

The Petrie symmetric functions

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Abstract. For any positive integer k and nonnegative integer m , we consider the symmetric function $G(k, m)$ defined as the sum of all monomials of degree m that contain no exponents larger than $k - 1$. We call $G(k, m)$ a *Petrie symmetric function* in honor of Flinders Petrie, as the coefficients in its expansion in the Schur basis are determinants of Petrie matrices (and thus belong to $\{-1, 0, 1\}$ by a classical result of Gordon and Wilkinson). More generally, we prove a Pieri-like rule for expanding a product of the form $G(k, m) \cdot s_\mu$ in the Schur basis whenever μ is a partition; all coefficients in this expansion belong to $\{-1, 0, 1\}$. We show a further formula for $G(k, m)$ and prove that $G(k, 1), G(k, 2), G(k, 3), \dots$ form an algebraically independent generating set for the symmetric functions when $1 - k$ is invertible in the base ring. We prove a conjecture of Liu and Polo about the expansion of $G(k, 2k - 1)$ in the Schur basis. We then take our Pieri-like rule as an impetus to pose a different question: What other symmetric functions f have the property that each product $f s_\mu$ expands in the Schur basis with all coefficients belonging to $\{-1, 0, 1\}$? We call this property *MNability* due to its most classical instance (besides the Pieri rules, which don't use -1 coefficients) being the Murnaghan–Nakayama rule. Surprisingly, we find a number of infinite families of MNable symmetric functions besides the classical ones.

Keywords: symmetric functions, Petrie matrices, Murnaghan–Nakayama rule, Pieri rules, Schur functions, determinants

1 Introduction

In the course of computing the cohomology of a line bundle in characteristic p , Liu and Polo [8] have encountered a symmetric function that can be defined as the sum of all monomials of degree $2p - 1$ that contain no exponents larger than $p - 1$. Using representation theory, they found a simple expansion of this function in the Schur basis [8, Corollary 1.4.4], which prompted them to ask whether this expansion also holds for non-prime p (in which case their argument no longer applies).

Indeed, it does (see Section 7 below). From a combinatorial point of view, it is natural to study an even more general family of symmetric functions: We fix a commutative ring \mathbf{k} . For any integers $k \geq 1$ and $m \geq 0$, we let $G(k, m)$ be the sum of all monomials (in x_1, x_2, x_3, \dots) of degree m that contain no exponents larger than $k - 1$. This $G(k, m)$

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is a symmetric function; moreover, if it is expanded in the Schur basis of the ring of symmetric functions, then the coefficients can be expressed as determinants of *Petrie matrices* (i.e., matrices with each column having the form $(0, 0, \dots, 0, 1, 1, \dots, 1, 0, 0, \dots, 0)^T$). Thus, we call $G(k, m)$ a *Petrie function* in honor of Flinders Petrie, whose invention of contextual seriation gave birth to the notion of Petrie matrices. By a result of Gordon and Wilkinson [5], Petrie matrices are unimodular; thus, the coefficients in the Schur expansion of $G(k, m)$ belong to $\{-1, 0, 1\}$.

More generally, if μ is any partition, then we can expand the product $G(k, m) \cdot s_\mu$ in the Schur basis; all coefficients in this Pieri-like expansion are determinants of Petrie matrices as well (and thus belong to $\{-1, 0, 1\}$). At least for $\mu = \emptyset$, the coefficients have a combinatorial interpretation.

We show a further formula for $G(k, m)$ in terms of Frobenius homomorphisms f_n (also known as plethysm by the power-sum function p_n), and we use it to show that $G(k, 1), G(k, 2), G(k, 3), \dots$ form an algebraically independent generating set for the symmetric functions when $1 - k$ is invertible in \mathbf{k} .

We then revisit our expansion of $G(k, m) \cdot s_\mu$ to ask a more general question (Section 8): What other symmetric functions f have the property that each product $f s_\mu$ expands in the Schur basis with all coefficients belonging to $\{-1, 0, 1\}$? We call such f *MNable*; examples of MNable symmetric functions are the classical functions h_m, e_m, p_m (by the Pieri and the Murnaghan–Nakayama rule, the latter of which gave MNability its name) and the Petrie functions $G(k, m)$ (by the above). Surprisingly, we have found several other MNable symmetric functions, such as the products $p_i p_j$ with $i \neq j$, or the differences $h_m - e_m, h_m - p_m$ and $h_m - p_m - e_m$ for even m .

