From the Vandermonde determinant to generalized factorials to greedoids and back

Darij Grinberg
joint work with Fedor Petrov

2022, New York Number Theory Zoom Seminar

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/greedtalk-ny2022.pdf
extended abstract with further references: http://www.cip.ifi.lmu.de/~grinberg/algebra/fps20gfv.pdf
solutions to exercises: at the end of the slides

1.

Bhargava's generalized factorials: an introduction

References:

- Manjul Bhargava, *P-orderings and polynomial functions on arbitrary subsets of Dedekind rings*, J. reine. angew. Math. **490** (1997), 101–127.
- Manjul Bhargava, *The Factorial Function and Generalizations*, Amer. Math. Month. **107** (2000), 783–799. (Recommended!)
- Manjul Bhargava, *On P-orderings, rings of integer-valued polynomials, and ultrametric analysis*, Journal of the AMS **22** (2009), 963–993.

- **Theorem** (classical exercise)**:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,
  $$0! \cdot 1! \cdot 2! \cdot \cdots \cdot n! \mid \prod_{i>j} (a_i - a_j) .$$
  (Here and in the following, $\prod\limits_{i>j}$ means $\prod\limits_{n \geq i > j \geq 0}$.)

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  $$\begin{aligned} 12 = &(1-0)(2-1)(2-0)(3-2)(3-1)(3-0) \\ &\mid (a_1 - a_0)(a_2 - a_1)(a_2 - a_0)(a_3 - a_2)(a_3 - a_1)(a_3 - a_0); \end{aligned}$$
  and so on.

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  This might remind you of the Vandermonde determinant, which says that
  $$\prod_{i>j} (a_i - a_j) = \det \left(a_i^j \right)_{i,j \in \{0,1,\ldots,n\}} = \det \begin{pmatrix} a_0^0 & a_0^1 & \cdots & a_0^n \\ a_1^0 & a_1^1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^0 & a_n^1 & \cdots & a_n^n \end{pmatrix} .$$

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Both are particular cases of the general ...
Theorem: If $a_0, a_1, \ldots, a_n$ are numbers, and $P_0, P_1, \ldots, P_n$ are polynomials with $\deg P_j \leq j$ for each j, then
$$\det\left(\left(P_j(a_i)\right)_{i,j\in\{0,1,\ldots,n\}}\right) = \ell_0 \ell_1 \cdots \ell_n \prod_{i>j} (a_i - a_j),$$
where ℓ_j is the x^j-coefficient of P_j. [**Exercise 1:** Prove this!]

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- **Hint to proof 2:** WLOG assume $0 \leq a_0 < a_1 < \cdots < a_n$.
  (Otherwise, move each a_i preserving $a_i \bmod \mathrm{LHS}$.)

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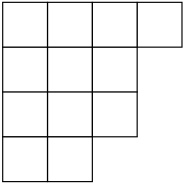
- **Hint to proof 2:** WLOG assume $0 \leq a_0 < a_1 < \cdots < a_n$.
  Consider an array of $n+1$ left-justified rows with lengths
  $a_0 - 0, a_1 - 1, \ldots, a_n - n$ from bottom to top:
  e.g., if $n = 3$ and $(a_0, a_1, \ldots, a_n) = (2, 4, 5, 7)$, then it is



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  Now fill this array with numbers $\in \{1, 2, \ldots, n+1\}$ that **increase weakly along rows** and **increase strictly down columns**, e.g.:

1	1	2	4
2	2	3	
3	4	4	
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(a "semistandard tableau") .

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("Weyl's character formula" in type A; see MathOverflow question #106606.)

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("Weyl's character formula" in type A; see MathOverflow question #106606.) **Question:** Bijective proof?

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  $$\prod_{i>j} (i-j) \mid \prod_{i>j} (a_i - a_j) .$$
- **Hint to proof 3:** To show that $u \mid v$, it suffices to prove that every prime p divides v at least as often as it does u.
  Now get your hands dirty.

What about squares?

- **Theorem** (Bhargava?)**:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,
  $$\frac{0! \cdot 2! \cdot \cdots \cdot (2n)!}{2^n} \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$
  (Typo in Bhargava corrected.)

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  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,
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- Analogues of the 3 above proofs work (I believe).
  [**Exercise 3:** Check this!]

What about cubes?

- **Question:** Do we also have

$$\prod_{i>j} \left(i^3 - j^3\right) \mid \prod_{i>j} \left(a_i^3 - a_j^3\right) \ ?$$

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- So what is

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- Already for $n = 6$, there is no choice of $a_0, a_1, \ldots, a_n$ that attains the gcd (such as $0, 1, \ldots, n$ was for first powers and for squares).

- **General question** (Bhargava, 1997)**:** Let S be a set of integers. Fix $n \geq 0$. What is

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- Enough to work out each prime p separately, because:

p-valuation

- Let p be a prime. Set $\mathbb{N} := \{0, 1, 2, \ldots\}$.
- For each nonzero $n \in \mathbb{Z}$, let $v_p(n)$ (the *p-valuation* of n) be the highest $k \in \mathbb{N}$ such that $p^k \mid n$.
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- **Examples:**

$v_3(18) = 2;$	$v_3(17) = 0;$
$v_2(14) = 1;$	$v_2(16) = 4.$

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- **Rules for p-valuations:**

$$\begin{array}{c|c} v_p(1) = 0; & v_p(ab) = v_p(a) + v_p(b); \\ v_p\left(p^k\right) = k; & v_p(a+b) \geq \min\{v_p(a), v_p(b)\}. \end{array}$$

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- Define the *p-distance $d_p(a, b)$* between two integers a and b by

$$d_p(a, b) = -v_p(a - b).$$

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The p-distance is not very geometric: For instance, 2 is closer to $p + 2$ than to 1, and even closer to $p^2 + 2$.
Cf. the p-adic solenoid. Also, artistic rendition by Fomenko.

