}} \rightarrow A, \quad [\pi]_{\text{st}} \mapsto u_\alpha,$$

where  $\alpha$  is the  $\text{st}$ -equivalence class of the composition  $\text{Comp } \pi$ , is an isomorphism of dendriform algebras  $\mathcal{A}_{\text{st}} \rightarrow A$ .

**Question 6.26.** Can the  $\mathbb{Q}$ -algebra  $\text{Pow } \mathcal{N}$  from Definition 2.20 be endowed with two binary operations  $\prec$  and  $\succeq$  that make it into a dendriform algebra? Can we then find an analogue of Proposition 2.25 along the following lines?

Let  $(P, \gamma)$ ,  $(Q, \delta)$  and  $(P \sqcup Q, \varepsilon)$  be as in Proposition 2.25. Assume that each of the posets  $P$  and  $Q$  has a minimum element; denote these elements by  $\min P$  and  $\min Q$ , respectively. Define two posets  $P \prec Q$  and  $P \succeq Q$  as in Proposition 6.8. Then, we hope to have

$$\begin{aligned} \Gamma_{\mathcal{Z}}(P, \gamma) \prec \Gamma_{\mathcal{Z}}(Q, \delta) &= \Gamma_{\mathcal{Z}}(P \prec Q, \varepsilon) & \text{and} \\ \Gamma_{\mathcal{Z}}(P, \gamma) \succeq \Gamma_{\mathcal{Z}}(Q, \delta) &= \Gamma_{\mathcal{Z}}(P \succeq Q, \varepsilon), \end{aligned}$$

assuming a simple condition on  $\min P$  and  $\min Q$  (say,  $\gamma(\min P) \prec_{\mathcal{Z}} \delta(\min Q)$ ).

Ideally, this would be a generalization of Proposition 6.8.

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## 6.6. Criteria for $\mathcal{K}_{\text{st}}$ to be a stack ideal

We have so far studied the combinatorial significance of when the kernel  $\mathcal{K}_{\text{st}}$  of a statistic  $\text{st}$  is a  $\prec$ -ideal or a  $\succeq$ -ideal of  $\text{QSym}$ . What about  $\Phi$ -ideals and  $\mathbb{X}$ -ideals? It turns out that the answer to this question is given (on the level of compositions) by the following (easily verified) proposition:

**Proposition 6.27.** Let  $\text{st}$  be a descent statistic.

(a) The set  $\mathcal{K}_{\text{st}}$  is a left  $\Phi$ -ideal of  $\text{QSym}$  if and only if  $\text{st}$  has the following property: If  $J$  and  $K$  are two  $\text{st}$ -equivalent nonempty compositions, and if  $G$  is any nonempty composition, then  $G \odot J$  and  $G \odot K$  are  $\text{st}$ -equivalent.

(b) The set  $\mathcal{K}_{\text{st}}$  is a right  $\Phi$ -ideal of  $\text{QSym}$  if and only if  $\text{st}$  has the following property: If  $J$  and  $K$  are two  $\text{st}$ -equivalent nonempty compositions, and if  $G$  is any nonempty composition, then  $J \odot G$  and  $K \odot G$  are  $\text{st}$ -equivalent.

(c) The set  $\mathcal{K}_{\text{st}}$  is a left  $\mathbb{X}$ -ideal of  $\text{QSym}$  if and only if  $\text{st}$  has the following property: If  $J$  and  $K$  are two  $\text{st}$ -equivalent nonempty compositions, and if  $G$  is any nonempty composition, then  $[G, J]$  and  $[G, K]$  are  $\text{st}$ -equivalent.

(d) The set  $\mathcal{K}_{\text{st}}$  is a right  $\mathbb{X}$ -ideal of  $\text{QSym}$  if and only if  $\text{st}$  has the following property: If  $J$  and  $K$  are two  $\text{st}$ -equivalent nonempty compositions, and if  $G$  is any nonempty composition, then  $[J, G]$  and  $[K, G]$  are  $\text{st}$ -equivalent.

Proposition 6.27 allows us to give a new proof of Theorem 6.15, which makes no use of Proposition 5.5. Instead, it will rely on analyzing  $\text{Epk}([A, B])$  and  $\text{Epk}(A \odot B)$  when  $A$  and  $B$  are two nonempty compositions.

First, we introduce a notation: If  $S$  is a set of integers, and  $p$  is an integer, then  $S + p$  shall denote the set  $\{s + p \mid s \in S\}$ .