Most results in this abstract are proved in the paper [6].

Some results below (in particular, Theorem 4.4 and Theorem 4.6 in an equivalent form) have been independently found by H. Fu and Z. Mei [3]. Also, Theorem 6.3 has been discovered by S. Doty and G. Walker in 1992 [2, Cor. 3.9].

2 Definitions

Our notations follow [7, Chapter 2]. We let $\mathbb{N} = \{0, 1, 2, \dots\}$.

We fix a commutative ring \mathbf{k} . We let Λ denote the ring of symmetric functions (i.e., symmetric power series of bounded degree) in infinitely many variables x_1, x_2, x_3, \dots over \mathbf{k} . This is a \mathbf{k} -subalgebra of the \mathbf{k} -algebra $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ of formal power series.

A *weak composition* means an infinite sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$ of nonnegative integers such that only finitely many i satisfy $\alpha_i \neq 0$. If α is a weak composition, then α_i shall mean the i -th entry of α (so that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$), and $|\alpha|$ shall mean the sum $\alpha_1 + \alpha_2 + \alpha_3 + \dots \in \mathbb{N}$ (and be called the *size* of α). We let WC denote the set of all weak compositions.

For any weak composition α , we let \mathbf{x}^α denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$. These monomials \mathbf{x}^α are all the monomials in $\mathbf{k}[[x_1, x_2, x_3, \dots]]$.

A weak composition α will be identified with the ℓ -tuple $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$ whenever $\ell \in \mathbb{N}$ satisfies $\alpha_{\ell+1} = \alpha_{\ell+2} = \alpha_{\ell+3} = \cdots = 0$.

A *partition* means a weak composition α such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$. We let Par denote the set of all partitions. For each $n \in \mathbb{N}$, we let Par_n denote the set of all partitions α satisfying $|\alpha| = n$.

The \mathbf{k} -module Λ has several bases indexed by the set Par . The simplest one is the *monomial basis* $(m_\lambda)_{\lambda \in \text{Par}}$, whose elements m_λ are the sums of the orbits of the monomials \mathbf{x}^α under the “permutation of variables” action of the infinite symmetric group. More precisely, for any partition λ , we can define the *monomial symmetric function* $m_\lambda \in \Lambda$ by

$$m_\lambda = \sum \mathbf{x}^\alpha,$$

where the sum ranges over all weak compositions $\alpha \in \text{WC}$ that can be obtained from λ by permuting entries. For example,

$$m_{(2,2,1)} = \sum_{i < j < k} x_i^2 x_j^2 x_k + \sum_{i < j < k} x_i^2 x_j x_k^2 + \sum_{i < j < k} x_i x_j^2 x_k^2.$$

As λ ranges over all of Par , the symmetric functions m_λ form a basis of the \mathbf{k} -module Λ . Other prominent symmetric functions in Λ are:

- the *complete homogeneous symmetric functions* h_n defined for all $n \in \mathbb{Z}$ by

$$h_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \text{WC}; \\ |\alpha| = n}} \mathbf{x}^\alpha = \sum_{\lambda \in \text{Par}_n} m_\lambda.$$

(Thus, $h_0 = 1$ and $h_n = 0$ for all $n < 0$.)

- the *elementary symmetric functions* e_n defined for all $n \in \mathbb{Z}$ by

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \text{WC} \cap \{0,1\}^\infty; \\ |\alpha| = n}} \mathbf{x}^\alpha.$$

(Thus, $e_0 = 1$ and $e_n = 0$ for all $n < 0$. If $n > 0$, then $e_n = m_{(1,1,\dots,1)}$, where $(1,1,\dots,1)$ is an n -tuple.)

