Double posets and the antipode of QSym (detailed version)

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Abstract

A quasisymmetric function is assigned to every double poset (that is, every finite set endowed with two partial orders) and any weight function on its ground set. This generalizes well-known objects such as monomial and fundamental quasisymmetric functions, (skew) Schur functions, dual immaculate functions, and quasisymmetric (P, ω)-partition enumerators. We prove a formula for the antipode of this function that holds under certain conditions (which are satisfied when the second order of the double poset is total, but also in some other cases); this restates (in a way that to us seems more natural) a result by Malvenuto and Reutenauer, but our proof is new and self-contained. We generalize it further to an even more comprehensive setting, where a group acts on the double poset by automorphisms.

Keywords: antipodes, double posets, Hopf algebras, posets, P-partitions, quasisymmetric functions.

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1. Introduction

Double posets and $\mathbf{E}$-partitions (for $\mathbf{E}$ a double poset) have been introduced by Claudia Malvenuto and Christophe Reutenauer [MalReu09]; their goal was to construct a combinatorial Hopf algebra which harbors a noticeable amount of structure, including an analogue of the Littlewood-Richardson rule and a lift of the internal product operation of the Malvenuto-Reutenauer Hopf algebra of permutations. In this note, we shall employ these same notions to restate in a simpler form, and reprove in a more elementary fashion, a formula for the antipode in the Hopf algebra QSym of quasisymmetric functions due to (the same) Malvenuto and Reutenauer [MalReu98, Theorem 3.1]. We then further generalize this formula to a setting in which a group acts on the double poset (a generalization inspired by Katharina Jochemko's [Joch13]).

The present version of the paper is the *detailed version*[1]. A standard version is also available[2]. The two versions differ in that the detailed version contains extra details in various proofs (although the level of detail is not always consistent).

[1] This version can be downloaded from `http://www.cip.ifi.lmu.de/~grinberg/algebra/dp-abstr-long.pdf` . It is also archived as an ancillary file on `http://arxiv.org/abs/1509.08355v3`, although the former website is more likely to be updated.

[2] at `http://www.cip.ifi.lmu.de/~grinberg/algebra/dp-abstr-long.pdf` and `http://arxiv.org/abs/1509.08355v3`

A short summary of this paper has been submitted to the FPSAC conference [Grin16b].

Acknowledgments

Katharina Jochemko's work [Joch13] provoked this research. I learnt a lot about QSym from Victor Reiner. The SageMath computer algebra system [Sage16] was used for some computations that suggested one of the proofs.

Note on the published version of this paper

The document you are reading is the detailed version of a preprint of a paper (of the same title) that was accepted for publication in the Electronic Journal of Combinatorics in 2017. The published version differs from the standard version of this preprint insubstantially[3].

2. Quasisymmetric functions

Let us first briefly introduce the notations that will be used in the following.

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. A *composition* means a finite sequence of positive integers. We let Comp be the set of all compositions. For $n \in \mathbb{N}$, a *composition of n* means a composition whose entries sum to n (that is, a composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$).

Let $\mathbf{k}$ be an arbitrary commutative ring. We shall keep $\mathbf{k}$ fixed throughout this paper. We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in infinitely many (commuting) indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. A *monomial* shall always mean a monomial (without coefficients) in the variables $x_1, x_2, x_3, \ldots$. [4]

[3]The main difference is that in the published version, the long footnote in Section 2 has been relegated into a separate subsection (§2.2), whereas the remainder of Section 2 has become §2.1. Other than this, the two versions differ in formatting and editorialization.

[4]For the sake of completeness, let us give a detailed definition of monomials and of the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. (This definition has been copied from [Grin14, §2], essentially unchanged.)

Let $x_1, x_2, x_3, \ldots$ be countably many distinct symbols. We let Mon be the free abelian monoid on the set $\{x_1, x_2, x_3, \ldots\}$ (written multiplicatively); it consists of elements of the form $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ for finitely supported $(a_1, a_2, a_3, \ldots) \in \mathbb{N}^\infty$ (where "finitely supported" means that all but finitely many positive integers i satisfy $a_i = 0$). A *monomial* will mean an element of Mon. Thus, a monomial is a combinatorial object, independent of $\mathbf{k}$; it does not carry a coefficient.

We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. By abuse of notation, we shall identify every monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots \in$ Mon with the corresponding element $x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \cdot \cdots$ of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ when necessary (e.g., when we speak of the sum of two monomials or when we multiply a monomial with an element of $\mathbf{k}$). (To be very pedantic, this identification is slightly dangerous, because it can happen that two distinct monomials in Mon get identified with two identical elements of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. However, this can only happen when the ring $\mathbf{k}$ is trivial, and even then it is not a real problem unless we infer the equality of monomials from the equality of their

Inside the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a subalgebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$ consisting of the *bounded-degree* formal power series; these are the power series f for which there exists a $d \in \mathbb{N}$ such that no monomial of degree $> d$ appears in f [5]. This $\mathbf{k}$-subalgebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$ becomes a topological $\mathbf{k}$-algebra, by inheriting the topology from $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent*[6] if they have the forms $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$ for two strictly increasing sequences $(i_1 < i_2 < \cdots < i_\ell)$ and $(j_1 < j_2 < \cdots < j_\ell)$ of positive integers and one (common) sequence $(a_1, a_2, \ldots, a_\ell)$ of positive integers.[7] A power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if it satisfies the following condition: If $\mathfrak{m}$ and $\mathfrak{n}$ are two pack-equivalent monomials, then the coefficient of $\mathfrak{m}$ in f equals the coefficient of $\mathfrak{n}$ in f.

It is easy to see that the quasisymmetric power series form a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. But usually one is interested in a subset of this $\mathbf{k}$-subalgebra: namely, the set of quasisymmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This latter set is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$, and is known as the *$\mathbf{k}$-algebra of quasisymmetric functions over* $\mathbf{k}$. It is denoted by QSym.

The symmetric functions (in the usual sense of this word in combinatorics – so, really, symmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$) form a $\mathbf{k}$-subalgebra of QSym. The quasisymmetric functions have a rich theory which is

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counterparts in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, which we are not going to do.)

We furthermore endow the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ with the following topology (as in [GriRei14, Section 2.6]):

We endow the ring $\mathbf{k}$ with the discrete topology. To define a topology on the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, we (temporarily) regard every power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ as the family of its coefficients (indexed by the set Mon). More precisely, we have a $\mathbf{k}$-module isomorphism

$$\prod_{\mathfrak{m} \in \mathrm{Mon}} \mathbf{k} \to \mathbf{k}[[x_1, x_2, x_3, \ldots]], \qquad (\lambda_{\mathfrak{m}})_{\mathfrak{m} \in \mathrm{Mon}} \mapsto \sum_{\mathfrak{m} \in \mathrm{Mon}} \lambda_{\mathfrak{m}} \mathfrak{m}.$$

We use this isomorphism to transport the product topology on $\prod_{\mathfrak{m} \in \mathrm{Mon}} \mathbf{k}$ to $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. The resulting topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ turns $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ into a topological $\mathbf{k}$-algebra; this is the topology that we will be using whenever we make statements about convergence in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ or write down infinite sums of power series. A sequence $(a_n)_{n \in \mathbb{N}}$ of power series converges to a power series a with respect to this topology if and only if for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the coefficient of } \mathfrak{m} \text{ in } a_n) = (\text{the coefficient of } \mathfrak{m} \text{ in } a).$$

Note that this topological $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is **not** the completion of the polynomial ring $\mathbf{k}[x_1, x_2, x_3, \ldots]$ with respect to the standard grading (in which all x_i have degree 1). (They are distinct even as sets.)

[5]The *degree* of a monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ is defined to be the nonnegative integer $a_1 + a_2 + a_3 + \cdots$. A monomial $\mathfrak{m}$ is said to *appear* in a power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ if and only if the coefficient of $\mathfrak{m}$ in f is nonzero.

[6]Pack-equivalence and the related notions of packed combinatorial objects that we will encounter below originate in work of Hivert, Novelli and Thibon [NovThi05]. Simple as they are, they are of great help in dealing with quasisymmetric functions.

[7]For instance, $x_2^2 x_3 x_4^2$ is pack-equivalent to $x_1^2 x_4 x_8^2$ but not to $x_2 x_3^2 x_4^2$.

related to, and often sheds new light on, the classical theory of symmetric functions; expositions can be found in [Stan99, §§7.19, 7.23] and [GriRei14, §§5-6] and other sources.[8]

As a **k**-module, QSym has a basis $(M_\alpha)_{\alpha\in\mathrm{Comp}}$ indexed by all compositions, where the quasisymmetric function M_α for a given composition α is defined as follows: Writing α as $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we set

$$M_\alpha = \sum_{i_1<i_2<\cdots<i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathfrak{m} \text{ is a monomial pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} \mathfrak{m}$$

(where the i_k in the first sum are positive integers). [9] This basis $(M_\alpha)_{\alpha\in\mathrm{Comp}}$ is known as the *monomial basis* of QSym, and is the simplest to define among many. (We shall briefly encounter another basis in Example 3.6.)

The **k**-algebra QSym can be endowed with a structure of a **k**-coalgebra which, combined with its **k**-algebra structure, turns it into a Hopf algebra. We refer to the literature both for the theory of coalgebras and Hopf algebras (see [Montg93], [GriRei14, §1], [Manchon04, §1-§2], [Abe77], [Sweed69], [DNR01] or [Fresse14, Chapter 7]) and for a deeper study of the Hopf algebra QSym (see [Malve93], [HaGuKi10, Chapter 6] or [GriRei14, §5]); in this note we shall need but the very basics of this structure, and so it is only them that we introduce.

In the following, all tensor products are over **k** by default (i.e., the sign $\otimes$ stands for $\otimes_{\mathbf{k}}$ unless it comes with a subscript).

Now, we define two **k**-linear maps Δ and ε as follows[10]:

- We define a **k**-linear map $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ by requiring that

$$\Delta\left(M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\right) = \sum_{k=0}^{\ell} M_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \otimes M_{(\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_\ell)} \qquad \text{for every } (\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \mathrm{Comp}. \tag{1}$$

- We define a **k**-linear map $\varepsilon : \mathrm{QSym} \to \mathbf{k}$ by requiring that

$$\varepsilon\left(M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\right) = \delta_{\ell,0} \qquad \text{for every } (\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \mathrm{Comp}.$$

(Here, $\delta_{u,v}$ is defined to be $\begin{cases} 1, & \text{if } u = v; \\ 0, & \text{if } u \neq v \end{cases}$ whenever u and v are two objects.)

[8] The notion of quasisymmetric functions goes back to Gessel in 1984 [Gessel84]; they have been studied by many authors, most significantly Malvenuto and Reutenauer [MalReu95].

[9] The second equality sign in this equality is proven in the Appendix (see Proposition 10.1).

[10] Both of their definitions rely on the fact that $\left(M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\right)_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\mathrm{Comp}} = (M_\alpha)_{\alpha\in\mathrm{Comp}}$ is a basis of the **k**-module QSym.

The map ε can also be defined in a simpler (equivalent) way: Namely, ε sends every power series $f \in \operatorname{QSym}$ to the result $f(0,0,0,\ldots)$ of substituting zeroes for the variables $x_1, x_2, x_3, \ldots$ in f. The map Δ can also be described in such terms, but with greater difficulty[11].

It is well-known that these maps Δ and ε make the three diagrams

$$\begin{tikzcd}
\operatorname{QSym} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \operatorname{QSym} \otimes \operatorname{QSym} \arrow[d, "\Delta \otimes \mathrm{id}"] \\
\operatorname{QSym} \otimes \operatorname{QSym} \arrow[r, "\mathrm{id} \otimes \Delta"'] & \operatorname{QSym} \otimes \operatorname{QSym} \otimes \operatorname{QSym}
\end{tikzcd},$$

$$\begin{tikzcd}
\operatorname{QSym} \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & \operatorname{QSym} \otimes \operatorname{QSym} \arrow[d, "\varepsilon \otimes \mathrm{id}"] \\
& \mathbf{k} \otimes \operatorname{QSym}
\end{tikzcd}, \qquad
\begin{tikzcd}
\operatorname{QSym} \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & \operatorname{QSym} \otimes \operatorname{QSym} \arrow[d, "\mathrm{id} \otimes \varepsilon"] \\
& \operatorname{QSym} \otimes \mathbf{k}
\end{tikzcd}$$

(where the $\cong$ arrows are the canonical isomorphisms) commutative, and so $(\operatorname{QSym}, \Delta, \varepsilon)$ is what is commonly called a $\mathbf{k}$-*coalgebra*. Furthermore, Δ and ε are $\mathbf{k}$-algebra homomorphisms, which is what makes this $\mathbf{k}$-coalgebra QSym into a $\mathbf{k}$-*bialgebra*. Finally, let $m : \operatorname{QSym} \otimes \operatorname{QSym} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map sending every pure tensor $a \otimes b$ to ab, and let $u : \mathbf{k} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map sending $1 \in \mathbf{k}$ to $1 \in \operatorname{QSym}$. Then, there exists a unique $\mathbf{k}$-linear map $S : \operatorname{QSym} \to \operatorname{QSym}$ making the diagram

$$\begin{tikzcd}
& \operatorname{QSym} \otimes \operatorname{QSym} \arrow[r, "S \otimes \mathrm{id}"] & \operatorname{QSym} \otimes \operatorname{QSym} \arrow[dr, "m"] & \\
\operatorname{QSym} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbf{k} \arrow[r, "u"] & \operatorname{QSym} \\
& \operatorname{QSym} \otimes \operatorname{QSym} \arrow[r, "\mathrm{id} \otimes S"] & \operatorname{QSym} \otimes \operatorname{QSym} \arrow[ur, "m"'] &
\end{tikzcd} \qquad (2)$$

commutative. This map S is known as the *antipode* of QSym. It is known to be an involution and an algebra automorphism of QSym, and its action on the various quasisymmetric functions defined combinatorially is the main topic of this note. The existence of the antipode S makes QSym into a *Hopf algebra*. [12]

3. Double posets

Next, we shall introduce the notion of a double poset, following Malvenuto and Reutenauer [MalReu09].

[11] See [GriRei14, (5.3)] for the details.

[12] See the Appendix (specifically, Proposition 10.61) for a proof of the fact that QSym is a Hopf algebra.

Definition 3.1. (a) We shall encode posets as pairs $(E, <)$, where E is a set and $<$ is a strict partial order (i.e., an irreflexive, transitive and antisymmetric binary relation) on the set E; this relation $<$ will be regarded as the smaller relation of the poset. All binary relations will be written in infix notation: i.e., we write "$a < b$" for "a is related to b by the relation $<$". (If you define binary relations as sets of pairs, then "a is related to b by the relation $<$" means that (a, b) is an element of the set $<$.)

(b) If $<$ is a strict partial order on a set E, and if a and b are two elements of E, then we say that a and b are *$<$-comparable* if we have either $a < b$ or $a = b$ or $b < a$. A strict partial order $<$ on a set E is said to be a *total order* if and only if every two elements of E are $<$-comparable.

(c) If $<$ is a strict partial order on a set E, and if a and b are two elements of E, then we say that a is *$<$-covered by b* if we have $a < b$ and there exists no $c \in E$ satisfying $a < c < b$. (For instance, if $<$ is the standard smaller relation on $\mathbb{Z}$, then each $i \in \mathbb{Z}$ is $<$-covered by $i + 1$.)

(d) A *double poset* is defined as a triple $(E, <_1, <_2)$ where E is a finite set and $<_1$ and $<_2$ are two strict partial orders on E.

(e) A double poset $(E, <_1, <_2)$ is said to be *special* if the relation $<_2$ is a total order.

(f) A double poset $(E, <_1, <_2)$ is said to be *semispecial* if every two $<_1$-comparable elements of E are $<_2$-comparable.

(g) A double poset $(E, <_1, <_2)$ is said to be *tertispecial* if it satisfies the following condition: If a and b are two elements of E such that a is $<_1$-covered by b, then a and b are $<_2$-comparable.

(h) If $<$ is a binary relation on a set E, then the *opposite relation* of $<$ is defined to be the binary relation $>$ on the set E that is defined as follows: For any $e \in E$ and $f \in E$, we have $e > f$ if and only if $f < e$. Notice that if $<$ is a strict partial order, then so is the opposite relation $>$ of $<$.

Clearly, every special double poset is semispecial, and every semispecial double poset is tertispecial.[13]

[13]The notions of a double poset and of a special double poset come from [MalReu09]. See [Foissy13] for further results on special double posets. The notion of a "tertispecial double poset" (Dog Latin for "slightly less special than semispecial"; in hindsight, "locally special" would have been better terminology) appears to be new and arguably sounds artificial, but is the most suitable setting for some of the results below (see, e.g., Remark 4.9 below); moreover, it appears in nature, beyond the particular case of special double posets (see Example 3.3). We shall not use semispecial double posets in the following; they were only introduced as a middle-ground notion between special and tertispecial double posets having a less daunting definition.

Definition 3.2. If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, then an $\mathbf{E}$-*partition* shall mean a map $\phi : E \to \{1, 2, 3, \ldots\}$ such that:

- every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$;
- every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

Example 3.3. The notion of an $\mathbf{E}$-partition (which was inspired by the earlier notions of P-partitions and (P, ω)-partitions as studied by Gessel and Stanley[14]) generalizes various well-known combinatorial concepts. For example:

- If $<_2$ is the same order as $<_1$ (or any extension of this order), then the $\mathbf{E}$-partitions are the weakly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.
- If $<_2$ is the opposite relation of $<_1$ (or any extension of this opposite relation), then the $\mathbf{E}$-partitions are the strictly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

(See the Appendix (specifically, Proposition 10.20 and Proposition 10.21) for the proofs of these two facts.)

For a more interesting example, let $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be two partitions such that $\mu \subseteq \lambda$. (See [GriRei14, §2] for the notations we are using here.) The skew Young diagram $Y(\lambda/\mu)$ is then defined as the set of all $(i, j) \in \{1, 2, 3, \ldots\}^2$ satisfying $\mu_i < j \leq \lambda_i$. On this set $Y(\lambda/\mu)$, we define two strict partial orders $<_1$ and $<_2$ by

$$(i,j) <_1 (i',j') \iff \left(i \leq i' \text{ and } j \leq j' \text{ and } (i,j) \neq (i',j')\right)$$

and

$$(i,j) <_2 (i',j') \iff \left(i \geq i' \text{ and } j \leq j' \text{ and } (i,j) \neq (i',j')\right).$$

The resulting double poset $\mathbf{Y}(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_2)$ has the property that the $\mathbf{Y}(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ. (Again, see [GriRei14, §2] for the meaning of these words. Also, see the Appendix (specifically, Proposition 10.23 (a)) for a proof of our claim that the $\mathbf{Y}(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ.)

This double poset $\mathbf{Y}(\lambda/\mu)$ is not special (in general), but it is tertispecial. (Indeed, if a and b are two elements of $Y(\lambda/\mu)$ such that a is $<_1$-covered by b, then a is either the left neighbor of b or the top neighbor of b, and thus we have either $a <_2 b$ (in the former case) or $b <_2 a$ (in the latter case).) Some authors prefer to use a special double poset instead, which is defined as follows: We define a total order $<_h$ on $Y(\lambda/\mu)$ by

$$(i,j) <_h (i',j') \iff \left(i > i' \text{ or } \left(i = i' \text{ and } j < j'\right)\right).$$

Then, $\mathbf{Y}_h(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_h)$ is a special double poset, and the $\mathbf{Y}_h(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ. (See the Appendix (specifically, Proposition 10.23 (b)) for a proof of the latter claim.)

We now assign a certain formal power series to every double poset:

Definition 3.4. If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ is a map, then we define a power series $\Gamma(\mathbf{E}, w) \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}, \qquad \text{where } \mathbf{x}_{\pi,w} = \prod_{e \in E} x_{\pi(e)}^{w(e)}.$$

(See the Appendix (specifically, Proposition 10.19) for a proof that this sum is well-defined.)

The following fact is easy to see (but will be reproven below):

Proposition 3.5. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then, $\Gamma(\mathbf{E}, w) \in \mathrm{QSym}$.

Example 3.6. The power series $\Gamma(\mathbf{E}, w)$ generalize various well-known quasisymmetric functions.

(a) If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ is the constant function sending everything to 1, then $\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_\pi$, where $\mathbf{x}_\pi = \prod_{e \in E} x_{\pi(e)}$. We shall denote this power series $\Gamma(\mathbf{E}, w)$ by $\Gamma(\mathbf{E})$; it is exactly what has been called $\Gamma(\mathbf{E})$ in [MalReu09, §2.2]. All results proven below for $\Gamma(\mathbf{E}, w)$ can be applied to $\Gamma(\mathbf{E})$, yielding simpler (but less general) statements.

(b) If $E = \{1, 2, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$, if $<_1$ is the usual total order inherited from $\mathbb{Z}$, and if $<_2$ is the opposite relation of $<_1$, then the special double poset $\mathbf{E} = (E, <_1, <_2)$ satisfies $\Gamma(\mathbf{E}, w) = M_\alpha$, where α is the composition $(w(1), w(2), \ldots, w(\ell))$. (See the Appendix (specifically, Proposition 10.26) for a proof of this.)

Note that every M_α can be obtained this way (by choosing ℓ and w appropriately). Thus, the elements of the monomial basis $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ are special cases of the functions $\Gamma(\mathbf{E}, w)$. This shows that the $\Gamma(\mathbf{E}, w)$ for varying $\mathbf{E}$ and w span the $\mathbf{k}$-module QSym.

(c) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Let $D(\alpha)$ be the set $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}$. Let E

[14]See [Gessel15] for the history of these notions, and see [Gessel84], [Stan71], [Stan11, §3.15] and [Stan99, §7.19] for some of their theory. Mind that these sources use different and sometimes incompatible notations – e.g., the P-partitions of [Stan11, §3.15] and [Gessel15] differ from those of [Gessel84] by a sign reversal.

be the set $\{1,2,\ldots,n\}$, and let $<_1$ be the total order inherited on E from $\mathbb{Z}$. Let $<_2$ be some partial order on E with the property that

$$i+1 <_2 i \qquad \text{for every } i \in D(\alpha)$$

and

$$i <_2 i+1 \qquad \text{for every } i \in \{1,2,\ldots,n-1\} \setminus D(\alpha).$$

(There are several choices for such an order; in particular, we can find one which is a total order. Indeed, this is proven in the Appendix (specifically, Proposition 10.62).) Then, a simple argument (explained in detail in the Appendix, in the proof of Proposition 10.65) shows that

$$\Gamma((E,<_1,<_2)) = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\beta \text{ is a composition of } n;\ D(\beta) \supseteq D(\alpha)} M_\beta.$$

This power series is known as the α-th *fundamental quasisymmetric function*, usually called F_α (in [Gessel84], [MalReu95, §2], [BBSSZ13, §2.4] and [Grin14, §2]) or L_α (in [Stan99, §7.19] or [GriRei14, Definition 5.15]).

(d) Let $\mathbf{E}$ be one of the two double posets $\mathbf{Y}(\lambda/\mu)$ and $\mathbf{Y}_h(\lambda/\mu)$ defined as in Example 3.3 for two partitions μ and λ. Then, $\Gamma(\mathbf{E})$ is the skew Schur function $s_{\lambda/\mu}$.

(e) Similarly, *dual immaculate functions* as defined in [BBSSZ13, §3.7] can be realized as $\Gamma(\mathbf{E})$ for conveniently chosen $\mathbf{E}$ (see [Grin14, Proposition 4.4]), which helped the author to prove one of their properties [Grin14]. (The $\mathbf{E}$-partitions here are the so-called *immaculate tableaux*.)

(f) When the relation $<_2$ of a double poset $\mathbf{E} = (E,<_1,<_2)$ is a total order (i.e., when the double poset $\mathbf{E}$ is special), the $\mathbf{E}$-partitions are precisely the reverse (P,ω)-partitions (for $P = (E,<_1)$ and ω being the unique bijection $E \to \{1,2,\ldots,|E|\}$ satisfying $\omega^{-1}(1) <_2 \omega^{-1}(2) <_2 \cdots <_2 \omega^{-1}(|E|)$) in the terminology of [Stan99, §7.19], and the power series $\Gamma(\mathbf{E})$ is the $K_{P,\omega}$ of [Stan99, §7.19]. This can also be rephrased using the notations of [GriRei14, §5.2]: When the relation $<_2$ of a double poset $\mathbf{E} = (E,<_1,<_2)$ is a total order, we can relabel the elements of E by the integers $1,2,\ldots,n$ (where $n = |E|$) in such a way that $1 <_2 2 <_2 \cdots <_2 n$; then, the $\mathbf{E}$-partitions are the P-partitions in the terminology of [GriRei14, Definition 5.12], where P is the labelled poset $(E,<_1)$; and furthermore, our $\Gamma(\mathbf{E})$ is the $F_P(\mathbf{x})$ of [GriRei14, Definition 5.12]. Conversely, if P is a labelled poset, then the $F_P(\mathbf{x})$ of [GriRei14, Definition 5.12] is our $\Gamma((P,<_P,<_\mathbb{Z}))$.

4. The antipode theorem

We now come to the main results of this note. We first state a theorem and a corollary which are not new, but will be reproven in a more self-contained way which allows them to take their (well-deserved) place as fundamental results rather than afterthoughts in the theory of QSym.

Definition 4.1. We let S denote the antipode of QSym.

Theorem 4.2. Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, $S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w)$, where $>_1$ denotes the opposite relation of $<_1$.

Corollary 4.3. Let $(E, <_1, <_2)$ be a tertispecial double poset. Then, $S(\Gamma((E, <_1, <_2))) = (-1)^{|E|} \Gamma((E, >_1, <_2))$, where $>_1$ denotes the opposite relation of $<_1$.

We shall give examples for consequences of these facts shortly (Example 4.8), but let us first explain where they have already appeared. Corollary 4.3 is equivalent to [GriRei14, Corollary 5.27][15] (a result found by Malvenuto and Reutenauer [MalReu98, Lemma 3.2]). Theorem 4.2 is equivalent to Malvenuto's and Reutenauer's [MalReu98, Theorem 3.1][16]. We nevertheless believe that our versions of these facts

[15] It is easiest to derive [GriRei14, Corollary 5.27] from our Corollary 4.3, as this only requires setting $\mathbf{E} = (P, <_P, <_{\mathbb{Z}})$ (this is a special double poset, thus in particular a tertispecial one) and noticing that $\Gamma((P, <_P, <_{\mathbb{Z}})) = F_P(\mathbf{x})$ and $\Gamma((P, >_P, <_{\mathbb{Z}})) = F_{P^{\mathrm{opp}}}(\mathbf{x})$, where all unexplained notations are defined in [GriRei14, Chapter 5]. But one can also proceed in the opposite direction (hint: replace the partial order $<_2$ by a linear extension, thus turning the tertispecial double poset $(E, <_1, <_2)$ into a special one; argue that this does not change $\Gamma((E, <_1, <_2))$ and $\Gamma((E, >_1, <_2))$).

[16] This equivalence requires some work to set up. First of all, Malvenuto and Reutenauer, in [MalReu98], do not work with the antipode S of QSym, but instead study a certain automorphism of QSym called ω. However, this automorphism is closely related to S (namely, for each $n \in \mathbb{N}$ and each homogeneous element $f \in \mathrm{QSym}$ of degree n, we have $\omega(f) = (-1)^n S(f)$); therefore, any statements about ω can be translated into statements about S and vice versa.

Let me sketch how to derive [MalReu98, Theorem 3.1] from our Theorem 4.2. Indeed, contract all undirected edges in G and G', denoting the (common) vertex set of the new graphs by E. Then, define two strict partial orders $<_1$ and $<_2$ on E by

$$(a <_1 b) \iff (a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G)$$

and

$$(a <_2 b) \iff \left(a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G'\right).$$

The map w sends every $e \in E$ to the number of vertices of G that became e when the edges were contracted. To show that the resulting double poset $(E, <_1, <_2)$ is tertispecial, we must notice that if a is $<_1$-covered by b, then G had an edge from one of the vertices that became a to one of the vertices that became b. The "x_i's in X satisfying a set of conditions" (in the language of [MalReu98, Section 3]) are in 1-to-1 correspondence with $(E, <_1, <_2)$-partitions (at least when $X = \{1, 2, 3, \ldots\}$); this is not immediately obvious but not hard to check either (the acyclicity of are slicker and simpler than the ones appearing in existing literature[17], and if not, then at least our proofs below are more natural.

To these known results, we add another, which seems to be unknown so far (probably because it is far harder to state in the terminologies of (P, ω)-partitions or equality-and-inequality conditions appearing in literature). First, we need to introduce some notation:

Definition 4.4. Let G be a group, and let E be a G-set.

(a) Let $<$ be a strict partial order on E. We say that G *preserves the relation* $<$ if the following holds: For every $g \in G$, $a \in E$ and $b \in E$ satisfying $a < b$, we have $ga < gb$.

(b) Let $w : E \to \{1, 2, 3, \ldots\}$. We say that G *preserves* w if every $g \in G$ and $e \in E$ satisfy $w(ge) = w(e)$.

(c) Let $g \in G$. Assume that the set E is finite. We say that g is *E-even* if the action of g on E (that is, the permutation of E that sends every $e \in E$ to ge) is an even permutation of E.

(d) If X is any set, then the set X^E of all maps $E \to X$ becomes a G-set in the following way: For any $\pi \in X^E$ and $g \in G$, we define the element $g\pi \in X^E$ to be the map sending each $e \in E$ to $\pi(g^{-1}e)$.

(e) Let F be a further G-set. Assume that the set E is finite. An element $\pi \in F$ is said to be *E-coeven* if every $g \in G$ satisfying $g\pi = \pi$ is E-even. A G-orbit O on F is said to be *E-coeven* if all elements of O are E-coeven.

Before we come to the promised result, let us state two simple facts:

Lemma 4.5. Let G be a group. Let F and E be G-sets such that E is finite. Let O be a G-orbit on F. Then, O is E-coeven if and only if at least one element of O is E-coeven.

Proposition 4.6. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $\operatorname{Par}\mathbf{E}$ denote the set of all $\mathbf{E}$-partitions. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$.

(a) Then, $\operatorname{Par}\mathbf{E}$ is a G-subset of the G-set $\{1, 2, 3, \ldots\}^E$ (see Definition 4.4 (d) for the definition of the latter).

G and G' is used in the proof). As a result, [MalReu98, Theorem 3.1] follows from Theorem 4.2 above. With some harder work, one can conversely derive our Theorem 4.2 from [MalReu98, Theorem 3.1].

[17] That said, we would not be surprised if Malvenuto and Reutenauer are aware of them; after all, they have discovered both the original version of Theorem 4.2 in [MalReu98] and the notion of double posets in [MalReu09].

(b) Let $w : E \to \{1, 2, 3, \ldots\}$. Assume that G preserves w. Let O be a G-orbit on $\operatorname{Par} \mathbf{E}$. Then, the values of $\mathbf{x}_{\pi,w}$ for all $\pi \in O$ are equal.

Theorem 4.7. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $\operatorname{Par} \mathbf{E}$ denote the set of all $\mathbf{E}$-partitions. Let $w : E \to \{1, 2, 3, \ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. Then, G acts also on the set $\operatorname{Par} \mathbf{E}$ of all $\mathbf{E}$-partitions; namely, $\operatorname{Par} \mathbf{E}$ is a G-subset of the G-set $\{1, 2, 3, \ldots\}^E$ (according to Proposition 4.6 (a)). For any G-orbit O on $\operatorname{Par} \mathbf{E}$, we define a monomial $\mathbf{x}_{O,w}$ by

$$\mathbf{x}_{O,w} = \mathbf{x}_{\pi,w} \qquad \text{for some element } \pi \text{ of } O.$$

(This is well-defined, since Proposition 4.6 (b) shows that $\mathbf{x}_{\pi,w}$ does not depend on the choice of $\pi \in O$.)

Let

$$\Gamma(\mathbf{E}, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par} \mathbf{E}} \mathbf{x}_{O,w}$$

and

$$\Gamma^+(\mathbf{E}, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par} \mathbf{E}} \mathbf{x}_{O,w}.$$

Then, $\Gamma(\mathbf{E}, w, G)$ and $\Gamma^+(\mathbf{E}, w, G)$ belong to QSym and satisfy

$$S(\Gamma(\mathbf{E}, w, G)) = (-1)^{|E|} \Gamma^+((E, >_1, <_2), w, G).$$

Here, $>_1$ denotes the opposite relation of $<_1$.

This theorem, which combines Theorem 4.2 with the ideas of Pólya enumeration, is inspired by Jochemko's reciprocity result for order polynomials [Joch13, Theorem 2.8], which can be obtained from it by specializations (see Section 8 for the details of how Jochemko's result follows from ours).

We shall now briefly review a number of particular cases of Theorem 4.2.

Example 4.8. (a) Corollary 4.3 follows from Theorem 4.2 by letting w be the function which is constantly 1.

(b) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n, and let $\mathbf{E} = (E, <_1, <_2)$ be the double poset defined in Example 3.6 (b). Let $w : \{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ be the map sending every i to α_i. As Example 3.6 (b) shows, we have $\Gamma(\mathbf{E}, w) = M_\alpha$. Thus, applying Theorem 4.2 to these $\mathbf{E}$ and w and performing some manipulations (see Proposition 10.70

in the Appendix for the details) yields

$$S(M_\alpha) = (-1)^\ell \Gamma((E, >_1, <_2), w) = (-1)^\ell \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$
$$= (-1)^\ell \sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1} = (-1)^\ell \sum_{\substack{\gamma \text{ is a composition of } n; \\ D(\gamma) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\gamma.$$

This is the formula for $S(M_\alpha)$ given in [Ehrenb96, Proposition 3.4], in [Malve93, (4.26)], in [GriRei14, Theorem 5.11], and in [BenSag14, Theorem 4.1] (originally due to Ehrenborg and to Malvenuto and Reutenauer).

(c) Applying Corollary 4.3 to the double poset of Example 3.6 (c) (where the relation $<_2$ is chosen to be a total order) yields the formula for the antipode of a fundamental quasisymmetric function ([Malve93, (4.27)], [GriRei14, (5.9)], [BenSag14, Theorem 5.1]).

(d) Let us use the notations of Example 3.3. For any partition λ, let λ^t denote the conjugate partition of λ. Let μ and λ be two partitions satisfying $\mu \subseteq \lambda$. Let $>_1$ and $>_2$ be the opposite relations of $<_1$ and $<_2$. Then, there is a bijection $\tau : Y(\lambda/\mu) \to Y(\lambda^t/\mu^t)$ sending each $(i,j) \in Y(\lambda/\mu)$ to (j,i). This bijection is an isomorphism of double posets from $(Y(\lambda/\mu), >_1, <_2)$ to $(Y(\lambda^t/\mu^t), >_1, >_2)$ (where the notion of an "isomorphism of double posets" is defined in the natural way – i.e., an isomorphism of double posets is a bijection ϕ between their ground sets such that each of the two maps ϕ and ϕ^{-1} preserves each of the two orders). Hence,

$$\Gamma((Y(\lambda/\mu), >_1, <_2)) = \Gamma((Y(\lambda^t/\mu^t), >_1, >_2)). \tag{3}$$

But applying Corollary 4.3 to the tertispecial double poset $\mathbf{Y}(\lambda/\mu)$, we obtain

$$S(\Gamma(\mathbf{Y}(\lambda/\mu))) = (-1)^{|\lambda/\mu|} \Gamma((Y(\lambda/\mu), >_1, <_2))$$
$$= (-1)^{|\lambda/\mu|} \Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) \tag{4}$$

(by (3)). But from Example 3.6 (d), we know that $\Gamma(\mathbf{Y}(\lambda/\mu)) = s_{\lambda/\mu}$. Moreover, a similar argument using [GriRei14, Remark 2.12] shows that $\Gamma((Y(\lambda/\mu), >_1, >_2)) = s_{\lambda/\mu}$. Applying this to λ^t and μ^t instead of λ and μ, we obtain $\Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) = s_{\lambda^t/\mu^t}$. Now, (4) rewrites as

$$S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t} \tag{5}$$

(since $\Gamma(\mathbf{Y}(\lambda/\mu)) = s_{\lambda/\mu}$ and $\Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) = s_{\lambda^t/\mu^t}$). This is a well-known formula, and is usually stated for S being the antipode of the Hopf algebra of symmetric (rather than quasisymmetric) functions; but this is an equivalent statement, since the latter antipode is a restriction of the antipode of QSym.

It is also possible (but more difficult) to derive (5) by using the double poset $\mathbf{Y}_h(\lambda/\mu)$ instead of $\mathbf{Y}(\lambda/\mu)$. (This boils down to what was done in [GriRei14, proof of Corollary 5.29].)

(e) A result of Benedetti and Sagan [BenSag14, Theorem 8.2] on the antipodes of immaculate functions can be obtained from Corollary 4.3 using dualization.

Remark 4.9. Corollary 4.3 has a sort of converse. Namely, let us assume that $\mathbf{k} = \mathbb{Z}$. If $(E, <_1, <_2)$ is a double poset satisfying $S(\Gamma((E, <_1, <_2))) = (-1)^{|E|}\Gamma((E, >_1, <_2))$, then $(E, <_1, <_2)$ is tertispecial.

More precisely, the following holds: Define the *length* $\ell(\alpha)$ of a composition α to be the number of entries of α. Define the *size* $|\alpha|$ of a composition α to be the sum of the entries of α. Let $\eta : \mathrm{QSym} \to \mathrm{QSym}$ be the $\mathbf{k}$-linear map defined by

$$\eta(M_\alpha) = \begin{cases} M_\alpha, & \text{if } \ell(\alpha) \geq |\alpha| - 1; \\ 0, & \text{if } \ell(\alpha) < |\alpha| - 1 \end{cases} \qquad \text{for every } \alpha \in \mathrm{Comp}.$$

Thus, η transforms a quasisymmetric function by removing all monomials $\mathfrak{m}$ for which the number of indeterminates appearing in $\mathfrak{m}$ is $< \deg \mathfrak{m} - 1$. We partially order the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by a coefficientwise order (i.e., two power series a and b satisfy $a \leq b$ if and only if each coefficient of a is $\leq$ to the corresponding coefficient of b). Now, every double poset $(E, <_1, <_2)$ satisfies

$$\eta\left((-1)^{|E|} S(\Gamma((E, <_1, <_2)))\right) \leq \eta(\Gamma((E, >_1, <_2))), \tag{6}$$

and equality holds if and only if the double poset $(E, <_1, <_2)$ is tertispecial. (If we omit η, then the inequality fails in general.)

The proof of (6) is somewhat technical, but not too difficult. I shall only give a rough outline, as the result is tangential to this paper. Fix a double poset $(E, <_1, <_2)$, and set $n = |E|$ and $[n] = \{1, 2, \ldots, n\}$. A *costrictor* will mean a bijection $\phi : [n] \to E$ whose inverse $\phi^{-1} : E \to [n]$ is a strictly increasing map from the poset $(E, <_1)$ to $([n], <_{\mathbb{Z}})$. (The costrictors are in 1-to-1 correspondence with the linear extensions of $(E, <_1)$.) For two elements e and f of E, we write $e \parallel_1 f$ if and only if e and f are not $<_1$-comparable. Whenever $k \in \mathbb{N}$, we shall use the notation 1^k for "k ones, written in a row"; thus, for example, $(3, 1^5, 4)$ is the composition $(3, 1, 1, 1, 1, 1, 4)$. Then, it is not hard to see that

$$\begin{aligned} &\Gamma((E, <_1, <_2)) \\ &= (\text{the number of all costrictors})\, M_{(1^n)} + \sum_{k=1}^{n-1} \gamma_{(E,<_1,<_2),k} M_{\left(1^{k-1}, 2, 1^{n-k-1}\right)} \\ &\qquad + (\text{a linear combination of } M_\alpha \text{ with } |\alpha| = n \text{ and } \ell(\alpha) < n-1), \end{aligned}$$

where

$$\begin{aligned}
&\gamma_{(E,<_1,<_2),k} \\
&= (\text{the number of all costrictors}) \\
&\qquad - \frac{1}{2} (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) \parallel_1 \phi(k+1)) \\
&\qquad - (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) <_1 \phi(k+1) \\
&\qquad\qquad \text{and } \phi(k) >_2 \phi(k+1)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\eta\left(\Gamma\left((E,<_1,<_2)\right)\right) \\
&= (\text{the number of all costrictors}) M_{(1^n)} + \sum_{k=1}^{n-1} \gamma_{(E,<_1,<_2),k} M_{\left(1^{k-1},2,1^{n-k-1}\right)}.
\end{aligned}$$

Using this (and the formula for $S(M_\alpha)$ in Example 4.8 (b)), it is easy to show that

$$\begin{aligned}
&\eta\left((-1)^{|E|} S\left(\Gamma\left((E,<_1,<_2)\right)\right)\right) \\
&= (\text{the number of all costrictors}) M_{(1^n)} \\
&\qquad + \sum_{k=1}^{n-1} \left((\text{the number of all costrictors}) - \gamma_{(E,<_1,<_2),k}\right) M_{\left(1^{n-k-1},2,1^{k-1}\right)}.
\end{aligned}$$

But we can also define an *anticostrictor* as a bijection $\phi : [n] \to E$ whose inverse $\phi^{-1} : E \to [n]$ is a strictly decreasing map from the poset $(E,<_1)$ to $([n], <_{\mathbb{Z}})$. Then, similarly to our formula for $\eta\left(\Gamma\left((E,<_1,<_2)\right)\right)$, we can derive the formula

$$\begin{aligned}
&\eta\left(\Gamma\left((E,>_1,<_2)\right)\right) \\
&= (\text{the number of all anticostrictors}) M_{(1^n)} + \sum_{k=1}^{n-1} \widetilde{\gamma}_{(E,<_1,<_2),k} M_{\left(1^{k-1},2,1^{n-k-1}\right)},
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{\gamma}_{(E,<_1,<_2),k} \\
&= (\text{the number of all anticostrictors}) \\
&\qquad - \frac{1}{2} (\text{the number of all anticostrictors } \phi \text{ satisfying } \phi(k) \parallel_1 \phi(k+1)) \\
&\qquad - (\text{the number of all anticostrictors } \phi \text{ satisfying } \phi(k) >_1 \phi(k+1) \\
&\qquad\qquad \text{and } \phi(k) >_2 \phi(k+1)).
\end{aligned}$$

Recall that we want to prove (6). In light of our formulas for $\eta\left((-1)^{|E|} S\left(\Gamma\left((E,<_1,<_2)\right)\right)\right)$ and $\eta\left(\Gamma\left((E,>_1,<_2)\right)\right)$, this boils down to proving the following two facts:

1. The number of all costrictors is $\leq$ to the number of all anticostrictors.

2. For each $k \in \{1, 2, \ldots, n-1\}$, we have

$$(\text{the number of all costrictors}) - \gamma_{(E,<_1,<_2),k} \leq \widetilde{\gamma}_{(E,<_1,<_2),n-k}.$$

But the first of these two facts is easy to see: Let w_0 be the involution $[n] \to [n]$, $i \mapsto n+1-i$. Then, w_0 is a poset isomorphism $([n], <_{\mathbb{Z}}) \to ([n], >_{\mathbb{Z}})$. Hence, there is a 1-to-1 correspondence between the costrictors and the anticostrictors, given by $\phi \mapsto \phi \circ w_0$. Thus, the number of all costrictors equals the number of all anticostrictors. This proves Fact 1.

Proving Fact 2 is harder. Fix $k \in \{1, 2, \ldots, n-1\}$. Recalling our definition of $\gamma_{(E,<_1,<_2),k}$ and $\widetilde{\gamma}_{(E,<_1,<_2),n-k}$, we notice that we must show

$$\begin{aligned}
&\frac{1}{2} (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) \parallel_1 \phi(k+1)) \\
&\qquad + (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) <_1 \phi(k+1) \\
&\qquad\qquad \text{and } \phi(k) >_2 \phi(k+1)) \\
&\leq (\text{the number of all anticostrictors}) \\
&\qquad - \frac{1}{2} (\text{the number of all anticostrictors } \phi \text{ satisfying } \phi(n-k) \parallel_1 \phi(n-k+1)) \\
&\qquad - (\text{the number of all anticostrictors } \phi \text{ satisfying } \phi(k) >_1 \phi(n-k+1) \\
&\qquad\qquad \text{and } \phi(k) >_2 \phi(n-k+1)).
\end{aligned}$$

Using the 1-to-1 correspondence between the costrictors and the anticostrictors (which we already used in the proof of Fact 1), we can rewrite this as

$$\begin{aligned}
&\frac{1}{2} (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) \parallel_1 \phi(k+1)) \\
&\qquad + (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) <_1 \phi(k+1) \\
&\qquad\qquad \text{and } \phi(k) >_2 \phi(k+1)) \\
&\leq (\text{the number of all costrictors}) \\
&\qquad - \frac{1}{2} (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) \parallel_1 \phi(k+1)) \\
&\qquad - (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) <_1 \phi(k+1) \\
&\qquad\qquad \text{and } \phi(k) <_2 \phi(k+1))
\end{aligned}$$

(here, we have used the fact that $(\phi \circ w_0)(n-k) = \phi\left(\underbrace{w_0(n-k)}_{=k+1}\right) = \phi(k+1)$

and $(\phi \circ w_0)(n-k+1) = \phi\left(\underbrace{w_0(n-k+1)}_{=k}\right) = \phi(k)$). This simplifies to

$$\begin{aligned}
&(\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) \parallel_1 \phi(k+1)) \\
&\qquad + (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) <_1 \phi(k+1) \\
&\qquad\qquad \text{and } \phi(k) >_2 \phi(k+1)) \\
&\qquad + (\text{the number of all costrictors } \phi \text{ satisfying } \phi(k) <_1 \phi(k+1) \\
&\qquad\qquad \text{and } \phi(k) <_2 \phi(k+1)) \\
&\leq (\text{the number of all costrictors}).
\end{aligned}$$

This inequality is clearly satisfied (since each costrictor ϕ satisfies at most one of the relations $\phi(k) \parallel_1 \phi(k+1)$, $(\phi(k) <_1 \phi(k+1)$ and $\phi(k) >_2 \phi(k+1))$ and $(\phi(k) <_1 \phi(k+1)$ and $\phi(k) <_2 \phi(k+1))$). Thus, the inequality (6) is proven. It now remains to show that equality holds only when the double poset $(E, <_1, <_2)$ is tertispecial.

Indeed, assume that $(E, <_1, <_2)$ is not tertispecial. Then, there exist two elements a and b of E such that a is $<_1$-covered by b but a and b are not $<_2$-comparable. Consider such a and b. There exists at least one pair (ϕ, k) of a costrictor ϕ and an element $k \in \{1, 2, \ldots, n-1\}$ satisfying $\phi(k) = a$ and $\phi(k+1) = b$. (In fact, in order to construct such a pair, we write our set E as the disjoint union $E = E_1 \cup \{a\} \cup \{b\} \cup E_2$, where $E_1 = \{e \in E \mid e <_1 b \text{ and } e \neq a\}$ and $E_2 = \{e \in E \mid \text{neither } e <_1 b \text{ nor } e = b\}$. Then, we set $k = |E_1| + 1$, and choose strictly increasing bijections $\alpha : E_1 \to [k-1]$ and $\beta : E_2 \to [n-k-1]$. Finally, we define a map $\gamma : E \to [n]$ by $\gamma(e) = \begin{cases} \alpha(e), & \text{if } e \in E_1; \\ k, & \text{if } e = a; \\ k+1, & \text{if } e = b; \\ k+1+\beta(e), & \text{if } e \in E_2 \end{cases}$, and define ϕ to be γ^{-1}. It is not hard to check that ϕ is a costrictor.) This causes one of the inequalities from which we obtained (6) to be strict. This completes the (outline of the) proof.

5. Lemmas: packed E-partitions and comultiplications

We shall now prepare for the proofs of our results. To this end, we introduce the notion of a *packed map*.

Definition 5.1. (a) An *initial interval* will mean a set of the form $\{1, 2, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$.

(b) If E is a set and $\pi : E \to \{1,2,3,\ldots\}$ is a map, then π is said to be *packed* if $\pi(E)$ is an initial interval. Clearly, this initial interval must be $\{1,2,\ldots,|\pi(E)|\}$. (Indeed, this follows from Proposition 5.2 (a), applied to $\ell = |\pi(E)|$.)

Proposition 5.2. Let E be a set. Let $\pi : E \to \{1,2,3,\ldots\}$ be a packed map. Let $\ell = |\pi(E)|$.

(a) We have $\pi(E) = \{1,2,\ldots,\ell\}$.

(b) Let $w : E \to \{1,2,3,\ldots\}$ be a map. For each $i \in \{1,2,\ldots,\ell\}$, define an integer α_i by $\alpha_i = \sum\limits_{e \in \pi^{-1}(i)} w(e)$. Then, $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition.

Proof of Proposition 5.2. The map π is packed. In other words, $\pi(E)$ is an initial interval (by the definition of "packed"). In other words, $\pi(E) = \{1,2,\ldots,k\}$ for some $k \in \mathbb{N}$. Consider this k. From $\pi(E) = \{1,2,\ldots,k\}$, we obtain $|\pi(E)| = |\{1,2,\ldots,k\}| = k$, so that $k = |\pi(E)| = \ell$. Now, $\pi(E) = \{1,2,\ldots,k\} = \{1,2,\ldots,\ell\}$ (since $k = \ell$). This proves Proposition 5.2 (a).

(b) Let $i \in \{1,2,\ldots,\ell\}$. Then, $i \in \{1,2,\ldots,\ell\} = \pi(E)$. Hence, there exists some $f \in E$ such that $i = \pi(f)$. Consider this f. We have $f \in \pi^{-1}(i)$ (since $\pi(f) = i$). Also, $w(f) \in \{1,2,3,\ldots\}$ (since w is a map $E \to \{1,2,3,\ldots\}$). Now,

$$\alpha_i = \sum_{e \in \pi^{-1}(i)} w(e) = \underbrace{w(f)}_{>0} + \sum_{\substack{e \in \pi^{-1}(i);\\ e \neq f}} \underbrace{w(e)}_{\substack{\geq 0 \\ (\text{since } w(e) \in \{1,2,3,\ldots\})}}$$
$$\left(\begin{array}{c}\text{here, we have split off the addend for } e = f \text{ from the sum} \\ (\text{since } f \in \pi^{-1}(i))\end{array}\right)$$
$$> 0 + \sum_{\substack{e \in \pi^{-1}(i);\\ e \neq f}} 0 = 0.$$

Thus, α_i is a positive integer.

Now, forget that we have fixed i. We thus have shown that α_i is a positive integer for each $i \in \{1,2,\ldots,\ell\}$. In other words, $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a finite list of positive integers, i.e., a composition. This proves Proposition 5.2 (b). $\square$

Definition 5.3. Let E be a set. Let $\pi : E \to \{1,2,3,\ldots\}$ be a packed map. Let $w : E \to \{1,2,3,\ldots\}$ be a map. Then, the composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ defined in Proposition 5.2 (b) will be denoted by $\operatorname{ev}_w \pi$.

Proposition 5.4. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1,2,3,\ldots\}$ be a map. Then,

$$\Gamma(\mathbf{E}, w) = \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\operatorname{ev}_w \varphi}. \tag{7}$$

Proof of Proposition 5.4. For every finite subset T of $\{1,2,3,\ldots\}$, there exists a unique strictly increasing bijection $\{1,2,\ldots,|T|\} \to T$. We shall denote this bijection by r_T. For every map $\pi : E \to \{1,2,3,\ldots\}$, we define the *packing of* π as the map $r_{\pi(E)}^{-1} \circ \pi : E \to \{1,2,3,\ldots\}$; this is a packed map (indeed, its image is $\{1,2,\ldots,|\pi(E)|\}$), and will be denoted by $\operatorname{pack}\pi$. This map $\operatorname{pack}\pi$ is an $\mathbf{E}$-partition if and only if π is an $\mathbf{E}$-partition[18]. Hence, $\operatorname{pack}\pi$ is a packed $\mathbf{E}$-partition for every $\mathbf{E}$-partition π.

We shall show that for every packed $\mathbf{E}$-partition φ, we have

$$\sum_{\pi \text{ is an } \mathbf{E}\text{-partition; } \operatorname{pack}\pi=\varphi} \mathbf{x}_{\pi,w} = M_{\operatorname{ev}_w \varphi}. \tag{8}$$

Once this is proven, it will follow that

$$\begin{aligned}\Gamma(\mathbf{E},w) &= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w} = \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition; } \operatorname{pack}\pi=\varphi} \mathbf{x}_{\pi,w}}_{\substack{=M_{\operatorname{ev}_w \varphi} \\ \text{(by (8))}}} \\ &\qquad (\text{since } \operatorname{pack}\pi \text{ is a packed } \mathbf{E}\text{-partition for every } \mathbf{E}\text{-partition } \pi) \\ &= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\operatorname{ev}_w \varphi},\end{aligned}$$

and Proposition 5.4 will be proven.

So it remains to prove (8). Let φ be a packed $\mathbf{E}$-partition. Hence, φ is a packed map $E \to \{1,2,3,\ldots\}$. Let $\ell = |\varphi(E)|$; thus $\varphi(E) = \{1,2,\ldots,\ell\}$ (by Proposition 5.2 (a) (applied to ψ instead of π)). Let $\alpha_i = \sum_{e\in\varphi^{-1}(i)} w(e)$ for every $i \in \{1,2,\ldots,\ell\}$; thus, $\operatorname{ev}_w \varphi = (\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ (by the definition of $\operatorname{ev}_w \varphi$). Hence, the definition of

[18] Indeed, $\operatorname{pack}\pi = r_{\pi(E)}^{-1} \circ \pi$. Since $r_{\pi(E)}$ is strictly increasing, we thus see that, for any given $e \in E$ and $f \in E$, the equivalences

$$((\operatorname{pack}\pi)(e) \le (\operatorname{pack}\pi)(f)) \iff (\pi(e) \le \pi(f))$$

and

$$((\operatorname{pack}\pi)(e) < (\operatorname{pack}\pi)(f)) \iff (\pi(e) < \pi(f))$$

hold. Hence, $\operatorname{pack}\pi$ is an $\mathbf{E}$-partition if and only if π is an $\mathbf{E}$-partition.

$M_{\operatorname{ev}_w \varphi}$ yields

$$M_{\operatorname{ev}_w \varphi} = \sum_{i_1<i_2<\cdots<i_\ell} \underbrace{x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}}_{=\prod_{k=1}^{\ell} x_{i_k}^{\alpha_k}} = \sum_{i_1<i_2<\cdots<i_\ell} \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\alpha_k}}_{\substack{=x_{i_k}^{\sum_{e\in\varphi^{-1}(k)} w(e)} \\ \text{(since } \alpha_k = \sum_{e\in\varphi^{-1}(k)} w(e))}}$$

$$= \sum_{i_1<i_2<\cdots<i_\ell} \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\sum_{e\in\varphi^{-1}(k)} w(e)}}_{\substack{=\prod_{e\in\varphi^{-1}(k)} x_{i_k}^{w(e)} \\ =\prod_{e\in E;\ \varphi(e)=k} x_{i_k}^{w(e)}}} = \sum_{i_1<i_2<\cdots<i_\ell} \prod_{k=1}^{\ell} \prod_{e\in E;\ \varphi(e)=k} \underbrace{x_{i_k}^{w(e)}}_{\substack{=x_{i_{\varphi(e)}}^{w(e)} \\ \text{(since } k=\varphi(e))}}$$

$$= \sum_{i_1<i_2<\cdots<i_\ell} \underbrace{\prod_{k=1}^{\ell} \prod_{e\in E;\ \varphi(e)=k} x_{i_{\varphi(e)}}^{w(e)}}_{=\prod_{e\in E} x_{i_{\varphi(e)}}^{w(e)}} = \sum_{i_1<i_2<\cdots<i_\ell} \prod_{e\in E} x_{i_{\varphi(e)}}^{w(e)}$$

$$= \sum_{T\subseteq\{1,2,3,\ldots\};\ |T|=\ell} \prod_{e\in E} x_{r_T(\varphi(e))}^{w(e)}$$

[19]. Hence,

$$M_{\operatorname{ev}_w \varphi} = \sum_{T\subseteq\{1,2,3,\ldots\};\ |T|=\ell} \underbrace{\prod_{e\in E} x_{r_T(\varphi(e))}^{w(e)}}_{\substack{=\prod_{e\in E} x_{(r_T\circ\varphi)(e)}^{w(e)} = \mathbf{x}_{r_T\circ\varphi,w} \\ \text{(by the definition of } \mathbf{x}_{r_T\circ\varphi,w})}} = \sum_{T\subseteq\{1,2,3,\ldots\};\ |T|=\ell} \mathbf{x}_{r_T\circ\varphi,w}. \tag{9}$$

On the other hand, recall that φ is an $\mathbf{E}$-partition. Hence, every map π satisfying $\operatorname{pack}\pi = \varphi$ is an $\mathbf{E}$-partition (because, as we know, $\operatorname{pack}\pi$ is an $\mathbf{E}$-partition if and only if π is an $\mathbf{E}$-partition). Thus, the $\mathbf{E}$-partitions π satisfying $\operatorname{pack}\pi = \varphi$ are precisely the maps $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$. Hence,

$$\sum_{\pi \text{ is an } \mathbf{E}\text{-partition};\ \operatorname{pack}\pi=\varphi} \mathbf{x}_{\pi,w} = \sum_{\pi:E\to\{1,2,3,\ldots\};\ \operatorname{pack}\pi=\varphi} \mathbf{x}_{\pi,w}$$
$$= \sum_{T\subseteq\{1,2,3,\ldots\};\ |T|=\ell} \ \sum_{\pi:E\to\{1,2,3,\ldots\};\ \operatorname{pack}\pi=\varphi;\ \pi(E)=T} \mathbf{x}_{\pi,w}$$

[19]In the last equality, we have used the fact that the strictly increasing sequences $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers are in bijection with the subsets $T \subseteq \{1,2,3,\ldots\}$ such that $|T| = \ell$. The bijection sends a sequence $(i_1 < i_2 < \cdots < i_\ell)$ to the set of its entries; its inverse map sends every T to the sequence $(r_T(1), r_T(2), \ldots, r_T(|T|))$.

(because if $\pi : E \to \{1,2,3,\ldots\}$ is a map satisfying $\operatorname{pack}\pi = \varphi$, then $|\pi(E)| = \ell$ [20]). But for every ℓ-element subset T of $\{1,2,3,\ldots\}$, there exists exactly one $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$: namely, $\pi = r_T \circ \varphi$ [21]. Therefore, for every ℓ-element subset T of $\{1,2,3,\ldots\}$, we have

$$\sum_{\pi:E\to\{1,2,3,\ldots\};\ \operatorname{pack}\pi=\varphi;\ \pi(E)=T} \mathbf{x}_{\pi,w} = \mathbf{x}_{r_T\circ\varphi,w}.$$

Hence,

$$\begin{aligned}\sum_{\pi \text{ is an } \mathbf{E}\text{-partition};\ \operatorname{pack}\pi=\varphi} \mathbf{x}_{\pi,w} &= \sum_{T\subseteq\{1,2,3,\ldots\};\ |T|=\ell} \underbrace{\sum_{\pi:E\to\{1,2,3,\ldots\};\ \operatorname{pack}\pi=\varphi;\ \pi(E)=T} \mathbf{x}_{\pi,w}}_{=\mathbf{x}_{r_T\circ\varphi,w}} \\ &= \sum_{T\subseteq\{1,2,3,\ldots\};\ |T|=\ell} \mathbf{x}_{r_T\circ\varphi,w} = M_{\operatorname{ev}_w\varphi}\end{aligned}$$

(by (9)). Thus, (8) is proven, and with it Proposition 5.4. □

Proof of Proposition 3.5. Proposition 3.5 follows immediately from Proposition 5.4 (since $M_\alpha \in \operatorname{QSym}$ for every composition α). □

We shall now describe the coproduct of $\Gamma(\mathbf{E}, w)$, essentially giving the proof that is left to the reader in [MalReu09, Theorem 2.2].

---

[20]*Proof.* Let $\pi : E \to \{1,2,3,\ldots\}$ be a map satisfying $\operatorname{pack}\pi = \varphi$. The definition of $\operatorname{pack}\pi$ yields $\operatorname{pack}\pi = r_{\pi(E)}^{-1} \circ \pi$. Hence, $|(\operatorname{pack}\pi)(E)| = \left|\left(r_{\pi(E)}^{-1}\circ\pi\right)(E)\right| = \left|r_{\pi(E)}^{-1}(\pi(E))\right| = |\pi(E)|$ (since $r_{\pi(E)}^{-1}$ is a bijection). Since $\operatorname{pack}\pi = \varphi$, this rewrites as $|\varphi(E)| = |\pi(E)|$. Hence, $|\pi(E)| = |\varphi(E)| = \ell$, qed.

[21]*Proof.* Let T be an ℓ-element subset of $\{1,2,3,\ldots\}$. We need to show that there exists exactly one $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$: namely, $\pi = r_T \circ \varphi$. In other words, we need to prove the following two claims:

Claim 1: The map $r_T \circ \varphi$ is a map $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$.

Claim 2: If $\pi : E \to \{1,2,3,\ldots\}$ is a map satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$, then $\pi = r_T \circ \varphi$.

Proof of Claim 1. We have $|T| = \ell$ (since the set T is ℓ-element), thus $\ell = |T|$. We have $(r_T\circ\varphi)(E) = r_T\left(\underbrace{\varphi(E)}_{=\{1,2,\ldots,\ell\}}\right) = r_T\left(\left\{1,2,\ldots,\underbrace{\ell}_{=|T|}\right\}\right) = r_T(\{1,2,\ldots,|T|\}) = T$ (by the definition of r_T). Now, the definition of $\operatorname{pack}(r_T\circ\varphi)$ shows that

$$\begin{aligned}\operatorname{pack}(r_T\circ\varphi) &= r_{(r_T\circ\varphi)(E)}^{-1} \circ (r_T\circ\varphi) = r_T^{-1}\circ(r_T\circ\varphi) \qquad (\text{since } (r_T\circ\varphi)(E) = T) \\ &= \varphi.\end{aligned}$$

Thus, the map $r_T\circ\varphi : E \to \{1,2,3,\ldots\}$ satisfies $\operatorname{pack}(r_T\circ\varphi) = \varphi$ and $(r_T\circ\varphi)(E) = T$. In other words, $r_T\circ\varphi$ is a map $\pi : E\to\{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$. This proves Claim 1.

Proof of Claim 2. Let $\pi : E \to \{1,2,3,\ldots\}$ be a map satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$. The definition of $\operatorname{pack}\pi$ shows that $\operatorname{pack}\pi = r_{\pi(E)}^{-1}\circ\pi = r_T^{-1}\circ\pi$ (since $\pi(E) = T$). Hence, $r_T^{-1}\circ\pi = \operatorname{pack}\pi = \varphi$, so that $\pi = r_T\circ\varphi$. This proves Claim 2.

Now, both Claims 1 and 2 are proven; hence, our proof is complete.

Definition 5.5. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset.

(a) Then, $\operatorname{Adm} \mathbf{E}$ will mean the set of all pairs (P, Q), where P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$ and having the property that no $p \in P$ and $q \in Q$ satisfy $q <_1 p$. These pairs (P, Q) are called the *admissible partitions* of $\mathbf{E}$. (In the terminology of [MalReu09], they are the *decompositions* of $(E, <_1)$.)

(b) For any subset T of E, we let $\mathbf{E} \mid_T$ denote the double poset $(T, <_1, <_2)$, where $<_1$ and $<_2$ (by abuse of notation) denote the restrictions of the relations $<_1$ and $<_2$ to T.

Proposition 5.6. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then,

$$\Delta\left(\Gamma\left(\mathbf{E}, w\right)\right) = \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} \Gamma\left(\mathbf{E} \mid_P, w \mid_P\right) \otimes \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right). \tag{10}$$

A particular case of Proposition 5.6 (namely, the case when $w(e) = 1$ for each $e \in E$) appears in [Malve93, Théorème 4.16].

The proof of Proposition 5.6 is based upon a simple bijection. We shall introduce it after some preparations.

Lemma 5.7. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset.

Let $\mathcal{S}$ be the set of all pairs (φ, k) consisting of a packed $\mathbf{E}$-partition φ and a $k \in \{0, 1, \ldots, |\varphi(E)|\}$.

Let $\mathcal{T}$ be the set of all triples $((P, Q), \sigma, \tau)$ consisting of a $(P, Q) \in \operatorname{Adm} \mathbf{E}$, a packed $\mathbf{E} \mid_P$-partition σ and a packed $\mathbf{E} \mid_Q$-partition τ.

For every $\ell \in \mathbb{Z}$, we let add_ℓ denote the bijective map $\mathbb{Z} \to \mathbb{Z}$, $z \mapsto z + \ell$.

Fix $(\varphi, k) \in \mathcal{S}$. Set

$$P = \varphi^{-1}\left(\{1, 2, \ldots, k\}\right), \qquad Q = \varphi^{-1}\left(\{k+1, k+2, \ldots, |\varphi(E)|\}\right), \tag{11}$$

$$\sigma = \varphi \mid_P \qquad \text{and} \qquad \tau = \operatorname{add}_{-k} \circ \left(\varphi \mid_Q\right). \tag{12}$$

Then, $((P, Q), \sigma, \tau) \in \mathcal{T}$.

