

The eta-basis of QSym (DRAFT)

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This is just a skeleton of what will hopefully become a paper.

The paper will construct a new family $(\eta_\alpha)_{\alpha \in \text{Comp}}$ of quasisymmetric functions that is a basis of QSym whenever 2 is invertible in the base ring. We will show a

formula for expanding products of the form $\eta_\alpha \eta_\beta$ as \mathbb{Z} -linear combinations of η_γ 's, and we will apply it to partially solve [Grinbe18, Question 2.51].

Once again: **this is nowhere near finished** (partially more of a todo list than a paper).

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1. Introduction

1.1. Formal power series and quasisymmetry

We will use some of the standard notations from [GriRei20, Chapter 5]. Namely:

- We let $\mathbb{N} = \{0, 1, 2, \dots\}$.
- We fix a commutative ring \mathbf{k} .
- We consider the ring $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ of formal power series in countably many commuting variables x_1, x_2, x_3, \dots . A *monomial* shall mean a formal expression of the form $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots) \in \mathbb{N}^\infty$ is a sequence of nonnegative integers such that only finitely many α_i are positive. Formal power series are formal infinite \mathbf{k} -linear combinations of such monomials.
- Each monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ has degree $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$.
- A formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$ is said to be of *bounded degree* if there exists some $d \in \mathbb{N}$ such that each monomial in f has degree $\leq d$ (that is, each monomial of degree $> d$ has coefficient 0 in f).

For example, the formal power series $(x_1 + x_2 + x_3 + \cdots)^3$ is of bounded degree, but the formal power series $\frac{1}{1 - x_1} = 1 + x_1 + x_1^2 + x_1^3 + \cdots$ is not.

Question: What do the monomials $x_1^4 x_3^7 x_4 x_9^2$ and $x_3^4 x_4^7 x_{10} x_{16}^2$ and $x_5^4 x_6^7 x_8 x_9^2$ have in common (but not in common with $x_1^7 x_3^4 x_4 x_9^2$) ?

Answer: They have the same sequence of nonzero exponents (when the variables are ordered in increasing order – i.e., if $i < j$, then x_i goes before x_j). Or, to put it differently, they all have the form $x_a^4 x_b^7 x_c x_d^2$ for some $a < b < c < d$. We shall call such monomials *pack-equivalent*.

Let us define this concept more rigorously:

Definition 1.1. Two monomials m and n are said to be *pack-equivalent* if they can be written in the forms

$$m = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell} \quad \text{and} \quad n = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$$

for some $\ell \in \mathbb{N}$, some positive integers a_1, a_2, \dots, a_ℓ and two strictly increasing ℓ -tuples $(i_1 < i_2 < \cdots < i_\ell)$ and $(j_1 < j_2 < \cdots < j_\ell)$ of positive integers.

For example, the monomials $x_1^4 x_3^7 x_4 x_9^2$ and $x_3^4 x_4^7 x_{10} x_{16}^2$ are pack-equivalent, since they can be written as $x_1^4 x_3^7 x_4 x_9^2 = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell}$ and $x_3^4 x_4^7 x_{10} x_{16}^2 = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$ for $\ell = 4$ and $(a_1, a_2, \dots, a_\ell) = (4, 7, 1, 2)$ and $(i_1 < i_2 < \cdots < i_\ell) = (1, 3, 4, 9)$ and $(j_1 < j_2 < \cdots < j_\ell) = (3, 4, 10, 16)$.

Definition 1.2. (a) A formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$ is *quasisymmetric* if it has the property that any two pack-equivalent monomials have the same coefficient in f (that is: if m and n are two pack-equivalent monomials, then the coefficient of m in f equals the coefficient of n in f).

(b) A *quasisymmetric function* means a formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$ that is quasisymmetric and of bounded degree.

Quasisymmetric functions are studied in [GriRei20, Chapters 5–6], [Stanle01, §7.19], [Sagan20, Chapter 8] and elsewhere.

It is known ([GriRei20, Proposition 5.1.3]) that the set of all quasisymmetric functions is a \mathbf{k} -subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \dots]]$. It is denoted by QSym and called the *ring of quasisymmetric functions*. It has several bases (as a \mathbf{k} -module), most of which are indexed by compositions.

1.2. Compositions

A *composition* means a finite list $(\alpha_1, \alpha_2, \dots, \alpha_k)$ of positive integers. The set of all compositions will be denoted by Comp. The *empty composition* \emptyset is the composition $()$, which is a 0-tuple.

The *length* $\ell(\alpha)$ of a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is defined to be the number k .

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a composition, then the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is called the *size* of α and is denoted by $|\alpha|$. For any $n \in \mathbb{N}$, we define a

composition of n to be a composition that has size n . We let Comp_n be the set of all compositions of n (for given $n \in \mathbb{N}$). The notation " $\alpha \models n$ " is short for " $\alpha \in \text{Comp}_n$ ". For example, $(1, 5, 2, 1)$ is a composition with size 9, so that $|(1, 5, 2, 1)| = 1 + 5 + 2 + 1 = 9$, so that $(1, 5, 2, 1) \in \text{Comp}_9$, or, in other words, $(1, 5, 2, 1) \models 9$.

For any $n \in \mathbb{Z}$, we let $[n]$ denote the set $\{1, 2, \dots, n\}$. This set is empty whenever $n \leq 0$, and otherwise has size n .

It is well-known that any positive integer n has exactly 2^{n-1} compositions. This has a standard bijective proof ("stars and bars") which is worth recalling in detail, as the bijection itself will be used a lot:

Definition 1.3. Let $n \in \mathbb{N}$. Let $\mathcal{P}([n-1])$ be the powerset of $[n-1]$ (that is, the set of all subsets of $[n-1]$).

(a) We define a map $D : \text{Comp}_n \rightarrow \mathcal{P}([n-1])$ by

$$D(\alpha_1, \alpha_2, \dots, \alpha_k) = \{\alpha_1 + \alpha_2 + \dots + \alpha_i \mid i \in [k-1]\} \\ = \{\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \dots < \alpha_1 + \alpha_2 + \dots + \alpha_{k-1}\}.$$

(b) We define a map $\text{comp} : \mathcal{P}([n-1]) \rightarrow \text{Comp}_n$ as follows: For any $I \in \mathcal{P}([n-1])$, we set

$$\text{comp}(I) = (i_1 - i_0, i_2 - i_1, \dots, i_m - i_{m-1}),$$

where i_0, i_1, \dots, i_m are the elements of the set $I \cup \{0, n\}$ in increasing order (so that $i_0 < i_1 < \dots < i_m$, therefore $i_0 = 0$ and $i_m = n$ and $\{i_1 < i_2 < \dots < i_{m-1}\} = I$).

The maps D and comp are mutually inverse bijections. (See [Grinbe15, detailed version, Proposition 10.17] for a detailed proof of this.)

The notation D we just introduced presumably originates in the word "descent", but the connection between D and actual descents is indirect and rather misleading. I prefer to call D the "partial sum map" (as $D(\alpha)$ consists of the partial sums of the composition α) and its inverse comp the "interstitial map" (as $\text{comp}(I)$ consists of the lengths of the intervals into which the elements of I split the interval $[n]$).

Note that every composition α of size $|\alpha| > 0$ satisfies $|D(\alpha)| = \ell(\alpha) - 1$, so that $|D(\alpha)| + 1 = \ell(\alpha)$. But this fails if α is the empty composition $\emptyset = ()$ (since $D(\emptyset) = \emptyset$ and $\ell(\emptyset) = 0$).

1.3. The monomial and fundamental bases of QSym

We will only need two bases of QSym: the monomial basis and the fundamental basis.

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ is a composition, then we define the *monomial quasisymmetric function* $M_\alpha \in \text{QSym}$ by

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathfrak{m} \text{ is a monomial pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \dots x_\ell^{\alpha_\ell}}} \mathfrak{m}. \quad (1)$$

For example,

$$M_{(2,1)} = \sum_{i < j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \dots .$$

The family $(M_\alpha)_{\alpha \in \text{Comp}}$ is a basis of the \mathbf{k} -module QSym, and is known as the *monomial basis* of QSym.

For any composition α , we define the *fundamental quasisymmetric function* $L_\alpha \in \text{QSym}$ by

$$L_\alpha = \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \supseteq D(\alpha)}} M_\beta, \tag{2}$$

where $n = |\alpha|$ (so that $\alpha \in \text{Comp}_n$). It is not hard to rewrite this as

$$L_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{3}$$

(again with $n = |\alpha|$). This quasisymmetric function L_α is sometimes denoted by F_α .

The family $(L_\alpha)_{\alpha \in \text{Comp}}$ is a basis of the \mathbf{k} -module QSym, and is known as the *fundamental basis* of QSym.

Using Möbius inversion on the Boolean lattice $\mathcal{P}([n - 1])$, the definition (2) of the fundamental basis can be turned around to obtain an expression of the M_α in the fundamental basis. Namely, if α is a composition, and if $n = |\alpha|$, then

$$M_\alpha = \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \supseteq D(\alpha)}} (-1)^{\ell(\beta) - \ell(\alpha)} L_\beta.$$

(See [GriRei20, Proposition 5.2.8] for more details of the proof. In a nutshell, the equality follows from Möbius inversion using the fact that $|D(\beta) \setminus D(\alpha)| = \ell(\beta) - \ell(\alpha)$ whenever $\alpha, \beta \in \text{Comp}_n$ satisfy $D(\beta) \supseteq D(\alpha)$.)