We shall use the following simple lemma:

**Lemma 6.28.** Let  $A$  and  $B$  be two nonempty compositions. Let  $n = |A|$ .

(a) We have  $\text{Epk}([A, B]) = (\text{Epk } A) \cup ((\text{Epk } B + n) \setminus \{n + 1\})$ .

(b) We have  $\text{Epk}(A \odot B) = ((\text{Epk } A) \setminus \{n\}) \cup (\text{Epk } B + n)$ .

*Proof of Lemma 6.28.* Let  $m = |B|$ . Consider any  $n$ -permutation  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  satisfying  $\text{Comp } \alpha = A$ . (Such  $\alpha$  exists, since  $n = |A|$ .) Consider any  $m$ -permutation  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$  satisfying  $\text{Comp } \beta = B$ . (Such  $\beta$  exists, since  $m = |B|$ .) From  $\text{Comp } \alpha = A$ , we obtain  $\text{Epk } \alpha = \text{Epk } A$ . Similarly,  $\text{Epk } \beta = \text{Epk } B$ .

(a) WLOG assume that  $\alpha_i > \beta_j$  for all  $i \in [n]$  and  $j \in [m]$ . (Indeed, we can achieve this by choosing a positive integer  $g$  that is larger than each entry of  $\beta$ , and adding  $g$  to each entry of  $\alpha$ .) Thus, in particular, the entries of  $\alpha$  are distinct from the entries of  $\beta$ . Also,  $\alpha_n > \beta_1$  (since  $\alpha_i > \beta_j$  for all  $i \in [n]$  and  $j \in [m]$ ).

Let  $\gamma$  be the  $(n + m)$ -permutation  $(\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m)$ . Then, the descents of  $\gamma$  are obtained as follows:

- Each descent of  $\alpha$  is a descent of  $\gamma$ .

- 
- The number  $n$  is a descent of  $\gamma$  (since  $\alpha_n > \beta_1$ ).
  - Adding  $n$  to each descent of  $\beta$  yields a descent of  $\gamma$  (that is, if  $i$  is a descent of  $\beta$ , then  $i + n$  is a descent of  $\gamma$ ).

These are all the descents of  $\gamma$ . Thus,

$$\text{Des } \gamma = \text{Des } \alpha \cup \{n\} \cup (\text{Des } \beta + n).$$

Hence,

$$\begin{aligned} \text{Comp } (\text{Des } \gamma) &= \text{Comp } (\text{Des } \alpha \cup \{n\} \cup (\text{Des } \beta + n)) \\ &= [\text{Comp } (\text{Des } \alpha), \text{Comp } (\text{Des } \beta)] \end{aligned}$$

(because of how  $\text{Comp } S$  is defined for a set  $S$ ). Since  $\text{Comp } (\text{Des } \pi) = \text{Comp } \pi$  for any permutation  $\pi$ , this rewrites as

$$\text{Comp } \gamma = [\text{Comp } \alpha, \text{Comp } \beta].$$

In view of  $\text{Comp } \alpha = A$  and  $\text{Comp } \beta = B$ , this rewrites as  $\text{Comp } \gamma = [A, B]$ . Thus,  $\text{Epk } ([A, B]) = \text{Epk } \gamma$ .

On the other hand, recall again that  $\gamma = (\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m)$  and  $\alpha_n > \beta_1$ . Thus, the exterior peaks of  $\gamma$  are obtained as follows:

- Each exterior peak of  $\alpha$  is an exterior peak of  $\gamma$ . (This includes  $n$ , if  $n$  is an exterior peak of  $\alpha$ , because  $\alpha_n > \beta_1$ .)
- Adding  $n$  to each exterior peak of  $\beta$  yields an exterior peak of  $\gamma$ , except for the number  $n + 1$ , which is **not** an exterior peak of  $\gamma$  (since  $\alpha_n > \beta_1$ ).

These are all the exterior peaks of  $\gamma$ . Thus,

$$\text{Epk } \gamma = (\text{Epk } \alpha) \cup ((\text{Epk } \beta + n) \setminus \{n + 1\}).$$

In view of  $\text{Epk } \alpha = \text{Epk } A$ ,  $\text{Epk } \beta = \text{Epk } B$  and  $\text{Epk } \gamma = \text{Epk } ([A, B])$ , this rewrites as

$$\text{Epk } ([A, B]) = (\text{Epk } A) \cup ((\text{Epk } B + n) \setminus \{n + 1\}).$$

This proves Lemma 6.28 **(a)**.