Lemma 5.8. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $\mathcal{S}$ and $\mathcal{T}$ be defined as in Lemma 5.7.

Fix $((P, Q), \sigma, \tau) \in \mathcal{T}$. Set $k = |\sigma(P)|$, and let φ be the map $E \to \{1, 2, 3, \ldots\}$ which sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$. Then, $(\varphi, k) \in \mathcal{S}$.

Lemma 5.9. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $\mathcal{S}$, $\mathcal{T}$ and add_ℓ be defined as in Lemma 5.7.

Define a map $\Phi : \mathcal{S} \to \mathcal{T}$ as follows: Let $(\varphi, k) \in \mathcal{S}$. Then, define P, Q, σ and τ by (11) and (12). From Lemma 5.7, we know that $((P,Q), \sigma, \tau) \in \mathcal{T}$. Define $\Phi(\varphi, k)$ to be $((P,Q), \sigma, \tau)$. Thus, a map $\Phi : \mathcal{S} \to \mathcal{T}$ is defined.

Define a map $\Psi : \mathcal{T} \to \mathcal{S}$ as follows: Let $((P,Q), \sigma, \tau) \in \mathcal{T}$. Set $k = |\sigma(P)|$, and let φ be the map $E \to \{1,2,3,\ldots\}$ which sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$. From Lemma 5.8, we know that $(\varphi, k) \in \mathcal{S}$. Set $\Psi((P,Q), \sigma, \tau) = (\varphi, k)$. Thus, a map $\Psi : \mathcal{T} \to \mathcal{S}$ is defined.

The maps $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{T} \to \mathcal{S}$ are mutually inverse.

The preceding three lemmas should be obvious if the reader has "the right picture in their mind". The following proof is merely a formalization of the argument that such a picture would straightforwardly produce; we are not sure whether it is actually worth reading (as opposed to trying to conjure "the right picture").

Proof of Lemma 5.7. We have $(\varphi, k) \in \mathcal{S}$. Thus, φ is a packed $\mathbf{E}$-partition, and k is an element of $\{0, 1, \ldots, |\varphi(E)|\}$ (by the definition of $\mathcal{S}$).

The map $\varphi : E \to \{1,2,3,\ldots\}$ is packed and satisfies $|\varphi(E)| = |\varphi(E)|$. Hence, $\varphi(E) = \{1, 2, \ldots, |\varphi(E)|\}$ (by Proposition 5.2 (a) (applied to φ and $|\varphi(E)|$ instead of π and ℓ)).

Now, $(P,Q) \in \operatorname{Adm}\mathbf{E}$ [22]. Furthermore, it is straightforward to see that for

[22]*Proof.* It is clear that P and Q are subsets of E. Also, from (11), we obtain

$$\begin{aligned} P \cap Q &= \varphi^{-1}(\{1,2,\ldots,k\}) \cap \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\}) \\ &= \varphi^{-1}\left(\underbrace{\{1,2,\ldots,k\} \cap \{k+1,k+2,\ldots,|\varphi(E)|\}}_{=\varnothing} \right) = \varphi^{-1}(\varnothing) = \varnothing \end{aligned}$$

and

$$\begin{aligned} P \cup Q &= \varphi^{-1}(\{1,2,\ldots,k\}) \cup \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\}) \\ &= \varphi^{-1}\left(\underbrace{\{1,2,\ldots,k\} \cup \{k+1,k+2,\ldots,|\varphi(E)|\}}_{\substack{=\{1,2,\ldots,|\varphi(E)|\}=\varphi(E) \\ \text{(since } \varphi \text{ is packed)}}} \right) = \varphi^{-1}(\varphi(E)) = E. \end{aligned}$$

Hence, in order to prove that $(P,Q) \in \operatorname{Adm}\mathbf{E}$, it remains to show that no $p \in P$ and $q \in Q$ satisfy $q <_1 p$.

Let us assume the contrary (for the sake of contradiction). Thus, let $p \in P$ and $q \in Q$ be such that $q <_1 p$. Since φ is an $\mathbf{E}$-partition, we have $\varphi(q) \leq \varphi(p)$ (because $q <_1 p$). But $p \in P = \varphi^{-1}(\{1,2,\ldots,k\})$, so that $\varphi(p) \leq k$. On the other hand, $q \in Q = \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\})$, so that $\varphi(q) > k$. This contradicts $\varphi(q) \leq \varphi(p) \leq k$. This contradiction shows that our assumption was false. Hence, the proof of $(P,Q) \in \operatorname{Adm}\mathbf{E}$ is complete.

every subset T of E,

$$\text{the map } \varphi\mid_T \text{ is an } \mathbf{E}\mid_T \text{-partition.} \tag{13}$$

Applying this to $T = P$, we conclude that $\varphi\mid_P$ is an $\mathbf{E}\mid_P$-partition.

Since $P = \varphi^{-1}(\{1,2,\ldots,k\})$, we have $\varphi(P) \subseteq \{1,2,\ldots,k\}$. Moreover, this inclusion is actually an equality (since $\varphi(E) = \{1,2,\ldots,|\varphi(E)|\}$) [23]. In other words, we have

$$\varphi(P) = \{1,2,\ldots,k\}. \tag{14}$$

Similarly,

$$\varphi(Q) = \{k+1,k+2,\ldots,|\varphi(E)|\}. \tag{15}$$

Hence,

$$\begin{aligned}\left(\operatorname{add}_{-k}\circ\left(\varphi\mid_Q\right)\right)(Q) &= \operatorname{add}_{-k}\left(\underbrace{\left(\varphi\mid_Q\right)(Q)}_{=\varphi(Q)=\{k+1,k+2,\ldots,|\varphi(E)|\}}\right)\\ &= \operatorname{add}_{-k}\left(\{k+1,k+2,\ldots,|\varphi(E)|\}\right)\\ &= \{1,2,\ldots,|\varphi(E)|-k\}\end{aligned}$$

(by the definition of add_{-k}). Since $(\varphi\mid_P)(P) = \varphi(P) = \{1,2,\ldots,k\}$ is an initial interval, we deduce that the $\mathbf{E}\mid_P$-partition $\varphi\mid_P$ is packed. Thus, $\sigma = \varphi\mid_P$ is a packed $\mathbf{E}\mid_P$-partition.

On the other hand, (13) (applied to $T = Q$) shows that $\varphi\mid_Q$ is an $\mathbf{E}\mid_Q$-partition. Hence, the map $\operatorname{add}_{-k}\circ\left(\varphi\mid_Q\right)$ is an $\mathbf{E}\mid_Q$-partition (since the map add_{-k} is strictly increasing, and since $\left(\operatorname{add}_{-k}\circ\left(\varphi\mid_Q\right)\right)(Q) = \{1,2,\ldots,|\varphi(E)|-k\} \subseteq \{1,2,3,\ldots\}$). This $\mathbf{E}\mid_Q$-partition $\operatorname{add}_{-k}\circ\left(\varphi\mid_Q\right)$ is packed (since $\left(\operatorname{add}_{-k}\circ\left(\varphi\mid_Q\right)\right)(Q) = \{1,2,\ldots,|\varphi(E)|-k\}$ is an initial interval). Thus, $\tau = \operatorname{add}_{-k}\circ\left(\varphi\mid_Q\right)$ is a packed $\mathbf{E}\mid_Q$-partition.

We now know that $(P,Q) \in \operatorname{Adm}\mathbf{E}$, that σ is a packed $\mathbf{E}\mid_P$-partition, and that τ is a packed $\mathbf{E}\mid_Q$-partition. In other words, we know that $((P,Q),\sigma,\tau) \in \mathcal{T}$. This proves Lemma 5.7. $\square$

Proof of Lemma 5.8. We have $((P,Q),\sigma,\tau) \in \mathcal{T}$. According to the definition of $\mathcal{T}$, this means that $(P,Q) \in \operatorname{Adm}\mathbf{E}$, that σ is a packed $\mathbf{E}\mid_P$-partition, and that τ is a packed $\mathbf{E}\mid_Q$-partition.

From $(P,Q) \in \operatorname{Adm}\mathbf{E}$, we conclude that P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$ and having the property that

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{16}$$

[23] The proof in more detail: Let $g \in \{1,2,\ldots,k\}$. Then, $g \in \{1,2,\ldots,k\} \subseteq \{1,2,\ldots,|\varphi(E)|\} = \varphi(E)$. Thus, there exists some $e \in E$ such that $g = \varphi(e)$. Consider this e. From $\varphi(e) = g \in \{1,2,\ldots,k\}$, we obtain $e \in \varphi^{-1}(\{1,2,\ldots,k\}) = P$. Thus, $\varphi(e) \in \varphi(P)$, so that $g = \varphi(e) \in \varphi(P)$. Now, let us forget that we fixed g. We thus have proven that $g \in \varphi(P)$ for every $g \in \{1,2,\ldots,k\}$. In other words, $\{1,2,\ldots,k\} \subseteq \varphi(P)$. Combining this with $\varphi(P) \subseteq \{1,2,\ldots,k\}$, we obtain $\varphi(P) = \{1,2,\ldots,k\}$, qed.

Using $P \cap Q = \varnothing$ and $P \cup Q = E$, we see that the map φ is well-defined. (Indeed, we defined it as the map $E \to \{1, 2, 3, \ldots\}$ which sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$.)

The map $\sigma : P \to \{1, 2, 3, \ldots\}$ is packed, and we have $k = |\sigma(P)|$. Hence, Proposition 5.2 (a) (applied to P, σ and k instead of E, π and ℓ) yields $\sigma(P) = \{1, 2, \ldots, k\}$.

The map $\tau : Q \to \{1, 2, 3, \ldots\}$ is packed, and we have $|\tau(Q)| = |\tau(Q)|$. Hence, Proposition 5.2 (a) (applied to Q, τ and $|\tau(Q)|$ instead of E, π and ℓ) yields $\tau(Q) = \{1, 2, \ldots, |\tau(Q)|\}$.

The definition of φ shows that

$$\varphi(e) = \sigma(e) \qquad \text{for every } e \in P. \tag{17}$$

Hence, $\varphi(P) = \sigma(P) = \{1, 2, \ldots, k\}$.

Also, the definition of φ shows that

$$\varphi(e) = \tau(e) + k \qquad \text{for every } e \in Q. \tag{18}$$

Thus,

$$\begin{aligned} \varphi(Q) &= \left\{ \underbrace{\varphi(e)}_{=\tau(e)+k} \mid e \in Q \right\} = \{\tau(e) + k \mid e \in Q\} \\ &= \left\{ u + k \mid u \in \underbrace{\tau(Q)}_{=\{1,2,\ldots,|\tau(Q)|\}} \right\} = \{u + k \mid u \in \{1, 2, \ldots, |\tau(Q)|\}\} \\ &= \{k + 1, k + 2, \ldots, k + |\tau(Q)|\}. \end{aligned}$$

Now, $E = P \cup Q$, so that

$$\begin{aligned} \varphi(E) = \varphi(P \cup Q) &= \underbrace{\varphi(P)}_{=\{1,2,\ldots,k\}} \cup \underbrace{\varphi(Q)}_{=\{k+1,k+2,\ldots,k+|\tau(Q)|\}} \\ &= \{1, 2, \ldots, k\} \cup \{k + 1, k + 2, \ldots, k + |\tau(Q)|\} \\ &= \{1, 2, \ldots, k + |\tau(Q)|\}. \end{aligned} \tag{19}$$

Thus, $\varphi(E)$ is an initial interval; in other words, the map φ is packed. Furthermore, (19) shows that $|\varphi(E)| = k + \underbrace{|\tau(Q)|}_{\geq 0} \geq k$, so that $k \in \{0, 1, \ldots, |\varphi(E)|\}$.

We shall now show that φ is an $\mathbf{E}$-partition. To do so, we must prove the following two claims:

Claim 1: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\varphi(e) \leq \varphi(f)$.

Claim 2: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\varphi(e) < \varphi(f)$.

We shall only prove Claim 1 (as the proof of Claim 2 is similar). So let $e \in E$ and $f \in E$ be such that $e <_1 f$. We need to show that $\varphi(e) \leq \varphi(f)$. We are in one of the following four cases:

Case 1: We have $e \in P$ and $f \in P$.

Case 2: We have $e \in P$ and $f \in Q$.

Case 3: We have $e \in Q$ and $f \in P$.

Case 4: We have $e \in Q$ and $f \in Q$.

In Case 1, our claim $\varphi(e) \leq \varphi(f)$ follows from the assumption that σ is an $\mathbf{E}\mid_P$-partition[24]. In Case 4, it follows from the assumption that τ is an $\mathbf{E}\mid_Q$-partition[25]. In Case 2, it clearly holds[26]. Finally, Case 3 is impossible[27]. Thus, we have proven the claim in each of the four cases, and consequently Claim 1 is proven. As we have said above, Claim 2 is proven similarly. Thus, we have proven that φ is an $\mathbf{E}$-partition.

Altogether, we now know that φ is a packed $\mathbf{E}$-partition, and that $k \in \{0, 1, \ldots, |\varphi(E)|\}$. In other words, $(\varphi, k) \in \mathcal{S}$. This proves Lemma 5.8. □

Proof of Lemma 5.9. We need to prove the following two claims:

Claim 1: We have $\Phi \circ \Psi = \mathrm{id}$.

Claim 2: We have $\Psi \circ \Phi = \mathrm{id}$.

Proof of Claim 1: Fix $((P,Q), \sigma, \tau) \in \mathcal{T}$. Set $k = |\sigma(P)|$, and let φ be the map $E \to \{1, 2, 3, \ldots\}$ which sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$. The definition

[24] *Proof.* Assume that we are in Case 1. Thus, we have $e \in P$ and $f \in P$. Thus, e and f are elements of P satisfying $e <_1 f$. Hence, $\sigma(e) \leq \sigma(f)$ (since σ is an $\mathbf{E}\mid_P$-partition). But the definition of φ yields $\varphi(e) = \begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases} = \sigma(e)$ (since $e \in P$) and $\varphi(f) = \sigma(f)$ (similarly). Hence, $\varphi(e) = \sigma(e) \leq \sigma(f) = \varphi(f)$. Qed.

[25] *Proof.* Assume that we are in Case 4. Thus, we have $e \in Q$ and $f \in Q$. Thus, e and f are elements of Q satisfying $e <_1 f$. Hence, $\tau(e) \leq \tau(f)$ (since τ is an $\mathbf{E}\mid_Q$-partition). But the definition of φ yields $\varphi(e) = \begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases} = \tau(e) + k$ (since $e \in Q$) and $\varphi(f) = \tau(f) + k$ (similarly). Hence, $\varphi(e) = \underbrace{\tau(e)}_{\leq \tau(f)} + k \leq \tau(f) + k = \varphi(f)$. Qed.

[26] *Proof.* Assume that we are in Case 2. Thus, we have $e \in P$ and $f \in Q$. The definition of φ yields $\varphi(e) = \begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases} = \sigma(e)$ (since $e \in P$) and $\varphi(f) = \begin{cases} \sigma(f), & \text{if } f \in P; \\ \tau(f) + k, & \text{if } f \in Q \end{cases} = \tau(f) + k$ (since $f \in Q$). But we have $\varphi(e) = \sigma\left(\underbrace{e}_{\in P}\right) \in \sigma(P) = \{1, 2, \ldots, k\}$, so that $\varphi(e) \leq k$. Meanwhile, $\varphi(f) = \underbrace{\tau(f)}_{>0} + k > k$. Thus, $\varphi(e) \leq k < \varphi(f)$, and therefore $\varphi(e) \leq \varphi(f)$. Qed.

[27] *Proof.* Assume that we are in Case 3. Thus, $e \in Q$ and $f \in P$. The elements $f \in P$ and $e \in Q$ satisfy $e <_1 f$. This contradicts (16) (applied to $p = f$ and $q = e$). Thus, we have obtained a contradiction; hence, our assumption (that we are in Case 3) was wrong. Therefore, Case 3 is impossible.

of Ψ thus yields $\Psi((P,Q),\sigma,\tau) = (\varphi,k)$. We shall now show that $\Phi(\varphi,k) = ((P,Q),\sigma,\tau)$.

Lemma 5.8 shows that $(\varphi,k) \in \mathcal{S}$. In other words, φ is a packed $\mathbf{E}$-partition, and we have $k \in \{0,1,\ldots,|\varphi(E)|\}$. The map $\varphi : E \to \{1,2,3,\ldots\}$ is packed and satisfies $|\varphi(E)| = |\varphi(E)|$. Hence, we have $\varphi(E) = \{1,2,\ldots,|\varphi(E)|\}$ (by Proposition 5.2 (a) (applied to φ and $|\varphi(E)|$ instead of π and ℓ)).

The map $\operatorname{add}_k : \mathbb{Z} \to \mathbb{Z}$ is a bijection, and its inverse is $(\operatorname{add}_k)^{-1} = \operatorname{add}_{-k}$.

The map $\sigma : P \to \{1,2,3,\ldots\}$ is packed, and we have $k = |\sigma(P)|$. Hence, Proposition 5.2 (a) (applied to P, σ and k instead of E, π and ℓ) yields $\sigma(P) = \{1,2,\ldots,k\}$.

From $(P,Q) \in \operatorname{Adm}\mathbf{E}$, we conclude that P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$. Hence, $Q = E \setminus P$ and $P = E \setminus Q$.

The definition of φ shows that

$$\varphi(e) = \sigma(e) \qquad \text{for every } e \in P. \tag{20}$$

Hence, $\varphi\mid_P = \sigma$. Also, the definition of φ shows that

$$\varphi(e) = \tau(e) + k \qquad \text{for every } e \in Q. \tag{21}$$

Thus, every $e \in Q$ satisfies

$$\begin{aligned} \varphi(e) &= \tau(e) + k = \operatorname{add}_k(\tau(e)) \qquad (\text{since } \operatorname{add}_k(\tau(e)) \text{ is defined to be } \tau(e)+k) \\ &= (\operatorname{add}_k \circ \tau)(e). \end{aligned} \tag{22}$$

Hence, $\varphi\mid_Q = \operatorname{add}_k \circ \tau$, so that $\tau = \underbrace{(\operatorname{add}_k)^{-1}}_{=\operatorname{add}_{-k}} \circ (\varphi\mid_Q) = \operatorname{add}_{-k} \circ (\varphi\mid_Q)$.

Furthermore, $P = \varphi^{-1}(\{1,2,\ldots,k\})$ [28] and $Q = \varphi^{-1}(\{k+1,k+2,\ldots,|\varphi(E)|\})$ [29]. Altogether, we thus know that

$$\begin{aligned} &P = \varphi^{-1}(\{1,2,\ldots,k\}), \qquad Q = \varphi^{-1}(\{k+1,k+2,\ldots,|\varphi(E)|\}), \\ &\sigma = \varphi\mid_P \qquad \text{and} \qquad \tau = \operatorname{add}_{-k} \circ (\varphi\mid_Q). \end{aligned}$$

[28] *Proof.* Let $e \in \varphi^{-1}(\{1,2,\ldots,k\})$. Thus, $e \in E$ and $\varphi(e) \in \{1,2,\ldots,k\}$. If we had $e \in Q$, then we would have

$$\begin{aligned} \varphi(e) &= \underbrace{\tau(e)}_{>0} + k \qquad (\text{by (21)}) \\ &> k, \end{aligned}$$

which would contradict $\varphi(e) \in \{1,2,\ldots,k\}$. Hence, we cannot have $e \in Q$. Thus, $e \in E \setminus Q = P$.

Now, let us forget that we fixed e. Thus we have proven that $e \in P$ for every $e \in \varphi^{-1}(\{1,2,\ldots,k\})$. In other words, $\varphi^{-1}(\{1,2,\ldots,k\}) \subseteq P$.

On the other hand, fix $p \in P$. Then, $\varphi(p) = \sigma(p)$ (by (20)). Hence, $\varphi(p) = \sigma(p) \in \sigma(P) = \{1,2,\ldots,k\}$, so that $p \in \varphi^{-1}(\{1,2,\ldots,k\})$.

Now, let us forget that we fixed p. Thus we have proven that $p \in \varphi^{-1}(\{1,2,\ldots,k\})$ for every $p \in P$. In other words, $P \subseteq \varphi^{-1}(\{1,2,\ldots,k\})$. Combining this with $\varphi^{-1}(\{1,2,\ldots,k\}) \subseteq P$, we obtain $P = \varphi^{-1}(\{1,2,\ldots,k\})$, qed.

[29] *Proof.* Let $e \in \varphi^{-1}(\{k+1,k+2,\ldots,|\varphi(E)|\})$. Thus, $e \in E$ and $\varphi(e) \in \{k+1,k+2,\ldots,|\varphi(E)|\}$.

These equations are identical with the equations (11) and (12) that were used in the definition of $\Phi(\varphi, k)$. Hence, the definition of Φ shows that $\Phi(\varphi, k) = ((P, Q), \sigma, \tau)$. Thus, $((P, Q), \sigma, \tau) = \Phi \underbrace{(\varphi, k)}_{=\Psi((P,Q),\sigma,\tau)} = \Phi(\Psi((P, Q), \sigma, \tau))$.

Now, let us forget that we fixed $((P, Q), \sigma, \tau)$. We thus have shown that $\Phi(\Psi((P, Q), \sigma, \tau)) = ((P, Q), \sigma, \tau)$ for every $((P, Q), \sigma, \tau) \in \mathcal{T}$. In other words, $\Phi \circ \Psi = \mathrm{id}$. This proves Claim 1.

Proof of Claim 2: Fix $(\varphi, k) \in \mathcal{S}$. Define P, Q, σ and τ by (11) and (12). The definition of Φ shows that $\Phi(\varphi, k) = ((P, Q), \sigma, \tau)$. From Lemma 5.7, we know that $((P, Q), \sigma, \tau) \in \mathcal{T}$. In other words, we know that $(P, Q) \in \operatorname{Adm} \mathbf{E}$, that σ is a packed $\mathbf{E}\mid_P$-partition, and that τ is a packed $\mathbf{E}\mid_Q$-partition.

From $(P, Q) \in \operatorname{Adm} \mathbf{E}$, we conclude that P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$.

We have $\varphi(P) = \{1, 2, \ldots, k\}$. (This was proven in our proof of Lemma 5.7 above; see the equality (14).)

We have $\sigma = \varphi\mid_P$. Thus, for every $e \in P$, we have $\sigma(e) = (\varphi\mid_P)(e) = \varphi(e)$. In other words, for every $e \in P$, we have

$$\varphi(e) = \sigma(e). \tag{23}$$

Also, $\tau = \operatorname{add}_{-k} \circ (\varphi\mid_Q)$. Hence, for every $e \in Q$, we have

$$\begin{aligned}
\tau(e) &= (\operatorname{add}_{-k} \circ (\varphi\mid_Q))(e) = \operatorname{add}_{-k}\left(\underbrace{(\varphi\mid_Q)(e)}_{=\varphi(e)}\right) \\
&= \operatorname{add}_{-k}(\varphi(e)) = \varphi(e) + (-k) \qquad (\text{by the definition of } \operatorname{add}_{-k}) \\
&= \varphi(e) - k.
\end{aligned}$$

If we had $e \in P$, then we would have

$$\begin{aligned}
\varphi(e) &= \sigma(e) \qquad (\text{by (20)}) \\
&\in \sigma(P) = \{1, 2, \ldots, k\}
\end{aligned}$$

and therefore $\varphi(e) \leq k$, which would contradict $\varphi(e) \in \{k+1, k+2, \ldots, |\varphi(E)|\}$. Hence, we cannot have $e \in P$. Thus, $e \in E \setminus P = Q$.

Now, let us forget that we fixed e. Thus we have proven that $e \in Q$ for every $e \in \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\})$. In other words, $\varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\}) \subseteq Q$.

On the other hand, fix $q \in Q$. Then, $\varphi(q) = \tau(q) + k$ (by (21)). Hence, $\varphi(q) = \underbrace{\tau(q)}_{>0} + k > k$. Combining this with $\varphi(q) \in \varphi(E) = \{1, 2, \ldots, |\varphi(E)|\}$, we obtain $\varphi(q) \in \{k+1, k+2, \ldots, |\varphi(E)|\}$. Hence, $q \in \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\})$.

Now, let us forget that we fixed q. Thus we have proven that $q \in \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\})$ for every $q \in Q$. In other words, $Q \subseteq \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\})$. Combining this with $\varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\}) \subseteq Q$, we obtain $Q = \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\})$, qed.

Thus, for every $e \in Q$, we have

$$\varphi(e) = \tau(e) + k.$$

Combining this with (23), we conclude that

$$\varphi(e) = \begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases} \qquad \text{for every } e \in E \tag{24}$$

[30]. Moreover, $\sigma : P \to \{1, 2, 3, \ldots\}$ is a packed map and satisfies $|\sigma(P)| = |\sigma(P)|$. Thus, Proposition 5.2 (a) (applied to P, σ and $|\sigma(P)|$ instead of E, π and ℓ) yields $\sigma(P) = \{1, 2, \ldots, |\sigma(P)|\}$. Hence,

$$\{1, 2, \ldots, |\sigma(P)|\} = \underbrace{\sigma}_{=\varphi|_P}(P) = (\varphi\mid_P)(P) = \varphi(P) = \{1, 2, \ldots, k\}.$$

Thus, $|\sigma(P)| = k$.

So we know that $k = |\sigma(P)|$, and that φ is the map $E \to \{1, 2, 3, \ldots\}$ which sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$ (because of (24)). Thus, our k and our φ are precisely the k and the φ in the definition of $\Psi((P, Q), \sigma, \tau)$. Hence, $\Psi((P, Q), \sigma, \tau) = (\varphi, k)$. Thus, $(\varphi, k) = \Psi\underbrace{((P, Q), \sigma, \tau)}_{=\Phi(\varphi,k)} = \Psi(\Phi(\varphi, k))$.

Now, let us forget that we fixed (φ, k). We thus have shown that $\Psi(\Phi(\varphi, k)) = (\varphi, k)$ for every $(\varphi, k) \in \mathcal{S}$. In other words, $\Psi \circ \Phi = \mathrm{id}$. This proves Claim 2.

Now, both Claims 1 and 2 are proven. Thus, the maps Φ and Ψ are mutually inverse. This proves Lemma 5.9. $\square$

Proof of Proposition 5.6. Define $\mathcal{S}$, $\mathcal{T}$, Φ and Ψ as in Lemma 5.9. From Lemma 5.9, we know that the maps Φ and Ψ are mutually inverse. Hence, Φ is a bijection from $\mathcal{S}$ to $\mathcal{T}$.

Whenever $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition and $k \in \{0, 1, \ldots, \ell\}$, we introduce the notation $\alpha[: k]$ for the composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, and the notation $\alpha[k :]$ for the composition $(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_\ell)$. Now, the formula (1) can be rewritten as follows:

$$\Delta(M_\alpha) = \sum_{k=0}^{\ell} M_{\alpha[:k]} \otimes M_{\alpha[k:]} \tag{25}$$

for every $\ell \in \mathbb{N}$ and every composition α with ℓ entries.

Let us observe a simple fact: For any $(\varphi, k) \in \mathcal{S}$, we have

$$(\mathrm{ev}_w \varphi)[: k] = \mathrm{ev}_{w|_P} \sigma \qquad \text{and} \qquad (\mathrm{ev}_w \varphi)[k :] = \mathrm{ev}_{w|_Q} \tau, \tag{26}$$

[30]The right hand side of this equality makes sense because $P \cap Q = \varnothing$ and $P \cup Q = E$.

where $((P,Q),\sigma,\tau) = \Phi(\varphi,k)$ [31].

If $\varphi : E \to \{1,2,3,\ldots\}$ is any packed map, then $\operatorname{ev}_w \varphi$ is a composition with $|\varphi(E)|$ entries (by the definition of $\operatorname{ev}_w \varphi$), and thus it satisfies

$$\Delta\left(M_{\operatorname{ev}_w \varphi}\right) = \sum_{k=0}^{|\varphi(E)|} M_{(\operatorname{ev}_w \varphi)[:k]} \otimes M_{(\operatorname{ev}_w \varphi)[k:]} \tag{32}$$

[31] *Proof of (26):* Let $(\varphi,k) \in \mathcal{S}$. Let $((P,Q),\sigma,\tau) = \Phi(\varphi,k)$. We must prove the equalities (26).

For every $\ell \in \mathbb{Z}$, define the map $\operatorname{add}_\ell : \mathbb{Z} \to \mathbb{Z}$ as in Lemma 5.7.

Let $\ell = |\varphi(E)|$. Thus, Proposition 5.2 (a) (applied to φ instead of π) yields $\varphi(E) = \{1,2,\ldots,\ell\}$ (since the map $\varphi : E \to \{1,2,3,\ldots\}$ is packed). For each $i \in \{1,2,\ldots,\ell\}$, define $\alpha_i \in \mathbb{N}$ by

$$\alpha_i = \sum_{e \in \varphi^{-1}(i)} w(e). \tag{27}$$

Then, $\operatorname{ev}_w \varphi = (\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ (by the definition of $\operatorname{ev}_w \varphi$).

We have $(\varphi,k) \in \mathcal{S}$. Thus, φ is a packed $\mathbf{E}$-partition, and k is an element of $\{0,1,\ldots,|\varphi(E)|\}$ (by the definition of $\mathcal{S}$). Thus, $k \in \{0,1,\ldots,|\varphi(E)|\} = \{0,1,\ldots,\ell\}$ (since $|\varphi(E)| = \ell$).

Now, from $\operatorname{ev}_w \varphi = (\alpha_1,\alpha_2,\ldots,\alpha_\ell)$, we obtain

$$(\operatorname{ev}_w \varphi)[:k] = (\alpha_1,\alpha_2,\ldots,\alpha_k) \qquad \text{and} \qquad (\operatorname{ev}_w \varphi)[k:] = (\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_\ell).$$

Recall that $((P,Q),\sigma,\tau) = \Phi(\varphi,k)$. Hence, P, Q, σ and τ are defined by (11) and (12) (according to the definition of Φ). We know (from Lemma 5.7) that $((P,Q),\sigma,\tau) \in \mathcal{T}$. In other words, we know that $(P,Q) \in \operatorname{Adm}\mathbf{E}$, that σ is a packed $\mathbf{E}\mid_P$-partition, and that τ is a packed $\mathbf{E}\mid_Q$-partition.

For every $e \in P$, we have

$$\underbrace{\sigma}_{\substack{=\varphi|_P \\ \text{(by (12))}}}(e) = (\varphi\mid_P)(e) = \varphi(e). \tag{28}$$

For every $e \in Q$, we have

$$\begin{aligned} \underbrace{\tau}_{\substack{=\operatorname{add}_{-k}\circ(\varphi|_Q) \\ \text{(by (12))}}}(e) &= \left(\operatorname{add}_{-k}\circ(\varphi\mid_Q)\right)(e) = \operatorname{add}_{-k}\left((\varphi\mid_Q)(e)\right) \\ &= \underbrace{(\varphi\mid_Q)(e)}_{=\varphi(e)} + (-k) \qquad \text{(by the definition of } \operatorname{add}_{-k}\text{)} \\ &= \varphi(e) - k. \end{aligned} \tag{29}$$

For every $i \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned} \sigma^{-1}(i) &= \left\{ e \in P \mid \underbrace{\sigma(e)}_{\substack{=\varphi(e) \\ \text{(by (28))}}} = i \right\} = \{e \in P \mid \varphi(e) = i\} \\ &= \underbrace{\{e \in E \mid \varphi(e) = i\}}_{=\varphi^{-1}(i)} \cap \underbrace{P}_{=\varphi^{-1}(\{1,2,\ldots,k\})} \\ &= \varphi^{-1}(i) \cap \varphi^{-1}(\{1,2,\ldots,k\}) = \varphi^{-1}(i) \end{aligned} \tag{30}$$

(since $\varphi^{-1}(i) \subseteq \varphi^{-1}(\{1,2,\ldots,k\})$ (because $i \in \{1,2,\ldots,k\}$)).

(by (25), applied to $\alpha = \operatorname{ev}_w \varphi$ and $\ell = |\varphi(E)|$).

For every $i \in \{1, 2, \ldots, \ell - k\}$, we have

$$\begin{aligned}
\tau^{-1}(i) &= \left\{ e \in Q \mid \underbrace{\tau(e)}_{\substack{=\varphi(e)-k \\ \text{(by (29))}}} = i \right\} = \left\{ e \in Q \mid \underbrace{\varphi(e) - k = i}_{\Longleftrightarrow\ (\varphi(e)=k+i)} \right\} \\
&= \{ e \in Q \mid \varphi(e) = k + i \} = \underbrace{\{ e \in E \mid \varphi(e) = k + i \}}_{=\varphi^{-1}(k+i)} \cap \underbrace{Q}_{\substack{=\varphi^{-1}(\{k+1,k+2,\ldots,|\varphi(E)|\}) \\ =\varphi^{-1}(\{k+1,k+2,\ldots,\ell\}) \\ \text{(since } |\varphi(E)|=\ell)}} \\
&= \varphi^{-1}(k+i) \cap \varphi^{-1}(\{k+1, k+2, \ldots, \ell\}) = \varphi^{-1}(k+i) \qquad (31)
\end{aligned}$$

(since $\varphi^{-1}(k+i) \subseteq \varphi^{-1}(\{k+1, k+2, \ldots, \ell\})$ (since $k + i \in \{k+1, k+2, \ldots, \ell\}$ (since $i \in \{1, 2, \ldots, \ell - k\}$))).

We have $\varphi(Q) = \{1, 2, \ldots, k\}$. (This was proven in our proof of Lemma 5.7 above; see the equality (14).) But $\sigma = \varphi|_P$, so that $\sigma(P) = (\varphi|_P)(P) = \varphi(P) = \{1, 2, \ldots, k\}$. Hence, $|\sigma(P)| = |\{1, 2, \ldots, k\}| = k$. Therefore, the definition of $\operatorname{ev}_{w|_P} \sigma$ shows that $\operatorname{ev}_{w|_P} \sigma = (\beta_1, \beta_2, \ldots, \beta_k)$, where each β_i is defined as $\sum_{e \in \sigma^{-1}(i)} (w|_P)(e)$. Thus, every $i \in \{1, 2, \ldots, k\}$ satisfies

$$\beta_i = \underbrace{\sum_{e \in \sigma^{-1}(i)}}_{\substack{= \sum_{e \in \varphi^{-1}(i)} \\ \text{(by (30))}}} \underbrace{(w|_P)(e)}_{=w(e)} = \sum_{e \in \varphi^{-1}(i)} w(e) = \alpha_i \qquad \text{(by (27))}.$$

Hence, $(\beta_1, \beta_2, \ldots, \beta_k) = (\alpha_1, \alpha_2, \ldots, \alpha_k) = (\operatorname{ev}_w \varphi)[:k]$, so that $(\operatorname{ev}_w \varphi)[:k] = (\beta_1, \beta_2, \ldots, \beta_k) = \operatorname{ev}_{w|_P} \sigma$.

We have $\varphi(Q) = \{k+1, k+2, \ldots, |\varphi(E)|\}$. (This was proven in our proof of Lemma 5.7 above; see the equality (15).) But

$$\tau(Q) = \left\{ \underbrace{\tau(e)}_{\substack{=\varphi(e)-k \\ \text{(by (29))}}} \mid e \in Q \right\} = \{ \varphi(e) - k \mid e \in Q \} = \{ u - k \mid u \in \varphi(Q) \}.$$

Thus,

$$\begin{aligned}
|\tau(Q)| &= |\varphi(Q)| = |\{k+1, k+2, \ldots, |\varphi(E)|\}| \qquad \text{(since } \varphi(Q) = \{k+1, k+2, \ldots, |\varphi(E)|\}) \\
&= \underbrace{|\varphi(E)|}_{=\ell} - k = \ell - k.
\end{aligned}$$

Therefore, the definition of $\operatorname{ev}_{w|_Q} \tau$ shows that $\operatorname{ev}_{w|_Q} \tau = (\gamma_1, \gamma_2, \ldots, \gamma_{\ell-k})$, where each γ_i is

Now, applying Δ to the equality (7) yields

$$\begin{aligned}
\Delta\left(\Gamma\left(\mathbf{E},w\right)\right) &= \Delta\left(\sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\operatorname{ev}_w \varphi}\right) \\
&= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \underbrace{\Delta\left(M_{\operatorname{ev}_w \varphi}\right)}_{\substack{=\sum_{k=0}^{\left|\varphi(E)\right|} M_{(\operatorname{ev}_w \varphi)[:k]} \otimes M_{(\operatorname{ev}_w \varphi)[k:]} \\ \text{(by (32))}}} \\
&= \underbrace{\sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \sum_{k=0}^{\left|\varphi(E)\right|}}_{\substack{=\sum_{(\varphi,k)\in\mathcal{S}} \\ \text{(by the definition of } \mathcal{S}\text{)}}} M_{(\operatorname{ev}_w \varphi)[:k]} \otimes M_{(\operatorname{ev}_w \varphi)[k:]} \\
&= \sum_{(\varphi,k)\in\mathcal{S}} M_{(\operatorname{ev}_w \varphi)[:k]} \otimes M_{(\operatorname{ev}_w \varphi)[k:]} .
\end{aligned} \tag{33}$$

On the other hand, rewriting each of the tensorands on the right hand side of

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defined as $\sum_{e\in\tau^{-1}(i)} \left(w\mid_Q\right)(e)$. Thus, every $i\in\{1,2,\ldots,\ell-k\}$ satisfies

$$\gamma_i = \underbrace{\sum_{e\in\tau^{-1}(i)}}_{\substack{=\sum_{e\in\varphi^{-1}(k+i)} \\ \text{(by (31))}}} \underbrace{\left(w\mid_Q\right)(e)}_{=w(e)} = \sum_{e\in\varphi^{-1}(k+i)} w(e) = \alpha_{k+i}$$
$$\left(\begin{array}{c}\text{since (27) (applied to } k+i \text{ instead of } i\text{)} \\ \text{shows that } \alpha_{k+i} = \sum_{e\in\varphi^{-1}(k+i)} w(e)\end{array}\right).$$

Hence, $\left(\gamma_1,\gamma_2,\ldots,\gamma_{\ell-k}\right) = \left(\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_\ell\right) = \left(\operatorname{ev}_w \varphi\right)[k:]$, so that $\left(\operatorname{ev}_w \varphi\right)[k:] = \left(\gamma_1,\gamma_2,\ldots,\gamma_{\ell-k}\right) = \operatorname{ev}_{w\mid_Q} \tau$.

Thus, both parts of (26) are proven.

(10) using (7), we obtain

$$\sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} \underbrace{\Gamma\left(\mathbf{E}\mid_P, w\mid_P\right)}_{\substack{=\sum_{\varphi \text{ is a packed } \mathbf{E}|_P\text{-partition}} M_{\operatorname{ev}_{w|_P}\varphi} \\ \text{(by (7))}}} \otimes \underbrace{\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right)}_{\substack{=\sum_{\varphi \text{ is a packed } \mathbf{E}|_Q\text{-partition}} M_{\operatorname{ev}_{w|_Q}\varphi} \\ \text{(by (7))}}}$$

$$= \sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} \underbrace{\left(\sum_{\varphi \text{ is a packed } \mathbf{E}|_P\text{-partition}} M_{\operatorname{ev}_{w|_P}\varphi}\right)}_{\substack{=\sum_{\sigma \text{ is a packed } \mathbf{E}|_P\text{-partition}} M_{\operatorname{ev}_{w|_P}\sigma} \\ \text{(here, we have renamed the} \\ \text{summation index } \varphi \text{ as } \sigma)}} \otimes \underbrace{\left(\sum_{\varphi \text{ is a packed } \mathbf{E}|_Q\text{-partition}} M_{\operatorname{ev}_{w|_Q}\varphi}\right)}_{\substack{=\sum_{\tau \text{ is a packed } \mathbf{E}|_Q\text{-partition}} M_{\operatorname{ev}_{w|_Q}\tau} \\ \text{(here, we have renamed the} \\ \text{summation index } \varphi \text{ as } \tau)}}$$

$$= \sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} \left(\sum_{\sigma \text{ is a packed } \mathbf{E}|_P\text{-partition}} M_{\operatorname{ev}_{w|_P}\sigma}\right) \otimes \left(\sum_{\tau \text{ is a packed } \mathbf{E}|_Q\text{-partition}} M_{\operatorname{ev}_{w|_Q}\tau}\right)$$

$$= \underbrace{\sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} \sum_{\sigma \text{ is a packed } \mathbf{E}|_P\text{-partition}} \sum_{\tau \text{ is a packed } \mathbf{E}|_Q\text{-partition}}}_{\substack{=\sum_{((P,Q),\sigma,\tau)\in\mathcal{T}} \\ \text{(by the definition of } \mathcal{T})}} M_{\operatorname{ev}_{w|_P}\sigma} \otimes M_{\operatorname{ev}_{w|_Q}\tau}$$

$$= \sum_{((P,Q),\sigma,\tau)\in\mathcal{T}} M_{\operatorname{ev}_{w|_P}\sigma} \otimes M_{\operatorname{ev}_{w|_Q}\tau}$$

$$= \sum_{(\varphi,k)\in\mathcal{S}} M_{(\operatorname{ev}_w\varphi)[:k]} \otimes M_{(\operatorname{ev}_w\varphi)[k:]}$$

(here, we have substituted $\Phi(\varphi, k)$ for $((P, Q), \sigma, \tau)$ in the sum, using the fact that Φ is a bijection from $\mathcal{S}$ to $\mathcal{T}$, and using the equalities (26) to rewrite the addend $M_{\operatorname{ev}_{w|_P}\sigma} \otimes M_{\operatorname{ev}_{w|_Q}\tau}$ as $M_{(\operatorname{ev}_w\varphi)[:k]} \otimes M_{(\operatorname{ev}_w\varphi)[k:]}$). Comparing this with (33), we obtain

$$\Delta\left(\Gamma\left(\mathbf{E}, w\right)\right) = \sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} \Gamma\left(\mathbf{E}\mid_P, w\mid_P\right) \otimes \Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right).$$

This proves Proposition 5.6. □

We note in passing that there is also a rule for multiplying quasisymmetric functions of the form $\Gamma(\mathbf{E}, w)$. Namely, if $\mathbf{E}$ and $\mathbf{F}$ are two double posets and u and v are corresponding maps, then $\Gamma(\mathbf{E}, u)\,\Gamma(\mathbf{F}, v) = \Gamma(\mathbf{EF}, w)$ for a map w which is defined to be u on the subset $\mathbf{E}$ of $\mathbf{EF}$, and v on the subset $\mathbf{F}$ of $\mathbf{EF}$. Here, $\mathbf{EF}$ is a double poset defined as in [MalReu09, §2.1]. [32] Combined with Proposition 3.5, this fact gives a combinatorial proof for the fact that QSym is a $\mathbf{k}$-algebra[33], as well as for

[32]See the Appendix (specifically, Corollary 10.46) for the exact statement of this rule (and a proof).

[33]This combinatorial proof is shown in detail in the Appendix (specifically, see the proof of Proposition 10.48).

Actually, we can do better: We can use these ideas to show that QSym is a Hopf algebra. See the proof of Proposition 10.61 for how this is done.

some standard formulas for multiplications of quasisymmetric functions; similarly, Proposition 5.6 can be used to derive the well-known formulas for ΔM_α, ΔL_α, $\Delta s_{\lambda/\mu}$ etc. (although, of course, we have already used the formula for ΔM_α in our proof of Proposition 5.6).

Finally, let us state one more almost-trivial lemma that will be used later:

Lemma 5.10. Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $>_1$ be the opposite relation of $<_1$. Then, $(E, >_1, <_2)$ is a tertispecial double poset.

Proof of Lemma 5.10. The relations $<_1$ and $<_2$ are strict partial orders (since $(E, <_1, <_2)$ is a double poset). The relation $>_1$ is the opposite relation of $<_1$, and thus is a strict partial order (since $<_1$ is a strict partial order). Now we know that both relations $>_1$ and $<_2$ are strict partial orders on the set E. Hence, $(E, >_1, <_2)$ is a double poset. It remains to prove that this double poset $(E, >_1, <_2)$ is tertispecial.

We know that the double poset $(E, <_1, <_2)$ is tertispecial. In other words, the following statement holds:

Statement 1: If a and b are two elements of E such that a is $<_1$-covered by b, then a and b are $<_2$-comparable.

On the other hand, the following statement holds:

Statement 2: Let a and b be two elements of E. Then, we have the following logical equivalence:

$$(a \text{ is } >_1 \text{-covered by } b) \iff (b \text{ is } <_1 \text{-covered by } a).$$

[*Proof of Statement 2:* We have the following chain of logical equivalences:

$$\begin{aligned}
&(a \text{ is } >_1 \text{-covered by } b) \\
\iff &\left(\text{we have } a >_1 b \text{, and there exists no } c \in E \text{ satisfying } \underbrace{a >_1 c >_1 b}_{\iff (a>_1c)\wedge(c>_1b)}\right) \\
&\qquad (\text{by the definition of the notion } ``>_1 \text{-covered by''}) \\
\iff &\left(\text{we have } \underbrace{a >_1 b}_{\substack{\iff (b<_1a) \\ (\text{since } >_1 \text{ is} \\ \text{the opposite} \\ \text{relation} \\ \text{of } <_1)}} \text{, and there exists no } c \in E \text{ satisfying } \underbrace{(a >_1 c)}_{\substack{\iff (c<_1a) \\ (\text{since } >_1 \text{ is} \\ \text{the opposite} \\ \text{relation} \\ \text{of } <_1)}} \wedge \underbrace{(c >_1 b)}_{\substack{\iff (b<_1c) \\ (\text{since } >_1 \text{ is} \\ \text{the opposite} \\ \text{relation} \\ \text{of } <_1)}}\right) \\
\iff &\left(\text{we have } b <_1 a \text{, and there exists no } c \in E \text{ satisfying } \underbrace{(c <_1 a) \wedge (b <_1 c)}_{\substack{\iff (b<_1c)\wedge(c<_1a) \\ \iff (b<_1c<_1a)}}\right) \\
\iff &\left(\text{we have } b <_1 a \text{, and there exists no } c \in E \text{ satisfying } b <_1 c <_1 a\right) . \qquad (34)
\end{aligned}$$

On the other hand, we have the following chain of logical equivalences:

$$\begin{aligned}
&(b \text{ is } <_1 \text{-covered by } a) \\
\iff &\left(\text{we have } b <_1 a \text{, and there exists no } c \in E \text{ satisfying } b <_1 c <_1 a\right) \\
&\qquad (\text{by the definition of the notion } ``<_1 \text{-covered by''}) \\
\iff &(a \text{ is } >_1 \text{-covered by } b) \qquad (\text{by (34)}) .
\end{aligned}$$

This proves Statement 2.]

Now, we shall prove the following statement:

Statement 3: If a and b are two elements of E such that a is $>_1$-covered by b, then a and b are $<_2$-comparable.

[*Proof of Statement 3:* Let a and b be two elements of E such that a is $>_1$-covered by b. We must show that a and b are $<_2$-comparable.

Statement 2 shows that we have the following logical equivalence:

$$(a \text{ is } >_1 \text{-covered by } b) \iff (b \text{ is } <_1 \text{-covered by } a) .$$

Hence, b is $<_1$-covered by a (since a is $>_1$-covered by b). Thus, Statement 1 (applied to b and a instead of a and b) yields that b and a are $<_2$-comparable. In other words, either $b <_2 a$ or $b = a$ or $a <_2 b$. In other words, either $a <_2 b$ or $b = a$ or $b <_2 a$. In other words, either $a <_2 b$ or $a = b$ or $b <_2 a$ (since $b = a$ is equivalent to $a = b$). In other words, a and b are $<_2$-comparable. This proves Statement 3.]

But the double poset $(E, >_1, <_2)$ is tertispecial if and only if Statement 3 holds (by the definition of "tertispecial"). Hence, the double poset $(E, >_1, <_2)$ is tertispecial (since Statement 3 holds). This completes the proof of Lemma 5.10. $\square$

6. Proof of Theorem 4.2

Before we come to the proof of Theorem 4.2, let us state five lemmas:

Lemma 6.1. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let P and Q be subsets of E such that $P \cap Q = \varnothing$ and $P \cup Q = E$. Assume that there exist no $p \in P$ and $q \in Q$ such that q is $<_1$-covered by p. Then, $(P, Q) \in \operatorname{Adm} \mathbf{E}$.

Proof of Lemma 6.1. For any $a \in E$ and $b \in E$, we let $[a, b]$ denote the subset $\{e \in E \mid a <_1 e <_1 b\}$ of E. It is easy to see that if a, b and c are three elements of E satisfying $a <_1 c <_1 b$, then both $[a, c]$ and $[c, b]$ are proper subsets of $[a, b]$, and therefore

$$\text{both numbers } |[a,c]| \text{ and } |[c,b]| \text{ are smaller than } |[a,b]|. \tag{35}$$

[*Proof of* (35)*:* Let a, b and c be three elements of E satisfying $a <_1 c <_1 b$.

The definition of $[a, b]$ yields $[a, b] = \{e \in E \mid a <_1 e <_1 b\}$. Hence, $c \in [a, b]$ (since $a <_1 c <_1 b$).

The definition of $[a, c]$ yields

$$[a,c] = \{e \in E \mid a <_1 e <_1 c\} \subseteq \{e \in E \mid a <_1 e <_1 b\}$$

(because every $e \in E$ satisfying $a <_1 e <_1 c$ must also satisfy $e <_1 c <_1 b$ and therefore $a <_1 e <_1 b$). Thus,

$$[a,c] \subseteq \{e \in E \mid a <_1 e <_1 b\} = [a,b].$$

If we had $[a, c] = [a, b]$, then we would have $c \in [a, b] = [a, c] = \{e \in E \mid a <_1 e <_1 c\}$ and therefore $a <_1 c <_1 c$; but this would contradict the fact that we don't have $c <_1 c$. Thus, we cannot have $[a, c] = [a, b]$. Thus, we have $[a, c] \neq [a, b]$. Combining this with $[a, c] \subseteq [a, b]$, we conclude that $[a, c]$ is a proper subset of $[a, b]$.

The definition of $[c, b]$ yields

$$[c,b] = \{e \in E \mid c <_1 e <_1 b\} \subseteq \{e \in E \mid a <_1 e <_1 b\}$$

(because every $e \in E$ satisfying $c <_1 e <_1 b$ must also satisfy $a <_1 c <_1 e$ and therefore $a <_1 e <_1 b$). Thus,

$$[c,b] \subseteq \{e \in E \mid a <_1 e <_1 b\} = [a,b].$$

If we had $[c,b] = [a,b]$, then we would have $c \in [a,b] = [c,b] = \{e \in E \mid c <_1 e <_1 b\}$ and therefore $c <_1 c <_1 b$; but this would contradict the fact that we don't have $c <_1 c$. Thus, we cannot have $[c,b] = [a,b]$. Thus, we have $[c,b] \neq [a,b]$. Combining this with $[c,b] \subseteq [a,b]$, we conclude that $[c,b]$ is a proper subset of $[a,b]$.

Thus, we have shown that both $[a,c]$ and $[c,b]$ are proper subsets of $[a,b]$. Hence, (35) follows (since $[a,b]$ is a finite set). This completes the proof of (35).]

A pair $(p,q) \in P \times Q$ is said to be a *malposition* if it satisfies $q <_1 p$. Now, let us assume (for the sake of contradiction) that there exists a malposition. Fix a malposition (u,v) for which the value $|[v,u]|$ is minimum. Thus, $(u,v) \in P \times Q$ and $v <_1 u$. From $(u,v) \in P \times Q$, we obtain $u \in P$ and $v \in Q$. Hence, v is not $<_1$-covered by u (since there exist no $p \in P$ and $q \in Q$ such that q is $<_1$-covered by p). Hence, there exists a $w \in E$ such that $v <_1 w <_1 u$ (since $v <_1 u$). Consider this w. Applying (35) to $a = v$, $c = w$ and $b = u$, we see that both numbers $|[v,w]|$ and $|[w,u]|$ are smaller than $|[v,u]|$, and therefore neither (w,v) nor (u,w) is a malposition (since we picked (u,v) to be a malposition with minimum $|[v,u]|$). But $w \in E = P \cup Q$, so that either $w \in P$ or $w \in Q$. If $w \in P$, then (w,v) is a malposition; if $w \in Q$, then (u,w) is a malposition. In either case, we obtain a contradiction to the fact that neither (w,v) nor (u,w) is a malposition. This contradiction shows that our assumption was wrong. Hence, there exists no malposition. In other words, there exists no $(p,q) \in P \times Q$ satisfying $q <_1 p$ (since this is what "malposition" means). In other words, no $p \in P$ and $q \in Q$ satisfy $q <_1 p$. Consequently, $(P,Q) \in \operatorname{Adm} \mathbf{E}$. This proves Lemma 6.1. □

Lemma 6.2. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $(P,Q) \in \operatorname{Adm} \mathbf{E}$. Then, $\mathbf{E} \mid_P$ is a tertispecial double poset.

Proof of Lemma 6.2. Recall that we are using the symbol $<_1$ to denote two different relations: a strict partial order on E, and its restriction to P. This abuse of notation is usually harmless, but in the current proof it is dangerous, because it causes the statement "a is $<_1$-covered by b" (for two elements a and b of P) to carry two meanings (depending on whether the symbol $<_1$ is interpreted as the strict partial order on E, or as its restriction to P). (These two meanings are actually equivalent, but their equivalence is not immediately obvious.)

Thus, for the duration of this proof, we shall revert to a less ambiguous notation. Namely, the notation $<_1$ shall only be used for the strict partial order on E which constitutes part of the double poset $\mathbf{E}$. The restriction of this partial order $<_1$ to the subset P will be denoted by $<_{1,P}$ (not by $<_1$). Similarly, the restriction of the partial order $<_2$ to the subset P will be denoted by $<_{2,P}$ (not by $<_2$). Thus, the double poset $\mathbf{E} \mid_P$ is defined as $\mathbf{E} \mid_P = (P, <_{1,P}, <_{2,P})$.

We need to show that the double poset $\mathbf{E} \mid_P = (P, <_{1,P}, <_{2,P})$ is tertispecial. In other words, we need to show that if a and b are two elements of P such that a is $<_{1,P}$-covered by b, then a and b are $<_{2,P}$-comparable.

Let a and b be two elements of P such that a is $<_{1,P}$-covered by b. Thus, $a <_{1,P} b$, and

$$\text{there exists no } c \in P \text{ satisfying } a <_{1,P} c <_{1,P} b. \tag{36}$$

We have $a <_{1,P} b$. In other words, $a <_1 b$ (since $<_{1,P}$ is the restriction of the relation $<_1$ to P).

Now, if $c \in E$ is such that $a <_1 c <_1 b$, then c must belong to P [34], and therefore satisfy $a <_{1,P} c <_{1,P} b$ [35], which entails a contradiction to (36). Thus, there is no $c \in E$ satisfying $a <_1 c <_1 b$. Therefore (and because we have $a <_1 b$), we see that a is $<_1$-covered by b. Since $\mathbf{E}$ is tertispecial, this yields that a and b are $<_2$-comparable. In other words, either $a <_2 b$ or $a = b$ or $b <_2 a$. Since a and b both belong to P, we can rewrite this by replacing the relation $<_2$ by its restriction $<_{2,P}$. We thus conclude that either $a <_{2,P} b$ or $a = b$ or $b <_{2,P} a$. In other words, a and b are $<_{2,P}$-comparable.

Now, forget that we fixed a and b. Thus, we have shown that if a and b are two elements of P such that a is $<_{1,P}$-covered by b, then a and b are $<_{2,P}$-comparable. This completes the proof of Lemma 6.2.

(We could similarly show that $\mathbf{E} \mid_Q$ is a tertispecial double poset; but we will not use this.) $\square$

Lemma 6.3. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map.

(a) If $E = \varnothing$, then $\Gamma(\mathbf{E}, w) = 1$.

(b) If $E \neq \varnothing$, then $\varepsilon(\Gamma(\mathbf{E}, w)) = 0$.

Proof of Lemma 6.3. (a) Part (a) is obvious (since there is only one $\mathbf{E}$-partition π when $E = \varnothing$, and since this $\mathbf{E}$-partition π satisfies $\mathbf{x}_{\pi,w} = 1$).

(b) Observe that $\Gamma(\mathbf{E}, w)$ is a homogeneous power series of degree $\sum\limits_{e \in E} w(e)$. When $E \neq \varnothing$, this degree is > 0 (since it is then a nonempty sum of positive integers), and thus the power series $\Gamma(\mathbf{E}, w)$ is annihilated by ε (since ε annihilates any homogeneous power series in QSym whose degree is > 0). $\square$

Lemma 6.4. Let $(E, <_1, <_2)$ be a double poset. Let $>_1$ be the opposite relation of $<_1$. Let P and Q be two subsets of E satisfying $P \cup Q = E$. Let $\pi : E \to \{1, 2, 3, \ldots\}$ be a map such that $\pi \mid_P$ is a $(P, >_1, <_2)$-partition. Let $f \in P$. Assume that

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{38}$$

[34] *Proof.* Assume the contrary. Thus, $c \notin P$. But $(P, Q) \in \operatorname{Adm} \mathbf{E}$. Thus, $P \cap Q = \varnothing$, $P \cup Q = E$, and

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{37}$$

From $c \in E$ and $c \notin P$, we obtain $c \in E \setminus P \subseteq Q$ (since $P \cup Q = E$). Applying (37) to $p = b$ and $q = c$, we thus conclude that we cannot have $c <_1 b$. This contradicts $c <_1 b$. This contradiction shows that our assumption was false, qed.

[35] *Proof.* Let $c \in E$ be such that $a <_1 c <_1 b$. Then, c must belong to P (as we have just proven). Now, $a <_1 c$. In light of $a \in P$ and $c \in P$, this rewrites as $a <_{1,P} c$ (since $<_{1,P}$ is the restriction of the relation $<_1$ to P). Similarly, $c <_1 b$ rewrites as $c <_{1,P} b$. Thus, $a <_{1,P} c <_{1,P} b$, qed.

Also, assume that

$$\pi(f) \leq \pi(h) \qquad \text{for every } h \in E. \tag{39}$$

Furthermore, assume that

$$\pi(f) < \pi(h) \qquad \text{for every } h \in E \text{ satisfying } h <_2 f. \tag{40}$$

(a) If $p \in P \setminus \{f\}$ and $q \in Q \cup \{f\}$ are such that $q <_1 p$, then we have neither $q <_2 p$ nor $p <_2 q$.

(b) If $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition, then $\pi \mid_{Q \cup \{f\}}$ is a $(Q \cup \{f\}, <_1, <_2)$-partition.

Proof of Lemma 6.4. From $P \cup Q = E$, we obtain $\underbrace{E}_{=P \cup Q} \setminus P = (P \cup Q) \setminus P \subseteq Q$.

(a) Let $p \in P \setminus \{f\}$ and $q \in Q \cup \{f\}$ be such that $q <_1 p$. We must show that we have neither $q <_2 p$ nor $p <_2 q$.

Indeed, assume the contrary. Thus, we have either $q <_2 p$ or $p <_2 q$.

We have $q <_1 p$ and $p \in P \setminus \{f\} \subseteq P$. Hence, if we had $q \in Q$, then we would obtain a contradiction to (38). Hence, we cannot have $q \in Q$. Therefore, $q = f$ (since $q \in Q \cup \{f\}$ but not $q \in Q$). Hence, $f = q <_1 p$, so that $p >_1 f$. Therefore, $\pi(p) \leq \pi(f)$ (since $\pi \mid_P$ is a $(P, >_1, <_2)$-partition, and since both f and p belong to P).

Now, recall that we have either $q <_2 p$ or $p <_2 q$. Since $q = f$, we can rewrite this as follows: We have either $f <_2 p$ or $p <_2 f$. But $p <_2 f$ cannot hold (because if we had $p <_2 f$, then (40) (applied to $h = p$) would lead to $\pi(f) < \pi(p)$, which would contradict $\pi(p) \leq \pi(f)$). Thus, we must have $f <_2 p$.

But $\pi \mid_P$ is a $(P, >_1, <_2)$-partition. Hence, $\pi(p) < \pi(f)$ (since $p >_1 f$ and $f <_2 p$, and since p and f both lie in P). But (39) (applied to $h = p$) shows that $\pi(f) \leq \pi(p)$. Hence, $\pi(p) < \pi(f) \leq \pi(p)$, a contradiction. Thus, our assumption was wrong. This completes the proof of Lemma 6.4 (a).

(b) Assume that $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition. We need to show that $\pi \mid_{Q \cup \{f\}}$ is a $(Q \cup \{f\}, <_1, <_2)$-partition. In order to prove this, we need to verify the following two claims:

Claim 1: Every $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ satisfying $a <_1 b$ satisfy $\pi(a) \leq \pi(b)$.

Claim 2: Every $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ satisfying $a <_1 b$ and $b <_2 a$ satisfy $\pi(a) < \pi(b)$.

Proof of Claim 1: Let $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ be such that $a <_1 b$. We need to prove that $\pi(a) \leq \pi(b)$. If $a = f$, then this follows immediately from (39) (applied to $h = b$). Hence, we WLOG assume that $a \neq f$. Thus, $a \in Q$ (since $a \in Q \cup \{f\}$). Now, if $b \in P$, then $a <_1 b$ contradicts (38) (applied to $p = b$ and $q = a$). Hence, we cannot have $b \in P$. Therefore, $b \in E \setminus P \subseteq Q$. Thus, $\pi(a) \leq \pi(b)$ follows immediately from the fact that $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition (since $a \in Q$ and

$b \in Q$ and $a <_1 b$). This proves Claim 1.

Proof of Claim 2: Let $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ be such that $a <_1 b$ and $b <_2 a$. We need to prove that $\pi(a) < \pi(b)$. If $a = f$, then this follows immediately from (40) (applied to $h = b$) (because if $a = f$, then $b <_2 a = f$). Hence, we WLOG assume that $a \neq f$. Thus, $a \in Q$ (since $a \in Q \cup \{f\}$). Now, if $b \in P$, then $a <_1 b$ contradicts (38) (applied to $p = b$ and $q = a$). Hence, we cannot have $b \in P$. Therefore, $b \in E \setminus P \subseteq Q$. Thus, $\pi(a) < \pi(b)$ follows immediately from the fact that $\pi\mid_Q$ is a $(Q, <_1, <_2)$-partition (since $a \in Q$ and $b \in Q$ and $a <_1 b$ and $b <_2 a$). This proves Claim 2.

Now, both Claim 1 and Claim 2 are proven. As already said, this completes the proof of Lemma 6.4 (b). □

Lemma 6.5. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset satisfying $|E| > 0$. Let $\pi : E \to \{1, 2, 3, \ldots\}$ be a map. Let $>_1$ denote the opposite relation of $<_1$. Then,

$$\sum_{\substack{(P,Q) \in \operatorname{Adm} \mathbf{E}; \\ \pi\mid_P \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \pi\mid_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} (-1)^{|P|} = 0. \tag{41}$$

Proof of Lemma 6.5. Our goal is to prove (41). To do so, we denote by Z the set of all $(P, Q) \in \operatorname{Adm} \mathbf{E}$ such that $\pi\mid_P$ is a $(P, >_1, <_2)$-partition and $\pi\mid_Q$ is a $(Q, <_1, <_2)$-partition. We are going to define an involution $T : Z \to Z$ of the set Z having the following property:

> *Property P:* Let $(P, Q) \in Z$. If we write $T((P, Q))$ in the form (P', Q'), then $(-1)^{|P'|} = -(-1)^{|P|}$.

Once such an involution T is found, the equality (41) will follow[36] and we will be done. It thus remains to find T.

[36] Here is the argument in detail:

Assume that we have found an involution $T : Z \to Z$ of the set Z satisfying Property P. Consider this T. Then, for any $(P, Q) \in Z$, if we write $T((P, Q))$ in the form (P', Q'), then

$$(-1)^{|P'|} = -(-1)^{|P|} \tag{42}$$

(because Property P is satisfied). We now need to prove the equality (41).

The map Z is an involution, and thus a bijection.

Now, let Z_0 be the subset $\{(P, Q) \in Z \mid |P| \text{ is even}\}$ of Z. Thus, for every $(P, Q) \in Z$, we have the following logical equivalence:

$$((P, Q) \in Z_0) \iff (|P| \text{ is even}). \tag{43}$$

Hence, for every $(P, Q) \in Z$, we have the following logical equivalence:

$$\begin{aligned} ((P, Q) \notin Z_0) &\iff (|P| \text{ is not even}) \\ &\qquad \left(\begin{array}{c} \text{this equivalence is obtained from (43)} \\ \text{by replacing each part by its negation} \end{array}\right) \\ &\iff (|P| \text{ is odd}). \end{aligned} \tag{44}$$

The definition of the map $T : Z \to Z$ is simple (although it will take us a while to prove that it is well-defined): Let F be the subset of E consisting of those $e \in E$

Now, for every $(P, Q) \in Z$, we have the following logical equivalence:

$$((P, Q) \in Z_0) \iff (T((P, Q)) \notin Z_0) . \tag{45}$$

Proof of (45): Let $(P, Q) \in Z$. Write $T((P, Q)) \in Z$ in the form (P', Q'). Then, (44) (applied to (P', Q') instead of (P, Q)) shows that we have the following logical equivalence:

$$((P', Q') \notin Z_0) \iff (|P'| \text{ is odd}) .$$

Thus, we have the following logical equivalence:

$$\begin{aligned} ((P', Q') \notin Z_0) &\iff (|P'| \text{ is odd}) \iff \left(\underbrace{(-1)^{|P'|}}_{\substack{=-(-1)^{|P|} \\ \text{(by (42))}}} = -1 \right) \iff \left(-(-1)^{|P|} = -1 \right) \\ &\iff \left((-1)^{|P|} = 1 \right) \iff (|P| \text{ is even}) \iff ((P, Q) \in Z_0) \\ &\qquad (\text{by (43)}) . \end{aligned}$$

Hence, we have the following logical equivalence:

$$((P, Q) \in Z_0) \iff \left(\underbrace{(P', Q')}_{=T((P,Q))} \notin Z_0 \right) \iff (T((P, Q)) \notin Z_0) .$$

This proves (45).