2. The η -basis of QSym

2.1. The η -functions

I shall now define a new family of quasisymmetric functions:

Definition 2.1. For any $n \in \mathbb{N}$ and any composition $\alpha \in \text{Comp}_n$, we define a quasisymmetric function $\eta_\alpha \in \text{QSym}$ by

$$\eta_\alpha = \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} 2^{\ell(\beta)} M_\beta. \tag{4}$$

Example 2.2. (a) Setting $n = 5$ and $\alpha = (1, 3, 1)$ in this definition, we obtain

$$\begin{aligned}\eta_{(1,3,1)} &= \sum_{\substack{\beta \in \text{Comp}_5; \\ D(\beta) \subseteq D(1,3,1)}} 2^{\ell(\beta)} M_\beta = \sum_{\substack{\beta \in \text{Comp}_5; \\ D(\beta) \subseteq \{1,4\}}} 2^{\ell(\beta)} M_\beta \quad (\text{since } D(1,3,1) = \{1,4\}) \\ &= 2^{\ell(5)} M_{(5)} + 2^{\ell(1,4)} M_{(1,4)} + 2^{\ell(4,1)} M_{(4,1)} + 2^{\ell(1,3,1)} M_{(1,3,1)}\end{aligned}$$

(since the compositions $\beta \in \text{Comp}_5$ satisfying $D(\beta) \subseteq \{1,4\}$ are (5) , $(1,4)$, $(4,1)$ and $(1,3,1)$). This simplifies to

$$\eta_{(1,3,1)} = 2M_{(5)} + 4M_{(1,4)} + 4M_{(4,1)} + 8M_{(1,3,1)}.$$

(b) For any positive integer n , we have $\eta_{(n)} = 2M_{(n)}$, because the only composition $\beta \in \text{Comp}_n$ satisfying $D(\beta) \subseteq D(n)$ is the composition (n) itself (since $D(n)$ is the empty set \emptyset) and has length $\ell(n) = 1$. Likewise, the empty composition $\emptyset = ()$ satisfies $\eta_\emptyset = M_\emptyset$.

When α is an odd composition (i.e., all entries of α are odd), our definition of η_α is precisely the one given in [AgBeSo14, (6.1)], and differs only in sign from the one given in [Hsiao07, (2.1)] (because of [Hsiao07, Proposition 2.1]). Our main innovation is extending this definition to arbitrary compositions α .

The following is easy to see:

Proposition 2.3. Let $n \in \mathbb{N}$ and $\alpha \in \text{Comp}_n$. Then,

$$\eta_\alpha = \sum_{\substack{g = (g_1 \leq g_2 \leq \dots \leq g_n); \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\alpha)}} 2^{|\{g_1, g_2, \dots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n},$$

where the sum is over all weakly increasing n -tuples $g = (g_1 \leq g_2 \leq \dots \leq g_n)$ of positive integers that satisfy $(g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\alpha))$.

Proof. TODO. The slickest way to prove this is using the definition of η_α and [Grinbe15, detailed version, Proposition 10.10]. \square

Proposition 2.4. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \text{Comp}$. Then,

$$\eta_\alpha = \sum_{h_1 \leq h_2 \leq \dots \leq h_k} 2^{|\{h_1, h_2, \dots, h_k\}|} x_{h_1}^{\alpha_1} x_{h_2}^{\alpha_2} \cdots x_{h_k}^{\alpha_k}.$$

Proof. TODO. The slickest way to prove this is by imitating [Grinbe15, detailed version, Proposition 10.69], after realizing that Proposition 2.3 rewrites as

$$\eta_\alpha = \sum_{\substack{g = (g_1 \leq g_2 \leq \dots \leq g_n); \\ \{j \in [n-1] \mid g_j < g_{j+1}\} \subseteq D(\alpha)}} 2^{|\{g_1, g_2, \dots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

□

We can also write the η_α in the fundamental basis:

Proposition 2.5. Let n be a positive integer. Let $\alpha \in \text{Comp}_n$. Then,

$$\eta_\alpha = 2 \sum_{\gamma \in \text{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} L_\gamma.$$

This proposition generalizes [Hsiao07, Proposition 2.2], and is a bit similar to the discrete Radon transform on a hypercube ([Stanle18, ca. Theorem 2.2]).

It can be proved using the following simple binomial identity:

Lemma 2.6. Let S and T be two finite sets. Then,

$$\sum_{I \subseteq S} (-1)^{|I \setminus T|} = \begin{cases} 2^{|S|}, & \text{if } S \subseteq T; \\ 0, & \text{otherwise} \end{cases}.$$

Proof of Lemma 2.6. Here is a sketch; a detailed proof can be found in [Grinbe20, solution to Exercise 2.9.11].

If $S \subseteq T$, then each subset I of S satisfies $I \setminus T = \emptyset$, and thus the sum $\sum_{I \subseteq S} (-1)^{|I \setminus T|}$

is a sum of $2^{|S|}$ many addends each equal to $(-1)^{|\emptyset|} = 1$. On the other hand, if $S \not\subseteq T$, then there exists some $v \in S$ such that $v \notin T$, and therefore the addends of the sum $\sum_{I \subseteq S} (-1)^{|I \setminus T|}$ cancel each other out in pairs (viz., for each subset K of

$S \setminus \{v\}$, the addend for $I = K$ cancels the addend for $I = K \cup \{v\}$); thus, the sum is 0 in this case. □

Proof of Proposition 2.5. We have

$$\begin{aligned} & 2 \sum_{\gamma \in \text{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} \\ & \qquad \qquad \qquad = \underbrace{\sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}}}_{\text{(by the definition of } L_\gamma)} L_\gamma M_\beta \\ & = 2 \sum_{\gamma \in \text{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} M_\beta \\ & = 2 \sum_{\beta \in \text{Comp}_n} \sum_{\substack{\gamma \in \text{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} (-1)^{|D(\gamma) \setminus D(\alpha)|} M_\beta. \end{aligned}$$

But every $\beta \in \text{Comp}_n$ satisfies

$$\begin{aligned} \sum_{\substack{\gamma \in \text{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} (-1)^{|D(\gamma) \setminus D(\alpha)|} &= \sum_{\substack{I \subseteq [n-1]; \\ D(\beta) \supseteq I}} (-1)^{|I \setminus D(\alpha)|} \\ &\left(\begin{array}{l} \text{here, we have substituted } I \text{ for } D(\gamma) \text{ in the sum,} \\ \text{since the map } D : \text{Comp}_n \rightarrow \mathcal{P}([n-1]) \text{ is a bijection} \end{array} \right) \\ &= \sum_{I \subseteq D(\beta)} (-1)^{|I \setminus D(\alpha)|} \\ &= \begin{cases} 2^{|D(\beta)|}, & \text{if } D(\beta) \subseteq D(\alpha); \\ 0, & \text{otherwise} \end{cases} \quad (\text{by Lemma 2.6}). \end{aligned}$$

Hence, this becomes

$$\begin{aligned} &2 \sum_{\gamma \in \text{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} L_\gamma \\ &= 2 \sum_{\beta \in \text{Comp}_n} \underbrace{\sum_{\substack{\gamma \in \text{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} (-1)^{|D(\gamma) \setminus D(\alpha)|}}_{\begin{cases} 2^{|D(\beta)|}, & \text{if } D(\beta) \subseteq D(\alpha); \\ 0, & \text{otherwise} \end{cases}} M_\beta \\ &= 2 \sum_{\beta \in \text{Comp}_n} \begin{cases} 2^{|D(\beta)|}, & \text{if } D(\beta) \subseteq D(\alpha); \\ 0, & \text{otherwise} \end{cases} M_\beta \\ &= 2 \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} 2^{|D(\beta)|} M_\beta = \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \underbrace{2^{|D(\beta)|+1}}_{=2^{\ell(\beta)} \text{ (since } |D(\beta)|+1=\ell(\beta))} M_\beta \\ &= \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} 2^{\ell(\beta)} M_\beta = \eta_\alpha \quad (\text{by the definition of } \eta_\alpha). \end{aligned}$$

□

2.2. The antipode of η_α

The *antipode* of QSym is a certain \mathbf{k} -linear map $S : \text{QSym} \rightarrow \text{QSym}$ that can be defined in terms of the Hopf algebra structure of QSym, which we have not defined so far. But there are various formulas for its values on certain quasisymmetric functions that can be used as alternative definitions. For example, for any $n \in \mathbb{N}$ and any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell) \in \text{Comp}_n$, we have

$$S(M_\alpha) = (-1)^\ell \sum_{\substack{\gamma \in \text{Comp}_n; \\ D(\gamma) \subseteq D(\alpha_\ell, \alpha_{\ell-1}, \dots, \alpha_1)}} M_\gamma.$$

This can be used to define S (since S is to be \mathbf{k} -linear). Also, for each composition α , we have $S(L_\alpha) = (-1)^{|\alpha|} L_{\omega(\alpha)}$, where $\omega(\alpha)$ is a certain composition known as the *complement* of α . See [GriRei20, Theorem 5.1.11 and Proposition 5.2.15] for details and proofs. Note that S is a \mathbf{k} -algebra homomorphism and an involution (that is, $S^2 = \text{id}$). (Again, this is derived from abstract algebraic properties of antipodes in [GriRei20], but can also be showed more directly.)

Definition 2.7. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ is a composition, then the *reversal* of α is defined to be the composition $(\alpha_\ell, \alpha_{\ell-1}, \dots, \alpha_1)$. It is denoted by $\text{rev } \alpha$.

Proposition 2.8. Let $\alpha \in \text{Comp}$. Then, the antipode S of QSym satisfies

$$S(\eta_\alpha) = (-1)^{\ell(\alpha)} \eta_{\text{rev } \alpha}.$$

Proof. TODO. (This follows easily from Proposition 2.5.) □

Proposition 2.8 generalizes [Hsiao07, Proposition 2.9].

2.3. The η_α as a basis

Theorem 2.9. Assume that 2 is invertible in \mathbf{k} . Then, the family $(\eta_\alpha)_{\alpha \in \text{Comp}}$ is a basis of the \mathbf{k} -module QSym.

Proof. TODO. Fix $n \in \mathbb{N}$. Consider the n -th graded component QSym_n of QSym.

