The Elser nuclei sum revisited

Version of April 26, 2021

(`arXiv:2009.11527v8`)

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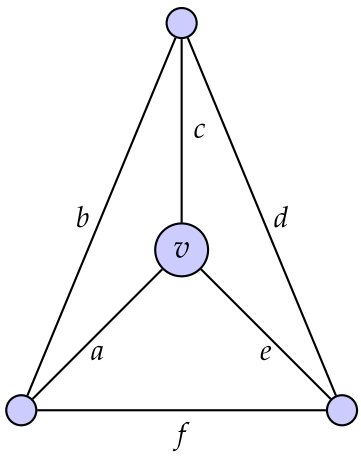

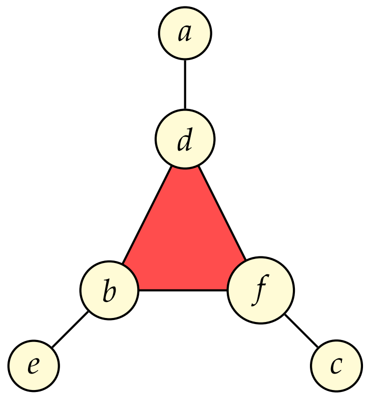

updated version of Oberwolfach Preprint OWP-2021-05

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Version of 26 April 2021,
equivalent to arXiv:2009.11527v8.

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Abstract. Fix a finite undirected graph Γ and a vertex v of Γ. Let E be the set of edges of Γ. We call a subset F of E *pandemic* if each edge of Γ has at least one endpoint that can be connected to v by an F-path (i.e., a path using edges from F only). In 1984, Elser showed that the sum of $(-1)^{|F|}$ over all pandemic subsets F of E is 0 if $E \neq \varnothing$. We give a simple proof of this result via a sign-reversing involution, and discuss variants, generalizations and refinements, revealing connections to abstract convexity (the notion of an antimatroid) and discrete Morse theory.

Keywords: graph theory, nuclei, simplicial complex, discrete Morse theory, alternating sum, enumerative combinatorics, inclusion/exclusion, convexity, antimatroids.

Future versions of this text will be available from the first author's website:
http://www.cip.ifi.lmu.de/~grinberg/algebra/elsersum.pdf
http://www.cip.ifi.lmu.de/~grinberg/algebra/elsersum-long.pdf (detailed version).

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In [Elser84], Veit Elser studied the probabilities of clusters forming when n points are sampled randomly in a d-dimensional volume. In the process, he found a purely graph-theoretical lemma [Elser84, Lemma 1], which served a crucial role in his work. For decades, the lemma stayed hidden from the eyes of combinatorialists in a physics journal, until it resurfaced in recent work [DHLetc19] by Dorpalen-Barry, Hettle, Livingston, Martin, Nasr, Vega and Whitlatch. In this note, I will show a simpler proof of the lemma using a sign-reversing involution. The proof also suggests multiple venues of generalization that I will explore in the later sections; one extends the lemma to a statement about arbitrary antimatroids (and even a wider setting). Finally, I will strengthen the lemma to a Morse-theoretical result, stating the collapsibility of a certain simplicial complex. Some open questions will be posed.

Note to the reader

The pictures on the title page illustrate the simplicial complex $\mathcal{A}$ from Proposition 5.2 on an example. The left picture is a graph Γ (with the vertex labelled v playing the role of v), whereas the right picture shows the corresponding simplicial complex $\mathcal{A}$ for $G = E$ (that is, the simplicial complex whose faces are the subsets of E that are not pandemic).

Much of this text has been conceived and written during a stay at the Mathematisches Forschungsinstitut Oberwolfach in 2020. This research was supported through the programme "Oberwolfach Leibniz Fellows" by the Mathematisches Forschungsinstitut Oberwolfach in 2020.

An older version of this text has been published as Oberwolfach Preprint OWP-2021-05.

Remark on alternative versions

This paper also has a detailed version [Grinbe20], which elaborates on the proofs.

1. Elser's result

Let us first introduce our setting, which is slightly more general (and perhaps also simpler) than that used in [Elser84]. (In Section 4, we will move to a more general setup.)

We fix an arbitrary graph Γ with vertex set V and edge set E. Here, "graph" means "finite undirected multigraph" – i.e., it can have self-loops and parallel edges, but it has finitely many vertices and edges, and its edges are undirected.

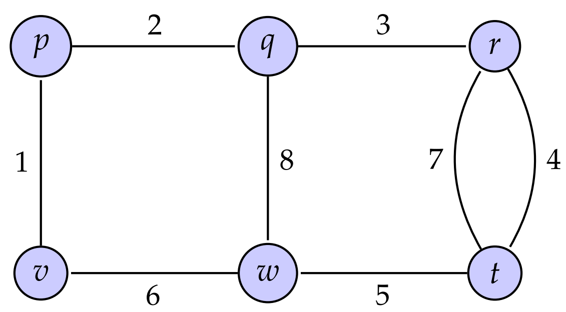
We fix a vertex $v \in V$.

If $F \subseteq E$, then an *F-path* shall mean a path of Γ such that all edges of the path belong to F.

If $e \in E$ is any edge and $F \subseteq E$ is any subset, then we say that F *infects* e if there exists an F-path from v to some endpoint of e. (The terminology is inspired by the idea of an infectious disease starting in the vertex v and being transmitted along edges.)[1]

A subset $F \subseteq E$ is said to be *pandemic* if it infects each edge $e \in E$.

Example 1.1. Let Γ be the following graph:



(where the vertex v is the vertex labelled v). Then, for example, the set $\{1,2\} \subseteq E$ infects edges $1,2,3,6,8$ (but none of the other edges). The set $\{1,2,5\}$ infects the same edges as $\{1,2\}$ (indeed, the additional edge 5 does not increase its infectiousness, since it is not on any $\{1,2,5\}$-path from v). The set $\{1,2,3\}$ infects every edge other than 5. The set $\{1,2,3,4\}$ infects each edge, and thus is pandemic.

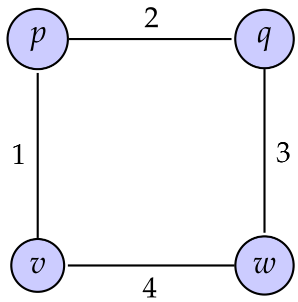
Now, we can state our version of [Elser84, Lemma 1]:

Theorem 1.2. Assume that $E \neq \varnothing$. Then,

$$\sum_{\substack{F \subseteq E \text{ is} \\ \text{pandemic}}} (-1)^{|F|} = 0. \tag{1}$$

[1]Note that if an edge e contains the vertex v, then any subset F of E (even the empty one) infects e, since there is a trivial (edgeless) F-path from v to v.

Example 1.3. Let Γ be the following graph:



(where the vertex v is the vertex labelled v). Then, the pandemic subsets of E are the sets

$$\{1,2\}, \quad \{1,4\}, \quad \{3,4\}, \quad \{1,2,3\}, \quad \{1,3,4\}, \quad \{1,2,4\}, \quad \{2,3,4\}, \quad \{1,2,3,4\}.$$

The sizes of these subsets are $2,2,2,3,3,3,3,4$, respectively. Hence, (1) says that

$$(-1)^2 + (-1)^2 + (-1)^2 + (-1)^3 + (-1)^3 + (-1)^3 + (-1)^3 + (-1)^4 = 0.$$

We note that the equality (1) can be restated as "there are equally many pandemic subsets $F \subseteq E$ of even size and pandemic subsets $F \subseteq E$ of odd size". Thus, in particular, the number of all pandemic subsets F of E is even (when $E \neq \varnothing$).

Remark 1.4. Theorem 1.2 is a bit more general than [Elser84, Lemma 1]. To see why, we assume that the graph Γ is connected and simple (i.e., has no self-loops and parallel edges). Then, a *nucleus* is defined in [Elser84] as a subgraph N of Γ with the properties that

1. the subgraph N is connected, and
2. each edge of Γ has at least one endpoint in N.

Given a subgraph N of Γ, we let $\operatorname{E}(N)$ denote the set of all edges of N. Now, [Elser84, Lemma 1] claims that if $E \neq \varnothing$, then

$$\sum_{\substack{N \text{ is a nucleus} \\ \text{containing } v}} (-1)^{|\operatorname{E}(N)|} = 0.$$

But this is equivalent to (1), because there is a bijection

$$\{\text{nuclei containing } v\} \to \{\text{pandemic subsets } F \subseteq E\}, \\ N \mapsto \operatorname{E}(N).$$

We leave it to the reader to check this in detail; what needs to be checked are the following three statements:

- If N is a nucleus containing v, then $\mathrm{E}(N)$ is a pandemic subset of E.
- Every nucleus N containing v is uniquely determined by the set $\mathrm{E}(N)$. (Indeed, since a nucleus has to be connected, each of its vertices must be an endpoint of one of its edges, unless its only vertex is v.)
- If F is a pandemic subset of E, then there is a nucleus N containing v such that $\mathrm{E}(N) = F$. (Indeed, N can be defined as the subgraph of Γ whose vertices are the endpoints of all edges in F as well as the vertex v, and whose edges are the edges in F. To see that this subgraph N is connected, it suffices to argue that each of its vertices has a path to v; but this follows from the definition of "pandemic", since each vertex of N other than v belongs to at least one edge in F.)

Thus, Theorem 1.2 is equivalent to [Elser84, Lemma 1] in the case when Γ is connected and simple.

Remark 1.5. It might appear more natural to talk about a subset $F \subseteq E$ infecting a vertex rather than an edge. (Namely, we can say that F infects a vertex w if there is an F-path from v to w.) However, the analogue of Theorem 1.2 in which pandemicity is defined via infecting all vertices is not true. The graph of Example 1.3 provides a counterexample.

2. The proof

2.1. Shades

Our proof of Theorem 1.2 will rest on a few notions. The first is that of a *shade*:

Definition 2.1. Let F be a subset of E. Then, we define a subset $\operatorname{Shade} F$ of E by

$$\operatorname{Shade} F = \{e \in E \mid F \text{ infects } e\}. \qquad (2)$$

We refer to $\operatorname{Shade} F$ as the *shade* of F.

Thus, the shade of a subset $F \subseteq E$ is the set of all edges of Γ that are infected by F. (In more standard graph-theoretical lingo, this means that $\operatorname{Shade} F$ is the set of edges that contain at least one vertex of the connected component containing v of the graph (V, F).)

Example 2.2. In Example 1.1, we have $\operatorname{Shade}\{1,2\} = \{1,2,3,6,8\}$ and $\operatorname{Shade}\{1\} = \{1,2,6\}$ and $\operatorname{Shade}\{8\} = \{1,6\}$.

The following property of shades is rather obvious:

Lemma 2.3. Let A and B be two subsets of E such that $A \subseteq B$. Then, $\operatorname{Shade} A \subseteq \operatorname{Shade} B$.

Proof of Lemma 2.3. We must show that each $q \in \operatorname{Shade} A$ satisfies $q \in \operatorname{Shade} B$. In other words, we must prove that if A infects some edge $q \in E$, then B also infects this edge q. But this is clear, since any A-path is a B-path. □

The major property of shades that we will need is the following:

Lemma 2.4. Let F be a subset of E. Let $u \in E$ be such that $u \notin \operatorname{Shade} F$. Then,

$$\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F \tag{3}$$

and

$$\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F. \tag{4}$$

Proof of Lemma 2.4. We shall prove (3) and (4) separately:

[*Proof of (3):* Let $q \in \operatorname{Shade}(F \cup \{u\})$. We shall show that $q \in \operatorname{Shade} F$.

We have assumed that $q \in \operatorname{Shade}(F \cup \{u\})$. In other words, q is an edge in E with the property that $F \cup \{u\}$ infects q (by the definition of $\operatorname{Shade}(F \cup \{u\})$).

We shall now show that F infects q. Indeed, assume the contrary. Thus, F does not infect q. In other words, there exists no F-path from v to any endpoint of q (by the definition of "infects").

We know that $F \cup \{u\}$ infects q. In other words, there exists an $(F \cup \{u\})$-path from v to some endpoint of q (by the definition of "infects"). Let π be this path. If this $(F \cup \{u\})$-path π did not contain the edge u, then it would be an F-path, which would contradict the fact that there exists no F-path from v to any endpoint of q. Hence, this $(F \cup \{u\})$-path π must contain the edge u. By removing u, we can thus cut this path π into two segments: The first segment is a path from v to some endpoint of u, while the second segment is a path from the other endpoint of u to some endpoint of q. Both segments are F-paths. Thus, in particular, the first segment is an F-path from v to some endpoint of u. Hence, there exists an F-path from v to some endpoint of u. In other words, F infects u (by the definition of "infects"). Hence, $u \in \operatorname{Shade} F$ (because of (2)). This contradicts $u \notin \operatorname{Shade} F$.

This contradiction shows that our assumption was false. Hence, we have proved that F infects q. In other words $q \in \operatorname{Shade} F$.

Forget that we fixed q. We thus have shown that $q \in \operatorname{Shade} F$ for each $q \in \operatorname{Shade}(F \cup \{u\})$. In other words, $\operatorname{Shade}(F \cup \{u\}) \subseteq \operatorname{Shade} F$. On the other hand, $F \subseteq F \cup \{u\}$; therefore, $\operatorname{Shade} F \subseteq \operatorname{Shade}(F \cup \{u\})$ (by Lemma 2.3).

Combining this with $\operatorname{Shade}(F \cup \{u\}) \subseteq \operatorname{Shade} F$, we obtain $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$. This proves (3).]

[*Proof of (4):* We must prove that $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$. This is obvious if $F \setminus \{u\} = F$. Thus, for the rest of this proof, we WLOG assume that $F \setminus \{u\} \neq F$. Hence, $u \in F$ and thus $(F \setminus \{u\}) \cup \{u\} = F$.