Now,

$$\sum_{\substack{(P,Q)\in Z; \\ |P| \text{ is even}}} \underbrace{(-1)^{|P|}}_{\substack{=1 \\ (\text{since } |P| \text{ is even})}} = \underbrace{\sum_{\substack{(P,Q)\in Z; \\ |P| \text{ is even}}}}_{\substack{= \sum_{\substack{(P,Q)\in Z; \\ (P,Q)\in Z_0}} \\ \text{(because for every } (P,Q)\in Z, \text{ the condition} \\ (|P| \text{ is even}) \text{ is equivalent to } ((P,Q)\in Z_0) \\ \text{(by (43)))}}} 1 = \sum_{\substack{(P,Q)\in Z; \\ (P,Q)\in Z_0}} 1$$

for which the value $\pi(e)$ is minimum. Then, F is a nonempty subposet[37] of the poset $(E, <_2)$, and hence has a minimal element[38] f (that is, an element f such that no $g \in F$ satisfies $g <_2 f$). Fix such an f. Now, the map T sends a $(P,Q) \in Z$ to $\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases}$.

In order to prove that the map T is well-defined, we need to prove that its output

and

$$\sum_{\substack{(P,Q)\in Z;\\ |P| \text{ is odd}}} \underbrace{(-1)^{|P|}}_{\substack{=-1\\ (\text{since } |P| \text{ is odd})}} = \underbrace{\sum_{\substack{(P,Q)\in Z;\\ |P| \text{ is odd}}}}_{\substack{= \sum\limits_{\substack{(P,Q)\in Z;\\ (P,Q)\notin Z_0}} \\ \text{(because for every } (P,Q)\in Z \text{, the condition} \\ (|P| \text{ is odd) is equivalent to } ((P,Q)\notin Z_0) \\ \text{(by (44)))}}} (-1) = \sum_{\substack{(P,Q)\in Z;\\ (P,Q)\notin Z_0}} (-1)$$

$$= \underbrace{\sum_{\substack{(P,Q)\in Z;\\ T((P,Q))\notin Z_0}}}_{\substack{= \sum\limits_{\substack{(P,Q)\in Z;\\ (P,Q)\in Z_0}} \\ \text{(because for every } (P,Q)\in Z \text{, the condition} \\ (T((P,Q))\notin Z_0) \text{ is equivalent to } ((P,Q)\in Z_0) \\ \text{(by (45)))}}} (-1)$$

$$\left(\begin{array}{c} \text{here, we have substituted } T((P,Q)) \text{ for } (P,Q) \\ \text{in the sum, since the map } T : Z \to Z \text{ is a bijection} \end{array} \right)$$

$$= \sum_{\substack{(P,Q)\in Z;\\ (P,Q)\in Z_0}} (-1) = - \sum_{\substack{(P,Q)\in Z;\\ (P,Q)\in Z_0}} 1.$$

Finally,

$$\underbrace{\sum_{\substack{(P,Q)\in \operatorname{Adm} \mathbf{E};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}}}_{\substack{= \sum\limits_{(P,Q)\in Z} \\ \text{(by the definition of } Z)}} (-1)^{|P|} = \sum_{(P,Q)\in Z} (-1)^{|P|} = \underbrace{\sum_{\substack{(P,Q)\in Z;\\ |P| \text{ is even}}} (-1)^{|P|}}_{= \sum\limits_{\substack{(P,Q)\in Z;\\ (P,Q)\in Z_0}} 1} + \underbrace{\sum_{\substack{(P,Q)\in Z;\\ |P| \text{ is odd}}} (-1)^{|P|}}_{= - \sum\limits_{\substack{(P,Q)\in Z;\\ (P,Q)\in Z_0}} 1}$$

$$= \sum_{\substack{(P,Q)\in Z;\\ (P,Q)\in Z_0}} 1 + \left(- \sum_{\substack{(P,Q)\in Z;\\ (P,Q)\in Z_0}} 1 \right) = 0.$$

Thus, (41) is proven.

[37]The nonemptiness of F follows from the nonemptiness of E (which, in turn, follows from $|E| > 0$).

[38]A *minimal element* of a poset $(P, \prec)$ is an element $p \in P$ such that no $g \in P$ satisfies $g \prec p$. It is well-known that every nonempty finite poset has at least one minimal element. We are using this fact here.

values all belong to Z. In other words, we need to prove that

$$\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} \in Z \tag{46}$$

for every $(P, Q) \in Z$.

Proof of (46)*:* Fix $(P, Q) \in Z$. Thus, (P, Q) is an element of $\operatorname{Adm} \mathbf{E}$ with the property that $\pi \mid_P$ is a $(P, >_1, <_2)$-partition and $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition (by the definition of Z).

From $(P, Q) \in \operatorname{Adm} \mathbf{E}$, we see that $P \cap Q = \varnothing$ and $P \cup Q = E$, and furthermore that

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{47}$$

We know that f belongs to the set F, which is the subset of E consisting of those $e \in E$ for which the value $\pi(e)$ is minimum. Thus,

$$\pi(f) \leq \pi(h) \qquad \text{for every } h \in E. \tag{48}$$

Moreover,

$$\pi(f) < \pi(h) \qquad \text{for every } h \in E \text{ satisfying } h <_2 f \tag{49}$$

[39].

We need to prove (46). We are in one of the following two cases:

Case 1: We have $f \in P$.

Case 2: We have $f \notin P$.

Let us first consider Case 1. In this case, we have $f \in P$.

Recall that $P \cap Q = \varnothing$ and $P \cup Q = E$. From this, we easily obtain $(P \setminus \{f\}) \cap (Q \cup \{f\}) = \varnothing$ and $(P \setminus \{f\}) \cup (Q \cup \{f\}) = E$.

Furthermore, there exist no $p \in P \setminus \{f\}$ and $q \in Q \cup \{f\}$ such that q is $<_1$-covered by p [40]. Hence, Lemma 6.1 (applied to $P \setminus \{f\}$ and $Q \cup \{f\}$ instead of P and Q) shows that $(P \setminus \{f\}, Q \cup \{f\}) \in \operatorname{Adm} \mathbf{E}$.

[39] *Proof of* (49)*:* Let $h \in E$ be such that $h <_2 f$. We must prove (49). Indeed, assume the contrary. Thus, $\pi(f) \geq \pi(h)$. But every $g \in E$ satisfies $\pi(f) \leq \pi(g)$ (by (48), applied to g instead of h). Hence, every $g \in E$ satisfies $\pi(g) \geq \pi(f) \geq \pi(h)$. In other words, h is one of those $e \in E$ for which the value $\pi(e)$ is minimum.

But recall that F is the subset of E consisting of those $e \in E$ for which the value $\pi(e)$ is minimum. Since h is one of these $e \in E$, we thus conclude that $h \in F$. Recall also that $h <_2 f$. Hence, there exists some $g \in F$ satisfying $g <_2 f$ (namely, $g = h$). But f is a minimal element of the subposet F of $(E, <_2)$. In other words, no $g \in F$ satisfies $g <_2 f$. This contradicts the fact that there exists some $g \in F$ satisfying $g <_2 f$. This contradiction proves that our assumption was wrong, qed.

[40] *Proof.* Assume the contrary. Thus, there exist $p \in P \setminus \{f\}$ and $q \in Q \cup \{f\}$ such that q is $<_1$-covered by p. Consider such p and q.

We know that q is $<_1$-covered by p, and thus we have $q <_1 p$. Hence, Lemma 6.4 (a) shows that we have neither $q <_2 p$ nor $p <_2 q$. On the other hand, q is $<_1$-covered by p. Hence, q and p are $<_2$-comparable (since $\mathbf{E}$ is tertispecial). In other words, we have either $q <_2 p$ or $q = p$ or $p <_2 q$. Hence, we must have $q = p$ (since we have neither $q <_2 p$ nor $p <_2 q$). But this contradicts $q <_1 p$. This contradiction shows that our assumption was wrong, qed.

Furthermore, $\pi\mid_P$ is a $(P,>_1,<_2)$-partition. Hence, $\pi\mid_{P\setminus\{f\}}$ is a $(P\setminus\{f\},>_1,<_2)$-partition (since $P\setminus\{f\}\subseteq P$).

Furthermore, $\pi\mid_{Q\cup\{f\}}$ is a $(Q\cup\{f\},<_1,<_2)$-partition[41].

Altogether, we now know that $(P\setminus\{f\},Q\cup\{f\})\in\operatorname{Adm}\mathbf{E}$, that $\pi\mid_{P\setminus\{f\}}$ is a $(P\setminus\{f\},>_1,<_2)$-partition, and that $\pi\mid_{Q\cup\{f\}}$ is a $(Q\cup\{f\},<_1,<_2)$-partition. In other words, $(P\setminus\{f\},Q\cup\{f\})\in Z$ (by the definition of Z). Thus,

$$\begin{aligned}\begin{cases}(P\cup\{f\},Q\setminus\{f\}), & \text{if } f\notin P;\\ (P\setminus\{f\},Q\cup\{f\}), & \text{if } f\in P\end{cases} &= (P\setminus\{f\},Q\cup\{f\}) \qquad (\text{since } f\in P)\\ &\in Z.\end{aligned}$$

Hence, (46) is proven in Case 1.

Let us next consider Case 2. In this case, we have $f\notin P$. Hence, $f\in E\setminus P=Q$ (since $P\cap Q=\varnothing$ and $P\cup Q=E$).

Recall that $P\cap Q=\varnothing$ and $P\cup Q=E$. From this, we easily obtain $(P\cup\{f\})\cap(Q\setminus\{f\})=\varnothing$ and $(P\cup\{f\})\cup(Q\setminus\{f\})=E$.

We have $f\in Q$ and $Q\cup P=P\cup Q=E$. Furthermore, $>_1$ is the opposite relation of $<_1$, and thus is a strict partial order (since $<_1$ is a strict partial order). Hence, $(E,>_1,<_2)$ is a double poset. Furthermore, the relation $<_1$ is the opposite relation of $>_1$ (since $>_1$ is the opposite relation of $<_1$). The map $\pi\mid_Q$ is a $(Q,<_1,<_2)$-partition. Moreover,

$$\text{no } p\in Q \text{ and } q\in P \text{ satisfy } q>_1 p \tag{50}$$

[42]. Hence, we can apply Lemma 6.4 to $(E,>_1,<_2)$, $<_1$, Q and P instead of $(E,<_1,<_2)$, $>_1$, P and Q.

There exist no $p\in P\cup\{f\}$ and $q\in Q\setminus\{f\}$ such that q is $<_1$-covered by p [43]. Hence, Lemma 6.1 (applied to $P\cup\{f\}$ and $Q\setminus\{f\}$ instead of P and Q) shows that $(P\cup\{f\},Q\setminus\{f\})\in\operatorname{Adm}\mathbf{E}$.

Furthermore, $\pi\mid_Q$ is a $(Q,<_1,<_2)$-partition. Hence, $\pi\mid_{Q\setminus\{f\}}$ is a $(Q\setminus\{f\},<_1,<_2)$-partition (since $Q\setminus\{f\}\subseteq Q$).

[41]This follows from Lemma 6.4 (b) (since $\pi\mid_Q$ is a $(Q,<_1,<_2)$-partition).

[42]*Proof.* Let $a\in Q$ and $b\in P$ be such that $b>_1 a$. We shall derive a contradiction.

We have $b>_1 a$. In other words, $a<_1 b$. Thus, $b\in P$ and $a\in Q$ satisfy $a<_1 b$. This contradicts (47) (applied to $p=b$ and $q=a$).

Now, forget that we fixed a and b. We thus have found a contradiction for every $a\in Q$ and $b\in P$ satisfying $b>_1 a$. Hence, no $a\in Q$ and $b\in P$ satisfy $b>_1 a$. Renaming a and b as p and q in this statement, we obtain the following: No $p\in Q$ and $q\in P$ satisfy $q>_1 p$. This proves (50).

[43]*Proof.* Assume the contrary. Thus, there exist $p\in P\cup\{f\}$ and $q\in Q\setminus\{f\}$ such that q is $<_1$-covered by p. Consider such p and q.

We know that q is $<_1$-covered by p, and thus we have $q<_1 p$. In other words, $p>_1 q$. Thus, Lemma 6.4 (a) (applied to $(E,>_1,<_2)$, $<_1$, Q, P, q and p instead of $(E,<_1,<_2)$, $>_1$, P, Q, p and q) yields that we have neither $p<_2 q$ nor $q<_2 p$. On the other hand, q is $<_1$-covered by p. Hence, q and p are $<_2$-comparable (since $\mathbf{E}$ is tertispecial). In other words, we have either $q<_2 p$ or $q=p$ or $p<_2 q$. Hence, we must have $q=p$ (since we have neither $p<_2 q$ nor $q<_2 p$). But this contradicts $q<_1 p$. This contradiction shows that our assumption was wrong, qed.

Furthermore, $\pi\mid_{P\cup\{f\}}$ is a $(P\cup\{f\}, >_1, <_2)$-partition[44].

Altogether, we now know that $(P\cup\{f\}, Q\setminus\{f\}) \in \operatorname{Adm}\mathbf{E}$, that $\pi\mid_{P\cup\{f\}}$ is a $(P\cup\{f\}, >_1, <_2)$-partition, and that $\pi\mid_{Q\setminus\{f\}}$ is a $(Q\setminus\{f\}, <_1, <_2)$-partition. In other words, $(P\cup\{f\}, Q\setminus\{f\}) \in Z$ (by the definition of Z). Thus,

$$\begin{cases} (P\cup\{f\}, Q\setminus\{f\}), & \text{if } f\notin P; \\ (P\setminus\{f\}, Q\cup\{f\}), & \text{if } f\in P \end{cases} = (P\cup\{f\}, Q\setminus\{f\}) \qquad (\text{since } f\notin P)$$
$$\in Z.$$

Hence, (46) is proven in Case 2.

We have now proven (46) in both Cases 1 and 2. Thus, (46) always holds. In other words, the map T is well-defined.

Every $\alpha\in Z$ satisfies $(T\circ T)(\alpha) = \operatorname{id}(\alpha)$ [45]. In other words, $T\circ T = \operatorname{id}$. In

[44] This follows from Lemma 6.4 (b) (applied to $(E, >_1, <_2)$, $<_1$, Q and P instead of $(E, <_1, <_2)$, $>_1$, P and Q), since $\pi\mid_P$ is a $(P, >_1, <_2)$-partition.

[45] *Proof.* Let $\alpha\in Z$. We want to show that $(T\circ T)(\alpha) = \operatorname{id}(\alpha)$.

We have $\alpha\in Z$. In other words, α can be written in the form $\alpha = (P,Q)$ for some $(P,Q)\in \operatorname{Adm}\mathbf{E}$ having the property that $\pi\mid_P$ is a $(P, >_1, <_2)$-partition and $\pi\mid_Q$ is a $(Q, <_1, <_2)$-partition (by the definition of Z). Write α in this form.

From $(P,Q)\in\operatorname{Adm}\mathbf{E}$, we see that $P\cap Q = \varnothing$ and $P\cup Q = E$, and furthermore that no $p\in P$ and $q\in Q$ satisfy $q <_1 p$. From $P\cap Q=\varnothing$ and $P\cup Q = E$, we conclude that $P = E\setminus Q$ and $Q = E\setminus P$.

We are in one of the following two cases:

Case 1: We have $f\in P$.

Case 2: We have $f\notin P$.

Let us first consider Case 1. In this case, we have $f\in P$. Hence, $f\notin E\setminus P = Q$. Clearly, $f\notin P\setminus\{f\}$ (since $f\in\{f\}$) and $\{f\}\subseteq P$ (since $f\in P$). Furthermore, the sets Q and $\{f\}$ are disjoint (since $f\notin Q$). Now,

$$T\left(\underbrace{\alpha}_{=(P,Q)}\right) = T((P,Q)) = \begin{cases} (P\cup\{f\}, Q\setminus\{f\}), & \text{if } f\notin P; \\ (P\setminus\{f\}, Q\cup\{f\}), & \text{if } f\in P \end{cases} \qquad (\text{by the definition of } T)$$
$$= (P\setminus\{f\}, Q\cup\{f\}) \qquad (\text{since } f\in P).$$

Now,

$$\begin{aligned}(T\circ T)(\alpha) &= T\left(\underbrace{T(\alpha)}_{=(P\setminus\{f\}, Q\cup\{f\})}\right) = T((P\setminus\{f\}, Q\cup\{f\})) \\ &= \begin{cases} ((P\setminus\{f\})\cup\{f\}, (Q\cup\{f\})\setminus\{f\}), & \text{if } f\notin P\setminus\{f\}; \\ ((P\setminus\{f\})\setminus\{f\}, (Q\cup\{f\})\cup\{f\}), & \text{if } f\in P\setminus\{f\} \end{cases} \qquad (\text{by the definition of } T) \\ &= \left(\underbrace{(P\setminus\{f\})\cup\{f\}}_{\substack{=P \\ (\text{since } \{f\}\subseteq P)}}, \underbrace{(Q\cup\{f\})\setminus\{f\}}_{\substack{=Q \\ (\text{since the sets } Q \text{ and } \{f\} \text{ are disjoint})}}\right) \qquad (\text{since } f\notin P\setminus\{f\}) \\ &= (P,Q) = \alpha = \operatorname{id}(\alpha).\end{aligned}$$

other words, the map T is an involution. Furthermore, this involution T satisfies Property P [46]. We thus have defined an involution $T : Z \to Z$ of the set Z satisfying Property P. This was precisely our goal. As we have already explained, this proves (41). Hence, Lemma 6.5 is proven. $\square$

Hence, $(T \circ T)(\alpha) = \mathrm{id}(\alpha)$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $f \notin P$. Hence, $f \in E \setminus P = Q$. Clearly, $f \in \{f\} \subseteq P \cup \{f\}$. Also, $\{f\} \subseteq Q$ (since $f \in Q$). Furthermore, the sets P and $\{f\}$ are disjoint (since $f \notin P$). Now,

$$T\left(\underbrace{\alpha}_{=(P,Q)}\right) = T((P,Q)) = \begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} \qquad \text{(by the definition of } T\text{)}$$
$$= (P \cup \{f\}, Q \setminus \{f\}) \qquad (\text{since } f \notin P).$$

Now,

$$(T \circ T)(\alpha) = T\left(\underbrace{T(\alpha)}_{=(P\cup\{f\},Q\setminus\{f\})}\right) = T((P \cup \{f\}, Q \setminus \{f\}))$$
$$= \begin{cases} ((P \cup \{f\}) \cup \{f\}, (Q \setminus \{f\}) \setminus \{f\}), & \text{if } f \notin P \cup \{f\}; \\ ((P \cup \{f\}) \setminus \{f\}, (Q \setminus \{f\}) \cup \{f\}), & \text{if } f \in P \cup \{f\} \end{cases} \qquad \text{(by the definition of } T\text{)}$$
$$= \left(\underbrace{(P \cup \{f\}) \setminus \{f\}}_{\substack{=P \\ \text{(since the sets } P \text{ and } \{f\} \text{ are disjoint)}}}, \underbrace{(Q \setminus \{f\}) \cup \{f\}}_{\substack{=Q \\ (\text{since } \{f\} \subseteq Q)}}\right) \qquad (\text{since } f \in P \cup \{f\})$$
$$= (P,Q) = \alpha = \mathrm{id}(\alpha).$$

Hence, $(T \circ T)(\alpha) = \mathrm{id}(\alpha)$ is proven in Case 2.

We have now proven $(T \circ T)(\alpha) = \mathrm{id}(\alpha)$ in both Cases 1 and 2. Thus, $(T \circ T)(\alpha) = \mathrm{id}(\alpha)$ always holds. Qed.

[46] *Proof.* Let $(P,Q) \in Z$. Write $T((P,Q))$ in the form (P',Q'). Then, we must prove that $(-1)^{|P'|} = -(-1)^{|P|}$.

We are in one of the following two cases:

Case 1: We have $f \in P$.

Case 2: We have $f \notin P$.

Let us first consider Case 1. In this case, we have $f \in P$. Now,

$$(P',Q') = T((P,Q)) = \begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} \qquad \text{(by the definition of } T\text{)}$$
$$= (P \setminus \{f\}, Q \cup \{f\}) \qquad (\text{since } f \in P).$$

In other words, $P' = P \setminus \{f\}$ and $Q' = Q \cup \{f\}$. Now, $\left|\underbrace{P'}_{=P\setminus\{f\}}\right| = |P \setminus \{f\}| = |P| - 1$ (since $f \in P$), and thus $(-1)^{|P'|} = (-1)^{|P|-1} = -(-1)^{|P|}$. Hence, $(-1)^{|P'|} = -(-1)^{|P|}$ is proven in Case 1.

Proof of Theorem 4.2. We shall prove Theorem 4.2 by strong induction over $|E|$. The induction step proceeds as follows: Consider a tertispecial double poset $\mathbf{E} = (E, <_1, <_2)$ and a map $w : E \to \{1, 2, 3, \ldots\}$, and assume (as the induction hypothesis) that Theorem 4.2 is proven for all tertispecial double posets of smaller size[47]. More precisely: Assume (as the induction hypothesis) that every tertispecial double poset $(P, \prec_1, \prec_2)$ satisfying $|P| < |E|$ and every map $x : P \to \{1, 2, 3, \ldots\}$ satisfy

$$S\left(\Gamma\left((P, \prec_1, \prec_2), x\right)\right) = (-1)^{|P|} \Gamma\left((P, \succ_1, \prec_2), x\right), \tag{51}$$

where $\succ_1$ denotes the opposite relation of $\prec_1$. Our goal is to show that $S\left(\Gamma\left((E, <_1, <_2), w\right)\right) = (-1)^{|E|} \Gamma\left((E, >_1, <_2), w\right)$. Here, as usual, $>_1$ denotes the opposite relation of $<_1$.

If $E = \varnothing$, then this is easy[48]. Thus, we WLOG assume that $E \neq \varnothing$. Hence, $|E| > 0$. Moreover, Lemma 6.3 (b) shows that $\varepsilon\left(\Gamma\left(\mathbf{E}, w\right)\right) = 0$. Thus, $(u \circ \varepsilon)\left(\Gamma\left(\mathbf{E}, w\right)\right) = u\left(\underbrace{\varepsilon\left(\Gamma\left(\mathbf{E}, w\right)\right)}_{=0}\right) = u(0) = 0$.

Using the induction hypothesis, we can see the following: If $(P, Q) \in \operatorname{Adm} \mathbf{E}$ is such that $(P, Q) \neq (E, \varnothing)$, then

$$S\left(\Gamma\left(\mathbf{E} \mid_P, w \mid_P\right)\right) = (-1)^{|P|} \Gamma\left((P, >_1, <_2), w \mid_P\right) \tag{52}$$

Let us now consider Case 2. In this case, we have $f \notin P$. Now,

$$\begin{aligned}(P', Q') = T\left((P, Q)\right) &= \begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} \qquad \text{(by the definition of } T\text{)} \\ &= (P \cup \{f\}, Q \setminus \{f\}) \qquad \text{(since } f \notin P\text{)}.\end{aligned}$$

In other words, $P' = P \cup \{f\}$ and $Q' = Q \setminus \{f\}$. Now, $\left|\underbrace{P'}_{=P \cup \{f\}}\right| = |P \cup \{f\}| = |P| + 1$ (since $f \notin P$), and thus $(-1)^{|P'|} = (-1)^{|P|+1} = -(-1)^{|P|}$. Hence, $(-1)^{|P'|} = -(-1)^{|P|}$ is proven in Case 2.

We have now proven $(-1)^{|P'|} = -(-1)^{|P|}$ in both Cases 1 and 2. Thus, $(-1)^{|P'|} = -(-1)^{|P|}$ always holds. This completes the proof of Property P.

[47]The *size* of a double poset $(P, <_1, <_2)$ means the nonnegative integer $|P|$.

[48]Hint: If $E = \varnothing$, then both $\Gamma\left((E, <_1, <_2), w\right)$ and $\Gamma\left((E, >_1, <_2), w\right)$ are equal to 1 (by Lemma 6.3 (a)), but the antipode S satisfies $S(1) = 1$ and $(-1)^{|\varnothing|} = 1$.

[49]. Furthermore, it is straightforward to see that $(E, \varnothing) \in \operatorname{Adm} \mathbf{E}$. Notice that

$$\Gamma\left(\underbrace{\mathbf{E}\mid_{\varnothing}}_{=(\varnothing,<_1,<_2)}, w\mid_{\varnothing}\right) = \Gamma\left(\left(\varnothing, <_1, <_2\right), w\mid_{\varnothing}\right) = 1$$

(by Lemma 6.3 (a)).

The upper commutative pentagon of (2) shows that $u \circ \varepsilon = m \circ (S \otimes \operatorname{id}) \circ \Delta$. Applying both sides of this equality to $\Gamma(\mathbf{E}, w)$, we obtain $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) =$

[49]*Proof of* (52)*:* Let $(P, Q) \in \operatorname{Adm} \mathbf{E}$ be such that $(P, Q) \neq (E, \varnothing)$. From $(P, Q) \in \operatorname{Adm} \mathbf{E}$, we conclude that P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$. Hence, $Q = E \setminus P$.

The double poset $\mathbf{E}\mid_P = (P, <_1, <_2)$ is tertispecial (by Lemma 6.2).

If we had $P = E$, then we would have $\left(\underbrace{P}_{=E}, \underbrace{Q}_{=E\setminus P}\right) = \left(E, E \setminus \underbrace{P}_{=E}\right) = \left(E, \underbrace{E \setminus E}_{=\varnothing}\right) =$ $(E, \varnothing)$, which would contradict $(P, Q) \neq (E, \varnothing)$. Hence, we cannot have $P = E$. Thus, P is a proper subset of E (since P is a subset of E). Hence, $|P| < |E|$. Therefore, (51) (applied to $(\prec_1) = (<_1)$, $(\prec_2) = (<_2)$, $(\succ_1) = (>_1)$, and $x = w\mid_P$) yields $S\left(\Gamma\left(\left(P, <_1, <_2\right), w\mid_P\right)\right) = (-1)^{|P|} \Gamma\left(\left(P, >_1, <_2\right), w\mid_P\right)$.

Now,

$$S\left(\Gamma\left(\underbrace{\mathbf{E}\mid_P}_{=(P,<_1,<_2)}, w\mid_P\right)\right) = S\left(\Gamma\left(\left(P, <_1, <_2\right), w\mid_P\right)\right) = (-1)^{|P|} \Gamma\left(\left(P, >_1, <_2\right), w\mid_P\right).$$

This proves (52).

$(m \circ (S \otimes \mathrm{id}) \circ \Delta)(\Gamma(\mathbf{E}, w))$. Since $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) = 0$, this becomes

$$\begin{aligned}
0 &= (m \circ (S \otimes \mathrm{id}) \circ \Delta)(\Gamma(\mathbf{E}, w)) = m((S \otimes \mathrm{id})(\Delta(\Gamma(\mathbf{E}, w)))) \\
&= m\left((S \otimes \mathrm{id})\left(\sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} \Gamma(\mathbf{E} \mid_P, w \mid_P) \otimes \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right)\right)\right) \qquad \text{(by (10))} \\
&= m\left(\sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} S(\Gamma(\mathbf{E} \mid_P, w \mid_P)) \otimes \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right)\right) \\
&\qquad \text{(by the definition of the map } S \otimes \mathrm{id}) \\
&= \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} S(\Gamma(\mathbf{E} \mid_P, w \mid_P)) \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right) \\
&\qquad \text{(by the definition of the map } m) \\
&= S\left(\Gamma\left(\underbrace{\mathbf{E} \mid_E}_{=\mathbf{E}}, \underbrace{w \mid_E}_{=w}\right)\right) \underbrace{\Gamma(\mathbf{E} \mid_\varnothing, w \mid_\varnothing)}_{=1} \\
&\qquad + \sum_{\substack{(P,Q) \in \operatorname{Adm} \mathbf{E}; \\ (P,Q) \neq (E,\varnothing)}} \underbrace{S(\Gamma(\mathbf{E} \mid_P, w \mid_P))}_{\substack{=(-1)^{|P|}\Gamma((P,>_1,<_2),w|_P) \\ \text{(by (52))}}} \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right) \\
&\qquad \left(\begin{array}{c} \text{here, we have split off the addend} \\ \text{for } (P,Q) = (E,\varnothing) \text{ from the sum} \end{array}\right) \\
&= S(\Gamma(\mathbf{E}, w)) + \sum_{\substack{(P,Q) \in \operatorname{Adm} \mathbf{E}; \\ (P,Q) \neq (E,\varnothing)}} (-1)^{|P|} \Gamma((P, >_1, <_2), w \mid_P) \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right).
\end{aligned}$$

Thus,

$$S(\Gamma(\mathbf{E}, w)) = -\sum_{\substack{(P,Q) \in \operatorname{Adm} \mathbf{E}; \\ (P,Q) \neq (E,\varnothing)}} (-1)^{|P|} \Gamma((P, >_1, <_2), w \mid_P) \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right). \tag{53}$$

For every subset P of E, we have

$$\begin{aligned}
\Gamma((P, >_1, <_2), w \mid_P) &= \sum_{\pi \text{ is a } (P,>_1,<_2)\text{-partition}} \mathbf{x}_{\pi, w|_P} \\
&\qquad \text{(by the definition of } \Gamma((P, >_1, <_2), w \mid_P)) \\
&= \sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}} \mathbf{x}_{\sigma, w|_P}
\end{aligned} \tag{54}$$

(here, we have renamed the summation index π as σ).

For every subset Q of E, we have

$$\Gamma\left(\underbrace{\mathbf{E}\mid_Q}_{=(Q,<_1,<_2)}, w\mid_Q\right) = \Gamma\left(\left(Q,<_1,<_2\right), w\mid_Q\right) = \sum_{\pi \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\pi,w|_Q}$$
$$\left(\text{by the definition of } \Gamma\left(\left(Q,<_1,<_2\right), w\mid_Q\right)\right)$$
$$= \sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\tau,w|_Q} \tag{55}$$

(here, we have renamed the summation index π as τ).

Now, for each $(P,Q) \in \operatorname{Adm}\mathbf{E}$, we have

$$\Gamma\left(\left(P,>_1,<_2\right), w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right) = \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\pi,w} \tag{56}$$

[50].

---

[50] *Proof of* (56). Let $(P,Q) \in \operatorname{Adm}\mathbf{E}$. Thus, P and Q are two subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$. Thus, the set E is the union of its two disjoint subsets P and Q.

If $\pi : E \to \{1,2,3,\ldots\}$ is a map, then

$$\underbrace{\mathbf{x}_{\pi|_P,w|_P}}_{\substack{=\prod_{e\in P} x_{(\pi|_P)(e)}^{(w|_P)(e)}\\ \text{(by the definition of } \mathbf{x}_{\pi|_P,w|_P})}} \underbrace{\mathbf{x}_{\pi|_Q,w|_Q}}_{\substack{=\prod_{e\in Q} x_{\left(\pi|_Q\right)(e)}^{\left(w|_Q\right)(e)}\\ \text{(by the definition of } \mathbf{x}_{\pi|_Q,w|_Q})}} = \left(\prod_{e\in P} \underbrace{x_{(\pi|_P)(e)}^{(w|_P)(e)}}_{\substack{=x_{\pi(e)}^{w(e)}\\ \text{(since } (\pi|_P)(e)=\pi(e)\\ \text{and } (w|_P)(e)=w(e))}}\right)\left(\prod_{e\in Q} \underbrace{x_{\left(\pi|_Q\right)(e)}^{\left(w|_Q\right)(e)}}_{\substack{=x_{\pi(e)}^{w(e)}\\ \text{(since } \left(\pi|_Q\right)(e)=\pi(e)\\ \text{and } \left(w|_Q\right)(e)=w(e))}}\right)$$
$$= \left(\prod_{e\in P} x_{\pi(e)}^{w(e)}\right)\left(\prod_{e\in Q} x_{\pi(e)}^{w(e)}\right) = \prod_{e\in E} x_{\pi(e)}^{w(e)}$$

(here, we have merged the two products, since the set E is the union of its two disjoint subsets P and Q).

But the set E is the union of its two disjoint subsets P and Q. Hence, every pair (σ,τ) consisting of a map $\sigma : P \to \{1,2,3,\ldots\}$ and a map $\tau : Q \to \{1,2,3,\ldots\}$ can be written as $\left(\pi\mid_P, \pi\mid_Q\right)$ for a unique $\pi : E \to \{1,2,3,\ldots\}$ (namely, for the $\pi : E \to \{1,2,3,\ldots\}$ that sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e), & \text{if } e \in Q \end{cases}$). Hence, we can substitute $\left(\pi\mid_P, \pi\mid_Q\right)$ for (σ,τ) in the sum

$\sum\limits_{\substack{(\sigma,\tau);\\ \sigma:P\to\{1,2,3,\ldots\};\\ \tau:Q\to\{1,2,3,\ldots\};\\ \sigma \text{ is a } (P,>_1,<_2)\text{-partition};\\ \tau \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\sigma,w|_P}\mathbf{x}_{\tau,w|_Q}$. We thus obtain

$$\sum_{\substack{(\sigma,\tau);\\ \sigma:P\to\{1,2,3,\ldots\};\\ \tau:Q\to\{1,2,3,\ldots\};\\ \sigma \text{ is a } (P,>_1,<_2)\text{-partition};\\ \tau \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\sigma,w|_P}\mathbf{x}_{\tau,w|_Q} = \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \underbrace{\mathbf{x}_{\pi|_P,w|_P}\mathbf{x}_{\pi|_Q,w|_Q}}_{\substack{=\prod_{e\in E} x_{\pi(e)}^{w(e)}=\mathbf{x}_{\pi,w}\\ \text{(since } \mathbf{x}_{\pi,w}=\prod_{e\in E} x_{\pi(e)}^{w(e)}\\ \text{(by the definition of } \mathbf{x}_{\pi,w}))}}$$
$$= \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\pi,w}.$$

Now,

$$\underbrace{\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)}_{\substack{=\sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}} \mathbf{x}_{\sigma,w|_P}\\ \text{(by (54))}}} \underbrace{\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)}_{\substack{=\sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\tau,w|_Q}\\ \text{(by (55))}}}$$
$$= \left(\sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}} \mathbf{x}_{\sigma,w|_P}\right)\left(\sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\tau,w|_Q}\right)$$
$$= \underbrace{\sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}}\sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}}}_{=\sum_{\substack{(\sigma,\tau);\\ \sigma:P\to\{1,2,3,\ldots\};\\ \tau:Q\to\{1,2,3,\ldots\};\\ \sigma \text{ is a } (P,>_1,<_2)\text{-partition};\\ \tau \text{ is a } (Q,<_1,<_2)\text{-partition}}}} \mathbf{x}_{\sigma,w|_P}\mathbf{x}_{\tau,w|_Q} = \sum_{\substack{(\sigma,\tau);\\ \sigma:P\to\{1,2,3,\ldots\};\\ \tau:Q\to\{1,2,3,\ldots\};\\ \sigma \text{ is a } (P,>_1,<_2)\text{-partition};\\ \tau \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\sigma,w|_P}\mathbf{x}_{\tau,w|_Q}$$
$$= \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\pi,w}.$$

This proves (56).

Now,

$$\sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} (-1)^{|P|} \underbrace{\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)}_{\substack{=\sum\limits_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\pi,w}\\ \text{(by (56))}}}$$
$$= \sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} (-1)^{|P|} \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\pi,w}$$
$$= \underbrace{\sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}}}_{=\sum\limits_{\pi:E\to\{1,2,3,\ldots\}} \sum\limits_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}}} (-1)^{|P|} \mathbf{x}_{\pi,w}$$
$$= \sum_{\pi:E\to\{1,2,3,\ldots\}} \underbrace{\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} (-1)^{|P|}}_{\substack{=0\\ \text{(by (41))}}} \mathbf{x}_{\pi,w} = \sum_{\pi:E\to\{1,2,3,\ldots\}} 0\mathbf{x}_{\pi,w} = 0.$$

Thus,

$$0 = \sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} (-1)^{|P|} \Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)$$
$$= (-1)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),\underbrace{w\mid_E}_{=w}\right)\underbrace{\Gamma\left(\mathbf{E}\mid_\varnothing,w\mid_\varnothing\right)}_{=1}$$
$$+ \sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ (P,Q)\neq(E,\varnothing)}} (-1)^{|P|} \Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)$$
$$\left(\begin{array}{c}\text{here, we have split off the addend}\\ \text{for } (P,Q)=(E,\varnothing) \text{ from the sum}\end{array}\right)$$
$$= (-1)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),w\right)$$
$$+ \sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ (P,Q)\neq(E,\varnothing)}} (-1)^{|P|} \Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right),$$

so that

$$\begin{aligned}(-1)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),w\right) &= -\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ (P,Q)\neq(E,\varnothing)}} (-1)^{|P|}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)\\ &= S\left(\Gamma\left(\underbrace{\mathbf{E}}_{=(E,<_1,<_2)},w\right)\right) \qquad \text{(by (53))}\\ &= S\left(\Gamma\left(\left(E,<_1,<_2\right),w\right)\right),\end{aligned}$$

and thus $S\left(\Gamma\left(\left(E,<_1,<_2\right),w\right)\right) = (-1)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),w\right)$. This completes the induction step and thus the proof of Theorem 4.2. $\square$

7. Proof of Theorem 4.7

Before we begin proving Theorem 4.7, we state a criterion for **E**-partitions that is less wasteful (in the sense that it requires fewer verifications) than the definition:

> **Lemma 7.1.** Let $\mathbf{E} = (E,<_1,<_2)$ be a tertispecial double poset. Let $\phi : E \to \{1,2,3,\ldots\}$ be a map. Assume that the following two conditions hold:
>
> - *Condition 1:* If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $e <_2 f$, then $\phi(e) \leq \phi(f)$.
> - *Condition 2:* If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $f <_2 e$, then $\phi(e) < \phi(f)$.
>
> Then, ϕ is an **E**-partition.

Proof of Lemma 7.1. For any $a \in E$ and $b \in E$, we define a subset $[a,b]$ of E as in the proof of Lemma 6.1.

We need to show that ϕ is an **E**-partition. In other words, we need to prove the following two claims:

Claim 1: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$.

Claim 2: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

Proof of Claim 1: Assume the contrary. Thus, there exists a pair $(e,f) \in E \times E$ satisfying $e <_1 f$ but not $\phi(e) \leq \phi(f)$. Such a pair will be called a *malrelation*. Fix a malrelation (u,v) for which the value $|[u,v]|$ is minimum (such a (u,v) exists, since there exists a malrelation). Thus, $u \in E$ and $v \in E$ and $u <_1 v$ but not $\phi(u) \leq \phi(v)$.

If u was $<_1$-covered by v, then we would obtain $\phi(u) \leq \phi(v)$ [51], which would

[51] *Proof.* Assume that u is $<_1$-covered by v. Thus, u and v are $<_2$-comparable (since the double poset **E** is tertispecial). In other words, we have either $u <_2 v$ or $u = v$ or $v <_2 u$. In the first of these three cases, we obtain $\phi(u) \leq \phi(v)$ by applying Condition 1 to $e = u$ and $f = v$. In the third of these cases, we obtain $\phi(u) < \phi(v)$ (and thus $\phi(u) \leq \phi(v)$) by applying Condition 2 to $e = u$ and $f = v$. The second of these cases cannot happen because $u <_1 v$. Thus, we always have $\phi(u) \leq \phi(v)$, qed.

contradict the fact that we do not have $\phi(u) \leq \phi(v)$. Hence, u is not $<_1$-covered by v. Consequently, there exists a $w \in E$ such that $u <_1 w <_1 v$ (since $u <_1 v$). Consider this w. Applying (35) to $a = u$, $c = w$ and $b = v$, we see that both numbers $|[u,w]|$ and $|[w,v]|$ are smaller than $|[u,v]|$. Hence, neither (u,w) nor (w,v) is a malrelation (since we picked (u,v) to be a malrelation with minimum $|[u,v]|$). Therefore, we have $\phi(u) \leq \phi(w)$ (since $u <_1 w$, but (u,w) is not a malrelation) and $\phi(w) \leq \phi(v)$ (since $w <_1 v$, but (w,v) is not a malrelation). Combining these two inequalities, we obtain $\phi(u) \leq \phi(w) \leq \phi(v)$. This contradicts the fact that we do not have $\phi(u) \leq \phi(v)$. This contradiction concludes the proof of Claim 1.

Instead of Claim 2, we shall prove the following stronger claim:

Claim 3: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and not $e <_2 f$ satisfy $\phi(e) < \phi(f)$.

Proof of Claim 3: Assume the contrary. Thus, there exists a pair $(e,f) \in E \times E$ satisfying $e <_1 f$ and not $e <_2 f$ but not $\phi(e) < \phi(f)$. Such a pair will be called a *malrelation*. Fix a malrelation (u,v) for which the value $|[u,v]|$ is minimum (such a (u,v) exists, since there exists a malrelation). Thus, $u \in E$ and $v \in E$ and $u <_1 v$ and not $u <_2 v$ but not $\phi(u) < \phi(v)$.

If u was $<_1$-covered by v, then we would obtain $\phi(u) < \phi(v)$ easily[52], which would contradict the fact that we do not have $\phi(u) < \phi(v)$. Hence, u is not $<_1$-covered by v. Consequently, there exists a $w \in E$ such that $u <_1 w <_1 v$ (since $u <_1 v$). Consider this w. Applying (35) to $a = u$, $c = w$ and $b = v$, we see that both numbers $|[u,w]|$ and $|[w,v]|$ are smaller than $|[u,v]|$. Hence, neither (u,w) nor (w,v) is a malrelation (since we picked (u,v) to be a malrelation with minimum $|[u,v]|$).

But $\phi(v) \leq \phi(u)$ (since we do not have $\phi(u) < \phi(v)$). On the other hand, $u <_1 w$ and therefore $\phi(u) \leq \phi(w)$ (by Claim 1, applied to $e = u$ and $f = w$). Furthermore, $w <_1 v$ and thus $\phi(w) \leq \phi(v)$ (by Claim 1, applied to $e = w$ and $f = v$). The chain of inequalities $\phi(v) \leq \phi(u) \leq \phi(w) \leq \phi(v)$ ends with the same term that it begins with; therefore, it must be a chain of equalities. In other words, we have $\phi(v) = \phi(u) = \phi(w) = \phi(v)$.

Now, using $\phi(w) = \phi(v)$, we can see that $w <_2 v$ [53]. The same argument (applied to u and w instead of w and v) shows that $u <_2 w$. Thus, $u <_2 w <_2 v$, which contradicts the fact that we do not have $u <_2 v$. This contradiction proves Claim 3.

Proof of Claim 2: The condition "$f <_2 e$" is stronger than "not $e <_2 f$". Thus, Claim 2 follows from Claim 3.

[52] *Proof.* Assume that u is $<_1$-covered by v. Thus, u and v are $<_2$-comparable (since the double poset $\mathbf{E}$ is tertispecial). In other words, we have either $u <_2 v$ or $u = v$ or $v <_2 u$. Since neither $u <_2 v$ nor $u = v$ can hold (indeed, $u <_2 v$ is ruled out by assumption, whereas $u = v$ is ruled out by $u <_1 v$), we thus have $v <_2 u$. Therefore, $\phi(u) < \phi(v)$ by Condition 2 (applied to $e = u$ and $f = v$), qed.

[53] *Proof.* Assume the contrary. Thus, we do not have $w <_2 v$. But $\phi(w) = \phi(v)$ shows that we do not have $\phi(w) < \phi(v)$. Hence, (w,v) is a malrelation (since $w <_1 v$ and not $w <_2 v$ but not $\phi(w) < \phi(v)$). This contradicts the fact that (w,v) is not a malrelation. This contradiction completes the proof.

Claims 1 and 2 are now both proven, and so Lemma 7.1 follows. □

Proof of Lemma 4.5. Consider the following three logical statements:

Statement 1: The G-orbit O is E-coeven.

Statement 2: All elements of O are E-coeven.

Statement 3: At least one element of O is E-coeven.

Statements 1 and 2 are equivalent (according to the definition of when a G-orbit is E-coeven). Our goal is to prove that Statements 1 and 3 are equivalent (because this is precisely what Lemma 4.5 says). Thus, it suffices to show that Statements 2 and 3 are equivalent (because we already know that Statements 1 and 2 are equivalent). Since Statement 2 obviously implies Statement 3 (in fact, the G-orbit O contains at least one element), we therefore only need to show that Statement 3 implies Statement 2. Thus, assume that Statement 3 holds. We need to prove that Statement 2 holds.

There exists at least one E-coeven $\phi \in O$ (because we assumed that Statement 3 holds). Consider this ϕ. Now, let $\pi \in O$ be arbitrary. We shall show that π is E-coeven.

We know that ϕ is E-coeven. In other words,

$$\text{every } g \in G \text{ satisfying } g\phi = \phi \text{ is } E\text{-even.} \tag{57}$$

Now, let $g \in G$ be such that $g\pi = \pi$. Since ϕ belongs to the G-orbit O, we have $O = G\phi$. Now, $\pi \in O = G\phi$. In other words, there exists some $h \in G$ such that $\pi = h\phi$. Consider this h. We have $g\pi = \pi$. Since $\pi = h\phi$, this rewrites as $gh\phi = h\phi$. In other words, $h^{-1}gh\phi = \phi$. Thus, (57) (applied to $h^{-1}gh$ instead of g) shows that $h^{-1}gh$ is E-even. In other words,

$$\text{the action of } h^{-1}gh \text{ on } E \text{ is an even permutation of } E. \tag{58}$$

Now, let ε be the group homomorphism from G to $\operatorname{Aut} E$ [54] which describes the G-action on E. Then, $\varepsilon\left(h^{-1}gh\right)$ is the action of $h^{-1}gh$ on E, and thus is an even permutation of E (by (58)).

But since ε is a group homomorphism, we have $\varepsilon\left(h^{-1}gh\right) = \varepsilon\left(h\right)^{-1}\varepsilon\left(g\right)\varepsilon\left(h\right)$. Thus, the permutations $\varepsilon\left(h^{-1}gh\right)$ and $\varepsilon\left(g\right)$ of E are conjugate. Since the permutation $\varepsilon\left(h^{-1}gh\right)$ is even, this shows that the permutation $\varepsilon\left(g\right)$ is even. In other words, the action of g on E is an even permutation of E. In other words, g is E-even.

Now, let us forget that we fixed g. We thus have shown that every $g \in G$ satisfying $g\pi = \pi$ is E-even. In other words, π is E-coeven.

Let us now forget that we fixed π. Thus, we have proven that every $\pi \in O$ is E-coeven. In other words, Statement 2 holds. We have thus shown that Statement 3 implies Statement 2. Consequently, Statements 2 and 3 are equivalent, and so the proof of Lemma 4.5 is complete. □

[54]We use the notation $\operatorname{Aut} E$ for the group of all permutations of the set E.

Proof of Proposition 4.6. (a) We need to prove that $g\pi \in \operatorname{Par}\mathbf{E}$ for each $g \in G$ and each $\pi \in \operatorname{Par}\mathbf{E}$. So let us fix $g \in G$ and $\pi \in \operatorname{Par}\mathbf{E}$. We must prove that $g\pi \in \operatorname{Par}\mathbf{E}$.

We know that π is an $\mathbf{E}$-partition (since $\pi \in \operatorname{Par}\mathbf{E}$). In other words, the following two claims hold:

Claim 1: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\pi(e) \leq \pi(f)$.

Claim 2: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\pi(e) < \pi(f)$.

But our goal is to prove that $g\pi \in \operatorname{Par}\mathbf{E}$. In other words, our goal is to prove that $g\pi$ is an $\mathbf{E}$-partition. In other words, we must show that the following two claims hold:

Claim 3: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $(g\pi)(e) \leq (g\pi)(f)$.

Claim 4: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $(g\pi)(e) < (g\pi)(f)$.

Let us first prove Claim 4:

Proof of Claim 4: Let $e \in E$ and $f \in E$ be such that $e <_1 f$ and $f <_2 e$. The definition of the G-action on the set $\{1,2,3,\ldots\}^E$ shows that $(g\pi)(e) = \pi\left(g^{-1}e\right)$ and $(g\pi)(f) = \pi\left(g^{-1}f\right)$.

Now, from $e <_1 f$, we obtain $g^{-1}e <_1 g^{-1}f$ (since G preserves the relation $<_1$). Also, from $f <_2 e$, we obtain $g^{-1}f <_2 g^{-1}e$ (since G preserves the relation $<_2$). Thus, we can apply Claim 2 to $g^{-1}e$ and $g^{-1}f$ instead of e and f. As a result, we conclude that $\pi\left(g^{-1}e\right) < \pi\left(g^{-1}f\right)$. In view of $(g\pi)(e) = \pi\left(g^{-1}e\right)$ and $(g\pi)(f) = \pi\left(g^{-1}f\right)$, this rewrites as $(g\pi)(e) < (g\pi)(f)$. Thus, Claim 4 is proven.

We thus have derived Claim 4 from Claim 2. Similarly, Claim 3 can be derived from Claim 1. Thus, Claim 3 and Claim 4 are proven. As explained above, this shows that $g\pi$ is an $\mathbf{E}$-partition. In other words, $g\pi \in \operatorname{Par}\mathbf{E}$. This completes our proof of Proposition 4.6 (a).

(b) We need to show that $\mathbf{x}_{\pi,w} = \mathbf{x}_{\psi,w}$ for any two elements π and ψ of O. So let π and ψ be two elements of O.

We know that G preserves w. In other words, every $g \in G$ and $e \in E$ satisfy

$$w(ge) = w(e). \tag{59}$$

The two elements π and ψ belong to one and the same G-orbit (namely, to O). Thus, there exists a $g \in G$ satisfying $g\pi = \psi$. Consider this g. For each $e \in E$, we have

$$\underbrace{\psi}_{=g\pi}(e) = (g\pi)(e) = \pi\left(g^{-1}e\right) \tag{60}$$

(by the definition of the G-action on the set $\{1,2,3,\ldots\}^E$).

The definition of $\mathbf{x}_{\pi,w}$ yields $\mathbf{x}_{\pi,w} = \prod_{e\in E} x_{\pi(e)}^{w(e)}$. The definition of $\mathbf{x}_{\psi,w}$ yields

$$\mathbf{x}_{\psi,w} = \prod_{e\in E} \underbrace{x_{\psi(e)}^{w(e)}}_{\substack{=x_{\pi(g^{-1}e)}^{w(e)} \\ \text{(by (60))}}} = \prod_{e\in E} x_{\pi\left(g^{-1}e\right)}^{w(e)} = \prod_{e\in E} \underbrace{x_{\pi\left(g^{-1}ge\right)}^{w(ge)}}_{\substack{=x_{\pi(e)}^{w(ge)} \\ \text{(since } g^{-1}ge=e\text{)}}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } ge \text{ for } e \text{ in the product,} \\ \text{since the map } E \to E,\ e \mapsto ge \text{ is a bijection}\end{array}\right)$$

$$= \prod_{e\in E} \underbrace{x_{\pi(e)}^{w(ge)}}_{\substack{=x_{\pi(e)}^{w(e)} \\ \text{(by (59))}}} = \prod_{e\in E} x_{\pi(e)}^{w(e)} = \mathbf{x}_{\pi,w}.$$

Thus, $\mathbf{x}_{\pi,w} = \mathbf{x}_{\psi,w}$ is proven. This completes the proof of Proposition 4.6 (b). $\square$

Next, we will show three simple properties of posets on which groups act. First, we introduce a notation:

Definition 7.2. Let G be a group. Let $g \in G$. Let E be a G-set. Then, the subgroup $\langle g \rangle$ of G (this is the subgroup of G generated by g) also acts on E. The $\langle g \rangle$-orbits on E will be called the *g-orbits* on E. When E is clear from the context, we shall simply call them the *g-orbits*.

We can also describe these g-orbits explicitly: For any given $e \in E$, the g-orbit of e (that is, the unique g-orbit that contains e) is $\langle g \rangle e = \left\{ g^k e \mid k \in \mathbb{Z} \right\}$.

Equivalently, the g-orbits on E can be characterized as follows: The action of g on E is a permutation of E. The cycles of this permutation are the g-orbits on E (at least when E is finite).

Proposition 7.3. Let E be a set. Let $<_1$ be a strict partial order on E. Let G be a finite group which acts on E. Assume that G preserves the relation $<_1$.

Let $g \in G$. Let E^g be the set of all g-orbits on E. Define a binary relation $<_1^g$ on E^g by

$$\left(u <_1^g v\right) \iff \left(\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b\right).$$

Then, $<_1^g$ is a strict partial order.

Proposition 7.3 is precisely [Joch13, Lemma 2.4], but let us outline the proof for the sake of completeness:

Proof of Proposition 7.3. Let us first show that the relation $<_1^g$ is irreflexive. Indeed, assume the contrary. Thus, there exists a $u \in E^g$ such that $u <_1^g u$. Consider this u.

We have $u \in E^g$. In other words, u is a g-orbit on E.

Since $u <_1^g u$, there exist $a \in u$ and $b \in u$ with $a <_1 b$ (by the definition of the relation $<_1^g$). Consider these a and b. There exists a $k \in \mathbb{Z}$ such that $b = g^k a$ (since a and b both lie in one and the same g-orbit u). Consider this k.

Each element of G has finite order (since G is a finite group). In particular, the element g of G has finite order. In other words, there exists a positive integer n such that $g^n = 1_G$. Consider this n. Every $p \in \mathbb{Z}$ satisfies $g^{np} = (g^n)^p = 1_G$ (since $g^n = 1_G$). Applying this to $p = k$, we obtain $g^{nk} = 1_G$.

Now, $a <_1 b = g^k a$. Since G preserves the relation $<_1$, this shows that $ha <_1 hg^k a$ for every $h \in G$. Thus, for every $\ell \in \mathbb{N}$, we have $g^{\ell k} a <_1 g^{\ell k} g^k a$ (by the inequality $ha <_1 hg^k a$, applied to $h = g^{\ell k}$). Hence, $g^{\ell k} a <_1 g^{\ell k} g^k a = g^{\ell k + k} a = g^{(\ell+1)k} a$ for every $\ell \in \mathbb{N}$. Consequently, $g^{0k} a <_1 g^{1k} a <_1 g^{2k} a <_1 \cdots <_1 g^{nk} a$. Thus, $g^{0k} a <_1 \underbrace{g^{nk}}_{=1_G} a = a$, which contradicts $\underbrace{g^{0k}}_{=g^0=1_G} a = 1_G a = a$. This contradiction proves that our assumption was wrong. Hence, the relation $<_1^g$ is irreflexive.

Let us next show that the relation $<_1^g$ is transitive. Indeed, let u, v and w be three elements of E^g such that $u <_1^g v$ and $v <_1^g w$. We must prove that $u <_1^g w$.

There exist $a \in u$ and $b \in v$ with $a <_1 b$ (since $u <_1^g v$). Consider these a and b.

There exist $a' \in v$ and $b' \in w$ with $a' <_1 b'$ (since $v <_1^g w$). Consider these a' and b'.

The set v is a g-orbit (since $v \in E^g$). The elements b and a' lie in one and the same g-orbit (namely, in v). Hence, there exists some $k \in \mathbb{Z}$ such that $a' = g^k b$. Consider this k. We have $a <_1 b$ and thus $g^k a <_1 g^k b$ (since G preserves the relation $<_1$). Hence, $g^k a <_1 g^k b = a' <_1 b'$. Since $g^k a \in u$ (because $a \in u$, and because u is a g-orbit) and $b' \in w$, this shows that $u <_1^g w$ (by the definition of the relation $<_1^g$). We thus have proven that the relation $<_1^g$ is transitive.

Now, we know that the relation $<_1^g$ is irreflexive and transitive, and thus also antisymmetric (since every irreflexive and transitive binary relation is antisymmetric). In other words, $<_1^g$ is a strict partial order. This proves Proposition 7.3. □

Remark 7.4. Proposition 7.3 can be generalized: Let E be a set. Let $<_1$ be a strict partial order on E. Let G be a finite group which acts on E. Assume that G preserves the relation $<_1$. Let H be a subgroup of G. Let E^H be the set of all H-orbits on E. Define a binary relation $<_1^H$ on E^H by

$$\left(u <_1^H v\right) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Then, $<_1^H$ is a strict partial order.

This result (whose proof is quite similar to that of Proposition 7.3) implicitly appears in [Stan84, p. 30].

Proposition 7.5. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$.

Let $g \in G$. Let E^g be the set of all g-orbits on E. Define a binary relation $<_1^g$ on E^g by

$$\left(u <_1^g v\right) \iff \left(\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b\right).$$

Define a binary relation $<_2^g$ on E^g by

$$\left(u <_2^g v\right) \iff \left(\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_2 b\right).$$

Let $\mathbf{E}^g$ be the triple $\left(E^g, <_1^g, <_2^g\right)$. Then, $\mathbf{E}^g$ is a tertispecial double poset.

Proof of Proposition 7.5. Both relations $<_1$ and $<_2$ are strict partial orders (since $\mathbf{E}$ is a double poset). Proposition 7.3 shows that $<_1^g$ is a strict partial order. Proposition 7.3 (applied to $<_2$ and $<_2^g$ instead of $<_1$ and $<_1^g$) shows that $<_2^g$ is a strict partial order. Thus, $\mathbf{E}^g = \left(E^g, <_1^g, <_2^g\right)$ is a double poset. It remains to show that this double poset $\mathbf{E}^g$ is tertispecial.

Let u and v be two elements of E^g such that u is $<_1^g$-covered by v. We shall prove that u and v are $<_2^g$-comparable.

We have $u <_1^g v$ (since u is $<_1^g$-covered by v). In other words, there exist $a \in u$ and $b \in v$ with $a <_1 b$ (by the definition of the relation $<_1^g$). Consider these a and b.

There exists no $c \in E$ satisfying $a <_1 c <_1 b$ [55]. In other words, a is $<_1$-covered by b (since we know that $a <_1 b$). Thus, a and b are $<_2$-comparable (since the double poset $\mathbf{E}$ is tertispecial). In other words, either $a <_2 b$ or $a = b$ or $b <_2 a$. Therefore, either $u <_2^g v$ or $u = v$ or $v <_2^g u$ [56]. In other words, u and v are $<_2^g$-comparable.

Now, let us forget that we fixed u and v. We thus have shown that if u and v are two elements of E^g such that u is $<_1^g$-covered by v, then u and v are $<_2^g$-

[55] *Proof.* Assume the contrary. Thus, there exists some $c \in E$ satisfying $a <_1 c <_1 b$. Consider this c. Let w be the g-orbit of c. Thus, $w \in E^g$ and $c \in w$.

Now, the elements $a \in u$ and $c \in w$ satisfy $a <_1 c$. Hence, $u <_1^g w$ (by the definition of the relation $<_1^g$).

Also, the elements $c \in w$ and $b \in v$ satisfy $c <_1 b$. Hence, $w <_1^g v$ (by the definition of the relation $<_1^g$).

Now, we have $u <_1^g w <_1^g v$. This contradicts the fact that u is $<_1^g$-covered by v. Thus, we have obtained a contradiction; hence, our assumption was wrong. Qed.

[56] Here, we have used the following facts:

- If $a <_2 b$, then $u <_2^g v$. (This follows from the definition of the relation $<_2^g$, since $a \in u$ and $b \in v$.)
- If $a = b$, then $u = v$. (This follows from the fact that u is the g-orbit of a (since u is a g-orbit and contains a) whereas v is the g-orbit of b (since v is a g-orbit and contains b).)
- If $b <_2 a$, then $v <_2^g u$. (This follows from the definition of the relation $<_2^g$, since $b \in v$ and $a \in u$.)

comparable. In other words, the double poset $\mathbf{E}^g = \left(E^g, <_1^g, <_2^g\right)$ is tertispecial. This proves Proposition 7.5. $\square$

Next, let us state a basic fact about G-sets:

Proposition 7.6. Let G be a finite group. Let E be a G-set. Let X be any set. Recall that the set X^E of all maps $E \to X$ becomes a G-set (according to Definition 4.4 (d)).

There is a bijection Φ between

- the maps $\pi : E \to X$ satisfying $g\pi = \pi$

and

- the maps $\overline{\pi} : E^g \to X$.

Namely, this bijection Φ sends any map $\pi : E \to X$ satisfying $g\pi = \pi$ to the map $\overline{\pi} : E^g \to X$ defined by

$$\overline{\pi}(u) = \pi(a) \qquad \text{for every } u \in E^g \text{ and } a \in u.$$

Proof of Proposition 7.6. Let $\mathfrak{A}$ be the set of all maps $\pi : E \to X$ satisfying $g\pi = \pi$. Thus,

$$\mathfrak{A} = \{\pi : E \to X \mid g\pi = \pi\} = \{\psi : E \to X \mid g\psi = \psi\}$$

(here, we have renamed the index π as ψ).

Let $\mathfrak{B}$ be the set of all maps $\overline{\pi} : E^g \to X$. Thus,

$$\mathfrak{B} = \{\overline{\pi} : E^g \to X\} = X^{E^g}.$$

Let $\pi \in \mathfrak{A}$. Thus, $\pi \in \mathfrak{A} = \{\psi : E \to X \mid g\psi = \psi\}$. In other words, π is a map $\psi : E \to X$ satisfying $g\psi = \psi$. In other words, π is a map $E \to X$ and satisfies $g\pi = \pi$. Now, we define a map $\pi^\circ : E^g \to X$ as follows: Let $u \in E^g$. Thus, u is a g-orbit. In particular, u is a nonempty set. Hence, we can pick some $a \in u$. Now, the element $\pi(a) \in X$ is independent on the choice of a [57]. Thus, we can define $\pi^\circ(u)$ as $\pi(a)$.

[57] *Proof.* We must show that if a_1 and a_2 are two elements of u, then $\pi(a_1) = \pi(a_2)$.

Indeed, let a_1 and a_2 be two elements of u. We must show that $\pi(a_1) = \pi(a_2)$.

The elements a_1 and a_2 lie in one and the same g-orbit (namely, in u). In other words, there exists some $k \in \mathbb{Z}$ such that $a_2 = g^k a_1$. Consider this k.

Recall that $g\pi = \pi$. Thus, g lies in the stabilizer of π. Since the stabilizer of π is a subgroup of G, we therefore conclude that every power of g must also lie in the stabilizer of π. In particular, g^k lies in the stabilizer of π (since g^k is a power of g). In other words, $g^k\pi = \pi$. But $a_2 = g^k a_1$, so that $a_1 = \left(g^k\right)^{-1} a_2$. Applying the map π to both sides of this equality, we obtain

$$\pi(a_1) = \pi\left(\left(g^k\right)^{-1} a_2\right).$$

Thus, we have defined an element $\pi^{\circ}(u)$ for each $u \in E^g$. In other words, we have defined a map $\pi^{\circ} : E^g \to X$. Furthermore, this map π° has the following property: If $u \in E^g$, then

$$\pi^{\circ}(u) = \pi(a) \qquad \text{for every } a \in u. \tag{61}$$

The element π° is a map $E^g \to X$, thus an element of X^{E^g}. In other words, $\pi^{\circ} \in X^{E^g} = \mathfrak{B}$.

Now, forget that we fixed π. Thus, for each $\pi \in \mathfrak{A}$, we have defined a $\pi^{\circ} \in \mathfrak{B}$, and this π° satisfies (61) for each $u \in E^g$.

Now, let Φ be the map

$$\mathfrak{A} \to \mathfrak{B}, \qquad \pi \mapsto \pi^{\circ}.$$

(This is well-defined, since $\pi^{\circ} \in \mathfrak{B}$ for each $\pi \in \mathfrak{A}$.)

Next, let us introduce one more notation: If e is an element of E, then $[e]$ shall mean the g-orbit of e. Furthermore, let $\mathfrak{o}$ be the map $E \to E^g$ that sends each element $e \in E$ to its g-orbit $[e]$.

For every $\overline{\pi} \in \mathfrak{B}$, we have $\overline{\pi} \circ \mathfrak{o} \in \mathfrak{A}$ [58].

Comparing this with

$$\left(g^k \pi\right)(a_2) = \pi\left(\left(g^k\right)^{-1} a_2\right) \qquad \left(\text{by the definition of the } G\text{-action on } X^E\right),$$

we obtain $\pi(a_1) = \underbrace{\left(g^k \pi\right)}_{=\pi}(a_2) = \pi(a_2)$. This completes our proof.