**(b)** WLOG assume that  $\alpha_i < \beta_j$  for all  $i \in [n]$  and  $j \in [m]$ . (Indeed, we can achieve this by choosing a positive integer  $g$  that is larger than each entry of  $\alpha$ , and adding  $g$  to each entry of  $\beta$ .) Thus, in particular, the entries of  $\alpha$  are distinct from the entries of  $\beta$ . Also,  $\alpha_n < \beta_1$  (since  $\alpha_i < \beta_j$  for all  $i \in [n]$  and  $j \in [m]$ ).

Let  $\gamma$  be the  $(n + m)$ -permutation  $(\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m)$ . Then, the descents of  $\gamma$  are obtained as follows:

- Each descent of  $\alpha$  is a descent of  $\gamma$ .



- Adding  $n$  to each descent of  $\beta$  yields a descent of  $\gamma$  (that is, if  $i$  is a descent of  $\beta$ , then  $i + n$  is a descent of  $\gamma$ ).

These are all the descents of  $\gamma$  (in particular,  $n$  is **not** a descent of  $\gamma$ , since  $\alpha_n < \beta_1$ ). Thus,

$$\text{Des } \gamma = \text{Des } \alpha \cup (\text{Des } \beta + n).$$

Hence,

$$\begin{aligned} \text{Comp } (\text{Des } \gamma) &= \text{Comp } (\text{Des } \alpha \cup (\text{Des } \beta + n)) \\ &= \text{Comp } (\text{Des } \alpha) \odot \text{Comp } (\text{Des } \beta) \end{aligned}$$

(because of how  $\text{Comp } S$  is defined for a set  $S$ ). Since  $\text{Comp } (\text{Des } \pi) = \text{Comp } \pi$  for any permutation  $\pi$ , this rewrites as

$$\text{Comp } \gamma = \text{Comp } \alpha \odot \text{Comp } \beta.$$

In view of  $\text{Comp } \alpha = A$  and  $\text{Comp } \beta = B$ , this rewrites as  $\text{Comp } \gamma = A \odot B$ . Thus,  $\text{Epk } (A \odot B) = \text{Epk } \gamma$ .

On the other hand, recall again that  $\gamma = (\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m)$  and  $\alpha_n < \beta_1$ . Thus, the exterior peaks of  $\gamma$  are obtained as follows:

- Each exterior peak of  $\alpha$  is an exterior peak of  $\gamma$ , except for the number  $n$ , which is **not** an exterior peak of  $\gamma$  (since  $\alpha_n < \beta_1$ ).
- Adding  $n$  to each exterior peak of  $\beta$  yields an exterior peak of  $\gamma$ . (This includes  $n + 1$ , if 1 is an exterior peak of  $\beta$ , because  $\alpha_n < \beta_1$ .)

These are all the exterior peaks of  $\gamma$ . Thus,

$$\text{Epk } \gamma = ((\text{Epk } \alpha) \setminus \{n\}) \cup (\text{Epk } \beta + n).$$

In view of  $\text{Epk } \alpha = \text{Epk } A$ ,  $\text{Epk } \beta = \text{Epk } B$  and  $\text{Epk } \gamma = \text{Epk } (A \odot B)$ , this rewrites as

$$\text{Epk } (A \odot B) = ((\text{Epk } A) \setminus \{n\}) \cup (\text{Epk } B + n).$$

This proves Lemma 6.28 **(b)**. □

We can now easily prove Theorem 6.15 again:

*Second proof of Theorem 6.15 (sketched).* Let  $A = \text{QSym}$ . Corollary 5.4 shows that  $\mathcal{K}_{\text{Epk}}$  is an ideal of  $\text{QSym}$ .

We now argue the following claim:<sup>78</sup>

*Claim 2:* We have  $A \rtimes \mathcal{K}_{\text{Epk}} \subseteq \mathcal{K}_{\text{Epk}}$ .

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<sup>78</sup>These claims are numbered Claim 2, Claim 4, Claim 6 and Claim 8, in order to match the numbering of the corresponding claims in the first proof of Theorem 6.15 above.