We have $F \setminus \{u\} \subseteq F$ and thus $\operatorname{Shade}(F \setminus \{u\}) \subseteq \operatorname{Shade} F$ (by Lemma 2.3). Hence, from $u \notin \operatorname{Shade} F$, we obtain $u \notin \operatorname{Shade}(F \setminus \{u\})$. Therefore, (3) (applied to $F \setminus \{u\}$ instead of F) yields $\operatorname{Shade}((F \setminus \{u\}) \cup \{u\}) = \operatorname{Shade}(F \setminus \{u\})$. Thus, $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} \underbrace{((F \setminus \{u\}) \cup \{u\})}_{=F} = \operatorname{Shade} F$. This proves (4).]

We have now proved both (3) and (4). Thus, Lemma 2.4 is proved. □

2.2. A slightly more general claim

Lemma 2.4 might not look very powerful, but it contains all we need to prove Theorem 1.2. Better yet, we shall prove the following slightly more general version of Theorem 1.2:

Theorem 2.5. Let G be any subset of E. Assume that $E \neq \varnothing$. Then,

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0.$$

We will soon prove Theorem 2.5 and explain how Theorem 1.2 follows from it. First, however, let us give an equivalent (but slightly easier to prove) version of Theorem 2.5:

Theorem 2.6. Let G be any subset of E. Then,

$$\sum_{\substack{F \subseteq E; \\ G \not\subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0.$$

Proof of Theorem 2.6. Let

$$\mathcal{A} = \{P \subseteq E \mid G \not\subseteq \operatorname{Shade} P\}. \tag{5}$$

Thus, $\mathcal{A}$ is a subset of the power set of E, and each $F \in \mathcal{A}$ satisfies $G \not\subseteq \operatorname{Shade} F$.

We equip the finite set E with a total order (chosen arbitrarily, but fixed henceforth). If $F \in \mathcal{A}$, then there exists a unique **smallest** edge $e \in G \setminus \operatorname{Shade} F$ (since $F \in \mathcal{A}$ entails $G \not\subseteq \operatorname{Shade} F$ and thus $G \setminus \operatorname{Shade} F \neq \varnothing$). This unique smallest edge e will be denoted by $\varepsilon(F)$.

We note that the edge $\varepsilon(F)$ (for a set $F \in \mathcal{A}$) depends only on $\operatorname{Shade} F$, but not on F itself (because it was defined as the smallest edge $e \in G \setminus \operatorname{Shade} F$). Thus, if two sets $F_1 \in \mathcal{A}$ and $F_2 \in \mathcal{A}$ satisfy $\operatorname{Shade}(F_1) = \operatorname{Shade}(F_2)$, then

$$\varepsilon(F_1) = \varepsilon(F_2). \tag{6}$$

We also notice the following simple fact: If F and F' are two subsets of E such that $F \in \mathcal{A}$ and $\operatorname{Shade}(F') = \operatorname{Shade} F$, then

$$F' \in \mathcal{A}. \tag{7}$$

(Indeed, $F \in \mathcal{A}$ means that $G \not\subseteq \operatorname{Shade} F$; but because of $\operatorname{Shade}(F') = \operatorname{Shade} F$, this entails $G \not\subseteq \operatorname{Shade}(F')$ as well, and therefore $F' \in \mathcal{A}$.)

We now define two subsets $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$ by

$$\mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} \qquad \text{and} \qquad \mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}.$$

Next, we claim the following:

> *Claim 1:* Let $F \in \mathcal{A}_+$. Set $F' = F \setminus \{\varepsilon(F)\}$. Then, $F' \in \mathcal{A}_-$ and $F' \cup \{\varepsilon(F')\} = F$ and $(-1)^{|F'|} = -(-1)^{|F|}$.

[*Proof of Claim 1:* We have $F \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$; in other words, $F \in \mathcal{A}$ and $\varepsilon(F) \in F$. From $F \in \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \operatorname{Shade} P\}$, we obtain $F \subseteq E$ and $G \not\subseteq \operatorname{Shade} F$.

The definition of F' yields $F' \subseteq F \subseteq E$.

Recall that $\varepsilon(F)$ is the smallest edge $e \in G \setminus \operatorname{Shade} F$ (by the definition of $\varepsilon(F)$). Hence, $\varepsilon(F) \in G \setminus \operatorname{Shade} F$. In other words, $\varepsilon(F) \in G$ and $\varepsilon(F) \notin \operatorname{Shade} F$. Thus, $\varepsilon(F) \in G \subseteq E$ and $\varepsilon(F) \notin \operatorname{Shade} F$. Therefore, (4) (applied to $u = \varepsilon(F)$) yields $\operatorname{Shade}(F \setminus \{\varepsilon(F)\}) = \operatorname{Shade} F$. This can be rewritten as $\operatorname{Shade}(F') = \operatorname{Shade} F$ (since $F' = F \setminus \{\varepsilon(F)\}$). Hence, (7) yields $F' \in \mathcal{A}$. In light of the preceding two sentences, (6) (applied to F' and F instead of F_1 and F_2) yields $\varepsilon(F') = \varepsilon(F)$. However, $\varepsilon(F) \notin F'$ (since $F' = F \setminus \{\varepsilon(F)\}$). In other words, $\varepsilon(F') \notin F'$ (since $\varepsilon(F') = \varepsilon(F)$). Hence, $F' \in \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}$ (since $F' \in \mathcal{A}$). In other words, $F' \in \mathcal{A}_-$ (since $\mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}$).

Moreover, from $\varepsilon(F') = \varepsilon(F)$, we obtain $F' \cup \{\varepsilon(F')\} = F' \cup \{\varepsilon(F)\} = F$ (since $F' = F \setminus \{\varepsilon(F)\}$ and $\varepsilon(F) \in F$).

Finally, the set $F' = F \setminus \{\varepsilon(F)\}$ has exactly one less element than the set F (since $\varepsilon(F) \in F$). That is, $|F'| = |F| - 1$. Hence, $(-1)^{|F'|} = -(-1)^{|F|}$. This completes the proof of Claim 1.]

> *Claim 2:* Let $F \in \mathcal{A}_-$. Set $F' = F \cup \{\varepsilon(F)\}$. Then, $F' \in \mathcal{A}_+$ and $F' \setminus \{\varepsilon(F')\} = F$.

[*Proof of Claim 2:* We have $F \in \mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}$; in other words, $F \in \mathcal{A}$ and $\varepsilon(F) \notin F$. From $F \in \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \operatorname{Shade} P\}$, we obtain $F \subseteq E$ and $G \not\subseteq \operatorname{Shade} F$.

As in the proof of Claim 1, we can see that $\varepsilon(F) \in G$ and $\varepsilon(F) \notin \operatorname{Shade} F$. Thus, $\varepsilon(F) \in G \subseteq E$ and $\varepsilon(F) \notin \operatorname{Shade} F$. Now, $F' = F \cup \{\varepsilon(F)\} \subseteq E$ (since $F \subseteq E$ and $\varepsilon(F) \in E$). Furthermore, (3) (applied to $u = \varepsilon(F)$) yields $\operatorname{Shade}(F \cup \{\varepsilon(F)\}) = \operatorname{Shade} F$ (since $\varepsilon(F) \notin \operatorname{Shade} F$). This can be rewritten as $\operatorname{Shade}(F') = \operatorname{Shade} F$ (since $F' = F \cup \{\varepsilon(F)\}$). Hence, (7) yields $F' \in \mathcal{A}$. In light of the preceding two sentences, (6) (applied to F' and F instead of F_1 and F_2) yields $\varepsilon(F') = \varepsilon(F) \in \{\varepsilon(F)\} \subseteq F \cup \{\varepsilon(F)\} = F'$. Thus, $F' \in \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$ (since $F' \in \mathcal{A}$). In other words, $F' \in \mathcal{A}_+$ (since $\mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$).

Moreover, from $\varepsilon(F') = \varepsilon(F)$, we obtain $F' \setminus \{\varepsilon(F')\} = F' \setminus \{\varepsilon(F)\} = F$ (since $F' = F \cup \{\varepsilon(F)\}$ and $\varepsilon(F) \notin F$). This completes the proof of Claim 2.]

Each $F \in \mathcal{A}_+$ satisfies $F \setminus \{\varepsilon(F)\} \in \mathcal{A}_-$ (by Claim 1, applied to $F' = F \setminus \{\varepsilon(F)\}$). Thus, we can define a map

$$\begin{aligned} \Phi : \mathcal{A}_+ &\to \mathcal{A}_-, \\ F &\mapsto F \setminus \{\varepsilon(F)\}. \end{aligned}$$

Each $F \in \mathcal{A}_-$ satisfies $F \cup \{\varepsilon(F)\} \in \mathcal{A}_+$ (by Claim 2, applied to $F' = F \cup \{\varepsilon(F)\}$). Thus, we can define a map

$$\begin{aligned} \Psi : \mathcal{A}_- &\to \mathcal{A}_+, \\ F &\mapsto F \cup \{\varepsilon(F)\}. \end{aligned}$$

We have $\Phi \circ \Psi = \operatorname{id}$ (this follows from the "$F' \setminus \{\varepsilon(F')\} = F$" part of Claim 2) and $\Psi \circ \Phi = \operatorname{id}$ (this follows from the "$F' \cup \{\varepsilon(F')\} = F$" part of Claim 1). Thus, the maps Φ and Ψ are mutually inverse. Hence, the map Φ is invertible, thus a bijection.

Moreover, each $F \in \mathcal{A}_+$ satisfies

$$(-1)^{|\Phi(F)|} = -(-1)^{|F|}. \tag{8}$$

(Indeed, this is just the "$(-1)^{|F'|} = -(-1)^{|F|}$" part of Claim 1.)

Now, the summation sign "$\sum\limits_{\substack{F \subseteq E; \\ G \not\subseteq \operatorname{Shade} F}}$" is equivalent to the summation sign "$\sum\limits_{F \in \mathcal{A}}$" (since the set of all subsets F of E satisfying $G \not\subseteq \operatorname{Shade} F$ is precisely $\mathcal{A}$).

Thus,

$$\begin{aligned}
&\sum_{\substack{F\subseteq E;\\ G\not\subseteq \operatorname{Shade} F}} (-1)^{|F|} \\
&= \sum_{F\in\mathcal{A}} (-1)^{|F|} = \underbrace{\sum_{\substack{F\in\mathcal{A};\\ \varepsilon(F)\in F}}}_{\substack{=\sum_{F\in\mathcal{A}_+} \\ (\text{since } \{P\in\mathcal{A} \mid \varepsilon(P)\in P\}=\mathcal{A}_+)}} (-1)^{|F|} + \underbrace{\sum_{\substack{F\in\mathcal{A};\\ \varepsilon(F)\notin F}}}_{\substack{=\sum_{F\in\mathcal{A}_-} \\ (\text{since } \{P\in\mathcal{A} \mid \varepsilon(P)\notin P\}=\mathcal{A}_-)}} (-1)^{|F|} \\
&= \sum_{F\in\mathcal{A}_+} (-1)^{|F|} + \sum_{F\in\mathcal{A}_-} (-1)^{|F|} = \sum_{F\in\mathcal{A}_+} (-1)^{|F|} + \sum_{F\in\mathcal{A}_+} \underbrace{(-1)^{|\Phi(F)|}}_{\substack{=-(-1)^{|F|} \\ (\text{by (8)})}} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \Phi(F) \text{ for } F \text{ in the second sum,} \\ \text{since the map } \Phi : \mathcal{A}_+ \to \mathcal{A}_- \text{ is a bijection}\end{array}\right) \\
&= \sum_{F\in\mathcal{A}_+} (-1)^{|F|} + \sum_{F\in\mathcal{A}_+} \left(-(-1)^{|F|}\right) = \sum_{F\in\mathcal{A}_+} (-1)^{|F|} - \sum_{F\in\mathcal{A}_+} (-1)^{|F|} = 0.
\end{aligned}$$

This proves Theorem 2.6. □

In order to derive Theorem 2.5 from Theorem 2.6, we need the following innocent lemma – which is one of the simplest facts in enumerative combinatorics:

Lemma 2.7. Let U be a finite set with $U \neq \varnothing$. Then,

$$\sum_{F\subseteq U} (-1)^{|F|} = 0.$$

Lemma 2.7 can easily be derived from the fact that $\sum\limits_{k=0}^{n} (-1)^k \dbinom{n}{k} = 0$ for any positive integer n (as follows readily from the binomial theorem). However, keeping true to the spirit of this paper, let us give a bijective proof for it:

Proof of Lemma 2.7. This is a standard argument that underlies many combinatorial proofs of alternating sum identities (see, for example, [Sagan20, proof of (2.4)] or [BenQui08, proof of (1)]). For the sake of completeness, let us nevertheless recall it.

We have $U \neq \varnothing$; hence, there exists some $u \in U$. Consider this u.

It is easy to see that the maps

$$\begin{aligned}\Phi : \{F \subseteq U \mid u \in F\} &\to \{F \subseteq U \mid u \notin F\}, \\ F &\mapsto F \setminus \{u\}\end{aligned}$$

and

$$\Psi : \{F \subseteq U \mid u \notin F\} \to \{F \subseteq U \mid u \in F\},$$
$$F \mapsto F \cup \{u\}$$

are well-defined.