[58] *Proof:* Let $\overline{\pi} \in \mathfrak{B}$. Thus, $\overline{\pi} \in \mathfrak{B} = X^{E^g}$. In other words, $\overline{\pi}$ is a map $E^g \to X$.

Let $e \in E$. Then, $\mathfrak{o}(e) = [e]$ (by the definition of $\mathfrak{o}$). Moreover, $(g(\overline{\pi} \circ \mathfrak{o}))(e) = (\overline{\pi} \circ \mathfrak{o})\left(g^{-1}e\right)$ (by the definition of the G-action on X^E). But the elements e and $g^{-1}e$ of E lie in one and the same g-orbit. In other words, $[e] = \left[g^{-1}e\right]$. Now,

$$(g(\overline{\pi} \circ \mathfrak{o}))(e) = (\overline{\pi} \circ \mathfrak{o})\left(g^{-1}e\right) = \overline{\pi}\left(\underbrace{\mathfrak{o}\left(g^{-1}e\right)}_{\substack{=[g^{-1}e] \\ \text{(by the definition of } \mathfrak{o})}}\right) = \overline{\pi}\left(\underbrace{\left[g^{-1}e\right]}_{=[e]}\right)$$
$$= \overline{\pi}\left(\underbrace{[e]}_{=\mathfrak{o}(e)}\right) = \overline{\pi}(\mathfrak{o}(e)) = (\overline{\pi} \circ \mathfrak{o})(e).$$

Now, forget that we fixed e. We thus have proven that $(g(\overline{\pi} \circ \mathfrak{o}))(e) = (\overline{\pi} \circ \mathfrak{o})(e)$ for each $e \in E$. In other words, $g(\overline{\pi} \circ \mathfrak{o}) = \overline{\pi} \circ \mathfrak{o}$.

Now, $\overline{\pi} \circ \mathfrak{o}$ is a map $E \to X$ and satisfies $g(\overline{\pi} \circ \mathfrak{o}) = \overline{\pi} \circ \mathfrak{o}$. In other words, $\overline{\pi} \circ \mathfrak{o}$ is a map $\psi : E \to X$ satisfying $g\psi = \psi$. In other words,

$$\overline{\pi} \circ \mathfrak{o} \in \{\psi : E \to X \mid g\psi = \psi\} = \mathfrak{A}.$$

Qed.

Now, let Ψ be the map

$$\mathfrak{B} \to \mathfrak{A}, \qquad \overline{\pi} \mapsto \overline{\pi} \circ \mathfrak{o}.$$

(This is well-defined, since $\overline{\pi} \circ \mathfrak{o} \in \mathfrak{A}$ for every $\overline{\pi} \in \mathfrak{B}$.)

Now, we have the equalities $\Phi \circ \Psi = \mathrm{id}$ [59] and $\Psi \circ \Phi = \mathrm{id}$ [60]. These two equalities show that the maps Φ and Ψ are mutually inverse. Hence, the map Φ is invertible. In other words, Φ is a bijection.

The map Φ is a bijection from $\mathfrak{A}$ to $\mathfrak{B}$. In other words, the map Φ is a bijection between

- the maps $\pi : E \to X$ satisfying $g\pi = \pi$

and

- the maps $\overline{\pi} : E^g \to X$

[59] *Proof.* Let $\beta \in \mathfrak{B}$. Then, $\Psi(\beta) = \beta \circ \mathfrak{o}$ (by the definition of Ψ). Let $\pi = \Psi(\beta)$. Thus, $\pi = \Psi(\beta) \in \mathfrak{A}$.

Now, let $u \in E^g$. Thus, u is a g-orbit; hence, u is nonempty. Thus, there exists some $a \in u$. Consider such an a. Now, (61) yields $\pi^\circ(u) = \pi(a)$. But a is an element of the g-orbit u; thus, the g-orbit of a is u. In other words, $[a] = u$. The definition of $\mathfrak{o}$ yields $\mathfrak{o}(a) = [a] = u$. Now, $\pi = \Psi(\beta) = \beta \circ \mathfrak{o}$, so that $\pi(a) = (\beta \circ \mathfrak{o})(a) = \beta\left(\underbrace{\mathfrak{o}(a)}_{=u}\right) = \beta(u)$. Now, $\pi^\circ(u) = \pi(a) = \beta(u)$.

Now, forget that we fixed u. We thus have proven that $\pi^\circ(u) = \beta(u)$ for each $u \in E^g$. In other words, $\pi^\circ = \beta$. But the definition of Φ yields $\Phi(\pi) = \pi^\circ = \beta$. Now, $(\Phi \circ \Psi)(\beta) = \Phi\left(\underbrace{\Psi(\beta)}_{=\pi}\right) = \Phi(\pi) = \beta = \mathrm{id}(\beta)$.

Now, forget that we fixed β. We thus have proven that $(\Phi \circ \Psi)(\beta) = \mathrm{id}(\beta)$ for each $\beta \in \mathfrak{B}$. In other words, $\Phi \circ \Psi = \mathrm{id}$, qed.

[60] *Proof.* Let $\alpha \in \mathfrak{A}$. Then, the definition of Φ yields $\Phi(\alpha) = \alpha^\circ$.

Now, let $a \in E$. Let $u = \mathfrak{o}(a)$. Thus, $u = \mathfrak{o}(a) = [a]$ (by the definition of $\mathfrak{o}$). In other words, u is the g-orbit of a. Thus, u is a g-orbit and contains a. So we know that $u \in E^g$ (since u is a g-orbit) and that $a \in u$ (since u contains a).

The equality (61) (applied to $\pi = \alpha$) yields $\alpha^\circ(u) = \alpha(a)$. We have $(\alpha^\circ \circ \mathfrak{o})(a) = \alpha^\circ\left(\underbrace{\mathfrak{o}(a)}_{=u}\right) = \alpha^\circ(u) = \alpha(a)$.

Now, forget that we fixed a. We thus have shown that $(\alpha^\circ \circ \mathfrak{o})(a) = \alpha(a)$ for each $a \in E$. In other words, $\alpha^\circ \circ \mathfrak{o} = \alpha$.

But $(\Psi \circ \Phi)(\alpha) = \Psi\left(\underbrace{\Phi(\alpha)}_{=\alpha^\circ}\right) = \Psi(\alpha^\circ) = \alpha^\circ \circ \mathfrak{o}$ (by the definition of Ψ). Hence, $(\Psi \circ \Phi)(\alpha) = \alpha^\circ \circ \mathfrak{o} = \alpha = \mathrm{id}(\alpha)$.

Now, forget that we fixed α. We thus have proven that $(\Psi \circ \Phi)(\alpha) = \mathrm{id}(\alpha)$ for each $\alpha \in \mathfrak{A}$. In other words, $\Psi \circ \Phi = \mathrm{id}$. Qed.

(because $\mathfrak{A}$ is the set of all maps $\pi : E \to X$ satisfying $g\pi = \pi$, whereas $\mathfrak{B}$ is the set of all maps $\overline{\pi} : E^g \to X$). Furthermore, this bijection Φ sends any map $\pi : E \to X$ satisfying $g\pi = \pi$ to the map $\overline{\pi} : E^g \to X$ defined by

$$\overline{\pi}(u) = \pi(a) \qquad \text{for every } u \in E^g \text{ and } a \in u$$

[61]. Hence, we have constructed the bijection Φ whose existence was claimed in Proposition 7.6. Thus, Proposition 7.6 is proven. $\square$

Proposition 7.7. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$.

Let $g \in G$. Define the set E^g, the relations $<_1^g$ and $<_2^g$ and the triple $\mathbf{E}^g$ as in Proposition 7.5. Thus, $\mathbf{E}^g$ is a tertispecial double poset (by Proposition 7.5).

Proposition 7.7 (applied to $X = \{1, 2, 3, \ldots\}$) shows the following:

There is a bijection Φ between

- the maps $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$

and

- the maps $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$.

Namely, this bijection Φ sends any map $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$ to the map $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$ defined by

$$\overline{\pi}(u) = \pi(a) \qquad \text{for every } u \in E^g \text{ and } a \in u.$$

Consider this bijection Φ. Let $\pi : E \to \{1, 2, 3, \ldots\}$ be a map satisfying $g\pi = \pi$.

(a) If π is an $\mathbf{E}$-partition, then $\Phi(\pi)$ is an $\mathbf{E}^g$-partition.

(b) If $\Phi(\pi)$ is an $\mathbf{E}^g$-partition, then π is an $\mathbf{E}$-partition.

(c) Let $w : E \to \{1, 2, 3, \ldots\}$ be map. Define a map $w^g : E^g \to \{1, 2, 3, \ldots\}$ by

$$w^g(u) = \sum_{a \in u} w(a) \qquad \text{for every } u \in E^g.$$

Then, $\mathbf{x}_{\Phi(\pi), w^g} = \mathbf{x}_{\pi, w}$.

[61] *Proof.* Let $\pi : E \to X$ be a map satisfying $g\pi = \pi$. We must prove that the bijection Φ sends π to the map $\overline{\pi} : E^g \to X$ defined by

$$\overline{\pi}(u) = \pi(a) \qquad \text{for every } u \in E^g \text{ and } a \in u. \tag{62}$$

In fact, (61) shows that $\pi^\circ(u) = \pi(a)$ for every $u \in E^g$ and $a \in u$. Thus, the map π° is the map $\overline{\pi} : E^g \to X$ defined by (62). Now, the bijection Φ sends π to π° (by the definition of Φ). In other words, the bijection Φ sends π to the map $\overline{\pi} : E^g \to X$ defined by (62) (since π° is the map $\overline{\pi} : E^g \to X$ defined by (62)). Qed.

Proof of Proposition 7.7 (sketched). The definition of Φ shows that

$$(\Phi(\pi))(u) = \pi(a) \qquad \text{for every } u \in E^g \text{ and } a \in u. \tag{63}$$

(a) Assume that π is an $\mathbf{E}$-partition. We want to show that $\Phi(\pi)$ is an $\mathbf{E}^g$-partition. In order to do so, we can use Lemma 7.1 (applied to $\mathbf{E}^g$, $\left(E^g, <_1^g, <_2^g\right)$ and $\Phi(\pi)$ instead of $\mathbf{E}$, $(E, <_1, <_2)$ and ϕ); we only need to check the following two conditions:

Condition 1: If $e \in E^g$ and $f \in E^g$ are such that e is $<_1^g$-covered by f, and if we have $e <_2^g f$, then $(\Phi(\pi))(e) \leq (\Phi(\pi))(f)$.

Condition 2: If $e \in E^g$ and $f \in E^g$ are such that e is $<_1^g$-covered by f, and if we have $f <_2^g e$, then $(\Phi(\pi))(e) < (\Phi(\pi))(f)$.

Proof of Condition 1: Let $e \in E^g$ and $f \in E^g$ be such that e is $<_1^g$-covered by f. Assume that we have $e <_2^g f$.

We have $e <_1^g f$ (because e is $<_1^g$-covered by f). In other words, there exist $a \in e$ and $b \in f$ satisfying $a <_1 b$. Consider these a and b. Since π is an $\mathbf{E}$-partition, we have $\pi(a) \leq \pi(b)$ (since $a <_1 b$). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(e) = \pi(a)$ (since $a \in e$) and $(\Phi(\pi))(f) = \pi(b)$ (since $b \in f$). Thus, $(\Phi(\pi))(e) = \pi(a) \leq \pi(b) = (\Phi(\pi))(f)$. Hence, Condition 1 is proven.

Proof of Condition 2: Let $e \in E^g$ and $f \in E^g$ be such that e is $<_1^g$-covered by f. Assume that we have $f <_2^g e$.

We have $e <_1^g f$ (because e is $<_1^g$-covered by f). In other words, there exist $a \in e$ and $b \in f$ satisfying $a <_1 b$. Consider these a and b.

There exists no $c \in E$ satisfying $a <_1 c <_1 b$ [62]. In other words, a is $<_1$-covered by b (since $a <_1 b$). Therefore, a and b are $<_2$-comparable (since $\mathbf{E}$ is tertispecial). In other words, we have either $a <_2 b$ or $a = b$ or $b <_2 a$. Since $a <_2 b$ is impossible (because if we had $a <_2 b$, then we would have $e <_2^g f$ (since $a \in e$ and $b \in f$), which would contradict $f <_2^g e$ (since $<_2^g$ is a strict partial order)), and since $a = b$ is impossible (because $a <_1 b$), we therefore must have $b <_2 a$. But since π is an $\mathbf{E}$-partition, we have $\pi(a) < \pi(b)$ (since $a <_1 b$ and $b <_2 a$). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(e) = \pi(a)$ (since $a \in e$) and $(\Phi(\pi))(f) = \pi(b)$ (since $b \in f$). Thus, $(\Phi(\pi))(e) = \pi(a) < \pi(b) = (\Phi(\pi))(f)$. Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 7.7 (a) is proven.

[62]*Proof.* Assume the contrary. Thus, there exists some $c \in E$ satisfying $a <_1 c <_1 b$. Consider this c. Let w be the g-orbit of c. Thus, $w \in E^g$ and $c \in w$.

Now, the elements $a \in e$ and $c \in w$ satisfy $a <_1 c$. Hence, $e <_1^g w$ (by the definition of the relation $<_1^g$).

Also, the elements $c \in w$ and $b \in f$ satisfy $c <_1 b$. Hence, $w <_1^g f$ (by the definition of the relation $<_1^g$).

Now, we have $e <_1^g w <_1^g f$. This contradicts the fact that e is $<_1^g$-covered by f. Thus, we have obtained a contradiction; hence, our assumption was wrong. Qed.

(b) Assume that $\Phi(\pi)$ is an $\mathbf{E}^g$-partition. We want to show that π is an $\mathbf{E}$-partition. In order to do so, we can use Lemma 7.1 (applied to $\phi = \pi$); we only need to check the following two conditions:

Condition 1: If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $e <_2 f$, then $\pi(e) \leq \pi(f)$.

Condition 2: If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $f <_2 e$, then $\pi(e) < \pi(f)$.

Proof of Condition 1: Let $e \in E$ and $f \in E$ be such that e is $<_1$-covered by f. Assume that we have $e <_2 f$.

We have $e <_1 f$ (since e is $<_1$-covered by f). Let u and v be the g-orbits of e and f, respectively. Thus, u and v belong to E^g, and satisfy $e \in u$ and $f \in v$. Hence, $u <_1^g v$ (since $e <_1 f$). Hence, $(\Phi(\pi))(u) \leq (\Phi(\pi))(v)$ (since $\Phi(\pi)$ is an $\mathbf{E}^g$-partition). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(u) = \pi(e)$ (since $e \in u$) and $(\Phi(\pi))(v) = \pi(f)$ (since $f \in v$). Thus, $\pi(e) = (\Phi(\pi))(u) \leq (\Phi(\pi))(v) = \pi(f)$. Hence, Condition 1 is proven.

Proof of Condition 2: Let $e \in E$ and $f \in E$ be such that e is $<_1$-covered by f. Assume that we have $f <_2 e$.

We have $e <_1 f$ (since e is $<_1$-covered by f). Let u and v be the g-orbits of e and f, respectively. Thus, u and v belong to E^g, and satisfy $e \in u$ and $f \in v$. Hence, $u <_1^g v$ (since $e <_1 f$) and $v <_2^g u$ (since $f <_2 e$). Hence, $(\Phi(\pi))(u) < (\Phi(\pi))(v)$ (since $\Phi(\pi)$ is an $\mathbf{E}^g$-partition). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(u) = \pi(e)$ (since $e \in u$) and $(\Phi(\pi))(v) = \pi(f)$ (since $f \in v$). Thus, $\pi(e) = (\Phi(\pi))(u) < (\Phi(\pi))(v) = \pi(f)$. Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 7.7 (b) is proven.

(c) The elements of E^g are the g-orbits on E. Hence, the elements of E^g are pairwise disjoint subsets of E, and their union is E. In other words, the set E is the union of its disjoint subsets $u \in E^g$. Therefore, $\prod\limits_{a \in E} = \prod\limits_{u \in E^g} \prod\limits_{a \in u}$ (an equality between product signs).

The definition of $\mathbf{x}_{\Phi(\pi),w^g}$ shows that

$$\begin{aligned}
\mathbf{x}_{\Phi(\pi),w^g} &= \prod_{e\in E^g} x_{(\Phi(\pi))(e)}^{w^g(e)} = \prod_{u\in E^g} \underbrace{x_{(\Phi(\pi))(u)}^{w^g(u)}}_{\substack{=x_{(\Phi(\pi))(u)}^{\sum_{a\in u} w(a)} \\ \text{(since } w^g(u)=\sum_{a\in u} w(a))}} \\
&\qquad \text{(here, we have renamed the index } e \text{ as } u \text{ in the product)} \\
&= \prod_{u\in E^g} \underbrace{x_{(\Phi(\pi))(u)}^{\sum_{a\in u} w(a)}}_{=\prod_{a\in u} x_{(\Phi(\pi))(u)}^{w(a)}} = \prod_{u\in E^g}\prod_{a\in u} \underbrace{x_{(\Phi(\pi))(u)}^{w(a)}}_{\substack{=x_{\pi(a)}^{w(a)} \\ \text{(since } (\Phi(\pi))(u)=\pi(a) \\ \text{(by (63)))}}} \\
&= \underbrace{\prod_{u\in E^g}\prod_{a\in u}}_{=\prod_{a\in E}} x_{\pi(a)}^{w(a)} = \prod_{a\in E} x_{\pi(a)}^{w(a)} = \prod_{e\in E} x_{\pi(e)}^{w(e)} \\
&\qquad \text{(here, we have renamed the index } a \text{ as } e \text{ in the product)} \\
&= \mathbf{x}_{\pi,w}
\end{aligned}$$

(since the definition of $\mathbf{x}_{\pi,w}$ yields $\mathbf{x}_{\pi,w} = \prod_{e\in E} x_{\pi(e)}^{w(e)}$). This proves Proposition 7.7 (c). □

Our next lemma is a standard argument in Pólya enumeration theory (compare it with the proof of Burnside's lemma):

Lemma 7.8. Let G be a finite group. Let F be a G-set. Let O be a G-orbit on F, and let $\pi \in O$.

(a) We have
$$\frac{1}{|O|} = \frac{1}{|G|} \sum_{\substack{g\in G;\\ g\pi=\pi}} 1. \tag{64}$$

(b) Let E be a finite G-set. For every $g \in G$, let $\operatorname{sign}_E g$ denote the sign of the permutation of E that sends every $e \in E$ to ge. (Thus, $g \in G$ is E-even if and only if $\operatorname{sign}_E g = 1$.) Then,
$$\begin{cases} \dfrac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} = \frac{1}{|G|} \sum_{\substack{g\in G;\\ g\pi=\pi}} \operatorname{sign}_E g. \tag{65}$$

Proof of Lemma 7.8. Let $\operatorname{Stab}_G \pi$ denote the stabilizer of π; this is the subgroup $\{g \in G \mid g\pi = \pi\}$ of G. (This subgroup is also known as the *stabilizer subgroup* or the *isotropy group* of π.) The G-orbit of π is O (since O is a G-orbit on F, and since $\pi \in O$). In other words, $O = G\pi$. Therefore, $|O| = |G\pi| = |G| / |\operatorname{Stab}_G \pi|$ (by the orbit-stabilizer theorem). Hence,

$$\frac{1}{|O|} = \frac{1}{|G| / |\operatorname{Stab}_G \pi|} = \frac{|\operatorname{Stab}_G \pi|}{|G|}. \tag{66}$$

(a) We have

$$\sum_{\substack{g \in G;\\ g\pi = \pi}} 1 = \left| \underbrace{\{g \in G \mid g\pi = \pi\}}_{= \operatorname{Stab}_G \pi} \right| = |\operatorname{Stab}_G \pi| .$$

Hence,

$$\frac{1}{|G|} \underbrace{\sum_{\substack{g \in G;\\ g\pi = \pi}} 1}_{= |\operatorname{Stab}_G \pi|} = \frac{1}{|G|} |\operatorname{Stab}_G \pi| = \frac{|\operatorname{Stab}_G \pi|}{|G|} = \frac{1}{|O|}$$

(by (66)). This proves Lemma 7.8 (a).

(b) We need to prove (65). Assume first that O is E-coeven. Thus, all elements of O are E-coeven (by the definition of what it means for O to be E-coeven). Hence, π is E-coeven (since $\pi \in O$). This means that every $g \in G$ satisfying $g\pi = \pi$ is E-even. Hence, every $g \in G$ satisfying $g\pi = \pi$ satisfies $\operatorname{sign}_E g = 1$ (since g is E-even if and only if $\operatorname{sign}_E g = 1$). Thus,

$$\begin{aligned} \frac{1}{|G|} \sum_{\substack{g \in G;\\ g\pi = \pi}} \underbrace{\operatorname{sign}_E g}_{=1} = \frac{1}{|G|} \sum_{\substack{g \in G;\\ g\pi = \pi}} 1 = \frac{1}{|O|} \qquad & (\text{by (64)}) \\ = \begin{cases} \dfrac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} \qquad & (\text{since } O \text{ is } E\text{-coeven}) . \end{aligned}$$

Thus, we have proven (65) under the assumption that O is E-coeven. We can therefore WLOG assume the opposite now. Thus, assume WLOG that O is not E-coeven. Hence, no element of O is E-coeven (due to the contrapositive of Lemma 4.5). In particular, π is not E-coeven (since $\pi \in O$). In other words, not every $g \in G$ satisfying $g\pi = \pi$ is E-even. In other words, not every $g \in \operatorname{Stab}_G \pi$ is E-even (since the elements $g \in G$ satisfying $g\pi = \pi$ are exactly the elements $g \in \operatorname{Stab}_G \pi$). In other words, not every $g \in \operatorname{Stab}_G \pi$ satisfies $\operatorname{sign}_E g = 1$ (since g is E-even if and only if $\operatorname{sign}_E g = 1$).

Now, the map

$$\operatorname{Stab}_G \pi \to \{1, -1\} , \qquad g \mapsto \operatorname{sign}_E g$$

is a group homomorphism (since the action of G on E is a group homomorphism $G \to \operatorname{Aut} E$, and since the sign of a permutation is multiplicative) and is not the trivial homomorphism (since not every $g \in \operatorname{Stab}_G \pi$ satisfies $\operatorname{sign}_E g = 1$). Hence, it must send exactly half the elements of $\operatorname{Stab}_G \pi$ to 1 and the other half to -1. Therefore, the addends in the sum $\sum_{g \in \operatorname{Stab}_G \pi} \operatorname{sign}_E g$ cancel each other out (one half of them are 1, and the others are -1). Therefore, $\sum_{g \in \operatorname{Stab}_G \pi} \operatorname{sign}_E g = 0$. Now,

$$\frac{1}{|G|} \underbrace{\sum_{\substack{g \in G; \\ g\pi = \pi}}}_{= \sum_{g \in \operatorname{Stab}_G \pi}} \operatorname{sign}_E g = \frac{1}{|G|} \underbrace{\sum_{g \in \operatorname{Stab}_G \pi} \operatorname{sign}_E g}_{=0} = 0 = \begin{cases} \dfrac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases}$$

(since O is not E-coeven). This proves (65). Lemma 7.8 (b) is thus proven. □

Proof of Theorem 4.7 (sketched). Let $g \in G$. Define the set E^g, the relations $<_1^g$ and $<_2^g$ and the triple $\mathbf{E}^g$ as in Proposition 7.5. Thus, $\mathbf{E}^g$ is a tertispecial double poset (by Proposition 7.5). In other words, $\left(E^g, <_1^g, <_2^g\right)$ is a tertispecial double poset (since $\mathbf{E}^g = \left(E^g, <_1^g, <_2^g\right)$).

Now, forget that we fixed g. We thus have constructed a tertispecial double poset $\mathbf{E}^g = \left(E^g, <_1^g, <_2^g\right)$ for every $g \in G$.

Moreover, for every $g \in G$, let us define $>_1^g$ to be the opposite relation of $<_1^g$.

Furthermore, for every $g \in G$, define a map $w^g : E^g \to \{1, 2, 3, \ldots\}$ by $w^g(u) = \sum_{a \in u} w(a)$. (Since G preserves w, the numbers $w(a)$ for all $a \in u$ are equal (for given u), and thus $\sum_{a \in u} w(a)$ can be rewritten as $|u| \cdot w(b)$ for any particular $b \in u$. But we shall not use this observation.) Now, every $g \in G$ satisfies

$$S\left(\Gamma\left(\left(E^g, <_1^g, <_2^g\right), w^g\right)\right) = (-1)^{|E^g|} \Gamma\left(\left(E^g, >_1^g, <_2^g\right), w^g\right). \qquad (67)$$

(Indeed, this follows from Theorem 4.2 (applied to $\left(E^g, <_1^g, <_2^g\right)$ and w^g instead of $(E, <_1, <_2)$ and w) since the double poset $\left(E^g, <_1^g, <_2^g\right)$ is tertispecial.)

For every $g \in G$, we have

$$\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi, w} = \Gamma\left(\mathbf{E}^g, w^g\right) \qquad (68)$$

[63].

[63]*Proof of* (68)*:* Let $g \in G$. The definition of $\Gamma\left(\mathbf{E}^g, w^g\right)$ yields

$$\Gamma\left(\mathbf{E}^g, w^g\right) = \sum_{\pi \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\pi, w^g} = \sum_{\overline{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\overline{\pi}, w^g} \qquad (69)$$

(here, we have renamed the summation index π as $\overline{\pi}$).

In Proposition 7.7, we have introduced a bijection Φ between

- the maps $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$

It is clearly sufficient to prove Theorem 4.7 for $\mathbf{k} = \mathbb{Z}$ (since all the power series that we are discussing are defined functorially in $\mathbf{k}$ (and so are the Hopf algebra QSym and its antipode S), and thus any identity between these series that holds over $\mathbb{Z}$ must hold over any $\mathbf{k}$). Therefore, it is sufficient to prove Theorem 4.7 for $\mathbf{k} = \mathbb{Q}$ (since $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ embeds into $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$, and using this embedding we have $\operatorname{QSym}_{\mathbb{Z}} = \operatorname{QSym}_{\mathbb{Q}} \cap \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ [64]). Thus, we WLOG assume that $\mathbf{k} = \mathbb{Q}$. This will allow us to divide by positive integers.

Every G-orbit O on $\operatorname{Par}\mathbf{E}$ satisfies

$$\frac{1}{|O|} \sum_{\pi \in O} \underbrace{\mathbf{x}_{\pi,w}}_{\substack{=\mathbf{x}_{O,w} \\ \text{(since } \mathbf{x}_{O,w} \text{ is defined} \\ \text{to be } \mathbf{x}_{\pi,w})}} = \frac{1}{|O|} \underbrace{\sum_{\pi \in O} \mathbf{x}_{O,w}}_{=|O|\mathbf{x}_{O,w}} = \frac{1}{|O|} |O| \, \mathbf{x}_{O,w} = \mathbf{x}_{O,w}. \tag{70}$$

and

- the maps $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$.

Parts (a) and (b) of Proposition 7.7 show that this bijection Φ restricts to a bijection between

- the $\mathbf{E}$-partitions $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$

and

- the $\mathbf{E}^g$-partitions $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$.

Hence, we can substitute $\Phi(\pi)$ for $\overline{\pi}$ in the sum $\sum_{\overline{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\overline{\pi},w^g}$. We thus obtain

$$\sum_{\overline{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\overline{\pi},w^g} = \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \underbrace{\mathbf{x}_{\Phi(\pi),w^g}}_{\substack{=\mathbf{x}_{\pi,w} \\ \text{(by Proposition 7.7 (c))}}} = \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi,w},$$

whence $\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi,w} = \sum_{\overline{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\overline{\pi},w^g} = \Gamma(\mathbf{E}^g, w^g)$ (by (69)). This proves (68).

[64]Here, we are using the notation $\operatorname{QSym}_{\mathbf{k}}$ for the Hopf algebra QSym defined over a base ring $\mathbf{k}$.

Now,

$$\begin{aligned}
\Gamma\left(\mathbf{E},w,G\right) &= \sum_{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E}} \underbrace{\mathbf{x}_{O,w}}_{\substack{=\frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\ \text{(by (70))}}} = \sum_{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E}} \frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w} \\
&= \sum_{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E}} \sum_{\pi\in O} \underbrace{\frac{1}{|O|}}_{\substack{=\frac{1}{|G|}\sum_{\substack{g\in G;\\ g\pi=\pi}}1\\ \text{(by (64), applied to }F=\operatorname{Par}\mathbf{E})}} \mathbf{x}_{\pi,w} \\
&= \underbrace{\sum_{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E}} \sum_{\pi\in O}}_{=\sum_{\pi\in\operatorname{Par}\mathbf{E}} = \sum_{\pi\text{ is an }\mathbf{E}\text{-partition}}} \left(\frac{1}{|G|}\sum_{\substack{g\in G;\\ g\pi=\pi}}1\right)\mathbf{x}_{\pi,w} \\
&= \sum_{\pi\text{ is an }\mathbf{E}\text{-partition}} \left(\frac{1}{|G|}\sum_{\substack{g\in G;\\ g\pi=\pi}}1\right)\mathbf{x}_{\pi,w} = \frac{1}{|G|}\underbrace{\sum_{\pi\text{ is an }\mathbf{E}\text{-partition}}\sum_{\substack{g\in G;\\ g\pi=\pi}}}_{=\sum_{g\in G}\sum_{\substack{\pi\text{ is an }\mathbf{E}\text{-partition};\\ g\pi=\pi}}}\mathbf{x}_{\pi,w} \\
&= \frac{1}{|G|}\sum_{g\in G}\underbrace{\sum_{\substack{\pi\text{ is an }\mathbf{E}\text{-partition};\\ g\pi=\pi}}\mathbf{x}_{\pi,w}}_{\substack{=\Gamma\left(\mathbf{E}^g,w^g\right)\\ \text{(by (68))}}} \\
&= \frac{1}{|G|}\sum_{g\in G}\Gamma\left(\underbrace{\mathbf{E}^g}_{=\left(E^g,<_1^g,<_2^g\right)},w^g\right) = \frac{1}{|G|}\sum_{g\in G}\Gamma\left(\left(E^g,<_1^g,<_2^g\right),w^g\right).
\end{aligned} \tag{71}$$

Hence, $\Gamma\left(\mathbf{E},w,G\right)\in\operatorname{QSym}$ (by Proposition 3.5).

Applying the map S to both sides of the equality (71), we obtain

$$\begin{aligned}
S\left(\Gamma\left(\mathbf{E},w,G\right)\right) &= S\left(\frac{1}{|G|}\sum_{g\in G}\Gamma\left(\left(E^g,<_1^g,<_2^g\right),w^g\right)\right) = \frac{1}{|G|}\sum_{g\in G}\underbrace{S\left(\Gamma\left(\left(E^g,<_1^g,<_2^g\right),w^g\right)\right)}_{\substack{=(-1)^{|E^g|}\Gamma\left(\left(E^g,>_1^g,<_2^g\right),w^g\right)\\ \text{(by (67))}}} \\
&= \frac{1}{|G|}\sum_{g\in G}(-1)^{|E^g|}\Gamma\left(\left(E^g,>_1^g,<_2^g\right),w^g\right).
\end{aligned} \tag{72}$$

On the other hand, for every $g \in G$, let $\operatorname{sign}_E g$ denote the sign of the permutation of E that sends every $e \in E$ to ge. Thus, $g \in G$ is E-even if and only if $\operatorname{sign}_E g = 1$. Now, every G-orbit O on $\operatorname{Par} \mathbf{E}$ and every $\pi \in O$ satisfy

$$\begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} = \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \operatorname{sign}_E g \tag{73}$$

(by (65), applied to $F = \operatorname{Par} \mathbf{E}$). Furthermore,

$$\operatorname{sign}_E g = (-1)^{|E| - |E^g|} \tag{74}$$

for every $g \in G$ [65].

[65] *Proof of* (74)*:* Let $g \in G$. Recall that $\operatorname{sign}_E g$ is the sign of the permutation of E that sends every $e \in E$ to ge. Denote this permutation by ζ. Thus, $\operatorname{sign}_E g$ is the sign of ζ.

The permutation ζ is the permutation of E that sends every $e \in E$ to ge. In other words, ζ is the action of g on E. Hence, the cycles of ζ are the g-orbits on E. Thus, the set of all cycles of ζ is the set of all g-orbits on E; this latter set is E^g. Hence, E^g is the set of all cycles of ζ.

But if σ is a permutation of a finite set X, then the sign of σ is $(-1)^{|X| - |X^\sigma|}$, where X^σ is the set of all cycles of σ. Applying this to $X = E$, $\sigma = \zeta$ and $X^\sigma = E^g$, we see that the sign of ζ is $(-1)^{|E| - |E^g|}$ (because E^g is the set of all cycles of ζ). In other words, $\operatorname{sign}_E g = (-1)^{|E| - |E^g|}$ (since $\operatorname{sign}_E g$ is the sign of ζ), qed.

Now,

$$\begin{aligned}
&\Gamma^{+}\left(\mathbf{E},w,G\right)\\
&=\sum_{O\text{ is an }E\text{-coeven }G\text{-orbit on }\operatorname{Par}\mathbf{E}}\underbrace{\mathbf{x}_{O,w}}_{\substack{=\frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\ \text{(by (70))}}}=\underbrace{\sum_{O\text{ is an }E\text{-coeven }G\text{-orbit on }\operatorname{Par}\mathbf{E}}}_{\substack{=\sum_{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is }E\text{-coeven}}}}\frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\
&=\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is }E\text{-coeven}}}\frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\
&=\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is }E\text{-coeven}}}\underbrace{\frac{1}{|O|}}_{\substack{=\begin{cases}\frac{1}{|O|}, & \text{if }O\text{ is }E\text{-coeven};\\ 0, & \text{if }O\text{ is not }E\text{-coeven}\end{cases}\\ \text{(since }O\text{ is }E\text{-coeven)}}}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\
&\qquad+\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is not }E\text{-coeven}}}\underbrace{0}_{\substack{=\begin{cases}\frac{1}{|O|}, & \text{if }O\text{ is }E\text{-coeven};\\ 0, & \text{if }O\text{ is not }E\text{-coeven}\end{cases}\\ \text{(since }O\text{ is not }E\text{-coeven)}}}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\
&\qquad\left(\begin{array}{c}\text{since }\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is }E\text{-coeven}}}\frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w}+\underbrace{\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is not }E\text{-coeven}}}0\sum_{\pi\in O}\mathbf{x}_{\pi,w}}_{=0}\\ =\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is }E\text{-coeven}}}\frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\end{array}\right)\\
&=\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is }E\text{-coeven}}}\begin{cases}\frac{1}{|O|}, & \text{if }O\text{ is }E\text{-coeven};\\ 0, & \text{if }O\text{ is not }E\text{-coeven}\end{cases}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\
&\qquad+\sum_{\substack{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E};\\ O\text{ is not }E\text{-coeven}}}\begin{cases}\frac{1}{|O|}, & \text{if }O\text{ is }E\text{-coeven};\\ 0, & \text{if }O\text{ is not }E\text{-coeven}\end{cases}\sum_{\pi\in O}\mathbf{x}_{\pi,w}\\
&=\sum_{O\text{ is a }G\text{-orbit on }\operatorname{Par}\mathbf{E}}\begin{cases}\frac{1}{|O|}, & \text{if }O\text{ is }E\text{-coeven};\\ 0, & \text{if }O\text{ is not }E\text{-coeven}\end{cases}\sum_{\pi\in O}\mathbf{x}_{\pi,w}
\end{aligned}$$

$$= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par} \mathbf{E}} \sum_{\pi \in O} \underbrace{\begin{cases} \dfrac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases}}_{\substack{= \frac{1}{|G|} \sum_{g \in G;\ g\pi = \pi} \operatorname{sign}_E g \\ \text{(by (73))}}} \mathbf{x}_{\pi,w}$$

$$= \underbrace{\sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par} \mathbf{E}} \sum_{\pi \in O}}_{= \sum_{\pi \in \operatorname{Par} \mathbf{E}} = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}} \left(\frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \operatorname{sign}_E g \right) \mathbf{x}_{\pi,w}$$

$$= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \left(\frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \operatorname{sign}_E g \right) \mathbf{x}_{\pi,w} = \frac{1}{|G|} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \sum_{\substack{g \in G; \\ g\pi = \pi}}}_{= \sum_{g \in G} \sum_{\pi \text{ is an } \mathbf{E}\text{-partition};\ g\pi = \pi}} \left(\operatorname{sign}_E g\right) \mathbf{x}_{\pi,w}$$

$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\operatorname{sign}_E g}_{\substack{= (-1)^{|E| - |E^g|} \\ \text{(by (74))}}} \underbrace{\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition}; \\ g\pi = \pi}} \mathbf{x}_{\pi,w}}_{\substack{= \Gamma(\mathbf{E}^g, w^g) \\ \text{(by (68))}}} = \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma \left(\underbrace{\mathbf{E}^g}_{= \left(E^g, <_1^g, <_2^g\right)}, w^g \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma \left(\left(E^g, <_1^g, <_2^g\right), w^g \right). \tag{75}$$

Hence, $\Gamma^+ (\mathbf{E}, w, G) \in \operatorname{QSym}$ (by Proposition 3.5).

The group G preserves the relation $>_1$ (since it preserves the relation $<_1$). Furthermore, Lemma 5.10 shows that $(E, >_1, <_2)$ is a tertispecial double poset. Hence, we can apply (75) to $(E, >_1, <_2)$, $>_1$ and $>_1^g$ instead of $\mathbf{E}$, $<_1$ and $<_1^g$. As a result, we obtain

$$\Gamma^+ ((E, >_1, <_2), w, G) = \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma \left(\left(E^g, >_1^g, <_2^g\right), w^g \right).$$

Multiplying both sides of this equality by $(-1)^{|E|}$, we transform it into

$$\begin{aligned}(-1)^{|E|}\,\Gamma^{+}\left(\left(E,>_1,<_2\right),w,G\right) &= (-1)^{|E|}\,\frac{1}{|G|}\sum_{g\in G}(-1)^{|E|-|E^g|}\,\Gamma\left(\left(E^g,>_1^g,<_2^g\right),w^g\right)\\ &= \frac{1}{|G|}\sum_{g\in G}\underbrace{(-1)^{|E|}\,(-1)^{|E|-|E^g|}}_{=(-1)^{|E^g|}}\,\Gamma\left(\left(E^g,>_1^g,<_2^g\right),w^g\right)\\ &= \frac{1}{|G|}\sum_{g\in G}(-1)^{|E^g|}\,\Gamma\left(\left(E^g,>_1^g,<_2^g\right),w^g\right)\\ &= S\left(\Gamma\left(\mathbf{E},w,G\right)\right) \qquad \text{(by (72))}.\end{aligned}$$

This completes the proof of Theorem 4.7. □

8. Application: Jochemko's theorem

We shall now demonstrate an application of Theorem 4.7: namely, we will use it to provide an alternative proof of [Joch13, Theorem 2.13]. The way we derive [Joch13, Theorem 2.13] from Theorem 4.7 is classical, and in fact was what originally motivated the discovery of Theorem 4.7 (although, of course, it cannot be conversely derived from [Joch13, Theorem 2.13], so it is an actual generalization).

An intermediate step between [Joch13, Theorem 2.13] and Theorem 4.7 will be the following fact:

Corollary 8.1. Let $\mathbf{E} = (E,<_1,<_2)$ be a tertispecial double poset. Let $w : E \to \{1,2,3,\ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. For every $q \in \mathbb{N}$, let $\operatorname{Par}_q \mathbf{E}$ denote the set of all $\mathbf{E}$-partitions whose image is contained in $\{1,2,\ldots,q\}$. Then, the group G also acts on $\operatorname{Par}_q \mathbf{E}$; namely, $\operatorname{Par}_q \mathbf{E}$ is a G-subset of the G-set $\{1,2,\ldots,q\}^E$ (see Definition 4.4 (d) for the definition of the latter).

(a) There exists a unique polynomial $\Omega_{\mathbf{E},G} \in \mathbb{Q}[X]$ such that every $q \in \mathbb{N}$ satisfies

$$\Omega_{\mathbf{E},G}(q) = \left(\text{the number of all } G\text{-orbits on } \operatorname{Par}_q \mathbf{E}\right). \tag{76}$$

(b) This polynomial satisfies

$$\begin{aligned}&\Omega_{\mathbf{E},G}(-q)\\ &= (-1)^{|E|}\left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q\left(E,>_1,<_2\right)\right)\\ &= (-1)^{|E|}\left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q\left(E,<_1,>_2\right)\right)\end{aligned} \tag{77}$$

for all $q \in \mathbb{N}$.

Proof of Corollary 8.1 (sketched). Set $\mathbf{k} = \mathbb{Q}$. For any $f \in \operatorname{QSym}$ and any $q \in \mathbb{N}$, we define an element $\operatorname{ps}^1(f)(q) \in \mathbb{Q}$ by

$$\operatorname{ps}^1(f)(q) = f\left(\underbrace{1,1,\ldots,1}_{q \text{ times}},0,0,0,\ldots\right)$$

(that is, $\operatorname{ps}^1(f)(q)$ is the result of substituting 1 for each of the variables $x_1, x_2, \ldots, x_q$ and 0 for each of the variables $x_{q+1}, x_{q+2}, x_{q+3}, \ldots$ in the power series f).

(a) Consider the elements $\Gamma(\mathbf{E}, w, G)$ and $\Gamma^+(\mathbf{E}, w, G)$ of QSym defined in Theorem 4.7. Observe that $\operatorname{Par}_q \mathbf{E}$ is a G-subset of $\operatorname{Par} \mathbf{E}$.

Now, [GriRei14, Proposition 7.7 (i)] shows that, for any given $f \in \operatorname{QSym}$, there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $\operatorname{ps}^1(f)(q)$. Applying this to $f = \Gamma(\mathbf{E}, w, G)$, we conclude that there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $\operatorname{ps}^1(\Gamma(\mathbf{E}, w, G))(q)$. But since every $q \in \mathbb{N}$ satisfies

$$\begin{aligned}
\operatorname{ps}^1(\Gamma(\mathbf{E}, w, G))(q) &= \underbrace{(\Gamma(\mathbf{E}, w, G))}_{=\sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w}}\left(\underbrace{1,1,\ldots,1}_{q \text{ times}},0,0,0,\ldots\right) \\
&= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \underbrace{\mathbf{x}_{O,w}\left(\underbrace{1,1,\ldots,1}_{q \text{ times}},0,0,0,\ldots\right)}_{=\begin{cases} 1, & \text{if } O \subseteq \operatorname{Par}_q \mathbf{E}; \\ 0, & \text{if } O \not\subseteq \operatorname{Par}_q \mathbf{E} \end{cases}} \\
&= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \begin{cases} 1, & \text{if } O \subseteq \operatorname{Par}_q \mathbf{E}; \\ 0, & \text{if } O \not\subseteq \operatorname{Par}_q \mathbf{E} \end{cases} \\
&= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}_q\mathbf{E}} 1 = \left(\text{the number of all } G\text{-orbits on } \operatorname{Par}_q \mathbf{E}\right),
\end{aligned} \tag{78}$$

this rewrites as follows: There exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $\left(\text{the number of all } G\text{-orbits on } \operatorname{Par}_q \mathbf{E}\right)$. This proves Corollary 8.1 (a).

(b) [GriRei14, Proposition 7.7 (i)] shows that, for any given $f \in \operatorname{QSym}$, there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $\operatorname{ps}^1(f)(q)$. This polynomial is denoted by $\operatorname{ps}^1(f)$ in [GriRei14, Proposition 7.7]. From our above proof of Corollary 8.1 (a), we see that

$$\Omega_{\mathbf{E},G} = \operatorname{ps}^1(\Gamma(\mathbf{E}, w, G)).$$

But [GriRei14, Proposition 7.7 (iii)] shows that, for any $f \in \operatorname{QSym}$ and $m \in \mathbb{N}$, we have $\operatorname{ps}^1(S(f))(m) = \operatorname{ps}^1(f)(-m)$. Applying this to $f = \Gamma(\mathbf{E}, w, G)$, we obtain

$$\operatorname{ps}^1(S(\Gamma(\mathbf{E}, w, G)))(m) = \underbrace{\operatorname{ps}^1(\Gamma(\mathbf{E}, w, G))}_{=\Omega_{\mathbf{E},G}}(-m) = \Omega_{\mathbf{E},G}(-m)$$

for any $m \in \mathbb{N}$. Thus, any $m \in \mathbb{N}$ satisfies

$$\begin{aligned}\Omega_{\mathbf{E},G}(-m) &= \operatorname{ps}^1\left(\underbrace{S(\Gamma(\mathbf{E}, w, G))}_{\substack{=(-1)^{|E|}\Gamma^+((E,>_1,<_2),w,G) \\ \text{(by Theorem 4.7)}}}\right)(m) \\ &= \operatorname{ps}^1\left((-1)^{|E|}\Gamma^+((E, >_1, <_2), w, G)\right)(m) \\ &= (-1)^{|E|}\operatorname{ps}^1\left(\Gamma^+((E, >_1, <_2), w, G)\right)(m).\end{aligned}$$

Renaming m as q in this equality, we see that every $q \in \mathbb{N}$ satisfies

$$\Omega_{\mathbf{E},G}(-q) = (-1)^{|E|}\operatorname{ps}^1\left(\Gamma^+((E, >_1, <_2), w, G)\right)(q). \tag{79}$$

But just as we proved (78), we can show that every $q \in \mathbb{N}$ satisfies

$$\operatorname{ps}^1\left(\Gamma^+(\mathbf{E}, w, G)\right)(q) = \left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q \mathbf{E}\right).$$

Applying this to $(E, >_1, <_2)$ instead of $\mathbf{E}$, we obtain

$$\begin{aligned}&\operatorname{ps}^1\left(\Gamma^+((E, >_1, <_2), w, G)\right)(q) \\ &= \left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q(E, >_1, <_2)\right).\end{aligned}$$

Now, (79) becomes

$$\begin{aligned}\Omega_{\mathbf{E},G}(-q) &= (-1)^{|E|} \underbrace{\operatorname{ps}^1\left(\Gamma^+((E, >_1, <_2), w, G)\right)(q)}_{=\left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q(E,>_1,<_2)\right)} \\ &= (-1)^{|E|}\left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q(E, >_1, <_2)\right).\end{aligned}$$

In order to prove Corollary 8.1 (b), it thus remains to show that

$$\begin{aligned}&\left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q(E, >_1, <_2)\right) \\ &= \left(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q(E, <_1, >_2)\right)\end{aligned} \tag{80}$$

for every $q \in \mathbb{N}$.

Proof of (80): Let $q \in \mathbb{N}$. Let $w_0 : \{1, 2, \ldots, q\} \to \{1, 2, \ldots, q\}$ be the map sending each $i \in \{1, 2, \ldots, q\}$ to $q + 1 - i$. Then, the map

$$\operatorname{Par}_q(E, >_1, <_2) \to \operatorname{Par}_q(E, <_1, >_2), \qquad \pi \mapsto w_0 \circ \pi$$

is an isomorphism of G-sets (this is easy to check). Thus, $\operatorname{Par}_q(E, >_1, <_2) \cong \operatorname{Par}_q(E, <_1, >_2)$ as G-sets. From this, (80) follows (by functoriality, if one wishes).

The proof of Corollary 8.1 (b) is now complete. $\square$

Now, the second formula of [Joch13, Theorem 2.13] follows from our (77), applied to $\mathbf{E} = (P, \prec, <_\omega)$ (where $<_\omega$ is the partial order on P given by $(p <_\omega q) \iff (\omega(p) < \omega(q))$). The first formula of [Joch13, Theorem 2.13] can also be derived from our above arguments. We leave the details to the reader.

9. A final question

With the results proven above (specifically, Theorems 4.2 and 4.7), we have obtained formulas for a large class of quasisymmetric generating functions for maps from a double poset to $\{1, 2, 3, \ldots\}$. At least one question arises:

Question 9.1. In [Grin16a], I have studied generalizations of Whitney's famous non-broken-circuit theorem for graphs and matroids. One of the cornerstones of that study is the bijection Φ in [Grin16a, proofs of Lemma 2.7 and Lemma 5.25], which is uncannily reminiscent of the involution T in the proof of Theorem 4.2. (Actually, this bijection Φ can be extended to an involution, thus making the analogy even more palpable.) Both Φ and T are defined by toggling a certain element in or out of a subset; and this element is chosen as the argmin or argmax of a function defined on the ground set. Is there a connection between the two results, or even a common generalization?

10. Appendix: Proofs of some basic properties of quasisymmetric functions

In this final section, we are going to restate and prove (in detail) some foundational facts that were stated without proof in the first few sections of this note. Most of these facts are well-known, and all are pretty obvious to anyone with some experience in this subject (although sometimes, formalizing the intuitively clear arguments is a nontrivial task); the reasons why I nevertheless have chosen to prove them here are twofold: One is to make this paper more self-contained (although this is not completely achieved, as some other results from places such as [GriRei14] are used without proof); another is to do (some of) the groundwork for an eventual formalization of the theory of quasisymmetric functions in a formal proof system (such as Coq). I do not expect much of the following to be useful to the reader; most likely, she will be able to reconstruct at least the proofs herself easily, if not the theorems as well.

10.1. Monomial quasisymmetric functions

We begin with a fact that was used in the definition of M_α given in Section 2:

Proposition 10.1. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. Then,

$$\sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathfrak{m} \text{ is a monomial pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} \mathfrak{m}.$$

Proof of Proposition 10.1. If $(i_1 < i_2 < \cdots < i_\ell)$ is a length-ℓ strictly increasing sequence of positive integers, then $(i_1 < i_2 < \cdots < i_\ell)$ can be uniquely reconstructed from the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ [66]. Hence, we conclude the following:

Claim 1: If a monomial $\mathfrak{m}$ can be written in the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ for some length-ℓ strictly increasing sequence $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers, then $\mathfrak{m}$ can be written in this form in a **unique** way.

On the other hand, the definition of "pack-equivalent" yields the following: The monomials which are pack-equivalent to $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$ are precisely the monomials of the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ where $(i_1 < i_2 < \cdots < i_\ell)$ is a length-ℓ strictly increasing sequence of positive integers. Hence,

$$\sum_{\substack{\mathfrak{m} \text{ is a monomial pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} = \sum_{\substack{\mathfrak{m} \text{ is a monomial of the form } x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \\ \text{for some length-}\ell\text{ strictly increasing sequence} \\ (i_1 < i_2 < \cdots < i_\ell) \text{ of positive integers}}}$$

(an equality between summation signs). Thus,

$$\begin{aligned} \sum_{\substack{\mathfrak{m} \text{ is a monomial pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} \mathfrak{m} &= \sum_{\substack{\mathfrak{m} \text{ is a monomial of the form } x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \\ \text{for some length-}\ell\text{ strictly increasing sequence} \\ (i_1 < i_2 < \cdots < i_\ell) \text{ of positive integers}}} \mathfrak{m} \\ &= \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \end{aligned}$$

(by Claim 1). This proves Proposition 10.1. □

Definition 10.2. Let $k \in \mathbb{Z}$. Then, $[k]$ will denote the subset $\{1, 2, \ldots, k\} = \{a \in \mathbb{Z} \mid 0 < a \leq k\}$ of $\{1, 2, 3, \ldots\}$. Notice that $|[k]| = k$ when $k \in \mathbb{N}$. For negative k, we have $[k] = \varnothing$.

We shall now prove a result that will be used further below. We recall the definition of $D(\alpha)$ given in Example 3.6 (c):

Definition 10.3. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Let $D(\alpha)$ denote the set

$$\begin{aligned} &\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\} \\ &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in [\ell - 1]\}. \end{aligned}$$

(We notice that this definition of $D(\alpha)$ is identical with that given in [GriRei14, Definition 5.10].)

Lemma 10.4. Let α be a composition of a nonnegative integer n. Then, $D(\alpha) \subseteq [n-1]$.

Proof of Lemma 10.4. Write α in the form $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Let $k \in D(\alpha)$.

We know that $(\alpha_1, \alpha_2, \ldots, \alpha_\ell) = \alpha$ is a composition of n. Thus, $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers, and their sum is $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$.

We have $k \in D(\alpha) = \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in [\ell - 1]\}$. In other words, there exists some $i \in [\ell - 1]$ such that $k = \alpha_1 + \alpha_2 + \cdots + \alpha_i$. Consider this i.

We have $i \in [\ell - 1]$, and thus $1 \leq i \leq \ell - 1$. From $i \leq \ell - 1$, we obtain

$$\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_\ell = \underbrace{(\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_{\ell-1})}_{\geq 0} + \alpha_\ell \geq \alpha_\ell > 0.$$

[66] Namely, $(i_1 < i_2 < \cdots < i_\ell)$ is the list of all positive integers j such that x_j appears in this monomial, written in increasing order.

Now,

$$\begin{aligned} n &= \alpha_1 + \alpha_2 + \cdots + \alpha_\ell \\ &= (\alpha_1 + \alpha_2 + \cdots + \alpha_i) + \underbrace{(\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_\ell)}_{>0} \qquad (\text{since } i \leq \ell - 1 \leq \ell) \\ &> \alpha_1 + \alpha_2 + \cdots + \alpha_i = k, \end{aligned}$$

so that $k < n$. Combining this with

$$\begin{aligned} k &= \alpha_1 + \alpha_2 + \cdots + \alpha_i = \alpha_1 + \underbrace{(\alpha_2 + \alpha_3 + \cdots + \alpha_i)}_{\geq 0} \qquad (\text{since } i \geq 1) \\ &\geq \alpha_1 > 0, \end{aligned}$$

we find that $0 < k < n$. In other words, $k \in [n-1]$.

Now, forget that we fixed k. We thus have proven that $k \in [n-1]$ for every $k \in D(\alpha)$. In other words, $D(\alpha) \subseteq [n-1]$. This proves Lemma 10.4. $\square$

Lemma 10.5. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. For every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

(a) We have $s_i \in [n]$ for every $i \in [\ell]$.

(b) We have $s_0 < s_1 < \cdots < s_\ell$.

(c) We have $D(\alpha) = \{s_1, s_2, \ldots, s_{\ell-1}\}$.

(d) We have $s_j - s_{j-1} = \alpha_j$ for every $j \in [\ell]$.

(e) We have $s_\ell = n$.

(f) We have $s_0 = 0$.

(g) For every $k \in [n]$, the element $\min\{p \in [\ell] \mid s_p \geq k\}$ is a well-defined element of $[\ell]$.

Proof of Lemma 10.5. We have $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Hence, $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of n (since α is a composition of n). Thus, $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers, and their sum is $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$.

(a) Let $i \in [\ell]$. Thus, $1 \leq i \leq \ell$. Now,

$$\begin{aligned} s_i &= \alpha_1 + \alpha_2 + \cdots + \alpha_i = \alpha_1 + \underbrace{(\alpha_2 + \alpha_3 + \cdots + \alpha_i)}_{\geq 0} \qquad (\text{since } i \geq 1) \\ &\geq \alpha_1 > 0. \end{aligned}$$

Also, $i \leq \ell$, so that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_i)}_{=s_i} + \underbrace{(\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_\ell)}_{\geq 0} \geq s_i,$$

and thus

$$s_i \leq \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n.$$

Combined with $s_i > 0$, this yields $s_i \in \{1, 2, \ldots, n\} = [n]$. This proves Lemma 10.5 (a).

(b) Let $k \in \{0, 1, \ldots, \ell - 1\}$. Then, the definition of s_k yields $s_k = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. Also, the definition of s_{k+1} yields

$$s_{k+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_{k+1} = \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_k)}_{=s_k} + \underbrace{\alpha_{k+1}}_{>0} > s_k.$$

In other words, $s_k < s_{k+1}$.

Now, let us forget that we fixed k. We thus have shown that $s_k < s_{k+1}$ for every $k \in \{0, 1, \ldots, \ell - 1\}$. In other words, $s_0 < s_1 < \cdots < s_\ell$. This proves Lemma 10.5 (b).

(c) We have

$$\begin{aligned} \{s_1, s_2, \ldots, s_{\ell-1}\} &= \left\{ \underbrace{s_i}_{=\alpha_1+\alpha_2+\cdots+\alpha_i} \mid i \in [\ell - 1] \right\} \\ &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in [\ell - 1]\} = D(\alpha) \end{aligned}$$

(because this is how $D(\alpha)$ is defined). This proves Lemma 10.5 (c).

(d) Let $j \in [\ell]$. The definition of s_{j-1} yields $s_{j-1} = \alpha_1 + \alpha_2 + \cdots + \alpha_{j-1}$. But the definition of s_j yields

$$s_j = \alpha_1 + \alpha_2 + \cdots + \alpha_j = \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_{j-1})}_{=s_{j-1}} + \alpha_j = s_{j-1} + \alpha_j.$$

Hence, $s_j - s_{j-1} = \alpha_j$. This proves Lemma 10.5 (d).

(e) The definition of s_ℓ yields $s_\ell = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. This proves Lemma 10.5 (e).

(f) The definition of s_0 yields $s_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_0 = (\text{empty sum}) = 0$. This proves Lemma 10.5 (f).

(g) Let $k \in [n]$. Hence, $1 \leq k \leq n$. Thus, $n \geq 1$, hence $n \neq 0$, so that $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n \neq 0$. If we had $\ell = 0$, then we would have $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = (\text{empty sum}) = 0$, which would contradict $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell \neq 0$. Thus, we cannot have $\ell = 0$. Therefore, we have $\ell > 0$, so that $\ell \in [\ell]$.

Lemma 10.5 (e) shows that $s_\ell = n \geq k$. Now, ℓ is an element of $[\ell]$ and satisfies $s_\ell \geq k$. In other words, ℓ is an element p of $[\ell]$ satisfying $s_p \geq k$. In other words, $\ell \in \{p \in [\ell] \mid s_p \geq k\}$. Hence, the set $\{p \in [\ell] \mid s_p \geq k\}$ is nonempty (since it contains ℓ) and finite, and thus has a minimum (since every nonempty finite set of integers has a minimum). In other words, the minimum $\min \{p \in [\ell] \mid s_p \geq k\}$ is well-defined. This minimum clearly is an element of $[\ell]$. This proves Lemma 10.5 (g). ☐

Let us next prove a basic lemma about integers:

Lemma 10.6. Let $\ell \in \mathbb{N}$. Let $s_0, s_1, \ldots, s_\ell$ be $\ell + 1$ integers satisfying $s_0 < s_1 < \cdots < s_\ell$. Let $a \in [\ell]$ and $b \in [\ell]$ and $u \in \mathbb{Z}$.

(a) If $s_{a-1} < u$ and $u \leq s_b$, then $a \leq b$.

(b) If $s_{a-1} < u \leq s_a$ and $s_{b-1} < u \leq s_b$, then $a = b$.

Proof of Lemma 10.6. (a) Assume that $s_{a-1} < u$ and $u \leq s_b$. We must prove that $a \leq b$.

Indeed, assume the contrary. Thus, we don't have $a \leq b$. Hence, we have $a > b$. In other words, $b < a$. Hence, $b \leq a - 1$ (since b and a are integers). Notice that $b \in [\ell] \subseteq \{0, 1, \ldots, \ell\}$. Also, from $a \in [\ell]$, we obtain $a - 1 \in \{0, 1, \ldots, \ell - 1\} \subseteq \{0, 1, \ldots, \ell\}$.

But $s_0 < s_1 < \cdots < s_\ell$. Hence, $s_u \leq s_v$ for any $u \in \{0, 1, \ldots, \ell\}$ and $v \in \{0, 1, \ldots, \ell\}$ satisfying $u \leq v$. Applying this to $u = b$ and $v = a - 1$, we obtain $s_b \leq s_{a-1}$ (since $b \leq a - 1$). Thus, $u \leq s_b \leq s_{a-1} < u$, which is absurd. This contradiction shows that our assumption was wrong. Thus, $a \leq b$ is proven. This proves Lemma 10.6 (a).

(b) Assume that $s_{a-1} < u \leq s_a$ and $s_{b-1} < u \leq s_b$. We must prove that $a = b$.

We have $s_{b-1} < u$ and $u \leq s_a$. Hence, Lemma 10.6 (a) (applied to b and a instead of a and b) yields $b \leq a$.

But $s_{a-1} < u$ and $u \leq s_b$. Hence, Lemma 10.6 (a) yields $a \leq b$. Combining this with $b \leq a$, we obtain $a = b$. This proves Lemma 10.6 (b). $\square$

Lemma 10.7. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. For every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

Lemma 10.5 (g) says the following: For every $k \in [n]$, the element $\min \{p \in [\ell] \mid s_p \geq k\}$ is a well-defined element of $[\ell]$. Hence, we can define a map $f : [n] \to [\ell]$ by

$$\left(f(k) = \min \{p \in [\ell] \mid s_p \geq k\} \qquad \text{for every } k \in [n] \right).$$

Consider this map f.

(a) We have
$$s_{f(k)-1} < k \leq s_{f(k)} \qquad \text{for every } k \in [n]. \tag{81}$$

(b) Moreover,
$$k = s_{f(k)} \qquad \text{for every } k \in D(\alpha). \tag{82}$$

(c) Also,
$$f(s_i) = i \qquad \text{for every } i \in [\ell]. \tag{83}$$

(d) Furthermore,
$$f(k) \leq f(k+1) \qquad \text{for every } k \in [n-1]. \tag{84}$$

(e) Also,
$$f(k) < f(k+1) \qquad \text{for every } k \in D(\alpha). \tag{85}$$

(f) Moreover,
$$f(k) = f(k+1) \qquad \text{for every } k \in [n-1] \setminus D(\alpha). \tag{86}$$

(g) We have
$$f^{-1}(j) = [s_j] \setminus [s_{j-1}] \qquad \text{for every } j \in [\ell]. \tag{87}$$

(h) We have
$$\left| f^{-1}(j) \right| = \alpha_j \qquad \text{for every } j \in [\ell]. \tag{88}$$

Proof of Lemma 10.7. We recall two fundamental properties of minima of sets:

- If A is a subset of $\mathbb{Z}$ for which $\min A$ is well-defined, then
$$\min A \in A. \tag{89}$$

- If A is a subset of $\mathbb{Z}$ for which $\min A$ is well-defined, and if a is an element of A, then
$$a \geq \min A. \tag{90}$$

(In other words, any element of A is greater or equal to the minimum of A.)

(a) Let $k \in [n]$. Hence, $0 < k \leq n$. We have

$$f(k) = \min \{p \in [\ell] \mid s_p \geq k\} \in \{p \in [\ell] \mid s_p \geq k\}$$

(by (89), applied to $A = \{p \in [\ell] \mid s_p \geq k\}$). In other words, $f(k)$ is an element of $[\ell]$ and satisfies $s_{f(k)} \geq k$.

On the other hand, let us prove that $s_{f(k)-1} < k$. Indeed, assume the contrary (for the sake of contradiction). Hence, $s_{f(k)-1} \geq k$. But Lemma 10.5 (f) shows that $s_0 = 0 < k$, so that $k > s_0$. Hence, $s_{f(k)-1} \geq k > s_0$, so that $s_{f(k)-1} \neq s_0$, and therefore $f(k) - 1 \neq 0$. In other words, $f(k) \neq 1$. Combined with $f(k) \in [\ell]$, this shows that $f(k) \in [\ell] \setminus \{1\}$. Hence, $f(k) - 1 \in [\ell - 1] \subseteq [\ell]$. Now, $f(k) - 1$ is an element of $[\ell]$ and satisfies $s_{f(k)-1} \geq k$. In other words, $f(k) - 1$ is an element of the set $\{p \in [\ell] \mid s_p \geq k\}$. Hence, (90) (applied to $A = \{p \in [\ell] \mid s_p \geq k\}$ and $a = f(k) - 1$) shows that

$$f(k) - 1 \geq \min\{p \in [\ell] \mid s_p \geq k\} = f(k).$$

In other words, $-1 \geq 0$. This is absurd. This contradiction proves that our assumption was wrong; thus, the proof of $s_{f(k)-1} < k$ is complete. Now, we know that $s_{f(k)-1} < k \leq s_{f(k)}$ (since $s_{f(k)} \geq k$). This proves Lemma 10.7 (a).

(b) Let $k \in D(\alpha)$. Thus, $k \in D(\alpha) \subseteq [n-1]$ (by Lemma 10.4). Hence, $k \in [n-1] \subseteq [n]$. Thus, $f(k) \in [\ell]$.

We have $k \in D(\alpha) = \{s_1, s_2, \ldots, s_{\ell-1}\}$ (by Lemma 10.5 (c)). In other words, $k = s_j$ for some $j \in [\ell - 1]$. Consider this j. Thus, $j \in [\ell - 1] \subseteq [\ell]$.

But (81) yields $s_{f(k)-1} < k \leq s_{f(k)}$.

Lemma 10.5 (b) shows that $s_0 < s_1 < \cdots < s_\ell$. Hence, $s_{j-1} < s_j$ (since $j \in [\ell]$). Hence, $s_{j-1} < s_j = k$ and $k \leq k = s_j$. Thus, we know that $s_{j-1} < k \leq s_j$ and $s_{f(k)-1} < k \leq s_{f(k)}$. Consequently, Lemma 10.6 (b) (applied to $a = j$, $b = f(k)$ and $u = k$) shows that $j = f(k)$. Hence, $s_j = s_{f(k)}$, so that $k = s_j = s_{f(k)}$. This proves Lemma 10.7 (b).