- the *power-sum symmetric functions* p_n defined for all positive integers n by

$$p_n = x_1^n + x_2^n + x_3^n + \cdots = m_{(n)}.$$

But most remarkable of all are the *Schur functions* s_λ for $\lambda \in \text{Par}$. One way to define the Schur function s_λ corresponding to a partition λ is as follows:

$$s_\lambda = \sum \mathbf{x}_T,$$

where the sum ranges over all semistandard tableaux T of shape λ , and where \mathbf{x}_T denotes the monomial obtained by multiplying the x_i for all entries i of T . The fact that $s_\lambda \in \Lambda$ is nontrivial (see, e.g., [7, Prop. 2.2.4]); the s_λ are rich in interesting and nontrivial properties ([7, Ch. 2], [9, Ch. 7], etc.). In particular, the family $(s_\lambda)_{\lambda \in \text{Par}}$ is a basis of the \mathbf{k} -module Λ , known as the *Schur basis*.

3 Definition of the Petrie functions

We are now ready to define the functions we will study:

Definition 3.1. (a) For any positive integer k , we let

$$G(k) = \sum_{\substack{\alpha \in \text{WC}; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha.$$

This is a symmetric formal power series in $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ (but does not lie in Λ in general, since it contains monomials of arbitrarily high degrees).

(b) For any positive integer k and any $m \in \mathbb{N}$, we let

$$G(k, m) = \sum_{\substack{\alpha \in \text{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha \in \Lambda.$$

For example,

$$\begin{aligned} G(3, 4) &= \sum_{i < j < k < \ell} x_i x_j x_k x_\ell + \sum_{i < j < k} x_i^2 x_j x_k + \sum_{i < j < k} x_i x_j^2 x_k + \sum_{i < j < k} x_i x_j x_k^2 + \sum_{i < j} x_i^2 x_j^2 \\ &= m_{(1,1,1,1)} + m_{(2,1,1)} + m_{(2,2)}. \end{aligned}$$

We suggest to name $G(k)$ and $G(k, m)$ the *Petrie functions*, for reasons that Theorem 4.4 and Corollary 4.5 will elucidate. (Doty and Walker, in [2, §3.3], call $G(k, m)$ a “modular complete symmetric function” instead.) We begin with some easy facts:

Proposition 3.2. Let k be a positive integer. Then,

$$G(k) = \sum_{\substack{\alpha \in \text{WC}; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \text{Par}; \\ \lambda_i < k \text{ for all } i}} m_\lambda = \prod_{i=1}^{\infty} (x_i^0 + x_i^1 + \dots + x_i^{k-1}).$$

Proposition 3.3. *Let k be a positive integer. Let $m \in \mathbb{N}$.*

- (a) *The symmetric function $G(k, m)$ is the m -th degree homogeneous component of $G(k)$.*
- (b) *We have*

$$G(k, m) = \sum_{\substack{\alpha \in \text{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \text{Par}; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}} m_\lambda.$$

- (c) *If $k > m$, then $G(k, m) = h_m$.*
- (d) *If $k = 2$, then $G(k, m) = e_m$.*

Parts (c) and (d) of Proposition 3.3 show that the Petrie functions $G(k, m)$ can be seen as interpolating between the h_m and the e_m . Another easily established identity is $G(m, m) = h_m - p_m$ for each positive integer m .

It is also not hard to see that the comultiplication Δ of the Hopf algebra Λ (see [7, Section 2.1] for its definition) satisfies

$$\Delta(G(k, m)) = \sum_{i=0}^m G(k, i) \otimes G(k, m - i)$$

for each $k > 0$ and $m \in \mathbb{N}$.

The Petrie function $G(3)$ has appeared in [9, Exercise 7.3], where it was expanded as a polynomial in e_1, e_2, e_3, \dots (a result of Gessel). We shall now expand $G(k)$ and $G(k, m)$ in terms of Schur functions. For this, we need to define some notations.

4 The Schur expansions of $G(k)$ and $G(k, m)$

If \mathcal{A} is any logical statement, then $[\mathcal{A}]$ shall denote the *truth value* of \mathcal{A} (that is, 1 if \mathcal{A} is true, and 0 if \mathcal{A} is false). We use the notation $(a_{i,j})_{1 \leq i \leq \ell, 1 \leq j \leq \ell}$ for the $\ell \times \ell$ -matrix whose (i, j) -th entry is $a_{i,j}$ for each $i, j \in \{1, 2, \dots, \ell\}$.