The maps Φ and Ψ we just defined are clearly mutually inverse. Thus, they are invertible, i.e., are bijections. Hence, in particular, Φ is a bijection. Thus, we can substitute $\Phi(F)$ for F in the sum $\sum_{\substack{F\subseteq U;\\ u\notin F}} (-1)^{|F|}$. We thus obtain

$$\sum_{\substack{F\subseteq U;\\ u\notin F}} (-1)^{|F|} = \sum_{\substack{F\subseteq U;\\ u\in F}} \underbrace{(-1)^{|\Phi(F)|}}_{\substack{=(-1)^{|F\setminus\{u\}|} \\ \text{(since } \Phi(F)=F\setminus\{u\} \\ \text{(by the definition of } \Phi\text{))}}} = \sum_{\substack{F\subseteq U;\\ u\in F}} \underbrace{(-1)^{|F\setminus\{u\}|}}_{\substack{=(-1)^{|F|-1} \\ \text{(since } |F\setminus\{u\}|=|F|-1 \\ \text{(because } u\in F\text{))}}}$$
$$= \sum_{\substack{F\subseteq U;\\ u\in F}} \underbrace{(-1)^{|F|-1}}_{=-(-1)^{|F|}} = -\sum_{\substack{F\subseteq U;\\ u\in F}} (-1)^{|F|}.$$

However, each $F \subseteq U$ satisfies either $u \in F$ or $u \notin F$ (but not both). Hence,

$$\sum_{F\subseteq U} (-1)^{|F|} = \sum_{\substack{F\subseteq U;\\ u\in F}} (-1)^{|F|} + \underbrace{\sum_{\substack{F\subseteq U;\\ u\notin F}} (-1)^{|F|}}_{=-\sum_{\substack{F\subseteq U;\\ u\in F}} (-1)^{|F|}} = \sum_{\substack{F\subseteq U;\\ u\in F}} (-1)^{|F|} - \sum_{\substack{F\subseteq U;\\ u\in F}} (-1)^{|F|} = 0.$$

This proves Lemma 2.7. $\square$

We can now easily derive Theorem 2.5 from Theorem 2.6:

Proof of Theorem 2.5. Each subset F of E satisfies either $G \subseteq \operatorname{Shade} F$ or $G \not\subseteq \operatorname{Shade} F$ (but not both at the same time). Hence,

$$\sum_{F\subseteq E} (-1)^{|F|} = \sum_{\substack{F\subseteq E;\\ G\subseteq \operatorname{Shade} F}} (-1)^{|F|} + \underbrace{\sum_{\substack{F\subseteq E;\\ G\not\subseteq \operatorname{Shade} F}} (-1)^{|F|}}_{\substack{=0 \\ \text{(by Theorem 2.6)}}} = \sum_{\substack{F\subseteq E;\\ G\subseteq \operatorname{Shade} F}} (-1)^{|F|}.$$

Therefore,

$$\sum_{\substack{F\subseteq E;\\ G\subseteq \operatorname{Shade} F}} (-1)^{|F|} = \sum_{F\subseteq E} (-1)^{|F|} = 0$$

(by Lemma 2.7, applied to $U = E$). This proves Theorem 2.5. $\square$

2.3. Proving Theorem 1.2

Theorem 1.2 is now a simple particular case of Theorem 2.5:

Proof of Theorem 1.2. Let G be the set E. Thus, $G = E$. Hence, for each subset F of E, we have the following chain of logical equivalences:

$$\begin{aligned}
(G \subseteq \operatorname{Shade} F) &\iff (E \subseteq \operatorname{Shade} F) \\
&\iff (\text{each } u \in E \text{ satisfies } u \in \operatorname{Shade} F) \\
&\iff (\text{each } u \in E \text{ has the property that } F \text{ infects } u) \\
&\qquad\qquad (\text{since } \operatorname{Shade} F = \{e \in E \mid F \text{ infects } e\}) \\
&\iff (F \text{ infects each } u \in E) \\
&\iff (F \text{ is pandemic}) \qquad (\text{by the definition of “pandemic”}).
\end{aligned}$$

Thus, the summation sign "$\sum\limits_{\substack{F \subseteq E;\\ G \subseteq \operatorname{Shade} F}}$" can be rewritten as "$\sum\limits_{\substack{F \subseteq E \text{ is}\\ \text{pandemic}}}$". Hence,

$$\sum_{\substack{F \subseteq E;\\ G \subseteq \operatorname{Shade} F}} (-1)^{|F|} = \sum_{\substack{F \subseteq E \text{ is}\\ \text{pandemic}}} (-1)^{|F|}.$$

But the left hand side of this equality is 0 (by Theorem 2.5). Hence, its right hand side is 0 as well. This proves Theorem 1.2. □

3. Vertex infection and other variants

In our study of graphs so far, we have barely ever mentioned vertices (even though they are, of course, implicit in the notion of a path). It may appear somewhat strange to talk about a subset infecting an edge, when the infection is spread from vertex to vertex. One might thus wonder if there is also a vertex counterpart of Theorem 1.2. So let us define analogues of our notions for vertices:

If $F \subseteq V$, then an *F-vertex-path* shall mean a path of Γ such that all vertices of the path except (possibly) for its two endpoints belong to F. (Thus, if a path has only one edge or none, then it automatically is an F-vertex-path.)

If $w \in V \setminus \{v\}$ is any vertex and $F \subseteq V \setminus \{v\}$ is any subset, then we say that *F vertex-infects w* if there exists an F-vertex-path from v to w. (This is always true when w is v or a neighbor of v.)

A subset $F \subseteq V \setminus \{v\}$ is said to be *vertex-pandemic* if it vertex-infects each vertex $w \in V \setminus \{v\}$.

Example 3.1. Let Γ be as in Example 1.3. Then, the path $v \xrightarrow{1} p \xrightarrow{2} q$ is an F-vertex-path for any subset $F \subseteq V$ that satisfies $p \in F$. The subset $\{p\}$

of $V \setminus \{v\}$ vertex-infects each vertex (for example, $v \xrightarrow{1} p \xrightarrow{2} q$ is a $\{p\}$-vertex-path from v to q, and $v \xrightarrow{4} w$ is a $\{p\}$-vertex-path from v to w), and thus is vertex-pandemic. The vertex-pandemic subsets of $V \setminus \{v\}$ are the sets

$$\{p\}, \quad \{w\}, \quad \{p,q\}, \quad \{p,w\}, \quad \{q,w\}, \quad \{p,q,w\}.$$

We now have the following analogue of Theorem 1.2:

Theorem 3.2. Assume that $V \setminus \{v\} \neq \varnothing$. Then,

$$\sum_{\substack{F \subseteq V \setminus \{v\} \text{ is} \\ \text{vertex-pandemic}}} (-1)^{|F|} = 0.$$

Proof of Theorem 3.2. With just a few easy modifications, our above proof of Theorem 1.2 can be repurposed as a proof of Theorem 3.2. Namely:

- We need to replace "edge" by "vertex" throughout the argument (including Definition 2.1, Lemma 2.3, Lemma 2.4, Theorem 2.5 and Theorem 2.6), as well as replace E by $V \setminus \{v\}$.
- The words "F-path", "infects" and "pandemic" have to be replaced by "F-vertex-path", "vertex-infects" and "vertex-pandemic", respectively.
- In the proofs of Lemma 2.3 and Lemma 2.4, the words "an endpoint of" (as well as "any endpoint of" and "some endpoint of") need to be removed (since the notion of "vertex-infects" is defined not in terms of paths to an endpoint of a given edge, but in terms of paths to a given vertex).
- In the proof of Lemma 2.4, specifically in the proof of (3), the path π is now cut not by removing the edge u, but by splitting the path π at the vertex u.

The reader may check that these changes result in a valid proof of Theorem 3.2. □

Another variant of Theorem 1.2 (and Theorem 2.5 and Theorem 2.6) is obtained by replacing the undirected graph Γ with a directed graph (while, of course, replacing paths by directed paths). More generally, we can replace Γ by a "hybrid" graph with some directed and some undirected edges.[2] No changes are required to the above proofs. Yet another variation can be obtained by replacing "endpoint" by "source" (for directed edges). We cannot, however, replace "endpoint" by "target".

[2]We understand that a directed edge still has two endpoints: its source and its target.

4. An abstract perspective

Seeing how little graph theory we have used in proving Theorem 1.2, and how easily the same argument adapted to Theorem 3.2, we get the impression that there might be some general theory lurking behind it. What follows is an attempt at building this theory.

4.1. Shade maps

Let $\mathcal{P}(E)$ denote the power set of E. In Definition 2.1, we have encoded the "infects" relation as a map $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ defined by

$$\operatorname{Shade} F = \{e \in E \mid F \text{ infects } e\}.$$

As we recall, Theorem 2.5 (a generalization of Theorem 1.2) states that

$$\sum_{\substack{F \subseteq E;\\ G \subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0 \tag{9}$$

for any $G \subseteq E$, under the assumption that $E \neq \varnothing$.

To generalize this, we forget about the graph Γ and the map Shade, and instead start with an **arbitrary** finite set E. (This set E corresponds to the set E in Theorem 1.2 and to the set $V \setminus \{v\}$ in Theorem 3.2.) Let $\mathcal{P}(E)$ be the power set of E. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be an arbitrary map (meant to generalize the map Shade from the previous paragraph). We may now ask:

Question 4.1. What (combinatorial) properties must Shade satisfy in order for (9) to hold for any $G \subseteq E$ under the assumption that $E \neq \varnothing$?

A partial answer to this question can be given by analyzing our above proof of Theorem 2.5 and extracting what was used:

Definition 4.2. Let E be a set. A *shade map* on E shall mean a map $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ that satisfies the following two axioms:

Axiom 1: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$.

Axiom 2: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$.

Theorem 4.3. Let E be a finite set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be a shade map on E.

Assume that $E \neq \varnothing$. Let G be any subset of E. Then,

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0.$$

Proof sketch. Again, the proof is analogous to our above proof of Theorem 2.5. (This time, in the proof of Lemma 2.4, the equalities (3) and (4) follow directly from Axiom 1 and Axiom 2, respectively.) □

How do shade maps relate to known concepts in the combinatorics of set families (such as topologies, clutters, matroids, or submodular functions)? Are they just one of these known concepts in disguise? We shall answer two versions of this question in the following subsections. Specifically:

- In Subsection 4.3, we will show that inclusion-reversing shade maps on E (i.e., shade maps Shade that satisfy $\operatorname{Shade} B \subseteq \operatorname{Shade} A$ whenever $A \subseteq B$) are in bijection with *antimatroidal quasi-closure operators* (a slight variant of antimatroids) on E.
- In Subsection 4.4, we will show that arbitrary shade maps are in bijection with *Boolean interval partitions* of $\mathcal{P}(E)$ (that is, set partitions of $\mathcal{P}(E)$ into intervals of the Boolean lattice $\mathcal{P}(E)$).

Before we come to these characterizations, we shall however make a few elementary remarks on shade maps.

First, we observe that Axioms 1 and 2 in Definition 4.2 can be weakened to the following statements:

Axiom 1′: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \cup \{u\}) \subseteq \operatorname{Shade} F$.

Axiom 2′: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \setminus \{u\}) \subseteq \operatorname{Shade} F$.

Axiom 1′ is weaker than Axiom 1, and likewise Axiom 2′ is weaker than Axiom 2. However, Axioms 1′ and 2′ combined are equivalent to Axioms 1 and 2 combined:

Proposition 4.4. Let E be a set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be any map. Then, Shade is a shade map on E if and only if Shade satisfies the two Axioms 1′ and 2′ stated above.

Proof. Easy exercise. □

Here is one further restatement of both Axioms 1′ and 2′: For any $F \subseteq E$ and any two elements $u, v \in E \setminus \operatorname{Shade} F$, we have $v \notin \operatorname{Shade}(F \cup \{u\})$ (Axiom 1′) and $v \notin \operatorname{Shade}(F \setminus \{u\})$ (Axiom 2′).

Axioms 1 and 2 can also be combined into one common axiom:

> *Axiom 3:* If $F \in \mathcal{P}(E)$ and $u \in E \setminus F$, then we have $\operatorname{Shade} F = \operatorname{Shade}(F \cup \{u\})$ or $u \in (\operatorname{Shade} F) \cap \operatorname{Shade}(F \cup \{u\})$.

Proposition 4.5. Let E be a set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be any map. Then, Shade is a shade map on E if and only if Shade satisfies Axiom 3.

Proof. Easy exercise. □

We will soon see some examples. First, let us introduce two more basic concepts that will help clarify these examples:

Definition 4.6. Let E be a set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be any map (not necessarily a shade map).

(a) We say that Shade is *inclusion-preserving* if it satisfies the following property: If A and B are two subsets of E such that $A \subseteq B$, then $\operatorname{Shade} A \subseteq \operatorname{Shade} B$.

(b) We say that Shade is *inclusion-reversing* if it satisfies the following property: If A and B are two subsets of E such that $A \subseteq B$, then $\operatorname{Shade} B \subseteq \operatorname{Shade} A$.

For instance, the map Shade from Definition 2.1 is inclusion-preserving (because of Lemma 2.3) and is a shade map (by Lemma 2.4). The same holds for the analogue of the map Shade that uses vertex-infection instead of infection. This does not mean that any shade map is inclusion-preserving. Indeed, we shall soon see some inclusion-reversing shade maps, and it is not hard to construct shade maps that are neither inclusion-preserving nor inclusion-reversing.

Let us observe that there is a simple bijection between inclusion-preserving and inclusion-reversing maps, and this bijection preserves shadeness:

Proposition 4.7. Let E be a set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be any map (not necessarily a shade map). Let $\operatorname{Shade}' : \mathcal{P}(E) \to \mathcal{P}(E)$ be the map that sends each $F \in \mathcal{P}(E)$ to $\operatorname{Shade}(E \setminus F) \in \mathcal{P}(E)$. Then:

(a) The map Shade is inclusion-preserving if and only if the map Shade' is inclusion-reversing.

(b) The map Shade is a shade map if and only if the map Shade' is a shade map.

Proof. This is easy and left to the reader. (For part **(b)**, observe that Axioms 1 and 2 for the map Shade translate into Axioms 2 and 1 for the map Shade'.) □

Definition 4.8. Let E, Shade and Shade$'$ be as in Proposition 4.7. We then say that the map Shade$'$ is *dual* to Shade.

4.2. Some examples of shade maps

As we already mentioned, Lemma 2.4 and its analogue for vertex-infection provide two examples of inclusion-preserving shade maps Shade. An example of an inclusion-reversing shade map comes from the theory of posets:

Example 4.9. Let E be a poset. For any $F \subseteq E$, we define

$$F_{\downarrow} = \{e \in E \mid \text{there exists an } f \in F \text{ with } e < f\}$$

and

$$\operatorname{Shade} F = E \setminus F_{\downarrow}.$$

Then, this map Shade : $\mathcal{P}(E) \to \mathcal{P}(E)$ is an inclusion-reversing shade map.