(c) Let $i \in [\ell]$. Clearly, i is an element of the set $\{p \in [\ell] \mid s_p \geq s_i\}$ (since $i \in [\ell]$ and $s_i \geq s_i$).

Now, the definition of $f(s_i)$ yields $f(s_i) = \min\{p \in [\ell] \mid s_p \geq s_i\}$. But (90) (applied to $A = \{p \in [\ell] \mid s_p \geq s_i\}$ and $a = i$) yields $i \geq \min\{p \in [\ell] \mid s_p \geq s_i\}$ (since i is an element of the set $\{p \in [\ell] \mid s_p \geq s_i\}$). In other words, $i \geq f(s_i)$ (since $f(s_i) = \min\{p \in [\ell] \mid s_p \geq s_i\}$).

Now, assume (for the sake of contradiction) that $i \neq f(s_i)$. Then, $i > f(s_i)$ (since $i \geq f(s_i)$). In other words, $f(s_i) < i$.

But Lemma 10.5 (b) shows that $s_0 < s_1 < \cdots < s_\ell$. In other words, $s_u < s_v$ for any $u \in \{0, 1, \ldots, \ell\}$ and $v \in \{0, 1, \ldots, \ell\}$ satisfying $u < v$. Applying this to $u = f(s_i)$ and $v = i$, we obtain $s_{f(s_i)} < s_i$.

From Lemma 10.5 (a), we obtain $s_i \in [n]$. Thus, (81) (applied to $k = s_i$) shows that $s_{f(s_i)-1} < s_i \leq s_{f(s_i)}$. Hence, $s_i \leq s_{f(s_i)} < s_i$. But this is absurd. This contradiction shows that our assumption (that $i \neq f(s_i)$) was false. We therefore have $i = f(s_i)$. This proves Lemma 10.7 (c).

(d) Let $k \in [n-1]$.

From $k \in [n-1]$, we see that both k and $k+1$ are elements of $[n]$. Thus, $f(k)$ and $f(k+1)$ are well-defined elements of $[\ell]$.

But (81) yields $s_{f(k)-1} < k \leq s_{f(k)}$. Also, (81) (applied to $k+1$ instead of k) yields $s_{f(k+1)-1} < k + 1 \leq s_{f(k+1)}$.

Now, $s_{f(k)-1} < k$ and $k \leq k+1 \leq s_{f(k+1)}$. Also, Lemma 10.5 (b) shows that $s_0 < s_1 < \cdots < s_\ell$. Thus, Lemma 10.6 (a) (applied to $a = f(k)$, $b = f(k+1)$ and $u = k$) yields $f(k) \leq f(k+1)$. This completes the proof of Lemma 10.7 (d).

(e) Let $k \in D(\alpha)$. Thus, $k \in D(\alpha) \subseteq [n-1]$ (by Lemma 10.4). Hence, $f(k) \leq f(k+1)$ (by (84)).

We want to prove that $f(k) < f(k+1)$. Indeed, assume the contrary (for the sake of contradiction). Thus, $f(k) \geq f(k+1)$. Combined with $f(k) \leq f(k+1)$, this shows that $f(k) = f(k+1)$.

We have $k = s_{f(k)}$ (by (82)).

But $k + 1 \in [n]$ (since $k \in [n-1]$). Hence, (81) (applied to $k+1$ instead of k) yields $s_{f(k+1)-1} < k + 1 \leq s_{f(k+1)}$. Hence, $k + 1 \leq s_{f(k+1)} = s_{f(k)}$ (since $f(k+1) = f(k)$). This contradicts $s_{f(k)} = k < k + 1$. This contradiction proves that our assumption was false. Hence, $f(k) < f(k+1)$ is proven. This completes the proof of Lemma 10.7 (e).

(f) Let $k \in [n-1] \setminus D(\alpha)$. Thus, $k \in [n-1]$ but $k \notin D(\alpha)$.

From $k \in [n-1]$, we see that both k and $k+1$ are elements of $[n]$. From (84), we obtain $f(k) \leq f(k+1)$.

We must prove that $f(k) = f(k+1)$. Indeed, assume the contrary (for the sake of contradiction). Thus, $f(k) \neq f(k+1)$. Combined with $f(k) \leq f(k+1)$, this shows that $f(k) < f(k+1)$.

From (81), we obtain $s_{f(k)-1} < k \leq s_{f(k)}$. From (81) (applied to $k+1$ instead of k), we obtain $s_{f(k+1)-1} < k+1 \leq s_{f(k+1)}$.

But $f(k) < f(k+1) \leq \ell$ (since $f(k+1) \in [\ell]$). Hence, $f(k) \leq \ell - 1$ (since $f(k)$ and ℓ are integers). Thus, $f(k) \in [\ell - 1]$. Hence, $s_{f(k)} \in \{s_1, s_2, \ldots, s_{\ell-1}\} = D(\alpha)$ (by Lemma 10.5 (c)).

Also, $f(k) < f(k+1)$, so that $f(k) \leq f(k+1) - 1$ (since $f(k)$ and $f(k+1)$ are integers). But Lemma 10.5 (b) shows that $s_0 < s_1 < \cdots < s_\ell$. Thus, $s_u \leq s_v$ for any $u \in \{0, 1, \ldots, \ell\}$ and $v \in \{0, 1, \ldots, \ell\}$ satisfying $u \leq v$. Applying this to $u = f(k)$ and $v = f(k+1) - 1$, we obtain $s_{f(k)} \leq s_{f(k+1)-1} < k+1$. In other words, $s_{f(k)} \leq (k+1) - 1$ (since $s_{f(k)}$ and $k+1$ are integers).

Now, combining $k \leq s_{f(k)}$ with $s_{f(k)} \leq (k+1) - 1 = k$, we obtain $k = s_{f(k)} \in D(\alpha)$. This contradicts $k \notin D(\alpha)$. This contradiction proves that our assumption was wrong. Hence, $f(k) = f(k+1)$ is proven. This completes the proof of Lemma 10.7 (f).

(g) Let $j \in [\ell]$.

From Lemma 10.5 (b), we have $s_0 < s_1 < \cdots < s_\ell$. Thus, $s_{j-1} < s_j$.

Let $k \in f^{-1}(j)$. Thus, $k \in [n]$ and $f(k) = j$. From (81), we obtain $s_{f(k)-1} < k \leq s_{f(k)}$. Since $f(k) = j$, this rewrites as follows: $s_{j-1} < k \leq s_j$. In other words, $k \in \{s_{j-1}+1, s_{j-1}+2, \ldots, s_j\} = [s_j] \setminus [s_{j-1}]$ (since $0 \leq s_{j-1} < s_j$).

Now, forget that we fixed k. We thus have shown that $k \in [s_j] \setminus [s_{j-1}]$ for every $k \in f^{-1}(j)$. In other words, $f^{-1}(j) \subseteq [s_j] \setminus [s_{j-1}]$.

On the other hand, let $g \in [s_j] \setminus [s_{j-1}]$. Thus, $g \in [s_j] \setminus [s_{j-1}] = \{s_{j-1}+1, s_{j-1}+2, \ldots, s_j\}$ (since $0 \leq s_{j-1} < s_j$). In other words, $s_{j-1} < g \leq s_j$.

Lemma 10.5 (a) (applied to $i = j$) yields $s_j \in [n]$. Hence, $s_j \leq n$. Now, $g \in [s_j] \subseteq [n]$ (since $s_j \leq n$). Hence, (81) (applied to $k = g$) shows that $s_{f(g)-1} < g \leq s_{f(g)}$.

We have $f(g) \in [\ell]$ and $j \in [\ell]$, and we have $s_{f(g)-1} < g \leq s_{f(g)}$ and $s_{j-1} < g \leq s_j$. Hence, Lemma 10.6 (b) (applied to $a = f(g)$, $b = j$ and $c = g$) yields $f(g) = j$. Hence, $g \in f^{-1}(j)$.

Now, forget that we fixed g. We thus have shown that $g \in f^{-1}(j)$ for every $g \in [s_j] \setminus [s_{j-1}]$. In other words, $[s_j] \setminus [s_{j-1}] \subseteq f^{-1}(j)$. Combined with $f^{-1}(j) \subseteq [s_j] \setminus [s_{j-1}]$, this yields $f^{-1}(j) = [s_j] \setminus [s_{j-1}]$. This proves Lemma 10.7 (g).

(h) Let $j \in [\ell]$. From Lemma 10.5 (b), we have $s_0 < s_1 < \cdots < s_\ell$. Thus, $s_{j-1} < s_j$. Consequently, $0 \leq s_{j-1} < s_j$. Hence, $[s_j] \setminus [s_{j-1}] = \{s_{j-1}+1, s_{j-1}+2, \ldots, s_j\}$, so that

$$\left| [s_j] \setminus [s_{j-1}] \right| = \left| \{s_{j-1}+1, s_{j-1}+2, \ldots, s_j\} \right| = s_j - s_{j-1} \qquad (\text{since } s_{j-1} < s_j).$$

But (87) shows that $f^{-1}(j) = [s_j] \setminus [s_{j-1}]$. Therefore, $\left| f^{-1}(j) \right| = \left| [s_j] \setminus [s_{j-1}] \right| = s_j - s_{j-1} = \alpha_j$ (by Lemma 10.5 (d)). This proves Lemma 10.7 (h). ☐

Lemma 10.8. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n.

For every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

Define a map $f : [n] \to [\ell]$ as in Lemma 10.7.

Let $(i_1, i_2, \ldots, i_n)$ be an element of $\{1, 2, 3, \ldots\}^n$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$ and $\{j \in [n-1] \mid i_j < i_{j+1}\} \subseteq D(\alpha)$. Then:

(a) We have $i_{s_{f(k)}} = i_k$ for every $k \in [n]$.

(b) We have $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \{1, 2, 3, \ldots\}^\ell$ and $i_{s_1} \leq i_{s_2} \leq \cdots \leq i_{s_\ell}$.

(c) If $\{j \in [n-1] \mid i_j < i_{j+1}\} = D(\alpha)$, then $i_{s_1} < i_{s_2} < \cdots < i_{s_\ell}$.

Proof of Lemma 10.8. Lemma 10.5 (b) shows that $s_0 < s_1 < \cdots < s_\ell$.

Recall that $i_1 \leq i_2 \leq \cdots \leq i_n$. In other words,

$$i_u \leq i_v \tag{91}$$

for any two elements u and v of $[n]$ satisfying $u \leq v$.

(a) First of all, we notice that $i_{s_{f(k)}}$ is well-defined for every $k \in [n]$ [67].

We must show that $i_{s_{f(k)}} = i_k$ for every $k \in [n]$.

Assume the contrary (for the sake of contradiction). Thus, $i_{s_{f(k)}} = i_k$ holds **not** for every $k \in [n]$. In other words, there exists some $k \in [n]$ satisfying $i_{s_{f(k)}} \neq i_k$. Let g be the **highest such** k. Thus, g is a $k \in [n]$ satisfying $i_{s_{f(k)}} \neq i_k$. In other words, g is an element of $[n]$ and satisfies $i_{s_{f(g)}} \neq i_g$. From $i_{s_{f(g)}} \neq i_g$, we obtain $s_{f(g)} \neq g$. If we had $g \in D(\alpha)$, then we would have

$$\begin{aligned} g &= s_{f(g)} \qquad \text{(by (82), applied to } k = g\text{)} \\ &\neq g; \end{aligned}$$

this would be absurd. Hence, we cannot have $g \in D(\alpha)$. We thus have $g \notin D(\alpha)$.

We have $g \in [n]$, so that $1 \leq g \leq n$. Hence, $n \geq 1$. Consequently, $n \neq 0$, so that $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n \neq 0$. If we had $\ell = 0$, then we would have $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = (\text{empty sum}) = 0$, which would contradict $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell \neq 0$. Thus, we cannot have $\ell = 0$. Therefore, we have $\ell > 0$, so that $\ell \in [\ell]$. Thus, $f(s_\ell) = \ell$ (by (83), applied to $i = \ell$). Also, $n \in [n]$ (since $n \geq 1$).

From Lemma 10.5 (e), we obtain $n = s_\ell$. If we had $g = n$, then we would have $f\left(\underbrace{g}_{=n=s_\ell}\right) = f(s_\ell) = \ell$ and therefore $s_{f(g)} = s_\ell = n = g$; but this would contradict $s_{f(g)} \neq g$. Hence, we cannot have $g = n$. Thus, $g \neq n$.

From $g \in [n]$ and $g \neq n$, we obtain $g \in [n] \setminus \{n\} = [n-1]$. Combining this with $g \notin D(\alpha)$, we find that $g \in [n-1] \setminus D(\alpha)$. Thus, (86) (applied to $k = g$) shows that $f(g) = f(g+1)$.

Recall that g is the **highest** $k \in [n]$ satisfying $i_{s_{f(k)}} \neq i_k$. Hence, for every $k \in [n]$ satisfying $i_{s_{f(k)}} \neq i_k$, we have

$$k \leq g. \tag{92}$$

From $g \in [n-1]$, we obtain $g+1 \in [n]$. Now, if we had $i_{s_{f(g+1)}} \neq i_{g+1}$, then we would have $g+1 \leq g$ (by (92), applied to $k = g+1$); but this would contradict $g+1 > g$. Hence, we cannot have $i_{s_{f(g+1)}} \neq i_{g+1}$. We therefore must have $i_{s_{f(g+1)}} = i_{g+1}$. But $f(g) = f(g+1)$, so that $i_{s_{f(g)}} = i_{s_{f(g+1)}} = i_{g+1}$. From $i_{s_{f(g)}} \neq i_g$, we now obtain $i_g \neq i_{s_{f(g)}} = i_{g+1}$.

Applying (91) to $u = g$ and $v = g+1$, we obtain $i_g \leq i_{g+1}$ (since $g \leq g+1$). Combined with $i_g \neq i_{g+1}$, this yields $i_g < i_{g+1}$.

Now, g is an element of $[n-1]$ and satisfies $i_g < i_{g+1}$. In other words, $g \in \{j \in [n-1] \mid i_j < i_{j+1}\}$. In other words, $g \in D(\alpha)$ (since $\{j \in [n-1] \mid i_j < i_{j+1}\} = D(\alpha)$). This contradicts $g \notin D(\alpha)$. This contradiction proves that our assumption was wrong. Hence, Lemma 10.8 (a) is proven.

(b) From Lemma 10.5 (a), we see that $s_i \in [n]$ for every $i \in [\ell]$. Thus, from $(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n$, we obtain $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \{1, 2, 3, \ldots\}^\ell$.

It remains to prove that $i_{s_1} \leq i_{s_2} \leq \cdots \leq i_{s_\ell}$.

Let $k \in [\ell - 1]$ be arbitrary. Then, both k and $k+1$ belong to $[\ell]$. Hence, both s_k and s_{k+1} belong to $[n]$ (by Lemma 10.5 (a)).

Now, $s_k < s_{k+1}$ (because of $s_0 < s_1 < \cdots < s_\ell$). Hence, $s_k \leq s_{k+1}$.

Applying (91) to $u = s_k$ and $v = s_{k+1}$, we obtain $i_{s_k} \leq i_{s_{k+1}}$ (since $s_k \leq s_{k+1}$).

[67] *Proof.* Let $k \in [n]$. Thus, $f(k) \in [\ell]$. Therefore, Lemma 10.5 (a) (applied to $i = f(k)$) yields $s_{f(k)} \in [n]$. Thus, $i_{s_{f(k)}}$ is well-defined.

Now, forget that we fixed k. We thus have proven that $i_{s_k} \leq i_{s_{k+1}}$ for every $k \in [\ell - 1]$. In other words, $i_{s_1} \leq i_{s_2} \leq \cdots \leq i_{s_\ell}$. This completes the proof of Lemma 10.8 (b).

(c) Assume that $\{j \in [n-1] \mid i_j < i_{j+1}\} = D(\alpha)$.

Now, let $k \in [\ell - 1]$ be arbitrary. Then, $k+1 \in [\ell]$ (since $k \in [\ell - 1]$). Thus, $s_{k+1} \in [n]$ (by Lemma 10.5 (a) (applied to $k+1$ instead of i)), and thus $s_{k+1} \leq n$.

Furthermore, $s_k < s_{k+1}$ (because of $s_0 < s_1 < \cdots < s_\ell$). Hence, $s_k + 1 \leq s_{k+1}$ (since s_k and s_{k+1} are integers). Hence, $s_k + 1 \leq s_{k+1} \leq n$. Combining this with $\underbrace{s_k}_{\geq 0} + 1 \geq 1$, we obtain $s_k + 1 \in [n]$.

Applying (91) to $u = s_k + 1$ and $v = s_{k+1}$, we obtain $i_{s_k+1} \leq i_{s_{k+1}}$ (since $s_k + 1 \leq s_{k+1}$).

On the other hand, $k \in [\ell - 1]$ and thus

$$\begin{aligned} s_k \in \{s_1, s_2, \ldots, s_{\ell-1}\} &= D(\alpha) \qquad \text{(by Lemma 10.5 (c))} \\ &= \{j \in [n-1] \mid i_j < i_{j+1}\}. \end{aligned}$$

In other words, s_k is an element of $[n-1]$ and satisfies $i_{s_k} < i_{s_k+1}$. Hence, $i_{s_k} < i_{s_k+1} \leq i_{s_{k+1}}$.

Now, forget that we fixed k. We thus have proven that $i_{s_k} < i_{s_{k+1}}$ for every $k \in [\ell - 1]$. In other words, $i_{s_1} < i_{s_2} < \cdots < i_{s_\ell}$. This proves Lemma 10.8 (c). $\square$

Lemma 10.9. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n.

For every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

Define a map $f : [n] \to [\ell]$ as in Lemma 10.7.

Let $(h_1, h_2, \ldots, h_\ell)$ be an ℓ-tuple in $\{1, 2, 3, \ldots\}^\ell$. Assume that $h_1 \leq h_2 \leq \cdots \leq h_\ell$. Then:

(a) We have $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \{1, 2, 3, \ldots\}^n$ and $h_{f(1)} \leq h_{f(2)} \leq \cdots \leq h_{f(n)}$.

(b) We have $\left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\} \subseteq D(\alpha)$.

(c) If $h_1 < h_2 < \cdots < h_\ell$, then $\left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\} = D(\alpha)$.

Proof of Lemma 10.9. We have $(h_1, h_2, \ldots, h_\ell) \in \{1, 2, 3, \ldots\}^\ell$. Hence, $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \{1, 2, 3, \ldots\}^n$.

(a) We have $h_1 \leq h_2 \leq \cdots \leq h_\ell$. Thus,

$$h_u \leq h_v \qquad \text{for any } u \in [\ell] \text{ and } v \in [\ell] \text{ satisfying } u \leq v. \tag{93}$$

Let $k \in [n-1]$. Then, $f(k) \leq f(k+1)$ (by (84)), and therefore $h_{f(k)} \leq h_{f(k+1)}$ (by (93), applied to $u = f(k)$ and $v = f(k+1)$). Now, let us forget that we fixed k. Thus we have shown that $h_{f(k)} \leq h_{f(k+1)}$ for every $k \in [n-1]$. In other words, $h_{f(1)} \leq h_{f(2)} \leq \cdots \leq h_{f(n)}$. Combined with $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \{1, 2, 3, \ldots\}^n$, this completes the proof of Lemma 10.9 (a).

(b) Let $k \in \left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\}$. Thus, k is an element of $[n-1]$ and satisfies $h_{f(k)} < h_{f(k+1)}$. If we had $k \notin D(\alpha)$, then we would have $k \in [n-1] \setminus D(\alpha)$ (since $k \in [n-1]$ and $k \notin D(\alpha)$) and therefore $f(k) = f(k+1)$ (by (86)), and thus $h_{f(k)} = h_{f(k+1)}$; but this would contradict $h_{f(k)} < h_{f(k+1)}$. Hence, we cannot have $k \notin D(\alpha)$. Thus, we have $k \in D(\alpha)$.

Now, let us forget that we fixed k. We thus have shown that $k \in D(\alpha)$ for every $k \in \left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\}$. In other words, $\left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\} \subseteq D(\alpha)$. This proves Lemma 10.9 (b).

(c) Assume that $h_1 < h_2 < \cdots < h_\ell$. Thus,

$$h_u < h_v \qquad \text{for any } u \in [\ell] \text{ and } v \in [\ell] \text{ satisfying } u < v. \tag{94}$$

From Lemma 10.9 (b), we obtain

$$\left\{ j \in [n-1] \ \mid \ h_{f(j)} < h_{f(j+1)} \right\} \subseteq D(\alpha). \tag{95}$$

On the other hand, we have

$$D(\alpha) \subseteq \left\{ j \in [n-1] \ \mid \ h_{f(j)} < h_{f(j+1)} \right\}. \tag{96}$$

[*Proof of (96):* Let $k \in D(\alpha)$. Then, $f(k) < f(k+1)$ (by (85)). Therefore, $h_{f(k)} < h_{f(k+1)}$ (by (94), applied to $u = f(k)$ and $v = f(k+1)$). Also, $k \in D(\alpha) \subseteq [n-1]$ (by Lemma 10.4). Thus, k is an element of $[n-1]$ and satisfies $h_{f(k)} < h_{f(k+1)}$. In other words, $k \in \left\{ j \in [n-1] \ \mid \ h_{f(j)} < h_{f(j+1)} \right\}$.

Now, let us forget that we fixed k. We thus have shown that $k \in \left\{ j \in [n-1] \ \mid \ h_{f(j)} < h_{f(j+1)} \right\}$ for every $k \in D(\alpha)$. In other words, $D(\alpha) \subseteq \left\{ j \in [n-1] \ \mid \ h_{f(j)} < h_{f(j+1)} \right\}$. This proves (96).]

Combining (95) with (96), we obtain $\left\{ j \in [n-1] \ \mid \ h_{f(j)} < h_{f(j+1)} \right\} = D(\alpha)$. This proves Lemma 10.9 (c). ☐

Proposition 10.10. Let α be a composition of a nonnegative integer n. Then,

$$M_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{ j \in [n-1] \ \mid \ i_j < i_{j+1} \right\} = D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Proof of Proposition 10.10. Let $\mathcal{J}$ denote the set of all length-ℓ strictly increasing sequences of positive integers. In other words,

$$\mathcal{J} = \left\{ (i_1, i_2, \ldots, i_\ell) \in \{1,2,3,\ldots\}^\ell \ \mid \ i_1 < i_2 < \cdots < i_\ell \right\}. \tag{97}$$

Renaming the index $(i_1, i_2, \ldots, i_\ell)$ as $(j_1, j_2, \ldots, j_\ell)$ in this formula, we obtain

$$\mathcal{J} = \left\{ (j_1, j_2, \ldots, j_\ell) \in \{1,2,3,\ldots\}^\ell \ \mid \ j_1 < j_2 < \cdots < j_\ell \right\}.$$

The definition of M_α yields

$$\begin{aligned} M_\alpha &= \underbrace{\sum_{i_1 < i_2 < \cdots < i_\ell}}_{\substack{= \sum_{\substack{(i_1,i_2,\ldots,i_\ell) \in \{1,2,3,\ldots\}^\ell; \\ i_1 < i_2 < \cdots < i_\ell}} = \sum_{(i_1,i_2,\ldots,i_\ell) \in \mathcal{J}} \\ \left(\text{since } \left\{ (i_1,i_2,\ldots,i_\ell) \in \{1,2,3,\ldots\}^\ell \ \mid \ i_1 < i_2 < \cdots < i_\ell \right\} = \mathcal{J}\right)}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \\ &= \sum_{(i_1,i_2,\ldots,i_\ell) \in \mathcal{J}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{(j_1,j_2,\ldots,j_\ell) \in \mathcal{J}} x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_\ell}^{\alpha_\ell} \end{aligned} \tag{98}$$

(here, we have renamed $(i_1, i_2, \ldots, i_\ell)$ as $(j_1, j_2, \ldots, j_\ell)$ in the sum).

Define a set $\mathcal{I}$ by

$$\begin{aligned} \mathcal{I} = \{ (i_1, i_2, \ldots, i_n) \in \{1,2,3,\ldots\}^n \ \mid \ & i_1 \leq i_2 \leq \cdots \leq i_n \\ & \text{and } \left\{ j \in [n-1] \ \mid \ i_j < i_{j+1} \right\} = D(\alpha) \}. \end{aligned} \tag{99}$$

Thus, $\sum\limits_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j\in[n-1] \ \mid \ i_j < i_{j+1}\right\} = D(\alpha)}} = \sum\limits_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}}$ (an equality between summation signs). Hence,

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j\in[n-1] \ \mid \ i_j < i_{j+1}\right\} = D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}} x_{i_1} x_{i_2} \cdots x_{i_n}. \qquad (100)$$

The definition of $\mathcal{I}$ shows that

$$\begin{aligned} \mathcal{I} &= \left\{ (i_1, i_2, \ldots, i_n) \in \{1,2,3,\ldots\}^n \ \mid \ i_1 \leq i_2 \leq \cdots \leq i_n \right. \\ &\qquad \left. \text{and } \left\{ j \in [n-1] \ \mid \ i_j < i_{j+1} \right\} = D(\alpha) \right\} \\ &= \left\{ (k_1, k_2, \ldots, k_n) \in \{1,2,3,\ldots\}^n \ \mid \ k_1 \leq k_2 \leq \cdots \leq k_n \right. \\ &\qquad \left. \text{and } \left\{ j \in [n-1] \ \mid \ k_j < k_{j+1} \right\} = D(\alpha) \right\} \end{aligned}$$

(here, we have renamed the index $(i_1, i_2, \ldots, i_n)$ as $(k_1, k_2, \ldots, k_n)$).

Now, for every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

Define a map $f : [n] \to [\ell]$ as in Lemma 10.7.

Now, for every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \mathcal{J}$ [68]. Hence, we can define a map $\Phi : \mathcal{I} \to \mathcal{J}$ by setting

$$(\Phi(i_1, i_2, \ldots, i_n) = (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \qquad \text{for every } (i_1, i_2, \ldots, i_n) \in \mathcal{I}).$$

Consider this Φ.

For every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have

$$i_{s_{f(k)}} = i_k \qquad \text{for every } k \in [n] \qquad (101)$$

[69].

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[68] *Proof.* Let $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$. Thus,

$$\begin{aligned} (i_1, i_2, \ldots, i_n) &\in \mathcal{I} \\ &= \left\{ (k_1, k_2, \ldots, k_n) \in \{1,2,3,\ldots\}^n \ \mid \ k_1 \leq k_2 \leq \cdots \leq k_n \right. \\ &\qquad \left. \text{and } \left\{ j \in [n-1] \ \mid \ k_j < k_{j+1} \right\} = D(\alpha) \right\}. \end{aligned}$$

In other words, $(i_1, i_2, \ldots, i_n)$ is an element of $\{1,2,3,\ldots\}^n$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$ and $\left\{ j \in [n-1] \ \mid \ i_j < i_{j+1} \right\} = D(\alpha)$.

Thus, $\left\{ j \in [n-1] \ \mid \ i_j < i_{j+1} \right\} = D(\alpha) \subseteq D(\alpha)$. Hence, Lemma 10.8 (b) shows that we have $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \{1,2,3,\ldots\}^\ell$ and $i_{s_1} \leq i_{s_2} \leq \cdots \leq i_{s_\ell}$. Furthermore, Lemma 10.8 (c) shows that $i_{s_1} < i_{s_2} < \cdots < i_{s_\ell}$ (since $\left\{ j \in [n-1] \ \mid \ i_j < i_{j+1} \right\} = D(\alpha)$).

Now, $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell})$ is an element of $\{1,2,3,\ldots\}^\ell$ (since $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \{1,2,3,\ldots\}^\ell$) and satisfies $i_{s_1} < i_{s_2} < \cdots < i_{s_\ell}$. In other words, $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell})$ is a $(j_1, j_2, \ldots, j_\ell) \in \{1,2,3,\ldots\}^\ell$ satisfying $j_1 < j_2 < \cdots < j_\ell$. In other words,

$$(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \left\{ (j_1, j_2, \ldots, j_\ell) \in \{1,2,3,\ldots\}^\ell \ \mid \ j_1 < j_2 < \cdots < j_\ell \right\} = \mathcal{J},$$

qed.

[69] *Proof of (101):* Fix $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$. Thus,

$$\begin{aligned} (i_1, i_2, \ldots, i_n) &\in \mathcal{I} \\ &= \left\{ (k_1, k_2, \ldots, k_n) \in \{1,2,3,\ldots\}^n \ \mid \ k_1 \leq k_2 \leq \cdots \leq k_n \right. \\ &\qquad \left. \text{and } \left\{ j \in [n-1] \ \mid \ k_j < k_{j+1} \right\} = D(\alpha) \right\}. \end{aligned}$$

Now, for every $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$, we have $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \mathcal{I}$ [70]. Hence, we can define a map $\Psi : \mathcal{J} \to \mathcal{I}$ by setting

$$\left(\Psi\left(h_1, h_2, \ldots, h_\ell\right) = \left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \qquad \text{for every } \left(h_1, h_2, \ldots, h_\ell\right) \in \mathcal{J}\right).$$

Consider this Ψ.

Now, $\Phi \circ \Psi = \mathrm{id}$ [71] and $\Psi \circ \Phi = \mathrm{id}$ [72]. Hence, the maps Φ and Ψ are mutually inverse. Thus,

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In other words, $(i_1, i_2, \ldots, i_n)$ is an element of $\{1, 2, 3, \ldots\}^n$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$ and $\{j \in [n-1] \mid i_j < i_{j+1}\} = D(\alpha)$.

Hence, Lemma 10.8 (a) shows that we have $i_{s_{f(k)}} = i_k$ for every $k \in [n]$. Hence, (101) is proven.

[70]*Proof.* Let $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$. Thus, $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J} = \left\{(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid i_1 < i_2 < \cdots < i_\ell\right\}$. In other words, $(h_1, h_2, \ldots, h_\ell)$ is an ℓ-tuple in $\{1, 2, 3, \ldots\}^\ell$ and satisfies $h_1 < h_2 < \cdots < h_\ell$.

From $h_1 < h_2 < \cdots < h_\ell$, we obtain $h_1 \leq h_2 \leq \cdots \leq h_\ell$. Hence, Lemma 10.9 (a) shows that we have $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \{1, 2, 3, \ldots\}^n$ and $h_{f(1)} \leq h_{f(2)} \leq \cdots \leq h_{f(n)}$. Furthermore, Lemma 10.9 (c) yields $\left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\} = D(\alpha)$.

Thus, $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right)$ is an element of $\{1, 2, 3, \ldots\}^n$ which satisfies $h_{f(1)} \leq h_{f(2)} \leq \cdots \leq h_{f(n)}$ and $\left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\} = D(\alpha)$. In other words,

$$\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \{(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \mid i_1 \leq i_2 \leq \cdots \leq i_n \\ \text{and } \{j \in [n-1] \mid i_j < i_{j+1}\} = D(\alpha)\}.$$

In light of (99), this rewrites as $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \mathcal{I}$. Qed.

[71]*Proof.* Let $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$. For every $i \in [\ell]$, we have $f(s_i) = i$ (by (83)) and thus $h_{f(s_i)} = h_i$. Now,

$$\begin{aligned}
(\Phi \circ \Psi)(h_1, h_2, \ldots, h_\ell) &= \Phi\left(\underbrace{\Psi(h_1, h_2, \ldots, h_\ell)}_{=\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right)}\right) = \Phi\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \\
&= \left(h_{f(s_1)}, h_{f(s_2)}, \ldots, h_{f(s_\ell)}\right) \qquad \text{(by the definition of } \Phi\text{)} \\
&= (h_1, h_2, \ldots, h_\ell)
\end{aligned}$$

(since $h_{f(s_i)} = h_i$ for every $i \in [\ell]$).

Now, forget that we fixed $(h_1, h_2, \ldots, h_\ell)$. We thus have shown that $(\Phi \circ \Psi)(h_1, h_2, \ldots, h_\ell) = (h_1, h_2, \ldots, h_\ell)$ for every $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$. In other words, $\Phi \circ \Psi = \mathrm{id}$, qed.

[72]*Proof.* For every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have

$$\begin{aligned}
(\Psi \circ \Phi)(i_1, i_2, \ldots, i_n) &= \Psi\left(\underbrace{\Phi(i_1, i_2, \ldots, i_n)}_{\substack{=\left(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}\right) \\ \text{(by the definition of } \Phi\text{)}}}\right) = \Psi\left(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}\right) \\
&= \left(i_{s_{f(1)}}, i_{s_{f(2)}}, \ldots, i_{s_{f(n)}}\right) \qquad \text{(by the definition of } \Psi\text{)} \\
&= (i_1, i_2, \ldots, i_n) \qquad \text{(by (101))}.
\end{aligned}$$

the map Φ is a bijection. In other words, the map

$$\mathcal{I} \to \mathcal{J}, \qquad (i_1, i_2, \ldots, i_n) \mapsto (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell})$$

is a bijection[73].

Now, for every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have

$$x_{i_1} x_{i_2} \cdots x_{i_n} = x_{i_{s_1}}^{\alpha_1} x_{i_{s_2}}^{\alpha_2} \cdots x_{i_{s_\ell}}^{\alpha_\ell} \tag{102}$$

[74]. But (98) becomes

$$M_\alpha = \sum_{(j_1,j_2,\ldots,j_\ell)\in\mathcal{J}} x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_\ell}^{\alpha_\ell} = \sum_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}} \underbrace{x_{i_{s_1}}^{\alpha_1} x_{i_{s_2}}^{\alpha_2} \cdots x_{i_{s_\ell}}^{\alpha_\ell}}_{\substack{=x_{i_1}x_{i_2}\cdots x_{i_n} \\ \text{(by (102))}}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \text{ for } (j_1, j_2, \ldots, j_\ell) \text{ in the} \\ \text{sum, since the map } \mathcal{I} \to \mathcal{J},\ (i_1, i_2, \ldots, i_n) \mapsto (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \\ \text{is a bijection}\end{array}\right)$$

$$= \sum_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n; \\ \{j\in[n-1] \mid i_j < i_{j+1}\} = D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

(by (100)). This proves Proposition 10.10. ◻

10.2. More on $D(\alpha)$

We shall now prove another property of the sets $D(\alpha)$ defined in Definition 10.3; this property will be used later on. Let us start with some definitions.

Definition 10.11. For every set A, we let $\mathcal{P}(A)$ denote the powerset of A (that is, the set of all subsets of A).

In other words, $\Psi \circ \Phi = \mathrm{id}$, qed.

[73]since Φ is the map

$$\mathcal{I} \to \mathcal{J}, \qquad (i_1, i_2, \ldots, i_n) \mapsto (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell})$$

(by the definition of Φ)

[74]*Proof of (102):* Let $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$. Then,

$$x_{i_1} x_{i_2} \cdots x_{i_n} = \prod_{k\in[n]} x_{i_k} = \prod_{j\in[\ell]} \underbrace{\prod_{\substack{k\in[n]; \\ f(k)=j}}}_{=\prod_{k\in f^{-1}(j)}} \underbrace{x_{i_k}}_{\substack{=x_{i_{s_j}} \\ \text{(because (101) shows that} \\ i_k = i_{s_{f(k)}} = i_{s_j} \text{ (since } f(k)=j))}} \qquad (\text{since } f(k) \in [\ell] \text{ for every } k \in [n])$$

$$= \prod_{j\in[\ell]} \underbrace{\prod_{k\in f^{-1}(j)} x_{i_{s_j}}}_{\substack{= x_{i_{s_j}}^{|f^{-1}(j)|} = x_{i_{s_j}}^{\alpha_j} \\ \text{(by (88))}}} = \prod_{j\in[\ell]} x_{i_{s_j}}^{\alpha_j} = x_{i_{s_1}}^{\alpha_1} x_{i_{s_2}}^{\alpha_2} \cdots x_{i_{s_\ell}}^{\alpha_\ell}.$$

This proves (102).

Definition 10.12. For every $n \in \mathbb{N}$, we let Comp_n denote the set of all compositions of n.

Definition 10.13. Let $n \in \mathbb{N}$. Let $\alpha \in \operatorname{Comp}_n$. Thus, α is a composition of n (since Comp_n is the set of all compositions of n). Hence, Lemma 10.4 shows that $D(\alpha) \subseteq [n-1]$. In other words, $D(\alpha) \in \mathcal{P}([n-1])$.

Now, forget that we fixed α. Thus, we have defined a $D(\alpha) \in \mathcal{P}([n-1])$ for every $\alpha \in \operatorname{Comp}_n$. In other words, we have defined a map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ which sends every $\alpha \in \operatorname{Comp}_n$ to $D(\alpha) \in \mathcal{P}([n-1])$. Consider this map D.

Definition 10.14. If J is a finite subset of $\mathbb{Z}$, then we let $\operatorname{ilis} J$ be the list of all elements of J in increasing order (with each element appearing only once). For example, $\operatorname{ilis}\{2,5,1\} = (1,2,5)$, whereas $\operatorname{ilis}\varnothing$ is the empty list.

Lemma 10.15. Let $n \in \mathbb{N}$. Let $I \in \mathcal{P}([n-1])$. Write the list $\operatorname{ilis}(I \cup \{0,n\})$ in the form $(i_0, i_1, \ldots, i_m)$ for some integer $m \geq -1$.

(a) We have $m \geq 0$.

(b) We have $i_0 = 0$.

(c) We have $i_m = n$.

(d) We have $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) \in \operatorname{Comp}_n$.

(e) We have $I = \{i_1, i_2, \ldots, i_{m-1}\}$.

(f) We have $D(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) = I$.

Proof of Lemma 10.15. We have $(i_0, i_1, \ldots, i_m) = \operatorname{ilis}(I \cup \{0,n\})$ (by the definition of $(i_0, i_1, \ldots, i_m)$). In other words, $(i_0, i_1, \ldots, i_m)$ is the list of all elements of $I \cup \{0,n\}$ in increasing order (since this is what we defined $\operatorname{ilis}(I \cup \{0,n\})$ to be). Thus,

$$\{i_0, i_1, \ldots, i_m\} = I \cup \{0,n\}$$

(since the list $(i_0, i_1, \ldots, i_m)$ is a list of all elements of $I \cup \{0,n\}$). Also,

$$i_0 < i_1 < \cdots < i_m$$

(since the list $(i_0, i_1, \ldots, i_m)$ is in increasing order). Thus,

$$i_u < i_v \qquad \text{for any two elements } u \text{ and } v \text{ of } \{0,1,\ldots,m\} \text{ satisfying } u < v. \tag{103}$$

But $I \in \mathcal{P}([n-1])$, so that $I \subseteq [n-1] \subseteq [n] \subseteq \{0,1,\ldots,n\}$. Hence,

$$\underbrace{I}_{\subseteq\{0,1,\ldots,n\}} \cup \underbrace{\{0,n\}}_{\subseteq\{0,1,\ldots,n\}} \subseteq \{0,1,\ldots,n\} \cup \{0,1,\ldots,n\} = \{0,1,\ldots,n\}.$$

(a) Assume the contrary. Thus, $m < 0$. Now, $0 \in \{0,n\} \subseteq I \cup \{0,n\} = \{i_0, i_1, \ldots, i_m\} = \{\}$ (since $m < 0$). This contradicts the fact that the set $\{\}$ is empty. This contradiction proves that our assumption was false. Hence, Lemma 10.15 (a) is proven.

(b) We have $0 \in \{0,n\} \subseteq I \cup \{0,n\} = \{i_0, i_1, \ldots, i_m\}$. Hence, there exists some $k \in \{0,1,\ldots,m\}$ such that $0 = i_k$. Consider this k.

From Lemma 10.15 (a), we have $m \geq 0$. Hence, $0 \in \{0,1,\ldots,m\}$. Thus, i_0 is well-defined. Assume (for the sake of contradiction) that $i_0 \neq 0$. Thus, $i_0 \neq 0 = i_k$, so that $0 \neq k$ and thus $0 < k$ (since $k \in \{0,1,\ldots,m\}$). Therefore, (103) (applied to $u = 0$ and $v = k$) shows that $i_0 < i_k = 0$.

But $i_0 \in \{i_0, i_1, \ldots, i_m\} = I \cup \{0, n\} \subseteq \{0, 1, \ldots, n\}$, so that $i_0 \geq 0$. This contradicts $i_0 < 0$. This contradiction proves that our assumption (that $i_0 \neq 0$) was false. Hence, we have $i_0 = 0$. This proves Lemma 10.15 (b).

(c) We have $n \in \{0, n\} \subseteq I \cup \{0, n\} = \{i_0, i_1, \ldots, i_m\}$. Hence, there exists some $k \in \{0, 1, \ldots, m\}$ such that $n = i_k$. Consider this k.

From Lemma 10.15 (a), we have $m \geq 0$. Hence, $m \in \{0, 1, \ldots, m\}$. Thus, i_m is well-defined. Assume (for the sake of contradiction) that $i_m \neq n$. Thus, $i_m \neq n = i_k$, so that $m \neq k$ and thus $k \neq m$. Hence, $k < m$ (since $k \in \{0, 1, \ldots, m\}$). Therefore, (103) (applied to $u = k$ and $v = m$) shows that $i_k < i_m$. Hence, $i_m > i_k = n$.

But $i_m \in \{i_0, i_1, \ldots, i_m\} = I \cup \{0, n\} \subseteq \{0, 1, \ldots, n\}$, so that $i_m \leq n$. This contradicts $i_m > n$. This contradiction proves that our assumption (that $i_m \neq n$) was false. Hence, we have $i_m = n$. This proves Lemma 10.15 (c).

(d) Lemma 10.15 (a) shows that $m \geq 0$. Lemma 10.15 (c) yields $i_m = n$. Lemma 10.15 (b) yields $i_0 = 0$. Now,

$$(i_1 - i_0) + (i_2 - i_1) + \ldots + (i_m - i_{m-1}) = n \tag{104}$$

[75].

Now, let $k \in \{1, 2, \ldots, m\}$. Then, $k - 1$ and k are two elements of $\{0, 1, \ldots, m\}$. These two elements satisfy $k - 1 < k$. Hence, (103) (applied to $u = k - 1$ and $v = k$) shows that $i_{k-1} < i_k$. Thus, $i_k - i_{k-1}$ is a positive integer.

Now, forget that we fixed k. Thus, we have shown that for every $k \in \{1, 2, \ldots, m\}$, the number $i_k - i_{k-1}$ is a positive integer. Hence, $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1})$ is a finite list of positive integers, i.e., a composition. The entries of this composition sum to n (because of (104)). Therefore, $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1})$ is a composition of n (by the definition of a "composition of n"). In other words, $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) \in \operatorname{Comp}_n$ (since Comp_n is the set of all compositions of n). This proves Lemma 10.15 (d).

(e) Let $g \in I$. We shall show that $g \in \{i_1, i_2, \ldots, i_{m-1}\}$.

---

[75] *Proof of (104):* If $m = 0$, then

$$\begin{aligned}(i_1 - i_0) + (i_2 - i_1) + \ldots + (i_m - i_{m-1}) &= (\text{empty sum}) = 0 = i_0 = i_m \qquad (\text{since } 0 = m) \\ &= n.\end{aligned}$$

Hence, (104) is proven in the case when $m = 0$. We thus WLOG assume that we don't have $m = 0$.

So we have $m \neq 0$ (since we don't have $m = 0$) but $m \geq 0$. Consequently, $m > 0$. Now,

$$\begin{aligned}
&(i_1 - i_0) + (i_2 - i_1) + \ldots + (i_m - i_{m-1}) \\
&= \sum_{r=1}^{m} (i_r - i_{r-1}) = \sum_{r=1}^{m} i_r - \sum_{r=1}^{m} i_{r-1} \\
&= \underbrace{\sum_{r=1}^{m} i_r}_{\substack{= \sum_{r=1}^{m-1} i_r + i_m \\ \text{(here, we have split off the addend for } r=m \\ \text{from the sum, since } m>0)}} - \underbrace{\sum_{r=0}^{m-1} i_r}_{\substack{= i_0 + \sum_{r=1}^{m-1} i_r \\ \text{(here, we have split off the addend for } r=0 \\ \text{from the sum, since } m>0)}} \\
&\qquad (\text{here, we have substituted } r \text{ for } r-1 \text{ in the second sum}) \\
&= \left(\sum_{r=1}^{m-1} i_r + i_m \right) - \left(i_0 + \sum_{r=1}^{m-1} i_r \right) = \underbrace{i_m}_{=n} - \underbrace{i_0}_{=0} = n.
\end{aligned}$$

This proves (104).

We have $g \in I \subseteq [n-1]$. Hence, $0 < g < n$. But $g \in I \subseteq I \cup \{0, n\} = \{i_0, i_1, \ldots, i_m\}$. Thus, there exists some $p \in \{0, 1, \ldots, m\}$ such that $g = i_p$. Consider this p.

We have $0 < g$, thus $g \neq 0$. Hence, $i_p = g \neq 0 = i_0$ (by Lemma 10.15 (b)), so that $p \neq 0$. Combining $p \in \{0, 1, \ldots, m\}$ with $p \neq 0$, we obtain $p \in \{0, 1, \ldots, m\} \setminus \{0\} = [m]$.

We have $g < n$, thus $g \neq n$. Hence, $i_p = g \neq n = i_m$ (by Lemma 10.15 (c)), so that $p \neq m$. Combining $p \in [m]$ with $p \neq m$, we obtain $p \in [m] \setminus \{m\} = [m-1]$. Therefore, $i_p \in \{i_1, i_2, \ldots, i_{m-1}\}$. Hence, $g = i_p \in \{i_1, i_2, \ldots, i_{m-1}\}$.

Now, forget that we fixed g. Thus, we have proven that $g \in \{i_1, i_2, \ldots, i_{m-1}\}$ for every $g \in I$. In other words,

$$I \subseteq \{i_1, i_2, \ldots, i_{m-1}\}. \tag{105}$$

On the other hand, fix $h \in \{i_1, i_2, \ldots, i_{m-1}\}$. Thus, $h = i_q$ for some $q \in [m-1]$. Consider this q. We shall show that $h \in I$.

We have $h \in \{i_1, i_2, \ldots, i_{m-1}\} \subseteq \{i_0, i_1, \ldots, i_m\} = I \cup \{0, n\}$.

But $q \in [m-1]$, so that $0 < q$. Therefore, (103) (applied to $u = 0$ and $v = q$) yields $i_0 < i_q$. But Lemma 10.15 (b) shows that $0 = i_0 < i_q = h$. Hence, $h \neq 0$.

Also, $q \in [m-1]$, so that $q < m$. Therefore, (103) (applied to $u = q$ and $v = m$) yields $i_q < i_m = n$ (by Lemma 10.15 (c)). Hence, $h = i_q < n$, so that $h \neq n$.

So h is none of the two elements 0 and n (since $h \neq 0$ and $h \neq n$). In other words, $h \notin \{0, n\}$. Combined with $h \in I \cup \{0, n\}$, this yields $h \in (I \cup \{0, n\}) \setminus \{0, n\} \subseteq I$.

Now, forget that we fixed h. We thus have proven that $h \in I$ for every $h \in \{i_1, i_2, \ldots, i_{m-1}\}$. In other words,

$$\{i_1, i_2, \ldots, i_{m-1}\} \subseteq I.$$

Combining this with (105), we obtain $I = \{i_1, i_2, \ldots, i_{m-1}\}$. This proves Lemma 10.15 (e).

(f) Lemma 10.15 (d) shows that $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) \in \operatorname{Comp}_n$. In other words, $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1})$ is a composition of n (since Comp_n is the set of all compositions of n).

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of n, then

$$\begin{aligned} D(\alpha) &= \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_i}_{= \sum_{r=1}^{i} \alpha_r} \mid i \in [\ell - 1] \right\} \qquad \text{(by the definition of } D(\alpha)\text{)} \\ &= \left\{ \sum_{r=1}^{i} \alpha_r \mid i \in [\ell - 1] \right\} = \left\{ \sum_{r=1}^{g} \alpha_r \mid g \in [\ell - 1] \right\} \end{aligned}$$

(here, we renamed the index i as g). Applying this to $\alpha = (i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1})$, $\ell = m$ and $\alpha_k = i_k - i_{k-1}$, we obtain the following:

$$D(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) = \left\{ \sum_{r=1}^{g} (i_r - i_{r-1}) \mid g \in [m-1] \right\}. \tag{106}$$

But Lemma 10.15 (b) yields $i_0 = 0$. Now, every $g \in [m-1]$ satisfies

$$\begin{aligned}
\sum_{r=1}^{g} (i_r - i_{r-1}) &= \sum_{r=1}^{g} i_r - \sum_{r=1}^{g} i_{r-1} \\
&= \underbrace{\sum_{r=1}^{g} i_r}_{\substack{= \sum_{r=1}^{g-1} i_r + i_g \\ \text{(here, we have split off the addend for } r=g \\ \text{from the sum, since } g>0)}} - \underbrace{\sum_{r=0}^{g-1} i_r}_{\substack{= i_0 + \sum_{r=1}^{g-1} i_r \\ \text{(here, we have split off the addend for } r=0 \\ \text{from the sum, since } g>0)}} \\
&\qquad \text{(here, we have substituted } r \text{ for } r-1 \text{ in the second sum)} \\
&= \left(\sum_{r=1}^{g-1} i_r + i_g \right) - \left(i_0 + \sum_{r=1}^{g-1} i_r \right) = i_g - \underbrace{i_0}_{=0} = i_g.
\end{aligned}$$

Thus, (106) becomes

$$\begin{aligned}
D(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) &= \left\{ \underbrace{\sum_{r=1}^{g} (i_r - i_{r-1})}_{=i_g} \mid g \in [m-1] \right\} \\
&= \{ i_g \mid g \in [m-1] \} = \{ i_1, i_2, \ldots, i_{m-1} \} = I
\end{aligned}$$

(by Lemma 10.15 (e)). This proves Lemma 10.15 (f). $\square$

Definition 10.16. Let $n \in \mathbb{N}$. We define a map $\operatorname{comp} : \mathcal{P}([n-1]) \to \operatorname{Comp}_n$ as follows: Let $I \in \mathcal{P}([n-1])$. Write the list $\operatorname{ilis}(I \cup \{0, n\})$ in the form $(i_0, i_1, \ldots, i_m)$ for some integer $m \geq -1$. Lemma 10.15 (d) shows that $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) \in \operatorname{Comp}_n$. Define $\operatorname{comp} I$ to be $(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1})$.

Hence, a map $\operatorname{comp} : \mathcal{P}([n-1]) \to \operatorname{Comp}_n$ is defined.

Proposition 10.17. Let $n \in \mathbb{N}$. Consider the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ defined in Definition 10.13. Consider the map $\operatorname{comp} : \mathcal{P}([n-1]) \to \operatorname{Comp}_n$ introduced in Definition 10.16.

These maps D and comp are mutually inverse.

Proof of Proposition 10.17. Let us first show that $\operatorname{comp} \circ D = \operatorname{id}$.

Indeed, let $\alpha \in \operatorname{Comp}_n$. Thus, α is a composition of n (since Comp_n is the set of all compositions of n). Write the composition α in the form $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Hence, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Lemma 10.4 shows that $D(\alpha) \subseteq [n-1]$, so that $D(\alpha) \in \mathcal{P}([n-1])$. Hence, $\operatorname{comp}(D(\alpha))$ is well-defined. The definition of $\operatorname{comp}(D(\alpha))$ shows that if the list $\operatorname{ilis}(D(\alpha) \cup \{0, n\})$ is written in the form $(i_0, i_1, \ldots, i_m)$ for some integer $m \geq -1$, then

$$\operatorname{comp}(D(\alpha)) = (i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}). \tag{107}$$

Now, for every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

Lemma 10.5 (d) yields $s_j - s_{j-1} = \alpha_j$ for every $j \in [\ell]$. In other words,

$$(s_1 - s_0, s_2 - s_1, \ldots, s_\ell - s_{\ell-1}) = (\alpha_1, \alpha_2, \ldots, \alpha_\ell). \tag{108}$$

Now, we have

$$D(\alpha) \cup \{0, n\} = \{s_0, s_1, \ldots, s_\ell\} \tag{109}$$

[76]. Thus, the list $(s_0, s_1, \ldots, s_\ell)$ contains precisely the elements of the set $D(\alpha) \cup \{0, n\}$. Since this list $(s_0, s_1, \ldots, s_\ell)$ is furthermore strictly increasing (because of Lemma 10.5 (b)), we thus see that the list $(s_0, s_1, \ldots, s_\ell)$ is the list of all elements of $D(\alpha) \cup \{0, n\}$ in increasing order (with each element appearing only once). In other words, the list $(s_0, s_1, \ldots, s_\ell)$ is $\operatorname{ilis}(D(\alpha) \cup \{0, n\})$ (since $\operatorname{ilis}(D(\alpha) \cup \{0, n\})$ is defined to be the list of all elements of $D(\alpha) \cup \{0, n\}$ in increasing order (with each element appearing only once)). In other words,

$$\operatorname{ilis}(D(\alpha) \cup \{0, n\}) = (s_0, s_1, \ldots, s_\ell).$$

Hence, (107) (applied to $m = \ell$ and $i_k = s_k$) yields

$$\begin{aligned} \operatorname{comp}(D(\alpha)) &= (s_1 - s_0, s_2 - s_1, \ldots, s_\ell - s_{\ell-1}) = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \qquad \text{(by (108))} \\ &= \alpha. \end{aligned}$$

Thus, $(\operatorname{comp} \circ D)(\alpha) = \operatorname{comp}(D(\alpha)) = \alpha$.

Now, forget that we fixed α. We thus have proven that $(\operatorname{comp} \circ D)(\alpha) = \alpha$ for every $\alpha \in \operatorname{Comp}_n$. In other words,

$$\operatorname{comp} \circ D = \operatorname{id}. \tag{110}$$

On the other hand, let us show that $D \circ \operatorname{comp} = \operatorname{id}$. Indeed, let $I \in \mathcal{P}([n-1])$. Write the list $\operatorname{ilis}(I \cup \{0, n\})$ in the form $(i_0, i_1, \ldots, i_m)$ for some integer $m \geq -1$. The definition of comp then shows that $\operatorname{comp} I = (i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1})$. Now,

$$(D \circ \operatorname{comp})(I) = D\left(\underbrace{\operatorname{comp} I}_{=(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1})}\right) = D(i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}) = I$$

(by Lemma 10.15 (f)).

Now, forget that we fixed I. We thus have proven that $(D \circ \operatorname{comp})(I) = I$ for every $I \in \mathcal{P}([n-1])$. In other words, $D \circ \operatorname{comp} = \operatorname{id}$. Combining this with (110), we conclude that the maps D and comp are mutually inverse. Proposition 10.17 is proven. $\square$

From what we have proven so far, we obtain the following corollary:

Corollary 10.18. Let α be a composition of a nonnegative integer n. Then,

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\beta \text{ is a composition of } n;\ D(\beta) \supseteq D(\alpha)} M_\beta.$$

(Here, we are using the notations of Definition 10.2 and Definition 10.3.)

[76] *Proof of (109):* From Lemma 10.5 (c), we have $D(\alpha) = \{s_1, s_2, \ldots, s_{\ell-1}\}$. Lemma 10.5 (f) yields $0 = s_0$. Lemma 10.5 (e) yields $n = s_\ell$. Thus,

$$\underbrace{D(\alpha)}_{=\{s_1, s_2, \ldots, s_{\ell-1}\}} \cup \left\{\underbrace{0}_{=s_0}, \underbrace{n}_{=s_\ell}\right\} = \{s_1, s_2, \ldots, s_{\ell-1}\} \cup \{s_0, s_\ell\} = \{s_0, s_1, \ldots, s_\ell\}.$$

This proves (109).

Proof of Corollary 10.18. Lemma 10.4 shows that $D(\alpha) \subseteq [n-1]$.

Proposition 10.17 yields that the maps D and comp are mutually inverse. Thus, the map D is invertible, i.e., a bijection.

If $(i_1, i_2, \ldots, i_n)$ is an n-tuple of positive integers, then the condition $\left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha)\right)$ is equivalent to the condition $\left(D(\alpha) \subseteq \left\{j \in [n-1] \mid i_j < i_{j+1}\right\}\right)$ [77]. Hence, we have the following equality between summation signs:

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ D(\alpha) \subseteq \left\{j \in [n-1] \mid i_j < i_{j+1}\right\}}} .$$

Thus,

$$\begin{aligned}
&\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ D(\alpha) \subseteq \left\{j \in [n-1] \mid i_j < i_{j+1}\right\}}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
&= \underbrace{\sum_{\substack{G \subseteq [n-1]; \\ D(\alpha) \subseteq G}}}_{= \sum_{\substack{G \subseteq [n-1]; \\ G \supseteq D(\alpha)}} = \sum_{\substack{G \in \mathcal{P}([n-1]); \\ G \supseteq D(\alpha)}}} \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \mid i_j < i_{j+1}\right\} = G}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
&\qquad \left(\text{since } \left\{j \in [n-1] \mid i_j < i_{j+1}\right\} \subseteq [n-1] \text{ for every } (i_1 \leq i_2 \leq \cdots \leq i_n)\right) \\
&= \sum_{\substack{G \in \mathcal{P}([n-1]); \\ G \supseteq D(\alpha)}} \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \mid i_j < i_{j+1}\right\} = G}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
&= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\alpha)}} \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \mid i_j < i_{j+1}\right\} = D(\beta)}} x_{i_1} x_{i_2} \cdots x_{i_n}
\end{aligned}$$

[77] *Proof.* Let $(i_1, i_2, \ldots, i_n)$ be an n-tuple of positive integers. Then, we have the following chain of logical equivalences:

$$\begin{aligned}
&\left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha)\right) \\
\iff &\left(i_h < i_{h+1} \text{ whenever } h \in D(\alpha)\right) \\
&\qquad (\text{here, we have renamed the index } j \text{ as } h) \\
\iff &\left(\text{every } h \in D(\alpha) \text{ satisfies } \underbrace{i_h < i_{h+1}}_{\substack{\text{this is equivalent to} \\ h \in \left\{j \in [n-1] \mid i_j < i_{j+1}\right\} \\ (\text{because } h \text{ is always an element of } [n-1] \\ (\text{since } h \in D(\alpha) \subseteq [n-1]))}}\right) \\
\iff &\left(\text{every } h \in D(\alpha) \text{ satisfies } h \in \left\{j \in [n-1] \mid i_j < i_{j+1}\right\}\right) \\
\iff &\left(D(\alpha) \subseteq \left\{j \in [n-1] \mid i_j < i_{j+1}\right\}\right),
\end{aligned}$$

Thus, the condition $\left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha)\right)$ is equivalent to the condition $\left(D(\alpha) \subseteq \left\{j \in [n-1] \mid i_j < i_{j+1}\right\}\right)$. Qed.

(here, we have substituted $D(\beta)$ for G in the outer sum, since the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection). Comparing this with

$$\underbrace{\sum_{\beta \text{ is a composition of } n;\ D(\beta)\supseteq D(\alpha)}}_{\substack{=\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\supseteq D(\alpha)}} \\ \text{(since } \operatorname{Comp}_n \text{ is the set} \\ \text{of all compositions of } n)}} \overbrace{\underbrace{M_\beta}}^{}_{\substack{=\sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ \{j\in[n-1]\ \mid\ i_j<i_{j+1}\}=D(\beta)}} x_{i_1}x_{i_2}\cdots x_{i_n} \\ \text{(by Proposition 10.10 (applied to } \beta \text{ instead of } \alpha))}}$$

$$= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\supseteq D(\alpha)}} \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ \{j\in[n-1]\ \mid\ i_j<i_{j+1}\}=D(\beta)}} x_{i_1}x_{i_2}\cdots x_{i_n},$$

we obtain

$$\sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_j<i_{j+1} \text{ whenever } j\in D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n} = \sum_{\beta \text{ is a composition of } n;\ D(\beta)\supseteq D(\alpha)} M_\beta .$$

This proves Corollary 10.18. □

10.3. $\Gamma(\mathbf{E}, w)$ is well-defined

Let us next show a really simple fact that was left unproven in Definition 3.4:

Proposition 10.19. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then, the sum $\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}$ in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ converges (with respect to the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$).

Proposition 10.19 shows that the power series $\Gamma(\mathbf{E}, w)$ in Definition 3.4 is well-defined.

Proof of Proposition 10.19. We know that $(E, <_1, <_2)$ is a double poset. Hence, E is a finite set.

For every power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and every monomial $\mathfrak{m}$, we denote by $[\mathfrak{m}](f)$ the coefficient of $\mathfrak{m}$ in f. Notice that any two monomials $\mathfrak{m}$ and $\mathfrak{n}$ satisfy

$$[\mathfrak{m}](\mathfrak{n}) = \begin{cases} 1, & \text{if } \mathfrak{m} = \mathfrak{n}; \\ 0, & \text{if } \mathfrak{m} \neq \mathfrak{n} \end{cases}. \tag{111}$$

The definition of the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ has the following consequence:

Fact 1: Let P be a set. Let $(\alpha_\pi)_{\pi\in P}$ be a family of elements of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. Assume that for every monomial $\mathfrak{m}$, all but finitely many $\pi \in P$ satisfy $[\mathfrak{m}](\alpha_\pi) = 0$. Then, the sum $\sum_{\pi\in P} \alpha_\pi$ converges.

Now, let $\mathfrak{m}$ be a monomial. Let Z be the set of all positive integers i such that the indeterminate x_i appears in the monomial $\mathfrak{m}$. Thus, Z is a finite subset of $\{1, 2, 3, \ldots\}$. Thus, $|Z| < \infty$. Also, $|E| < \infty$ (since E is finite).

If A and B are sets, and if C is a subset of B, then

$$\left\{\pi \in B^A \ \mid\ \pi(A) \subseteq C\right\} \cong C^A \text{ as sets} \tag{112}$$

[78].

[78]*Proof of (112):* This is a well-known fact about sets. The proof proceeds as follows:

Let $\operatorname{Par}\mathbf{E}$ be the set of all $\mathbf{E}$-partitions. Thus, every element of $\operatorname{Par}\mathbf{E}$ is an $\mathbf{E}$-partition, hence a map $E \to \{1,2,3,\ldots\}$, hence an element of $\{1,2,3,\ldots\}^E$. In other words, $\operatorname{Par}\mathbf{E} \subseteq \{1,2,3,\ldots\}^E$.

Let Q be the subset
$$\{\pi \in \operatorname{Par}\mathbf{E} \mid \pi(E) \subseteq Z\}$$
of $\operatorname{Par}\mathbf{E}$. The set Q is finite[79].

We have $[\mathfrak{m}]\left(\mathbf{x}_{\phi,w}\right) = 0$ for every $\phi \in (\operatorname{Par}\mathbf{E}) \setminus Q$ [80]. Since Q is a finite set, this shows that we have $[\mathfrak{m}]\left(\mathbf{x}_{\phi,w}\right) = 0$ for all but finitely many $\phi \in \operatorname{Par}\mathbf{E}$. If we rename ϕ as π in this statement, we

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Let A and B be sets. Let C be a subset of B. Let $\iota : C \to B$ be the canonical inclusion map.

- Define a map $\Phi : \left\{\pi \in B^A \mid \pi(A) \subseteq C\right\} \to C^A$ as follows:

  Let $f \in \left\{\pi \in B^A \mid \pi(A) \subseteq C\right\}$. Thus, f is an element of B^A and satisfies $f(A) \subseteq C$. Hence, we can define a map $f' : A \to C$ by $(f'(a) = f(a)$ for every $a \in A)$. Define $\Phi(f)$ to be this map f'. Hence, a map $\Phi : \left\{\pi \in B^A \mid \pi(A) \subseteq C\right\} \to C^A$ is defined.

- Define a map $\Psi : C^A \to \left\{\pi \in B^A \mid \pi(A) \subseteq C\right\}$ by
$$\left(\Psi(g) = \iota \circ g \qquad \text{for each } g \in C^A\right).$$

It is easy to see that the maps Φ and Ψ are mutually inverse bijections. Hence, there is a bijection from $\left\{\pi \in B^A \mid \pi(A) \subseteq C\right\}$ to C^A (namely, the map Φ). This proves (112).

[79] *Proof.* From (112) (applied to $A = E$, $B = \{1,2,3,\ldots\}$ and $C = Z$), we obtain the fact that
$$\left\{\pi \in \{1,2,3,\ldots\}^E \mid \pi(E) \subseteq Z\right\} \cong Z^E \text{ as sets.}$$
Hence,
$$\left|\left\{\pi \in \{1,2,3,\ldots\}^E \mid \pi(E) \subseteq Z\right\}\right| = \left|Z^E\right| = |Z|^{|E|} < \infty$$
(since $|Z| < \infty$ and $|E| < \infty$). But
$$Q = \left\{\pi \in \underbrace{\operatorname{Par}\mathbf{E}}_{\subseteq\{1,2,3,\ldots\}^E} \mid \pi(E) \subseteq Z\right\} \subseteq \left\{\pi \in \{1,2,3,\ldots\}^E \mid \pi(E) \subseteq Z\right\}$$
and thus
$$|Q| \leq \left|\left\{\pi \in \{1,2,3,\ldots\}^E \mid \pi(E) \subseteq Z\right\}\right| < \infty.$$
Hence, the set Q is finite, qed.