Definition 4.1. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \in \text{Par}$, and let k be a positive integer. Then, the k -Petrie number $\text{pet}_k(\lambda)$ of λ is the integer defined by*

$$\text{pet}_k(\lambda) = \det \left(([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right).$$

Note that this integer does not depend on the choice of ℓ (in the sense that it does not change if we enlarge ℓ by adding trailing zeroes to the representation of λ).

Example 4.2. *Let λ be the partition $(3, 1, 1) \in \text{Par}$, let $\ell = 3$, and let k be a positive integer.*

Then, the definition of $\text{pet}_k(\lambda)$ yields

$$\begin{aligned} \text{pet}_k(\lambda) &= \det \begin{pmatrix} [0 \leq \lambda_1 < k] & [0 \leq \lambda_1 + 1 < k] & [0 \leq \lambda_1 + 2 < k] \\ [0 \leq \lambda_2 - 1 < k] & [0 \leq \lambda_2 < k] & [0 \leq \lambda_2 + 1 < k] \\ [0 \leq \lambda_3 - 2 < k] & [0 \leq \lambda_3 - 1 < k] & [0 \leq \lambda_3 < k] \end{pmatrix} \\ &= \det \begin{pmatrix} [0 \leq 3 < k] & [0 \leq 4 < k] & [0 \leq 5 < k] \\ [0 \leq 0 < k] & [0 \leq 1 < k] & [0 \leq 2 < k] \\ [0 \leq -1 < k] & [0 \leq 0 < k] & [0 \leq 1 < k] \end{pmatrix} \end{aligned}$$

(since $\lambda_1 = 3$ and $\lambda_2 = 1$ and $\lambda_3 = 1$). Thus, taking $k = 4$, we obtain

$$\text{pet}_4(\lambda) = \det \begin{pmatrix} [0 \leq 3 < 4] & [0 \leq 4 < 4] & [0 \leq 5 < 4] \\ [0 \leq 0 < 4] & [0 \leq 1 < 4] & [0 \leq 2 < 4] \\ [0 \leq -1 < 4] & [0 \leq 0 < 4] & [0 \leq 1 < 4] \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 0.$$

On the other hand, taking $k = 5$, we obtain

$$\text{pet}_5(\lambda) = \det \begin{pmatrix} [0 \leq 3 < 5] & [0 \leq 4 < 5] & [0 \leq 5 < 5] \\ [0 \leq 0 < 5] & [0 \leq 1 < 5] & [0 \leq 2 < 5] \\ [0 \leq -1 < 5] & [0 \leq 0 < 5] & [0 \leq 1 < 5] \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -1.$$

Proposition 4.3. *Let $\lambda \in \text{Par}$, and let k be a positive integer. Then, $\text{pet}_k(\lambda) \in \{-1, 0, 1\}$.*

Proof sketch. We will use the concept of Petrie matrices (see [5, Theorem 1]). Each row of the matrix $([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell, 1 \leq j \leq \ell}$ has the form

$$\underbrace{(0, 0, \dots, 0)}_{a \text{ zeroes}} \underbrace{(1, 1, \dots, 1)}_{b \text{ ones}} \underbrace{(0, 0, \dots, 0)}_{c \text{ zeroes}} \quad \text{for some } a, b, c \in \mathbb{N} \text{ (where any of } a, b, c \text{ can be 0)}.$$

Thus, the matrix $([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell, 1 \leq j \leq \ell}$ is the transpose of a Petrie matrix. Hence, its determinant belongs to $\{-1, 0, 1\}$ (since [5, Theorem 1] shows that the determinant of any square Petrie matrix belongs to $\{-1, 0, 1\}$). \square

We can now expand the Petrie symmetric functions $G(k)$ in the basis $(s_\lambda)_{\lambda \in \text{Par}}$ of Λ :

Theorem 4.4. *Let k be a positive integer. Then,*

$$G(k) = \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda) s_\lambda.$$

Corollary 4.5. *Let k be a positive integer. Let $m \in \mathbb{N}$. Then,*

$$G(k, m) = \sum_{\lambda \in \text{Par}_m} \text{pet}_k(\lambda) s_\lambda.$$

These two results are particular cases of more general facts stated below (Theorem 5.3 and Corollary 5.4).

