Gaussian elimination greedoids from ultrametric spaces

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joint work with Fedor Petrov

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/greedtalk-iml2020.pdf
extended abstract with further references: http://www.cip.ifi.lmu.de/~grinberg/algebra/fps20gfv.pdf

1.

Bhargava's generalized factorials: an introduction

References:

- Manjul Bhargava, *P-orderings and polynomial functions on arbitrary subsets of Dedekind rings*, J. reine. angew. Math. **490** (1997), 101–127.
- Manjul Bhargava, *The Factorial Function and Generalizations*, Amer. Math. Month. **107** (2000), 783–799. (Recommended!)
- Manjul Bhargava, *On P-orderings, rings of integer-valued polynomials, and ultrametric analysis*, Journal of the AMS **22** (2009), 963–993.

It begins with a Vandermonde

- **Theorem** (classical exercise)**:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,

$$0! \cdot 1! \cdot 2! \cdot \cdots \cdot n! \mid \prod_{i>j} (a_i - a_j) .$$

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- **Theorem** (classical exercise, slightly restated)**:**
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- **Hint to proof 1:** Show that
  $$\frac{\text{RHS}}{\text{LHS}} = \det \left(\binom{a_i}{j} \right)_{i,j \in \{0,1,\ldots,n\}} .$$

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- **Hint to proof 2:** WLOG assume that $0 \leq a_0 < a_1 < \cdots < a_n$. (Otherwise, move a_i preserving $a_i \bmod \mathrm{LHS}$.)
Then, the partition $\lambda := (a_n - n, a_{n-1} - (n-1), \ldots, a_0 - 0)$ satisfies
$$\frac{\mathrm{RHS}}{\mathrm{LHS}} = s_\lambda \Big(\underbrace{1, 1, \ldots, 1}_{n+1 \text{ times}} \Big) \qquad \text{(Schur function)}$$
$$= (\# \text{ of semistandard tableaux of shape } \lambda \text{ with entries } \in \{1, 2, \ldots, n+1\}) .$$
(Weyl's character formula in type A.)

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- **Hint to proof 3:** To show that $u \mid v$, it suffices to prove that every prime p divides v at least as often as it does u.
  Now get your hands dirty.

What about squares?

- **Theorem:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,

$$\frac{0! \cdot 2! \cdot \cdots \cdot (2n)!}{2^n} \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

  (Typo in Bhargava corrected.)

What about squares?

- **Theorem** (slightly restated)**:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,

$$\prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

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  $$\prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$
- Analogues of the 3 above proofs work (I believe). In particular, $\dfrac{\text{RHS}}{\text{LHS}}$ is the dimension of an $\operatorname{Sp}(n)$-irrep.

What about cubes?

- **Question:** Do we also have

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- **Answer:** No. For example, $n = 2$ and $(a_0, a_1, a_2) = (0, 1, 3)$.
- So what is

$$\gcd \left\{ \prod_{i>j} \left(a_i^3 - a_j^3\right) \ \middle| \ a_0, a_1, \ldots, a_n \in \mathbb{Z} \right\} \ ?$$

- **General question** (Bhargava, 1997)**:** Let S be a set of integers. What is

$$\gcd\left\{\prod_{i>j}(a_i - a_j) \;\middle|\; a_0, a_1, \ldots, a_n \in S\right\}\ ?$$

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  And when is it attained?

- Enough to work out each prime p separately, because:

p-valuation

- Let p be a prime.
- For each nonzero $n \in \mathbb{Z}$, let $v_p(n)$ (the *p-valuation* of n) be the highest $k \in \mathbb{N}$ such that $p^k \mid n$. (We use $\mathbb{N} := \{0, 1, 2, \ldots\}$.)
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- **Rules for p-valuations:**

$v_p(1) = 0;$	$v_p(ab) = v_p(a) + v_p(b);$
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- Define the *p-distance $d_p(a, b)$* between two integers a and b by

$$d_p(a, b) = -v_p(a - b).$$

  Then, the last rule rewrites as

$$d_p(a, c) \leq \max\{d_p(a, b), d_p(b, c)\}.$$

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- Two integers u and v satisfy $u \mid v$ if and only if

$$v_p(u) \leq v_p(v) \qquad \text{for each prime } p.$$

Thus, checking divisibility is reduced to a "local" problem.

Equivalent restatement of the problem

- **Equivalent problem:** Let S be a set of integers. Let p be a prime. What is

$$\min \left\{ v_p \left(\prod_{i>j} (a_i - a_j) \right) \;\middle|\; a_0, a_1, \ldots, a_n \in S \right\} ?$$

And when is it attained?