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- Two integers u and v satisfy $u \mid v$ if and only if

$$v_p(u) \leq v_p(v) \qquad \text{for each prime } p.$$

  Thus, checking divisibility is reduced to a "local" problem.

- **Equivalent problem:** Let S be a set of integers. Let p be a prime. Fix $n \geq 0$. What is

$$\min \left\{ v_p \left(\prod_{i>j} (a_i - a_j) \right) \;\middle|\; a_0, a_1, \ldots, a_n \in S \right\} ?$$

Equivalent restatement of the problem

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- We can WLOG assume that $a_0, a_1, \ldots, a_n$ are distinct.

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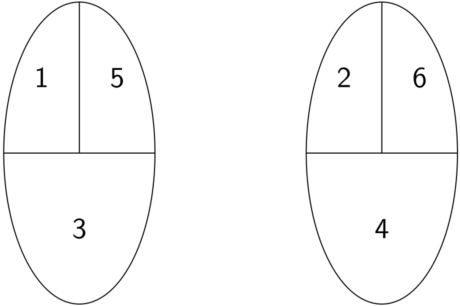

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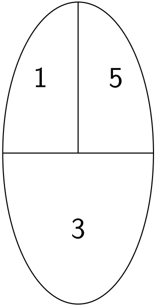

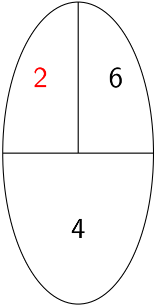

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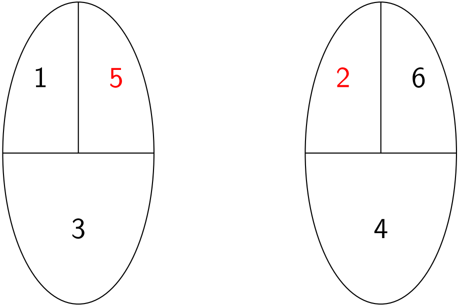

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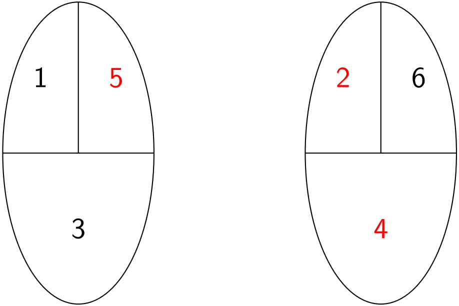

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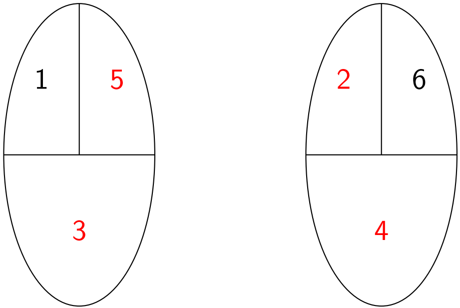

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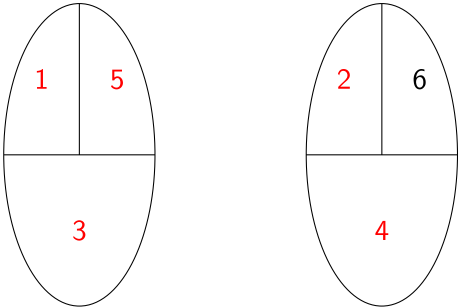

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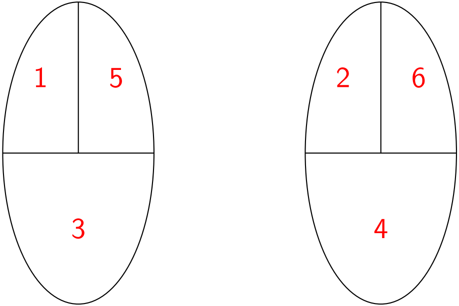

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- **Note:** There is such a sequence for each prime p, but there isn't always such a sequence that works for all p simultaneously.

- In his first (1997) paper on the subject, Bhargava already noticed that p is a red herring: The properties of d_p are all that is needed.

  > "*We note that the above results (i.e. Theorem 1, Lemmas 1 and 2) do not rely on any special properties of P or R; they depend only on the fact that R becomes an ultrametric space when given the P-adic metric. Hence these results could be viewed more generally as statements about certain special sequences in ultrametric spaces. For convenience, however, we have chosen to present these statements only in the relevant context. In particular, we note that our proof of Theorem 1 shall be a purely algebraic one, involving no inequalities.*"

  (Theorem 1 is a slight generalization of the above Theorem.)

2.

Ultra triples

References:

- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
- Darij Grinberg, *The Bhargava greedoid as a Gaussian elimination greedoid*, arXiv:2001.05535.
- Alex J. Lemin, *The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices LAT**, Algebra univers. **50** (2003), pp. 35–49.

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- More generally, we can replace $\mathbb{R}$ by any totally ordered abelian group $\mathbb{V}$.

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- We will only consider ultra triples with **finite** ground set E. (Bhargava's E is infinite, but results adapt easily.)