For the straightforward proof of Example 4.9, see the detailed version [Grinbe20] of this paper.

Another example of a shade map comes from discrete geometry:

Example 4.10. Let A be an affine space over $\mathbb{R}$. If S is a finite subset of A, then a *nontrivial convex combination* of S will mean a point of the form $\sum\limits_{s \in S} \lambda_s s \in A$, where the coefficients λ_s are nonnegative reals smaller than 1 and satisfying $\sum\limits_{s \in S} \lambda_s = 1$.

Fix a finite subset E of A. For any $F \subseteq E$, we define

$$\operatorname{Shade} F = \{e \in E \mid e \text{ is } \textbf{not} \text{ a nontrivial convex combination of } F\}.$$

Then, this map Shade : $\mathcal{P}(E) \to \mathcal{P}(E)$ is an inclusion-reversing shade map.

For a (not very difficult) proof of Example 4.10, see the detailed version [Grinbe20] of this paper.

As a contrast to Example 4.10, let us mention a not-quite-example (satisfying only one of the two axioms in Theorem 4.3):

Example 4.11. Let V be a vector space over $\mathbb{R}$. If S is a finite subset of V, then a *nontrivial conic combination* of S will mean a vector of the form $\sum\limits_{s \in S} \lambda_s s \in V$, where the coefficients λ_s are nonnegative reals with the property that at least two elements $s \in S$ satisfy $\lambda_s > 0$.

Fix a finite subset E of V. For any $F \subseteq E$, we define

$$\operatorname{Shade} F = \{e \in E \mid e \text{ is } \textbf{not} \text{ a nontrivial conic combination of } F\}.$$

It can be shown that this map Shade : $\mathcal{P}(E) \to \mathcal{P}(E)$ satisfies Axiom 1 in Definition 4.2. In general, it does not satisfy Axiom 2. Thus, it is not a shade map in general.

4.3. Antimatroids and inclusion-reversing shade maps

Examples 4.9 and 4.10 are instances of a general class of examples: shade maps coming from *antimatroids*. Not unlike matroids, antimatroids are a combinatorial concept with many equivalent avatars (see, e.g., [KoLoSc91, Chapter III]). Here we shall view them through one of these avatars: that of *antimatroidal quasi-closure operators* (roughly equivalent to *convex geometries*). We begin by defining the notions we need:

Definition 4.12. Let E be any set.

(a) A *quasi-closure operator* on E means a map $\tau : \mathcal{P}(E) \to \mathcal{P}(E)$ with the following properties:

1. We have $A \subseteq \tau(A)$ for any $A \subseteq E$.
2. If A and B are two subsets of E satisfying $A \subseteq B$, then $\tau(A) \subseteq \tau(B)$.
3. We have $\tau(\tau(A)) = \tau(A)$ for any $A \subseteq E$.

(b) A quasi-closure operator τ on E is said to be *antimatroidal* if it has the following additional property:

4. If X is a subset of E, and if y and z are two distinct elements of $E \setminus \tau(X)$ satisfying $z \in \tau(X \cup \{y\})$, then $y \notin \tau(X \cup \{z\})$.

(c) A *closure operator* on E means a quasi-closure operator τ on E that satisfies $\tau(\varnothing) = \varnothing$.

(d) If τ is an antimatroidal closure operator on E, then the pair (E, τ) is called a *convex geometry*.

Here are some examples of antimatroidal quasi-closure operators:

Example 4.13. Let E be a poset. For any $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid \text{there exists an } f \in F \text{ with } e \leqslant f\}.$$

Then, τ is an antimatroidal closure operator on E. (This example is the "downset alignment" from [EdeJam85, §3, Example II], and is equivalent to the "poset antimatroid" from [KoLoSc91, §III.2.3].)

Example 4.14. Let E be a poset. For any $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid \text{there exist } f \in F \text{ and } g \in F \text{ with } g \leqslant e \leqslant f\}.$$

Then, τ is an antimatroidal closure operator on E. (This example is the "order convex alignment" from [EdeJam85, §3, Example II], and is the "double shelling of a poset" example from [KoLoSc91, §III.2.4].)

Example 4.15. Let A be an affine space over $\mathbb{R}$. If S is a finite subset of A, then a *convex combination* of S will mean a point of the form $\sum\limits_{s \in S} \lambda_s s \in A$, where the coefficients λ_s are nonnegative reals satisfying $\sum\limits_{s \in S} \lambda_s = 1$.

Fix a finite subset E of A. For any $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid e \text{ is a convex combination of } F\}.$$

Then, τ is an antimatroidal closure operator on E. (This example is [EdeJam85, §3, Example I]; it gave the name "convex geometry" to the notion defined in Definition 4.12 **(d)**.)

Example 4.16. Let Γ be any graph with edge set E. Fix a vertex v of Γ. We say that a subset $F \subseteq E$ *blocks* an edge $e \in E$ if each path of Γ that contains v and e must contain at least one edge of F. (In particular, this is automatically the case when $e \in F$.) For each $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid F \text{ blocks } e\}.$$

Then, τ is an antimatroidal quasi-closure operator on E. (This example is the "line-search antimatroid" from [KoLoSc91, §III.2.11].)

If Γ is connected, then τ is actually a closure operator.

Further examples of antimatroidal closure operators can be found in [KoLoSc91, §III.2] and [EdeJam85, §3].

We shall be dealing with quasi-closure operators rather than closure operators most of the time. However, since the latter concept is somewhat more widespread, let us comment on the connection between the two. Roughly speaking, the relation between quasi-closure and closure operators is comparable to the relation between semigroups and monoids, or between nonunital rings and unital rings, or (perhaps the best analogue) between simplicial complexes in general and simplicial complexes without ghost vertices (i.e., simplicial complexes for which every element of the ground set is a dimension-0 face). More concretely, specifying a quasi-closure operator on a set E is tantamount to specifying a subset of E and a closure operator on this subset. To wit:

Proposition 4.17. Let E be a set. Let L be a subset of E.

(a) Then, there is a bijection from

$$\{\text{quasi-closure operators } \tau \text{ on } E \text{ satisfying } \tau(\varnothing) = L\}$$

to

$$\{\text{closure operators } \sigma \text{ on } E \setminus L\}$$

that is defined as follows: It sends each quasi-closure operator τ to the closure operator σ that sends each $F \subseteq E \setminus L$ to $\tau(F) \setminus L$.

(b) This bijection restricts to a bijection from

$$\{\text{antimatroidal quasi-closure operators } \tau \text{ on } E \text{ satisfying } \tau(\varnothing) = L\}$$

to

$$\{\text{antimatroidal closure operators } \sigma \text{ on } E \setminus L\}.$$

Proposition 4.17 will not be important to what follows, so we omit the (rather straightforward) proof.

Now, we claim the following:

Theorem 4.18. Let E be a set. Let $\tau : \mathcal{P}(E) \to \mathcal{P}(E)$ be an antimatroidal quasi-closure operator on E. For any $F \subseteq E$, we define

$$\operatorname{Shade} F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}. \tag{10}$$

Then, this map $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ is an inclusion-reversing shade map.

Theorem 4.18 generalizes Examples 4.9 and 4.10. Indeed, applying Theorem 4.18 to the setting of Example 4.13, we can easily recover the claim of Example 4.9. Likewise, applying Theorem 4.18 to the setting of Example 4.15, we can recover the claim of Example 4.10. Less directly, Lemma 2.4 and its vertex-infection analogue are particular cases of Theorem 4.18 as well (even though they involve shade maps that are inclusion-preserving rather than inclusion-reversing). Indeed, if we apply Theorem 4.18 to the setting of Example 4.16, then we obtain the claim of Lemma 2.4 with $\operatorname{Shade} F$ replaced by $\operatorname{Shade}(E \setminus F)$; this is easily seen to be equivalent to Lemma 2.4 (by the duality stated in Proposition 4.7).

We shall soon prove Theorem 4.18; first we set up two lemmas:

Lemma 4.19. Let E be a set. Let τ be a quasi-closure operator on E. Let X be a subset of E, and let $z \in \tau(X)$. Then, $\tau(X \cup \{z\}) = \tau(X)$.

Proof of Lemma 4.19. Well-known and thus omitted (see [Grinbe20]). □

Lemma 4.20. Let E be a set. Let τ be an antimatroidal quasi-closure operator on E. Let X be a subset of E, and let y and z be two distinct elements of E satisfying $z \in \tau(X \cup \{y\})$ and $y \in \tau(X \cup \{z\})$. Then, $y \in \tau(X)$.

Proof of Lemma 4.20. Assume the contrary. Thus, $y \notin \tau(X)$, so that $y \in E \setminus \tau(X)$.

If we had $z \in \tau(X)$, then Lemma 4.19 would yield $\tau(X \cup \{z\}) = \tau(X)$ and therefore $y \in \tau(X \cup \{z\}) = \tau(X)$, which would contradict $y \notin \tau(X)$. Hence, we cannot have $z \in \tau(X)$. Thus, we have $z \notin \tau(X)$. Therefore, $z \in E \setminus \tau(X)$.

Now, we know that y and z are two distinct elements of $E \setminus \tau(X)$ satisfying $z \in \tau(X \cup \{y\})$. Hence, Property 4 in Definition 4.12 **(b)** shows that $y \notin \tau(X \cup \{z\})$. This contradicts $y \in \tau(X \cup \{z\})$. This contradiction shows that our assumption was false. Thus, Lemma 4.20 is proven. $\square$

Note that Lemma 4.20 has a converse: If τ is a quasi-closure operator on E satisfying the claim of Lemma 4.20, then τ is antimatroidal. This is easy to see but will not be used in what follows.

Proof of Theorem 4.18. We shall prove the following three statements:

> *Statement 0:* If A and B are two subsets of E such that $A \subseteq B$, then $\operatorname{Shade} B \subseteq \operatorname{Shade} A$.
>
> *Statement 1:* If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$.
>
> *Statement 2:* If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$.

[*Proof of Statement 0:* Let A and B be two subsets of E such that $A \subseteq B$. We must prove that $\operatorname{Shade} B \subseteq \operatorname{Shade} A$.

Let $u \in \operatorname{Shade} B$. Thus,

$$u \in \operatorname{Shade} B = \{e \in E \mid e \notin \tau(B \setminus \{e\})\}$$

(by the definition of $\operatorname{Shade} B$). In other words, $u \in E$ and $u \notin \tau(B \setminus \{u\})$.

However, $A \setminus \{u\} \subseteq B \setminus \{u\}$ (since $A \subseteq B$) and thus $\tau(A \setminus \{u\}) \subseteq \tau(B \setminus \{u\})$ (by Property 2 in Definition 4.12 **(a)**, applied to $A \setminus \{u\}$ and $B \setminus \{u\}$ instead of A and B). Hence, from $u \notin \tau(B \setminus \{u\})$, we obtain $u \notin \tau(A \setminus \{u\})$. Therefore, $u \in \{e \in E \mid e \notin \tau(A \setminus \{e\})\} = \operatorname{Shade} A$ (by the definition of $\operatorname{Shade} A$).

Forget that we fixed u. We thus have shown that $u \in \operatorname{Shade} A$ for each $u \in \operatorname{Shade} B$. In other words, $\operatorname{Shade} B \subseteq \operatorname{Shade} A$. This proves Statement 0.]

[*Proof of Statement 2:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$. We must prove that $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$.

We have $u \in E \setminus \operatorname{Shade} F = \{e \in E \mid e \in \tau(F \setminus \{e\})\}$ (since $\operatorname{Shade} F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}$). In other words, $u \in E$ and $u \in \tau(F \setminus \{u\})$.

We have $F \setminus \{u\} \subseteq F$ and thus $\operatorname{Shade} F \subseteq \operatorname{Shade}(F \setminus \{u\})$ (by Statement 0, applied to $A = F \setminus \{u\}$ and $B = F$).

Now, let $v \in \operatorname{Shade}(F \setminus \{u\}) \setminus \operatorname{Shade} F$. We shall derive a contradiction.

We have

$$v \in \underbrace{\operatorname{Shade}(F \setminus \{u\})}_{\subseteq E} \setminus \operatorname{Shade} F \subseteq E \setminus \operatorname{Shade} F = \{e \in E \mid e \in \tau(F \setminus \{e\})\}.$$

In other words, $v \in E$ and $v \in \tau(F \setminus \{v\})$.

On the other hand,

$$\begin{aligned} v &\in \operatorname{Shade}(F \setminus \{u\}) \setminus \operatorname{Shade} F \\ &\subseteq \operatorname{Shade}(F \setminus \{u\}) = \{e \in E \mid e \notin \tau((F \setminus \{u\}) \setminus \{e\})\} \end{aligned}$$

(by the definition of $\operatorname{Shade}(F \setminus \{u\})$). In other words,

$$v \in E \qquad \text{and} \qquad v \notin \tau((F \setminus \{u\}) \setminus \{v\}).$$

Let $X = (F \setminus \{u\}) \setminus \{v\}$. Then, $X \cup \{v\} \supseteq F \setminus \{u\}$, so that $F \setminus \{u\} \subseteq X \cup \{v\}$ and therefore $\tau(F \setminus \{u\}) \subseteq \tau(X \cup \{v\})$ (by Property 2 in Definition 4.12 **(a)**, applied to $A = F \setminus \{u\}$ and $B = X \cup \{v\}$). Hence, $u \in \tau(F \setminus \{u\}) \subseteq \tau(X \cup \{v\})$.

Also, from $X = (F \setminus \{u\}) \setminus \{v\} = (F \setminus \{v\}) \setminus \{u\}$, we obtain $X \cup \{u\} \supseteq F \setminus \{v\}$, so that $F \setminus \{v\} \subseteq X \cup \{u\}$ and therefore $\tau(F \setminus \{v\}) \subseteq \tau(X \cup \{u\})$ (by Property 2 in Definition 4.12 **(a)**, applied to $A = F \setminus \{v\}$ and $B = X \cup \{u\}$). Hence, $v \in \tau(F \setminus \{v\}) \subseteq \tau(X \cup \{u\})$.