[80] *Proof.* Let $\phi \in (\operatorname{Par}\mathbf{E}) \setminus Q$. We shall first show that $\mathfrak{m} \neq \mathbf{x}_{\phi,w}$ (as monomials).

Indeed, assume the contrary. Thus, $\mathfrak{m} = \mathbf{x}_{\phi,w}$ (as monomials). But the definition of $\mathbf{x}_{\phi,w}$ shows that $\mathbf{x}_{\phi,w} = \prod_{e\in E} x_{\phi(e)}^{w(e)}$.

Now, let $f \in E$. Then, $w(f) \in \{1,2,3,\ldots\}$ (since w is a map $E \to \{1,2,3,\ldots\}$). Hence, $x_{\phi(f)} \mid x_{\phi(f)}^{w(f)}$ (as monomials). But $x_{\phi(f)}^{w(f)}$ is a factor in the product $\prod_{e\in E} x_{\phi(e)}^{w(e)}$. Therefore, $x_{\phi(f)}^{w(f)} \mid \prod_{e\in E} x_{\phi(e)}^{w(e)}$ (as monomials). Hence, $x_{\phi(f)} \mid x_{\phi(f)}^{w(f)} \mid \prod_{e\in E} x_{\phi(e)}^{w(e)} = \mathfrak{m}$ (as monomials). Thus, the indeterminate $x_{\phi(f)}$ appears in the monomial $\mathfrak{m}$. Thus, $\phi(f)$ is a positive integer i such that the indeterminate x_i appears in the monomial $\mathfrak{m}$. In other words, $\phi(f) \in Z$ (since Z is the set of all positive integers i such that the indeterminate x_i appears in the monomial $\mathfrak{m}$).

Now, forget that we fixed f. We thus have shown that $\phi(f) \in Z$ for each $f \in E$. In other words, $\phi(E) \subseteq Z$.

Now, $\phi \in (\operatorname{Par}\mathbf{E}) \setminus Q$. In other words, $\phi \in \operatorname{Par}\mathbf{E}$ but $\phi \notin Q$. We now know that ϕ is an obtain the following: We have $[\mathfrak{m}]\left(\mathbf{x}_{\pi,w}\right)=0$ for all but finitely many $\pi \in \operatorname{Par}\mathbf{E}$. In other words, all but finitely many $\pi \in \operatorname{Par}\mathbf{E}$ satisfy $[\mathfrak{m}]\left(\mathbf{x}_{\pi,w}\right)=0$.

Let us now forget that we fixed $\mathfrak{m}$. We therefore have shown that for every monomial $\mathfrak{m}$, all but finitely many $\pi \in \operatorname{Par}\mathbf{E}$ satisfy $[\mathfrak{m}]\left(\mathbf{x}_{\pi,w}\right)=0$. Thus, Fact 1 (applied to $P=\operatorname{Par}\mathbf{E}$ and $\alpha_\pi=\mathbf{x}_{\pi,w}$) shows that the sum $\sum\limits_{\pi\in\operatorname{Par}\mathbf{E}} \mathbf{x}_{\pi,w}$ converges. Since

$$\sum_{\pi\in\operatorname{Par}\mathbf{E}} = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}$$

(because $\operatorname{Par}\mathbf{E}$ is the set of all $\mathbf{E}$-partitions), this rewrites as follows: The sum $\sum\limits_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}$ converges. This proves Proposition 10.19. □

Let us also show a more detailed proof of Lemma 6.3:

Proof of Lemma 6.3. (a) Assume that $E=\varnothing$. We need to show that $\Gamma\left(\mathbf{E},w\right)=1$.

Let $\operatorname{Par}\mathbf{E}$ be the set of all $\mathbf{E}$-partitions. Let g be the unique map $\varnothing\to\{1,2,3,\ldots\}$. Then, $g\in\operatorname{Par}\mathbf{E}$ [81]. Thus, $\{g\}\subseteq\operatorname{Par}\mathbf{E}$.

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element of $\operatorname{Par}\mathbf{E}$ satisfying $\phi(E)\subseteq Z$. In other words, $\phi\in\{\pi\in\operatorname{Par}\mathbf{E}\ \mid\ \pi(E)\subseteq Z\}$. In other words, $\phi\in Q$ (since $Q=\{\pi\in\operatorname{Par}\mathbf{E}\ \mid\ \pi(E)\subseteq Z\}$). This contradicts $\phi\notin Q$.

This contradiction proves that our assumption was wrong. Hence, $\mathfrak{m}\neq\mathbf{x}_{\phi,w}$ is proven. Now, (111) (applied to $\mathfrak{n}=\mathbf{x}_{\phi,w}$) shows that

$$[\mathfrak{m}]\left(\mathbf{x}_{\phi,w}\right)=\begin{cases}1, & \text{if } \mathfrak{m}=\mathbf{x}_{\phi,w};\\ 0, & \text{if } \mathfrak{m}\neq\mathbf{x}_{\phi,w}\end{cases}=0 \qquad \left(\text{since } \mathfrak{m}\neq\mathbf{x}_{\phi,w}\right),$$

qed.

[81] *Proof.* Recall the definition of an $\mathbf{E}$-partition. This definition shows that g is an $\mathbf{E}$-partition if and only if g is a map $E\to\{1,2,3,\ldots\}$ satisfying the following two assertions:

Assertion $\mathcal{A}_1$: Every $e\in E$ and $f\in E$ satisfying $e<_1 f$ satisfy $g(e)\leq g(f)$.

Assertion $\mathcal{A}_2$: Every $e\in E$ and $f\in E$ satisfying $e<_1 f$ and $f<_2 e$ satisfy $g(e)<g(f)$.

Now, g is a map $\varnothing\to\{1,2,3,\ldots\}$. In other words, g is a map $E\to\{1,2,3,\ldots\}$ (since $E=\varnothing$). Also, there exists no $e\in E$ (since $E=\varnothing$). Hence, Assertion $\mathcal{A}_1$ is vacuously true. Also, Assertion $\mathcal{A}_2$ is vacuously true (for the same reason). Thus, g is a map $E\to\{1,2,3,\ldots\}$ satisfying the two Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$. In other words, g is an $\mathbf{E}$-partition (since g is an $\mathbf{E}$-partition if and only if g is a map $E\to\{1,2,3,\ldots\}$ satisfying the two Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$). In other words, g belongs to the set of all $\mathbf{E}$-partitions. In other words, g belongs to $\operatorname{Par}\mathbf{E}$ (since $\operatorname{Par}\mathbf{E}$ is the set of all $\mathbf{E}$-partitions). In other words, $g\in\operatorname{Par}\mathbf{E}$. Qed.

Now, $\operatorname{Par}\mathbf{E} = \{g\}$ [82] and $\mathbf{x}_{g,w} = 1$ [83]. Now, the definition of $\Gamma(\mathbf{E}, w)$ yields

$$\begin{aligned}
\Gamma(\mathbf{E}, w) &= \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}}_{\substack{=\sum_{\pi\in\operatorname{Par}\mathbf{E}} \\ \text{(since } \operatorname{Par}\mathbf{E} \text{ is the set of all } \mathbf{E}\text{-partitions)}}} \mathbf{x}_{\pi,w} = \sum_{\pi\in\operatorname{Par}\mathbf{E}} \mathbf{x}_{\pi,w} \\
&= \sum_{\pi\in\{g\}} \mathbf{x}_{\pi,w} \qquad (\text{since } \operatorname{Par}\mathbf{E} = \{g\}) \\
&= \mathbf{x}_{g,w} = 1.
\end{aligned}$$

This proves Lemma 6.3 (a).

(b) Assume that $E \neq \varnothing$. We need to show that $\varepsilon(\Gamma(\mathbf{E}, w)) = 0$.

For every $\mathbf{E}$-partition π, we have

$$(\text{the constant term of } \mathbf{x}_{\pi,w}) = 0 \qquad (113)$$

[84].

It is well-known that

$$\varepsilon(f) = (\text{the constant term of } f) \qquad \text{for every } f \in \operatorname{QSym}$$

(where the constant term of f makes sense because f is a power series). Applying this to $f =$

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[82] *Proof.* Let $\phi \in \operatorname{Par}\mathbf{E}$. We have $\operatorname{Par}\mathbf{E} \subseteq \{1,2,3,\ldots\}^E$ (indeed, we have shown this in the proof of Proposition 10.19). Since $E = \varnothing$, this rewrites as $\operatorname{Par}\mathbf{E} \subseteq \{1,2,3,\ldots\}^{\varnothing} = \{g\}$. Combining this with $\{g\} \subseteq \operatorname{Par}\mathbf{E}$, we obtain $\operatorname{Par}\mathbf{E} = \{g\}$, qed.

[83] *Proof.* The definition of $\mathbf{x}_{g,w}$ yields

$$\begin{aligned}
\mathbf{x}_{g,w} &= \prod_{e\in E} x_{g(e)}^{w(e)} = \prod_{e\in\varnothing} x_{g(e)}^{w(e)} \qquad (\text{since } E = \varnothing) \\
&= (\text{empty product}) = 1,
\end{aligned}$$

qed.

[84] *Proof of (113):* Let π be an $\mathbf{E}$-partition. The definition of $\mathbf{x}_{\pi,w}$ yields $\mathbf{x}_{\pi,w} = \prod_{e\in E} x_{\pi(e)}^{w(e)}$. Thus, $\mathbf{x}_{\pi,w}$ is a monomial, and its degree is

$$\deg\left(\underbrace{\mathbf{x}_{\pi,w}}_{=\prod_{e\in E} x_{\pi(e)}^{w(e)}}\right) = \deg\left(\prod_{e\in E} x_{\pi(e)}^{w(e)}\right) = \sum_{e\in E} \underbrace{\deg\left(x_{\pi(e)}^{w(e)}\right)}_{=w(e)} = \sum_{e\in E} w(e).$$

Now, the sum $\sum_{e\in E} w(e)$ is a nonempty sum (since $E \neq \varnothing$), and all its addends are positive integers (since $w(e) \in \{1,2,3,\ldots\}$ for each $e \in E$). Thus, $\sum_{e\in E} w(e)$ is a nonempty sum of positive integers, and therefore is itself a positive integer (since any nonempty sum of positive integers is a positive integer). In other words, $\deg(\mathbf{x}_{\pi,w})$ is a positive integer (since $\deg(\mathbf{x}_{\pi,w}) = \sum_{e\in E} w(e)$). Hence, $\deg(\mathbf{x}_{\pi,w}) \neq 0 = \deg 1$.

But if $\mathfrak{m}$ is a monomial such that $\mathfrak{m} \neq 1$, then $(\text{the constant term of } \mathfrak{m}) = 0$. Applying this to $\mathfrak{m} = \mathbf{x}_{\pi,w}$, we obtain $(\text{the constant term of } \mathbf{x}_{\pi,w}) = 0$ (since $\mathbf{x}_{\pi,w} \neq 1$). This proves (113).

$\Gamma(\mathbf{E}, w)$, we obtain

$$\varepsilon(\Gamma(\mathbf{E}, w)) = \left(\text{the constant term of } \underbrace{\Gamma(\mathbf{E}, w)}_{= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}}\right)$$
$$= \left(\text{the constant term of } \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}\right)$$
$$= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \underbrace{(\text{the constant term of } \mathbf{x}_{\pi,w})}_{\substack{=0 \\ (\text{by (113)})}} = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} 0 = 0.$$

This proves Lemma 6.3 (b). $\square$

10.4. Increasing and strictly increasing maps as E-partitions

Now, we shall prove two claims that were left unproven in Example 3.3:

Proposition 10.20. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Assume that the order $<_2$ is an extension of the order $<_1$ (that is, we have $u <_2 v$ for every two elements u and v of E satisfying $u <_1 v$). Then, the $\mathbf{E}$-partitions are precisely the weakly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

Proposition 10.21. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $>_1$ denote the opposite relation of $<_1$. Assume that the order $<_2$ is an extension of the order $>_1$ (that is, we have $u <_2 v$ for every two elements u and v of E satisfying $u >_1 v$). Then, the $\mathbf{E}$-partitions are precisely the strictly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

Proof of Proposition 10.20. Let $\phi : E \to \{1, 2, 3, \ldots\}$ be a map. We need to show the following logical equivalence:

$$(\phi \text{ is an } \mathbf{E}\text{-partition}) \iff (\phi \text{ is a weakly increasing map from } (E, <_1) \text{ to } \{1, 2, 3, \ldots\}). \tag{114}$$

Recall the definition of an $\mathbf{E}$-partition. This definition shows that ϕ is an $\mathbf{E}$-partition if and only if it satisfies the following two assertions:

Assertion $\mathcal{A}_1$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$.

Assertion $\mathcal{A}_2$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

In other words, we have the following logical equivalence:

$$(\phi \text{ is an } \mathbf{E}\text{-partition}) \iff (\text{Assertions } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ hold}).$$

But Assertion $\mathcal{A}_2$ always holds[85]. Now, we have the following chain of equivalences:

$$\begin{aligned}
&(\phi \text{ is an } \mathbf{E}\text{-partition}) \\
\iff &(\text{Assertions } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ hold}) \\
\iff &(\text{Assertion } \mathcal{A}_1 \text{ holds}) \qquad (\text{since Assertion } \mathcal{A}_2 \text{ always holds}) \\
\iff &(\text{every } e \in E \text{ and } f \in E \text{ satisfying } e <_1 f \text{ satisfy } \phi(e) \leq \phi(f)) \\
&\quad (\text{because this is Assertion } \mathcal{A}_1) \\
\iff &(\phi \text{ is a weakly increasing map from } (E, <_1) \text{ to } \{1, 2, 3, \ldots\}) \\
&\quad (\text{by the definition of a ``weakly increasing map''}).
\end{aligned}$$

Thus, (114) is proven. This concludes the proof of Proposition 10.20. □

Proof of Proposition 10.21. Let $\phi : E \to \{1, 2, 3, \ldots\}$ be a map. We need to show the following logical equivalence:

$$\begin{aligned}
&(\phi \text{ is an } \mathbf{E}\text{-partition}) \\
\iff &(\phi \text{ is a strictly increasing map from } (E, <_1) \text{ to } \{1, 2, 3, \ldots\}).
\end{aligned} \tag{115}$$

Recall the definition of an $\mathbf{E}$-partition. This definition shows that ϕ is an $\mathbf{E}$-partition if and only if it satisfies the following two assertions:

Assertion $\mathcal{A}_1$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$.

Assertion $\mathcal{A}_2$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

In other words, we have the following logical equivalence:

$$(\phi \text{ is an } \mathbf{E}\text{-partition}) \iff (\text{Assertions } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ hold}).$$

Now, consider the following assertion:

Assertion $\mathcal{A}_3$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) < \phi(f)$.

The following logical implication is obvious:

$$(\text{Assertion } \mathcal{A}_3 \text{ holds}) \implies (\text{Assertions } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ hold}). \tag{116}$$

On the other hand, we also have the following logical implication:

$$(\text{Assertions } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ hold}) \implies (\text{Assertion } \mathcal{A}_3 \text{ holds}) \tag{117}$$

[86]. Combining this implication with (116), we obtain the equivalence

$$(\text{Assertions } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ hold}) \iff (\text{Assertion } \mathcal{A}_3 \text{ holds}).$$

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[85] *Proof.* We shall show that Assertion $\mathcal{A}_2$ is vacuously true. In other words, we shall show that there exist no $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$. Indeed, assume the contrary. Thus, there exist two $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$. Consider these e and f. From $e <_1 f$, we obtain $e <_2 f$ (since the order $<_2$ is an extension of the order $<_1$). This contradicts $f <_2 e$ (since $<_2$ is a strict partial order). This contradiction proves that our assumption was false. Hence, there exist no $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$. Thus, Assertion $\mathcal{A}_2$ is vacuously true, qed.

[86] *Proof of (117):* Assume that Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold. We need to show that Assertion $\mathcal{A}_3$ holds.

Let $e \in E$ and $f \in E$ be such that $e <_1 f$. From $e <_1 f$, we conclude that $f >_1 e$, so that $f <_2 e$ (since the order $<_2$ is an extension of the order $>_1$). Hence, Assertion $\mathcal{A}_2$ shows that $\phi(e) < \phi(f)$.

Now, forget that we fixed e and f. We thus have proven that every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) < \phi(f)$. In other words, Assertion $\mathcal{A}_3$ holds. This proves the implication (117).

Now, we have the following chain of equivalences:

$$\begin{aligned}
& (\phi \text{ is an } \mathbf{E}\text{-partition}) \\
\iff & (\text{Assertions } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ hold}) \\
\iff & (\text{Assertion } \mathcal{A}_3 \text{ holds}) \\
\iff & (\text{every } e \in E \text{ and } f \in E \text{ satisfying } e <_1 f \text{ satisfy } \phi(e) < \phi(f)) \\
& \quad (\text{because this is Assertion } \mathcal{A}_3) \\
\iff & (\phi \text{ is a strictly increasing map from } (E, <_1) \text{ to } \{1, 2, 3, \ldots\}) \\
& \quad (\text{by the definition of a "strictly increasing map"}) .
\end{aligned}$$

Thus, (115) is proven. This concludes the proof of Proposition 10.21. □

10.5. Semistandard tableaux as E-partitions

Let us now verify two further claims made in Example 3.3 – namely, the claims about semistandard tableaux. We recall the definition of a semistandard tableau:

Definition 10.22. Let μ and λ be two partitions such that $\mu \subseteq \lambda$. Define the set $Y(\lambda/\mu)$ as in Example 3.3. A *semistandard tableau of shape* λ/μ means a map $\phi : Y(\lambda/\mu) \to \{1, 2, 3, \ldots\}$ satisfying the following two assertions:

Assertion $\mathcal{T}_1$: For any $(i, a) \in Y(\lambda/\mu)$ and $(i, b) \in Y(\lambda/\mu)$ with $a < b$, we have $\phi(i, a) \leq \phi(i, b)$.

Assertion $\mathcal{T}_2$: For any $(a, j) \in Y(\lambda/\mu)$ and $(b, j) \in Y(\lambda/\mu)$ with $a < b$, we have $\phi(a, j) < \phi(b, j)$.

(It is usual to visualize the set $Y(\lambda/\mu)$ as a set of 1×1-squares on the integer lattice $\mathbb{Z}^2$; then, a map $\phi : Y(\lambda/\mu) \to \{1, 2, 3, \ldots\}$ can be regarded as a filling of these squares with numbers in $\{1, 2, 3, \ldots\}$. In this visual representation, Assertion $\mathcal{T}_1$ claims that the entries of the filling ϕ are weakly increasing from left to right along each row of the tableau, whereas Assertion $\mathcal{T}_2$ says that the entries of ϕ are strictly increasing from top to bottom along each column of the tableau. See [GriRei14, §2.2] for more about semistandard tableaux (which are called column-strict tableaux in [GriRei14]) as well as for examples of this visual representation.)

The following proposition contains two unproven claims made in Example 3.3:

Proposition 10.23. Let μ and λ be two partitions such that $\mu \subseteq \lambda$. Define the double posets $\mathbf{Y}(\lambda/\mu)$ and $\mathbf{Y}_h(\lambda/\mu)$ as in Example 3.3. Then:

(a) The $\mathbf{Y}(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ.

(b) The $\mathbf{Y}_h(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ.

We shall prove a slightly more general fact:

Proposition 10.24. Let μ and λ be two partitions such that $\mu \subseteq \lambda$. Define the set $Y(\lambda/\mu)$ and the relation $<_1$ as in Example 3.3. Let $\prec$ be a strict partial order on the set $Y(\lambda/\mu)$ such that the following two conditions hold:

Condition $\mathcal{O}_1$: For any $(i, a) \in Y(\lambda/\mu)$ and $(i, b) \in Y(\lambda/\mu)$ with $a < b$, we have $(i, a) \prec (i, b)$.

Condition $\mathcal{O}_2$: For any $(a, j) \in Y(\lambda/\mu)$ and $(b, j) \in Y(\lambda/\mu)$ with $a < b$, we have $(b, j) \prec (a, j)$.

Then, the $(Y(\lambda/\mu), <_1, \prec)$-partitions are precisely the semistandard tableaux of shape λ/μ.

Before we begin proving this, let us however reach back and verify a really basic result about partitions:

Lemma 10.25. Let μ and λ be two partitions such that $\mu \subseteq \lambda$. Let (a_1, b_1), (a_2, b_2) and (a_3, b_3) be three elements of $\mathbb{Z}^2$ such that $(a_1, b_1) \in Y(\lambda/\mu)$ and $(a_3, b_3) \in Y(\lambda/\mu)$ and $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$. Then,

$$(a_2, b_2) \in Y(\lambda/\mu). \tag{118}$$

Proof of Lemma 10.25. Write the partition μ in the form $(\mu_1, \mu_2, \mu_3, \ldots)$.

Write the partition λ in the form $(\lambda_1, \lambda_2, \lambda_3, \ldots)$.

Now, $Y(\lambda/\mu)$ is the set of all $(i,j) \in \{1,2,3,\ldots\}^2$ satisfying $\mu_i < j \leq \lambda_i$ (by the definition of $Y(\lambda/\mu)$). In other words,

$$Y(\lambda/\mu) = \left\{(i,j) \in \{1,2,3,\ldots\}^2 \mid \mu_i < j \leq \lambda_i\right\}.$$

Now,

$$(a_1, b_1) \in Y(\lambda/\mu) = \left\{(i,j) \in \{1,2,3,\ldots\}^2 \mid \mu_i < j \leq \lambda_i\right\}.$$

In other words, (a_1, b_1) is an element of $\{1,2,3,\ldots\}^2$ and satisfies $\mu_{a_1} < b_1 \leq \lambda_{a_1}$.

Also,

$$(a_3, b_3) \in Y(\lambda/\mu) = \left\{(i,j) \in \{1,2,3,\ldots\}^2 \mid \mu_i < j \leq \lambda_i\right\}.$$

In other words, (a_3, b_3) is an element of $\{1,2,3,\ldots\}^2$ and satisfies $\mu_{a_3} < b_3 \leq \lambda_{a_3}$.

From $(a_1, b_1) \in \{1,2,3,\ldots\}^2$, we $a_1 \geq 1$ and $b_1 \geq 1$. Now, $(a_2, b_2) \in \{1,2,3,\ldots\}^2$ (since $a_2 \geq a_1 \geq 1$ and $b_2 \geq b_1 \geq 1$). Hence, λ_{a_2} and μ_{a_2} are well-defined.

The sequence $(\mu_1, \mu_2, \mu_3, \ldots) = \mu$ is a partition. Thus, $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$. In other words, any two positive integers u and v satisfying $u \leq v$ satisfy $\mu_u \geq \mu_v$. Applying this to $u = a_1$ and $v = a_2$, we obtain $\mu_{a_1} \geq \mu_{a_2}$ (since $a_1 \leq a_2$). Thus, $\mu_{a_2} \leq \mu_{a_1} < b_1 \leq b_2$.

The sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots) = \lambda$ is a partition. Hence, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. In other words, any two positive integers u and v satisfying $u \leq v$ satisfy $\lambda_u \geq \lambda_v$. Applying this to $u = a_2$ and $v = a_3$, we obtain $\lambda_{a_2} \geq \lambda_{a_3}$ (since $a_2 \leq a_3$). Thus, $\lambda_{a_3} \leq \lambda_{a_2}$, so that $b_2 \leq b_3 \leq \lambda_{a_3} \leq \lambda_{a_2}$. Hence, $\mu_{a_2} < b_2 \leq \lambda_{a_2}$.

Now, we know that (a_2, b_2) is an element of $\{1,2,3,\ldots\}^2$ and satisfies $\mu_{a_2} < b_2 \leq \lambda_{a_2}$. Hence,

$$(a_2, b_2) \in \left\{(i,j) \in \{1,2,3,\ldots\}^2 \mid \mu_i < j \leq \lambda_i\right\} = Y(\lambda/\mu).$$

This proves Lemma 10.25. $\square$

Proof of Proposition 10.24. Both $(Y(\lambda/\mu), <_1, \prec)$-partitions and semistandard tableaux of shape λ/μ are maps from $Y(\lambda/\mu)$ to $\{1,2,3,\ldots\}$. Our goal is to show that the $(Y(\lambda/\mu), <_1, \prec)$-partitions are precisely the semistandard tableaux of shape λ/μ. In other words, our goal is to prove that, for every map $\phi : Y(\lambda/\mu) \to \{1,2,3,\ldots\}$, we have the following equivalence:

$$\begin{aligned} &(\phi \text{ is a } (Y(\lambda/\mu), <_1, \prec)\text{-partition}) \\ &\Longleftrightarrow (\phi \text{ is a semistandard tableau of shape } \lambda/\mu). \end{aligned} \tag{119}$$

Let $\phi : Y(\lambda/\mu) \to \{1,2,3,\ldots\}$ be a map.

Recall the definition of a $(Y(\lambda/\mu), <_1, \prec)$-partition. This definition shows that ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition if and only if it satisfies the following two assertions:

Assertion $\mathcal{P}_1$: Every $e \in Y(\lambda/\mu)$ and $f \in Y(\lambda/\mu)$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$.

Assertion $\mathcal{P}_2$: Every $e \in Y(\lambda/\mu)$ and $f \in Y(\lambda/\mu)$ satisfying $e <_1 f$ and $f \prec e$ satisfy $\phi(e) < \phi(f)$.

On the other hand, recall the definition of a semistandard tableau of shape λ/μ. This definition shows that ϕ is a semistandard tableau of shape λ/μ if and only if it satisfies the following two assertions:

Assertion $\mathcal{T}_1$: For any $(i,a) \in Y(\lambda/\mu)$ and $(i,b) \in Y(\lambda/\mu)$ with $a < b$, we have $\phi(i,a) \leq \phi(i,b)$.

Assertion $\mathcal{T}_2$: For any $(a,j) \in Y(\lambda/\mu)$ and $(b,j) \in Y(\lambda/\mu)$ with $a < b$, we have $\phi(a,j) < \phi(b,j)$.

Now, we shall prove the logical implication

$$\begin{aligned}&(\phi \text{ is a } (Y(\lambda/\mu), <_1, \prec)\text{-partition}) \\ \Longrightarrow\ &(\phi \text{ is a semistandard tableau of shape } \lambda/\mu).\end{aligned} \tag{120}$$

Proof of (120): Assume that ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition. We shall now prove that ϕ is a semistandard tableau of shape λ/μ.

We know that ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition if and only if ϕ satisfies Assertions $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus, ϕ satisfies Assertions $\mathcal{P}_1$ and $\mathcal{P}_2$ (since ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition).

We now notice that ϕ satisfies Assertion $\mathcal{T}_1$ [87] and satisfies Assertion $\mathcal{T}_2$ [88].

Recall that ϕ is a semistandard tableau of shape λ/μ if and only if it satisfies Assertions $\mathcal{T}_1$ and $\mathcal{T}_2$. Thus, ϕ is a semistandard tableau of shape λ/μ (since ϕ satisfies Assertions $\mathcal{T}_1$ and $\mathcal{T}_2$).

Now, forget that we have assumed that ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition. We thus have shown that if ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition, then ϕ is a semistandard tableau of shape λ/μ. Thus, the implication (120) is proven.

Let us next prove the logical implication

$$\begin{aligned}&(\phi \text{ is a semistandard tableau of shape } \lambda/\mu) \\ \Longrightarrow\ &(\phi \text{ is a } (Y(\lambda/\mu), <_1, \prec)\text{-partition}).\end{aligned} \tag{121}$$

Proof of (121): Assume that ϕ is a semistandard tableau of shape λ/μ. We shall now prove that ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition.

Recall that ϕ is a semistandard tableau of shape λ/μ if and only if it satisfies Assertions $\mathcal{T}_1$ and $\mathcal{T}_2$. Thus, ϕ satisfies Assertions $\mathcal{T}_1$ and $\mathcal{T}_2$ (since ϕ is a semistandard tableau of shape λ/μ).

Let us make some simple observations:

- For any $(i,a) \in Y(\lambda/\mu)$ and $(i,b) \in Y(\lambda/\mu)$ with $a \leq b$, we have

$$\phi(i,a) \leq \phi(i,b) \tag{122}$$

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[87] *Proof.* Let $(i,a) \in Y(\lambda/\mu)$ and $(i,b) \in Y(\lambda/\mu)$ be such that $a < b$. From $a < b$, we obtain $a \neq b$, hence $(i,a) \neq (i,b)$. Also, $a \leq b$ (since $a < b$).

The definition of the relation $<_1$ shows that $(i,a) <_1 (i,b)$ holds if and only if $(i \leq i \text{ and } a \leq b \text{ and } (i,a) \neq (i,b))$. Thus, $(i,a) <_1 (i,b)$ holds (since we have $(i \leq i \text{ and } a \leq b \text{ and } (i,a) \neq (i,b))$). But we know that ϕ satisfies Assertion $\mathcal{P}_1$. Thus, Assertion $\mathcal{P}_1$ (applied to $e = (i,a)$ and $f = (i,b)$) shows that $\phi(i,a) \leq \phi(i,b)$ (since $(i,a) <_1 (i,b)$).

Now, forget that we fixed (i,a) and (i,b). We thus have shown that for any $(i,a) \in Y(\lambda/\mu)$ and $(i,b) \in Y(\lambda/\mu)$ with $a < b$, we have $\phi(i,a) \leq \phi(i,b)$. In other words, ϕ satisfies Assertion $\mathcal{T}_1$.

[88] *Proof.* Let $(a,j) \in Y(\lambda/\mu)$ and $(b,j) \in Y(\lambda/\mu)$ be such that $a < b$. From $a < b$, we obtain $a \neq b$, hence $(a,j) \neq (b,j)$. Also, $a \leq b$ (since $a < b$).

The definition of the relation $<_1$ shows that $(a,j) <_1 (b,j)$ holds if and only if $(a \leq b \text{ and } j \leq j \text{ and } (a,j) \neq (b,j))$. Thus, $(a,j) <_1 (b,j)$ holds (since we have $(a \leq b \text{ and } j \leq j \text{ and } (a,j) \neq (b,j))$). Also, $(b,j) \prec (a,j)$ (because of Condition $\mathcal{O}_2$ in the statement of Proposition 10.24). But we know that ϕ satisfies Assertion $\mathcal{P}_2$. Thus, Assertion $\mathcal{P}_2$ (applied to $e = (a,j)$ and $f = (b,j)$) shows that $\phi(a,j) < \phi(b,j)$ (since $(a,j) <_1 (b,j)$ and $(b,j) \prec (a,j)$).

Now, forget that we fixed (a,j) and (b,j). We thus have shown that for any $(a,j) \in Y(\lambda/\mu)$ and $(b,j) \in Y(\lambda/\mu)$ with $a < b$, we have $\phi(a,j) < \phi(b,j)$. In other words, ϕ satisfies Assertion $\mathcal{T}_2$.

[89].

- For any $(a,j) \in Y(\lambda/\mu)$ and $(b,j) \in Y(\lambda/\mu)$ with $a \leq b$, we have
$$\phi(a,j) \leq \phi(b,j) \tag{123}$$
[90].

Now, we can easily see that ϕ satisfies Assertion $\mathcal{P}_1$ [91] and Assertion $\mathcal{P}_2$ [92].

Recall that ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition if and only if ϕ satisfies Assertions $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus, ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition (since ϕ satisfies Assertions $\mathcal{P}_1$ and $\mathcal{P}_2$).

[89] *Proof of (122):* Let $(i,a) \in Y(\lambda/\mu)$ and $(i,b) \in Y(\lambda/\mu)$ be such that $a \leq b$. We must prove the inequality (122). If $a = b$, then this inequality holds (because if $a = b$, then $\phi\left(i, \underbrace{a}_{=b}\right) = \phi(i,b) \leq \phi(i,b)$). Hence, for the rest of this proof, we can WLOG assume that we don't have $a = b$. Assume this.

We have $a \neq b$ (since we don't have $a = b$). Combining this with $a \leq b$, we obtain $a < b$. Now, recall that ϕ satisfies Assertion $\mathcal{T}_1$. Hence, Assertion $\mathcal{T}_1$ shows that $\phi(i,a) \leq \phi(i,b)$. This proves (122).

[90] *Proof of (123):* Let $(a,j) \in Y(\lambda/\mu)$ and $(b,j) \in Y(\lambda/\mu)$ be such that $a \leq b$. We must prove the inequality (123). If $a = b$, then this inequality holds (because if $a = b$, then $\phi\left(\underbrace{a}_{=b}, j\right) = \phi(b,j) \leq \phi(b,j)$). Hence, for the rest of this proof, we can WLOG assume that we don't have $a = b$. Assume this.

We have $a \neq b$ (since we don't have $a = b$). Combining this with $a \leq b$, we obtain $a < b$. Now, recall that ϕ satisfies Assertion $\mathcal{T}_2$. Hence, Assertion $\mathcal{T}_2$ shows that $\phi(a,j) < \phi(b,j)$. Thus, $\phi(a,j) \leq \phi(b,j)$. This proves (123).

[91] *Proof.* Let $e \in Y(\lambda/\mu)$ and $f \in Y(\lambda/\mu)$ be such that $e <_1 f$. We shall prove that $\phi(e) \leq \phi(f)$.

We have $e \in Y(\lambda/\mu) \subseteq \{1,2,3,\ldots\}^2$. Hence, there exist two positive integers a_1 and b_1 such that $e = (a_1, b_1)$. Consider these a_1 and b_1. Thus, $(a_1,b_1) = e \in Y(\lambda/\mu)$.

We have $f \in Y(\lambda/\mu) \subseteq \{1,2,3,\ldots\}^2$. Hence, there exist two positive integers a_3 and b_3 such that $f = (a_3, b_3)$. Consider these a_3 and b_3. Thus, $(a_3,b_3) = f \in Y(\lambda/\mu)$.

We have $(a_1,b_1) = e <_1 f = (a_3,b_3)$. On the other hand, $(a_1,b_1) <_1 (a_3,b_3)$ holds if and only if ($a_1 \leq a_3$ and $b_1 \leq b_3$ and $(a_1,b_1) \neq (a_3,b_3)$) (by the definition of the relation $<_1$). Thus, we have ($a_1 \leq a_3$ and $b_1 \leq b_3$ and $(a_1,b_1) \neq (a_3,b_3)$) (since $(a_1,b_1) <_1 (a_3,b_3)$ holds).

Now, $a_1 \leq a_3 \leq a_3$ and $b_1 \leq b_1 \leq b_3$. Hence, (118) (applied to $(a_2,b_2) = (a_3,b_1)$) shows that $(a_3,b_1) \in Y(\lambda/\mu)$.

Now, (122) (applied to a_3, b_1 and b_3 instead of i, a and b) shows that $\phi(a_3,b_1) \leq \phi(a_3,b_3)$ (since $(a_3,b_1) \in Y(\lambda/\mu)$, $(a_3,b_3) \in Y(\lambda/\mu)$ and $b_1 \leq b_3$). Thus, $\phi(a_3,b_1) \leq \phi\underbrace{(a_3,b_3)}_{=f} = \phi(f)$.

Also, (123) (applied to a_1, a_3 and b_1 instead of a, b and j) shows that $\phi(a_1,b_1) \leq \phi(a_3,b_1)$ (since $(a_1,b_1) \in Y(\lambda/\mu)$, $(a_3,b_1) \in Y(\lambda/\mu)$ and $a_1 \leq a_3$). Thus, $\phi\underbrace{(e)}_{=(a_1,b_1)} = \phi(a_1,b_1) \leq \phi(a_3,b_1)$.

Now, $\phi(e) \leq \phi(a_3,b_1) \leq \phi(f)$.

Now, forget that we fixed e and f. We thus have shown that every $e \in Y(\lambda/\mu)$ and $f \in Y(\lambda/\mu)$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$. In other words, ϕ satisfies Assertion $\mathcal{P}_1$.

[92] *Proof.* Let $e \in Y(\lambda/\mu)$ and $f \in Y(\lambda/\mu)$ be such that $e <_1 f$ and $f \prec e$. We shall prove that $\phi(e) < \phi(f)$.

Assume the contrary. Thus, $\phi(e) \geq \phi(f)$.

We have $e \in Y(\lambda/\mu) \subseteq \{1,2,3,\ldots\}^2$. Hence, there exist two positive integers a_1 and b_1 such that $e = (a_1, b_1)$. Consider these a_1 and b_1. Thus, $(a_1,b_1) = e \in Y(\lambda/\mu)$.

Now, forget that we have assumed that ϕ is a semistandard tableau of shape λ/μ. We thus have shown that if ϕ is a semistandard tableau of shape λ/μ, then ϕ is a $(Y(\lambda/\mu), <_1, \prec)$-partition. Thus, the implication (121) is proven.

Now, combining the implications (120) and (121), we obtain the equivalence

$$\begin{aligned}&(\phi \text{ is a } (Y(\lambda/\mu), <_1, \prec)\text{-partition})\\ \iff &(\phi \text{ is a semistandard tableau of shape } \lambda/\mu).\end{aligned}$$

Thus, the equivalence (119) is proven.

Now, forget that we fixed ϕ. Thus, for every map $\phi : Y(\lambda/\mu) \to \{1,2,3,\ldots\}$, we have proven the equivalence (119). Thus, a map $\phi : Y(\lambda/\mu) \to \{1,2,3,\ldots\}$ is a $(Y(\lambda/\mu), <_1, \prec)$-partition if and only if it is a semistandard tableau of shape λ/μ. Hence, the $(Y(\lambda/\mu), <_1, \prec)$-partitions are

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We have $f \in Y(\lambda/\mu) \subseteq \{1,2,3,\ldots\}^2$. Hence, there exist two positive integers a_3 and b_3 such that $f = (a_3, b_3)$. Consider these a_3 and b_3. Thus, $(a_3, b_3) = f \in Y(\lambda/\mu)$.

We have $(a_1, b_1) = e <_1 f = (a_3, b_3)$. On the other hand, $(a_1, b_1) <_1 (a_3, b_3)$ holds if and only if $(a_1 \leq a_3 \text{ and } b_1 \leq b_3 \text{ and } (a_1, b_1) \neq (a_3, b_3))$ (by the definition of the relation $<_1$). Thus, we have $(a_1 \leq a_3 \text{ and } b_1 \leq b_3 \text{ and } (a_1, b_1) \neq (a_3, b_3))$ (since $(a_1, b_1) <_1 (a_3, b_3)$ holds).

Now, $a_1 \leq a_3 \leq a_3$ and $b_1 \leq b_1 \leq b_3$. Hence, (118) (applied to $(a_2, b_2) = (a_3, b_1)$) shows that $(a_3, b_1) \in Y(\lambda/\mu)$.

Now, (122) (applied to a_3, b_1 and b_3 instead of i, a and b) shows that $\phi(a_3, b_1) \leq \phi(a_3, b_3)$ (since $(a_3, b_1) \in Y(\lambda/\mu)$, $(a_3, b_3) \in Y(\lambda/\mu)$ and $b_1 \leq b_3$). Thus,

$$\begin{aligned}\phi(a_3, b_1) &\leq \phi\underbrace{(a_3, b_3)}_{=f} = \phi(f) \leq \phi\underbrace{(e)}_{=(a_1,b_1)} \qquad (\text{since } \phi(e) \geq \phi(f))\\ &= \phi(a_1, b_1).\end{aligned}$$

Now, assume (for the sake of contradiction) that $a_1 < a_3$. Recall that ϕ satisfies Assertion $\mathcal{T}_2$. Hence, Assertion $\mathcal{T}_2$ (applied to a_1, a_3 and b_1 instead of a, b and j) shows that $\phi(a_1, b_1) < \phi(a_3, b_1)$ (since $(a_1, b_1) \in Y(\lambda/\mu)$, $(a_3, b_1) \in Y(\lambda/\mu)$ and $a_1 < a_3$). This contradicts $\phi(a_3, b_1) \leq \phi(a_1, b_1)$. This contradiction shows that our assumption (that $a_1 < a_3$) was false. Hence, $a_1 \geq a_3$. Combining this with $a_1 \leq a_3$, we obtain $a_1 = a_3$. Thus, $\left(\underbrace{a_1}_{=a_3}, b_3\right) = (a_3, b_3) = f \in Y(\lambda/\mu)$.

If we had $b_1 = b_3$, then we would have $\left(\underbrace{a_1}_{=a_3}, \underbrace{b_1}_{=b_3}\right) = (a_3, b_3)$, which would contradict $(a_1, b_1) \neq (a_3, b_3)$. Hence, we cannot have $b_1 = b_3$. Thus, we have $b_1 \neq b_3$. Combined with $b_1 \leq b_3$, this shows that $b_1 < b_3$.

The relation $\prec$ is a strict partial order, and thus is antisymmetric.

However, recall that the relation $\prec$ satisfies Condition $\mathcal{O}_1$ in the statement of Proposition 10.24. This Condition $\mathcal{O}_1$ (applied to a_1, b_1 and b_3 instead of i, a and b) shows that $(a_1, b_1) \prec (a_1, b_3)$ (since $(a_1, b_1) \in Y(\lambda/\mu)$, $(a_1, b_3) \in Y(\lambda/\mu)$ and $b_1 < b_3$). Thus, $e = (a_1, b_1) \prec \left(\underbrace{a_1}_{=a_3}, b_3\right) = (a_3, b_3) = f$. Hence, we cannot have $f \prec e$ (since the relation $\prec$ is antisymmetric). This contradicts $f \prec e$. This contradiction proves that our assumption was wrong. Hence, the proof of $\phi(e) < \phi(f)$ is complete.

Now, forget that we fixed e and f. We thus have shown that every $e \in Y(\lambda/\mu)$ and $f \in Y(\lambda/\mu)$ satisfying $e <_1 f$ and $f \prec e$ satisfy $\phi(e) < \phi(f)$. In other words, ϕ satisfies Assertion $\mathcal{P}_2$.

precisely the semistandard tableaux of shape λ/μ (because both $(Y(\lambda/\mu), <_1, \prec)$-partitions and semistandard tableaux of shape λ/μ are maps from $Y(\lambda/\mu)$ to $\{1,2,3,\ldots\}$). $\square$

Proof of Proposition 10.23. Define the set $Y(\lambda/\mu)$ as in Example 3.3.

Define the relations $<_1$, $<_2$ and $<_h$ as in Example 3.3. It is straightforward to see that $<_1$, $<_2$ and $<_h$ are strict partial orders. The definition of $\mathbf{Y}(\lambda/\mu)$ yields $\mathbf{Y}(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_2)$. The definition of $\mathbf{Y}_h(\lambda/\mu)$ yields $\mathbf{Y}_h(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_h)$.

(a) It is straightforward to see that the relation $<_2$ satisfies the Conditions $\mathcal{O}_1$ and $\mathcal{O}_2$ in the statement of Proposition 10.24 (with $\prec$ replaced by $<_2$). Hence, Proposition 10.24 (applied to $<_2$ instead of $\prec$) shows that the $(Y(\lambda/\mu), <_1, <_2)$-partitions are precisely the semistandard tableaux of shape λ/μ. In other words, the $\mathbf{Y}(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ (since $\mathbf{Y}(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_2)$). This proves Proposition 10.23 (a).

(b) The proof of Proposition 10.23 (b) is analogous to that of Proposition 10.23 (a) (but now, the relation $<_h$ and the double poset $\mathbf{Y}_h(\lambda/\mu)$ replace the relation $<_2$ and the double poset $\mathbf{Y}(\lambda/\mu)$). $\square$

10.6. M_α as $\Gamma(\mathbf{E}, w)$

Next, let us prove the claim of Example 3.6 (b):

Proposition 10.26. Let $\ell \in \mathbb{N}$. Let $E = \{1, 2, \ldots, \ell\}$. Let $<_1$ be the restriction of the standard relation $<$ on $\mathbb{Z}$ to the subset E. (Thus, two elements e and f of E satisfy $e <_1 f$ if and only if $e < f$.) Let $>_1$ be the opposite relation of $<_1$. (Thus, two elements e and f of E satisfy $e >_1 f$ if and only if $f <_1 e$.) Let $\mathbf{E} = (E, <_1, >_1)$.

(a) Then, $\mathbf{E}$ is a special double poset.

(b) Let $w : E \to \{1, 2, 3, \ldots\}$ be any map. Set $\alpha = (w(1), w(2), \ldots, w(\ell))$. Then, α is a composition and satisfies $\Gamma(\mathbf{E}, w) = M_\alpha$.

Proof of Proposition 10.26. (a) The relation $<_1$ is a total order (since it is a restriction of the relation $<$ on $\mathbb{Z}$, which is a total order). Hence, the relation $>_1$ is a total order as well (since it is the opposite relation of the total order $<_1$). Thus, $(E, <_1, >_1)$ is a special double poset (by the definition of "special"). In other words, $\mathbf{E}$ is a special double poset (since $\mathbf{E} = (E, <_1, >_1)$). This proves Proposition 10.26 (a).

(b) The map w is a map $E \to \{1, 2, 3, \ldots\}$. In other words, the map w is a map $\{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, \ell\}$). Thus, $(w(1), w(2), \ldots, w(\ell))$ is a sequence of positive integers, i.e., a composition. In other words, α is a composition (since $\alpha = (w(1), w(2), \ldots, w(\ell))$).

We have $\alpha = (w(1), w(2), \ldots, w(\ell))$. Thus, the definition of M_α yields

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{w(1)} x_{i_2}^{w(2)} \cdots x_{i_\ell}^{w(\ell)}. \tag{124}$$

It remains to prove that $\Gamma(\mathbf{E}, w) = M_\alpha$. The order $>_1$ is an extension of the order $>_1$ (obviously). Thus, Proposition 10.21 (applied to $>_1$ instead of $<_2$) shows that the $\mathbf{E}$-partitions are precisely the strictly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

On the other hand, let $\mathcal{J}$ denote the set of all length-ℓ strictly increasing sequences of positive integers. In other words,

$$\mathcal{J} = \left\{ (i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid i_1 < i_2 < \cdots < i_\ell \right\}.$$

Thus,

$$\sum_{(i_1, i_2, \ldots, i_\ell) \in \mathcal{J}} = \sum_{\substack{(i_1, i_2, \ldots, i_\ell) \in \{1,2,3,\ldots\}^\ell; \\ i_1 < i_2 < \cdots < i_\ell}} = \sum_{i_1 < i_2 < \cdots < i_\ell}$$

(an equality between summation signs).

Let Z denote the set of all **E**-partitions.

For every $\phi \in Z$, we have $(\phi(1), \phi(2), \ldots, \phi(\ell)) \in \mathcal{J}$ [93]. Hence, we can define a map $\Phi : Z \to \mathcal{J}$ by

$$(\Phi(\phi) = (\phi(1), \phi(2), \ldots, \phi(\ell)) \qquad \text{for every } \phi \in Z).$$

Consider this map Φ. This map Φ is injective[94] and surjective[95]. In other words, the map Φ is

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[93] *Proof.* Let $\phi \in Z$. Thus, ϕ is an element of Z. In other words, ϕ is an **E**-partition (since Z is the set of all **E**-partitions). In other words, ϕ is a strictly increasing map from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$ (since the **E**-partitions are precisely the strictly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$). In other words, ϕ is a map $E \to \{1, 2, 3, \ldots\}$ which has the property that if e and f are two elements of E satisfying $e <_1 f$, then

$$\phi(e) < \phi(f) \tag{125}$$

(by the definition of a "strictly increasing map").

The map ϕ is a map $E \to \{1, 2, 3, \ldots\}$. In other words, the map ϕ is a map $\{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, \ell\}$). Hence, $(\phi(1), \phi(2), \ldots, \phi(\ell))$ is an element of $\{1, 2, 3, \ldots\}^\ell$.

Now, let i and j be two elements of $\{1, 2, \ldots, \ell\}$ satisfying $i < j$. The elements $\phi(i)$ and $\phi(j)$ are well-defined (since i and j belong to $\{1, 2, \ldots, \ell\} = E$). We furthermore have $i < j$. In other words, $i <_1 j$ (since the relation $<_1$ is the restriction of the standard relation $<$ on $\mathbb{Z}$ to the subset E). Thus, (125) (applied to $e = i$ and $f = j$) shows that $\phi(i) < \phi(j)$.

Now, forget that we fixed i and j. We thus have shown that if i and j are two elements of $\{1, 2, \ldots, \ell\}$ satisfying $i < j$, then $\phi(i) < \phi(j)$. In other words, $\phi(1) < \phi(2) < \cdots < \phi(\ell)$.

Now, $(\phi(1), \phi(2), \ldots, \phi(\ell))$ is an element of $\{1, 2, 3, \ldots\}^\ell$ and satisfies $\phi(1) < \phi(2) < \cdots < \phi(\ell)$. Hence,

$$(\phi(1), \phi(2), \ldots, \phi(\ell)) \in \left\{(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid i_1 < i_2 < \cdots < i_\ell\right\} = \mathcal{J},$$

qed.

[94] *Proof.* Let ϕ_1 and ϕ_2 be two elements of Z such that $\Phi(\phi_1) = \Phi(\phi_2)$. We shall show that $\phi_1 = \phi_2$.

The definition of Φ shows that $\Phi(\phi_1) = (\phi_1(1), \phi_1(2), \ldots, \phi_1(\ell))$. The definition of Φ shows that $\Phi(\phi_2) = (\phi_2(1), \phi_2(2), \ldots, \phi_2(\ell))$. Hence,

$$(\phi_1(1), \phi_1(2), \ldots, \phi_1(\ell)) = \Phi(\phi_1) = \Phi(\phi_2) = (\phi_2(1), \phi_2(2), \ldots, \phi_2(\ell)).$$

In other words, $\phi_1(i) = \phi_2(i)$ for each $i \in \{1, 2, \ldots, \ell\}$. In other words, $\phi_1(i) = \phi_2(i)$ for each $i \in E$ (since $E = \{1, 2, \ldots, \ell\}$). In other words, $\phi_1 = \phi_2$.

Now, forget that we fixed ϕ_1 and ϕ_2. We thus have shown that if ϕ_1 and ϕ_2 are two elements of Z such that $\Phi(\phi_1) = \Phi(\phi_2)$, then $\phi_1 = \phi_2$. In other words, the map Φ is injective, qed.

[95] *Proof.* Let $\mathbf{j} \in \mathcal{J}$. We shall show that $\mathbf{j} \in \Phi(Z)$.

We have $\mathbf{j} \in \mathcal{J} = \left\{(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid i_1 < i_2 < \cdots < i_\ell\right\}$. In other words, $\mathbf{j}$ has the form $(i_1, i_2, \ldots, i_\ell)$ for some $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell$ satisfying $i_1 < i_2 < \cdots < i_\ell$. Consider this $(i_1, i_2, \ldots, i_\ell)$. Thus, $\mathbf{j} = (i_1, i_2, \ldots, i_\ell)$.

We have $i_e \in \{1, 2, 3, \ldots\}$ for every $e \in \{1, 2, \ldots, \ell\}$ (since $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell$). In other words, $i_e \in \{1, 2, 3, \ldots\}$ for every $e \in E$ (since $E = \{1, 2, \ldots, \ell\}$). Thus, we can define a map $\phi : E \to \{1, 2, 3, \ldots\}$ by $(\phi(e) = i_e$ for every $e \in E)$. Consider this map ϕ.

We have $i_1 < i_2 < \cdots < i_\ell$. In other words, if e and f are two elements of $\{1, 2, \ldots, \ell\}$ such that $e < f$, then

$$i_e < i_f. \tag{126}$$

Let e and f be two elements of E satisfying $e <_1 f$. From $e <_1 f$, we obtain $e < f$ (since the relation $<_1$ is the restriction of the standard relation $<$ on $\mathbb{Z}$ to the subset E). Thus, (126) bijective. Thus, Φ is a bijection. In other words, the map

$$Z \to \mathcal{J}, \qquad \phi \mapsto (\phi(1), \phi(2), \ldots, \phi(\ell)) \tag{127}$$

is a bijection[96].

For every $\pi \in Z$, we have

$$\mathbf{x}_{\pi,w} = x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)} \tag{128}$$

[97].

shows that $i_e < i_f$. But the definition of ϕ shows that $\phi(e) = i_e$ and $\phi(f) = i_f$. Hence, $\phi(e) = i_e < i_f = \phi(f)$.

Now, forget that we fixed e and f. We thus have shown that if e and f are two elements of E satisfying $e <_1 f$, then $\phi(e) < \phi(f)$. In other words, ϕ is a strictly increasing map from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$ (by the definition of a "strictly increasing map"). In other words, ϕ is an $\mathbf{E}$-partition (since the $\mathbf{E}$-partitions are precisely the strictly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$). In other words, $\phi \in Z$ (since Z is the set of all $\mathbf{E}$-partitions).

We have $\phi(e) = i_e$ for every $e \in E$ (by the definition of ϕ). In other words, $\phi(e) = i_e$ for every $e \in \{1, 2, \ldots, \ell\}$ (since $E = \{1, 2, \ldots, \ell\}$).

Now, the definition of Φ yields

$$\begin{aligned} \Phi(\phi) &= (\phi(1), \phi(2), \ldots, \phi(\ell)) = (i_1, i_2, \ldots, i_\ell) \\ &\qquad (\text{since } \phi(e) = i_e \text{ for every } e \in \{1, 2, \ldots, \ell\}) \\ &= \mathbf{j}. \end{aligned}$$

Thus, $\mathbf{j} = \Phi\left(\underbrace{\phi}_{\in Z}\right) \in \Phi(Z)$.

Now, forget that we fixed $\mathbf{j}$. We thus have shown that $\mathbf{j} \in \Phi(Z)$ for every $\mathbf{j} \in \mathcal{J}$. In other words, $\mathcal{J} \subseteq \Phi(Z)$. In other words, the map Φ is surjective, qed.

[96]since the map (127) is the map Φ (because $\Phi(\phi) = (\phi(1), \phi(2), \ldots, \phi(\ell))$ for every $\phi \in Z$)

[97]*Proof of (128):* Let $\pi \in Z$. Then, the definition of $\mathbf{x}_{\pi,w}$ yields

$$\begin{aligned} \mathbf{x}_{\pi,w} &= \prod_{e \in E} x_{\pi(e)}^{w(e)} = \underbrace{\prod_{e \in \{1,2,\ldots,\ell\}}}_{=\prod_{e=1}^{\ell}} x_{\pi(e)}^{w(e)} \qquad (\text{since } E = \{1, 2, \ldots, \ell\}) \\ &= \prod_{e=1}^{\ell} x_{\pi(e)}^{w(e)} = x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)}. \end{aligned}$$

This proves (128).

Now, the definition of $\Gamma(\mathbf{E}, w)$ yields

$$\begin{aligned}
\Gamma(\mathbf{E}, w) &= \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}}_{\substack{=\sum_{\pi\in Z} \\ \text{(since } Z \text{ is the set of} \\ \text{all } \mathbf{E}\text{-partitions)}}} \mathbf{x}_{\pi,w} = \sum_{\pi\in Z} \underbrace{\mathbf{x}_{\pi,w}}_{\substack{= x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)} \\ \text{(by (128))}}} = \sum_{\pi\in Z} x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)} \\
&= \underbrace{\sum_{(i_1,i_2,\ldots,i_\ell)\in\mathcal{J}}}_{=\sum_{i_1<i_2<\cdots<i_\ell}} x_{i_1}^{w(1)} x_{i_2}^{w(2)} \cdots x_{i_\ell}^{w(\ell)} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } (i_1, i_2, \ldots, i_\ell) \text{ for } (\pi(1), \pi(2), \ldots, \pi(\ell)), \\ \text{since the map } Z \to \mathcal{J},\ \phi \mapsto (\phi(1), \phi(2), \ldots, \phi(\ell)) \\ \text{is a bijection} \end{array}\right) \\
&= \sum_{i_1<i_2<\cdots<i_\ell} x_{i_1}^{w(1)} x_{i_2}^{w(2)} \cdots x_{i_\ell}^{w(\ell)} = M_\alpha \qquad \text{(by (124))}.
\end{aligned}$$

This completes the proof of Proposition 10.26 (b). ☐

For future reference, let us state a consequence of Proposition 10.26:

Corollary 10.27. Let α be a composition. Then, there exist a set E, a special double poset $\mathbf{E} = (E, <_1, >_1)$, and a map $w : E \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}, w) = M_\alpha$.

Proof of Corollary 10.27. Write the composition α in the form $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Hence, $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition (since α is a composition). Therefore, $\alpha_i \in \{1, 2, 3, \ldots\}$ for every $i \in \{1, 2, \ldots, \ell\}$.

Define the set E, the relations $<_1$ and $<_1$, and the double poset $\mathbf{E} = (E, <_1, >_1)$ as in Proposition 10.26 (a). Then, Proposition 10.26 (a) shows that $\mathbf{E}$ is a special double poset.

Define a map $w : \{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ by $(w(i) = \alpha_i$ for every $i \in \{1, 2, \ldots, \ell\})$. (This is well-defined, since $\alpha_i \in \{1, 2, 3, \ldots\}$ for every $i \in \{1, 2, \ldots, \ell\}$). Then, w is a map $\{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$. In other words, w is a map $E \to \{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, \ell\}$).

Recall that $w(i) = \alpha_i$ for every $i \in \{1, 2, \ldots, \ell\}$. Thus,

$$(w(1), w(2), \ldots, w(\ell)) = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) = \alpha.$$

In other words, $\alpha = (w(1), w(2), \ldots, w(\ell))$. Proposition 10.26 (b) thus shows that α is a composition and satisfies $\Gamma(\mathbf{E}, w) = M_\alpha$.

We thus have constructed a set E, a special double poset $\mathbf{E} = (E, <_1, >_1)$, and a map $w : E \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}, w) = M_\alpha$. Hence, there exist a set E, a special double poset $\mathbf{E} = (E, <_1, >_1)$, and a map $w : E \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}, w) = M_\alpha$. This proves Corollary 10.27. ☐

10.7. Disjoint unions of double posets, and the algebra QSym

We shall now study disjoint unions of double posets. As a result of this study, we will give a new proof of the fact that QSym is a $\mathbf{k}$-algebra.

Let us first recall the classical definition of the disjoint union of several sets:

Definition 10.28. Let I be a set. For each $i \in I$, let E_i be a set. Then, we define a set $\bigsqcup_{i\in I} E_i$ by

$$\bigsqcup_{i\in I} E_i = \bigcup_{i\in I} (\{i\} \times E_i).$$

(Notice that the sets $\{i\} \times E_i$ for distinct $i \in I$ are disjoint.) The set $\bigsqcup_{i \in I} E_i$ is called the *disjoint union* of the sets E_i. Thus, an element of this disjoint union $\bigsqcup_{i \in I} E_i$ is a pair (i, e), where i is an element of I and where e is an element of E_i.

For each $j \in I$, we let inc_j be the map

$$E_j \to \bigsqcup_{i \in I} E_i, \qquad e \mapsto (j, e).$$

This map inc_j is injective, and satisfies $\operatorname{inc}_j(E_j) = \{j\} \times E_j$. This map inc_j is called the *canonical inclusion of the j-th term into the disjoint union $\bigsqcup_{i \in I} E_i$*.

When the sets E_i are disjoint, their disjoint union $\bigsqcup_{i \in I} E_i$ is often identified with their union $\bigcup\limits_{i \in I} E_i$ via the bijection

$$\bigsqcup_{i \in I} E_i \to \bigcup_{i \in I} E_i, \qquad (i, e) \mapsto e. \tag{129}$$

We shall not make this identification, however.

When the sets E_i are not disjoint, the map (129) is no longer a bijection (but just a surjection), and thus our above definition of $\bigsqcup_{i \in I} E_i$ really is the simplest way to define a "disjoint union" of these sets E_i (i.e., a big set into which each E_i is canonically embedded in such a way that all the embeddings have disjoint images).

We have thus defined the disjoint union of arbitrarily many sets. As a particular case of this construction, we can define the disjoint union of two sets:

Definition 10.29. Let E and F be two sets. Then, a set $E \sqcup F$ is defined as follows: Define a family $(E_i)_{i \in \{0,1\}}$ of sets by setting $E_0 = E$ and $E_1 = F$. Then, define $E \sqcup F$ to be the set $\bigsqcup_{i \in \{0,1\}} E_i$. Thus, explicitly, we have $E \sqcup F = (\{0\} \times E) \cup (\{1\} \times F)$. Again, this set $E \sqcup F$ is often identified with $E \cup F$ when E and F are already disjoint; we shall not make this identification, however.

Here are some fundamental properties of the disjoint union of sets:

Remark 10.30. Let I be a set. For each $i \in I$, let E_i be a set.

(a) The sets $\{i\} \times E_i$ for distinct $i \in I$ are disjoint.

(b) If $j \in I$ and $e \in E_j$, then $(j, e) \in \bigsqcup_{i \in I} E_i$.

(c) Let $x \in \bigsqcup_{i \in I} E_i$. Then, there exist an $i \in I$ and an $e \in E_i$ such that $x = (i, e)$.

A basic map associated to any disjoint union is its *index map*:

Definition 10.31. Let I be a set. For each $i \in I$, let E_i be a set. Define a map $\operatorname{ind} : \bigsqcup_{i \in I} E_i \to I$ as follows: Let $x \in \bigsqcup_{i \in I} E_i$. Then, there exist an $i \in I$ and an $e \in E_i$ such that $x = (i, e)$ (according to Remark 10.30 (c)). Consider these i and e. Clearly, (i, e) is uniquely determined by x (since $(i, e) = x$). Hence, both i and e are uniquely determined by x (since i and e are the two entries of the pair (i, e)). Set $\operatorname{ind}(x) = i$. Thus, a map $\operatorname{ind} : \bigsqcup_{i \in I} E_i \to I$ is defined.

We call ind the *index map* of the family $(E_i)_{i \in I}$.

Remark 10.32. Let I be a set. For each $i \in I$, let E_i be a set. For each $j \in I$ and $e \in E_j$, we have $\operatorname{ind}(j, e) = j$.

Proposition 10.33. Let I be a set. For each $i \in I$, let E_i be a set. Consider the map $\operatorname{ind} : \bigsqcup_{i \in I} E_i \to I$ defined in Definition 10.31.

Let $j \in I$. Then, the map

$$E_j \to \operatorname{ind}^{-1}(j), \qquad e \mapsto (j, e)$$

is well-defined and a bijection.

Remark 10.30, Remark 10.32 and Proposition 10.33 are standard facts about sets, and their proofs are straightforward; we will not dwell on them here any longer.

Here is a sample application of Proposition 10.33 that we will actually use:

Proposition 10.34. Let I be a finite set. For each $i \in I$, let E_i be a finite set.

(a) The set $\bigsqcup_{i \in I} E_i$ is finite.

(b) Let A be a commutative ring. Let $E = \bigsqcup_{i \in I} E_i$. Let $a : E \to A$ be a map. Then,

$$\prod_{e \in E} a(e) = \prod_{i \in I} \prod_{e \in E_i} a(\operatorname{inc}_i(e)).$$

An analogue of Proposition 10.34 (b) holds for sums instead of products, of course.

Proof of Proposition 10.34. (a) For each $i \in I$, the set $\{i\} \times E_i$ is finite (since it is the Cartesian product of the two finite sets $\{i\}$ and E_i). Hence, $\bigcup_{i \in I} (\{i\} \times E_i)$ is a finite union of finite sets (since I is finite), and thus itself a finite set. In other words, $\bigsqcup_{i \in I} E_i$ is a finite set (since $\bigsqcup_{i \in I} E_i = \bigcup_{i \in I} (\{i\} \times E_i)$ (by the definition of $\bigsqcup_{i \in I} E_i$)). This proves Proposition 10.34 (a).

(b) Let $i \in I$. Then, Proposition 10.33 (applied to $j = i$) shows that the map

$$E_i \to \operatorname{ind}^{-1}(i), \qquad e \mapsto (i, e)$$

is well-defined and a bijection. Thus, we can substitute (i, e) for g in the product $\prod_{g \in \operatorname{ind}^{-1}(i)} a(g)$. We thus obtain

$$\prod_{g \in \operatorname{ind}^{-1}(i)} a(g) = \prod_{e \in E_i} a \underbrace{(i, e)}_{\substack{= \operatorname{inc}_i(e) \\ (\text{since } \operatorname{inc}_i(e) = (i,e) \\ (\text{by the definition of } \operatorname{inc}_i))}} = \prod_{e \in E_i} a(\operatorname{inc}_i(e)). \tag{130}$$

Now, forget that we fixed i. We thus have proven (130) for every $i \in I$.

We have

$$\begin{aligned}
\prod_{e \in E} a(e) &= \prod_{g \in E} a(g) \qquad (\text{here, we have renamed the index } e \text{ as } g \text{ in the product}) \\
&= \prod_{i \in I} \underbrace{\prod_{\substack{g \in E; \\ \operatorname{ind} g = i}}}_{= \prod_{g \in \operatorname{ind}^{-1}(i)}} a(g) \qquad (\text{since } \operatorname{ind} g \in I \text{ for each } g \in E) \\
&= \prod_{i \in I} \underbrace{\prod_{g \in \operatorname{ind}^{-1}(i)} a(g)}_{\substack{= \prod_{e \in E_i} a(\operatorname{inc}_i(e)) \\ (\text{by (130)})}} = \prod_{i \in I} \prod_{e \in E_i} a(\operatorname{inc}_i(e)).
\end{aligned}$$

This proves Proposition 10.34 (b). ◻

The following fact is the universal property of disjoint unions:

Proposition 10.35. Let I be a set. For each $i \in I$, let E_i be a set. Let F be a set. For each $i \in I$, let $f_i : E_i \to F$ be a map. Then, there exists a unique map $f : \bigsqcup_{i \in I} E_i \to F$ such that every $j \in I$ satisfies $f_j = f \circ \operatorname{inc}_j$.