The k -Petrie numbers can be described more explicitly:

Theorem 4.6. *Let $\lambda \in \text{Par}$, and let k be a positive integer. Let $\mu = \lambda^t$ be the conjugate of λ (that is, the partition μ defined by setting $\mu_i = |\{j \geq 1 \mid \lambda_j \geq i\}|$ for all i).*

(a) *If $\mu_k \neq 0$, then $\text{pet}_k(\lambda) = 0$.*

From now on, let us assume that $\mu_k = 0$. For each $i \in \{1, 2, \dots, k-1\}$, let γ_i be the unique element of $\{1, 2, \dots, k\}$ that is congruent to $\mu_i - i$ modulo k .

(b) *If the $k-1$ numbers $\gamma_1, \gamma_2, \dots, \gamma_{k-1}$ are not distinct, then $\text{pet}_k(\lambda) = 0$.*

(c) *If the $k-1$ numbers $\gamma_1, \gamma_2, \dots, \gamma_{k-1}$ are distinct, then $\text{pet}_k(\lambda)$ is a certain power of -1 (see [6] for details).*

5 A ‘‘Pieri’’ rule

It turns out that Theorem 4.4 can be generalized. For that, we need to define a ‘‘relative’’ version of Petrie numbers:

Definition 5.1. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \in \text{Par}$ and $\mu = (\mu_1, \mu_2, \dots, \mu_\ell) \in \text{Par}$, and let k be a positive integer. Then, the k -Petrie number $\text{pet}_k(\lambda, \mu)$ of λ and μ is the integer defined by*

$$\text{pet}_k(\lambda, \mu) = \det \left(([0 \leq \lambda_i - \mu_j - i + j < k])_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right).$$

Note that this integer does not depend on the choice of ℓ (in the sense that it does not change if we enlarge ℓ by adding trailing zeroes to the representations of λ and μ).

The following proposition generalizes (and is proved similarly to) Proposition 4.3:

Proposition 5.2. *Let $\lambda \in \text{Par}$ and $\mu \in \text{Par}$, and let k be a positive integer. Then, $\text{pet}_k(\lambda, \mu) \in \{-1, 0, 1\}$.*

Now, we have the following generalizations of Theorem 4.4 and Corollary 4.5:

Theorem 5.3. *Let k be a positive integer. Let $\mu \in \text{Par}$. Then,*

$$G(k) \cdot s_\mu = \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda, \mu) s_\lambda.$$

Corollary 5.4. *Let k be a positive integer. Let $m \in \mathbb{N}$. Let $\mu \in \text{Par}$. Then,*

$$G(k, m) \cdot s_\mu = \sum_{\lambda \in \text{Par}_{m+|\mu|}} \text{pet}_k(\lambda, \mu) s_\lambda.$$

We have two proofs of Theorem 5.3: one using the skew Cauchy and the Jacobi–Trudi identities, and another using the approach to Schur polynomials via alternants. (See [6] for the second proof.) Corollary 5.4 easily follows from Theorem 5.3.

We are not aware of any combinatorial rules for $\text{pet}_k(\lambda, \mu)$ other than the (general, but recursive and rather indirect) algorithmic description given in [5] for determinants of arbitrary square Petrie matrices.

6 The Frobenius formula and Petrie generating sets

We shall next state another formula for the Petrie symmetric functions $G(k, m)$. For this formula, we need the following definition ([7, Exercise 2.9.9]):

Definition 6.1. *Let n be a positive integer. We define a map $\mathbf{f}_n : \Lambda \rightarrow \Lambda$ by setting*

$$\mathbf{f}_n(a) = a(x_1^n, x_2^n, x_3^n, \dots) \quad \text{for each } a \in \Lambda.$$

This map \mathbf{f}_n is called the n -th Frobenius endomorphism of Λ .

It is known (e.g., [7, Exercise 2.9.9(d)]) that this map $\mathbf{f}_n : \Lambda \rightarrow \Lambda$ is a \mathbf{k} -algebra endomorphism of Λ (and even a Hopf algebra endomorphism, using the appropriate Hopf structure). In terms of plethysm ([9, Ch. 7, Definition A.2.6]), it is simply described by $\mathbf{f}_n(a) = a[p_n]$ for each $a \in \Lambda$ (and also by $\mathbf{f}_n(a) = p_n[a]$ if $\mathbf{k} = \mathbb{Z}$).