Define a partial order \prec on the finite set Comp_n by setting $\beta \prec \alpha$ if and only if $\ell(\beta) < \ell(\alpha)$

The definition of η_α shows that

$$\begin{aligned} \eta_\alpha &= 2^{\ell(\alpha)} M_\alpha + (\text{a linear combination of } M_\beta \text{ with } \beta \in \text{Comp}_n \text{ satisfying } \ell(\beta) < \ell(\alpha)) \\ &= 2^{\ell(\alpha)} M_\alpha + (\text{a linear combination of } M_\beta \text{ with } \beta \in \text{Comp}_n \text{ satisfying } \beta \prec \alpha) \end{aligned}$$

for each $\alpha \in \text{Comp}_n$. Thus, the family $(\eta_\alpha)_{\alpha \in \text{Comp}_n}$ expands invertibly triangularly in the family $(M_\alpha)_{\alpha \in \text{Comp}_n}$ with respect to the partial order \prec (where we are using the terminology from [GriRei20, §11.1]). Hence, [GriRei20, Corollary 11.1.19(e)] shows that the family $(\eta_\alpha)_{\alpha \in \text{Comp}_n}$ is a basis of the \mathbf{k} -module QSym_n (since the family $(M_\alpha)_{\alpha \in \text{Comp}_n}$ is a basis of QSym_n).

Forget that we fixed n . Thus, we have shown that the family $(\eta_\alpha)_{\alpha \in \text{Comp}_n}$ is a basis of the \mathbf{k} -module QSym_n for each $n \in \mathbb{N}$. Hence, the family $(\eta_\alpha)_{\alpha \in \text{Comp}}$ is a basis of the \mathbf{k} -module $\bigoplus_{n \in \mathbb{N}} \text{QSym}_n = \text{QSym}$. This proves Theorem 2.9. □

We can explicitly expand the monomial quasisymmetric functions M_β in the basis $(\eta_\alpha)_{\alpha \in \text{Comp}}$:

Proposition 2.10. Let $n \in \mathbb{N}$. Let $\beta \in \text{Comp}_n$ be a composition. Then,

$$2^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \text{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} \eta_\alpha.$$

Proof. TODO. (Follows from (4) using Möbius inversion.) \square

2.4. The product rule

Next comes a fairly nontrivial result: Given two compositions α and β , the product $\eta_\alpha \eta_\beta$ is a \mathbf{k} -linear combination of the family $(\eta_\gamma)_{\gamma \in \text{Comp}}$. If 2 is invertible in \mathbf{k} , this follows from Theorem 2.9, but in the general case (thus, e.g., for $\mathbf{k} = \mathbb{Z}$), I don't see any simple reasons why this should hold. Nevertheless it does, and there is a combinatorial expression. To state it, we need a weird variant of shuffles that I have never seen in the literature. First, as inspiration, let me cite the analogous rule for products of the form $M_\alpha M_\beta$:

Definition 2.11. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ be two compositions.

Fix two chains (i.e., totally ordered sets) $\{p_1 < p_2 < \dots < p_\ell\}$ and $\{q_1 < q_2 < \dots < q_m\}$, and let

$$D = \{p_1 < p_2 < \dots < p_\ell\} \sqcup \{q_1 < q_2 < \dots < q_m\}$$

be their disjoint union. This D is a poset with $\ell + m$ elements $p_1, p_2, \dots, p_\ell, q_1, q_2, \dots, q_m$, whose relations are given by $p_1 < p_2 < \dots < p_\ell$ and $q_1 < q_2 < \dots < q_m$ (while each p_i is incomparable to each q_j).

A *stuffer* for α and β shall mean a surjective and strictly order-preserving map

$$f : D \rightarrow \{1 < 2 < \dots < k\} \quad \text{for some } k \in \mathbb{N}.$$

("Strictly order-preserving" means that if u and v are two elements of the poset D satisfying $u < v$, then $f(u) < f(v)$.)

If $f : D \rightarrow \{1 < 2 < \dots < k\}$ is a stuffer for α and β , then we define the *weight* $\text{wt}(f)$ of the stuffer f to be the composition $(\text{wt}_1(f), \text{wt}_2(f), \dots, \text{wt}_k(f))$, where

$$\text{wt}_s(f) = \sum_{\substack{u \in [\ell]; \\ f(p_u) = s}} \alpha_u + \sum_{\substack{v \in [m]; \\ f(q_v) = s}} \beta_v \quad \text{for each } s \in [k].$$

Note that each of the two sums on the right hand side has at most 1 addend.

Example 2.12. Let $\alpha = (4, 2)$ and $\beta = (1, 3, 1)$ be two compositions. Then, the poset D in Definition 2.11 is $D = \{p_1 < p_2\} \sqcup \{q_1 < q_2 < q_3\}$. The following maps (written in two-line notation) are stufflers for α and β :

$$\begin{array}{cc} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 4 & 2 & 3 & 5 \end{pmatrix}, \\ \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 3 & 4 \end{pmatrix}, \\ \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 3 & 2 & 3 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 3 & 1 & 2 & 3 \end{pmatrix}. \end{array}$$

(The list is not exhaustive – there are many more stufflers for α and β .) On the other hand, $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 4 & 2 & 2 & 3 \end{pmatrix}$ is not a stuffler for α and β (since it fails the “strictly order-preserving” condition, by way of sending q_1 and q_2 to the same number), and $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 4 & 2 & 3 & 4 \end{pmatrix}$ is not a stuffler either (since it fails to be surjective onto $\{1 < 2 < 3 < 4\}$).

Here are the weights of the above listed stufflers:

$$\begin{array}{l} \text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (4, 2, 1, 3, 1), \\ \text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 4 & 2 & 3 & 5 \end{pmatrix} = (4, 1, 3, 2, 1), \\ \text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} = (1, 4, 3, 1, 2), \\ \text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 3 & 4 \end{pmatrix} = (4 + 1, 2, 3, 1) = (5, 2, 3, 1), \\ \text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 3 & 2 & 3 & 4 \end{pmatrix} = (4, 1, 2 + 3, 1) = (4, 1, 5, 1), \\ \text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 3 & 1 & 2 & 3 \end{pmatrix} = (4 + 1, 3, 2 + 1) = (5, 3, 3). \end{array}$$

The composition $\text{wt}(f)$ in Definition 2.11 is called a *shuffle* (or *overlapping shuffle*) of α with β . Each of its entries is either an entry of α or an entry of β or a sum of an entry of α with an entry of β ; moreover, each of the entries of α and of β is used in exactly one entry of $\text{wt}(f)$, and the entries of α appear in their original order in the entries of $\text{wt}(f)$, and so do the entries of β .

Now we can state the multiplication rule for products of the form $M_\alpha M_\beta$ ([GriRei20, Proposition 5.1.3]):

Theorem 2.13. Let α and β be two compositions. Then,

$$M_\alpha M_\beta = \sum_{f \text{ is a stuffer for } \alpha \text{ and } \beta} M_{\text{wt}(f)}.$$

Example 2.14. Let $\alpha = (a, b)$ and $\beta = (c, d)$ be two compositions of length 2. Let us compute $M_{(a,b)}M_{(c,d)}$ using Theorem 2.13. The stuffers for α and β are the maps (written here in two-line notation)

$$\begin{array}{ccc} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 3 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 2 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \\ \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 4 & 1 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \\ \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 1 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 2 & 3 \end{pmatrix}, \\ \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 2 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 3 \end{pmatrix}, \\ \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 2 \end{pmatrix}. & & \end{array}$$

Their respective weights are

$$\begin{array}{cccc} (a, b, c, d), & (a, c, b, d), & (a, c, d, b), & (c, a, b, d), \\ (c, a, d, b), & (c, d, a, b), & (a + c, b, d), & (a + c, d, b), \\ (a, b + c, d), & (c, a + d, b), & (a, c, b + d), & (c, a, b + d), \\ (a + c, b + d). & & & \end{array}$$

Thus, Theorem 2.13 yields

$$\begin{aligned} M_{(a,b)}M_{(c,d)} &= M_{(a,b,c,d)} + M_{(a,c,b,d)} + M_{(a,c,d,b)} + M_{(c,a,b,d)} + M_{(c,a,d,b)} + M_{(c,d,a,b)} \\ &\quad + M_{(a+c,b,d)} + M_{(a+c,d,b)} + M_{(a,b+c,d)} + M_{(c,a+d,b)} \\ &\quad + M_{(a,c,b+d)} + M_{(c,a,b+d)} + M_{(a+c,b+d)}. \end{aligned}$$

The formula for $\eta_\alpha \eta_\beta$ is similar but subtler. Instead of stuffers, we need what I call the *liminal stuffers*, which are defined as follows:

Definition 2.15. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ be two compositions.

Fix two chains (i.e., totally ordered sets) $\{p_1 < p_2 < \dots < p_\ell\}$ and $\{q_1 < q_2 < \dots < q_m\}$, and define a poset D as in Definition 2.11.

A *liminal stuffer* for α and β shall mean a surjective and weakly order-preserving map

$$f : D \rightarrow \{1 < 2 < \dots < k\} \quad \text{for some } k \in \mathbb{N}$$

with the property that each $s \in \{1 < 2 < \dots < k\}$ satisfies

$$|\{u \in [\ell] \mid f(p_u) = s\}| - |\{v \in [m] \mid f(q_v) = s\}| \in \{1, -1\}. \tag{5}$$

(“Weakly order-preserving” means that if u and v are two elements of the poset D satisfying $u < v$, then $f(u) \leq f(v)$.)

If $f : D \rightarrow \{1 < 2 < \dots < k\}$ is a liminal stuffer for α and β , then we define the *weight* $\text{wt}(f)$ of f to be the composition $(\text{wt}_1(f), \text{wt}_2(f), \dots, \text{wt}_k(f))$, where

$$\text{wt}_s(f) = \sum_{\substack{u \in [\ell]; \\ f(p_u) = s}} \alpha_u + \sum_{\substack{v \in [m]; \\ f(q_v) = s}} \beta_v \quad \text{for each } s \in [k].$$

(This time, the sums on the right hand side can have more than 1 addend. But (5) ensures that one of the two sums has exactly 1 more addend than the other.)

If $f : D \rightarrow \{1 < 2 < \dots < k\}$ is a liminal stuffer for α and β , then the *loss* of f is defined to be the nonnegative integer

$$\sum_{s=1}^k \min \{ |\{u \in [\ell] \mid f(p_u) = s\}|, |\{v \in [m] \mid f(q_v) = s\}| \}.$$

This is denoted by $\text{loss}(f)$. It is easy to see that $k = \ell + m - 2 \text{loss}(f)$.