- **Equivalent problem:** Let S be a set of integers. Let p be a prime. What is

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  And when is it attained?
- We can WLOG assume that $a_0, a_1, \ldots, a_n$ are distinct.

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  - ... (Ad infinitum, or until S is exhausted.)

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- Thus, the choice of a_n tactically maximizes $\sum_{n \geq i > j} d_p(a_i, a_j)$ for fixed $a_0, a_1, \ldots, a_{n-1}$. (Thus "greedy".) But is it strategically optimal?

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- **Theorem (Bhargava):** Yes. Any such sequence $(a_0, a_1, a_2, \ldots)$ will always maximize $\sum_{n \geq i > j} d_p(a_i, a_j)$ for each n.

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- **Theorem (Bhargava):** Yes. Any such sequence $(a_0, a_1, a_2, \ldots)$ will always maximize $\sum_{n \geq i > j} d_p(a_i, a_j)$ for each n.
- **Note:** There is such a sequence for each prime p, but there might not be such a sequence that works for all p simultaneously.

A cryptic hint

- Bhargava, 1997:

  > *"We note that the above results (i.e. Theorem 1, Lemmas 1 and 2) do not rely on any special properties of P or R; they depend only on the fact that R becomes an ultrametric space when given the P-adic metric. Hence these results could be viewed more generally as statements about certain special sequences in ultrametric spaces. For convenience, however, we have chosen to present these statements only in the relevant context. In particular, we note that our proof of Theorem 1 shall be a purely algebraic one, involving no inequalities."*

  (Theorem 1 is a slight generalization of the above Theorem.)

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  (Theorem 1 is a slight generalization of the above Theorem.)
- In other news, the properties of d_p are all that is needed.

2.

Ultra triples

References:

- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
- Darij Grinberg, *The Bhargava greedoid as a Gaussian elimination greedoid*, arXiv:2001.05535.
- Alex J. Lemin, *The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices LAT**, Algebra univers. **50** (2003), pp. 35–49.

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- More generally, we can replace $\mathbb{R}$ by any totally ordered abelian group $\mathbb{V}$.

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- We will only consider ultra triples with **finite** ground set E. (Bhargava's E is infinite, but results adapt easily.)

Ultra triples, examples: 1 (congruence)

- **Example:** Let $E \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$. Define a map $w : E \to \mathbb{R}$ arbitrarily. Define a map $d : E \underline{\times} E \to \mathbb{R}$ by

$$d(a, b) = \begin{cases} 0, & \text{if } a \equiv b \mod n; \\ 1, & \text{if } a \not\equiv b \mod n \end{cases} \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

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where ε and α are fixed reals with $\varepsilon \leq \alpha$. Then, (E, w, d) is an ultra triple.

Ultra triples, examples: 2 (p-adic distance)

- Let p be a prime. Let $E \subseteq \mathbb{Z}$. Define the weights $w(e) \in \mathbb{R}$ arbitrarily. Then, (E, w, d_p) is an ultra triple.
  Here, d_p is as before:

$$d_p(a, b) = -v_p(a - b).$$

- This is the case of relevance to Bhargava's problem!
  Thus, we call such a triple (E, w, d_p) a *Bhargava-type ultra triple*.

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  Thus, we call such a triple (E, w, d_p) a *Bhargava-type ultra triple*.
- Lots of other distance functions also give ultra triples:
  Compose d_p with any weakly increasing function $\mathbb{R} \to \mathbb{R}$. For example,

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- More generally, we can replace $p^0, p^1, p^2, \ldots$ with any unbounded sequence $r_0 \mid r_1 \mid r_2 \mid \cdots$ of integers.

Ultra triples, examples: 3 (Linnaeus)

- Let E be the set of all living organisms. Let

$$d(e,f) = \begin{cases} 0, & \text{if } e = f; \\ 1, & \text{if } e \text{ and } f \text{ belong to the same species;} \\ 2, & \text{if } e \text{ and } f \text{ belong to the same genus;} \\ 3, & \text{if } e \text{ and } f \text{ belong to the same family;} \\ \ldots \end{cases}$$

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  Then, (E, w, d) is an ultra triple (for any $w : E \to \mathbb{R}$).
- More generally, any "nested" family of equivalence relations on E gives a distance function for an ultra triple.

Ultra triples, examples: 4 (Darwin)

- Let T be a (finite, undirected) tree. For each edge e of T, let $\lambda(e) \geq 0$ be a real. We shall call this real the *weight* of e.