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[*Proof of Claim 2:* We must show that  $\mathcal{K}_{\text{Epk}}$  is a left  $\ast$ -ideal of  $\text{QSym}$ . According to Proposition 6.27 (c), this boils down to proving that if  $J$  and  $K$  are two Epk-equivalent nonempty compositions, and if  $G$  is any nonempty composition, then  $[G, J]$  and  $[G, K]$  are Epk-equivalent.

So let  $J$  and  $K$  be two Epk-equivalent nonempty compositions. Thus,  $|J| = |K| > 0$  and  $\text{Epk } J = \text{Epk } K$ . Let  $G$  be any nonempty composition.

Define a positive integer  $n$  by  $n = |G|$ . Lemma 6.28 (a) (applied to  $A = G$  and  $B = J$ ) yields

$$\text{Epk}([G, J]) = (\text{Epk } G) \cup ((\text{Epk } J + n) \setminus \{n + 1\}). \quad (96)$$

Similarly,

$$\text{Epk}([G, K]) = (\text{Epk } G) \cup ((\text{Epk } K + n) \setminus \{n + 1\}). \quad (97)$$

The right hand sides of (96) and (97) are equal (since  $\text{Epk } J = \text{Epk } K$ ). Hence, the left hand sides are equal as well. In other words,  $\text{Epk}([G, J]) = \text{Epk}([G, K])$ . Combining this with

$$|[G, J]| = |G| + \underbrace{|J|}_{=|K|} = |G| + |K| = |[G, K]|,$$

we conclude that  $[G, J]$  and  $[G, K]$  are Epk-equivalent. As we have said, this concludes the proof of Claim 2.]

Similarly to Claim 2, we can show the following three claims:

*Claim 4:* We have  $\mathcal{K}_{\text{Epk}} \ast A \subseteq \mathcal{K}_{\text{Epk}}$ .

*Claim 6:* We have  $A \phi \mathcal{K}_{\text{Epk}} \subseteq \mathcal{K}_{\text{Epk}}$ .

*Claim 8:* We have  $\mathcal{K}_{\text{Epk}} \phi A \subseteq \mathcal{K}_{\text{Epk}}$ .

(Of course, in proving Claims 4, 6 and 8, we need to use the other three parts of Proposition 6.27 instead of Proposition 6.27 (c), and we occasionally need to use Lemma 6.28 (b) instead of Lemma 6.28 (a).)

Combining Claim 2 and Claim 4, we conclude that  $\mathcal{K}_{\text{Epk}}$  is a  $\ast$ -ideal of  $A$ .

Combining Claim 6 and Claim 8, we conclude that  $\mathcal{K}_{\text{Epk}}$  is a  $\phi$ -ideal of  $A$ .

Thus, Theorem 6.12 (c) (applied to  $M = \mathcal{K}_{\text{Epk}}$ ) shows that  $\mathcal{K}_{\text{Epk}}$  is a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ . This proves Theorem 6.15 again.  $\square$

## 6.7. Left/right-shuffle-compatibility of other statistics

Let us now briefly analyze the kernels  $\mathcal{K}_{\text{st}}$  of some other descent statistics, following the same approach that we took in our above second proof of Theorem 6.15 again. Much of what follows will merely reproduce results from Section 3.

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### 6.7.1. The descent set Des

First of all, the following is obvious:

**Proposition 6.29.** The ideal  $\mathcal{K}_{\text{Des}}$  of  $\text{QSym}$  is the trivial ideal 0, and is a  $\ast$ -ideal, a  $\phi$ -ideal, a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

**Corollary 6.30.** The descent statistic Des is left-shuffle-compatible and right-shuffle-compatible.

*Proof of Corollary 6.30.* Corollary 6.30 can be derived from Proposition 6.29 in the same way as Corollary 6.22 was derived from Theorem 6.15.  $\square$

### 6.7.2. The descent number des

The permutation statistic des (called the *descent number*) is defined as follows: For each permutation  $\pi$ , we set  $\text{des } \pi = |\text{Des } \pi|$  (that is, des  $\pi$  is the number of all descents of  $\pi$ ). It was proven in [GesZhu17, Theorem 4.6 (a)] that this statistic des is shuffle-compatible. Furthermore, des is clearly a descent statistic. Hence, Proposition 5.3 (applied to  $\text{st} = \text{des}$ ) shows that  $\mathcal{K}_{\text{des}}$  is an ideal of  $\text{QSym}$ . We now claim the following:

**Proposition 6.31.** The ideal  $\mathcal{K}_{\text{des}}$  of  $\text{QSym}$  is a  $\ast$ -ideal, a  $\phi$ -ideal, a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

**Corollary 6.32.** The descent statistic des is left-shuffle-compatible and right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 6.28):

**Lemma 6.33.** Let  $A$  and  $B$  be two nonempty compositions. Let  $n = |A|$ .

- (a) We have  $\text{des}([A, B]) = \text{des } A + \text{des } B + 1$ .
- (b) We have  $\text{des}(A \odot B) = \text{des } A + \text{des } B$ .