Ultra triples, examples: 1 (congruence)

- **Example:** Let $E \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$. Define a map $w : E \to \mathbb{R}$ arbitrarily. Define a map $d : E \underline{\times} E \to \mathbb{R}$ by

  $$d(a, b) = \begin{cases} 0, & \text{if } a \equiv b \mod n; \\ 1, & \text{if } a \not\equiv b \mod n \end{cases} \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

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where ε and α are fixed reals with $\varepsilon \leq \alpha$. Then, (E, w, d) is an ultra triple.

Ultra triples, examples: 2 (p-adic distance)

- Let p be a prime. Let $E \subseteq \mathbb{Z}$. Define the weights $w(e) \in \mathbb{R}$ arbitrarily. Then, (E, w, d_p) is an ultra triple.
  Here, d_p is as before:
  $$d_p(a, b) = -v_p(a - b).$$
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  Thus, we call such a triple (E, w, d_p) a *Bhargava-type ultra triple*.

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- More generally, we can replace $p^0, p^1, p^2, \ldots$ with any unbounded sequence $r_0 \mid r_1 \mid r_2 \mid \cdots$ of integers.

- Let E be the set of all living organisms. Let

$$d(e,f) = \begin{cases} 0, & \text{if } e = f; \\ 1, & \text{if } e \text{ and } f \text{ belong to the same species;} \\ 2, & \text{if } e \text{ and } f \text{ belong to the same genus;} \\ 3, & \text{if } e \text{ and } f \text{ belong to the same family;} \\ \ldots \end{cases}$$

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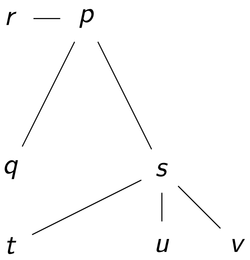
- More generally, any "nested" family of equivalence relations on E gives a distance function for an ultra triple.

- Let T be a (finite, undirected) tree. For each edge e of T, let $\lambda(e) \geq 0$ be a real. We shall call this real the *weight* of e.



Ultra triples, examples: 4 (Darwin)

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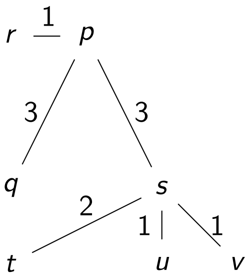

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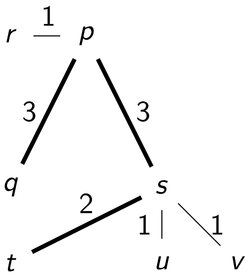

$$\Longrightarrow \qquad \lambda(q, t) = 3 + 3 + 2.$$

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- Fix any vertex r of T. Let E be any subset of the vertex set of T. Set

  $$d(x, y) = \lambda(x, y) - \lambda(x, r) - \lambda(y, r) \qquad \text{for each } (x, y) \in E \underline{\times} E.$$

  Then, (E, w, d) is an ultra triple for any $w : E \to \mathbb{R}$.

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- **Hint to proof:** Use the well-known fact ("four-point condition") saying that if x, y, z, w are four vertices of T, then the two largest of the three numbers

  $$\lambda(x, y) + \lambda(z, w), \quad \lambda(x, z) + \lambda(y, w), \quad \lambda(x, w) + \lambda(y, z)$$

  are equal. [**Exercise 4:** Prove this!]

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- This is particularly useful when T is a *phylogenetic tree* and E is a set of its leaves.
  Actually, this is the general case: Any (finite) ultra triple can be translated back into a phylogenetic tree. It is "essentially" an inverse operation.
  (The idea is not new; see, e.g., Lemin 2003.)

- Let (E, w, d) be an ultra triple, and $S \subseteq E$ be any subset. Then, the *perimeter* of S is defined to be

$$\mathrm{PER}(S) := \underbrace{\sum_{x \in S} w(x)}_{|S| \text{ addends}} + \underbrace{\sum_{\substack{\{x,y\} \subseteq S; \\ x \neq y}} d(x,y)}_{\binom{|S|}{2} \text{ addends}}.$$

Perimeters in ultra triples

- Let (E, w, d) be an ultra triple, and $S \subseteq E$ be any subset. Then, the *perimeter* of S is defined to be

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- Thus,

$$\begin{aligned}
\operatorname{PER} \varnothing &= 0; \\
\operatorname{PER}\{x\} &= w(x); \\
\operatorname{PER}\{x, y\} &= w(x) + w(y) + d(x, y); \\
\operatorname{PER}\{x, y, z\} &= w(x) + w(y) + w(z) \\
&\quad + d(x, y) + d(x, z) + d(y, z).
\end{aligned}$$

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- **Bhargava's problem (generalized):** Given an ultra triple (E, w, d) and an $n \in \mathbb{N}$, find the maximum perimeter of an n-element subset of E, and find the subsets that attain it. (The n here corresponds to the $n + 1$ before.)

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$$\mathrm{PER}(S) \quad := \quad \underbrace{\sum_{x \in S} w(x)}_{|S| \text{ addends}} \quad + \quad \underbrace{\sum_{\substack{\{x,y\} \subseteq S; \\ x \neq y}} d(x, y)}_{\binom{|S|}{2} \text{ addends}} .$$

- **Bhargava's problem (generalized):** Given an ultra triple (E, w, d) and an $n \in \mathbb{N}$, find the maximum perimeter of an n-element subset of E, and find the subsets that attain it. (The n here corresponds to the $n+1$ before.)
- For $E \subseteq \mathbb{Z}$ and $w(e) = 0$ and $d_p(a, b) = -v_p(a - b)$, this is Bhargava's problem.