Proof of Proposition 10.35. Proposition 10.35 is another basic property of sets that we are not going to prove; let us merely exhibit the unique map $f : \bigsqcup_{i \in I} E_i \to F$ such that every $j \in I$ satisfies $f_j = f \circ \operatorname{inc}_j$. Namely, this is the map

$$\bigsqcup_{i \in I} E_i \to F, \qquad (i, e) \mapsto f_i(e).$$

□

Definition 10.36. Let I be a set. For each $i \in I$, let E_i be a set. Let F be a set. We define a map

$$\operatorname{Restr} : F^{\bigsqcup_{i \in I} E_i} \to \prod_{i \in I} F^{E_i}$$

by

$$\left(\operatorname{Restr}(f) = (f \circ \operatorname{inc}_i)_{i \in I} \qquad \text{for every } f \in F^{\bigsqcup_{i \in I} E_i}\right).$$

Corollary 10.37. Let I be a set. For each $i \in I$, let E_i be a set. Let F be a set. Then, the map $\operatorname{Restr} : F^{\bigsqcup_{i \in I} E_i} \to \prod_{i \in I} F^{E_i}$ is a bijection.

Proof of Corollary 10.37. The map $\operatorname{Restr} : F^{\bigsqcup_{i\in I} E_i} \to \prod_{i\in I} F^{E_i}$ is injective[98] and surjective[99]. Thus, this map Restr is bijective, i.e., a bijection. This proves Corollary 10.37. □

Next, let us define the direct sum of relations:

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[98] *Proof.* Let α and β be two elements of $F^{\bigsqcup_{i\in I} E_i}$ such that $\operatorname{Restr}\alpha = \operatorname{Restr}\beta$. We shall prove that $\alpha = \beta$.

Both α and β are elements of $F^{\bigsqcup_{i\in I} E_i}$, hence are maps $\bigsqcup_{i\in I} E_i \to F$.

We have $\operatorname{Restr}\alpha = (\alpha \circ \operatorname{inc}_i)_{i\in I}$ (by the definition of Restr) and $\operatorname{Restr}\beta = (\beta \circ \operatorname{inc}_i)_{i\in I}$ (by the definition of Restr). Hence,

$$(\alpha \circ \operatorname{inc}_i)_{i\in I} = \operatorname{Restr}\alpha = \operatorname{Restr}\beta = (\beta \circ \operatorname{inc}_i)_{i\in I}.$$

In other words,

$$\alpha \circ \operatorname{inc}_i = \beta \circ \operatorname{inc}_i \qquad \text{for every } i \in I. \tag{131}$$

Now, let $x \in \bigsqcup_{i\in I} E_i$. We shall prove that $\alpha(x) = \beta(x)$.

There exist an $i \in I$ and an $e \in E_i$ such that $x = (i, e)$ (by Remark 10.30 (c)). Consider these i and e. The definition of inc_i shows that $\operatorname{inc}_i(e) = (i, e) = x$. Now,

$$(\alpha \circ \operatorname{inc}_i)(e) = \alpha\left(\underbrace{\operatorname{inc}_i(e)}_{=x}\right) = \alpha(x),$$

so that

$$\alpha(x) = \underbrace{(\alpha \circ \operatorname{inc}_i)}_{\substack{=\beta\circ\operatorname{inc}_i \\ \text{(by (131))}}}(e) = (\beta \circ \operatorname{inc}_i)(e) = \beta\left(\underbrace{\operatorname{inc}_i(e)}_{=x}\right) = \beta(x).$$

Now, forget that we fixed x. We thus have proven that $\alpha(x) = \beta(x)$ for every $x \in \bigsqcup_{i\in I} E_i$. In other words, $\alpha = \beta$.

Now, forget that we fixed α and β. We thus have shown that if α and β are two elements of $F^{\bigsqcup_{i\in I} E_i}$ such that $\operatorname{Restr}\alpha = \operatorname{Restr}\beta$, then $\alpha = \beta$. In other words, the map Restr is injective. Qed.

[99] *Proof.* Let $\sigma \in \prod_{i\in I} F^{E_i}$. Thus, σ can be written in the form $(f_i)_{i\in I}$, where f_i is an element of F^{E_i} for every $i \in I$. Consider these f_i. Thus, $\sigma = (f_i)_{i\in I}$.

For each $i \in I$, the element f_i is an element of F^{E_i}, thus a map from E_i to F. Thus, Proposition 10.35 shows that there exists a unique map $f : \bigsqcup_{i\in I} E_i \to F$ such that every $j \in I$ satisfies $f_j = f \circ \operatorname{inc}_j$. Consider this f.

Every $j \in I$ satisfies $f_j = f \circ \operatorname{inc}_j$. Renaming j as i in this statement, we obtain the following: Every $i \in I$ satisfies $f_i = f \circ \operatorname{inc}_i$. In other words, $(f_i)_{i\in I} = (f \circ \operatorname{inc}_i)_{i\in I}$.

But f is a map $\bigsqcup_{i\in I} E_i \to F$, thus an element of $F^{\bigsqcup_{i\in I} E_i}$. Hence, $\operatorname{Restr}(f)$ is well-defined. The definition of Restr shows that $\operatorname{Restr}(f) = (f \circ \operatorname{inc}_i)_{i\in I}$. Comparing this with $\sigma = (f_i)_{i\in I} = (f \circ \operatorname{inc}_i)_{i\in I}$, we obtain $\sigma = \operatorname{Restr}(f) \in \operatorname{Restr}\left(F^{\bigsqcup_{i\in I} E_i}\right)$.

Now, forget that we fixed σ. We thus have shown that $\sigma \in \operatorname{Restr}\left(F^{\bigsqcup_{i\in I} E_i}\right)$ for every $\sigma \in \prod_{i\in I} F^{E_i}$. In other words, $\prod_{i\in I} F^{E_i} \subseteq \operatorname{Restr}\left(F^{\bigsqcup_{i\in I} E_i}\right)$. In other words, the map Restr is surjective. Qed.

Definition 10.38. Let I be a set. For each $i \in I$, let E_i be a set. For each $i \in I$, let ρ_i be a binary relation on the set E_i. We shall write the relation ρ_i in infix notation (i.e., we shall write $e \rho_i f$ instead of $(e, f) \in \rho_i$ when we want to say that two elements e and f of E_i are related by ρ). We define a binary relation ρ on the set $\bigsqcup_{i \in I} E_i$ (again, written in infix notation) by the rule

$$\left(\begin{array}{c} ((j, e) \, \rho \, (k, f)) \iff (j = k \text{ and } e \rho_j f) \\ \text{for any two elements } (j, e) \text{ and } (k, f) \text{ of } \bigsqcup_{i \in I} E_i \end{array} \right).$$

This relation ρ will be denoted by $\bigoplus_{i \in I} \rho_i$ and called the *direct sum* of the relations ρ_i.

Notice that we are denoting the relation ρ in Definition 10.38 by $\bigoplus_{i \in I} \rho_i$ and not by $\bigsqcup_{i \in I} \rho_i$. The (rather pedantic) reason for this is that the relations ρ_i are sets (of pairs of elements of E_i), and thus the expression $\bigsqcup_{i \in I} \rho_i$ already has a meaning, which is not the meaning we want to give $\bigoplus_{i \in I} \rho_i$.

Proposition 10.39. Let I be a set. For each $i \in I$, let E_i be a set. For each $i \in I$, let ρ_i be a strict partial order on the set E_i. Then, $\bigoplus_{i \in I} \rho_i$ is a strict partial order on the set $\bigsqcup_{i \in I} E_i$.

Proof of Proposition 10.39. Let us denote the relation $\bigoplus_{i \in I} \rho_i$ by ρ. Thus, ρ is a binary relation on the set $\bigsqcup_{i \in I} E_i$.

We shall write the relations ρ_i and also the relations $\bigoplus_{i \in I} \rho_i$ and ρ in infix notation.

The definition of $\bigoplus_{i \in I} \rho_i$ shows that we have the equivalence

$$\left((j, e) \left(\bigoplus_{i \in I} \rho_i \right) (k, f) \right) \iff (j = k \text{ and } e \rho_j f)$$

for any two elements (j, e) and (k, f) of $\bigsqcup_{i \in I} E_i$. In other words, we have the equivalence

$$((j, e) \, \rho \, (k, f)) \iff (j = k \text{ and } e \rho_j f) \qquad (132)$$

for any two elements (j, e) and (k, f) of $\bigsqcup_{i \in I} E_i$ (since $\rho = \bigoplus_{i \in I} \rho_i$).

The relation ρ is transitive[100], irreflexive[101] and antisymmetric[102]. Thus, ρ is an irreflexive, tran-

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[100] *Proof.* Let u, v and w be three elements of $\bigsqcup_{i\in I} E_i$ such that $u\rho v$ and $v\rho w$. We shall prove that $u\rho w$.

Remark 10.30 (c) (applied to $x = u$) shows that there exist an $i \in I$ and an $e \in E_i$ such that $u = (i, e)$. Denote these i and e by a and α. Thus, $a \in I$ and $\alpha \in E_a$ are such that $u = (a, \alpha)$. Thus, $(a, \alpha) = u \in \bigsqcup_{i\in I} E_i$.

Remark 10.30 (c) (applied to $x = v$) shows that there exist an $i \in I$ and an $e \in E_i$ such that $v = (i, e)$. Denote these i and e by b and β. Thus, $b \in I$ and $\beta \in E_b$ are such that $v = (b, \beta)$. Thus, $(b, \beta) = v \in \bigsqcup_{i\in I} E_i$.

Remark 10.30 (c) (applied to $x = w$) shows that there exist an $i \in I$ and an $e \in E_i$ such that $w = (i, e)$. Denote these i and e by c and γ. Thus, $c \in I$ and $\gamma \in E_c$ are such that $w = (c, \gamma)$. Thus, $(c, \gamma) = w \in \bigsqcup_{i\in I} E_i$.

We have $u\rho v$. This rewrites as $(a, \alpha)\,\rho\,(b, \beta)$ (since $u = (a, \alpha)$ and $v = (b, \beta)$). But (132) (applied to $(j, e) = (a, \alpha)$ and $(k, f) = (b, \beta)$) shows that we have the equivalence

$$((a, \alpha)\,\rho\,(b, \beta)) \iff (a = b \text{ and } \alpha\rho_a\beta).$$

Thus, we have $(a = b \text{ and } \alpha\rho_a\beta)$ (since we have $(a, \alpha)\,\rho\,(b, \beta)$).

We have $v\rho w$. This rewrites as $(b, \beta)\,\rho\,(c, \gamma)$ (since $v = (b, \beta)$ and $w = (c, \gamma)$). But (132) (applied to $(j, e) = (b, \beta)$ and $(k, f) = (c, \gamma)$) shows that we have the equivalence

$$((b, \beta)\,\rho\,(c, \gamma)) \iff (b = c \text{ and } \beta\rho_b\gamma).$$

Thus, we have $(b = c \text{ and } \beta\rho_b\gamma)$ (since we have $(b, \beta)\,\rho\,(c, \gamma)$). This rewrites as $(a = c \text{ and } \beta\rho_a\gamma)$ (since $a = b$).

The relation ρ_a is a strict partial order on the set E_a (since ρ_i is a strict partial order on the set E_i for each $i \in I$), and thus is transitive. Hence, from $\alpha\rho_a\beta$ and $\beta\rho_a\gamma$, we obtain $\alpha\rho_a\gamma$. Hence, $(a = c \text{ and } \alpha\rho_a\gamma)$. But (132) (applied to $(j, e) = (a, \alpha)$ and $(k, f) = (c, \gamma)$) shows that we have the equivalence

$$((a, \alpha)\,\rho\,(c, \gamma)) \iff (a = c \text{ and } \alpha\rho_a\gamma).$$

Thus, we have $(a, \alpha)\,\rho\,(c, \gamma)$ (since we have $(a = c \text{ and } \alpha\rho_a\gamma)$). This rewrites as $u\rho w$ (since $u = (a, \alpha)$ and $w = (c, \gamma)$).

Now, forget that we fixed u, v and w. We thus have proven that if u, v and w are three elements of $\bigsqcup_{i\in I} E_i$ such that $u\rho v$ and $v\rho w$, then $u\rho w$. In other words, the relation ρ is transitive, qed.

[101] *Proof.* Let $u \in \bigsqcup_{i\in I} E_i$ be such that $u\rho u$. We shall derive a contradiction.

Remark 10.30 (c) (applied to $x = u$) shows that there exist an $i \in I$ and an $e \in E_i$ such that $u = (i, e)$. Denote these i and e by a and α. Thus, $a \in I$ and $\alpha \in E_a$ are such that $u = (a, \alpha)$. Thus, $(a, \alpha) = u \in \bigsqcup_{i\in I} E_i$.

We have $u\rho u$. This rewrites as $(a, \alpha)\,\rho\,(a, \alpha)$ (since $u = (a, \alpha)$). Hence, (132) (applied to $(j, e) = (a, \alpha)$ and $(k, f) = (a, \alpha)$) shows that we have the equivalence

$$((a, \alpha)\,\rho\,(a, \alpha)) \iff (a = a \text{ and } \alpha\rho_a\alpha).$$

Thus, we have $(a = a \text{ and } \alpha\rho_a\alpha)$ (since we have $(a, \alpha)\,\rho\,(a, \alpha)$). Hence, $\alpha\rho_a\alpha$. Thus, there exists an $e \in E_a$ such that $e\rho_a e$ (namely, $e = \alpha$).

The relation ρ_a is a strict partial order on the set E_a (since ρ_i is a strict partial order on the set E_i for each $i \in I$), and thus is irreflexive. Hence, there exists no $e \in E_a$ such that $e\rho_a e$. This contradicts the fact that there exists an $e \in E_a$ such that $e\rho_a e$.

Now, forget that we fixed u. We thus have derived a contradiction for each $u \in \bigsqcup_{i\in I} E_i$ satisfying $u\rho u$. Hence, there exists no $u \in \bigsqcup_{i\in I} E_i$ satisfying $u\rho u$. In other words, the relation ρ is irreflexive, qed.

sitive and antisymmetric binary relation on the set $\bigsqcup_{i\in I} E_i$. In other words, ρ is a strict partial order on the set $\bigsqcup_{i\in I} E_i$ (because this is how strict partial orders are defined). In other words, $\bigoplus_{i\in I} \rho_i$ is a strict partial order on the set $\bigsqcup_{i\in I} E_i$ (since $\rho = \bigoplus_{i\in I} \rho_i$). This proves Proposition 10.39. □

Now, we can define the disjoint union of double posets:

Proposition 10.40. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. Then, $\left(\bigsqcup_{i\in I} E_i, \bigoplus_{i\in I} (<_{1,i}), \bigoplus_{i\in I} (<_{2,i})\right)$ is a double poset.

Proof of Proposition 10.40. For each $i \in I$, we know that E_i is a finite set (since $(E_i, <_{1,i}, <_{2,i})$ is a double poset), and that $<_{1,i}$ and $<_{2,i}$ are two strict partial orders on the set E_i (for the same reason).

For each $i \in I$, the relation $<_{1,i}$ is a strict partial order on the set E_i. Thus, $\bigoplus_{i\in I} (<_{1,i})$ is a strict partial order on the set $\bigsqcup_{i\in I} E_i$ (by Proposition 10.39, applied to $<_{1,i}$ instead of ρ_i). Similarly, $\bigoplus_{i\in I} (<_{2,i})$ is a strict partial order on the set $\bigsqcup_{i\in I} E_i$.

Proposition 10.34 (a) shows that the set $\bigsqcup_{i\in I} E_i$ is finite. Thus, $\bigsqcup_{i\in I} E_i$ is a finite set, and $\bigoplus_{i\in I} (<_{1,i})$ and $\bigoplus_{i\in I} (<_{2,i})$ are two strict partial orders on the set $\bigsqcup_{i\in I} E_i$. In other words, $\left(\bigsqcup_{i\in I} E_i, \bigoplus_{i\in I} (<_{1,i}), \bigoplus_{i\in I} (<_{2,i})\right)$ is a double poset (by the definition of a "double poset"). This proves Proposition 10.40. □

Definition 10.41. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. Proposition 10.40 shows that $\left(\bigsqcup_{i\in I} E_i, \bigoplus_{i\in I} (<_{1,i}), \bigoplus_{i\in I} (<_{2,i})\right)$ is a double poset. We denote this double poset by $\bigsqcup_{i\in I} \mathbf{E}_i$. We call it the *disjoint union* of the double posets $\mathbf{E}_i$.

In particular, let us define the disjoint union of two double posets:

Definition 10.42. Let $\mathbf{E}$ and $\mathbf{F}$ be two double posets. Then, a double poset $\mathbf{E} \sqcup \mathbf{F}$ is defined as follows: Define a family $(\mathbf{E}_i)_{i\in\{0,1\}}$ of double posets by setting $\mathbf{E}_0 = \mathbf{E}$ and $\mathbf{E}_1 = \mathbf{F}$. Then, define $\mathbf{E} \sqcup \mathbf{F}$ to be the double poset $\bigsqcup_{i\in\{0,1\}} \mathbf{E}_i$.

The next proposition characterizes the $\bigsqcup_{i\in I} \mathbf{E}_i$-partitions when $(\mathbf{E}_i)_{i\in I}$ is a finite family of double posets:

Proposition 10.43. For every double poset $\mathbf{E}$, let $\operatorname{Par}\mathbf{E}$ be the set of all $\mathbf{E}$-partitions.

Let $F = \{1, 2, 3, \ldots\}$. Thus, $\operatorname{Par}\mathbf{E} \subseteq F^E$ for each double poset $\mathbf{E} = (E, <_1, <_2)$.

Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. Corollary 10.37 shows that the map $\operatorname{Restr} : F^{\bigsqcup_{i\in I} E_i} \to \prod_{i\in I} F^{E_i}$ is a bijection. Let $\phi \in F^{\bigsqcup_{i\in I} E_i}$.

(a) Then, we have the following logical equivalence:

$$\left(\phi \in \operatorname{Par}\left(\bigsqcup_{i\in I} \mathbf{E}_i\right)\right) \iff \left(\operatorname{Restr}(\phi) \in \prod_{i\in I} \operatorname{Par}(\mathbf{E}_i)\right).$$

(b) Let $w : \bigsqcup_{i\in I} E_i \to \{1, 2, 3, \ldots\}$ be a map. Then,

$$\mathbf{x}_{\phi,w} = \prod_{i\in I} \mathbf{x}_{\phi\circ\operatorname{inc}_i, w\circ\operatorname{inc}_i}.$$

[102]*Proof.* Every binary relation which is transitive and irreflexive must be antisymmetric (this is well-known). Applying this to the binary relation ρ, we conclude that ρ is antisymmetric (since ρ is transitive and irreflexive). Qed.

Proof of Proposition 10.43. Let $\mathbf{E}$ denote the double poset $\bigsqcup_{i \in I} \mathbf{E}_i$. Thus, $\mathbf{E} = \bigsqcup_{i \in I} \mathbf{E}_i = \left(\bigsqcup_{i \in I} E_i, \bigoplus_{i \in I} (<_{1,i}), \bigoplus_{i \in I} (<_{2,i})\right)$ (by the definition of $\bigsqcup_{i \in I} \mathbf{E}_i$).

Let E be the set $\bigsqcup_{i \in I} E_i$. The element ϕ is an element of $F^{\bigsqcup_{i \in I} E_i} = F^E$ (since $\bigsqcup_{i \in I} E_i = E$), thus a map from E to F. In other words, ϕ is a map from E to $\{1, 2, 3, \ldots\}$ (since $F = \{1, 2, 3, \ldots\}$).

Let $<_1$ denote the binary relation $\bigoplus_{i \in I} (<_{1,i})$. Thus, $\bigoplus_{i \in I} (<_{1,i}) = (<_1)$.

Let $<_2$ denote the binary relation $\bigoplus_{i \in I} (<_{2,i})$. Thus, $\bigoplus_{i \in I} (<_{2,i}) = (<_2)$.

(a) Now, ϕ is a map $E \to \{1, 2, 3, \ldots\}$, and we have

$$\mathbf{E} = \left(\underbrace{\bigsqcup_{i \in I} E_i}_{= E}, \underbrace{\bigoplus_{i \in I} (<_{1,i})}_{= (<_1)}, \underbrace{\bigoplus_{i \in I} (<_{2,i})}_{= (<_2)} \right) = (E, <_1, <_2).$$

Hence, the definition of an "$\mathbf{E}$-partition" shows that ϕ is an $\mathbf{E}$-partition if and only if ϕ satisfies the following two conditions:

Condition $\mathcal{P}_1$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$.

Condition $\mathcal{P}_2$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

We have the following chain of logical equivalences:

$$\begin{aligned}
(\phi \in \operatorname{Par} \mathbf{E}) &\iff (\phi \text{ belongs to } \operatorname{Par} \mathbf{E}) \\
&\iff (\phi \text{ is an } \mathbf{E}\text{-partition}) \\
&\qquad (\text{since } \operatorname{Par} \mathbf{E} \text{ is the set of all } \mathbf{E}\text{-partitions}) \\
&\iff (\phi \text{ satisfies Conditions } \mathcal{P}_1 \text{ and } \mathcal{P}_2)
\end{aligned}$$

(since ϕ is an $\mathbf{E}$-partition if and only if ϕ satisfies Conditions $\mathcal{P}_1$ and $\mathcal{P}_2$). Thus, we have the following chain of logical equivalences:

$$\begin{aligned}
&(\phi \in \operatorname{Par} \mathbf{E}) \\
&\iff (\phi \text{ satisfies Conditions } \mathcal{P}_1 \text{ and } \mathcal{P}_2) \\
&\iff (\text{Conditions } \mathcal{P}_1 \text{ and } \mathcal{P}_2 \text{ hold}) \\
&\iff (\text{Condition } \mathcal{P}_1 \text{ holds}) \wedge (\text{Condition } \mathcal{P}_2 \text{ holds}). \qquad (133)
\end{aligned}$$

On the other hand, let $j \in I$. Then, inc_j is a map $E_j \to \bigsqcup_{i \in I} E_i$. In other words, inc_j is a map $E_j \to E$ (since $\bigsqcup_{i \in I} E_i = E$). Hence, $\phi \circ \operatorname{inc}_j$ is a map $E_j \to \{1, 2, 3, \ldots\}$ (since inc_j is a map $E_j \to E$, and since ϕ is a map $E \to \{1, 2, 3, \ldots\}$).

Furthermore, $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ is a double poset for each $i \in I$. Applying this to $i = j$, we see that $\mathbf{E}_j = (E_j, <_{1,j}, <_{2,j})$ is a double poset.

So we know that $\phi \circ \operatorname{inc}_j$ is a map $E_j \to \{1, 2, 3, \ldots\}$, and we have $\mathbf{E}_j = (E_j, <_{1,j}, <_{2,j})$. Hence, the definition of an "$\mathbf{E}_j$-partition" shows that $\phi \circ \operatorname{inc}_j$ is an $\mathbf{E}_j$-partition if and only if $\phi \circ \operatorname{inc}_j$ satisfies the following two conditions:

Condition $\mathcal{Q}_1(j)$: Every $e \in E_j$ and $f \in E_j$ satisfying $e <_{1,j} f$ satisfy $(\phi \circ \operatorname{inc}_j)(e) \leq (\phi \circ \operatorname{inc}_j)(f)$.

Condition $\mathcal{Q}_2(j)$: Every $e \in E_j$ and $f \in E_j$ satisfying $e <_{1,j} f$ and $f <_{2,j} e$ satisfy $(\phi \circ \operatorname{inc}_j)(e) < (\phi \circ \operatorname{inc}_j)(f)$.

We have the following chain of logical equivalences:

$$\begin{aligned}
&(\phi \circ \operatorname{inc}_j \in \operatorname{Par}(\mathbf{E}_j)) \\
&\iff (\phi \circ \operatorname{inc}_j \text{ belongs to } \operatorname{Par}(\mathbf{E}_j)) \\
&\iff (\phi \circ \operatorname{inc}_j \text{ is an } \mathbf{E}_j\text{-partition}) \\
&\qquad (\text{since } \operatorname{Par}(\mathbf{E}_j) \text{ is the set of all } \mathbf{E}_j\text{-partitions}) \\
&\iff (\phi \circ \operatorname{inc}_j \text{ satisfies Conditions } \mathcal{Q}_1(j) \text{ and } \mathcal{Q}_2(j)) \qquad (134)
\end{aligned}$$

(since $\phi \circ \operatorname{inc}_j$ is an $\mathbf{E}$-partition if and only if $\phi \circ \operatorname{inc}_j$ satisfies Conditions $\mathcal{Q}_1(j)$ and $\mathcal{Q}_2(j)$).

Now, let us forget that we fixed j. We thus have introduced two Conditions $\mathcal{Q}_1(j)$ and $\mathcal{Q}_2(j)$ for each $j \in I$, and we have proven the logical equivalence (134) for each $j \in I$.

The definition of Restr yields $\operatorname{Restr}(\phi) = (\phi \circ \operatorname{inc}_i)_{i \in I}$. Now, we have the following chain of logical equivalences:

$$\begin{aligned}
& \left(\underbrace{\operatorname{Restr}(\phi)}_{=(\phi \circ \operatorname{inc}_i)_{i \in I}} \in \prod_{i \in I} \operatorname{Par}(\mathbf{E}_i) \right) \\
\Longleftrightarrow\ & \left((\phi \circ \operatorname{inc}_i)_{i \in I} \in \prod_{i \in I} \operatorname{Par}(\mathbf{E}_i) \right) \\
\Longleftrightarrow\ & \left(\phi \circ \operatorname{inc}_i \in \operatorname{Par}(\mathbf{E}_i) \text{ for each } i \in I \right) \\
\Longleftrightarrow\ & \left(\underbrace{\phi \circ \operatorname{inc}_j \in \operatorname{Par}(\mathbf{E}_j)}_{\substack{\Longleftrightarrow (\phi \circ \operatorname{inc}_j \text{ satisfies Conditions } \mathcal{Q}_1(j) \text{ and } \mathcal{Q}_2(j)) \\ \text{(by (134))}}} \text{ for each } j \in I \right) \\
& \text{(here, we have renamed the index } i \text{ as } j\text{)} \\
\Longleftrightarrow\ & \left(\underbrace{\phi \circ \operatorname{inc}_j \text{ satisfies Conditions } \mathcal{Q}_1(j) \text{ and } \mathcal{Q}_2(j)}_{\Longleftrightarrow (\text{Conditions } \mathcal{Q}_1(j) \text{ and } \mathcal{Q}_2(j) \text{ hold})} \text{ for each } j \in I \right) \\
\Longleftrightarrow\ & \left(\text{Conditions } \mathcal{Q}_1(j) \text{ and } \mathcal{Q}_2(j) \text{ hold for each } j \in I \right) \\
\Longleftrightarrow\ & \left(\text{Condition } \mathcal{Q}_1(j) \text{ holds for each } j \in I \right) \\
& \wedge \left(\text{Condition } \mathcal{Q}_2(j) \text{ holds for each } j \in I \right).
\end{aligned} \tag{135}$$

But for any two elements (j, e) and (k, f) of E, we have the equivalence

$$((j, e) <_1 (k, f)) \Longleftrightarrow (j = k \text{ and } e <_{1,j} f) \tag{136}$$

[103]. Similarly, for any two elements (j, e) and (k, f) of E, we have the equivalence

$$((j, e) <_2 (k, f)) \Longleftrightarrow (j = k \text{ and } e <_{2,j} f). \tag{137}$$

We observe that every $j \in I$ and every $e \in E_j$ satisfy

$$\phi(j, e) = (\phi \circ \operatorname{inc}_j)(e) \tag{138}$$

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[103] *Proof of (136):* Recall that $\bigoplus_{i \in I} (<_{1,i}) = (<_1)$.

Now, the definition of the relation $\bigoplus_{i \in I} (<_{1,i})$ shows that

$$\left(\begin{array}{c} ((j, e) (\bigoplus_{i \in I} (<_{1,i})) (k, f)) \Longleftrightarrow (j = k \text{ and } e <_{1,j} f) \\ \text{for any two elements } (j, e) \text{ and } (k, f) \text{ of } \bigsqcup_{i \in I} E_i \end{array} \right).$$

In other words,

$$\left(\begin{array}{c} ((j, e) <_1 (k, f)) \Longleftrightarrow (j = k \text{ and } e <_{1,j} f) \\ \text{for any two elements } (j, e) \text{ and } (k, f) \text{ of } E \end{array} \right)$$

(since $\bigoplus_{i \in I} (<_{1,i}) = (<_1)$ and $\bigsqcup_{i \in I} E_i = E$). This proves (136).

[104].

Now, let us prove the implication

$$(\text{Condition } \mathcal{P}_1 \text{ holds}) \Longrightarrow (\text{Condition } \mathcal{Q}_1(j) \text{ holds for each } j \in I). \tag{139}$$

[*Proof of (139):* Assume that Condition $\mathcal{P}_1$ holds. We must prove that Condition $\mathcal{Q}_1(j)$ holds for each $j \in I$.

Indeed, fix $j \in I$. Let $e \in E_j$ and $f \in E_j$ be such that $e <_{1,j} f$. We shall show that $(\phi \circ \mathrm{inc}_j)(e) \leq (\phi \circ \mathrm{inc}_j)(f)$.

The definition of inc_j yields $\mathrm{inc}_j(e) = (j,e)$ and $\mathrm{inc}_j(f) = (j,f)$.

Recall that inc_j is a map $E_j \to \bigsqcup_{i\in I} E_i$. In other words, inc_j is a map $E_j \to E$ (since $\bigsqcup_{i\in I} E_i = E$).

We have $(j,e) = \mathrm{inc}_j(e) \in E$ (since inc_j is a map $E_j \to E$). Also, $(j,f) = \mathrm{inc}_j(f) \in E$ (since inc_j is a map $E_j \to E$). Thus, (136) (applied to j instead of k) shows that we have the equivalence

$$((j,e) <_1 (j,f)) \iff (j = j \text{ and } e <_{1,j} f).$$

Thus, we have $(j,e) <_1 (j,f)$ (since we have $(j = j \text{ and } e <_{1,j} f)$).

Now, recall that Condition $\mathcal{P}_1$ holds. Hence, Condition $\mathcal{P}_1$ (applied to (j,e) and (j,f) instead of e and f) shows that $\phi(j,e) \leq \phi(j,f)$ (since $(j,e) <_1 (j,f)$). But (138) yields $\phi(j,e) = (\phi \circ \mathrm{inc}_j)(e)$. Also, (138) (applied to f instead of e) yields $\phi(j,f) = (\phi \circ \mathrm{inc}_j)(f)$. Now,

$$(\phi \circ \mathrm{inc}_j)(e) = \phi(j,e) \leq \phi(j,f) = (\phi \circ \mathrm{inc}_j)(f).$$

Now, forget that we fixed e and f. We thus have shown that every $e \in E_j$ and $f \in E_j$ satisfying $e <_{1,j} f$ satisfy $(\phi \circ \mathrm{inc}_j)(e) \leq (\phi \circ \mathrm{inc}_j)(f)$. In other words, Condition $\mathcal{Q}_1(j)$ holds.

Now, forget that we fixed j. We thus have shown that Condition $\mathcal{Q}_1(j)$ holds for each $j \in I$. This completes the proof of the implication (139).]

Next, let us prove the implication

$$(\text{Condition } \mathcal{Q}_1(j) \text{ holds for each } j \in I) \Longrightarrow (\text{Condition } \mathcal{P}_1 \text{ holds}). \tag{140}$$

[*Proof of (140):* Assume that Condition $\mathcal{Q}_1(j)$ holds for each $j \in I$. We must prove that Condition $\mathcal{P}_1$ holds.

We have assumed that

$$\text{Condition } \mathcal{Q}_1(j) \text{ holds for each } j \in I. \tag{141}$$

Now, let $u \in E$ and $v \in E$ be such that $u <_1 v$. We shall show that $\phi(u) \leq \phi(v)$.

We have $v \in E = \bigsqcup_{i\in I} E_i$. Thus, Remark 10.30 (c) (applied to $x = v$) shows that there exist an $i \in I$ and an $e \in E_i$ such that $v = (i,e)$. Denote these i and e by k and f. Thus, $k \in I$ and $f \in E_k$ are such that $v = (k,f)$.

We have $u \in E = \bigsqcup_{i\in I} E_i$. Thus, Remark 10.30 (c) (applied to $x = u$) shows that there exist an $i \in I$ and an $e \in E_i$ such that $u = (i,e)$. Denote these i and e by j and e. Thus, $j \in I$ and $e \in E_j$ are such that $u = (j,e)$.

Now, $u <_1 v$. In other words, $(j,e) <_1 (k,f)$ (since $u = (j,e)$ and $v = (k,f)$).

We have $(j,e) = u \in E$ and $(k,f) = v \in E$. Thus, (136) shows that we have the equivalence

$$((j,e) <_1 (k,f)) \iff (j = k \text{ and } e <_{1,j} f).$$

Thus, we have $(j = k \text{ and } e <_{1,j} f)$ (since we have $(j,e) <_1 (k,f)$). Hence, $j = k$ and $e <_{1,j} f$. Now, $f \in E_k = E_j$ (since $k = j$).

[104] *Proof of (138):* Let $j \in I$ and $e \in E_j$. The definition of inc_j yields $\mathrm{inc}_j(e) = (j,e)$. Now, $(\phi \circ \mathrm{inc}_j)(e) = \phi\left(\underbrace{\mathrm{inc}_j(e)}_{=(j,e)}\right) = \phi(j,e)$. This proves (138).

But (141) shows that Condition $\mathcal{Q}_1(j)$ holds. Thus, this condition shows that $\left(\phi \circ \operatorname{inc}_j\right)(e) \leq \left(\phi \circ \operatorname{inc}_j\right)(f)$ (since $e <_{1,j} f$). But (138) yields $\phi(j,e) = \left(\phi \circ \operatorname{inc}_j\right)(e)$. Also, (138) (applied to f instead of e) yields $\phi(j,f) = \left(\phi \circ \operatorname{inc}_j\right)(f)$. Now,

$$\phi\left(\underbrace{u}_{=(j,e)}\right) = \phi(j,e) = \left(\phi \circ \operatorname{inc}_j\right)(e) \leq \left(\phi \circ \operatorname{inc}_j\right)(f) = \phi\left(\underbrace{j}_{=k}, f\right)$$
$$= \phi\underbrace{(k,f)}_{=v} = \phi(v).$$

Now, forget that we fixed u and v. We thus have shown that every $u \in E$ and $v \in E$ satisfying $u <_1 v$ satisfy $\phi(u) \leq \phi(v)$. Renaming u and v as e and f in this statement, we obtain the following: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$. In other words, Condition $\mathcal{P}_1$ holds. This proves the implication (140).]

Combining the two implications (139) and (140), we obtain the logical equivalence

$$(\text{Condition } \mathcal{P}_1 \text{ holds}) \Longleftrightarrow (\text{Condition } \mathcal{Q}_1(j) \text{ holds for each } j \in I). \tag{142}$$

Next, we shall prove the implication

$$(\text{Condition } \mathcal{P}_2 \text{ holds}) \Longrightarrow (\text{Condition } \mathcal{Q}_2(j) \text{ holds for each } j \in I). \tag{143}$$

[*Proof of (143):* Assume that Condition $\mathcal{P}_2$ holds. We must prove that Condition $\mathcal{Q}_2(j)$ holds for each $j \in I$.

Indeed, fix $j \in I$. Let $e \in E_j$ and $f \in E_j$ be such that $e <_{1,j} f$ and $f <_{2,j} e$. We shall show that $\left(\phi \circ \operatorname{inc}_j\right)(e) < \left(\phi \circ \operatorname{inc}_j\right)(f)$.

Just as in the proof of (139), we can

- show that $(j,e) \in E$ and $(j,f) \in E$ and $(j,e) <_1 (j,f)$;
- show that $\phi(j,e) = \left(\phi \circ \operatorname{inc}_j\right)(e)$ and $\phi(j,f) = \left(\phi \circ \operatorname{inc}_j\right)(f)$.

Furthermore, recall that $(j,f) \in E$ and $(j,e) \in E$. Thus, (137) (applied to j, f, j and e instead of j, e, k and f) shows that we have the equivalence

$$((j,f) <_2 (j,e)) \iff \left(j = j \text{ and } f <_{2,j} e\right).$$

Thus, we have $(j,f) <_2 (j,e)$ (since we have $\left(j = j \text{ and } f <_{2,j} e\right)$).

Now, recall that Condition $\mathcal{P}_2$ holds. Hence, Condition $\mathcal{P}_2$ (applied to (j,e) and (j,f) instead of e and f) shows that $\phi(j,e) < \phi(j,f)$ (since $(j,e) <_1 (j,f)$ and $(j,f) <_2 (j,e)$). Now,

$$\left(\phi \circ \operatorname{inc}_j\right)(e) = \phi(j,e) < \phi(j,f) = \left(\phi \circ \operatorname{inc}_j\right)(f).$$

Now, forget that we fixed e and f. We thus have shown that every $e \in E_j$ and $f \in E_j$ satisfying $e <_{1,j} f$ and $f <_{2,j} e$ satisfy $\left(\phi \circ \operatorname{inc}_j\right)(e) < \left(\phi \circ \operatorname{inc}_j\right)(f)$. In other words, Condition $\mathcal{Q}_2(j)$ holds.

Now, forget that we fixed j. We thus have shown that Condition $\mathcal{Q}_2(j)$ holds for each $j \in I$. This proves the implication (143).]

Next, let us prove the implication

$$(\text{Condition } \mathcal{Q}_2(j) \text{ holds for each } j \in I) \Longrightarrow (\text{Condition } \mathcal{P}_2 \text{ holds}). \tag{144}$$

[*Proof of (144):* Assume that Condition $\mathcal{Q}_2(j)$ holds for each $j \in I$. We must prove that Condition $\mathcal{P}_2$ holds.

We have assumed that

$$\text{Condition } \mathcal{Q}_2(j) \text{ holds for each } j \in I. \tag{145}$$

Now, let $u \in E$ and $v \in E$ be such that $u <_1 v$ and $v <_2 u$. We shall show that $\phi(u) < \phi(v)$.

Just as in the proof of (140), we can

- construct $k \in I$ and $f \in E_k$ such that $v = (k, f)$;
- construct $j \in I$ and $e \in E_j$ such that $u = (j, e)$;
- show that $(j, e) = u \in E$ and $(k, f) = v \in E$;
- show that $j = k$ and $e <_{1,j} f$ and $f \in E_j$;
- show that $\phi(j, e) = (\phi \circ \operatorname{inc}_j)(e)$ and $\phi(j, f) = (\phi \circ \operatorname{inc}_j)(f)$.

On the other hand, $v <_2 u$. In other words, $(k, f) <_2 (j, e)$ (since $u = (j, e)$ and $v = (k, f)$).

We have $(k, f) \in E$ and $(j, e) \in E$. Thus, (137) (applied to k, f, j and e instead of j, e, k and f) shows that we have the equivalence

$$((k, f) <_2 (j, e)) \iff (k = j \text{ and } f <_{2,k} e).$$

Thus, we have ($k = j$ and $f <_{2,k} e$) (since we have $(k, f) <_2 (j, e)$). Hence, $f <_{2,k} e$. In other words, $f <_{2,j} e$ (since $k = j$).

But (145) shows that Condition $\mathcal{Q}_2(j)$ holds. Thus, this condition shows that $(\phi \circ \operatorname{inc}_j)(e) < (\phi \circ \operatorname{inc}_j)(f)$ (since $e <_{1,j} f$ and $f <_{2,j} e$). Now,

$$\begin{aligned}
\phi\left(\underbrace{u}_{=(j,e)}\right) &= \phi(j, e) = (\phi \circ \operatorname{inc}_j)(e) < (\phi \circ \operatorname{inc}_j)(f) = \phi\left(\underbrace{j}_{=k}, f\right) \\
&= \phi\underbrace{(k, f)}_{=v} = \phi(v).
\end{aligned}$$

Now, forget that we fixed u and v. We thus have shown that every $u \in E$ and $v \in E$ satisfying $u <_1 v$ and $v <_2 u$ satisfy $\phi(u) < \phi(v)$. Renaming u and v as e and f in this statement, we obtain the following: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$. In other words, Condition $\mathcal{P}_2$ holds. This proves the implication (144).]

Combining the two implications (143) and (144), we obtain the logical equivalence

$$(\text{Condition } \mathcal{P}_2 \text{ holds}) \iff (\text{Condition } \mathcal{Q}_2(j) \text{ holds for each } j \in I). \tag{146}$$

Now, we have the following chain of logical equivalences:

$$\begin{aligned}
&\left(\phi \in \operatorname{Par}\underbrace{\left(\bigsqcup_{i \in I} \mathbf{E}_i\right)}_{=\mathbf{E}}\right) \\
\iff &\ (\phi \in \operatorname{Par} \mathbf{E}) \\
\iff &\ \underbrace{(\text{Condition } \mathcal{P}_1 \text{ holds})}_{\substack{\iff (\text{Condition } \mathcal{Q}_1(j) \text{ holds for each } j \in I) \\ (\text{by (142)})}} \wedge \underbrace{(\text{Condition } \mathcal{P}_2 \text{ holds})}_{\substack{\iff (\text{Condition } \mathcal{Q}_2(j) \text{ holds for each } j \in I) \\ (\text{by (146)})}} \\
&\ (\text{by (133)}) \\
\iff &\ (\text{Condition } \mathcal{Q}_1(j) \text{ holds for each } j \in I) \\
&\ \wedge (\text{Condition } \mathcal{Q}_2(j) \text{ holds for each } j \in I) \\
\iff &\ \left(\operatorname{Restr}(\phi) \in \prod_{i \in I} \operatorname{Par}(\mathbf{E}_i)\right) \qquad (\text{by (135)}).
\end{aligned}$$

This proves Proposition 10.43 (a).

(b) We know that w is a map $\bigsqcup_{i\in I} E_i \to \{1,2,3,\ldots\}$. In other words, w is a map $E \to \{1,2,3,\ldots\}$ (since $E = \bigsqcup_{i\in I} E_i$).

Both w and ϕ are maps $E \to \{1,2,3,\ldots\}$. Thus, $w(e)$ and $\phi(e)$ are well-defined elements of $\{1,2,3,\ldots\}$ for each $e \in E$.

Let A be the commutative ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. Define a map $a : E \to A$ by

$$\left(a(e) = x_{\phi(e)}^{w(e)} \qquad \text{for every } e \in E \right). \tag{147}$$

The definition of $\mathbf{x}_{\phi,w}$ yields

$$\begin{aligned}
\mathbf{x}_{\phi,w} &= \prod_{e\in E} \underbrace{x_{\phi(e)}^{w(e)}}_{\substack{=a(e) \\ \text{(by (147))}}} = \prod_{e\in E} a(e) \\
&= \prod_{i\in I} \prod_{e\in E_i} \underbrace{a(\mathrm{inc}_i(e))}_{\substack{=x_{\phi(\mathrm{inc}_i(e))}^{w(\mathrm{inc}_i(e))} \\ \text{(by the definition} \\ \text{of the map } a)}} \qquad \text{(by Proposition 10.34 (b))} \\
&= \prod_{i\in I} \prod_{e\in E_i} \underbrace{x_{\phi(\mathrm{inc}_i(e))}^{w(\mathrm{inc}_i(e))}}_{=x_{(\phi\circ\mathrm{inc}_i)(e)}^{(w\circ\mathrm{inc}_i)(e)}} = \prod_{i\in I} \prod_{e\in E_i} x_{(\phi\circ\mathrm{inc}_i)(e)}^{(w\circ\mathrm{inc}_i)(e)}.
\end{aligned}$$

Comparing this with

$$\prod_{i\in I} \underbrace{\mathbf{x}_{\phi\circ\mathrm{inc}_i, w\circ\mathrm{inc}_i}}_{\substack{=\prod_{e\in E_i} x_{(\phi\circ\mathrm{inc}_i)(e)}^{(w\circ\mathrm{inc}_i)(e)} \\ \text{(by the definition of } \mathbf{x}_{\phi\circ\mathrm{inc}_i, w\circ\mathrm{inc}_i})}} = \prod_{i\in I} \prod_{e\in E_i} x_{(\phi\circ\mathrm{inc}_i)(e)}^{(w\circ\mathrm{inc}_i)(e)},$$

we obtain

$$\mathbf{x}_{\phi,w} = \prod_{i\in I} \mathbf{x}_{\phi\circ\mathrm{inc}_i, w\circ\mathrm{inc}_i}.$$

This proves Proposition 10.43 (b). □

Now, we can prove the following fact:

Proposition 10.44. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. Every map $w : \bigsqcup_{i\in I} E_i \to \{1,2,3,\ldots\}$ satisfies

$$\Gamma\left(\bigsqcup_{i\in I} \mathbf{E}_i, w \right) = \prod_{i\in I} \Gamma(\mathbf{E}_i, w \circ \mathrm{inc}_i).$$

Proof of Proposition 10.44. For every double poset $\mathbf{E}$, let $\operatorname{Par}\mathbf{E}$ be the set of all $\mathbf{E}$-partitions. For every double poset $\mathbf{E} = (E, <_1, <_2)$ and every map $w : E \to \{1,2,3,\ldots\}$, we have

$$\begin{aligned}
\Gamma(\mathbf{E}, w) &= \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}}_{\substack{=\sum_{\pi\in\operatorname{Par}\mathbf{E}} \\ \text{(since } \operatorname{Par}\mathbf{E} \text{ is the} \\ \text{set of all } \mathbf{E}\text{-partitions)}}} \mathbf{x}_{\pi,w} \qquad \text{(by the definition of } \Gamma(\mathbf{E}, w)) \\
&= \sum_{\pi\in\operatorname{Par}\mathbf{E}} \mathbf{x}_{\pi,w}.
\end{aligned} \tag{148}$$

Let $F = \{1,2,3,\ldots\}$. Thus, $\operatorname{Par}\mathbf{E} \subseteq F^E$ for each double poset $\mathbf{E} = (E, <_1, <_2)$. In particular, $\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right) \subseteq F^{\bigsqcup_{i\in I}E_i}$, and every $i \in I$ satisfies $\operatorname{Par}(\mathbf{E}_i) \subseteq F^{E_i}$.

Let $w : \bigsqcup_{i\in I} E_i \to \{1,2,3,\ldots\}$ be any map.

Corollary 10.37 shows that the map $\operatorname{Restr} : F^{\bigsqcup_{i\in I}E_i} \to \prod\limits_{i\in I} F^{E_i}$ is a bijection. Every $\pi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$ satisfies $\operatorname{Restr}(\pi) \in \prod\limits_{i\in I}\operatorname{Par}(\mathbf{E}_i)$ [105]. Hence, we can define a map $\rho : \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right) \to \prod\limits_{i\in I}\operatorname{Par}(\mathbf{E}_i)$ by $\left(\rho(\pi) = \operatorname{Restr}(\pi) \text{ for every } \pi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\right)$. Consider this map ρ.

Every $\pi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$ satisfies

$$\rho(\pi) = \operatorname{Restr}(\pi) = (\pi \circ \operatorname{inc}_i)_{i\in I} \tag{149}$$

(by the definition of Restr). Thus, ρ is the map

$$\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right) \to \prod_{i\in I}\operatorname{Par}(\mathbf{E}_i), \qquad \pi \mapsto (\pi \circ \operatorname{inc}_i)_{i\in I}.$$

The map $\rho : \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right) \to \prod\limits_{i\in I}\operatorname{Par}(\mathbf{E}_i)$ is injective[106] and surjective[107]. Hence, this map ρ is

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[105] *Proof.* Let $\pi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$. Then, $\pi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right) \subseteq F^{\bigsqcup_{i\in I}E_i}$. Proposition 10.43 (a) (applied to $\phi = \pi$) thus shows that we have the following logical equivalence:

$$\left(\pi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\right) \iff \left(\operatorname{Restr}(\pi) \in \prod_{i\in I}\operatorname{Par}(\mathbf{E}_i)\right).$$

Thus, we have $\operatorname{Restr}(\pi) \in \prod\limits_{i\in I}\operatorname{Par}(\mathbf{E}_i)$ (since we have $\pi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$). Qed.

[106] *Proof.* Let π_1 and π_2 be two elements of $\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$ such that $\rho(\pi_1) = \rho(\pi_2)$. We must show that $\pi_1 = \pi_2$.

The map Restr is a bijection, thus injective.

The definition of ρ yields $\rho(\pi_1) = \operatorname{Restr}(\pi_1)$ and $\rho(\pi_2) = \operatorname{Restr}(\pi_2)$. Thus, $\operatorname{Restr}(\pi_1) = \rho(\pi_1) = \rho(\pi_2) = \operatorname{Restr}(\pi_2)$. Hence, $\pi_1 = \pi_2$ (since the map Restr is injective).

Now, forget that we fixed π_1 and π_2. We thus have shown that if π_1 and π_2 are two elements of $\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$ such that $\rho(\pi_1) = \rho(\pi_2)$, then $\pi_1 = \pi_2$. In other words, the map ρ is injective. Qed.

[107] *Proof.* Let $\gamma \in \prod\limits_{i\in I}\operatorname{Par}(\mathbf{E}_i)$. We shall prove that $\gamma \in \rho\left(\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\right)$.

The map $\operatorname{Restr} : F^{\bigsqcup_{i\in I}E_i} \to \prod\limits_{i\in I} F^{E_i}$ is a bijection, thus surjective. Hence, there exists a $\phi \in F^{\bigsqcup_{i\in I}E_i}$ such that $\gamma = \operatorname{Restr}(\phi)$ (since $\gamma \in \prod\limits_{i\in I}\underbrace{\operatorname{Par}(\mathbf{E}_i)}_{\subseteq F^{E_i}} \subseteq \prod\limits_{i\in I} F^{E_i}$). Consider this ϕ.

Proposition 10.43 (a) shows that we have the following logical equivalence:

$$\left(\phi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\right) \iff \left(\operatorname{Restr}(\phi) \in \prod_{i\in I}\operatorname{Par}(\mathbf{E}_i)\right).$$

Thus, we have $\phi \in \operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$ (since we have $\operatorname{Restr}(\phi) = \gamma \in \prod\limits_{i\in I}\operatorname{Par}(\mathbf{E}_i)$).

The definition of ρ yields $\rho(\phi) = \operatorname{Restr}(\phi) = \gamma$, so that $\gamma = \rho\left(\underbrace{\phi}_{\in\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)}\right) \in \rho\left(\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\right)$.

Now, forget that we fixed γ. We thus have shown that $\gamma \in \rho\left(\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\right)$ for each bijective, i.e., a bijection. In other words, the map

$$\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\to\prod_{i\in I}\operatorname{Par}\left(\mathbf{E}_i\right),\qquad \pi\mapsto\left(\pi\circ\operatorname{inc}_i\right)_{i\in I}$$

is a bijection[108].

Every $\pi\in\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$ satisfies

$$\mathbf{x}_{\pi,w}=\prod_{i\in I}\mathbf{x}_{\pi\circ\operatorname{inc}_i,w\circ\operatorname{inc}_i} \tag{150}$$

[109].

Now, recall that $\bigsqcup_{i\in I}\mathbf{E}_i=\left(\bigsqcup_{i\in I}E_i,\bigoplus_{i\in I}\left(<_{1,i}\right),\bigoplus_{i\in I}\left(<_{2,i}\right)\right)$ (by the definition of $\bigsqcup_{i\in I}\mathbf{E}_i$). Hence, (148) (applied to $\bigsqcup_{i\in I}\mathbf{E}_i$, $\bigsqcup_{i\in I}E_i$, $\bigoplus_{i\in I}\left(<_{1,i}\right)$ and $\bigoplus_{i\in I}\left(<_{2,i}\right)$ instead of $\mathbf{E}$, E, $<_1$ and $<_2$) shows that

$$\begin{aligned}\Gamma\left(\bigsqcup_{i\in I}\mathbf{E}_i,w\right)&=\sum_{\pi\in\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)}\underbrace{\mathbf{x}_{\pi,w}}_{\substack{=\prod_{i\in I}\mathbf{x}_{\pi\circ\operatorname{inc}_i,w\circ\operatorname{inc}_i}\\ \text{(by (150))}}}=\sum_{\pi\in\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)}\prod_{i\in I}\mathbf{x}_{\pi\circ\operatorname{inc}_i,w\circ\operatorname{inc}_i}\\&=\sum_{(\pi_i)_{i\in I}\in\prod_{i\in I}\operatorname{Par}(\mathbf{E}_i)}\prod_{i\in I}\mathbf{x}_{\pi_i,w\circ\operatorname{inc}_i}\end{aligned}$$

(here, we have substituted $(\pi_i)_{i\in I}$ for $(\pi\circ\operatorname{inc}_i)_{i\in I}$ in the sum, since the map $\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\to\prod_{i\in I}\operatorname{Par}\left(\mathbf{E}_i\right)$, $\pi\mapsto\left(\pi\circ\operatorname{inc}_i\right)_{i\in I}$ is a bijection). Comparing this with

$$\prod_{i\in I}\underbrace{\Gamma\left(\mathbf{E}_i,w\circ\operatorname{inc}_i\right)}_{\substack{=\sum_{\pi\in\operatorname{Par}(\mathbf{E}_i)}\mathbf{x}_{\pi,w\circ\operatorname{inc}_i}\\ \text{(by (148), applied to}\\ \mathbf{E}_i,\ E_i,\ <_{1,i},\ <_{2,i}\text{ and }w\circ\operatorname{inc}_i\\ \text{instead of }\mathbf{E},\ E,\ <_1,\ <_2\text{ and }w)}}=\prod_{i\in I}\sum_{\pi\in\operatorname{Par}(\mathbf{E}_i)}\mathbf{x}_{\pi,w\circ\operatorname{inc}_i}=\sum_{(\pi_i)_{i\in I}\in\prod_{i\in I}\operatorname{Par}(\mathbf{E}_i)}\prod_{i\in I}\mathbf{x}_{\pi_i,w\circ\operatorname{inc}_i}$$
$$(\text{by the product rule}),$$

we obtain $\Gamma\left(\bigsqcup_{i\in I}\mathbf{E}_i,w\right)=\prod_{i\in I}\Gamma\left(\mathbf{E}_i,w\circ\operatorname{inc}_i\right)$. This proves Proposition 10.44. □

Corollary 10.45. Let I be a finite set. For each $i\in I$, let $\mathbf{E}_i=\left(E_i,<_{1,i},<_{2,i}\right)$ be a double poset. For each $i\in I$, let $w_i:E_i\to\{1,2,3,\ldots\}$ be a map.

Let $F=\{1,2,3,\ldots\}$. For each $i\in I$, we have $w_i\in\{1,2,3,\ldots\}^{E_i}=F^{E_i}$ (since $\{1,2,3,\ldots\}=F$). Thus, $(w_i)_{i\in I}\in\prod_{i\in I}F^{E_i}$.

$\gamma\in\prod_{i\in I}\operatorname{Par}\left(\mathbf{E}_i\right)$. In other words, $\prod_{i\in I}\operatorname{Par}\left(\mathbf{E}_i\right)\subseteq\rho\left(\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\right)$. In other words, the map ρ is surjective. Qed.

[108]since ρ is the map

$$\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\to\prod_{i\in I}\operatorname{Par}\left(\mathbf{E}_i\right),\qquad \pi\mapsto\left(\pi\circ\operatorname{inc}_i\right)_{i\in I}$$

[109]*Proof of (150):* Let $\pi\in\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$. Then, $\pi\in\operatorname{Par}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)\subseteq F^{\bigsqcup_{i\in I}E_i}$. Hence, Proposition 10.43 (b) (applied to $\phi=\pi$) yields

$$\mathbf{x}_{\pi,w}=\prod_{i\in I}\mathbf{x}_{\pi\circ\operatorname{inc}_i,w\circ\operatorname{inc}_i}.$$

Corollary 10.37 shows that the map $\operatorname{Restr} : F^{\bigsqcup_{i\in I} E_i} \to \prod_{i\in I} F^{E_i}$ is a bijection. Hence, $\operatorname{Restr}^{-1}\left((w_i)_{i\in I}\right)$ is a well-defined element of $F^{\bigsqcup_{i\in I} E_i}$. Denote this element by w. Then,

$$\prod_{i\in I} \Gamma\left(\mathbf{E}_i, w_i\right) = \Gamma\left(\bigsqcup_{i\in I} \mathbf{E}_i, w\right).$$

Proof of Corollary 10.45. We have $w = \operatorname{Restr}^{-1}\left((w_i)_{i\in I}\right)$ (by the definition of w) and thus

$$(w_i)_{i\in I} = \operatorname{Restr}(w) = (w \circ \operatorname{inc}_i)_{i\in I} \qquad \text{(by the definition of Restr)}.$$

In other words,

$$w_i = w \circ \operatorname{inc}_i \qquad \text{for each } i \in I. \tag{151}$$

We have $w \in F^{\bigsqcup_{i\in I} E_i}$. Thus, w is a map $\bigsqcup_{i\in I} E_i \to F$. In other words, w is a map $\bigsqcup_{i\in I} E_i \to \{1,2,3,\ldots\}$ (since $F = \{1,2,3,\ldots\}$). Hence, Proposition 10.44 yields

$$\Gamma\left(\bigsqcup_{i\in I} \mathbf{E}_i, w\right) = \prod_{i\in I} \Gamma\left(\mathbf{E}_i, \underbrace{w \circ \operatorname{inc}_i}_{\substack{=w_i \\ \text{(by (151))}}}\right) = \prod_{i\in I} \Gamma\left(\mathbf{E}_i, w_i\right).$$

This proves Corollary 10.45. □

As a consequence of Corollary 10.45 in the case when $I = \{0,1\}$, we obtain the following:

Corollary 10.46. Let $\mathbf{E} = (E, <_{1,0}, <_{2,0})$ and $\mathbf{F} = (F, <_{1,1}, <_{2,1})$ be two double posets. Recall that a double poset $\mathbf{E} \sqcup \mathbf{F}$ is defined (in Definition 10.42).

Define a family $(E_i)_{i\in\{0,1\}}$ of sets by setting $E_0 = E$ and $E_1 = F$. Recall that $E \sqcup F = \bigsqcup_{i\in\{0,1\}} E_i$ (by the definition of $E \sqcup F$).

Recall that there is a map $\operatorname{inc}_j : E_j \to \bigsqcup_{i\in\{0,1\}} E_i$ defined for each $j \in \{0,1\}$. Thus, we have two maps $\operatorname{inc}_0 : E_0 \to \bigsqcup_{i\in\{0,1\}} E_i$ and $\operatorname{inc}_1 : E_1 \to \bigsqcup_{i\in\{0,1\}} E_i$. In other words, we have two maps $\operatorname{inc}_0 : E \to E \sqcup F$ and $\operatorname{inc}_1 : F \to E \sqcup F$ (since $E_0 = E$, $E_1 = F$ and $E \sqcup F = \bigsqcup_{i\in\{0,1\}} E_i$). (Explicitly, these maps are given as follows: The map inc_0 sends each $e \in E$ to $(0,e) \in E \sqcup F$; the map inc_1 sends each $f \in F$ to $(1,f) \in E \sqcup F$.)

Every map $w : E \sqcup F \to \{1,2,3,\ldots\}$ satisfies

$$\Gamma\left(\mathbf{E} \sqcup \mathbf{F}, w\right) = \Gamma\left(\mathbf{E}, w \circ \operatorname{inc}_0\right) \Gamma\left(\mathbf{F}, w \circ \operatorname{inc}_1\right).$$

Corollary 10.46 is the "rule for multiplying quasisymmetric functions of the form $\Gamma(\mathbf{E}, w)$" mentioned at the end of Section 5 (but here we are denoting by $\mathbf{E} \sqcup \mathbf{F}$ what had been called $\mathbf{EF}$ back there).

Proof of Corollary 10.46. Define a family $(\mathbf{E}_i)_{i\in\{0,1\}}$ of double posets by setting $\mathbf{E}_0 = \mathbf{E}$ and $\mathbf{E}_1 = \mathbf{F}$. Then, $\mathbf{E} \sqcup \mathbf{F} = \bigsqcup_{i\in\{0,1\}} \mathbf{E}_i$ (by the definition of $\mathbf{E} \sqcup \mathbf{F}$).

We have $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ for every $i \in \{0,1\}$ [110]. Also, w is a map $E \sqcup F \to \{1,2,3,\ldots\}$. In other words, w is a map $\bigsqcup_{i\in\{0,1\}} E_i \to \{1,2,3,\ldots\}$ (since $E \sqcup F = \bigsqcup_{i\in\{0,1\}} E_i$). Thus, Proposition

[110] *Proof:*

- For $i = 0$, this follows from $\mathbf{E}_0 = \mathbf{E} = \left(\underbrace{E}_{=E_0}, <_{1,0}, <_{2,0}\right) = (E_0, <_{1,0}, <_{2,0})$.

10.44 (applied to $I = \{0,1\}$) yields

$$\Gamma\left(\bigsqcup_{i\in\{0,1\}} \mathbf{E}_i, w\right) = \prod_{i\in\{0,1\}} \Gamma\left(\mathbf{E}_i, w\circ \operatorname{inc}_i\right) = \Gamma\left(\underbrace{\mathbf{E}_0}_{=\mathbf{E}}, w\circ\operatorname{inc}_0\right)\Gamma\left(\underbrace{\mathbf{E}_1}_{=\mathbf{F}}, w\circ\operatorname{inc}_1\right)$$
$$= \Gamma\left(\mathbf{E}, w\circ\operatorname{inc}_0\right)\Gamma\left(\mathbf{F}, w\circ\operatorname{inc}_1\right).$$

Thus,

$$\Gamma\left(\underbrace{\mathbf{E}\sqcup\mathbf{F}}_{=\bigsqcup_{i\in\{0,1\}}\mathbf{E}_i}, w\right) = \Gamma\left(\bigsqcup_{i\in\{0,1\}} \mathbf{E}_i, w\right) = \Gamma\left(\mathbf{E}, w\circ\operatorname{inc}_0\right)\Gamma\left(\mathbf{F}, w\circ\operatorname{inc}_1\right).$$

This proves Corollary 10.46. □

Here is a further corollary of Corollary 10.45:

Corollary 10.47. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. For each $i \in I$, let $w_i : E_i \to \{1,2,3,\ldots\}$ be a map. Then,

$$\prod_{i\in I} \Gamma\left(\mathbf{E}_i, w_i\right) \in \operatorname{QSym}.$$

Proof of Corollary 10.47. Let $F = \{1,2,3,\ldots\}$. For each $i \in I$, we have $w_i \in \{1,2,3,\ldots\}^{E_i} = F^{E_i}$ (since $\{1,2,3,\ldots\} = F$). Thus, $(w_i)_{i\in I} \in \prod_{i\in I} F^{E_i}$.

Corollary 10.37 shows that the map $\operatorname{Restr} : F^{\bigsqcup_{i\in I} E_i} \to \prod_{i\in I} F^{E_i}$ is a bijection. Hence, $\operatorname{Restr}^{-1}\left((w_i)_{i\in I}\right)$ is a well-defined element of $F^{\bigsqcup_{i\in I} E_i}$. Denote this element by w. Thus, w is an element of $F^{\bigsqcup_{i\in I} E_i}$. In other words, w is a map $\bigsqcup_{i\in I} E_i \to F$. In other words, w is a map $\bigsqcup_{i\in I} E_i \to \{1,2,3,\ldots\}$ (since $F = \{1,2,3,\ldots\}$).

Recall that $\bigsqcup_{i\in I} \mathbf{E}_i$ is a double poset. Its definition shows that

$$\bigsqcup_{i\in I} \mathbf{E}_i = \left(\bigsqcup_{i\in I} E_i, \bigoplus_{i\in I} (<_{1,i}), \bigoplus_{i\in I} (<_{2,i})\right).$$

Proposition 3.5 (applied to $\bigsqcup_{i\in I} \mathbf{E}_i$, $\bigsqcup_{i\in I} E_i$, $\bigoplus_{i\in I} (<_{1,i})$ and $\bigoplus_{i\in I} (<_{2,i})$ instead of $\mathbf{E}$, E, $<_1$ and $<_2$) thus yields that $\Gamma\left(\bigsqcup_{i\in I} \mathbf{E}_i, w\right) \in \operatorname{QSym}$.

But Corollary 10.45 yields

$$\prod_{i\in I} \Gamma\left(\mathbf{E}_i, w_i\right) = \Gamma\left(\bigsqcup_{i\in I} \mathbf{E}_i, w\right) \in \operatorname{QSym}.$$

This proves Corollary 10.47. □

Now, we can prove the following basic fact:

Proposition 10.48. The subset QSym of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

- For $i = 1$, this follows from $\mathbf{E}_1 = \mathbf{F} = \left(\underbrace{F}_{=E_1}, <_{1,1}, <_{2,1}\right) = (E_1, <_{1,1}, <_{2,1})$.