Example 2.16. Let $\alpha = (4, 2)$ and $\beta = (1, 3, 1)$ be two compositions. Then, the poset D in Definition 2.11 is $D = \{p_1 < p_2\} \sqcup \{q_1 < q_2 < q_3\}$. The following maps (written in two-line notation) are liminal stuffers for α and β :

$$\begin{aligned} & \left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{array} \right), & & \left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{array} \right), \\ & \left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 2 & 3 \end{array} \right), & & \left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 2 & 3 \end{array} \right), \\ & \left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 1 & 2 & 3 \end{array} \right), & & \left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right). \end{aligned}$$

(The list is not exhaustive – there are many more liminal stuffers for α and β . In particular, any **injective** stuffer for α and β is a liminal stuffer for α and β as well.) On the other hand, $\left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 2 \end{array} \right)$ is not a liminal stuffer for α and β (since (5) fails for $s = 1$), and $\left(\begin{array}{ccccc} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 2 & 2 & 2 \end{array} \right)$ is not a liminal

stuffer either (since it fails to be surjective onto $\{1 < 2 < 3 < 4\}$).

Here are the weights of the above listed liminal stuffers:

$$\text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (4, 2, 1, 3, 1),$$

$$\text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} = (1, 4, 3, 1, 2),$$

$$\text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix} = (4 + 2 + 1, 3, 1) = (7, 3, 1),$$

$$\text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} = (4, 2 + 1 + 3, 1) = (4, 6, 1),$$

$$\text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 1 & 2 & 3 \end{pmatrix} = (1, 4 + 2 + 3, 1) = (1, 9, 1),$$

$$\text{wt} \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = (4 + 2 + 1 + 3 + 1) = (11).$$

The losses of these liminal stuffers are 0, 0, 1, 1, 1 and 2, respectively.

Intuitively, the composition $\text{wt}(f)$ in Definition 2.15 can be thought of as a variant of a stuffle of α with β , but instead of adding an entry of α with an entry of β , it allows adding i consecutive entries of α and j consecutive entries of β whenever i and j are integers satisfying $i - j \in \{1, -1\}$. The statistic $\text{loss}(f)$ tells how much is being added, i.e., how far this “stuffle” deviates from a shuffle.

Now we can state the multiplication rule for products of the form $\eta_\alpha \eta_\beta$:

Theorem 2.17. Let α and β be two compositions. Then,

$$\eta_\alpha \eta_\beta = \sum_{\substack{f \text{ is a liminal stuffer} \\ \text{for } \alpha \text{ and } \beta}} (-1)^{\text{loss}(f)} \eta_{\text{wt}(f)}.$$

Example 2.18. Let $\alpha = (a, b)$ and $\beta = (c, d)$ be two compositions of length 2. Let us compute $\eta_{(a,b)} \eta_{(c,d)}$ using Theorem 2.17. The liminal stuffers for α and β are

the maps (written here in two-line notation)

$$\begin{array}{ccc}
 \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 3 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 2 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \\
 \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 4 & 1 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \\
 \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 2 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 2 & 1 & 2 \end{pmatrix}, & \\
 \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 1 & 1 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 1 \end{pmatrix}. &
 \end{array}$$

Their respective weights are

$$\begin{array}{cccc}
 (a, b, c, d), & (a, c, b, d), & (a, c, d, b), & (c, a, b, d), \\
 (c, a, d, b), & (c, d, a, b), & (a, b + c + d), & \\
 (c, a + b + d), & (a + b + c, d), & (a + c + d, b), &
 \end{array}$$

and their respective losses are 0, 0, 0, 0, 0, 0, 1, 1, 1, 1. Thus, Theorem 2.17 yields

$$\begin{aligned}
 \eta_{(a,b)}\eta_{(c,d)} &= \eta_{(a,b,c,d)} + \eta_{(a,c,b,d)} + \eta_{(a,c,d,b)} + \eta_{(c,a,b,d)} + \eta_{(c,a,d,b)} + \eta_{(c,d,a,b)} \\
 &\quad - \eta_{(a,b+c+d)} - \eta_{(c,a+b+d)} - \eta_{(a+b+c,d)} - \eta_{(a+c+d,b)}.
 \end{aligned}$$

We will prove Theorem 2.17 further below.

Question 2.19. Is there a direct/combinatorial proof of Theorem 2.17, possibly using Proposition 2.4 and a sign-reversing involution?

Theorem 2.17 can be seen to generalize [Hsiao07, Corollary 2.5].

2.5. The dual basis η_α^* in NSym

In order to prove Theorem 2.17, it suffices to prove it when $\mathbf{k} = \mathbb{Q}$ (because all identities that hold in QSym over \mathbb{Q} but involve no denominators will automatically hold in QSym over \mathbb{Z} , and therefore also in QSym over any commutative ring \mathbf{k}). Thus, the following convention is harmless:

Convention 2.20. For the rest of this section, we WLOG assume that 2 is invertible in \mathbf{k} .

Recall the Hopf algebra NSym defined in [GriRei20, §5.4]. It is the graded dual of the Hopf algebra QSym.

We will use only one basis of NSym, namely the basis $(H_\alpha)_{\alpha \in \text{Comp}}$. This is the basis of NSym dual to the basis $(M_\alpha)_{\alpha \in \text{Comp}}$ of QSym. We write H_n for $H_{(n)}$ whenever n is a positive integer. We also set $H_0 = 1$ and $H_n = 0$ for all $n < 0$.

(What I call H_β is called S_β in [GKLLRT94].)

Definition 2.21. For each $n \in \mathbb{N}$ and each composition α of n , we define

$$\eta_\alpha^* = \sum_{\substack{\beta \in \text{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} \frac{1}{2^{\ell(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} H_\beta \in \text{NSym}.$$

Then, it is straightforward to see the following:

Proposition 2.22. The family $(\eta_\alpha^*)_{\alpha \in \text{Comp}}$ is the basis of NSym dual to the basis $(\eta_\alpha)_{\alpha \in \text{Comp}}$ of QSym.

Proof. TODO. (Follows from Proposition 2.10 by dualization.) □

Definition 2.23. For each positive integer n , let

$$\eta_n^* = \eta_{(n)}^* = \sum_{\beta \in \text{Comp}_n} \frac{1}{2^{\ell(\beta)}} (-1)^{\ell(\beta) - 1} H_\beta \in \text{NSym}.$$

It turns out that we can easily express η_α^* for any composition α using these η_n^* :

Proposition 2.24. We have

$$\eta_\alpha^* = \eta_{\alpha_1}^* \eta_{\alpha_2}^* \cdots \eta_{\alpha_k}^* \quad \text{for each composition } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_k).$$

The main idea of the proof of Proposition 2.24 is to recognize that if $n = |\alpha|$, then the compositions $\beta \in \text{Comp}_n$ satisfying $D(\alpha) \subseteq D(\beta)$ are precisely the compositions obtained from α by breaking up each entry of α into pieces. A slicker way to formalize this proof proceeds using the notion of concatenation:

Definition 2.25. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_k)$ are two compositions, then the composition $\alpha\beta$ is defined by

$$\alpha\beta = (\alpha_1, \alpha_2, \dots, \alpha_\ell, \beta_1, \beta_2, \dots, \beta_k).$$

This composition $\alpha\beta$ is called the *concatenation* of α and β . The operation of concatenation (sending any two compositions α and β to $\alpha\beta$) is associative, and the empty composition \emptyset is a neutral element for it; thus, the set of all compositions is an abelian monoid under this operation.

The following proposition is saying (in the jargon of combinatorial Hopf algebras) that the basis $(\eta_\alpha^*)_{\alpha \in \text{Comp}}$ of NSym is multiplicative:

Proposition 2.26. Let α and β be two compositions. Then,

$$\eta_\alpha^* \eta_\beta^* = \eta_{\alpha\beta}^*.$$

Proof of Proposition 2.26. TODO. Sketch: The definition of a concatenation easily yields that $\ell(\gamma) + \ell(\delta) = \ell(\gamma\delta)$ for any two compositions γ and δ . Thus, in particular, $\ell(\alpha\beta) = \ell(\alpha) + \ell(\beta)$.

Let $n = |\alpha|$ and $m = |\beta|$. Thus, $\alpha \in \text{Comp}_n$ and $\beta \in \text{Comp}_m$, so that $\alpha\beta \in \text{Comp}_{n+m}$.

The definitions of η_α^* and η_β^* yield

$$\eta_\alpha^* = \sum_{\substack{\gamma \in \text{Comp}_n \\ D(\alpha) \subseteq D(\gamma)}} \frac{1}{2^{\ell(\gamma)}} (-1)^{\ell(\gamma) - \ell(\alpha)} H_\gamma \quad \text{and}$$

$$\eta_\beta^* = \sum_{\substack{\delta \in \text{Comp}_m \\ D(\beta) \subseteq D(\delta)}} \frac{1}{2^{\ell(\delta)}} (-1)^{\ell(\delta) - \ell(\beta)} H_\delta.$$

Multiplying these two equalities, we obtain

$$\begin{aligned} \eta_\alpha^* \eta_\beta^* &= \left(\sum_{\substack{\gamma \in \text{Comp}_n \\ D(\alpha) \subseteq D(\gamma)}} \frac{1}{2^{\ell(\gamma)}} (-1)^{\ell(\gamma) - \ell(\alpha)} H_\gamma \right) \left(\sum_{\substack{\delta \in \text{Comp}_m \\ D(\beta) \subseteq D(\delta)}} \frac{1}{2^{\ell(\delta)}} (-1)^{\ell(\delta) - \ell(\beta)} H_\delta \right) \\ &= \sum_{\substack{(\gamma, \delta) \in \text{Comp}_n \times \text{Comp}_m \\ D(\alpha) \subseteq D(\gamma) \text{ and } D(\beta) \subseteq D(\delta)}} \underbrace{\frac{1}{2^{\ell(\gamma)}} \cdot \frac{1}{2^{\ell(\delta)}}}_{= \frac{1}{2^{\ell(\gamma) + \ell(\delta)}} = \frac{1}{2^{\ell(\gamma\delta)}}} \underbrace{(-1)^{\ell(\gamma) - \ell(\alpha)} (-1)^{\ell(\delta) - \ell(\beta)}}_{= (-1)^{(\ell(\gamma) - \ell(\alpha)) + (\ell(\delta) - \ell(\beta))} = (-1)^{(\ell(\gamma) + \ell(\delta)) - (\ell(\alpha) + \ell(\beta))} = (-1)^{\ell(\gamma\delta) - \ell(\alpha\beta)}} H_\gamma H_\delta \\ &= \sum_{\substack{(\gamma, \delta) \in \text{Comp}_n \times \text{Comp}_m \\ D(\alpha) \subseteq D(\gamma) \text{ and } D(\beta) \subseteq D(\delta)}} \frac{1}{2^{\ell(\gamma\delta)}} (-1)^{\ell(\gamma\delta) - \ell(\alpha\beta)} H_{\gamma\delta}. \end{aligned} \quad (6)$$