- Let (E, w, d) be an ultra triple, and $S \subseteq E$ be any subset. Then, the *perimeter* of S is defined to be

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- For Linnaeus or Darwin ultra triples, this is a "Noah's Ark" problem: What choices of n organisms maximize biodiversity? A similar problem has been studied in: Vincent Moulton, Charles Semple, Mike Steel, *Optimizing phylogenetic diversity under constraints*, J. Theor. Biol. **246** (2007), pp. 186–194.

3.

Solving the problem

References:

- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
- Darij Grinberg, *The Bhargava greedoid as a Gaussian elimination greedoid*, arXiv:2001.05535.

Greedy permutations: definition

- Fix an ultra triple (E, w, d).

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- Let $m \in \mathbb{N}$. A *greedy m-permutation* of E is a list $(c_1, c_2, \ldots, c_m)$ of m distinct elements of E such that for each $i \in \{1, 2, \ldots, m\}$ and each $x \in E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$, we have

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- In other words, a greedy m-permutation of E is what you obtain if you try to greedily construct a maximum-perimeter m-element subset of E, by starting with $\varnothing$ and adding new points one at a time.

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Greedy permutations: examples

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- In Example 1 (congruence modulo n), a greedy m-permutation is one in which all congruence classes (that appear in S) are "represented as equitably as possible".
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Greedy permutations: theorems

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- **Theorem (Petrov, G.):** Let $m, k \in \mathbb{N}$ with $|E| \geq m \geq k$. Let A be a k-element subset of E that has maximum perimeter among all such.
  Then, there exists a greedy m-permutation $(c_1, c_2, \ldots, c_m)$ of E such that $A = \{c_1, c_2, \ldots, c_k\}$.

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- **Exercise 5:** Use this to prove

$$\prod_{i>j} (i-j) \mid \prod_{i>j} (a_i - a_j) \qquad \text{and} \qquad \prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

4.

Greedoids

References:

- Bernhard Korte, László Lovász, Rainer Schrader, *Greedoids*, Algorithms and Combinatorics #4, Springer 1991.
- Anders Björner, Günter M. Ziegler, *Introd. to Greedoids*, in: Neil White (ed.), *Matroid applications*, CUP 1992.
- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
- Darij Grinberg, *The Bhargava greedoid as a Gaussian elimination greedoid*, arXiv:2001.05535.
- Victor Bryant, Ian Sharpe, *Gaussian, Strong and Transversal Greedoids*, Europ. J. Comb. **20** (1999), pp. 259–262.

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- This is characteristic of a *greedoid* – a "noncommutative analogue" of a matroid.
- I will now define greedoids.
  Warning: some abstraction to follow.

Greedoids

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- **Example:** $E = \{1, 2, 3, 4, 5\}$ and

$$\mathcal{F} = \Big\{\varnothing, \{1\}, \{2\}, \{5\}, \{1,2\}, \{1,5\}, \{2,5\}, \{4,5\}, \{1,2,3\}, \{1,2,5\}, \{2,4,5\}, \{3,4,5\}\Big\}.$$

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Axiom 3. holds for $A = \{4, 5\}$ and $B = \{2, 4, 5\}$, since we can take $b = 2$.
(More generally, Axiom 3. always holds if $A \subseteq B$.)

- If you have seen matroids:
  Let M be a matroid on a ground set E. Then,

  $$\{\text{independent sets of } M\}$$

  is a greedoid on E.
  We shall call this a *matroid greedoid*.

- Let A be an $m \times n$-matrix over a field $\mathbb{K}$. Let $E = \{1, 2, \ldots, n\}$. Then,

  $$\left\{ F \subseteq E \ \mid \ \text{we have } |F| \leq n \text{ and } \det\left(\operatorname{sub}_{\{1,2,\ldots,|F|\}}^{F} A\right) \neq 0 \right\}$$

  is a greedoid on E, where $\operatorname{sub}_F^G A$ means the submatrix of A with rows indexed by F and columns indexed by G.

Greedoids, examples: 2 (Gaussian elimination)

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- This is called a *Gaussian elimination greedoid* over $\mathbb{K}$. We denote it by $\mathrm{GEG}(A)$.
  [**Exercise 6:** Prove that it is a greedoid!]

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- For example, if $\mathbb{K} = \mathbb{Q}$ and $m = 5$ and $n = 5$ and
$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ then}$$

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$$\{2, 5\} \in \mathrm{GEG}(A), \qquad \text{since } \det\left(\operatorname{sub}^{\{2,5\}}_{\{1,2\}} A\right) \neq 0.$$

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$$\{1, 2, 3, 5\} \in \operatorname{GEG}(A), \qquad \text{since } \det\left(\operatorname{sub}_{\{1,2,3,4\}}^{\{1,2,3,5\}} A\right) \neq 0.$$

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$$\{3, 5\} \notin \operatorname{GEG}(A), \qquad \text{since } \det\left(\operatorname{sub}^{\{3,5\}}_{\{1,2\}} A\right) = 0.$$

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$$\operatorname{GEG}(A) = \Big\{ \varnothing, \{2\}, \{3\}, \{5\}, \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}, \{1,2,3\}, \{1,2,5\}, \{1,2,3,5\} \Big\}.$$

- Let P be a finite poset. Let J be the set of all *order ideals* of P (that is, of all subsets I of P such that $(b \in I) \wedge (a \leq b) \Longrightarrow (a \in I)$).
- Then, J is a greedoid on P. [**Exercise 7:** Prove this!] We shall call this an *order ideal greedoid*.