We now have a new formula for $G(k, m)$:

Theorem 6.2. *Let k be a positive integer. Let $m \in \mathbb{N}$. Then,*

$$G(k, m) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i).$$

(The sum on the right hand side of this equality is well-defined, since all sufficiently large $i \in \mathbb{N}$ satisfy $m - ki < 0$ and thus $h_{m-ki} = 0$.)

This theorem can be proved by an inclusion-exclusion-like computation or using generating functions (the latter proof is given in [6]).

Theorem 6.2 can be used to derive the following ([2, Corollary 3.9]):

Theorem 6.3. *Fix a positive integer k . Assume that $1 - k$ is invertible in \mathbf{k} . Then, the family $(G(k, m))_{m \geq 1} = (G(k, 1), G(k, 2), G(k, 3), \dots)$ is an algebraically independent generating set of the commutative \mathbf{k} -algebra Λ .*

Thus, products of several elements of this family form a basis of Λ (if $1 - k$ is invertible in \mathbf{k}). These bases remain to be studied.

7 The Liu–Polo conjecture

We now sketch the answer to the question posed in [8, Remark 1.4.5] by Liu and Polo. By studying cohomology in positive characteristic, they have proved the following identity for all prime numbers n :

Theorem 7.1. *Let n be an integer such that $n > 1$. Then,*

$$\sum_{\substack{\lambda \in \text{Par}_{2n-1}; \\ (n-1, n-1, 1) \triangleright \lambda}} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{(n-1, n-1-i, 1^{i+1})}. \quad (7.1)$$

Here, the symbol \triangleright stands for *dominance* of partitions (also known as majorization); i.e., for two partitions λ and μ satisfying $|\lambda| = |\mu|$, we have

$$\lambda \triangleright \mu \quad \text{if and only if} \quad (\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \text{ for all } i).$$

Also, the power-like notation “ 1^{i+1} ” in the partition on the right hand side of (7.1) stands for a sequence of $i + 1$ entries all equal to 1. Thus, $(n - 1, n - 1 - i, 1^{i+1}) = (n - 1, n - 1 - i, \underbrace{1, 1, \dots, 1}_{i+1 \text{ times}})$.

We can prove Theorem 7.1 for all n as follows. The first step is to recognize that the left hand side of (7.1) is $G(n, 2n - 1)$, because the partitions $\lambda \in \text{Par}_{2n-1}$ satisfying $(n - 1, n - 1, 1) \triangleright \lambda$ are precisely the partitions $\lambda \in \text{Par}_{2n-1}$ satisfying $\lambda_i < n$ for all i . Theorem 4.4 gives an expansion of $G(n, 2n - 1)$ in the Schur basis, if we content ourselves with knowing that the coefficients are n -Petrie numbers. However, we want to know their exact values in order to prove (7.1). Thus, we proceed differently. An application of Theorem 6.2 (or a simple combinatorial argument) yields

$$G(n, n + k) = h_{n+k} - h_k p_n \quad \text{for each } k \in \{0, 1, \dots, n - 1\}.$$

Thus, in particular, $G(n, 2n - 1) = h_{2n-1} - h_{n-1} p_n$.

Now, we recall the *skewing operations* $f^\perp : \Lambda \rightarrow \Lambda$ for all $f \in \Lambda$ as defined in [7, §2.8] (and in various other places). All we need to know about them is that for each $i \in \mathbb{N}$, the skewing operation $e_i^\perp : \Lambda \rightarrow \Lambda$ is the \mathbf{k} -linear map that sends each Schur function s_λ to the skew Schur function $s_{\lambda/(1^i)}$.