*Proof of Lemma 6.33.* If  $I$  is a nonempty composition, then  $\text{des } I$  equals the length of  $I$  minus 1. Lemma 6.33 follows easily from this.  $\square$

*Proof of Proposition 6.31.* Analogous to the above second proof of Theorem 6.15, but using Lemma 6.33 instead of Lemma 6.28.  $\square$

*Proof of Corollary 6.32.* Corollary 6.32 can be derived from Proposition 6.31 in the same way as Corollary 6.22 was derived from Theorem 6.15.  $\square$

### 6.7.3. The major index maj

The permutation statistic maj (called the *major index*) is defined as follows: For each permutation  $\pi$ , we set  $\text{maj } \pi = \sum_{i \in \text{Des } \pi} i$  (that is, maj  $\pi$  is the sum of all descents of  $\pi$ ). It was proven in [GesZhu17, Theorem 3.1 (a)] that this statistic maj is shuffle-compatible. Furthermore, maj is clearly a descent statistic. Hence, Proposition 5.3 (applied to  $\text{st} = \text{maj}$ ) shows that  $\mathcal{K}_{\text{maj}}$  is an ideal of QSym. We now claim the following:

**Proposition 6.34.** The ideal  $\mathcal{K}_{\text{maj}}$  of QSym is a right  $\ast$ -ideal and a right  $\phi$ -ideal, but neither a  $\prec$ -ideal nor a  $\succeq$ -ideal of QSym.

**Corollary 6.35.** The descent statistic maj is neither left-shuffle-compatible nor right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 6.28):

**Lemma 6.36.** Let  $A$  and  $B$  be two nonempty compositions. Let  $n = |A|$ .

- (a) We have  $\text{maj}([A, B]) = \text{maj } A + \text{maj } B + n \cdot (\text{des } B + 1)$ .
- (b) We have  $\text{maj}(A \odot B) = \text{maj } A + \text{maj } B + n \cdot \text{des } B$ .

*Proof of Lemma 6.36.* If  $I = (i_1, i_2, \dots, i_k)$  is a nonempty composition, then

$$\begin{aligned} \text{maj } I &= i_1 + (i_1 + i_2) + (i_1 + i_2 + i_3) + \cdots + (i_1 + i_2 + \cdots + i_{k-1}) \\ &= (k-1)i_1 + (k-2)i_2 + \cdots + (k-k)i_k. \end{aligned}$$

Lemma 6.36 follows easily from this. □

*Proof of Proposition 6.34.* To prove that  $\mathcal{K}_{\text{maj}}$  is a right  $\ast$ -ideal of QSym, we proceed as in the proof of Claim 2 in the second proof of Theorem 6.15, but using Lemma 6.36 instead of Lemma 6.28. Similarly, we can show that  $\mathcal{K}_{\text{maj}}$  is a right  $\phi$ -ideal of QSym.

To prove that  $\mathcal{K}_{\text{maj}}$  is not a  $\prec$ -ideal of QSym (and not even a left  $\prec$ -ideal of QSym), it suffices to find some  $m \in \mathcal{K}_{\text{maj}}$  and some  $a \in \text{QSym}$  such that  $a \prec m \notin \mathcal{K}_{\text{maj}}$ . For example, we can take  $m = F_{(1,1,2)} - F_{(3,1)}$  and  $a = F_{(1)}$ ; then,  $a \prec m = F_{(1,1,1,2)} - F_{(1,3,1)} \notin \mathcal{K}_{\text{maj}}$ . The same values of  $m$  and  $a$  also satisfy  $a \succeq m \notin \mathcal{K}_{\text{maj}}$ ,  $m \prec a \notin \mathcal{K}_{\text{maj}}$  and  $m \succeq a \notin \mathcal{K}_{\text{maj}}$ ; thus,  $\mathcal{K}_{\text{maj}}$  is not a  $\succeq$ -ideal of QSym either. Proposition 6.34 is now proven. □