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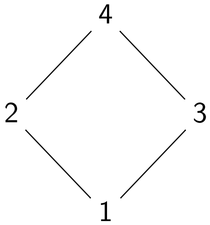
- **Example:** If P is the poset with Hasse diagram



then

$$G = \left\{ \varnothing, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\} \right\}.$$

- Let T be a tree with vertex set V. Let G be the set of all subsets $U \subseteq V$ such that the induced subgraph on $V \setminus U$ is connected (i.e., no vertex in U lies on the path between two vertices in $V \setminus U$).
- Then, G is a greedoid on V. [**Exercise 8:** Prove this!]

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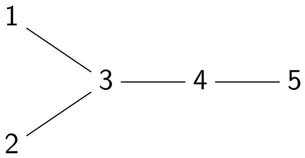
- **Example:** If T is the tree



then

$$G = \Big\{ \varnothing, \{1\}, \{2\}, \{5\}, \{1,2\}, \{1,5\}, \{2,5\}, \{4,5\}, \\ \{1,2,3\}, \{1,2,5\}, \{2,4,5\}, \{1,2,3,4\}, \\ \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\} \Big\}.$$

- Back to our setting: For any ultra triple (E, w, d), define

$$\begin{aligned}\mathcal{B}(E, w, d) &= \{A \subseteq E \mid A \text{ has maximum perimeter among} \\ &\qquad \text{all } |A| \text{-element subsets of } E\} \\ &= \{A \subseteq E \mid \mathrm{PER}(A) \geq \mathrm{PER}(B) \text{ for} \\ &\qquad \text{all } B \subseteq E \text{ satisfying } |B| = |A|\}.\end{aligned}$$

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- **Theorem (G., Petrov):** This Bhargava greedoid $\mathcal{B}(E, w, d)$ is a greedoid indeed.

- Recall: A *greedoid* on a set E means a set system $\mathcal{F}$ on E such that
  1. We have $\varnothing \in \mathcal{F}$.
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- **Remark:** In axiom 3., we can replace the condition "$|B| = |A| + 1$" by the weaker "$|B| > |A|$"; the axiom stays equivalent.
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- Strong greedoids are also known as "*Gauss greedoids*" (not to be confused with Gaussian elimination greedoids).

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- **Theorem (Bryant, Sharpe):** Let $\mathcal{F}$ be a strong greedoid, and $k \in \mathbb{N}$. Then, the k-element sets that belong to $\mathcal{F}$ are the bases of a matroid (unless there are none of them).
  If $\mathcal{F}$ is a Gaussian elimination greedoid, then the latter matroid is representable.

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- Note that this Theorem yields the previous one, which is thus proved twice.
- **Stronger theorem (G.):** Let (E, w, d) be an ultra triple. Let $\mathbb{K}$ be any field of size $|\mathbb{K}| \geq \operatorname{mcs}(E, w, d)$, where $\operatorname{mcs}(E, w, d)$ is the *maximum clique size* of E (that is, the maximum size of a subset $C \subseteq E$ such that $d \mid_{C \underline{\times} C}$ is constant).
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- Note that this Theorem yields the previous one, which is thus proved twice.
- **Converse theorem (G.):** Assume that the map w is constant. Let $\mathbb{K}$ be a field. Then, the Bhargava greedoid $\mathcal{B}(E, w, d)$ is (up to renaming the elements of E) a Gaussian elimination greedoid over $\mathbb{K}$ **if and only if** $|\mathbb{K}| \geq \operatorname{mcs}(E, w, d)$.

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- **1st step:** If (E, w, d) is a Bhargava-type ultra triple (E, w, d_p) for some prime p and some $E \subseteq \mathbb{Z}$, then we can explicitly find a matrix A over $\mathbb{F}_p$ that gives $\mathcal{B}(E, w, d)$ as its Gaussian elimination greedoid.

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  Even better, this matrix A is the projection of a matrix $\widetilde{A}$ over $\mathbb{Z}$ that satisfies
  $$v_p\left(\det\left(\operatorname{sub}^F_{\{1,2,\ldots,|F|\}} \widetilde{A}\right)\right) = (\text{max. possible perimeter}) - \operatorname{PER}(F)$$
  for each subset F of E.
  (The matrix $\widetilde{A}$ is a Vandermonde-like matrix, with entries $\dfrac{1}{p^{\text{something}}}(a_i - e_1)(a_i - e_2)\cdots(a_i - e_j)$.)

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- **2nd step:** So we know how to deal with Bhargava-type ultra triples. It would be nice if all ultra triples were isomorphic to some of them!

  I'm not sure this is true, but I can prove something close that suffices:

- **2nd step, continued:** Replace $\mathbb{Z}$ by the "polynomial ring" $\mathbb{K}[t]$, except that all powers t^a with $a \in \mathbb{R}_+$ are allowed (not just for integer a).
  For example,
  $$3 + 2t^{0.5} - 7t^{0.8} + 4t^{3.2} \qquad \text{lies in this ring.}$$

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- **2nd step, continued:** Replace $\mathbb{Z}$ by the "polynomial ring" $\mathbb{K}[t]$, except that all powers t^a with $a \in \mathbb{R}_+$ are allowed (not just for integer a).
  Replace v_p by v_t (which sends any polynomial to the lowest exponent of t that appears in it).
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  $$v_t\left(3t^{0.2} + 2t^{0.5} - 7t^{0.8} + 4t^{3.2}\right) = 0.2.$$

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  (The proof proceeds by strong induction, decomposing the ultra triple into smaller ones. Iterating this decomposition again reveals the connection to phylogenetic trees.)
- **Note:** In proving the general case, we had to come back to our original example, the (generalized) Vandermonde determinant!