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- Fix any vertex r of T. Let E be any subset of the vertex set of T. Set

  $$d(x, y) = \lambda(x, y) - \lambda(x, r) - \lambda(y, r) \qquad \text{for each } (x, y) \in E \underline{\times} E.$$

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  Then, (E, w, d) is an ultra triple for any $w : E \to \mathbb{R}$.
- **Hint to proof:** Use the well-known fact ("four-point condition") saying that if x, y, z, w are four vertices of T, then the two largest of the three numbers

  $$\lambda(x, y) + \lambda(z, w), \quad \lambda(x, z) + \lambda(y, w), \quad \lambda(x, w) + \lambda(y, z)$$

  are equal.

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  Then, (E, w, d) is an ultra triple for any $w : E \to \mathbb{R}$.
- This is particularly useful when T is a *phylogenetic tree* and E is a set of its leaves.
  Actually, this is the general case: Any (finite) ultra triple can be translated back into a phylogenetic tree. It is "essentially" an inverse operation.
  (The idea is not new; see, e.g., Lemin 2003.)

- Let (E, w, d) be an ultra triple, and $S \subseteq E$ be any subset. Then, the *perimeter* of S is defined to be

$$\operatorname{PER}(S) := \underbrace{\sum_{x \in S} w(x)}_{|S| \text{ addends}} + \underbrace{\sum_{\substack{\{x,y\} \subseteq S; \\ x \neq y}} d(x, y)}_{\binom{|S|}{2} \text{ addends}} .$$

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- Thus,

$$\begin{aligned} \operatorname{PER} \varnothing &= 0; \\ \operatorname{PER} \{x\} &= w(x); \\ \operatorname{PER} \{x, y\} &= w(x) + w(y) + d(x, y); \\ \operatorname{PER} \{x, y, z\} &= w(x) + w(y) + w(z) \\ &\qquad + d(x, y) + d(x, z) + d(y, z). \end{aligned}$$

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- **Bhargava's problem (generalized):** Given an ultra triple (E, w, d) and an $n \in \mathbb{N}$, find the maximum perimeter of an n-element subset of E, and find the subsets that attain it. (The n here corresponds to the $n + 1$ before.)

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- For $E \subseteq \mathbb{Z}$ and $w(e) = 0$ and $d_p(a, b) = -v_p(a - b)$, this is Bhargava's problem.

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- For Linnaeus or Darwin ultra triples, this is a "Noah's Ark" problem: What choices of n organisms maximize biodiversity? A similar problem has been studied in: Vincent Moulton, Charles Semple, Mike Steel, *Optimizing phylogenetic diversity under constraints*, J. Theor. Biol. **246** (2007), pp. 186–194.

3.

Solving the problem

References:

- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
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Greedy permutations: definition

- Fix an ultra triple (E, w, d).

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- Let $m \in \mathbb{N}$. A *greedy m-permutation* of E is a list $(c_1, c_2, \ldots, c_m)$ of m distinct elements of E such that for each $i \in \{1, 2, \ldots, m\}$ and each $x \in E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$, we have

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- In other words, a greedy m-permutation of E is what you obtain if you try to greedily construct a maximum-perimeter m-element subset of E, by starting with $\varnothing$ and adding new points one at a time.

Greedy permutations: examples

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  Note: The greedy m-permutations for $\left(E, w, d'_p\right)$ are different. The values of $d(e, f)$ matter, not just their relative order!

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- **Exercise:** Use this to prove

$$\prod_{i>j} (i - j) \mid \prod_{i>j} (a_i - a_j) \qquad \text{and} \qquad \prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

4.

Greedoids

References:

- Bernhard Korte, László Lovász, Rainer Schrader, *Greedoids*, Algorithms and Combinatorics #4, Springer 1991.
- Anders Björner, Günter M. Ziegler, *Introd. to Greedoids*, in: Neil White (ed.), *Matroid applications*, CUP 1992.
- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
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- Victor Bryant, Ian Sharpe, *Gaussian, Strong and Transversal Greedoids*, Europ. J. Comb. **20** (1999), pp. 259–262.

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- This is characteristic of a *greedoid* – a "noncommutative analogue" of a matroid.
- Matroids have several "cryptomorphic" definitions. ("Cryptomorphism" = isomorphism of species, to my understanding.)
- For greedoids, we will give two cryptomorphic definitions: one as languages, one as set systems. See Korte/Lovász/Schrader for details.