But it is easy to see that every two compositions $\gamma \in \text{Comp}_n$ and $\delta \in \text{Comp}_m$ satisfy $D(\gamma\delta) = D(\gamma) \cup \{n\} \cup (D(\delta) + n)$, where $D(\delta) + n$ denotes the set $\{d + n \mid d \in D(\delta)\}$. Using this fact, it is easy to see that the map

$$\begin{aligned} &\{(\gamma, \delta) \in \text{Comp}_n \times \text{Comp}_m \mid D(\alpha) \subseteq D(\gamma) \text{ and } D(\beta) \subseteq D(\delta)\} \\ &\rightarrow \{\zeta \in \text{Comp}_{n+m} \mid D(\alpha\beta) \subseteq D(\zeta)\}, \\ &(\gamma, \delta) \mapsto \gamma\delta \end{aligned}$$

is well-defined and is a bijection. Hence, we can substitute ζ for $\gamma\delta$ in the sum on

the right hand side of (6). Thus, (6) rewrites as

$$\eta_\alpha^* \eta_\beta^* = \sum_{\substack{\zeta \in \text{Comp}_{n+m} \\ D(\alpha\beta) \subseteq D(\zeta)}} \frac{1}{2^{\ell(\zeta)}} (-1)^{\ell(\zeta) - \ell(\alpha\beta)} H_\zeta.$$

Comparing this with

$$\eta_{\alpha\beta}^* = \sum_{\substack{\zeta \in \text{Comp}_{n+m} \\ D(\alpha\beta) \subseteq D(\zeta)}} \frac{1}{2^{\ell(\zeta)}} (-1)^{\ell(\zeta) - \ell(\alpha\beta)} H_\zeta \quad \left(\text{by the definition of } \eta_{\alpha\beta}^* \right),$$

we obtain $\eta_\alpha^* \eta_\beta^* = \eta_{\alpha\beta}^*$. This proves Proposition 2.26. □

Corollary 2.27. Let $\beta_1, \beta_2, \dots, \beta_k$ be finitely many compositions. Then,

$$\eta_{\beta_1}^* \eta_{\beta_2}^* \cdots \eta_{\beta_k}^* = \eta_{\beta_1 \beta_2 \cdots \beta_k}^*.$$

Proof. TODO. (This follows by induction on k using Proposition 2.26.) □

Proof of Proposition 2.24. This follows from applying Corollary 2.27 to the 1-element compositions $\beta_i = (\alpha_i)$ (since $\eta_n^* = \eta_{(n)}^*$ for each $n > 0$). □

Define the formal power series

$$H(t) = \sum_{n \geq 0} H_n t^n \in \text{NSym}[[t]]$$

and

$$G(t) = \sum_{n \geq 1} \eta_n^* t^n \in \text{NSym}[[t]].$$

Then, it is easy to see that:

Proposition 2.28. We have

$$G(t) = 1 - \frac{1}{1 + \frac{H(t) - 1}{2}} = \frac{H(t) - 1}{H(t) + 1}.$$

Proof. TODO. (Indeed, the first equality sign follows from the geometric series expansion, and the second is simple manipulation.) □

Consider the comultiplication $\Delta : \text{NSym} \rightarrow \text{NSym} \otimes \text{NSym}$ of the Hopf algebra NSym. The following formula for $\Delta(\eta_n^*)$ is a dual of Theorem 2.17:

Theorem 2.29. Let $n \in \mathbb{N}$. Then,

$$\Delta(\eta_n^*) = \sum_{\substack{\beta, \gamma \in \text{Comp}; \\ |\beta| + |\gamma| = n; \\ \ell(\beta) - \ell(\gamma) \in \{1, -1\}}} (-1)^{\min\{\ell(\beta), \ell(\gamma)\}} \eta_\beta^* \otimes \eta_\gamma^*.$$

Proof. From $G(t) = \sum_{n \geq 1} \eta_n^* t^n$, we obtain

$$\begin{aligned} G(t)^k &= \left(\sum_{n \geq 1} \eta_n^* t^n \right)^k = \sum_{n_1, n_2, \dots, n_k \geq 1} \eta_{n_1}^* \eta_{n_2}^* \cdots \eta_{n_k}^* t^{n_1 + n_2 + \cdots + n_k} \\ &= \sum_{\substack{\beta = (\beta_1, \beta_2, \dots, \beta_k) \in \text{Comp}; \\ \ell(\beta) = k}} \underbrace{\eta_{\beta_1}^* \eta_{\beta_2}^* \cdots \eta_{\beta_k}^*}_{= \eta_\beta^*} \underbrace{t^{\beta_1 + \beta_2 + \cdots + \beta_k}}_{= t^{|\beta|}} \\ &= \sum_{\substack{\beta = (\beta_1, \beta_2, \dots, \beta_k) \in \text{Comp}; \\ \ell(\beta) = k}} \eta_\beta^* t^{|\beta|} = \sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta) = k}} \eta_\beta^* t^{|\beta|} \end{aligned} \quad (7)$$

for every $k \in \mathbb{N}$.

The comultiplication $\Delta : \text{NSym} \rightarrow \text{NSym} \otimes \text{NSym}$ induces a \mathbf{k} -algebra homomorphism

$$\Delta_t : \text{NSym}[[t]] \rightarrow (\text{NSym} \otimes \text{NSym})[[t]]$$

that sends each formal power series $\sum_{i \in \mathbb{N}} a_i t^i$ to $\sum_{i \in \mathbb{N}} \Delta(a_i) t^i$. Note that there is a canonical \mathbf{k} -algebra homomorphism

$$\begin{aligned} \iota : \text{NSym}[[t]] \otimes_{\mathbf{k}[[t]]} \text{NSym}[[t]] &\rightarrow (\text{NSym} \otimes \text{NSym})[[t]], \\ \left(\sum_{i \in \mathbb{N}} a_i t^i \right) \otimes \left(\sum_{j \in \mathbb{N}} b_j t^j \right) &\mapsto \left(\sum_{i \in \mathbb{N}} (a_i \otimes 1) t^i \right) \left(\sum_{j \in \mathbb{N}} (1 \otimes b_j) t^j \right). \end{aligned}$$

Authors often treat ι as an embedding. We won't, but we will still use the fact that ι is a \mathbf{k} -algebra homomorphism a lot without saying.

It is easy to see that

$$\Delta_t(H(t)) = \iota(H(t) \otimes H(t)) \quad (8)$$

(this follows from [GriRei20, (5.4.2)]).

Define four elements h_1, h_2, g_1 and g_2 of $(\text{NSym} \otimes \text{NSym})[[t]]$ by

$$\begin{aligned} h_1 &= \iota(H(t) \otimes 1) & \text{and} & & h_2 &= \iota(1 \otimes H(t)) & \text{and} \\ g_1 &= \iota(G(t) \otimes 1) & \text{and} & & g_2 &= \iota(1 \otimes G(t)). \end{aligned}$$

The elements h_1 and h_2 commute (since $H(t) \otimes 1$ and $1 \otimes H(t)$ commute). The elements $\frac{1}{h_1 + 1}$, $\frac{1}{h_2 + 1}$ and $\frac{1}{h_1 h_2 + 1}$ are rational functions in h_1 and h_2 and therefore

also commute with them (and with each other). Thus, $h_1, h_2, \frac{1}{h_1+1}, \frac{1}{h_2+1}$ and $\frac{1}{h_1h_2+1}$ generate a commutative \mathbf{k} -subalgebra \mathcal{H} of $(NSym \otimes NSym)[[t]]$.

Moreover, from $G(t) = \frac{H(t)-1}{H(t)+1}$, we obtain

$$g_1 = \frac{h_1-1}{h_1+1} \quad \text{and} \quad g_2 = \frac{h_2-1}{h_2+1}. \quad (9)$$

Thus, the elements g_1 and g_2 also belong to the commutative \mathbf{k} -subalgebra \mathcal{H} generated by $h_1, h_2, \frac{1}{h_1+1}, \frac{1}{h_2+1}$ and $\frac{1}{h_1h_2+1}$.

The equality (8) becomes

$$\begin{aligned} \Delta_t(H(t)) &= \iota \left(\underbrace{H(t) \otimes H(t)}_{=(H(t) \otimes 1)(1 \otimes H(t))} \right) = \iota((H(t) \otimes 1)(1 \otimes H(t))) \\ &= \underbrace{\iota(H(t) \otimes 1)}_{=h_1} \cdot \underbrace{\iota(1 \otimes H(t))}_{=h_2} \\ &= h_1 h_2. \end{aligned} \quad (10)$$

Hence,

$$\frac{\Delta_t(H(t)) - 1}{\Delta_t(H(t)) + 1} = \frac{h_1 h_2 - 1}{h_1 h_2 + 1} = \frac{g_1 + g_2}{1 + g_1 g_2}. \quad (11)$$

(Indeed, the last equality sign can easily be verified by straightforward computation in the commutative \mathbf{k} -algebra \mathcal{H} , using the equalities (9).)