Questions

- If w is constant, then we have a necessary and sufficient condition for $\mathcal{B}(E, w, d)$ to be a Gaussian elimination greedoid over $\mathbb{K}$.
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- It is not too hard to define a multiset analogue of greedoids (e.g., by lifting the "simple" requirement on greedoid languages). How much of the theory adapts?

Thank you!

- **Fedor Petrov** for getting this started by answering my MathOverflow question #314130.
- **Alexander Postnikov** for interesting conversations.
- **Melvyn Nathanson** for the invitation.
- **you** for your patience and typo hunting.

Hints to Exercise 1

Exercise 1: Prove the following **Theorem:** If $a_0, a_1, \ldots, a_n$ are numbers, and $P_0, P_1, \ldots, P_n$ are polynomials with $\deg P_j \leq j$ for each j, then

$$\det\left(\left(P_j\left(a_i\right)\right)_{i,j\in\{0,1,\ldots,n\}}\right) = \ell_0 \ell_1 \cdots \ell_n \prod_{i>j}\left(a_i - a_j\right),$$

where ℓ_j is the x^j-coefficient of P_j.

Hints to solution:

One approach is by root identification: WLOG assume that $\ell_0 = \ell_1 = \cdots = \ell_n = 1$ (that is, the polynomials P_j are monic), and consider both the a_i's and the remaining coefficients of all P_j's as indeterminates. Argue that $\det\left(\left(P_j\left(a_i\right)\right)_{i,j\in\{0,1,\ldots,n\}}\right)$ is a polynomial of total degree $\leq n\left(n-1\right)/2$ that vanishes whenever two of the a_i's are equal. A standard divisibility argument then finishes the job.

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Another approach is explicit: Write each P_j as $P_j\left(X\right) = \sum_{k=0}^{j} \ell_{j,k} X^k$ (so that $\ell_{j,j} = \ell_j$), and observe that

$$\left(P_j\left(a_i\right) \right)_{i,j \in \{0,1,\ldots,n\}} = \underbrace{\begin{pmatrix} a_0^0 & a_0^1 & \cdots & a_0^n \\ a_1^0 & a_1^1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^0 & a_n^1 & \cdots & a_n^n \end{pmatrix}}_{\substack{\text{Vandermonde matrix} \\ \text{with known determinant}}} \cdot \underbrace{\begin{pmatrix} \ell_{0,0} & \ell_{1,0} & \cdots & \ell_{n,0} \\ 0 & \ell_{1,1} & \cdots & \ell_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \ell_{n,n} \end{pmatrix}}_{\substack{\text{triangular matrix} \\ \text{with determinant} \\ \ell_{0,0}\ell_{1,1}\cdots\ell_{n,n}=\ell_0\ell_1\cdots\ell_n}}.$$

See Theorem 2 in

https://www.cip.ifi.lmu.de/~grinberg/hyperfactorialBRIEF.pdf for a more self-contained variant of this proof.

Hints to Exercise 2

Exercise 2: Prove that $\dfrac{0!\cdot 2!\cdot\cdots\cdot(2n)!}{2^n} = \prod_{n\geq i>j\geq 0}\left(i^2-j^2\right)$ for any $n\geq 0$.

Hints to solution:

$$\prod_{n\geq i>j\geq 0}\left(i^2-j^2\right) = \prod_{i=1}^{n}\prod_{j=0}^{i-1}\underbrace{\left(i^2-j^2\right)}_{=(i-j)(i+j)} = \prod_{i=1}^{n}\underbrace{\prod_{j=0}^{i-1}(i-j)(i+j)}_{\substack{=(2i)!/2\\ \text{(check this!)}}}$$

$$= \prod_{i=1}^{n}\left((2i)!/2\right) = \frac{2!\cdot 4!\cdot\cdots\cdot(2n)!}{2^n} = \frac{0!\cdot 2!\cdot\cdots\cdot(2n)!}{2^n}.$$

Hints to Exercise 3

Exercise 3: Prove Bhargava's theorem: Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,

$$\prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

Hints to solution: We can imitate each of the three proofs of $\prod_{i>j} (i-j) \mid \prod_{i>j} (a_i - a_j)$.

Analogue of proof 1: For each $a \in \mathbb{R}$ and $k \in \mathbb{N}$, define

$$\binom{a}{k}' := \frac{(a(a+1)(a+2)\cdots(a+k-1)) \cdot (a(a-1)(a-2)\cdots(a-k+1))}{k \cdot (2k-1)!}$$

(read the fraction as 1 if $k = 0$). This is an even polynomial in a of degree $2k$ with leading coefficient $\dfrac{2}{(2k)!}$ (or 1 if $k = 0$). Thus, it is a polynomial in a^2 of degree k with leading coefficient $\dfrac{2}{(2k)!}$ (or 1 if $k = 0$). Hence, Exercise 1 yields

$$\det\left(\left(\binom{a_i}{j}'\right)_{i,j\in\{0,1,\ldots,n\}}\right) = \prod_{k=1}^{n} \frac{2}{(2k)!} \cdot \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

Hints to Exercise 3

Exercise 3: Prove Bhargava's theorem: Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,

$$\prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

Hints to solution: We can imitate each of the three proofs of $\prod_{i>j} (i - j) \mid \prod_{i>j} (a_i - a_j)$.