Greedoids as languages

- A *language* on a set E means a set $\mathcal{L}$ of finite tuples of elements of E.
- A tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in E^k$ is *simple* if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct.
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- A *greedoid language* on a set E means a simple language $\mathcal{L}$ on E such that
  1. The empty tuple $() \in \mathcal{L}$.
  2. If $\alpha\beta \in \mathcal{L}$, then $\alpha \in \mathcal{L}$.
  3. (to be revealed...)

  Here,
  - The *concatenation* $\alpha\beta$ of two tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ is the tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_\ell)$.

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  Here,
  - any $x \in E$ is identified with the 1-tuple (x).
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  Here,
  - any $x \in E$ is identified with the 1-tuple (x).
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- This is analogous to the definition of a matroid (as a system of independent sets), but using "ordered sets" (i.e., simple tuples) instead of sets.

Greedoids as set systems

- A *set system* on a set E means a set of subsets of E.
- A *greedoid* on a set E means a set system $\mathcal{F}$ on E such that
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- There is a canonical bijection

$$\{\text{greedoid languages}\} \to \{\text{greedoids}\},$$
$$\mathcal{L} \mapsto \{\operatorname{set} \alpha \mid \alpha \in \mathcal{L}\},$$

where $\operatorname{set}(\alpha_1, \alpha_2, \ldots, \alpha_k) := \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

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  where $\operatorname{set}(\alpha_1, \alpha_2, \ldots, \alpha_k) := \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.
- In the reverse direction, send a greedoid $\mathcal{F}$ to the set of all simple tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that all $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $m \leq k$ belong to $\mathcal{F}$.

- Let M be a matroid on a ground set E. Then,

$$\{\text{independent sets of } M\}$$

  is a greedoid on E.
  We shall call this a *matroid greedoid*.

Greedoids, examples: 2 (Gaussian elimination)

- Let A be an $m \times n$-matrix over a field $\mathbb{K}$. Let $E = \{1, 2, \ldots, n\}$. Then,

  $$\left\{ F \subseteq E \mid \text{ we have } |F| \leq n \text{ and } \det\left(\operatorname{sub}_{\{1,2,\ldots,|F|\}}^{F} A\right) \neq 0 \right\}$$

  is a greedoid on E, where $\operatorname{sub}_F^G A$ means the submatrix of A with rows indexed by F and columns indexed by G.

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- This is called a *Gaussian elimination greedoid* over $\mathbb{K}$.
- For example, if $\mathbb{K} = \mathbb{Q}$ and $m = 5$ and $n = 5$ and

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then this Gaussian elimination greedoid is

$$\left\{ \varnothing, \{2\}, \{3\}, \{5\}, \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}, \{1,2,3\}, \{1,2,5\}, \{1,2,3,5\} \right\}.$$

- Let P be a finite poset. Let J be the set of all *order ideals* of P (that is, of all subsets I of P such that $(b \in I) \wedge (a \leq b) \Longrightarrow (a \in I)$).
- Then, J is a greedoid on P.
  We shall call this an *order ideal greedoid*.

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  We shall call this an *order ideal greedoid*.
- The corresponding greedoid language consists of all linear extensions of all order ideals of P.

- Back to our setting: For any ultra triple (E, w, d), define

$$\begin{aligned}\mathcal{B}(E, w, d) &= \{A \subseteq E \mid A \text{ has maximum perimeter among} \\ &\qquad \text{all } |A| \text{-element subsets of } E\} \\ &= \{A \subseteq E \mid \text{PER}(A) \geq \text{PER}(B) \text{ for} \\ &\qquad \text{all } B \subseteq E \text{ satisfying } |B| = |A|\}.\end{aligned}$$

We call this the *Bhargava greedoid* of (E, w, d).

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  We call this the *Bhargava greedoid* of (E, w, d).
- **Theorem (G., Petrov):** This Bhargava greedoid $\mathcal{B}(E, w, d)$ is a greedoid indeed.

Strong greedoids: definition

- Recall: A *greedoid* on a set E means a set system $\mathcal{F}$ on E such that
  1. We have $\varnothing \in \mathcal{F}$.
  2. If $B \in \mathcal{F}$ satisfies $|B| > 0$, then there exists $b \in B$ such that $B \setminus \{b\} \in \mathcal{F}$.
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  3. If $A, B \in \mathcal{F}$ satisfy $|B| = |A| + 1$, then there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$.
- A *strong greedoid* on E means a greedoid $\mathcal{F}$ on E that also satisfies
  4. If $A, B \in \mathcal{F}$ satisfy $|B| = |A| + 1$, then there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$ and $B \setminus \{b\} \in \mathcal{F}$.