*Proof of Corollary 6.35.* Again, this follows from Proposition 6.34. □

### 6.7.4. The joint statistic (des, maj)

The next permutation statistic we shall study is the so-called joint statistic (des, maj). This statistic is defined as the permutation statistic that sends each permutation  $\pi$  to the ordered pair (des  $\pi$ , maj  $\pi$ ). (Calling it (des, maj) is thus a slight abuse

of notation.) It was proven in [GesZhu17, Theorem 4.5 (a)] that this statistic  $(\text{des}, \text{maj})$  is shuffle-compatible. Furthermore,  $(\text{des}, \text{maj})$  is clearly a descent statistic. Hence, Proposition 5.3 (applied to  $\text{st} = (\text{des}, \text{maj})$ ) shows that  $\mathcal{K}_{(\text{des}, \text{maj})}$  is an ideal of  $\text{QSym}$ . We now claim the following:

**Proposition 6.37.** The ideal  $\mathcal{K}_{(\text{des}, \text{maj})}$  of  $\text{QSym}$  is a  $\ast$ -ideal, a  $\phi$ -ideal, a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

**Corollary 6.38.** The descent statistic  $(\text{des}, \text{maj})$  is left-shuffle-compatible and right-shuffle-compatible.

*Proof of Proposition 6.37.* Analogous to the above second proof of Theorem 6.15, but using Lemma 6.33 together with Lemma 6.36 instead of Lemma 6.28.  $\square$

*Proof of Corollary 6.38.* Corollary 6.38 can be derived from Proposition 6.37 in the same way as Corollary 6.22 was derived from Theorem 6.15.  $\square$

### 6.7.5. The left peak set $\text{Lpk}$

Recall the permutation statistic  $\text{Lpk}$  (the left peak set) defined in Definition 1.8. It was proven in [GesZhu17, Theorem 4.9 (a)] that this statistic  $\text{Lpk}$  is shuffle-compatible. Furthermore,  $\text{Lpk}$  is clearly a descent statistic. Hence, Proposition 5.3 (applied to  $\text{st} = \text{Lpk}$ ) shows that  $\mathcal{K}_{\text{Lpk}}$  is an ideal of  $\text{QSym}$ . We now claim the following:

**Proposition 6.39.** The ideal  $\mathcal{K}_{\text{Lpk}}$  of  $\text{QSym}$  is a left  $\ast$ -ideal, a  $\phi$ -ideal, a  $\prec$ -ideal and a  $\succeq$ -ideal of  $\text{QSym}$ .

**Corollary 6.40.** The descent statistic  $\text{Lpk}$  is left-shuffle-compatible and right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 6.28):

**Lemma 6.41.** Let  $A$  and  $B$  be two nonempty compositions. Let  $n = |A|$ .

(a) We have  $\text{Lpk}([A, B]) = (\text{Lpk } A) \cup ((\text{Lpk } B + n) \setminus \{n + 1\}) \cup \{n \mid n - 1 \notin \text{Des } A\}$ .

(b) We have  $\text{Lpk}(A \odot B) = (\text{Lpk } A) \cup (\text{Lpk } B + n)$ .

*Proof of Lemma 6.41.* Not unlike the proof of Lemma 6.28 (but left to the reader).  $\square$

*Proof of Proposition 6.39.* Analogous to the above second proof of Theorem 6.15, but using Lemma 6.41 instead of Lemma 6.28. This time, however, the analogue of Claim 4 will be false (i.e., we don't have  $\mathcal{K}_{\text{Lpk}} \ast A \subseteq \mathcal{K}_{\text{Lpk}}$ ), because the formula for  $\text{Lpk}([A, B])$  in Lemma 6.41 (a) depends on  $\text{Des } A$ . Thus,  $\mathcal{K}_{\text{Lpk}}$  is merely a left  $\ast$ -ideal, not a  $\ast$ -ideal. (But this does not prevent us from applying Theorem 6.12 (c), because that theorem does not require  $M \ast A \subseteq M$ .)  $\square$

*Proof of Corollary 6.40.* Corollary 6.40 can be derived from Proposition 6.39 in the same way as Corollary 6.22 was derived from Theorem 6.15.  $\square$