If we had $v = u$, then we would have $(F \setminus \{u\}) \setminus \{v\} = (F \setminus \{u\}) \setminus \{u\} = F \setminus \{u\} = F \setminus \{v\}$ (since $u = v$) and therefore $v \notin \tau\left(\underbrace{(F \setminus \{u\}) \setminus \{v\}}_{=F \setminus \{v\}}\right) = \tau(F \setminus \{v\})$, which would contradict $v \in \tau(F \setminus \{v\})$. Thus, we cannot have $v = u$. Hence, v and u are distinct.

Thus, Lemma 4.20 (applied to $y = v$ and $z = u$) yields $v \in \tau(X)$ (since $v \in \tau(X \cup \{u\})$ and $u \in \tau(X \cup \{v\})$). In other words, $v \in \tau((F \setminus \{u\}) \setminus \{v\})$ (since $X = (F \setminus \{u\}) \setminus \{v\}$). But this contradicts $v \notin \tau((F \setminus \{u\}) \setminus \{v\})$.

Forget that we fixed v. We thus have found a contradiction for each $v \in \operatorname{Shade}(F \setminus \{u\}) \setminus \operatorname{Shade} F$. Hence, there exists no such v. In other words, the set $\operatorname{Shade}(F \setminus \{u\}) \setminus \operatorname{Shade} F$ is empty. Hence, $\operatorname{Shade}(F \setminus \{u\}) \subseteq \operatorname{Shade} F$. Combining this with $\operatorname{Shade} F \subseteq \operatorname{Shade}(F \setminus \{u\})$, we obtain $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$. This proves Statement 2.]

[*Proof of Statement 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$. We must prove that $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$. If $u \in F$, then this is obvious (since $F \cup \{u\} = F$ in this case). Thus, we WLOG assume that $u \notin F$. Hence, $(F \cup \{u\}) \setminus \{u\} = F$.

We have $F \subseteq F \cup \{u\}$ and thus $\operatorname{Shade}(F \cup \{u\}) \subseteq \operatorname{Shade} F$ (by Statement 0, applied to $A = F$ and $B = F \cup \{u\}$). Hence, $E \setminus \underbrace{\operatorname{Shade}(F \cup \{u\})}_{\subseteq \operatorname{Shade} F} \supseteq E \setminus \operatorname{Shade} F$, so that $E \setminus \operatorname{Shade} F \subseteq E \setminus \operatorname{Shade}(F \cup \{u\})$.

Now, $u \in E \setminus \operatorname{Shade} F \subseteq E \setminus \operatorname{Shade}(F \cup \{u\})$. Hence, Statement 2 (applied to $F \cup \{u\}$ instead of F) yields $\operatorname{Shade}((F \cup \{u\}) \setminus \{u\}) = \operatorname{Shade}(F \cup \{u\})$. In view of $(F \cup \{u\}) \setminus \{u\} = F$, this rewrites as $\operatorname{Shade} F = \operatorname{Shade}(F \cup \{u\})$. Hence, $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$. This proves Statement 1.]

Now, we have proved Statements 1 and 2. Thus, the map $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ satisfies the two axioms in Definition 4.2. In other words, this map is a shade map. Moreover, this map is inclusion-reversing (by Statement 0). Thus, Theorem 4.18 is proved. ☐

We note that the quasi-closure operator τ in Theorem 4.18 can be reconstructed from the map Shade. This does not even require τ to be antimatroidal; the following holds for any quasi-closure operator:

Proposition 4.21. Let E be a set. Let $\tau : \mathcal{P}(E) \to \mathcal{P}(E)$ be a quasi-closure operator on E. For any $F \subseteq E$, we define

$$\operatorname{Shade} F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}.$$

Then, each $F \subseteq E$ satisfies

$$\tau(F) = F \cup (E \setminus \operatorname{Shade} F). \tag{11}$$

Proof. Exercise (see [Grinbe20] for details). ☐

It turns out that if one applies the formula (11) to an inclusion-reversing shade map Shade, then the resulting map τ is an antimatroidal quasi-closure operator, at least when E is finite. In fact, we have the following:

Proposition 4.22. Let E be a finite set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be an inclusion-reversing shade map. Define a map $\tau : \mathcal{P}(E) \to \mathcal{P}(E)$ by setting

$$\tau(F) = F \cup (E \setminus \operatorname{Shade} F) \tag{12}$$

for each $F \subseteq E$. Then, τ is an antimatroidal quasi-closure operator on E.

The proof of this proposition rests on the following lemma:

Lemma 4.23. Let E be a set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be a shade map. Let A and B be two subsets of E such that B is finite and $B \cap \operatorname{Shade} A = \varnothing$. Then, $\operatorname{Shade}(A \cup B) = \operatorname{Shade} A$.

Proof of Lemma 4.23. Induction on $|B|$, using Axiom 1 from Definition 2.1. ☐

Proof of Proposition 4.22. Again, see [Grinbe20]. (The proof of Property 3 relies on Lemma 4.23.) □

Proposition 4.21 has a (sort of) converse:

Proposition 4.24. Let E be a set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be an inclusion-reversing shade map. For any $F \subseteq E$, we define

$$\tau(F) = F \cup (E \setminus \operatorname{Shade} F).$$

Then, each $F \subseteq E$ satisfies

$$\operatorname{Shade} F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}. \tag{13}$$

Proof. Again, we omit the proof (see [Grinbe20]). □

Combining many of the results in this section, we obtain the following description of inclusion-reversing shade maps:

Theorem 4.25. Let E be a finite set. Then, there is a bijection from the set

$$\{\text{inclusion-reversing shade maps } \operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)\}$$

to the set

$$\{\text{antimatroidal quasi-closure operators } \tau : \mathcal{P}(E) \to \mathcal{P}(E)\}.$$

It sends each map Shade to the map τ defined by (12). Its inverse map sends each map τ to the map Shade defined by (10).

Proof. This follows from Theorem 4.18, Proposition 4.22, Proposition 4.21 and Proposition 4.24. □

Theorem 4.25 classifies **inclusion-reversing** shade maps in terms of antimatroidal quasi-closure operators[3]. The latter can in turn be described in terms of antimatroidal closure operators (by Proposition 4.17), i.e., in terms of antimatroids. Thus, inclusion-reversing shade maps "boil down" to antimatroids. The same can be said of **inclusion-preserving** shade maps (because Proposition 4.7 establishes a bijection between them and the inclusion-reversing ones). In the next subsection, we shall classify **arbitrary** shade maps in terms of what we will call *Boolean interval partitions*.

[3] In the parlance of matroid theorists, it shows that inclusion-reversing shade maps are *cryptomorphic* to antimatroidal quasi-closure operators.

4.4. Boolean interval partitions and arbitrary shade maps

Let us first define Boolean interval partitions:

Definition 4.26. Let E be a set.

(a) If U and V are two subsets of E satisfying $U \subseteq V$, then $[U, V]$ shall denote the subset $\{I \in \mathcal{P}(E) \mid U \subseteq I \subseteq V\}$ of $\mathcal{P}(E)$. This is the set of all subsets of E that lie between U and V (meaning that they contain U as a subset, but in turn are contained in V as subsets).

(b) A *Boolean interval* of $\mathcal{P}(E)$ shall mean a subset of $\mathcal{P}(E)$ that has the form $[U, V]$ for two subsets U and V of E satisfying $U \subseteq V$. Note that each Boolean interval $[U, V]$ of $\mathcal{P}(E)$ is nonempty (as it contains U and V), and the two subsets U and V can easily be reconstructed from it (namely, U is the intersection of all $I \in [U, V]$, whereas V is the union of all $I \in [U, V]$).

(c) A *Boolean interval partition* of $\mathcal{P}(E)$ means a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$ whose union is $\mathcal{P}(E)$.

(d) If $\mathbf{P}$ is a Boolean interval partition of $\mathcal{P}(E)$, then the elements of $\mathbf{P}$ (that is, the Boolean intervals that belong to $\mathbf{P}$) are called the *blocks* of $\mathbf{P}$.

Example 4.27. For this example, let $E = \{1, 2, 3\}$. We shall use the shorthand $i_1 i_2 \cdots i_k$ for a subset $\{i_1, i_2, \ldots, i_k\}$ of E. (For example, 13 means the subset $\{1, 3\}$.)

(a) We have

$$[1,\ 123] = \{I \in \mathcal{P}(123) \mid 1 \subseteq I \subseteq 123\} = \{1,\ 12,\ 13,\ 123\}$$

and

$$[1,\ 13] = \{I \in \mathcal{P}(123) \mid 1 \subseteq I \subseteq 13\} = \{1,\ 13\}.$$

(b) There are $3^3 = 27$ Boolean intervals of $\mathcal{P}(E)$. (More generally, if E is an n-element set, then there are 3^n Boolean intervals of $\mathcal{P}(E)$.)

(c) Here is one of many Boolean interval partitions of $\mathcal{P}(E)$ (where E is still $\{1, 2, 3\}$):

$$\left\{ \underbrace{\{\varnothing\}}_{=[\varnothing,\ \varnothing]},\ \underbrace{\{1,\ 13\}}_{=[1,\ 13]},\ \underbrace{\{3\}}_{=[3,\ 3]},\ \underbrace{\{2,\ 12,\ 23,\ 123\}}_{=[2,\ 123]} \right\}.$$

Here is another:

$$\left\{ \underbrace{\{\varnothing,\ 1\}}_{=[\varnothing,\ 1]},\ \underbrace{\{3,\ 13\}}_{=[3,\ 13]},\ \underbrace{\{2,\ 23\}}_{=[2,\ 23]},\ \underbrace{\{12\}}_{=[12,\ 12]},\ \underbrace{\{123\}}_{=[123,\ 123]} \right\}.$$

The former has four blocks; the latter has five.

Here are two ways to think of Boolean interval partitions of $\mathcal{P}(E)$:

- The following is just a slick restatement of Definition 4.26 **(c)** using standard combinatorial lingo: A Boolean interval partition of $\mathcal{P}(E)$ is a set partition of the Boolean lattice $\mathcal{P}(E)$ into intervals.

- It is well-known that the set partitions of a given set are in a canonical bijection with the equivalence relations on this set. In light of this, the Boolean interval partitions of $\mathcal{P}(E)$ can be viewed as the equivalence relations on $\mathcal{P}(E)$ whose equivalence classes are Boolean intervals. In other words, they can be viewed as the equivalence relations $\sim$ on $\mathcal{P}(E)$ satisfying the axiom "if $U, V, I \in \mathcal{P}(E)$ satisfy $U \sim V$ and $U \cap V \subseteq I \subseteq U \cup V$, then $U \sim I \sim V$". The reader can prove this alternative characterization as an easy exercise in Boolean algebra.

Boolean interval partitions have come up in combinatorics before (e.g., [BrOlNo09], [Dawson80], [DedTit20], [GorMah97]).

We shall now construct a shade map from any Boolean interval partition:

Theorem 4.28. Let E be a set. Let $\mathbf{P}$ be a Boolean interval partition of $\mathcal{P}(E)$.

For any $F \in \mathcal{P}(E)$, let $[\alpha(F), \tau(F)]$ denote the (unique) block of $\mathbf{P}$ that contains F.

We define a map $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ by setting

$$\operatorname{Shade} F = E \setminus (\tau(F) \setminus \alpha(F)) \qquad \text{for any } F \in \mathcal{P}(E).$$

Then:

(a) The map Shade is a shade map on E.

(b) We have $\alpha(F) = F \cap \operatorname{Shade} F$ and $\tau(F) = F \cup (E \setminus \operatorname{Shade} F)$ for any $F \in \mathcal{P}(E)$.

(c) We have $\mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$.

The proof of Theorem 4.28 is rather easy. We lighten our burden somewhat with a simple lemma:

Lemma 4.29. Let X, Y, Z and E be four sets such that $X \subseteq Y \subseteq Z \subseteq E$. Then:

(a) We have $Y \cup (Z \setminus X) = Z$.

(b) We have $Y \cap (E \setminus (Z \setminus X)) = X$.

Proof of Lemma 4.29. This is straightforward set theory; it is easily checked (e.g.) using Venn diagrams. $\square$

Proof of Theorem 4.28. **(a)** We shall prove the following two statements:

> *Statement 1:* If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$.

Statement 2: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$, then $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$.

[*Proof of Statement 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$. We must prove that $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$.

The definition of $\alpha(F)$ and $\tau(F)$ reveals that $[\alpha(F), \tau(F)]$ is the (unique) block of $\mathbf{P}$ that contains F. Thus, $[\alpha(F), \tau(F)]$ is a block of $\mathbf{P}$, so that $\alpha(F)$ and $\tau(F)$ are subsets of E. Hence, $\tau(F) \setminus \alpha(F)$ is a subset of E as well.

The definition of Shade yields $\operatorname{Shade} F = E \setminus (\tau(F) \setminus \alpha(F))$. Now,

$$u \in E \setminus \underbrace{\operatorname{Shade} F}_{=E \setminus (\tau(F) \setminus \alpha(F))} = E \setminus (E \setminus (\tau(F) \setminus \alpha(F))) = \tau(F) \setminus \alpha(F)$$

(since $\tau(F) \setminus \alpha(F)$ is a subset of E). In other words, $u \in \tau(F)$ and $u \notin \alpha(F)$.

On the other hand, the Boolean interval $[\alpha(F), \tau(F)]$ contains F (since $[\alpha(F), \tau(F)]$ is the (unique) block of $\mathbf{P}$ that contains F). In other words, $\alpha(F) \subseteq F \subseteq \tau(F)$.

Now, set $F' = F \cup \{u\}$. From $F \subseteq \tau(F)$ and $u \in \tau(F)$, we thus obtain $F' \subseteq \tau(F)$. Combined with $\alpha(F) \subseteq F \subseteq F'$, this entails $F' \in [\alpha(F), \tau(F)]$. Hence, $[\alpha(F), \tau(F)]$ is a block of $\mathbf{P}$ that contains F' (since we already know that $[\alpha(F), \tau(F)]$ is a block of $\mathbf{P}$).