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- **Remark:** Axiom 4. implies axiom 3.

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- **Remark:** In axiom 3., we can replace the condition "$|B| = |A| + 1$" by the weaker "$|B| > |A|$"; the axiom stays equivalent.
  But we cannot do the same in axiom 4. (it would become much stronger, forcing $\mathcal{F}$ to be a matroid greedoid).

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  4. If $A, B \in \mathcal{F}$ satisfy $|B| = |A| + 1$, then there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$ and $B \setminus \{b\} \in \mathcal{F}$.
- Strong greedoids are also known as "*Gauss greedoids*" (not to be confused with Gaussian elimination greedoids).

Strong greedoids: examples

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  If $\mathcal{F}$ is a Gaussian elimination greedoid, then the latter matroid is representable.

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- **Stronger theorem (G.):** Let (E, w, d) be an ultra triple. Let $\mathbb{K}$ be any field of size $|\mathbb{K}| \geq \operatorname{mcs}(E, w, d)$, where $\operatorname{mcs}(E, w, d)$ is the *maximum clique size* of E (that is, the maximum size of a subset $C \subseteq E$ such that $d\mid_{C \underline{\times} C}$ is constant).
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- Note that this Theorem yields the previous one, which is thus proved twice.
- **Converse theorem (G.):** Assume that the map w is constant. Let $\mathbb{K}$ be a field. Then, the Bhargava greedoid $\mathcal{B}(E, w, d)$ is (up to renaming the elements of E) a Gaussian elimination greedoid over $\mathbb{K}$ **if and only if** $|\mathbb{K}| \geq \operatorname{mcs}(E, w, d)$.

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- **1st step:** If (E, w, d) is a Bhargava-type ultra triple (E, w, d_p) for some prime p and some $E \subseteq \mathbb{Z}$, then we can explicitly find a matrix A over $\mathbb{F}_p$ that gives $\mathcal{B}(E, w, d)$ as its Gaussian elimination greedoid.

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  Even better, this matrix A is the projection of a matrix $\widetilde{A}$ over $\mathbb{Z}$ that satisfies
  $$v_p \left(\det\left(\operatorname{sub}^{F}_{\{1,2,\ldots,|F|\}} \widetilde{A}\right)\right) = (\text{max. possible perimeter}) - \operatorname{PER}(F)$$
  for each subset F of E.
  (The matrix $\widetilde{A}$ is a Vandermonde-like matrix, with entries $\dfrac{1}{p^{\text{something}}} (a_i - e_1)(a_i - e_2) \cdots (a_i - e_j)$.)

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- **2nd step:** So we know how to deal with Bhargava-type ultra triples. It would be nice if any ultra triple was isomorphic to one of them!

  I'm not sure this is true, but I can prove something close that suffices:

A few words about the proofs, 2

- **2nd step, continued:** Replace $\mathbb{Z}$ by an arbitrary valuation ring with value group $\mathbb{R}$, and replace v_p by its valuation.

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  (The proof proceeds by strong induction, decomposing the ultra triple into smaller ones. Iterating this decomposition again reveals the connection to phylogenetic trees.)

Questions

- If w is constant, then we have a necessary and sufficient condition for $\mathcal{B}(E, w, d)$ to be a Gaussian elimination greedoid over $\mathbb{K}$.
  What about the general case? ($|\mathbb{K}| \geq \operatorname{mcs}(E, w, d)$ is still sufficient, but no longer necessary.)

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- It is not too hard to define a multiset analogue of greedoids (e.g., by lifting the “simple” requirement on greedoid languages). How much of the theory adapts?

$$\frac{\bullet}{\textbf{Bonus problem: stalagmic greedoids}}$$

References:

- to be written (contact me).

- **Proposition (G., easy consequence of known facts):** Let E and $U = \{u_1, u_2, \ldots, u_n\}$ be two disjoint finite sets (with $u_1, u_2, \ldots, u_n$ distinct).
Let $\mathcal{B}$ be the set of bases of a matroid on ground set $E \cup U$.
Assume that $\{u_1, u_2, \ldots, u_n\} \in \mathcal{B}$. Let

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- **If yes**, then we have found a machine for deriving properties of strong greedoids from properties of matroids.

Thank you!

- **Fedor Petrov** for getting this started by answering my MathOverflow question #314130.
- **Alexander Postnikov** for interesting conversations.
- **you** for your patience and typo hunting.