From $g_1 = \iota(G(t) \otimes 1)$ and $g_2 = \iota(1 \otimes G(t))$, we obtain

$$g_1 + g_2 = \iota(G(t) \otimes 1) + \iota(1 \otimes G(t)) = \iota(G(t) \otimes 1 + 1 \otimes G(t)) \quad (12)$$

(since ι is a \mathbf{k} -algebra homomorphism) and

$$\begin{aligned} g_1 g_2 &= \iota(G(t) \otimes 1) \cdot \iota(1 \otimes G(t)) = \iota \left(\underbrace{(1 \otimes G(t)) \cdot (G(t) \otimes 1)}_{=G(t) \otimes G(t)} \right) \\ &\quad \text{(since } \iota \text{ is a } \mathbf{k}\text{-algebra homomorphism)} \\ &= \iota(G(t) \otimes G(t)). \end{aligned} \quad (13)$$

Now, Proposition 2.28 yields

$$\begin{aligned}
 \Delta_t(G(t)) &= \Delta_t\left(\frac{H(t)-1}{H(t)+1}\right) = \frac{\Delta_t(H(t))-1}{\Delta_t(H(t))+1} \\
 &\quad \text{(since } \Delta_t \text{ is a continuous } \mathbf{k}\text{-algebra homomorphism)} \\
 &= \frac{g_1 + g_2}{1 + g_1 g_2} \quad \text{(by (11))} \\
 &= \sum_{i \in \mathbb{N}} (-1)^i (g_1 g_2)^i (g_1 + g_2) \\
 &= \sum_{i \in \mathbb{N}} (-1)^i \underbrace{(\iota(G(t) \otimes G(t)))^i \cdot \iota(G(t) \otimes 1 + 1 \otimes G(t))}_{= \iota\left(\underbrace{(G(t)^i \otimes G(t)^i)}_{= G(t)^{i+1} \otimes G(t)^i + G(t)^i \otimes G(t)^{i+1}}\right)} \\
 &\quad \text{(since } \iota \text{ is a } \mathbf{k}\text{-algebra homomorphism)} \\
 &\quad \text{(by (13) and (12))} \\
 &= \sum_{i \in \mathbb{N}} (-1)^i \iota \left(\underbrace{(G(t)^i \otimes G(t)^i) (G(t) \otimes 1 + 1 \otimes G(t))}_{= G(t)^{i+1} \otimes G(t)^i + G(t)^i \otimes G(t)^{i+1}} \right) \\
 &= \sum_{i \in \mathbb{N}} (-1)^i \iota \left(G(t)^{i+1} \otimes G(t)^i + G(t)^i \otimes G(t)^{i+1} \right) \\
 &= \sum_{\substack{i \in \mathbb{N}; \\ j \in \mathbb{N}; \\ i-j \in \{1, -1\}}} (-1)^{\min\{i, j\}} \iota \left(\underbrace{\underbrace{G(t)^i}_{= \sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta)=i \\ \text{(by (7))}} \eta_\beta^* t^{|\beta|}} \otimes \underbrace{G(t)^j}_{= \sum_{\substack{\gamma \in \text{Comp}; \\ \ell(\gamma)=j \\ \text{(by (7))}} \eta_\gamma^* t^{|\gamma|}}}}_{= \sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta)=i}} \sum_{\substack{\gamma \in \text{Comp}; \\ \ell(\gamma)=j}} \eta_\beta^* \otimes \eta_\gamma^* t^{|\beta|+|\gamma|}} \right) \\
 &= \sum_{\substack{i \in \mathbb{N}; \\ j \in \mathbb{N}; \\ i-j \in \{1, -1\}}} (-1)^{\min\{i, j\}} \iota \left(\underbrace{\left(\sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta)=i}} \eta_\beta^* t^{|\beta|} \right) \otimes \left(\sum_{\substack{\gamma \in \text{Comp}; \\ \ell(\gamma)=j}} \eta_\gamma^* t^{|\gamma|} \right)}_{= \sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta)=i}} \sum_{\substack{\gamma \in \text{Comp}; \\ \ell(\gamma)=j}} \eta_\beta^* \otimes \eta_\gamma^* t^{|\beta|+|\gamma|}} \right) \\
 &= \sum_{\substack{i \in \mathbb{N}; \\ j \in \mathbb{N}; \\ i-j \in \{1, -1\}}} (-1)^{\min\{i, j\}} \sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta)=i}} \sum_{\substack{\gamma \in \text{Comp}; \\ \ell(\gamma)=j}} \eta_\beta^* \otimes \eta_\gamma^* t^{|\beta|+|\gamma|} \\
 &= \sum_{\substack{\beta, \gamma \in \text{Comp}; \\ \ell(\beta) - \ell(\gamma) \in \{1, -1\}}} (-1)^{\min\{\ell(\beta), \ell(\gamma)\}} \eta_\beta^* \otimes \eta_\gamma^* t^{|\beta|+|\gamma|}.
 \end{aligned}$$

Comparing coefficients of t^n here, we obtain

$$\Delta(\eta_n^*) = \sum_{\substack{\beta, \gamma \in \text{Comp}; \\ |\beta| + |\gamma| = n; \\ \ell(\beta) - \ell(\gamma) \in \{1, -1\}}} (-1)^{\min\{\ell(\beta), \ell(\gamma)\}} \eta_\beta^* \otimes \eta_\gamma^*$$

for every positive integer n . □

2.6. The proof of the product rule

Proof of Theorem 2.17. TODO. (This needs some fleshing out.)

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be any composition. Then,

$$\begin{aligned}
\Delta(\eta_\alpha^*) &= \Delta\left(\eta_{\alpha_1}^* \eta_{\alpha_2}^* \cdots \eta_{\alpha_k}^*\right) && \text{(by Proposition 2.24)} \\
&= \Delta\left(\eta_{\alpha_1}^*\right) \Delta\left(\eta_{\alpha_2}^*\right) \cdots \Delta\left(\eta_{\alpha_k}^*\right) && \text{(since } \Delta \text{ is a } \mathbf{k}\text{-algebra homomorphism)} \\
&= \left(\sum_{\substack{\beta_1, \gamma_1 \in \text{Comp}; \\ |\beta_1| + |\gamma_1| = \alpha_1; \\ \ell(\beta_1) - \ell(\gamma_1) \in \{1, -1\}}} (-1)^{\min\{\ell(\beta_1), \ell(\gamma_1)\}} \eta_{\beta_1}^* \otimes \eta_{\gamma_1}^* \right) \\
&\quad \cdot \left(\sum_{\substack{\beta_2, \gamma_2 \in \text{Comp}; \\ |\beta_2| + |\gamma_2| = \alpha_2; \\ \ell(\beta_2) - \ell(\gamma_2) \in \{1, -1\}}} (-1)^{\min\{\ell(\beta_2), \ell(\gamma_2)\}} \eta_{\beta_2}^* \otimes \eta_{\gamma_2}^* \right) \\
&\quad \cdots \left(\sum_{\substack{\beta_k, \gamma_k \in \text{Comp}; \\ |\beta_k| + |\gamma_k| = \alpha_k; \\ \ell(\beta_k) - \ell(\gamma_k) \in \{1, -1\}}} (-1)^{\min\{\ell(\beta_k), \ell(\gamma_k)\}} \eta_{\beta_k}^* \otimes \eta_{\gamma_k}^* \right) \\
&\quad \text{(by Theorem 2.29)} \\
&= \sum_{\substack{\beta_1, \beta_2, \dots, \beta_k \in \text{Comp}; \\ \gamma_1, \gamma_2, \dots, \gamma_k \in \text{Comp}; \\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for each } s; \\ \ell(\beta_s) - \ell(\gamma_s) \in \{1, -1\} \text{ for each } s}} \left((-1)^{\min\{\ell(\beta_1), \ell(\gamma_1)\}} \eta_{\beta_1}^* \otimes \eta_{\gamma_1}^* \right) \\
&\quad \cdot \left((-1)^{\min\{\ell(\beta_2), \ell(\gamma_2)\}} \eta_{\beta_2}^* \otimes \eta_{\gamma_2}^* \right) \cdots \left((-1)^{\min\{\ell(\beta_k), \ell(\gamma_k)\}} \eta_{\beta_k}^* \otimes \eta_{\gamma_k}^* \right) \\
&= \sum_{\substack{\beta_1, \beta_2, \dots, \beta_k \in \text{Comp}; \\ \gamma_1, \gamma_2, \dots, \gamma_k \in \text{Comp}; \\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for each } s; \\ \ell(\beta_s) - \ell(\gamma_s) \in \{1, -1\} \text{ for each } s}} (-1)^{\min\{\ell(\beta_1), \ell(\gamma_1)\} + \min\{\ell(\beta_2), \ell(\gamma_2)\} + \cdots + \min\{\ell(\beta_k), \ell(\gamma_k)\}} \\
&\quad \underbrace{\left(\eta_{\beta_1}^* \eta_{\beta_2}^* \cdots \eta_{\beta_k}^* \right)}_{= \eta_{\beta_1 \beta_2 \cdots \beta_k}^* \text{ (by Corollary 2.27)}} \otimes \underbrace{\left(\eta_{\gamma_1}^* \eta_{\gamma_2}^* \cdots \eta_{\gamma_k}^* \right)}_{= \eta_{\gamma_1 \gamma_2 \cdots \gamma_k}^* \text{ (by Corollary 2.27)}} \\
&= \sum_{\substack{\beta_1, \beta_2, \dots, \beta_k \in \text{Comp}; \\ \gamma_1, \gamma_2, \dots, \gamma_k \in \text{Comp}; \\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for each } s; \\ \ell(\beta_s) - \ell(\gamma_s) \in \{1, -1\} \text{ for each } s}} (-1)^{\min\{\ell(\beta_1), \ell(\gamma_1)\} + \min\{\ell(\beta_2), \ell(\gamma_2)\} + \cdots + \min\{\ell(\beta_k), \ell(\gamma_k)\}} \\
&\quad \eta_{\beta_1 \beta_2 \cdots \beta_k}^* \otimes \eta_{\gamma_1 \gamma_2 \cdots \gamma_k}^*
\end{aligned}$$

$$\begin{aligned}
 = & \sum_{\delta, \varepsilon \in \text{Comp}} \sum_{\substack{\beta_1, \beta_2, \dots, \beta_k \in \text{Comp}; \\ \gamma_1, \gamma_2, \dots, \gamma_k \in \text{Comp}; \\ \beta_1 \beta_2 \cdots \beta_k = \delta; \\ \gamma_1 \gamma_2 \cdots \gamma_k = \varepsilon; \\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for each } s; \\ \ell(\beta_s) - \ell(\gamma_s) \in \{1, -1\} \text{ for each } s}} (-1)^{\min\{\ell(\beta_1), \ell(\gamma_1)\} + \min\{\ell(\beta_2), \ell(\gamma_2)\} + \cdots + \min\{\ell(\beta_k), \ell(\gamma_k)\}} \\
 & \eta_\delta^* \otimes \eta_\varepsilon^*. \tag{14}
 \end{aligned}$$