However, the definition of $\alpha(F')$ and $\tau(F')$ shows that $[\alpha(F'), \tau(F')]$ is the (unique) block of $\mathbf{P}$ that contains F'. Since we know that $[\alpha(F), \tau(F)]$ is a block of $\mathbf{P}$ that contains F', we therefore conclude that $[\alpha(F'), \tau(F')] = [\alpha(F), \tau(F)]$. Hence, we have

$$\alpha(F') = \alpha(F) \qquad \text{and} \qquad \tau(F') = \tau(F)$$

(because a Boolean interval $[U, V]$ uniquely determines both U and V). Now, the definition of Shade yields

$$\operatorname{Shade}(F') = E \setminus \left(\underbrace{\tau(F')}_{=\tau(F)} \setminus \underbrace{\alpha(F')}_{=\alpha(F)} \right) = E \setminus (\tau(F) \setminus \alpha(F)) = \operatorname{Shade} F.$$

In view of $F' = F \cup \{u\}$, this rewrites as $\operatorname{Shade}(F \cup \{u\}) = \operatorname{Shade} F$. This proves Statement 1.]

[*Proof of Statement 2:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \operatorname{Shade} F$. We must prove that $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$.

We proceed exactly as in our above proof of Statement 1 up until the point where we define F'. Insead of setting $F' = F \cup \{u\}$, we now set $F' = F \setminus \{u\}$. Combining $\alpha(F) \subseteq F$ with $u \notin \alpha(F)$, we obtain $\alpha(F) \subseteq F \setminus \{u\} = F'$. Combining this with $F' = F \setminus \{u\} \subseteq F \subseteq \tau(F)$, we see that $F' \in [\alpha(F), \tau(F)]$. From this, we can obtain $\operatorname{Shade}(F') = \operatorname{Shade} F$ by the same argument that we used back in the proof of Statement 1. In view of $F' = F \setminus \{u\}$, this rewrites as $\operatorname{Shade}(F \setminus \{u\}) = \operatorname{Shade} F$. This proves Statement 2.]

Now, we have proved Statements 1 and 2. Thus, the map $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ satisfies the two axioms in Definition 4.2. In other words, this map is a shade map. This proves Theorem 4.28 **(a)**.

(b) Let $F \in \mathcal{P}(E)$. We must prove that $\alpha(F) = F \cap \operatorname{Shade} F$ and $\tau(F) = F \cup (E \setminus \operatorname{Shade} F)$.

The definition of $\alpha(F)$ and $\tau(F)$ reveals that $[\alpha(F), \tau(F)]$ is the (unique) block of $\mathbf{P}$ that contains F. Hence, $[\alpha(F), \tau(F)]$ contains F. In other words, $\alpha(F) \subseteq F \subseteq \tau(F)$. Moreover, $[\alpha(F), \tau(F)]$ is a block of $\mathbf{P}$ (since $[\alpha(F), \tau(F)]$ is the (unique) block of $\mathbf{P}$ that contains F), therefore a Boolean interval of $\mathcal{P}(E)$ (since $\mathbf{P}$ is a Boolean interval partition of $\mathcal{P}(E)$). Hence, $\alpha(F)$ and $\tau(F)$ belong to $\mathcal{P}(E)$, hence are subsets of E. Hence, $\tau(F) \setminus \alpha(F)$ is a subset of E as well.

The definition of Shade yields $\operatorname{Shade} F = E \setminus (\tau(F) \setminus \alpha(F))$. Hence,

$$E \setminus \underbrace{\operatorname{Shade} F}_{=E \setminus (\tau(F) \setminus \alpha(F))} = E \setminus (E \setminus (\tau(F) \setminus \alpha(F))) = \tau(F) \setminus \alpha(F)$$

(since $\tau(F) \setminus \alpha(F)$ is a subset of E).

Now, $\alpha(F) \subseteq F \subseteq \tau(F) \subseteq E$ (since $\tau(F)$ is a subset of E). Hence, Lemma 4.29 **(b)** (applied to $X = \alpha(F)$ and $Y = F$ and $Z = \tau(F)$) yields that $F \cap (E \setminus (\tau(F) \setminus \alpha(F))) = \alpha(F)$. In view of $\operatorname{Shade} F = E \setminus (\tau(F) \setminus \alpha(F))$, this rewrites as $F \cap \operatorname{Shade} F = \alpha(F)$. In other words, $\alpha(F) = F \cap \operatorname{Shade} F$.

Furthermore, Lemma 4.29 **(a)** (applied to $X = \alpha(F)$ and $Y = F$ and $Z = \tau(F)$) yields that $F \cup (\tau(F) \setminus \alpha(F)) = \tau(F)$. In view of $E \setminus \operatorname{Shade} F = \tau(F) \setminus \alpha(F)$, this rewrites as $F \cup (E \setminus \operatorname{Shade} F) = \tau(F)$. In other words, $\tau(F) = F \cup (E \setminus \operatorname{Shade} F)$. Thus, Theorem 4.28 **(b)** is proven.

(c) Each block of $\mathbf{P}$ has the form $[\alpha(F), \tau(F)]$ for some $F \in \mathcal{P}(E)$ (since it is a Boolean interval, thus nonempty, therefore contains some $F \in \mathcal{P}(E)$; but then it must be $[\alpha(F), \tau(F)]$ for this F). Conversely, any set of the form $[\alpha(F), \tau(F)]$ is a block of $\mathbf{P}$ (by the definition of $[\alpha(F), \tau(F)]$). Combining these two facts, we conclude that the blocks of $\mathbf{P}$ are precisely the sets of the form $[\alpha(F), \tau(F)]$ with $F \in \mathcal{P}(E)$. But this is precisely the claim of Theorem 4.28 **(c)**. □

A converse to Theorem 4.28 is provided by the following theorem:

Theorem 4.30. Let E be a finite set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be a shade map on E. Define a map $\alpha : \mathcal{P}(E) \to \mathcal{P}(E)$ by setting

$$\alpha(F) = F \cap \operatorname{Shade} F \qquad \text{for any } F \in \mathcal{P}(E).$$

Define a map $\tau : \mathcal{P}(E) \to \mathcal{P}(E)$ by setting

$$\tau(F) = F \cup (E \setminus \operatorname{Shade} F) \qquad \text{for any } F \in \mathcal{P}(E).$$

Let

$$\mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}. \tag{14}$$

Then:

(a) We have $F \in [\alpha(F), \tau(F)]$ for any $F \in \mathcal{P}(E)$.

(b) If $F \in \mathcal{P}(E)$ and $G \in [\alpha(F), \tau(F)]$, then $[\alpha(F), \tau(F)] = [\alpha(G), \tau(G)]$.

(c) The set $\mathbf{P}$ is a Boolean interval partition of $\mathcal{P}(E)$.

(d) We have $\operatorname{Shade} F = E \setminus (\tau(F) \setminus \alpha(F))$ for any $F \in \mathcal{P}(E)$.

To prove this theorem, we will need the following variant of Lemma 4.23:

Lemma 4.31. Let E be a set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be a shade map. Let A and B be two subsets of E such that B is finite and $B \cap \operatorname{Shade} A = \varnothing$. Then, $\operatorname{Shade}(A \setminus B) = \operatorname{Shade} A$.

Proof of Lemma 4.31. Induction on $|B|$, using Axiom 2 from Definition 2.1. □

We will also use another simple set-theoretical lemma:

Lemma 4.32. Let E be a set. Let X and Y be two subsets of E. Then,

$$E \setminus Y = (X \cup (E \setminus Y)) \setminus (X \cap Y).$$

Proof of Lemma 4.32. This is straightforward set theory; it is easily checked (e.g.) using Venn diagrams. □

Proof of Theorem 4.30. **(a)** Let $F \in \mathcal{P}(E)$. The definition of α yields $\alpha(F) = F \cap \operatorname{Shade} F \subseteq F$. The definition of τ yields $\tau(F) = F \cup (E \setminus \operatorname{Shade} F) \supseteq F$, so that $F \subseteq \tau(F)$. Thus, we have the chain of inclusions $\alpha(F) \subseteq F \subseteq \tau(F)$. In other words, $F \in [\alpha(F), \tau(F)]$. This proves Theorem 4.30 **(a)**.

(b) Let $F \in \mathcal{P}(E)$ and $G \in [\alpha(F), \tau(F)]$. From $G \in [\alpha(F), \tau(F)]$, we obtain the chain of inclusions $\alpha(F) \subseteq G \subseteq \tau(F)$. Thus, $G \subseteq \tau(F) = F \cup (E \setminus \operatorname{Shade} F)$ (by the definition of τ). Hence,

$$\underbrace{G}_{= F \cup (E \setminus \operatorname{Shade} F)} \setminus F = (F \cup (E \setminus \operatorname{Shade} F)) \setminus F \subseteq E \setminus \operatorname{Shade} F$$

(since $(X \cup Y) \setminus X \subseteq Y$ for any two sets X and Y). Therefore,

$$\underbrace{(G \setminus F)}_{\subseteq E \setminus \operatorname{Shade} F} \cap \operatorname{Shade} F \subseteq (E \setminus \operatorname{Shade} F) \cap \operatorname{Shade} F = \varnothing$$

(since $(X \setminus Y) \cap Y = \varnothing$ for any two sets X and Y). Thus,

$$(G \setminus F) \cap \operatorname{Shade} F = \varnothing. \tag{15}$$

Note that $G \setminus F \subseteq G \subseteq \tau(F) \subseteq E$ (since $\tau(F) \in \mathcal{P}(E)$) and $F \cap G \subseteq F \subseteq E$ (since $F \in \mathcal{P}(E)$) and $F \setminus G \subseteq F \subseteq E$.

Furthermore, we have $\alpha(F) = F \cap \operatorname{Shade} F$ (by the definition of α). Hence, $F \cap \operatorname{Shade} F = \alpha(F) \subseteq G$. Thus, $G \supseteq F \cap \operatorname{Shade} F$, so that

$$F \setminus \underbrace{G}_{\supseteq F \cap \operatorname{Shade} F} \subseteq F \setminus (F \cap \operatorname{Shade} F) = F \setminus \operatorname{Shade} F$$

(since $X \setminus (X \cap Y) = X \setminus Y$ for any two sets X and Y). Hence,

$$\underbrace{(F \setminus G)}_{\subseteq F \setminus \operatorname{Shade} F} \cap \operatorname{Shade} F \subseteq (F \setminus \operatorname{Shade} F) \cap \operatorname{Shade} F = \varnothing$$

(since $(X \setminus Y) \cap Y = \varnothing$ for any two sets X and Y). Thus,

$$(F \setminus G) \cap \operatorname{Shade} F = \varnothing.$$

Hence, Lemma 4.31 (applied to $A = F$ and $B = F \setminus G$) yields $\operatorname{Shade}(F \setminus (F \setminus G)) = \operatorname{Shade} F$. In view of $F \setminus (F \setminus G) = F \cap G$, this rewrites as $\operatorname{Shade}(F \cap G) = \operatorname{Shade} F$. Hence, (15) rewrites as

$$(G \setminus F) \cap \operatorname{Shade}(F \cap G) = \varnothing.$$

Thus, Lemma 4.23 (applied to $A = F \cap G$ and $B = G \setminus F$) yields $\operatorname{Shade}((F \cap G) \cup (G \setminus F)) = \operatorname{Shade}(F \cap G) = \operatorname{Shade} F$. In view of $(F \cap G) \cup (G \setminus F) = G$, this rewrites as $\operatorname{Shade} G = \operatorname{Shade} F$. In other words,

$$\operatorname{Shade} F = \operatorname{Shade} G.$$

The definition of α yields $\alpha(F) = F \cap \operatorname{Shade} F$ and $\alpha(G) = G \cap \operatorname{Shade} G$. Now, $\alpha(F) = F \cap \operatorname{Shade} F \subseteq \operatorname{Shade} F$, so that

$$\alpha(F) = \underbrace{\alpha(F)}_{\subseteq G} \cap \underbrace{\operatorname{Shade} F}_{= \operatorname{Shade} G} \subseteq G \cap \operatorname{Shade} G = \alpha(G).$$

Combining this with

$$\begin{aligned}
\alpha(G) &= \underbrace{G}_{\substack{\subseteq \tau(F) \\ = F \cup (E \setminus \operatorname{Shade} F)}} \cap \underbrace{\operatorname{Shade} G}_{= \operatorname{Shade} F} \subseteq (F \cup (E \setminus \operatorname{Shade} F)) \cap \operatorname{Shade} F \\
&= \underbrace{(F \cap \operatorname{Shade} F)}_{= \alpha(F)} \cup \underbrace{((E \setminus \operatorname{Shade} F) \cap \operatorname{Shade} F)}_{= \varnothing} \\
&\qquad \left(\text{since } (X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z) \text{ for any three sets } X, Y, Z\right) \\
&= \alpha(F) \cup \varnothing = \alpha(F),
\end{aligned}$$

we obtain $\alpha(F) = \alpha(G)$.

The definition of τ yields $\tau(F) = F \cup (E \setminus \operatorname{Shade} F)$ and $\tau(G) = G \cup (E \setminus \operatorname{Shade} G)$. Now, $\tau(F) = F \cup (E \setminus \operatorname{Shade} F) \supseteq E \setminus \operatorname{Shade} F$, so that

$$\tau(F) = \underbrace{\tau(F)}_{\substack{\supseteq G \\ (\text{since } G \subseteq \tau(F))}} \cup \left(E \setminus \underbrace{\operatorname{Shade} F}_{= \operatorname{Shade} G} \right) \supseteq G \cup (E \setminus \operatorname{Shade} G) = \tau(G).$$

Combining this with

$$\begin{aligned}
\tau(F) &= \underbrace{F}_{\substack{= (F \cap \operatorname{Shade} F) \cup (F \setminus \operatorname{Shade} F) \\ (\text{since } X = (X \cap Y) \cup (X \setminus Y) \\ \text{for any two sets } X \text{ and } Y)}} \cup (E \setminus \operatorname{Shade} F) \\
&= (F \cap \operatorname{Shade} F) \cup \left(\underbrace{F}_{\subseteq E} \setminus \operatorname{Shade} F \right) \cup (E \setminus \operatorname{Shade} F) \\
&\subseteq \underbrace{(F \cap \operatorname{Shade} F)}_{= \alpha(F)} \cup \underbrace{(E \setminus \operatorname{Shade} F) \cup (E \setminus \operatorname{Shade} F)}_{= E \setminus \operatorname{Shade} F} \\
&= \underbrace{\alpha(F)}_{\subseteq G} \cup \left(E \setminus \underbrace{\operatorname{Shade} F}_{= \operatorname{Shade} G} \right) \subseteq G \cup (E \setminus \operatorname{Shade} G) = \tau(G),
\end{aligned}$$

we obtain $\tau(F) = \tau(G)$.