However, for any $a \in \mathbb{Z}$, we have

$$\binom{a}{k}' = \binom{a+k}{2k} + \binom{a+k-1}{2k} \in \mathbb{Z}$$

(check this!). Thus, the determinant on the LHS is an integer, so the RHS is an integer too. In other words, $\prod_{k=1}^{n} \dfrac{(2k)!}{2} \mid \prod_{i>j} \left(a_i^2 - a_j^2\right)$. Now use Exercise 2.

Exercise 3: Prove Bhargava's theorem: Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,

$$\prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

Hints to solution: We can imitate each of the three proofs of $\prod_{i>j} (i - j) \mid \prod_{i>j} (a_i - a_j)$.

Analogue of proof 2: The number $\dfrac{\text{RHS}}{\text{LHS}}$ counts a variant of semistandard tableaux called *King tableaux*. See Exercise 24.42 in Fulton/Harris, *Representation Theory: A First Course*, 2004 for a proof that $\dfrac{\text{RHS}}{\text{LHS}}$ is the dimension of an irreducible representation of $\mathfrak{so}_{2(n+1)}(\mathbb{C})$, and see Robert A. Proctor, *Young Tableaux, Gelfand Patterns, and Branching Rules for Classical Groups*, 1994 for the notion of tableaux that is counted by this dimension. (Thanks to Travis Scrimshaw for the latter reference.)

Analogue of proof 3: This becomes a lot easier using the results of Chapter 2 (see Exercise 5 below).

Hints to Exercise 4

Exercise 4: Let T be a tree. Let x, y, z, w be four vertices of T. Prove that the two largest of the three numbers

$$\lambda(x, y) + \lambda(z, w), \qquad \lambda(x, z) + \lambda(y, w), \qquad \lambda(x, w) + \lambda(y, z)$$

are equal.

Hints to solution: This is a well-known fact known as the **four-point condition**, although few authors bother to prove it. Three proofs can be found in my Spring 2017 graph theory course, where it was exercise 6 on midterm #2. Sasha Pevzner's solution is probably the nicest. The idea is to induct on the size of T; in the induction step, remove a leaf from T and distinguish cases based on how many of x, y, z, w are equal to this leaf.

Exercise 5: Prove

$$\prod_{i>j} (i-j) \mid \prod_{i>j} (a_i - a_j) \qquad \text{and} \qquad \prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

using greedy m-permutations.

Hints to solution: Let p be a prime. Let E be a finite subset of $\mathbb{Z}$ that contains $0, 1, \ldots, n$ as well as $a_0, a_1, \ldots, a_n$. Equip this set E with the p-adic distance $d = d_p$ and the zero weight function w (so that $w(e) = 0$ for all $e \in E$). Thus, (E, w, d) is an ultra triple.

Check that the tuple $(0, 1, \ldots, n)$ is a greedy $(n+1)$-permutation of E (this uses the fact that binomial coefficients are $\in \mathbb{Z}$). Hence, by the first Petrov-G. result, the set $\{0, 1, \ldots, n\}$ has maximum perimeter among all $(n+1)$-element subsets of E. Thus, $\operatorname{PER}\{0, 1, \ldots, n\} \geq \operatorname{PER}\{a_0, a_1, \ldots, a_n\}$. In other words, $v_p\left(\prod_{i>j}(i-j)\right) \leq v_p\left(\prod_{i>j}(a_i - a_j)\right)$. Since this holds for all primes p, we thus conclude that $\prod_{i>j}(i-j) \mid \prod_{i>j}(a_i - a_j)$.

A similar argument shows that $\prod_{i>j}\left(i^2 - j^2\right) \mid \prod_{i>j}\left(a_i^2 - a_j^2\right)$. Here, we need to show that the tuple $\left(0^2, 1^2, \ldots, n^2\right)$ is a greedy $(n+1)$-permutation of E. (Again, use binomial coefficients.)

Exercise 6: Prove that the Gaussian elimination greedoid of a matrix is a greedoid.

Hints to solution: Even better, we'll show that it is a strong greedoid (defined near the end of the talk). So we need to verify Axioms 1, 2 and 4 (since Axiom 3 follows from 4).

Axiom 1 is obvious. Axiom 2 follows from Laplace expansion. For Axiom 4, we need the following determinantal identity:

Plücker identity (one of many): Let n be a positive integer. Let X be an $n \times (n-1)$-matrix, and Y an $n \times n$-matrix. Let $i \in \{1, 2, \ldots, n\}$. Then,

$$\det\left(X_{\sim i,\bullet}\right) \det Y = \sum_{q=1}^{n} (-1)^{n+q} \det\left(X \mid Y_{\bullet,q}\right) \det\left(Y_{\sim i,\sim q}\right).$$

Here,

- $X_{\sim i,\bullet}$ means the matrix X without its i-th row;
- $(X \mid Y_{\bullet,q})$ means the matrix X with the q-th column of Y appended to it on the right;
- $Y_{\sim i,\sim q}$ means the matrix Y without its i-th row and its q-th column.

See the Appendix in *The Bhargava greedoid as a Gaussian elimination greedoid* for details.