Now, let us take a closer look at the inner sum on the right hand side. For a given pair (δ, ε) of compositions, what are the pairs

$$((\beta_1, \beta_2, \dots, \beta_k), (\gamma_1, \gamma_2, \dots, \gamma_k))$$

satisfying the six conditions

$$\begin{aligned}
 & \beta_1, \beta_2, \dots, \beta_k \in \text{Comp}; & \gamma_1, \gamma_2, \dots, \gamma_k \in \text{Comp}; \\
 & \beta_1 \beta_2 \cdots \beta_k = \delta; & \gamma_1 \gamma_2 \cdots \gamma_k = \varepsilon; \\
 & |\beta_s| + |\gamma_s| = \alpha_s \text{ for each } s; \\
 & \ell(\beta_s) - \ell(\gamma_s) \in \{1, -1\} \text{ for each } s
 \end{aligned}$$

? I claim that they are in bijection with the liminal stufflers f for δ and ε that satisfy $\text{wt}(f) = \alpha$. Indeed, if we write the compositions δ and ε as $\delta = (\delta_1, \delta_2, \dots, \delta_\ell)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$, and if we define D as in Definition 2.11, then the bijection sends any such liminal stuffler f to the pair

$$((\beta_1, \beta_2, \dots, \beta_k), (\gamma_1, \gamma_2, \dots, \gamma_k)),$$

where

$$\begin{aligned}
 \beta_s &= (\text{the composition consisting of the } \delta_u \text{ for all } u \in [\ell] \text{ satisfying } f(p_u) = s \\
 &\quad \text{(in the order of increasing } u)) \\
 \gamma_s &= (\text{the composition consisting of the } \varepsilon_v \text{ for all } v \in [m] \text{ satisfying } f(q_v) = s \\
 &\quad \text{(in the order of increasing } v))
 \end{aligned}$$

(We are here using the fact that our liminal stuffler f must necessarily be a map from D to $\{1 < 2 < \cdots < k\}$, because its weight $\text{wt}(f) = \alpha$ is a composition of length k .)

Using this bijection, we can rewrite (14) as

$$\Delta(\eta_\alpha^*) = \sum_{\delta, \varepsilon \in \text{Comp}} \sum_{\substack{f \text{ is a liminal stuffler} \\ \text{for } \delta \text{ and } \varepsilon; \\ \text{wt}(f) = \alpha}} (-1)^{\text{loss}(f)} \eta_\delta^* \otimes \eta_\varepsilon^*. \tag{15}$$

(Here, we have used the fact that if our bijection sends a liminal stuffler f to a pair $((\beta_1, \beta_2, \dots, \beta_k), (\gamma_1, \gamma_2, \dots, \gamma_k))$, then

$$\min\{\ell(\beta_1), \ell(\gamma_1)\} + \min\{\ell(\beta_2), \ell(\gamma_2)\} + \cdots + \min\{\ell(\beta_k), \ell(\gamma_k)\} = \text{loss}(f).$$

This is easily seen from the definition of the bijection and of the loss.)

Dualizing the equality (15), we find precisely the claim of Theorem 2.17. (This needs some formalization.) \square

2.7. The coproduct of η_α

Consider the coproduct $\Delta : \text{QSym} \rightarrow \text{QSym} \otimes \text{QSym}$ of the Hopf algebra QSym. (See [GriRei20, §5.1] for its definition.) We claim the following simple formula for $\Delta(\eta_\alpha)$:

Theorem 2.30. Let $\alpha \in \text{Comp}$. Then,

$$\Delta(\eta_\alpha) = \sum_{\substack{\beta, \gamma \in \text{Comp}; \\ \alpha = \beta\gamma}} \eta_\beta \otimes \eta_\gamma.$$

This generalizes [Hsiao07, Corollary 2.7].

Proof of Theorem 2.30. TODO. (This follows by dualizing Proposition 2.26.) \square

3. The liminal stuffle algebra

Here is a different way of rewriting Theorem 2.17.

Let \mathcal{F} be the free \mathbf{k} -algebra with generators x_1, x_2, x_3, \dots . It has a basis consisting of all words over the alphabet $\{x_1, x_2, x_3, \dots\}$; these words are in bijection with the compositions.

For any $k \geq 0$, we let $r_k : \mathcal{F} \rightarrow \mathcal{F}$ be the linear operator defined recursively by

$$\begin{aligned} r_k(1) &= 0; \\ r_k(x_i w) &= x_{i+k} w \quad \text{for each } i \geq 1 \text{ and any word } w. \end{aligned}$$

(Thus, explicitly: r_k sends 1 to 0, and transforms any nonempty word by adding k to the subscript of its first letter. For example, $r_k(x_a x_b x_c) = x_{a+k} x_b x_c$.)

Let $\# : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ be the bilinear map on \mathcal{F} defined recursively by

$$\begin{aligned} 1\#w &= w \quad \text{for any word } w; \\ w\#1 &= w \quad \text{for any word } w; \\ (x_i u)\#(x_j v) &= x_i (u\#(x_j v)) + x_j ((x_i u)\#v) - r_{i+j}(u\#v). \end{aligned}$$

I call this bilinear map $\#$ the *liminal stuffle*. Thus, if $u = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_\ell}$ and $v = x_{\beta_1} x_{\beta_2} \cdots x_{\beta_m}$ are two words in \mathcal{F} , then

$$u\#v = \sum_{\substack{f \text{ is a liminal stuffler} \\ \text{for } \alpha \text{ and } \beta}} (-1)^{\text{loss}(f)} x_{\text{wt}(f)},$$

where we set $x_\gamma = x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_k}$ for every composition $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$. (This needs to be formally proven.)

Let $\text{eta} : \mathcal{F} \rightarrow \text{QSym}$ be the \mathbf{k} -linear map that sends $x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k}$ to η_α for each composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$. Then, the claim of Theorem 2.17 is that $\text{eta}(u\#v) = \text{eta } u \cdot \text{eta } v$.

Proposition 3.1. The liminal stuffle $\#$ on \mathcal{F} is commutative and associative and has identity 1.

Proof. Only associativity is nontrivial. It follows from Theorem 2.17, since eta is injective when 2 is invertible in \mathbf{k} . Is there a direct proof? I haven't found one so far, but it is likely an induction argument. \square

Thus, $(\mathcal{F}, \#)$ is a \mathbf{k} -algebra with unity 1. Hence, eta is a \mathbf{k} -algebra homomorphism from $(\mathcal{F}, \#)$ to QSym.

Let $\Delta : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$ be the \mathbf{k} -linear map that sends each word $w_1 w_2 \cdots w_n$ to $\sum_{i=0}^n w_1 w_2 \cdots w_i \otimes w_{i+1} w_{i+2} \cdots w_n$. This map Δ is called the *deconcatenation coproduct* (or the *cut coproduct*). This coproduct turns \mathcal{F} into a Hopf algebra (with counit $\varepsilon : \mathcal{F} \rightarrow \mathbf{k}$ sending each word $w_1 w_2 \cdots w_n$ to $\begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases}$). The map eta : $\mathcal{F} \rightarrow$ QSym is then easily seen to be a \mathbf{k} -coalgebra homomorphism (by Theorem 2.30).

The liminal stuffle $\#$ on \mathcal{F} respects the deconcatenation coproduct Δ of \mathcal{F} , in the following sense:

Proposition 3.2. We have $\Delta(u) \# \Delta(v) = \Delta(u \# v)$ for any $u, v \in \mathcal{F}$. Here, the “ $\#$ ” sign on the left hand side stands for the multiplication in the \mathbf{k} -algebra $(\mathcal{F}, \#) \otimes (\mathcal{F}, \#)$ (so, explicitly, is given by $(u_1 \otimes u_2) \# (v_1 \otimes v_2) = (u_1 \# v_1) \otimes (u_2 \# v_2)$ for any $u_1, u_2, v_1, v_2 \in \mathcal{F}$).

Proof. This follows from the corresponding property of QSym, since eta : $\mathcal{F} \rightarrow$ QSym is an injective \mathbf{k} -coalgebra homomorphism when 2 is invertible in \mathbf{k} .

Probably an inductive proof also exists. \square

4. The original problem

4.1. 2-phobic compositions

Definition 4.1. We say that a composition α is *2-phobic* if none of the entries of α equals 2.

Proposition 4.2. The span $\text{span}(\eta_\alpha)_\alpha$ is a 2-phobic composition is a \mathbf{k} -subalgebra of QSym.

Proof. This follows from Theorem 2.17, because if α and β are 2-phobic compositions, then $\text{wt}(f)$ is a 2-phobic composition whenever f is a limited stuffer for α and β . (Note that this is a special property of 2. Nothing like this holds for 4-phobic compositions.) \square

Remark 4.3. TODO! Proposition 4.2 seems to be the same as [BMSW00, Theorem 5.7], if my η_α are (proportional to) their θ_α (possibly for different α). Check this and say this.

4.2. The watered-down original problem

Now, we define another family of quasisymmetric functions (not a basis this time):

Definition 4.4. Let $n \in \mathbb{N}$, and let $\Lambda \subseteq [n]$. Then, we set

$$\bar{K}_{n,\Lambda} = \sum_{\substack{(g_1, g_2, \dots, g_n) \in \{1, 2, 3, \dots\}^n; \\ g_1 \leq g_2 \leq \dots \leq g_n; \\ \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1} \\ \text{(where we set } g_0 = 0 \text{ and } g_{n+1} = \infty)}} 2^{|\{g_1, g_2, \dots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

Theorem 4.5. (a) These $\bar{K}_{n,\Lambda}$ are quasisymmetric functions.