Now, $\left[\underbrace{\alpha(F)}_{= \alpha(G)}, \underbrace{\tau(F)}_{= \tau(G)} \right] = [\alpha(G), \tau(G)]$. Theorem 4.30 **(b)** is thus proven.

(c) We have $\mathbf{P} = \{ [\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E) \}$. Thus, $\mathbf{P}$ is a set of Boolean intervals of $\mathcal{P}(E)$ (since $[\alpha(F), \tau(F)]$ is a Boolean interval of $\mathcal{P}(E)$ whenever $F \in \mathcal{P}(E)$). We shall refer to the elements of $\mathbf{P}$ as the *blocks* of $\mathbf{P}$ (even though we don't know yet that $\mathbf{P}$ is a Boolean interval partition).

Next, we shall show that the blocks of $\mathbf{P}$ are pairwise disjoint.

[*Proof.* Let Q and R be two distinct blocks of $\mathbf{P}$.

Let $G \in Q \cap R$. Now, we have $Q = [\alpha(F), \tau(F)]$ for some $F \in \mathcal{P}(E)$ (since $Q \in \mathbf{P} = \{ [\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E) \}$). Consider this F. We have $G \in Q \cap R \subseteq Q = [\alpha(F), \tau(F)]$. Therefore, Theorem 4.30 **(b)** yields $[\alpha(F), \tau(F)] = [\alpha(G), \tau(G)]$. Hence, $Q = [\alpha(F), \tau(F)] = [\alpha(G), \tau(G)]$. The same argument (applied to R instead of Q) yields $R = [\alpha(G), \tau(G)]$. Comparing these two equalities, we obtain $Q = R$; this contradicts the assumption that Q and R are distinct.

Forget that we fixed G. We thus have obtained a contradiction for each $G \in Q \cap R$. Hence, there exists no $G \in Q \cap R$. In other words, $Q \cap R = \varnothing$. In other words, Q and R are disjoint.

Forget that we fixed Q and R. We thus have shown that any two distinct blocks Q and R of $\mathbf{P}$ are disjoint. In other words, the blocks of $\mathbf{P}$ are pairwise disjoint.]

Recall that $\mathbf{P}$ is a set of Boolean intervals of $\mathcal{P}(E)$. Hence, $\mathbf{P}$ is a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$ (since the blocks of $\mathbf{P}$ are pairwise disjoint).

Now, let $G \in \mathcal{P}(E)$. Then, $[\alpha(G), \tau(G)] \in \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\} = \mathbf{P}$. Thus, the set $[\alpha(G), \tau(G)]$ is one of the sets that are being united in the union $\bigcup_{R \in \mathbf{P}} R$. Therefore, $[\alpha(G), \tau(G)] \subseteq \bigcup_{R \in \mathbf{P}} R$. However, Theorem 4.30 **(a)** (applied to $F = G$) yields $G \in [\alpha(G), \tau(G)] \subseteq \bigcup_{R \in \mathbf{P}} R$.

Forget that we fixed G. We thus have shown that $G \in \bigcup_{R \in \mathbf{P}} R$ for each $G \in \mathcal{P}(E)$. In other words, $\mathcal{P}(E) \subseteq \bigcup_{R \in \mathbf{P}} R$. Combining this inclusion with the inclusion $\bigcup_{R \in \mathbf{P}} R \subseteq \mathcal{P}(E)$ (which is obvious, since each $R \in \mathbf{P}$ is a subset of $\mathcal{P}(E)$), we obtain $\bigcup_{R \in \mathbf{P}} R = \mathcal{P}(E)$. In other words, the union of all blocks of $\mathbf{P}$ is $\mathcal{P}(E)$.

Now, recall that $\mathbf{P}$ is a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$. Hence, $\mathbf{P}$ is a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$ whose union is $\mathcal{P}(E)$ (since the union of all blocks of $\mathbf{P}$ is $\mathcal{P}(E)$). In other words, $\mathbf{P}$ is a Boolean interval partition of $\mathcal{P}(E)$. This proves Theorem 4.30 **(c)**.

(d) Let $F \in \mathcal{P}(E)$. Then, $\operatorname{Shade} F$ belongs to $\mathcal{P}(E)$ as well (since Shade is a map from $\mathcal{P}(E)$ to $\mathcal{P}(E)$). In other words, $\operatorname{Shade} F$ is a subset of E.

Also, F is a subset of E (since $F \in \mathcal{P}(E)$). Hence, Lemma 4.32 (applied to $X = F$ and $Y = \operatorname{Shade} F$) yields

$$E \setminus \operatorname{Shade} F = \underbrace{(F \cup (E \setminus \operatorname{Shade} F))}_{\substack{=\tau(F) \\ (\text{since } \tau(F) = F \cup (E \setminus \operatorname{Shade} F))}} \setminus \underbrace{(F \cap \operatorname{Shade} F)}_{\substack{=\alpha(F) \\ (\text{since } \alpha(F) = F \cap \operatorname{Shade} F)}} = \tau(F) \setminus \alpha(F).$$

However, $\operatorname{Shade} F$ is a subset of E; thus, $E \setminus (E \setminus \operatorname{Shade} F) = \operatorname{Shade} F$. Therefore,

$$\operatorname{Shade} F = E \setminus \underbrace{(E \setminus \operatorname{Shade} F)}_{=\tau(F) \setminus \alpha(F)} = E \setminus (\tau(F) \setminus \alpha(F)).$$

This proves Theorem 4.30 **(d)**. ☐

Combining Theorem 4.28 with Theorem 4.30, we obtain the following:

Theorem 4.33. Let E be a finite set. Then, there is a bijection from the set

$$\{\text{shade maps } \operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)\}$$

to the set

$$\{\text{Boolean interval partitions of } \mathcal{P}(E)\}.$$

It sends each map Shade to the Boolean interval partition **P** defined in Theorem 4.30 **(c)**. Its inverse map sends each Boolean interval partition **P** to the map Shade defined in Theorem 4.28.

Proof of Theorem 4.33. This follows easily from Theorem 4.28 and Theorem 4.30. □

Question 4.34. According to Theorem 4.18, any antimatroid gives rise to an inclusion-reversing shade map, which in turn gives rise to a Boolean interval partition by Theorem 4.30. Is this construction equivalent to the construction of a Boolean interval partition from an antimatroid described by Gordon and McMahon in [GorMah97, Theorem 2.5]?

5. The topological viewpoint

Now we return to the setting of Section 1. We aim to reinterpret Theorem 2.6 in the terms of combinatorial topology (specifically, finite simplicial complexes) and strengthen it. We recall the definition of a *simplicial complex*:[4]

Definition 5.1. Let E be a finite set. A *simplicial complex* on ground set E means a subset $\mathcal{A}$ of the power set of E with the following property:

If $P \in \mathcal{A}$ and $Q \subseteq P$, then $Q \in \mathcal{A}$.

Thus, in terms of posets, a simplicial complex on ground set E means a down-closed subset of the Boolean lattice on E. Note that a simplicial complex contains the empty set $\varnothing$ unless it is empty itself.

We refer to [Kozlov20] for context and theory about simplicial complexes. We shall restrict ourselves to the few definitions relevant to what we will prove. The following is fairly simple:

Proposition 5.2. Let us use the notations from Section 1 as well as Definition 2.1. Let G be any subset of E. Let

$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\} . \tag{16}$$

Then, $\mathcal{A}$ is a simplicial complex on ground set E.

Proof of Proposition 5.2. Clearly, $\mathcal{A}$ is a subset of the power set of E. Thus, we only need to verify the following claim:

Claim 1: If $P \in \mathcal{A}$ and $Q \subseteq P$, then $Q \in \mathcal{A}$.

[4] We forget all the conventions we have introduced so far. (Thus, for example, E no longer means the edge set of a graph Γ.)

[*Proof of Claim 1:* Let $P \in \mathcal{A}$ and let $Q \subseteq P$. We must show that $Q \in \mathcal{A}$.

We have $P \in \mathcal{A} = \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\}$. In other words, $P \subseteq E$ and $G \not\subseteq \operatorname{Shade} P$. But $Q \subseteq P$ and thus $\operatorname{Shade} Q \subseteq \operatorname{Shade} P$ (by Lemma 2.3). Hence, from $G \not\subseteq \operatorname{Shade} P$, we obtain $G \not\subseteq \operatorname{Shade} Q$. Thus, $Q \in \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\}$. This can be rewritten as $Q \in \mathcal{A}$ (by (16)). Thus, Claim 1 is proved.] □

To state the main result of this section, we need a few more notions:

Definition 5.3. Let A and B be two sets. Then, we say that $A \prec B$ if we have $B = A \cup \{b\}$ for some $b \in B \setminus A$.

Equivalently, two sets A and B satisfy $A \prec B$ if and only if $A \subseteq B$ and $|B \setminus A| = 1$.

Definition 5.4. Let E be a finite set. Let $\mathcal{A}$ be a simplicial complex on ground set E.

(a) A *complete matching* of $\mathcal{A}$ means a triple $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$, where $\mathcal{A}_-$ and $\mathcal{A}_+$ are two disjoint subsets of $\mathcal{A}$ satisfying $\mathcal{A}_- \cup \mathcal{A}_+ = \mathcal{A}$, and where $\Phi : \mathcal{A}_+ \to \mathcal{A}_-$ is a bijection with the property that

$$\text{each } F \in \mathcal{A}_+ \text{ satisfies } \Phi(F) \prec F. \tag{17}$$

(b) A complete matching $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ of $\mathcal{A}$ is said to be *acyclic* if there exists no tuple $(B_1, B_2, \ldots, B_n)$ of distinct sets $B_1, B_2, \ldots, B_n \in \mathcal{A}_+$ with the property that $n \geqslant 2$ and that

$$\Phi(B_i) \prec B_{i+1} \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}$$

and

$$\Phi(B_n) \prec B_1.$$

(c) The simplicial complex $\mathcal{A}$ is said to be *collapsible* if it has an acyclic complete matching.

These definitions are essentially equivalent to the definitions in [Kozlov20], although it takes a bit of work to match them up precisely. Our notion of a "complete matching" as defined in Definition 5.4 **(a)** is a particular case of the notion introduced in [Kozlov20, Chapter 10], as we restrict ourselves to simplicial complexes (i.e., down-closed sets of Boolean lattices) instead of arbitrary posets. To be fully precise, our complete matchings are triples $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$, whereas the complete matchings of [Kozlov20, Chapter 10] are certain fixed-point-free involutions[5] $\mu : \mathcal{A} \to \mathcal{A}$; the equivalence between these two objects is fairly easy to see (in particular, if $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is a complete matching in our sense, then the corresponding complete matching $\mu : \mathcal{A} \to \mathcal{A}$ in the sense of

[5] A *fixed-point-free involution* means an involution (i.e., a map that is inverse to itself) that has no fixed point.

[Kozlov20, Chapter 10] is the map that sends each $B \in \mathcal{A}_+$ to $\Phi(B) \in \mathcal{A}_-$ and sends each $B \in \mathcal{A}_-$ to $\Phi^{-1}(B) \in \mathcal{A}_+$). Our notion of "collapsible" as defined in Definition 5.4 **(c)** is equivalent to the classical notion of "collapsible" (even though the latter is usually defined differently) because of [Kozlov20, Theorem 10.9].

We now claim:

Theorem 5.5. Let us use the notations from Section 1 as well as Definition 2.1. Let G be any subset of E. Define $\mathcal{A}$ as in (16). Then, the simplicial complex $\mathcal{A}$ is collapsible.

Collapsible simplicial complexes are well-behaved in various ways – in particular, they are contractible ([Kozlov20, Corollary 9.19]), and thus have trivial homotopy and homology groups (in positive degrees). Moreover, the reduced Euler characteristic of any collapsible simplicial complex is 0 (for obvious reasons: having a complete matching suffices, even if it is not acyclic); thus, Theorem 2.6 follows from Theorem 5.5.

Our proof of Theorem 5.5 will rely on the following simple lemma:

Lemma 5.6. Let X and Y be two sets. Let $u \in X \cap Y$. If $X \setminus \{u\} \prec Y$, then $X = Y$.

Proof of Lemma 5.6. Set $X' = X \setminus \{u\}$. Thus, $\underbrace{X'}_{=X\setminus\{u\}} \cup \{u\} = (X \setminus \{u\}) \cup \{u\} = X$ (since $u \in X \cap Y \subseteq X$). Also, clearly, $u \notin X \setminus \{u\}$ (since $u \notin P \setminus \{u\}$ for any set P). In other words, $u \notin X'$ (since $X' = X \setminus \{u\}$).

Now, assume that $X \setminus \{u\} \prec Y$. In other words, $X' \prec Y$ (since $X' = X \setminus \{u\}$). In other words, we have

$$Y = X' \cup \{b\} \tag{18}$$

for some $b \in Y \setminus X'$ (by Definition 5.3). Consider this b. Combining $u \in X \cap Y \subseteq Y = X' \cup \{b\}$ with $u \notin X'$, we obtain

$$u \in \left(X' \cup \{b\}\right) \setminus X' \subseteq \{b\}.$$

In other words, $u = b$. Thus, (18) can be rewritten as $Y = X' \cup \{u\}$. Hence, $Y = X' \cup \{u\} = X$, so that $X = Y$. This proves Lemma 5.6. □

Proof of Theorem 5.5. We know from Proposition 5.2 that $\mathcal{A}$ is a simplicial complex. It remains to show that $\mathcal{A}$ is collapsible.