### 6.7.6. The right peak set $\text{Rpk}$

Recall the permutation statistic  $\text{Rpk}$  (the right peak set) defined in Definition 1.8. It follows from [GesZhu17, Theorem 4.9 (a) and Theorem 3.5] that this statistic  $\text{Rpk}$  is shuffle-compatible (since  $\text{Lpk}$  and  $\text{Rpk}$  are  $r$ -equivalent, using the terminology of [GesZhu17]). Furthermore,  $\text{Rpk}$  is clearly a descent statistic. Hence, Proposition 5.3 (applied to  $\text{st} = \text{Rpk}$ ) shows that  $\mathcal{K}_{\text{Rpk}}$  is an ideal of  $\text{QSym}$ . We now claim the following:

**Proposition 6.42.** The ideal  $\mathcal{K}_{\text{Rpk}}$  of  $\text{QSym}$  is a  $\ast$ -ideal, a right  $\phi$ -ideal, a left  $\prec$ -ideal and a left  $\succeq$ -ideal, but neither a  $\prec$ -ideal nor a  $\succeq$ -ideal of  $\text{QSym}$ .

**Corollary 6.43.** The descent statistic  $\text{Rpk}$  is neither left-shuffle-compatible nor right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 6.28):

**Lemma 6.44.** Let  $A$  and  $B$  be two nonempty compositions. Let  $n = |A|$  and  $m = |B|$ .

(a) We have  $\text{Rpk}([A, B]) = (\text{Rpk } A) \cup (\text{Rpk } B + n)$ .

(b) We have  $\text{Rpk}(A \odot B) = ((\text{Rpk } A) \setminus \{n\}) \cup (\text{Rpk } B + n) \cup \{n + 1 \mid 1 \in \text{Des } B \text{ or } m = 1\}$ .

*Proof of Lemma 6.44.* Not unlike the proof of Lemma 6.28 (but left to the reader).  $\square$

*Proof of Proposition 6.42.* To prove that  $\mathcal{K}_{\text{Rpk}}$  is a  $\ast$ -ideal and a right  $\phi$ -ideal, we proceed as in the above second proof of Theorem 6.15, but using Lemma 6.44 instead of Lemma 6.28. This time, however, the analogue of Claim 6 will be false (i.e., we don't have  $A \phi \mathcal{K}_{\text{Rpk}} \subseteq \mathcal{K}_{\text{Rpk}}$ ), because the formula for  $\text{Rpk}(A \odot B)$  in Lemma 6.44 (b) depends on  $\text{Des } B$ . Thus,  $\mathcal{K}_{\text{Rpk}}$  is merely a right  $\phi$ -ideal, not a  $\phi$ -ideal. This prevents us from applying Theorem 6.12 (c). However, we can apply Theorem 6.12 (b) instead, and obtain  $\text{QSym} \succeq \mathcal{K}_{\text{Rpk}} \subseteq \mathcal{K}_{\text{Rpk}}$ . In other words,  $\mathcal{K}_{\text{Rpk}}$  is a left  $\succeq$ -ideal of  $\text{QSym}$ . Using (73), we thus easily see that  $\mathcal{K}_{\text{Rpk}}$  is a left  $\prec$ -ideal of  $\text{QSym}$  as well.

To prove that  $\mathcal{K}_{\text{Rpk}}$  is not a  $\prec$ -ideal of  $\text{QSym}$  (and not even a right  $\prec$ -ideal of  $\text{QSym}$ ), it suffices to find some  $m \in \mathcal{K}_{\text{Rpk}}$  and some  $a \in \text{QSym}$  such that  $m \prec a \notin \mathcal{K}_{\text{Rpk}}$ . For example, we can take  $m = F_{(1,2)} - F_{(3)}$  and  $a = F_{(1)}$ ; then,

$$m \prec a = F_{(3,2)} + F_{(2,3)} + F_{(2,2,1)} - F_{(1,2,2)} - F_{(1,1,3)} - F_{(1,1,2,1)} \notin \mathcal{K}_{\text{Rpk}}.$$

The same values of  $m$  and  $a$  also satisfy  $m \succeq a \notin \mathcal{K}_{\text{Rpk}}$ ; thus,  $\mathcal{K}_{\text{Rpk}}$  is not a  $\succeq$ -ideal of  $\text{QSym}$  either. Proposition 6.42 is now proven.  $\square$

*Proof of Corollary 6.43.* Follows from Proposition 6.42.  $\square$

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### 6.7.7. The peak set $\text{Pk}$