(b) The span $\text{span}(\bar{K}_{n,\Lambda})_{n \in \mathbb{N}; \Lambda \subseteq [n]}$ is a \mathbf{k} -subalgebra of QSym.

Proof of Theorem 4.5. Set

$$\bar{L}_{n,\Lambda} = \sum_{\substack{(g_1, g_2, \dots, g_n) \in \{1, 2, 3, \dots\}^n; \\ g_1 \leq g_2 \leq \dots \leq g_n; \\ \text{each } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1} \\ \text{(where we set } g_0 = 0 \text{ and } g_{n+1} = \infty)}} 2^{|\{g_1, g_2, \dots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}$$

for each $\Lambda \subseteq [n]$. Then, a standard inclusion/exclusion computation yields

$$\bar{K}_{n,\Lambda} = \sum_{M \subseteq \Lambda} (-1)^{|M|} \bar{L}_{n,M}$$

for each $\Lambda \subseteq [n]$. Thus,

$$\text{span}(\bar{K}_{n,\Lambda})_{n \in \mathbb{N}; \Lambda \subseteq [n]} = \text{span}(\bar{L}_{n,\Lambda})_{n \in \mathbb{N}; \Lambda \subseteq [n]}$$

(by the standard triangularity argument).

But $\bar{L}_{n,\Lambda}$ is 0 if 1 or n belongs to Λ (since neither $g_0 = g_1$ nor $g_n = g_{n+1}$ can hold), and otherwise equals η_α for the 2-phobic composition α defined by $[n-1] \setminus D(\alpha) = \Lambda \cup (\Lambda - 1)$. Conversely, for each 2-phobic composition α , we can find a $\Lambda \subseteq [n]$ containing neither 1 nor n that satisfies $\bar{L}_{n,\Lambda} = \eta_\alpha$. Thus, the families $(\bar{L}_{n,\Lambda})_{n \in \mathbb{N}; \Lambda \subseteq [n]}$ and $(\eta_\alpha)_\alpha$ is a 2-phobic composition contain the same nonzero elements. Hence,

$$\text{span}(\bar{L}_{n,\Lambda})_{n \in \mathbb{N}; \Lambda \subseteq [n]} = \text{span}(\eta_\alpha)_{\alpha \text{ is a 2-phobic composition}}.$$

Hence,

$$\begin{aligned} \text{span} (\bar{K}_{n,\Lambda})_{n \in \mathbb{N}; \Lambda \subseteq [n]} &= \text{span} (\bar{L}_{n,\Lambda})_{n \in \mathbb{N}; \Lambda \subseteq [n]} \\ &= \text{span} (\eta_\alpha)_{\alpha \text{ is a 2-phobic composition}} \cdot \end{aligned}$$

Thus, Theorem 4.5 (a) follows immediately (since $\text{span} (\eta_\alpha)_{\alpha \text{ is a 2-phobic composition}} \subseteq \text{QSym}$), and Theorem 4.5 (b) follows from Proposition 4.2. \square

4.3. Note on the Φ and Ψ bases

Remark 4.6. Consider the Φ and Ψ bases of $\text{NSym}_{\mathbb{Q}}$ from [GKLLRT94], and consider their dual bases ϕ and ψ of $\text{QSym}_{\mathbb{Q}}$ ([BDHMN17]). Then, over \mathbb{Q} , we have

$$\begin{aligned} \text{span} (\phi_\alpha)_{\alpha \text{ is a 2-phobic composition}} &= \text{span} (\psi_\alpha)_{\alpha \text{ is a 2-phobic composition}} \\ &= \text{span} (\eta_\alpha)_{\alpha \text{ is a 2-phobic composition}} \cdot \end{aligned}$$

This follows from the fact that $\Phi_2 = \Psi_2 = 4\eta_2^*$.

4.4. The original problem

Here comes the original problem ([Grinbe18, Question 2.51]) that all the above was invented to solve.

Let \mathcal{N} be the totally ordered set $\{0, 1, 2, \dots\} \cup \{\infty\}$, with total order given by $0 < 1 < 2 < \dots < \infty$.

Let $\text{Pow } \mathcal{N}$ be the ring of formal power series $\mathbb{Z} [[x_0, x_1, x_2, \dots, x_\infty]]$.

Let $n \in \mathbb{N}$. For any map $g : [n] \rightarrow \mathcal{N}$, we define a subset $\text{FE}(g)$ of $[n]$ as follows:

$$\begin{aligned} \text{FE}(g) &= \left\{ \min(g^{-1}(h)) \mid h \in \{1, 2, 3, \dots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} \\ &\quad \cup \left\{ \max(g^{-1}(h)) \mid h \in \{0, 1, 2, 3, \dots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}. \end{aligned}$$

In other words, $\text{FE}(g)$ is the set comprising the smallest elements of all nonempty fibers of g except for $g^{-1}(0)$ as well as the largest elements of all nonempty fibers of g except for $g^{-1}(\infty)$. We shall refer to the elements of $\text{FE}(g)$ as the *fiber-ends* of g .

If Λ is any subset of $[n]$, then we define a power series $K_{n,\Lambda}^{\mathbb{Z}} \in \text{Pow } \mathcal{N}$ by

$$\begin{aligned} K_{n,\Lambda}^{\mathbb{Z}} &= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \subseteq \text{FE}(g)}} 2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g \\ &= \sum_{\substack{g:[n] \rightarrow \mathcal{N} \text{ is weakly increasing;} \\ \text{no } i \in \Lambda \text{ satisfies } g(i-1)=g(i)=g(i+1) \\ \text{(where we set } g(0)=0 \text{ and } g(n+1)=\infty)}} 2^{|\{g([n]) \cap \{1,2,3,\dots\}\}|} \mathbf{x}_g \\ &= \sum_{\substack{(g_1, g_2, \dots, g_n) \in \mathcal{N}^n; \\ 0 \leq g_1 \leq g_2 \leq \dots \leq g_n \leq \infty; \\ \text{no } i \in \Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ \text{(where we set } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1, g_2, \dots, g_n\} \cap \{1,2,3,\dots\}\}|} x_{g_1} x_{g_2} \cdots x_{g_n}, \end{aligned}$$

where $\mathbf{x}_g := x_{g(1)} x_{g(2)} \cdots x_{g(n)}$.

Question 4.7. Is the span $\text{span} \left(K_{n,\Lambda}^{\mathbb{Z}} \right)_{n \in \mathbb{N}; \Lambda \subseteq [n]}$ a \mathbb{Z} -subalgebra of $\text{Pow } \mathcal{N}$?

In other words, is it true that $K_{n,\Lambda}^{\mathbb{Z}} \cdot K_{m,\Omega}^{\mathbb{Z}}$ is a \mathbb{Z} -linear combination of $K_{n+m,\Xi}^{\mathbb{Z}}$'s whenever $\Lambda \subseteq [n]$ and $\Omega \subseteq [m]$ are sets?

In [Grinbe18, Corollary 2.42], I have shown a formula for rewriting a product of the form $K_{n,\Lambda}^{\mathbb{Z}} \cdot K_{m,\Omega}^{\mathbb{Z}}$ as a \mathbb{Z} -linear combination of $K_{n+m,\Xi}^{\mathbb{Z}}$'s when $\Lambda \subseteq [n]$ and $\Omega \subseteq [m]$ are **lacunar nonempty** sets. But can it also be rewritten in such a way if Λ and Ω are arbitrary subsets of $[n]$ and $[m]$? Computations with SageMath suggest that the answer is “yes”. For example,

$$\begin{aligned} K_{2,\{1,2\}}^{\mathbb{Z}} \cdot K_{1,\{1\}}^{\mathbb{Z}} &= K_{3,\{2\}}^{\mathbb{Z}} + 2 \cdot K_{3,\{1,3\}}^{\mathbb{Z}} \quad \text{and} \\ K_{2,\emptyset}^{\mathbb{Z}} \cdot K_{1,\{1\}}^{\mathbb{Z}} &= K_{3,\emptyset}^{\mathbb{Z}} + K_{3,\{2\}}^{\mathbb{Z}} + K_{3,\{1,3\}}^{\mathbb{Z}} = K_{3,\{1\}}^{\mathbb{Z}} + K_{3,\{2\}}^{\mathbb{Z}} + K_{3,\{3\}}^{\mathbb{Z}}. \end{aligned}$$

But in general, the coefficients cannot be taken to be nonnegative!

Nevertheless, do they have combinatorial interpretations?

Note that the \mathbb{Q} -linear span of the $K_{n+m,\Xi}^{\mathbb{Z}}$'s for all $\Xi \subseteq [n+m]$ is (generally) larger than that of the $K_{n+m,\Xi}^{\mathbb{Z}}$'s with Ξ lacunar nonempty. It has dimension Sequence A005251 in the OEIS (I believe):

$$\begin{aligned} a(0) &= 0, \quad a(1) = a(2) = a(3) = 1; \\ \text{thereafter, } a(n) &= a(n-1) + a(n-2) + a(n-4). \end{aligned}$$

Theorem 4.5 (b) shows that Question 4.7 has a positive answer if we set the indeterminates x_0 and x_∞ to 0. Indeed, the $\bar{K}_{n,\Lambda}$ from Definition 4.4 can be rewritten as follows:

$$\bar{K}_{n,\Lambda} = K_{n,\Lambda}^{\mathbb{Z}} \Big|_{x_0=0 \text{ and } x_\infty=0}.$$

Now let us try to go back to the original Question 4.7, where 0 and ∞ are allowed as g_i -values (and x_0 and x_∞ are not set to 0).

Can we transform the duality argument (from the proof of Theorem 2.17) into a “Cauchy kernel” style computation in $\text{NSym}_{\mathbb{Q}} [[x_0, x_1, x_2, \dots, x_\infty]]$?

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