For any $m \in \mathbb{N}$, we define a map $\mathbf{B}_m : \Lambda \rightarrow \Lambda$ by setting

$$\mathbf{B}_m(f) = \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp f \quad \text{for all } f \in \Lambda.$$

It is known ([7, Exercise 2.9.1(a)]) that this map \mathbf{B}_m is well-defined and \mathbf{k} -linear. Moreover, [7, Exercise 2.9.1(b)] shows that if $\lambda \in \text{Par}$ and $m \in \mathbb{Z}$ satisfy $m \geq \lambda_1$, then

$$\mathbf{B}_m(s_\lambda) = s_{(m, \lambda_1, \lambda_2, \lambda_3, \dots)}. \quad (7.2)$$

(This map \mathbf{B}_m is known as the m -th Bernstein operator [11, §4.20(a)] or — in honor of (7.2) — a Schur row-adder [4].) On the other hand, it is not hard to see that

$$\mathbf{B}_m(h_n) = h_m h_n - h_{m+1} h_{n-1} \quad \text{and} \quad \mathbf{B}_m(p_n) = h_m p_n - h_{m+n}$$

for each positive integer n and each $m \in \{0, 1, \dots, n\}$. Using these two equalities, we readily see that

$$\mathbf{B}_{n-1}(h_n - p_n) = h_{2n-1} - h_{n-1} p_n = G(n, 2n - 1). \quad (7.3)$$

On the other hand, [7, Exercise 5.4.12(g)] (or the Murnaghan–Nakayama rule) yields

$$p_n = \sum_{i=0}^{n-1} (-1)^i s_{(n-i, 1^i)}.$$

Subtracting this from $h_n = s_{(n)} = s_{(n-0, 1^0)}$, we find

$$h_n - p_n = \sum_{i=0}^{n-2} (-1)^i s_{(n-1-i, 1^{i+1})}.$$

Applying the map \mathbf{B}_{n-1} to this equality, we obtain

$$\mathbf{B}_{n-1}(h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i \mathbf{B}_{n-1}(s_{(n-1-i, 1^{i+1})}) = \sum_{i=0}^{n-2} (-1)^i s_{(n-1, n-1-i, 1^{i+1})}$$

(by (7.2)). Comparing this with (7.3), we find

$$G(n, 2n - 1) = \sum_{i=0}^{n-2} (-1)^i s_{(n-1, n-1-i, 1^{i+1})}.$$

Since the left hand side of (7.1) is $G(n, 2n - 1)$, we have thus proved Theorem 7.1.

8 MNable symmetric functions

Let us now take Corollary 5.4 as inspiration to identify a property of some symmetric functions that appears to have been hitherto unstudied.

Let $\mathbf{k} = \mathbb{Z}$ throughout this section. We recall the Hall inner product $(\cdot, \cdot) : \Lambda \times \Lambda \rightarrow \mathbf{k}$; it is the unique \mathbf{k} -bilinear form on Λ that satisfies $(s_\lambda, s_\mu) = \delta_{\lambda, \mu}$ for all $\lambda, \mu \in \text{Par}$. (In other words, it is the unique \mathbf{k} -bilinear form on Λ that makes $(s_\lambda)_{\lambda \in \text{Par}}$ into an orthonormal basis.) See [7, Definition 2.5.12] or [9, §7.9] (where it is denoted by $\langle \cdot, \cdot \rangle$) for its further properties.

Definition 8.1. (a) A symmetric function $f \in \Lambda$ will be called signed multiplicity-free if f can be expanded as a linear combination of distinct Schur functions with all coefficients in $\{-1, 0, 1\}$. (That is, if the Hall inner product (f, s_μ) is -1 or 0 or 1 for each partition μ .)

(b) A symmetric function $f \in \Lambda$ will be called MNable if for each partition μ , the product fs_μ is signed multiplicity-free.

For example, the symmetric function h_3p_2 is signed multiplicity-free, since $h_3p_2 = s_{(5)} + s_{(3,2)} - s_{(3,1,1)}$; but it is not MNable, since the product

$$h_3p_2s_{(2)} = -s_{(3,2,1,1)} + s_{(3,2,2)} - s_{(4,1,1,1)} + s_{(4,3)} - s_{(5,1,1)} + 2s_{(5,2)} + s_{(6,1)} + s_{(7)}$$

is not signed multiplicity-free (due to the coefficient of $s_{(5,2)}$ being 2).

Roughly speaking, an $f \in \Lambda$ is MNable if and only if there is a Murnaghan–Nakayama-like rule for multiplying Schur functions by f . Thus, the name “MNable”.