Recall the permutation statistic  $\text{Pk}$  (the peak set) defined in Definition 1.8. It was proven in [GesZhu17, Theorem 4.7 (a)] that this statistic  $\text{Pk}$  is shuffle-compatible. Furthermore,  $\text{Pk}$  is clearly a descent statistic. Hence, Proposition 5.3 (applied to  $\text{st} = \text{Pk}$ ) shows that  $\mathcal{K}_{\text{Pk}}$  is an ideal of  $\text{QSym}$ . We now claim the following:

**Proposition 6.45.** The ideal  $\mathcal{K}_{\text{Pk}}$  of  $\text{QSym}$  is a left  $\ast$ -ideal, a right  $\phi$ -ideal, a left  $\prec$ -ideal and a left  $\succeq$ -ideal, but neither a  $\prec$ -ideal nor a  $\succeq$ -ideal of  $\text{QSym}$ .

**Corollary 6.46.** The descent statistic  $\text{Pk}$  is neither left-shuffle-compatible nor right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 6.28):

**Lemma 6.47.** Let  $A$  and  $B$  be two nonempty compositions. Let  $n = |A|$  and  $m = |B|$ .

(a) We have  $\text{Pk}([A, B]) = (\text{Pk } A) \cup (\text{Pk } B + n) \cup \{n \mid n - 1 \notin \text{Des } A \text{ and } n > 1\}$ .

(b) We have  $\text{Pk}(A \odot B) = (\text{Pk } A) \cup (\text{Pk } B + n) \cup \{n + 1 \mid 1 \in \text{Des } B\}$ .

*Proof of Lemma 6.47.* Not unlike the proof of Lemma 6.28 (but left to the reader).  $\square$

*Proof of Proposition 6.45.* To prove that  $\mathcal{K}_{\text{Pk}}$  is a left  $\ast$ -ideal and a right  $\phi$ -ideal, we proceed as in the above second proof of Theorem 6.15, but using Lemma 6.47 instead of Lemma 6.28. This time, however, the analogues of Claim 4 and Claim 6 will be false (i.e., neither  $\mathcal{K}_{\text{Pk}} \ast A \subseteq \mathcal{K}_{\text{Pk}}$  nor  $A \phi \mathcal{K}_{\text{Pk}} \subseteq \mathcal{K}_{\text{Pk}}$  will hold), because the formula for  $\text{Pk}([A, B])$  in Lemma 6.47 (a) depends on  $\text{Des } A$  whereas the formula for  $\text{Pk}(A \odot B)$  in Lemma 6.47 (b) depends on  $\text{Des } B$ . Again, this prevents us from applying Theorem 6.12 (c). However, we can apply Theorem 6.12 (b) instead, and obtain  $\text{QSym} \succeq \mathcal{K}_{\text{Pk}} \subseteq \mathcal{K}_{\text{Pk}}$ . In other words,  $\mathcal{K}_{\text{Pk}}$  is a left  $\succeq$ -ideal of  $\text{QSym}$ . Using (73), we thus easily see that  $\mathcal{K}_{\text{Pk}}$  is a left  $\prec$ -ideal of  $\text{QSym}$  as well.

To prove that  $\mathcal{K}_{\text{Pk}}$  is not a  $\prec$ -ideal of  $\text{QSym}$  (and not even a right  $\prec$ -ideal of  $\text{QSym}$ ), it suffices to find some  $m \in \mathcal{K}_{\text{Pk}}$  and some  $a \in \text{QSym}$  such that  $m \prec a \notin \mathcal{K}_{\text{Pk}}$ . For example, we can take  $m = F_{(1,2)} - F_{(3)}$  and  $a = F_{(1)}$ ; then,

$$m \prec a = F_{(3,2)} + F_{(2,3)} + F_{(2,2,1)} - F_{(1,2,2)} - F_{(1,1,3)} - F_{(1,1,2,1)} \notin \mathcal{K}_{\text{Pk}}.$$

The same values of  $m$  and  $a$  also satisfy  $m \succeq a \notin \mathcal{K}_{\text{Pk}}$ ; thus,  $\mathcal{K}_{\text{Pk}}$  is not a  $\succeq$ -ideal of  $\text{QSym}$  either. Proposition 6.45 is now proven.  $\square$

*Proof of Corollary 6.46.* Follows from Proposition 6.45.  $\square$

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