The easiest way to prove the Plücker identity is to expand $\det\left(X \mid Y_{\bullet,q}\right)$ along the last column and use $Y \cdot \operatorname{adj} Y = \det Y \cdot I_n$.

Exercise 7: Let P be a finite poset. Let J be the set of all *order ideals* of P (that is, of all subsets I of P such that $(b \in I) \wedge (a \leq b) \Longrightarrow (a \in I)$).
Prove that J is a greedoid on P.
Hints to solution: Axioms 1 and 2 are easy. For Axiom 3, let A and B be two order ideals of P satisfying $|B| = |A| + 1$. We must prove that there exists $b \in B \setminus A$ such that $A \cup \{b\}$ is again an order ideal of P.
This is easy: Just let b be a minimal element of $B \setminus A$.

Exercise 8: Let T be a tree with vertex set V. Let G be the set of all subsets $U \subseteq V$ such that the induced subgraph on $V \setminus U$ is connected (i.e., no vertex in U lies on the path between two vertices in $V \setminus U$).
Prove that G is a greedoid on V.
Hints to solution: It is best to restate Axioms 1, 2, 3 in terms of the complements $\overline{A}$ of the sets A. Thus, we must prove:

1. The induced subgraph on $V \setminus \varnothing$ is connected.
2. If a proper subset $\overline{B} \subseteq V$ is such that the induced subgraph on $\overline{B}$ is connected, then there exists $b \notin \overline{B}$ such that the induced subgraph on $\overline{B} \cup \{b\}$ is again connected.
3. If $\overline{A}, \overline{B} \subseteq V$ are such that the induced subgraphs on $\overline{A}$ and on $\overline{B}$ are both connected, and if $\left|\overline{B}\right| = \left|\overline{A}\right| - 1$, then there exists $b \in \overline{A} \setminus \overline{B}$ such that the induced subgraph on $\overline{A} \setminus \{b\}$ is connected.

Item 1 is obvious. For item 2, just add a neighbor of a $\overline{B}$-vertex that is not in $\overline{B}$. For item 3, observe that the induced subgraph on $\overline{A}$ is itself a tree. Call this tree T'. Each vertex of a tree lies on a path connecting two leaves (why?). Thus, if all leaves of T' were contained in $\overline{B}$, then $\overline{A}$ would be contained in $\overline{B}$, contradicting $\left|\overline{B}\right| = \left|\overline{A}\right| - 1$. Hence, there exists a leaf of T' that doesn't lie in $\overline{B}$. Let b be such a leaf.

Hints to Exercise 9

Exercise 9: Show that the order ideal greedoid of a finite poset P is a strong greedoid if and only if P is a disjoint union of subsets $P_0, P_1, P_2, \ldots$ such that any two elements $p \in P_i$ and $q \in P_j$ satisfy $p < q$ if and only if $i < j$.
Hints to solution: $\Longrightarrow$: Assume that the greedoid is a strong greedoid.
For each $i \geq 0$, let P_i be the set of all $p \in P$ with the following property: The largest chain of P with largest element p has exactly $i + 1$ elements. (Thus, P_0 consists of the minimal elements of P.)
We must show that any two elements $p \in P_i$ and $q \in P_j$ satisfy $p < q$ if and only if $i < j$. The "only if" part is clear, so we need to prove the "if" part. We prove it by strong induction on j. So fix a j and two elements $p \in P_i$ and $q \in P_j$ such that $i < j$. We must show that $p < q$, assuming that the same has been showed "for all smaller j's".
Let r be the second-highest element of a maximum-length chain ending in q; thus, $r \in P_{j-1}$. If $i < j - 1$, then the induction hypothesis yields $p < r$, hence $p < r < q$, and we are done. Hence, WLOG we don't have $i < j - 1$. Thus, $i = j - 1$ and $r \in P_{j-1} = P_i$.
Let $J := P_0 \cup P_1 \cup \cdots \cup P_i$. This is an order ideal of P. So is $J \setminus \{r\}$ (why?). So is $(J \setminus \{p\}) \cup \{q\}$ unless $p < q$ (why?). Thus, unless $p < q$, then Axiom 4 of a strong greedoid (applied to $A = J \setminus \{r\}$ and $B = (J \setminus \{p\}) \cup \{q\}$) yields that there is some $b \in \{q, r\}$ such that $(J \setminus \{r\}) \cup \{b\}$ and $((J \setminus \{p\}) \cup \{q\}) \setminus \{b\}$ are order ideals. But this is easily seen to be false.
Hence, we must have $p < q$, and the induction is complete.

Hints to Exercise 9

Exercise 9: Show that the order ideal greedoid of a finite poset P is a strong greedoid if and only if P is a disjoint union of subsets $P_0, P_1, P_2, \ldots$ such that any two elements $p \in P_i$ and $q \in P_j$ satisfy $p < q$ if and only if $i < j$.

Hints to solution:

$\Longleftarrow$: Straightforward (characterize the order ideals of this disjoint union).

Exercise 10: Show that the greedoid of a tree T is a strong greedoid if and only if T is a star.

Hints to solution: $\Longrightarrow$: Assume that the greedoid is a strong greedoid. If T has a path $u - v - w - t$ of length 3, then the two sets $A = V \setminus \{u, v\}$ and $B = V \setminus \{t\}$ contradict Axiom 4 of strong greedoids. Thus, T has no path of length 3. This easily yields that T is a star (just pick any path $u - v - w$ of length 2 and argue that each vertex $\neq v$ must be a neighbor of v).

$\Longleftarrow$: Straightforward.