Question 8.2. Which symmetric functions are MNable?

It is not clear whether a full characterization of MNable symmetric functions is even possible. However, there are many. Here is a non-exhaustive list:

Theorem 8.3. (a) The functions h_i and e_i are MNable for each $i \in \mathbb{N}$.

(b) The function p_i is MNable for each positive integer i .

(c) The Petrie function $G(k, m)$ and the difference $G(k, m) - h_m$ are MNable for any integers $k \geq 1$ and $m \geq 0$.

(d) The differences $h_i - p_i$ and $h_i - e_i$ are MNable for each positive integer i . (This includes $h_1 - e_1 = 0$.)

(e) The difference $h_i - p_i - e_i$ is MNable for each even positive integer i .

(f) The product $p_i p_j$ is MNable whenever $i > j > 0$.

(g) The function $m_{(k^n)}$ as well as the differences $h_{nk} - m_{(k^n)}$ and $e_{nk} - (-1)^{(k-1)n} m_{(k^n)}$ are MNable for any positive integers n and k (where (k^n) denotes the n -tuple (k, k, \dots, k)).

(h) If some $f \in \Lambda$ is MNable, then so are $-f$ and $\omega(f)$, where $\omega : \Lambda \rightarrow \Lambda$ is the fundamental involution of Λ (see [7, §2.4] or [9, §7.6]).

(i) A symmetric function $f \in \Lambda$ is MNable if and only if all its homogeneous components are MNable.

(j) If $f \in \Lambda$ is MNable and k is a positive integer, then $\mathbf{f}_k(f)$ is MNable. (See Definition 6.1 for the meaning of \mathbf{f}_k .)

(k) A symmetric function $f \in \Lambda$ is MNable if and only if $(f, s_{\lambda/\mu}) \in \{-1, 0, 1\}$ for each skew partition λ/μ .

A few telegraphic remarks on the proofs are in order. Part (a) of Theorem 8.3 follows from the Pieri and dual Pieri rules, as part (b) does from the Murnaghan–Nakayama rule. The $G(k, m)$ claim in part (c) follows from Corollary 5.4; the $G(k, m) - h_m$ claim

relies on the fact that $\text{pet}_k(\lambda, \mu) \in \{0, 1\}$ if λ/μ is a horizontal strip. Parts **(d)** and **(e)** can be shown by analyzing the rare cases in which a skew partition can be two of “horizontal strip”, “vertical strip” and “rim hook” at once. Part **(f)** follows from a study of rim hook tableaux. Part **(h)** follows from the facts that ω is an algebra automorphism and sends s_λ to s_{λ^t} . Part **(k)** is easy to see using skewing operators (or simply using the fact that the same Littlewood–Richardson coefficients appear in the formulas $s_\mu s_\nu = \sum_{\lambda \in \text{Par}} c_{\mu, \nu}^\lambda s_\lambda$ and $s_{\lambda/\mu} = \sum_{\nu \in \text{Par}} c_{\mu, \nu}^\lambda s_\nu$). Part **(i)** is easy. Part **(j)** follows from part **(k)** and the SXP algorithm in [1]. The $m_{(k^n)}$ claim in part **(g)** follows from part **(j)** (since $m_{(k^n)} = \mathbf{f}_k(e_n)$); the rest of **(g)** follows by studying skew partitions again.

Note that Theorem 8.3 **(k)** shows that the MNability of a symmetric function can be tested in finite time: For each $d \in \mathbb{N}$, there are only finitely many skew Schur functions $s_{\lambda/\mu}$ of degree d .

The families in parts **(a)–(h)** and **(j)** of Theorem 8.3 cover all MNable homogeneous symmetric functions of degree < 4 . In degree 4, we have two further MNable symmetric functions that we were unable to “explain” (i.e., embed in any infinite family):

$$s_{(1,1,1,1)} - s_{(3,1)} + s_{(4)} \quad \text{and} \quad s_{(4)} - s_{(2,2)}.$$

While Question 8.2 seems wide open, several particular cases appear manageable: for example, which products of h_i 's (or p_i 's) are MNable? Note that the only MNable Schur functions are $h_i = s_{(i)}$ and $e_i = s_{(1,1,\dots,1)}$.

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