We have

$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\} = \{P \subseteq E \mid G \not\subseteq \operatorname{Shade} P\}$$

(here, we have renamed the index F as P). Thus, our set $\mathcal{A}$ is precisely the set $\mathcal{A}$ defined in the proof of Theorem 2.6 above.

We equip the finite set E with a total order (chosen arbitrarily, but fixed henceforth).

If $F \in \mathcal{A}$, then we define the edge $\varepsilon(F) \in G \setminus \operatorname{Shade} F$ as in the proof of Theorem 2.6. That is, we define $\varepsilon(F)$ as the **smallest** edge $e \in G \setminus \operatorname{Shade} F$.

If two sets $F_1 \in \mathcal{A}$ and $F_2 \in \mathcal{A}$ satisfy $\operatorname{Shade}(F_1) = \operatorname{Shade}(F_2)$, then

$$\varepsilon(F_1) = \varepsilon(F_2). \tag{19}$$

(Indeed, this is precisely the equality (6) from the above proof of Theorem 2.6.)

We define two subsets $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$ as in the proof of Theorem 2.6. That is, we set

$$\mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} \qquad \text{and} \qquad \mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}.$$

Thus, each $P \in \mathcal{A}$ satisfies either $P \in \mathcal{A}_-$ or $P \in \mathcal{A}_+$ but not both at the same time (since it satisfies either $\varepsilon(P) \notin P$ or $\varepsilon(P) \in P$ but not both at the same time). Hence, $\mathcal{A}_-$ and $\mathcal{A}_+$ are two disjoint subsets of $\mathcal{A}$ satisfying $\mathcal{A}_- \cup \mathcal{A}_+ = \mathcal{A}$.

We define a map $\Phi : \mathcal{A}_+ \to \mathcal{A}_-$ as in the proof of Theorem 2.6. That is, we set

$$\Phi(F) = F \setminus \{\varepsilon(F)\} \qquad \text{for each } F \in \mathcal{A}_+.$$

We know (from the proof of Theorem 2.6) that the map Φ is a bijection. Moreover, it is clear that each $F \in \mathcal{A}_+$ satisfies $\Phi(F) \prec F$ [6]. Hence, the triple $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is a complete matching of $\mathcal{A}$.

We shall now prove that this complete matching $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is acyclic. Indeed, let $(B_1, B_2, \ldots, B_n)$ be a tuple of distinct sets $B_1, B_2, \ldots, B_n \in \mathcal{A}_+$ with the property that $n \geqslant 2$ and that

$$\Phi(B_i) \prec B_{i+1} \qquad \text{for each } i \in \{1, 2, \ldots, n-1\} \tag{20}$$

and

$$\Phi(B_n) \prec B_1. \tag{21}$$

We shall derive a contradiction.

Set $B_{n+1} = B_1$. Then, (21) can be rewritten as $\Phi(B_n) \prec B_{n+1}$. In other words, we have $\Phi(B_i) \prec B_{i+1}$ for $i = n$. Combining this with (20), we conclude that

$$\Phi(B_i) \prec B_{i+1} \qquad \text{for each } i \in \{1, 2, \ldots, n\}. \tag{22}$$

Now, set $A_i = \operatorname{Shade}(B_i)$ for each $i \in \{1, 2, \ldots, n+1\}$. Then, $A_{n+1} = A_1$ (since $B_{n+1} = B_1$).

We now claim the following:

[6] *Proof.* Let $F \in \mathcal{A}_+$. Thus, $F \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, F is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, $F \in \mathcal{A}$ and $\varepsilon(F) \in F$. From $\varepsilon(F) \in F$, we obtain $F = (F \setminus \{\varepsilon(F)\}) \cup \{\varepsilon(F)\}$ and $\varepsilon(F) \in F \setminus (F \setminus \{\varepsilon(F)\})$. Hence, $F \setminus \{\varepsilon(F)\} \prec F$ (by Definition 5.3). In other words, $\Phi(F) \prec F$ (since the definition of Φ yields $\Phi(F) = F \setminus \{\varepsilon(F)\}$). Qed.

Claim 1: We have $A_i \subseteq A_{i+1}$ for each $i \in \{1, 2, \ldots, n\}$.

[*Proof of Claim 1:* Let $i \in \{1, 2, \ldots, n\}$. Then, the definition of A_i yields $A_i = \operatorname{Shade}(B_i)$. Likewise, $A_{i+1} = \operatorname{Shade}(B_{i+1})$.

We have $B_i \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, B_i is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, B_i is an element of $\mathcal{A}$ and satisfies $\varepsilon(B_i) \in B_i$.

We set $u = \varepsilon(B_i)$. The definition of Φ yields $\Phi(B_i) = B_i \setminus \{\varepsilon(B_i)\} = B_i \setminus \{u\}$ (since $\varepsilon(B_i) = u$).

Recall that $\varepsilon(B_i)$ is the smallest edge $e \in G \setminus \operatorname{Shade}(B_i)$ (by the definition of $\varepsilon(B_i)$). Hence, $\varepsilon(B_i) \in G \setminus \operatorname{Shade}(B_i)$. In other words, $u \in G \setminus \operatorname{Shade}(B_i)$ (since $u = \varepsilon(B_i)$). In other words, $u \in G$ and $u \notin \operatorname{Shade}(B_i)$. Thus, $u \in G \subseteq E$ and $u \notin \operatorname{Shade}(B_i)$. Therefore, (4) (applied to $F = B_i$) yields $\operatorname{Shade}(B_i \setminus \{u\}) = \operatorname{Shade}(B_i)$. This can be rewritten as $\operatorname{Shade}(\Phi(B_i)) = A_i$ (since $\Phi(B_i) = B_i \setminus \{u\}$ and $A_i = \operatorname{Shade}(B_i)$).

But (22) yields $\Phi(B_i) \prec B_{i+1}$, so that $\Phi(B_i) \subseteq B_{i+1}$ and thus $\operatorname{Shade}(\Phi(B_i)) \subseteq \operatorname{Shade}(B_{i+1})$ (by Lemma 2.3, applied to $A = \Phi(B_i)$ and $B = B_{i+1}$). In view of $\operatorname{Shade}(\Phi(B_i)) = A_i$ and $\operatorname{Shade}(B_{i+1}) = A_{i+1}$, this can be rewritten as $A_i \subseteq A_{i+1}$. This proves Claim 1.]

Claim 1 shows that $A_i \subseteq A_{i+1}$ for each $i \in \{1, 2, \ldots, n\}$. In other words,

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1}.$$

This is a chain of inclusions, but its last entry equals the first: indeed, $A_{n+1} = A_1$. Thus, all inclusions in this chain must be equalities. That is, we have

$$A_1 = A_2 = \cdots = A_n = A_{n+1}. \tag{23}$$

Hence, in particular, $A_n = A_1$. However, $A_n = \operatorname{Shade}(B_n)$ (by the definition of A_n) and $A_1 = \operatorname{Shade}(B_1)$ (by the definition of A_1). Hence, $\operatorname{Shade}(B_n) = A_n = A_1 = \operatorname{Shade}(B_1)$. Thus, (19) (applied to $F_1 = B_n$ and $F_2 = B_1$) yields $\varepsilon(B_n) = \varepsilon(B_1)$ (since $B_n \in \mathcal{A}_+ \subseteq \mathcal{A}$ and $B_1 \in \mathcal{A}_+ \subseteq \mathcal{A}$).

Set $u = \varepsilon(B_n)$. Thus, $u = \varepsilon(B_n) = \varepsilon(B_1)$.

Recall that the sets $B_1, B_2, \ldots, B_n$ are distinct. Hence, $B_n \neq B_1$ (since $n \geqslant 2$).

The definition of Φ yields $\Phi(B_n) = B_n \setminus \{\varepsilon(B_n)\} = B_n \setminus \{u\}$ (since $\varepsilon(B_n) = u$). However, (21) says that $\Phi(B_n) \prec B_1$. In other words, $B_n \setminus \{u\} \prec B_1$ (since $\Phi(B_n) = B_n \setminus \{u\}$).

On the other hand, $B_1 \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, B_1 is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, B_1 is an element of $\mathcal{A}$ and satisfies $\varepsilon(B_1) \in B_1$. Now, $u = \varepsilon(B_1) \in B_1$. The same argument (applied to B_n instead of B_1) yields $u \in B_n$ (since $u = \varepsilon(B_n)$). Combining $u \in B_n$ with $u \in B_1$, we obtain $u \in B_n \cap B_1$.

Thus, Lemma 5.6 (applied to $X = B_n$ and $Y = B_1$) yields $B_n = B_1$ (since $B_n \setminus \{u\} \prec B_1$). This contradicts $B_n \neq B_1$.

Forget that we fixed $(B_1, B_2, \ldots, B_n)$. We thus have found a contradiction whenever $(B_1, B_2, \ldots, B_n)$ is a tuple of distinct sets $B_1, B_2, \ldots, B_n \in \mathcal{A}_+$ with the property that $n \geqslant 2$ and that (20) and (21). Hence, there exists no such tuple. In other words, the complete matching $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is acyclic. Therefore, the simplicial complex $\mathcal{A}$ has an acyclic complete matching, and thus is collapsible (by Definition 5.4 **(c)**). This finishes the proof of Theorem 5.5. □

The analogues of Proposition 5.2 and of Theorem 5.5 for vertex-infection (instead of usual infection) also hold (with the same proofs). More generally, Proposition 5.2 and Theorem 5.5 can be generalized to any inclusion-preserving shade map:

Theorem 5.7. Let E be any set. Let $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ be an inclusion-preserving shade map on E. Let G be any subset of E. Let

$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\}.$$

Then:

(a) This $\mathcal{A}$ is a simplicial complex on ground set E.

(b) This simplicial complex $\mathcal{A}$ is collapsible.

Proof sketch. Part **(a)** is a straightforward generalization of Proposition 5.2, while part **(b)** is a straightforward generalization of Theorem 5.5. The proofs we gave above generalize (mutatis mutandis). □

However, Theorem 5.7 cannot be lifted to the full generality of arbitrary shade maps, since $\mathcal{A}$ will generally not be a simplicial complex unless the shade map is inclusion-preserving. (However, for inclusion-reversing shade maps, we can obtain a variant of Theorem 5.7 by applying Theorem 5.7 to the dual shade map Shade' from Proposition 4.7.)

6. Open questions

I shall now comment on two natural directions of research so far unexplored.

6.1. The Alexander dual

Any simplicial complex has an *Alexander dual*, which is defined as follows:

Definition 6.1. Let E be a finite set. Let $\mathcal{A}$ be a simplicial complex on ground set E. Then, we define a new simplicial complex $\mathcal{A}^\vee$ on ground set E by

$$\mathcal{A}^\vee = \{F \subseteq E \mid E \setminus F \notin \mathcal{A}\}.$$

(That is, $\mathcal{A}^\vee$ consists of those subsets of E whose complements don't belong to $\mathcal{A}$.) This simplicial complex $\mathcal{A}^\vee$ is called the *Alexander dual* of $\mathcal{A}$.

It is well-known that a simplicial complex $\mathcal{A}$ and its Alexander dual $\mathcal{A}^\vee$ share many properties; in particular, the reduced homology of $\mathcal{A}$ is isomorphic to the reduced cohomology of $\mathcal{A}^\vee$ (see, e.g., [BjoTan09, Theorem 1.1]). However, the collapsibility and the homotopy types of $\mathcal{A}$ and $\mathcal{A}^\vee$ are not always related. Thus, the following question is suggested but not answered by Theorem 5.5:

Question 6.2. Let us use the notations from Section 1 as well as Definition 2.1. Let G be any subset of E. Define $\mathcal{A}$ as in (16). Is the simplicial complex

$$\mathcal{A}^\vee = \{F \subseteq E \mid G \subseteq \operatorname{Shade}(E \setminus F)\}$$

collapsible? Is it contractible?

6.2. Several vertices v

Elser's nuclei-based viewpoint in [Elser84] (and [DHLetc19, Conjecture 9.1]) suggests yet another question.

Our definition of Shade F (Definition 2.1), and the underlying notion of "infecting" an edge, implicitly relied on the choice of vertex v. It thus is advisable to rename the set Shade F as $\operatorname{Shade}_v F$ and combine such sets for different values of v. In particular, we can define

$$\mathcal{A}_U = \{F \subseteq E \mid G \not\subseteq \operatorname{Shade}_v F \text{ for some } v \in U\}$$

for any subset U of V. This $\mathcal{A}_U$ is a simplicial complex (being the union of a family of simplicial complexes), and thus we can ask the same questions about it as we did about $\mathcal{A}$:

Question 6.3. What can we say about the homotopy and discrete Morse theory of $\mathcal{A}_U$? What about its Alexander dual?

For $G = E$ and $|U| > 0$, this simplicial complex $\mathcal{A}_U$ is the Alexander dual of the "U-nucleus complex" Δ_U^G from [DHLetc19, Definition 3.2] (when G is connected). If [DHLetc19, Conjecture 9.1 for $|U| > 1$] is correct, then the homology of $\mathcal{A}_U$ with real coefficients should be concentrated in a single degree; this suggests the possible existence of an acyclic partial matching with all critical faces in one degree.

Acknowledgments

I thank Anders Björner, Galen Dorpalen-Barry, Dmitry Feichtner-Kozlov, Patricia Hersh, Sam Hopkins, Vic Reiner, Tom Roby and Richard Stanley for insightful conversations. A referee has greatly simplified the proof of Theorem 5.5. This research was supported through the programme "Oberwolfach Leibniz Fellows" by the Mathematisches Forschungsinstitut Oberwolfach in 2020. I am deeply grateful to the Institute for its hospitality during a none-too-hospitable time.

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