Proof of Proposition 10.48. It is well-known that QSym is a $\mathbf{k}$-submodule of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, and that $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of this $\mathbf{k}$-module QSym.

We shall now prove that

$$M_\alpha M_\beta \in \operatorname{QSym} \qquad \text{for any } \alpha \in \operatorname{Comp} \text{ and } \beta \in \operatorname{Comp}. \tag{152}$$

[*Proof of (152):* Let $\alpha \in \operatorname{Comp}$ and $\beta \in \operatorname{Comp}$.

We know that $\beta \in \operatorname{Comp}$. In other words, β is a composition (since Comp is the set of all compositions). Corollary 10.27 (applied to β instead of α) thus shows that there exist a set E, a special double poset $\mathbf{E} = (E, <_1, >_1)$, and a map $w : E \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}, w) = M_\beta$. Renaming E, $\mathbf{E}$, $<_1$, $>_1$ and w as E_1, $\mathbf{E}_1$, $<_{1,1}$, $>_{1,1}$ and w_1 in this statement, we obtain the following: There exist a set E_1, a special double poset $\mathbf{E}_1 = (E_1, <_{1,1}, >_{1,1})$, and a map $w_1 : E_1 \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}_1, w_1) = M_\beta$. Consider these E_1, $\mathbf{E}_1$, $<_{1,1}$, $>_{1,1}$ and w_1.

We know that $\alpha \in \operatorname{Comp}$. In other words, α is a composition (since Comp is the set of all compositions). Corollary 10.27 thus shows that there exist a set E, a special double poset $\mathbf{E} = (E, <_1, >_1)$, and a map $w : E \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}, w) = M_\alpha$. Renaming E, $\mathbf{E}$, $<_1$, $>_1$ and w as E_0, $\mathbf{E}_0$, $<_{1,0}$, $>_{1,0}$ and w_0 in this statement, we obtain the following: There exist a set E_0, a special double poset $\mathbf{E}_0 = (E_0, <_{1,0}, >_{1,0})$, and a map $w_0 : E_0 \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}_0, w_0) = M_\alpha$. Consider these E_0, $\mathbf{E}_0$, $<_{1,0}$, $>_{1,0}$ and w_0.

Now, $\mathbf{E}_i = (E_i, <_{1,i}, >_{1,i})$ for every $i \in \{0, 1\}$ [111]. Moreover, w_i is a map $E_i \to \{1, 2, 3, \ldots\}$ for each $i \in \{0, 1\}$ [112]. Hence, Corollary 10.47 (applied to $\{0, 1\}$ and $>_{1,i}$ instead of I and $<_{2,i}$) shows that

$$\prod_{i \in \{0,1\}} \Gamma(\mathbf{E}_i, w_i) \in \operatorname{QSym}.$$

Since

$$\prod_{i \in \{0,1\}} \Gamma(\mathbf{E}_i, w_i) = \underbrace{\Gamma(\mathbf{E}_0, w_0)}_{=M_\alpha} \underbrace{\Gamma(\mathbf{E}_1, w_1)}_{=M_\beta} = M_\alpha M_\beta,$$

this rewrites as $M_\alpha M_\beta \in \operatorname{QSym}$. Thus, (152) is proven.]

Now, we can see that

$$ab \in \operatorname{QSym} \qquad \text{for any } a \in \operatorname{QSym} \text{ and } b \in \operatorname{QSym}. \tag{153}$$

[*Proof of (153):* Let $a \in \operatorname{QSym}$ and $b \in \operatorname{QSym}$. We must prove the relation $ab \in \operatorname{QSym}$.

This relation is $\mathbf{k}$-linear in b (since QSym is a $\mathbf{k}$-submodule of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$). Hence, we can WLOG assume that b belongs to the basis $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ of the $\mathbf{k}$-module QSym. Assume this. Thus, $b = M_\beta$ for some $\beta \in \operatorname{Comp}$. Consider this β.

We must prove the relation $ab \in \operatorname{QSym}$. This relation is $\mathbf{k}$-linear in a (since QSym is a $\mathbf{k}$-submodule of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$). Hence, we can WLOG assume that a belongs to the basis $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ of the $\mathbf{k}$-module QSym. Assume this. Thus, $a = M_\alpha$ for some $\alpha \in \operatorname{Comp}$. Consider this α.

Now, $\underbrace{a}_{=M_\alpha} \underbrace{b}_{=M_\beta} = M_\alpha M_\beta \in \operatorname{QSym}$ (by (152)). This proves (153).]

[111] *Proof:*

- For $i = 0$, this follows from $\mathbf{E}_0 = (E_0, <_{1,0}, <_{2,0})$.
- For $i = 1$, this follows from $\mathbf{E}_1 = (E_1, <_{1,1}, <_{2,1})$.

[112] *Proof:*

- For $i = 0$, this holds because w_0 is a map $E_0 \to \{1, 2, 3, \ldots\}$.
- For $i = 1$, this holds because w_1 is a map $E_1 \to \{1, 2, 3, \ldots\}$.

Finally, let us prove that

$$1 \in \operatorname{QSym}. \tag{154}$$

[*Proof of (154):* There are many reasons why (154) is obvious (for example, it follows from Lemma 6.3 (a) or from $M_{\varnothing} = 1$), but let us derive (154) from Corollary 10.47:

For each $i \in \varnothing$, we define a double poset $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ and a map $w_i : E_i \to \{1, 2, 3, \ldots\}$ as follows: There is nothing to define, because there exists no $i \in \varnothing$.

Thus, Corollary 10.47 (applied to $I = \varnothing$) yields

$$\prod_{i \in \varnothing} \Gamma(\mathbf{E}_i, w_i) \in \operatorname{QSym}.$$

Since $\prod_{i \in \varnothing} \Gamma(\mathbf{E}_i, w_i) = (\text{empty product}) = 1$, this rewrites as $1 \in \operatorname{QSym}$. Thus, (154) is proven.]

Now, recall that QSym is a $\mathbf{k}$-submodule of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. Combining this with (153) and (154), we conclude that QSym is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This proves Proposition 10.48. $\square$

10.8. Restrictions and disjoint unions

In this short section, we shall prove the following straightforward fact, which will be used in the next section.

Proposition 10.49. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset.

Let $\mathbf{E}$ be the double poset $\bigsqcup_{i \in I} \mathbf{E}_i$.

For each $i \in I$, let T_i be a subset of E_i.

Then, $\mathbf{E} \mid_{\bigsqcup_{i \in I} T_i} = \bigsqcup_{i \in I} \left(\mathbf{E}_i \mid_{T_i}\right)$.

The proof will rely on a rather pedantic notation:

Definition 10.50. Let X be a set. Let ρ be a binary relation on the set X. Let Y be a subset of X. Then, $\rho \mid_Y$ shall denote the restriction of the relation ρ to Y.

Using this notation, we can state the following trivial fact:

Proposition 10.51. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let T be a subset of E. Then, $\mathbf{E} \mid_T = (T, (<_1) \mid_T, (<_2) \mid_T)$.

Proof of Proposition 10.51. Recall that the double poset $\mathbf{E} \mid_T$ is defined as the double poset $(T, <_1, <_2)$, where $<_1$ and $<_2$ (by abuse of notation) denote the restrictions of the relations $<_1$ and $<_2$ to T. Avoiding the abuse of notation, this rewrites as follows: The double poset $\mathbf{E} \mid_T$ is defined as the double poset $(T, (<_1) \mid_T, (<_2) \mid_T)$. This proves Proposition 10.51. $\square$

Lemma 10.52. Let I be a set. For each $i \in I$, let E_i be a set. For each $i \in I$, let T_i be a subset of E_i.

(a) The set $\bigsqcup_{i \in I} T_i$ is a subset of $\bigsqcup_{i \in I} E_i$.

(b) For each $i \in I$, let ρ_i be a binary relation on the set E_i. Then, $\left(\bigoplus_{i \in I} \rho_i\right) \mid_{\bigsqcup_{i \in I} T_i} = \bigoplus_{i \in I} \left(\rho_i \mid_{T_i}\right)$.

Proof of Lemma 10.52. This is a straightforward consequence of the definitions. $\square$

Proof of Proposition 10.49. Let E denote the set $\bigsqcup_{i\in I} E_i$. Let T denote the set $\bigsqcup_{i\in I} T_i$.

Let $<_1$ denote the relation $\bigoplus_{i\in I} (<_{1,i})$. Let $<_2$ denote the relation $\bigoplus_{i\in I} (<_{2,i})$.

We have

$$\mathbf{E} = \bigsqcup_{i\in I} \mathbf{E}_i = \left(\underbrace{\bigsqcup_{i\in I} E_i}_{=E}, \underbrace{\bigoplus_{i\in I} (<_{1,i})}_{=(<_1)}, \underbrace{\bigoplus_{i\in I} (<_{2,i})}_{=(<_2)} \right) \qquad \left(\text{by the definition of } \bigsqcup_{i\in I} \mathbf{E}_i\right)$$
$$= (E, <_1, <_2).$$

Lemma 10.52 (a) shows that the set $\bigsqcup_{i\in I} T_i$ is a subset of $\bigsqcup_{i\in I} E_i$. In other words, the set T is a subset of E (since $\bigsqcup_{i\in I} T_i = T$ and $\bigsqcup_{i\in I} E_i = E$). Hence, Proposition 10.51 shows that

$$\mathbf{E} \mid_T = (T, (<_1) \mid_T, (<_2) \mid_T). \tag{155}$$

We have

$$\bigoplus_{i\in I} \left((<_{1,i}) \mid_{T_i}\right) = (<_1) \mid_T \tag{156}$$

[113]. The same argument (applied to $<_{2,i}$ and $<_2$ instead of $<_{1,i}$ and $<_1$) shows that

$$\bigoplus_{i\in I} \left((<_{2,i}) \mid_{T_i}\right) = (<_2) \mid_T. \tag{157}$$

Furthermore, for each $i \in I$, we have $\mathbf{E}_i \mid_{T_i} = \left(T_i, (<_{1,i}) \mid_{T_i}, (<_{2,i}) \mid_{T_i}\right)$ [114]. Hence, the definition of $\bigsqcup_{i\in I} \left(\mathbf{E}_i \mid_{T_i}\right)$ yields

$$\bigsqcup_{i\in I} \left(\mathbf{E}_i \mid_{T_i}\right) = \left(\underbrace{\bigsqcup_{i\in I} T_i}_{=T}, \underbrace{\bigoplus_{i\in I} \left((<_{1,i}) \mid_{T_i}\right)}_{\substack{=(<_1)\mid_T \\ \text{(by (156))}}}, \underbrace{\bigoplus_{i\in I} \left((<_{2,i}) \mid_{T_i}\right)}_{\substack{=(<_2)\mid_T \\ \text{(by (157))}}} \right)$$
$$= (T, (<_1) \mid_T, (<_2) \mid_T) = \mathbf{E} \mid_T = \mathbf{E} \mid_{\bigsqcup_{i\in I} T_i}$$

(since $T = \bigsqcup_{i\in I} T_i$). This proves Proposition 10.49. $\square$

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[113]*Proof of (156):* For each $i \in I$, the relation $<_{1,i}$ is a binary relation on the set E_i (since $(E_i, <_{1,i}, <_{2,i})$ is a double poset). Recall furthermore that T_i is a subset of E_i for each $i \in I$. Hence, Lemma 10.52 (b) (applied to $<_{1,i}$ instead of ρ_i) shows that $\left(\bigoplus_{i\in I} (<_{1,i})\right) \mid_{\bigsqcup_{i\in I} T_i} = \bigoplus_{i\in I} \left((<_{1,i}) \mid_{T_i}\right)$. Thus,

$$\bigoplus_{i\in I} \left((<_{1,i}) \mid_{T_i}\right) = \underbrace{\left(\bigoplus_{i\in I} (<_{1,i})\right)}_{\substack{=(<_1) \\ \text{(since } <_1 \text{ was defined} \\ \text{as } \bigoplus_{i\in I}(<_{1,i}))}} \mid_{\bigsqcup_{i\in I} T_i} = (<_1) \mid_{\bigsqcup_{i\in I} T_i} = (<_1) \mid_T \qquad \left(\text{since } \bigsqcup_{i\in I} T_i = T\right).$$

This proves (156).

[114]*Proof.* Let $i \in I$. Then, $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ is a double poset, whereas T_i is a subset of E_i. Hence, Proposition 10.51 (applied to $\mathbf{E}_i$, E_i, $<_{1,i}$, $<_{2,i}$ and T_i instead of $\mathbf{E}$, E, $<_1$, $<_2$ and T) shows that $\mathbf{E}_i \mid_{T_i} = \left(T_i, (<_{1,i}) \mid_{T_i}, (<_{2,i}) \mid_{T_i}\right)$.

10.9. $\operatorname{Adm}\left(\bigsqcup_{i\in I}\mathbf{E}_i\right)$ and the bialgebra QSym

In this section, we shall continue analyzing the disjoint union of several double posets. This will result in a new proof of the fact that QSym is a $\mathbf{k}$-bialgebra.

We begin with some simple facts:

Proposition 10.53. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. Let $\mathbf{E}$ be the double poset $\bigsqcup_{i\in I}\mathbf{E}_i$.

(a) For each $(P, Q) \in \operatorname{Adm}\mathbf{E}$, we have

$$\left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i\in I} \in \prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i).$$

(b) For each $((P_i, Q_i))_{i\in I} \in \prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)$, we have

$$\left(\bigsqcup_{i\in I}P_i, \bigsqcup_{i\in I}Q_i\right) \in \operatorname{Adm}\mathbf{E}.$$

Proof of Proposition 10.53. Write the double poset $\mathbf{E}$ as $(E, <_1, <_2)$. Thus,

$$(E, <_1, <_2) = \mathbf{E} = \bigsqcup_{i\in I}\mathbf{E}_i = \left(\bigsqcup_{i\in I}E_i, \bigoplus_{i\in I}(<_{1,i}), \bigoplus_{i\in I}(<_{2,i})\right)$$

(by the definition of the double poset $\bigsqcup_{i\in I}\mathbf{E}_i$). In other words,

$$E = \bigsqcup_{i\in I}E_i, \qquad (<_1) = \bigoplus_{i\in I}(<_{1,i}) \qquad \text{and} \qquad (<_2) = \bigoplus_{i\in I}(<_{2,i}).$$

For any two elements (j, e) and (k, f) of E, we have the equivalence

$$((j,e) <_1 (k,f)) \iff (j = k \text{ and } e <_{1,j} f) \tag{158}$$

[115].

(a) Let $(P, Q) \in \operatorname{Adm}\mathbf{E}$. We shall show that $\left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i\in I} \in \prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)$.

Indeed, fix $i \in I$. We shall prove that $\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right) \in \operatorname{Adm}(\mathbf{E}_i)$.

The sets $(\operatorname{inc}_i)^{-1}(P)$ and $(\operatorname{inc}_i)^{-1}(Q)$ are subsets of E (since inc_i is a map $E_i \to E$).

[115] *Proof of (158):* Recall that $(<_1) = \bigoplus_{i\in I}(<_{1,i})$.

Now, the definition of the relation $\bigoplus_{i\in I}(<_{1,i})$ shows that

$$\left(\begin{array}{c}((j,e)\left(\bigoplus_{i\in I}(<_{1,i})\right)(k,f)) \iff (j = k \text{ and } e <_{1,j} f) \\ \text{for any two elements } (j,e) \text{ and } (k,f) \text{ of } \bigsqcup_{i\in I}E_i\end{array}\right).$$

In other words,

$$\left(\begin{array}{c}((j,e) <_1 (k,f)) \iff (j = k \text{ and } e <_{1,j} f) \\ \text{for any two elements } (j,e) \text{ and } (k,f) \text{ of } E\end{array}\right)$$

(since $\bigoplus_{i\in I}(<_{1,i}) = (<_1)$ and $\bigsqcup_{i\in I}E_i = E$). This proves (158).

We have $(P,Q) \in \operatorname{Adm}\mathbf{E}$. In other words, P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$ and having the property that

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p \tag{159}$$

(by the definition of $\operatorname{Adm}\mathbf{E}$ (since $\mathbf{E} = (E, <_1, <_2)$).

If $p \in P$ and $q \in Q$, then

$$\text{we do not have } q <_1 p. \tag{160}$$

(Indeed, this is merely a restatement of (159).)

We have

$$(\operatorname{inc}_i)^{-1}(P) \cap (\operatorname{inc}_i)^{-1}(Q) = (\operatorname{inc}_i)^{-1}\left(\underbrace{P \cap Q}_{=\varnothing}\right) = (\operatorname{inc}_i)^{-1}(\varnothing) = \varnothing$$

and

$$(\operatorname{inc}_i)^{-1}(P) \cup (\operatorname{inc}_i)^{-1}(Q) = (\operatorname{inc}_i)^{-1}\left(\underbrace{P \cup Q}_{=E}\right) = (\operatorname{inc}_i)^{-1}(E) = E_i.$$

Moreover, no $p \in (\operatorname{inc}_i)^{-1}(P)$ and $q \in (\operatorname{inc}_i)^{-1}(Q)$ satisfy $q <_{1,i} p$ [116].

Thus, $(\operatorname{inc}_i)^{-1}(P)$ and $(\operatorname{inc}_i)^{-1}(Q)$ are subsets of E_i satisfying $(\operatorname{inc}_i)^{-1}(P) \cap (\operatorname{inc}_i)^{-1}(Q) = \varnothing$ and $(\operatorname{inc}_i)^{-1}(P) \cup (\operatorname{inc}_i)^{-1}(Q) = E_i$ and having the property that no $p \in (\operatorname{inc}_i)^{-1}(P)$ and $q \in (\operatorname{inc}_i)^{-1}(Q)$ satisfy $q <_{1,i} p$. In other words, $\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right) \in \operatorname{Adm}(\mathbf{E}_i)$ (by the definition of $\operatorname{Adm}(\mathbf{E}_i)$ (since $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$)).

Now, forget that we fixed i. We thus have shown that $\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right) \in \operatorname{Adm}(\mathbf{E}_i)$ for each $i \in I$. In other words, $\left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i \in I} \in \prod_{i \in I} \operatorname{Adm}(\mathbf{E}_i)$. This proves Proposition 10.53 (a).

(b) Let $((P_i, Q_i))_{i \in I} \in \prod_{i \in I} \operatorname{Adm}(\mathbf{E}_i)$. We shall show that $\left(\bigsqcup_{i \in I} P_i, \bigsqcup_{i \in I} Q_i\right) \in \operatorname{Adm}\mathbf{E}$.

We have $((P_i, Q_i))_{i \in I} \in \prod_{i \in I} \operatorname{Adm}(\mathbf{E}_i)$. In other words,

$$(P_i, Q_i) \in \operatorname{Adm}(\mathbf{E}_i) \qquad \text{for each } i \in I. \tag{161}$$

From this, we can easily obtain the following observation:

Observation 1: Let $i \in I$. Then, the sets P_i and Q_i are subsets of E_i satisfying $P_i \cap Q_i = \varnothing$ and $P_i \cup Q_i = E_i$ and having the property that

$$\text{no } p \in P_i \text{ and } q \in Q_i \text{ satisfy } q <_{1,i} p. \tag{162}$$

[116] *Proof.* Assume the contrary. Thus, there exist $p \in (\operatorname{inc}_i)^{-1}(P)$ and $q \in (\operatorname{inc}_i)^{-1}(Q)$ satisfying $q <_{1,i} p$. Consider these p and q.

We have $p \in (\operatorname{inc}_i)^{-1}(P)$. In other words, p is an element of E_i satisfying $\operatorname{inc}_i(p) \in P$. The definition of inc_i yields $\operatorname{inc}_i(p) = (i,p)$; thus, $(i,p) = \operatorname{inc}_i(p) \in P \subseteq E$.

We have $q \in (\operatorname{inc}_i)^{-1}(Q)$. In other words, q is an element of E_i satisfying $\operatorname{inc}_i(q) \in Q$. The definition of inc_i yields $\operatorname{inc}_i(q) = (i,q)$; thus, $(i,q) = \operatorname{inc}_i(q) \in Q \subseteq E$.

Now, (158) (applied to $(j,e) = (i,q)$ and $(k,f) = (i,p)$) yields the equivalence

$$((i,q) <_1 (i,p)) \iff (i = i \text{ and } q <_{1,i} p).$$

Hence, we have $(i,q) <_1 (i,p)$ (since we have $(i = i$ and $q <_{1,i} p)$). Moreover, recall that $(i,p) \in P$ and $(i,q) \in Q$. Hence, (160) (applied to (i,p) and (i,q) instead of p and q) shows that we do not have $(i,q) <_1 (i,p)$. This contradicts $(i,q) <_1 (i,p)$. This contradiction shows that our assumption was false. This completes the proof.

[*Proof of Observation 1:* From (161), we obtain $(P_i, Q_i) \in \operatorname{Adm}(\mathbf{E}_i)$. From this, Observation 1 immediately follows by the definition of $\operatorname{Adm}(\mathbf{E}_i)$.]

Now, $\bigsqcup_{i\in I} P_i$ and $\bigsqcup_{i\in I} Q_i$ are subsets of E [117] satisfying $\left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right) = \varnothing$ [118] and $\left(\bigsqcup_{i\in I} P_i\right) \cup \left(\bigsqcup_{i\in I} Q_i\right) = E$ [119] and having the property that

$$\text{no } p \in \bigsqcup_{i\in I} P_i \text{ and } q \in \bigsqcup_{i\in I} Q_i \text{ satisfy } q <_1 p$$

[120]. In other words, $\left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right) \in \operatorname{Adm} \mathbf{E}$ (by the definition of $\operatorname{Adm} \mathbf{E}$ (since $\mathbf{E} = (E, <_1, <_2)$). This proves Proposition 10.53 (b). □

[117] *Proof.* For each $i \in I$, the set P_i is a subset of E_i (by Observation 1). In other words, for each $i \in I$, we have $P_i \subseteq E_i$. Thus, $\bigsqcup_{i\in I} \underbrace{P_i}_{\subseteq E_i} \subseteq \bigsqcup_{i\in I} E_i = E$. The same argument (but applied to Q_i instead of P_i) shows that $\bigsqcup_{i\in I} Q_i \subseteq E$. Thus, $\bigsqcup_{i\in I} P_i$ and $\bigsqcup_{i\in I} Q_i$ are subsets of E.

[118] *Proof.* Let $x \in \left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right)$. We shall derive a contradiction.

We have $x \in \left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right) \subseteq \bigsqcup_{i\in I} P_i$. Hence, Remark 10.30 (c) (applied to P_i instead of E_i) shows that there exist an $i \in I$ and an $e \in P_i$ such that $x = (i, e)$. Denote these i and e by j and f. Thus, $j \in I$ and $f \in P_j$ satisfy $x = (j, f)$.

We have $x \in \left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right) \subseteq \bigsqcup_{i\in I} Q_i$. Hence, Remark 10.30 (c) (applied to Q_i instead of E_i) shows that there exist an $i \in I$ and an $e \in Q_i$ such that $x = (i, e)$. Denote these i and e by k and g. Thus, $k \in I$ and $g \in Q_k$ satisfy $x = (k, g)$.

We have $(j, f) = x = (k, g)$. In other words, $j = k$ and $f = g$. Hence, $f = g \in Q_k = Q_j$ (since $k = j$).

Observation 1 (applied to $i = j$) yields $P_j \cap Q_j = \varnothing$. But combining $f \in P_j$ with $f \in Q_j$, we obtain $f \in P_j \cap Q_j = \varnothing$. This shows that the set $\varnothing$ is nonempty (because it contains the element f). But this is absurd.

Now, forget that we have fixed x. We thus have derived a contradiction for each $x \in \left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right)$. Hence, there exists no $x \in \left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right)$. In other words, the set $\left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right)$ is empty. In other words, $\left(\bigsqcup_{i\in I} P_i\right) \cap \left(\bigsqcup_{i\in I} Q_i\right) = \varnothing$.

[119] *Proof.* We have $E = \bigsqcup_{i\in I} E_i = \bigcup_{i\in I} (\{i\} \times E_i)$ (by the definition of $\bigsqcup_{i\in I} E_i$) and $\bigsqcup_{i\in I} P_i = \bigcup_{i\in I} (\{i\} \times P_i)$ (by the definition of $\bigsqcup_{i\in I} P_i$) and $\bigsqcup_{i\in I} Q_i = \bigcup_{i\in I} (\{i\} \times Q_i)$ (by the definition of $\bigsqcup_{i\in I} Q_i$). Now,

$$\underbrace{\left(\bigsqcup_{i\in I} P_i\right)}_{=\bigcup_{i\in I}(\{i\}\times P_i)} \cup \underbrace{\left(\bigsqcup_{i\in I} Q_i\right)}_{=\bigcup_{i\in I}(\{i\}\times Q_i)} = \left(\bigcup_{i\in I} (\{i\} \times P_i)\right) \cup \left(\bigcup_{i\in I} (\{i\} \times Q_i)\right) = \bigcup_{i\in I} \underbrace{\left((\{i\} \times P_i) \cup (\{i\} \times Q_i)\right)}_{\substack{=\{i\}\times(P_i\cup Q_i) \\ \text{(since any three sets } X, Y \text{ and } Z \\ \text{satisfy } (X\times Y)\cup(X\times Z)=X\times(Y\cup Z))}}$$

$$= \bigcup_{i\in I} \left(\{i\} \times \underbrace{(P_i \cup Q_i)}_{\substack{=E_i \\ \text{(by Observation 1)}}}\right) = \bigcup_{i\in I} (\{i\} \times E_i) = E,$$

qed.

[120] *Proof.* Assume the contrary. Thus, there exist $p \in \bigsqcup_{i\in I} P_i$ and $q \in \bigsqcup_{i\in I} Q_i$ satisfying $q <_1 p$. Fix these p and q.

We have $p \in \bigsqcup_{i\in I} P_i$. Hence, Remark 10.30 (c) (applied to p and P_i instead of x and E_i) shows that there exist an $i \in I$ and an $e \in P_i$ such that $p = (i, e)$. Denote these i and e by k and f. Thus, $k \in I$ and $f \in P_k$ satisfy $p = (k, f)$.

We have $q \in \bigsqcup_{i\in I} Q_i$. Hence, Remark 10.30 (c) (applied to q and Q_i instead of x and E_i) shows that there exist an $i \in I$ and an $e \in Q_i$ such that $q = (i, e)$. Denote these i and e by j and e. Thus,

Definition 10.54. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset.
Let $\mathbf{E}$ be the double poset $\bigsqcup_{i \in I} \mathbf{E}_i$.

(a) We define a map

$$\operatorname{Split} : \operatorname{Adm} \mathbf{E} \to \prod_{i \in I} \operatorname{Adm}(\mathbf{E}_i)$$

by

$$\left(\begin{array}{c} \operatorname{Split}((P,Q)) = \left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q) \right) \right)_{i \in I} \\ \text{for each } (P,Q) \in \operatorname{Adm} \mathbf{E} \end{array} \right).$$

(This is well-defined, because of Proposition 10.53 (a).)

(b) We define a map

$$\operatorname{Combine} : \prod_{i \in I} \operatorname{Adm}(\mathbf{E}_i) \to \operatorname{Adm} \mathbf{E}$$

by

$$\left(\begin{array}{c} \operatorname{Combine}\left(((P_i, Q_i))_{i \in I} \right) = \left(\bigsqcup_{i \in I} P_i, \bigsqcup_{i \in I} Q_i \right) \\ \text{for every } ((P_i, Q_i))_{i \in I} \in \prod_{i \in I} \operatorname{Adm}(\mathbf{E}_i) \end{array} \right).$$

(This is well-defined, because of Proposition 10.53 (b).)

Next, we need another basic lemma about sets:

Lemma 10.55. Let I be a finite set. For each $i \in I$, let E_i be a set. For each $j \in I$, consider the map $\operatorname{inc}_j : E_j \to \bigsqcup_{i \in I} E_i$.

(a) If R is a subset of $\bigsqcup_{i \in I} E_i$, then $\bigsqcup_{i \in I} (\operatorname{inc}_i)^{-1}(R) = R$.

(b) For each $i \in I$, let R_i be a subset of E_i. Let $j \in I$. Then, $(\operatorname{inc}_j)^{-1} \left(\bigsqcup_{i \in I} R_i \right) = R_j$.

Proof of Lemma 10.55. This is, again, a straightforward fact about sets, and its proof is left to the reader. □

Proposition 10.56. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset.
Let $\mathbf{E}$ be the double poset $\bigsqcup_{i \in I} \mathbf{E}_i$.
The maps Split and Combine are mutually inverse bijections.

$j \in I$ and $e \in Q_j$ satisfy $q = (j, e)$.
We have $(k, f) = p \in \bigsqcup_{i \in I} P_i \subseteq E$ and $(j, e) = q \in \bigsqcup_{i \in I} Q_i \subseteq E$. Also, $(j, e) = q <_1 p = (k, f)$. But (158) yields the equivalence

$$((j, e) <_1 (k, f)) \iff (j = k \text{ and } e <_{1,j} f).$$

Hence, we have $(j = k \text{ and } e <_{1,j} f)$ (since we have $(j, e) <_1 (k, f)$).
We have $e \in Q_j = Q_k$ (since $j = k$). Also, we have $e <_{1,j} f$. This rewrites as $e <_{1,k} f$ (since $j = k$).
But (162) (applied to $i = k$) shows that no $p \in P_k$ and $q \in Q_k$ satisfy $q <_{1,k} p$. In other words, if $p \in P_k$ and $q \in Q_k$, then we do not have $q <_{1,k} p$. Applying this to $p = f$ and $q = e$, we conclude that we do not have $e <_{1,k} f$ (since $f \in P_k$ and $e \in Q_k$). This contradicts $e <_{1,k} f$. This contradiction shows that our assumption was wrong. This completes the proof.

Proof of Proposition 10.56. For each $(P,Q) \in \operatorname{Adm}\mathbf{E}$, we have

$$\operatorname{Split}((P,Q)) = \left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i\in I} \qquad \text{(by the definition of Split)}$$
$$= \left(\left((\operatorname{inc}_j)^{-1}(P), (\operatorname{inc}_j)^{-1}(Q)\right)\right)_{j\in I} \tag{163}$$

(here, we have renamed the index i as j).

We have $\operatorname{Split}\circ\operatorname{Combine} = \operatorname{id}$ [121] and $\operatorname{Combine}\circ\operatorname{Split} = \operatorname{id}$ [122]. Hence, the maps Split and Combine are mutually inverse. Thus, these maps Split and Combine are mutually inverse bijections. This proves Proposition 10.56. □

[121] *Proof.* Let $\alpha \in \prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)$. Thus, α has the form $\alpha = (\alpha_i)_{i\in I}$, where each α_i is an element of $\operatorname{Adm}(\mathbf{E}_i)$. Consider these α_i.

For each $i \in I$, the element $\alpha_i \in \operatorname{Adm}(\mathbf{E}_i)$ has the form $\alpha_i = (P_i, Q_i)$ for two subsets P_i and Q_i of E_i (by the definition of $\operatorname{Adm}(\mathbf{E}_i)$ (since $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$)). Consider these P_i and Q_i.

Now, $\alpha = (\alpha_i)_{i\in I} = ((P_i,Q_i))_{i\in I}$ (since $\alpha_i = (P_i,Q_i)$ for all $i \in I$). Applying the map Combine to both sides of this equality, we find

$$\operatorname{Combine}(\alpha) = \operatorname{Combine}\left(((P_i,Q_i))_{i\in I}\right) = \left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right)$$

(by the definition of Combine). Hence, $\left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right) = \operatorname{Combine}(\alpha) \in \operatorname{Adm}\mathbf{E}$. Thus, (163) (applied to $(P,Q) = \left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right)$) yields

$$\operatorname{Split}\left(\left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right)\right) = \left(\left(\underbrace{(\operatorname{inc}_j)^{-1}\left(\bigsqcup_{i\in I} P_i\right)}_{\substack{=P_j \\ \text{(by Lemma 10.55 (b)} \\ \text{(applied to } R_i=P_i))}}, \underbrace{(\operatorname{inc}_j)^{-1}\left(\bigsqcup_{i\in I} Q_i\right)}_{\substack{=Q_j \\ \text{(by Lemma 10.55 (b)} \\ \text{(applied to } R_i=Q_i))}}\right)\right)_{j\in I} = \left((P_j,Q_j)\right)_{j\in I}$$
$$= ((P_i,Q_i))_{i\in I} \qquad \text{(here, we have renamed the index } j \text{ as } i)$$
$$= \alpha.$$

But

$$(\operatorname{Split}\circ\operatorname{Combine})(\alpha) = \operatorname{Split}\left(\underbrace{\operatorname{Combine}(\alpha)}_{=\left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right)}\right) = \operatorname{Split}\left(\left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right)\right) = \alpha = \operatorname{id}(\alpha).$$

Now, forget that we fixed α. We thus have shown that $(\operatorname{Split}\circ\operatorname{Combine})(\alpha) = \operatorname{id}(\alpha)$ for each $\alpha \in \prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)$. In other words, $\operatorname{Split}\circ\operatorname{Combine} = \operatorname{id}$.

[122] *Proof.* Let $\alpha \in \operatorname{Adm}\mathbf{E}$. Recall that

$$\mathbf{E} = \bigsqcup_{i\in I}\mathbf{E}_i = \left(\bigsqcup_{i\in I} E_i, \bigoplus_{i\in I}(<_{1,i}), \bigoplus_{i\in I}(<_{2,i})\right)$$

(by the definition of the double poset $\bigsqcup_{i\in I}\mathbf{E}_i$).

The element $\alpha \in \operatorname{Adm}\mathbf{E}$ has the form $\alpha = (P,Q)$ for two subsets P and Q of $\bigsqcup_{i\in I} E_i$ (by the definition of $\operatorname{Adm}\mathbf{E}$ (since $\mathbf{E} = \left(\bigsqcup_{i\in I} E_i, \bigoplus_{i\in I}(<_{1,i}), \bigoplus_{i\in I}(<_{2,i})\right)$)). Consider these P and Q.

Lemma 10.57. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. For each $i \in I$, let $w_i : E_i \to \{1, 2, 3, \ldots\}$ be a map. For each $i \in I$, let T_i be a subset of E_i.

Let $\mathbf{E}$ be the double poset $\bigsqcup_{i \in I} \mathbf{E}_i$. Then,

$$\Gamma\left(\mathbf{E} \mid_{\bigsqcup_{i\in I} T_i}, w \mid_{\bigsqcup_{i\in I} T_i}\right) = \prod_{i \in I} \Gamma\left(\mathbf{E}_i \mid_{T_i}, (w \circ \operatorname{inc}_i) \mid_{T_i}\right).$$

Proof of Lemma 10.57. We notice that the notation inc_j (for $j \in I$) is slightly ambiguous: It may mean both the map $\operatorname{inc}_j : E_j \to \bigsqcup_{i \in I} E_i$ and the map $\operatorname{inc}_j : T_j \to \bigsqcup_{i \in I} T_i$. In order to resolve this ambiguity, let us agree to denote the latter map by $\operatorname{inc}_{j,T}$ (instead of just calling it inc_j). Thus, inc_j shall only mean the map $\operatorname{inc}_j : E_j \to \bigsqcup_{i \in I} E_i$.

Set $T = \bigsqcup_{i \in I} T_i$.

Each $j \in I$ satisfies

$$(w \mid_T) \circ \operatorname{inc}_{j,T} = \left(w \circ \operatorname{inc}_j\right) \mid_{T_j} \tag{164}$$

[123].

But for each $i \in I$, we have $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ and therefore $\mathbf{E}_i \mid_{T_i} = (T_i, <_{1,i}, <_{2,i})$ (by the definition of $\mathbf{E}_i \mid_{T_i}$). Furthermore, $w \mid_T$ is a map from $\bigsqcup_{i \in I} T_i$ to $\{1, 2, 3, \ldots\}$ (because the domain of

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Now, $\alpha = (P, Q)$. Applying the map Split to both sides of this equality, we find

$$\operatorname{Split}(\alpha) = \operatorname{Split}((P, Q)) = \left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i \in I}$$

(by the definition of Split). Hence, $\left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i \in I} = \operatorname{Split}(\alpha) \in \prod_{i \in I} \operatorname{Adm}(\mathbf{E}_i)$. Thus, the definition of the map Combine yields

$$\operatorname{Combine}\left(\left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i \in I}\right) = \left(\underbrace{\bigsqcup_{i \in I} (\operatorname{inc}_i)^{-1}(P)}_{\substack{=P \\ \text{(by Lemma 10.55 (a)} \\ \text{(applied to } R=P))}}, \underbrace{\bigsqcup_{i \in I} (\operatorname{inc}_i)^{-1}(Q)}_{\substack{=Q \\ \text{(by Lemma 10.55 (a)} \\ \text{(applied to } R=Q))}}\right) = (P, Q) = \alpha.$$

But

$$(\operatorname{Combine} \circ \operatorname{Split})(\alpha) = \operatorname{Combine}\left(\underbrace{\operatorname{Split}(\alpha)}_{=\left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i \in I}}\right) = \operatorname{Combine}\left(\left(\left((\operatorname{inc}_i)^{-1}(P), (\operatorname{inc}_i)^{-1}(Q)\right)\right)_{i \in I}\right) = \alpha = \operatorname{id}(\alpha).$$

Now, forget that we fixed α. We thus have shown that $(\operatorname{Combine} \circ \operatorname{Split})(\alpha) = \operatorname{id}(\alpha)$ for each $\alpha \in \operatorname{Adm} \mathbf{E}$. In other words, $\operatorname{Combine} \circ \operatorname{Split} = \operatorname{id}$.

[123] *Proof of (164):* Let $j \in I$.

Notice that the map $(w \mid_T) \circ \operatorname{inc}_{j,T}$ is well-defined, since $\operatorname{inc}_{j,T}$ is a map from T_j to $\bigsqcup_{i \in I} T_i = T$.

Let $g \in T_j$. Then, $\operatorname{inc}_j(g) = (j, g)$ (by the definition of the map inc_j). Applying the map w to both sides of this equality, we obtain $w\left(\operatorname{inc}_j(g)\right) = w((j, g))$. On the other hand, $\operatorname{inc}_{j,T}(g) = (j, g)$ (by the definition of the map $\operatorname{inc}_{j,T}$). Applying the map w to both sides of this equality, we

$w\mid_T$ is $T = \bigsqcup_{i\in I} T_i$). Now,

$$\Gamma\left(\underbrace{\mathbf{E}\mid_{\bigsqcup_{i\in I} T_i}}_{\substack{=\bigsqcup_{i\in I}\left(\mathbf{E}_i\mid_{T_i}\right)\\ \text{(by Proposition 10.49)}}}, \underbrace{w\mid_{\bigsqcup_{i\in I} T_i}}_{\substack{=w\mid_T\\ \text{(since } \bigsqcup_{i\in I} T_i=T)}}\right)$$

$$= \Gamma\left(\bigsqcup_{i\in I}\left(\mathbf{E}_i\mid_{T_i}\right), w\mid_T\right) = \prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{T_i}, \underbrace{(w\mid_T)\circ \operatorname{inc}_{i,T}}_{\substack{=(w\circ \operatorname{inc}_i)\mid_{T_i}\\ \text{(by (164))}\\ \text{(applied to } j=i))}}\right)$$

$$\left(\begin{array}{c}\text{by Proposition 10.44 (applied to } \mathbf{E}_i\mid_{T_i},\ T_i,\ <_{1,i},\ <_{2,i} \text{ and } w\mid_T\\ \text{instead of } \mathbf{E}_i,\ E_i,\ <_{1,i},\ <_{2,i} \text{ and } w)\end{array}\right)$$

$$= \prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{T_i}, (w\circ \operatorname{inc}_i)\mid_{T_i}\right).$$

This proves Lemma 10.57. □

Corollary 10.58. Let I be a finite set. For each $i \in I$, let $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ be a double poset. For each $i \in I$, let $w_i : E_i \to \{1,2,3,\ldots\}$ be a map. Then,

$$\Delta\left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i, w_i\right)\right) = \prod_{i\in I}\Delta\left(\Gamma\left(\mathbf{E}_i, w_i\right)\right).$$

Proof of Corollary 10.58. Let $\mathbf{E}$ be the double poset $\bigsqcup_{i\in I}\mathbf{E}_i$. Then,

$$\mathbf{E} = \bigsqcup_{i\in I}\mathbf{E}_i = \left(\bigsqcup_{i\in I} E_i, \bigoplus_{i\in I}(<_{1,i}), \bigoplus_{i\in I}(<_{2,i})\right)$$

(by the definition of $\bigsqcup_{i\in I}\mathbf{E}_i$).

Proposition 10.56 shows that the maps Split and Combine are mutually inverse bijections. In particular, the map Combine is a bijection. In other words, the map

$$\prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i) \to \operatorname{Adm}\mathbf{E}, \qquad ((P_i, Q_i))_{i\in I} \mapsto \left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right) \tag{165}$$

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obtain $w\left(\operatorname{inc}_{j,T}(g)\right) = w((j,g))$. Now,

$$\left((w\circ \operatorname{inc}_j)\mid_{T_j}\right)(g) = \left(w\circ \operatorname{inc}_j\right)(g) = w\left(\operatorname{inc}_j(g)\right) = w((j,g)).$$

Comparing this with

$$\left((w\mid_T)\circ \operatorname{inc}_{j,T}\right)(g) = (w\mid_T)\left(\operatorname{inc}_{j,T}(g)\right) = w\left(\operatorname{inc}_{j,T}(g)\right) = w((j,g)),$$

we obtain $\left((w\circ \operatorname{inc}_j)\mid_{T_j}\right)(g) = \left((w\mid_T)\circ \operatorname{inc}_{j,T}\right)(g)$.

Now, forget that we fixed g. We thus have proven that $\left((w\circ \operatorname{inc}_j)\mid_{T_j}\right)(g) = \left((w\mid_T)\circ \operatorname{inc}_{j,T}\right)(g)$ for each $g \in T_j$. In other words, $(w\circ \operatorname{inc}_j)\mid_{T_j} = (w\mid_T)\circ \operatorname{inc}_{j,T}$. Thus, the proof of (164) is complete.

is a bijection (since this map is the map Combine [124]).

Let $F = \{1, 2, 3, \ldots\}$. For each $i \in I$, we have $w_i \in \{1, 2, 3, \ldots\}^{E_i} = F^{E_i}$ (since $\{1, 2, 3, \ldots\} = F$). Thus, $(w_i)_{i\in I} \in \prod_{i\in I} F^{E_i}$.

Corollary 10.37 shows that the map $\operatorname{Restr} : F^{\bigsqcup_{i\in I} E_i} \to \prod_{i\in I} F^{E_i}$ is a bijection. Hence, $\operatorname{Restr}^{-1}\left((w_i)_{i\in I}\right)$ is a well-defined element of $F^{\bigsqcup_{i\in I} E_i}$. Denote this element by w. Then, Corollary 10.45 shows that

$$\prod_{i\in I} \Gamma\left(\mathbf{E}_i, w_i\right) = \Gamma\left(\underbrace{\bigsqcup_{i\in I} \mathbf{E}_i}_{=\mathbf{E}}, w\right) = \Gamma\left(\mathbf{E}, w\right).$$

Applying the map Δ to both sides of this equality, we obtain

$$\begin{aligned}
&\Delta\left(\prod_{i\in I} \Gamma\left(\mathbf{E}_i, w_i\right)\right) \\
&= \Delta\left(\Gamma\left(\mathbf{E}, w\right)\right) \\
&= \sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} \Gamma\left(\mathbf{E}\mid_P, w\mid_P\right) \otimes \Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right) \qquad \text{(by Proposition 5.6)} \\
&= \sum_{((P_i,Q_i))_{i\in I}\in\prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)} \Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} P_i}, w\mid_{\bigsqcup_{i\in I} P_i}\right) \otimes \Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} Q_i}, w\mid_{\bigsqcup_{i\in I} Q_i}\right) \qquad (166) \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right) \text{ for } (P,Q) \text{ in the sum,} \\ \text{since the map (165) is a bijection}\end{array}\right).
\end{aligned}$$

We have $w = \operatorname{Restr}^{-1}\left((w_i)_{i\in I}\right)$ (by the definition of w) and thus

$$(w_i)_{i\in I} = \operatorname{Restr}(w) = (w \circ \operatorname{inc}_i)_{i\in I} \qquad \text{(by the definition of Restr)}.$$

In other words,

$$w_i = w \circ \operatorname{inc}_i \qquad \text{for each } i \in I. \tag{167}$$

But the following holds:

Observation 1: Let $((P_i, Q_i))_{i\in I} \in \prod_{i\in I} \operatorname{Adm}(\mathbf{E}_i)$. Then,

$$\Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} P_i}, w\mid_{\bigsqcup_{i\in I} P_i}\right) = \prod_{i\in I} \Gamma\left(\mathbf{E}_i\mid_{P_i}, w_i\mid_{P_i}\right) \tag{168}$$

and

$$\Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} Q_i}, w\mid_{\bigsqcup_{i\in I} Q_i}\right) = \prod_{i\in I} \Gamma\left(\mathbf{E}_i\mid_{Q_i}, w_i\mid_{Q_i}\right) \tag{169}$$

[*Proof of Observation 1:* Let $((P_i, Q_i))_{i\in I} \in \prod_{i\in I} \operatorname{Adm}(\mathbf{E}_i)$. Thus, for each $i \in I$, we have $(P_i, Q_i) \in \operatorname{Adm}(\mathbf{E}_i)$. Hence, for each $i \in I$, the sets P_i and Q_i are two subsets of E_i (by the definition of $\operatorname{Adm}(\mathbf{E}_i)$) (since $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$)). Hence, Lemma 10.57 (applied to $T_i = P_i$) yields

$$\Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} P_i}, w\mid_{\bigsqcup_{i\in I} P_i}\right) = \prod_{i\in I} \Gamma\left(\mathbf{E}_i\mid_{P_i}, w_i\mid_{P_i}\right).$$

[124] because $\operatorname{Combine}\left(((P_i, Q_i))_{i\in I}\right) = \left(\bigsqcup_{i\in I} P_i, \bigsqcup_{i\in I} Q_i\right)$ for every $((P_i, Q_i))_{i\in I} \in \prod_{i\in I} \operatorname{Adm}(\mathbf{E}_i)$

Furthermore, Lemma 10.57 (applied to $T_i = Q_i$) yields

$$\Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} Q_i}, w\mid_{\bigsqcup_{i\in I} Q_i}\right) = \prod_{i\in I} \Gamma\left(\mathbf{E}_i\mid_{Q_i}, w_i\mid_{Q_i}\right).$$

This completes the proof of Observation 1.]

Now, (166) rewrites as follows:

$$\begin{aligned}
&\Delta\left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i, w_i\right)\right) \\
&= \sum_{((P_i,Q_i))_{i\in I}\in\prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)} \underbrace{\Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} P_i}, w\mid_{\bigsqcup_{i\in I} P_i}\right)}_{\substack{=\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{P_i}, w_i\mid_{P_i}\right) \\ \text{(by (168))}}} \otimes \underbrace{\Gamma\left(\mathbf{E}\mid_{\bigsqcup_{i\in I} Q_i}, w\mid_{\bigsqcup_{i\in I} Q_i}\right)}_{\substack{=\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{Q_i}, w_i\mid_{Q_i}\right) \\ \text{(by (169))}}} \\
&= \sum_{((P_i,Q_i))_{i\in I}\in\prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)} \left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{P_i}, w_i\mid_{P_i}\right)\right)\otimes\left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{Q_i}, w_i\mid_{Q_i}\right)\right).
\end{aligned} \tag{170}$$

On the other hand, for each $i \in I$, we have

$$\Delta\left(\Gamma\left(\mathbf{E}_i, w_i\right)\right) = \sum_{(P,Q)\in\operatorname{Adm}(\mathbf{E}_i)} \Gamma\left(\mathbf{E}_i\mid_P, w_i\mid_P\right)\otimes\Gamma\left(\mathbf{E}_i\mid_Q, w_i\mid_Q\right) \tag{171}$$

[125]. Hence,

$$\begin{aligned}
&\prod_{i\in I} \underbrace{\Delta\left(\Gamma\left(\mathbf{E}_i, w_i\right)\right)}_{\substack{=\sum_{(P,Q)\in\operatorname{Adm}(\mathbf{E}_i)}\Gamma\left(\mathbf{E}_i\mid_P, w_i\mid_P\right)\otimes\Gamma\left(\mathbf{E}_i\mid_Q, w_i\mid_Q\right) \\ \text{(by (171))}}} \\
&= \prod_{i\in I}\sum_{(P,Q)\in\operatorname{Adm}(\mathbf{E}_i)} \Gamma\left(\mathbf{E}_i\mid_P, w_i\mid_P\right)\otimes\Gamma\left(\mathbf{E}_i\mid_Q, w_i\mid_Q\right) \\
&= \sum_{((P_i,Q_i))_{i\in I}\in\prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)} \underbrace{\prod_{i\in I}\left(\Gamma\left(\mathbf{E}_i\mid_{P_i}, w_i\mid_{P_i}\right)\otimes\Gamma\left(\mathbf{E}_i\mid_{Q_i}, w_i\mid_{Q_i}\right)\right)}_{\substack{=\left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{P_i}, w_i\mid_{P_i}\right)\right)\otimes\left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{Q_i}, w_i\mid_{Q_i}\right)\right) \\ \text{(by the definition of the multiplication on } \operatorname{QSym}\otimes\operatorname{QSym}\text{)}}} \\
&\qquad \text{(by the product rule)} \\
&= \sum_{((P_i,Q_i))_{i\in I}\in\prod_{i\in I}\operatorname{Adm}(\mathbf{E}_i)} \left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{P_i}, w_i\mid_{P_i}\right)\right)\otimes\left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i\mid_{Q_i}, w_i\mid_{Q_i}\right)\right) \\
&= \Delta\left(\prod_{i\in I}\Gamma\left(\mathbf{E}_i, w_i\right)\right) \qquad \text{(by (170))}.
\end{aligned}$$

This proves Corollary 10.58. □

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[125] *Proof of (171):* Let $i \in I$. Recall that $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ is a double poset. Hence, Proposition 5.6 (applied to $\mathbf{E}_i$, $<_{1,i}$, $<_{2,i}$ and w_i instead of $\mathbf{E}$, $<_1$, $<_2$ and w) yields

$$\Delta\left(\Gamma\left(\mathbf{E}_i, w_i\right)\right) = \sum_{(P,Q)\in\operatorname{Adm}(\mathbf{E}_i)} \Gamma\left(\mathbf{E}_i\mid_P, w_i\mid_P\right)\otimes\Gamma\left(\mathbf{E}_i\mid_Q, w_i\mid_Q\right).$$

This proves (171).

Corollary 10.59. Let $a \in \operatorname{QSym}$ and $b \in \operatorname{QSym}$. Then, $\Delta(ab) = \Delta(a)\Delta(b)$.

Proof of Corollary 10.59. It is well-known that QSym is a $\mathbf{k}$-submodule of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, and that $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of this $\mathbf{k}$-module QSym.

We shall now prove that

$$\Delta\left(M_\alpha M_\beta\right) = \Delta\left(M_\alpha\right)\Delta\left(M_\beta\right) \qquad \text{for any } \alpha \in \operatorname{Comp} \text{ and } \beta \in \operatorname{Comp}. \tag{172}$$

[*Proof of (172):* Let $\alpha \in \operatorname{Comp}$ and $\beta \in \operatorname{Comp}$.

As in the proof of (152), we can

- find a set E_1, a special double poset $\mathbf{E}_1 = (E_1, <_{1,1}, >_{1,1})$, and a map $w_1 : E_1 \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}_1, w_1) = M_\beta$;
- find a set E_0, a special double poset $\mathbf{E}_0 = (E_0, <_{1,0}, >_{1,0})$, and a map $w_0 : E_0 \to \{1, 2, 3, \ldots\}$ satisfying $\Gamma(\mathbf{E}_0, w_0) = M_\alpha$;
- show that $\mathbf{E}_i = (E_i, <_{1,i}, >_{1,i})$ for every $i \in \{0, 1\}$;
- show that w_i is a map $E_i \to \{1, 2, 3, \ldots\}$ for each $i \in \{0, 1\}$.

Now, Corollary 10.58 (applied to $\{0, 1\}$ and $>_{1,i}$ instead of I and $<_{2,i}$) shows that

$$\Delta\left(\prod_{i \in \{0,1\}} \Gamma(\mathbf{E}_i, w_i)\right) = \prod_{i \in \{0,1\}} \Delta(\Gamma(\mathbf{E}_i, w_i)).$$

Since

$$\prod_{i \in \{0,1\}} \Gamma(\mathbf{E}_i, w_i) = \underbrace{\Gamma(\mathbf{E}_0, w_0)}_{= M_\alpha} \underbrace{\Gamma(\mathbf{E}_1, w_1)}_{= M_\beta} = M_\alpha M_\beta,$$

this rewrites as $\Delta\left(M_\alpha M_\beta\right) = \prod\limits_{i \in \{0,1\}} \Delta(\Gamma(\mathbf{E}_i, w_i))$. Hence,

$$\Delta\left(M_\alpha M_\beta\right) = \prod_{i \in \{0,1\}} \Delta(\Gamma(\mathbf{E}_i, w_i)) = \Delta\left(\underbrace{\Gamma(\mathbf{E}_0, w_0)}_{= M_\alpha}\right) \Delta\left(\underbrace{\Gamma(\mathbf{E}_1, w_1)}_{= M_\beta}\right) = \Delta\left(M_\alpha\right)\Delta\left(M_\beta\right).$$

Thus, (172) is proven.]

Now, we can see that

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \text{for any } a \in \operatorname{QSym} \text{ and } b \in \operatorname{QSym}. \tag{173}$$

[*Proof of (173):* Let $a \in \operatorname{QSym}$ and $b \in \operatorname{QSym}$. We must prove the relation $\Delta(ab) = \Delta(a)\Delta(b)$.

This relation is $\mathbf{k}$-linear in b (since Δ is a $\mathbf{k}$-linear map). Hence, we can WLOG assume that b belongs to the basis $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ of the $\mathbf{k}$-module QSym. Assume this. Thus, $b = M_\beta$ for some $\beta \in \operatorname{Comp}$. Consider this β.

We must prove the relation $\Delta(ab) = \Delta(a)\Delta(b)$. This relation is $\mathbf{k}$-linear in a (since Δ is a $\mathbf{k}$-linear map). Hence, we can WLOG assume that a belongs to the basis $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ of the $\mathbf{k}$-module QSym. Assume this. Thus, $a = M_\alpha$ for some $\alpha \in \operatorname{Comp}$. Consider this α.

Now, (172) yields $\Delta\left(M_\alpha M_\beta\right) = \Delta\left(M_\alpha\right)\Delta\left(M_\beta\right)$. Since $a = M_\alpha$ and $b = M_\beta$, this rewrites as $\Delta(ab) = \Delta(a)\Delta(b)$. This proves (173).]

Corollary 10.59 immediately follows from (173). $\square$

Now, we can prove that QSym is a bialgebra:

Proposition 10.60. The $\mathbf{k}$-algebra QSym, equipped with the comultiplication Δ and the counit ε, becomes a $\mathbf{k}$-bialgebra.

Proof of Proposition 10.60. Let ε' be the map

$$\mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}, \qquad f \mapsto f(0,0,0,\ldots).$$

Then, ε' is a substitution homomorphism (since it substitutes $(0,0,0,\ldots)$ for the indeterminates $x_1, x_2, x_3, \ldots$), and therefore is a $\mathbf{k}$-algebra homomorphism.

Now, $\varepsilon = \varepsilon' \mid_{\mathrm{QSym}}$ [126]. Hence, ε is a restriction of the map ε'. Thus, ε is a restriction of a $\mathbf{k}$-algebra homomorphism (since ε' is a $\mathbf{k}$-algebra homomorphism), and thus itself is a $\mathbf{k}$-algebra homomorphism.

It is easy to see that $\Delta(1) = 1 \otimes 1$ [127]. Thus, the map Δ sends the unity of the $\mathbf{k}$-algebra QSym to the unity of the $\mathbf{k}$-algebra QSym $\otimes$ QSym (since the latter unity is $1 \otimes 1$). Combining this with Corollary 10.59, we conclude that Δ is a $\mathbf{k}$-algebra homomorphism (since Δ is a $\mathbf{k}$-linear map).

Now, the $\mathbf{k}$-algebra QSym, equipped with the comultiplication Δ and the counit ε, becomes a $\mathbf{k}$-bialgebra (because it is a $\mathbf{k}$-algebra and a $\mathbf{k}$-coalgebra, and because Δ and ε are $\mathbf{k}$-algebra homomorphisms). This proves Proposition 10.60. ☐

Finally, using [GriRei14, Proposition 1.36], we can leverage Proposition 10.60 to a proof of the following fact:

Proposition 10.61. The $\mathbf{k}$-bialgebra QSym is a Hopf algebra (i.e., it has an antipode).

Proof of Proposition 10.61 (sketched). It is not hard to see that the $\mathbf{k}$-bialgebra QSym is a connected graded $\mathbf{k}$-bialgebra, where the grading is given by the degree of the power series (i.e., for each $n \in \mathbb{N}$, the n-th graded component of QSym is the $\mathbf{k}$-submodule

$$\{f \in \mathrm{QSym} \mid \text{the power series } f \text{ is homogeneous of degree } n\}$$

of QSym [128]).

A well-known result states that every connected graded $\mathbf{k}$-bialgebra is a Hopf algebra (i.e., it has an antipode).[129] Applying this to the connected graded $\mathbf{k}$-bialgebra QSym, we thus conclude that the $\mathbf{k}$-bialgebra QSym is a Hopf algebra (i.e., it has an antipode). This proves Proposition 10.61. ☐

[126] *Proof.* Recall that the map ε sends every power series $f \in \mathrm{QSym}$ to the result $f(0,0,0,\ldots)$ of substituting zeroes for the variables $x_1, x_2, x_3, \ldots$ in f. Thus, for each $f \in \mathrm{QSym}$, we have

$$\begin{aligned}\varepsilon(f) &= f(0,0,0,\ldots) = \varepsilon'(f) \qquad \left(\text{since } \varepsilon'(f) = f(0,0,0,\ldots) \text{ (by the definition of } \varepsilon')\right) \\ &= \left(\varepsilon' \mid_{\mathrm{QSym}}\right)(f).\end{aligned}$$

In other words, $\varepsilon = \varepsilon' \mid_{\mathrm{QSym}}$.

[127] *Proof.* There are many ways to prove $\Delta(1) = 1 \otimes 1$ (for example, it follows from $1 = M_{\varnothing}$ using the definition of Δ), but let us derive $\Delta(1) = 1 \otimes 1$ from Corollary 10.58:

For each $i \in \varnothing$, we define a double poset $\mathbf{E}_i = (E_i, <_{1,i}, <_{2,i})$ and a map $w_i : E_i \to \{1,2,3,\ldots\}$ as follows: There is nothing to define, because there exists no $i \in \varnothing$.

Thus, Corollary 10.58 (applied to $I = \varnothing$) yields

$$\begin{aligned}\Delta\left(\prod_{i \in \varnothing} \Gamma(\mathbf{E}_i, w_i)\right) &= \prod_{i \in \varnothing} \Delta(\Gamma(\mathbf{E}_i, w_i)) = (\text{empty product}) \\ &= (\text{the unity of the } \mathbf{k}\text{-algebra } \mathrm{QSym} \otimes \mathrm{QSym}) = 1 \otimes 1.\end{aligned}$$

Since $\prod_{i \in \varnothing} \Gamma(\mathbf{E}_i, w_i) = (\text{empty product}) = 1$, this rewrites as $\Delta(1) = 1 \otimes 1$. Qed.

[128] This $\mathbf{k}$-submodules has basis $(M_\alpha)_{\alpha \in \mathrm{Comp}_n}$.

[129] This is proven, for example, in [GriRei14, Proposition 1.36].

There is also an alternative way to prove Proposition 10.61, by constructing the antipode explicitly (e.g., using Proposition 10.70 as the **definition** of the antipode) and then showing that it satisfies the axioms of an antipode. We shall not dwell on this.

10.10. F_α as $\Gamma(\mathbf{E})$

Next, let us prove a claim made in Example 3.6 (c):

Proposition 10.62. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Define a set $D(\alpha)$ as in Definition 10.3. Let $E = \{1, 2, \ldots, n\}$. Then, there exists a total order $<_2$ on the set E satisfying

$$(i + 1 <_2 i \qquad \text{for every } i \in D(\alpha)) \tag{174}$$

and

$$(i <_2 i + 1 \qquad \text{for every } i \in \{1, 2, \ldots, n-1\} \setminus D(\alpha)). \tag{175}$$

We shall actually prove the following fact first (which is easily seen to be equivalent to Proposition 10.62):

Proposition 10.63. Let $n \in \mathbb{N}$. Let $E = \{1, 2, \ldots, n\}$. Let Q be a subset of $\{1, 2, \ldots, n-1\}$. Then, there exists a total order $<_2$ on the set E satisfying

$$(i + 1 <_2 i \qquad \text{for every } i \in Q) \tag{176}$$

and

$$(i <_2 i + 1 \qquad \text{for every } i \in \{1, 2, \ldots, n-1\} \setminus Q). \tag{177}$$

Usually, there are, in fact, several total orders $<_2$ satisfying the conditions of Proposition 10.63, but of course it suffices to construct one of them in order to prove the proposition.

Before we prove Proposition 10.63, let us record a really simple fact:

Proposition 10.64. Let E and F be two sets. Let $\omega : E \to F$ be a map. Let $<$ be a strict partial order on the set F. We define a binary relation $\prec$ on the set E as follows: For any $a \in E$ and $b \in E$, we set $a \prec b$ if and only if $\omega(a) < \omega(b)$.

(a) The relation $\prec$ is a strict partial order on E.

(b) Assume that the relation $<$ is a total order. Assume that the map ω is injective. Then, the relation $\prec$ is a total order.

Proposition 10.64 is a basic fact about partial and total orders; its easy proof is left to the reader.

Proof of Proposition 10.63. We shall use the notation introduced in Definition 10.2 (that is, we shall write $[k]$ for $\{1, 2, \ldots, k\}$ when $k \in \mathbb{Z}$). In particular, $[n] = \{1, 2, \ldots, n\} = E$.

We shall also use the so-called *Iverson bracket notation*: If $\mathcal{A}$ is any logical statement, then we shall write $[\mathcal{A}]$ for the integer $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$.

Now, we define a map $\rho : [n] \to \mathbb{Z}$ by

$$(\rho(i) = |Q \cap [i-1]| \qquad \text{for every } i \in [n]).$$

We also define a map $\omega : [n] \to \mathbb{Z}$ by

$$(\omega(i) = -n\rho(i) + i \qquad \text{for every } i \in [n]).$$

The map ω is injective[130]. Notice that ω is a map $[n] \to \mathbb{Z}$. In other words, ω is a map $E \to \mathbb{Z}$ (since $[n] = E$).

Consider the total order $<$ on the set $\mathbb{Z}$ (that is, the usual smaller relation on $\mathbb{Z}$). Define a binary relation $\prec$ on the set E as follows: For any $a \in E$ and $b \in E$, we set $a \prec b$ if and only if $\omega(a) < \omega(b)$. Proposition 10.64 (b) (applied to $F = \mathbb{Z}$) thus shows that the relation $\prec$ is a total order.

Now, it is easy to show that

$$\rho(i+1) - \rho(i) = [i \in Q] \qquad \text{for every } i \in [n-1] \tag{178}$$

[131].

We now claim that

$$(i+1 \prec i \qquad \text{for every } i \in Q). \tag{179}$$

[130] *Proof.* Let i and j be two elements of $[n]$ such that $\omega(i) = \omega(j)$. We shall show that $i = j$.

Assume the contrary. Thus, $i \neq j$. We can WLOG assume that $i \geq j$ (since otherwise, we can just switch i with j). Assume this. Combining $i \geq j$ with $i \neq j$, we obtain $i > j$, so that $i - j > 0$.

We have $i \in [n]$, so that $1 \leq i \leq n$. Thus, $n \geq 1$, so that $n > 0$. Combined with $i - j > 0$, this yields $\dfrac{i-j}{n} > 0$.

The definition of ω shows that $\omega(i) = \underbrace{-n\rho(i)}_{\equiv 0 \bmod n} + i \equiv i \bmod n$. The same argument (applied to j instead of i) yields $\omega(j) \equiv j \bmod n$. Thus, $i \equiv \omega(i) = \omega(j) \equiv j \bmod n$. In other words, $n \mid i - j$. Thus, $\dfrac{i-j}{n}$ is an integer. Hence, $\dfrac{i-j}{n} \geq 1$ (since $\dfrac{i-j}{n} > 0$), so that $i - j \geq n$ (since $n > 0$). Hence, $i \geq n + \underbrace{j}_{\substack{>0 \\ (\text{since } j \in [n])}} > n$. This contradicts $i \leq n$. This contradiction proves that our assumption was wrong. Hence, $i = j$ is proven.

Now, forget that we fixed i and j. We thus have shown that if i and j are two elements of $[n]$ such that $\omega(i) = \omega(j)$, then $i = j$. In other words, the map ω is injective. Qed.

[131] *Proof of (178):* Let $i \in [n-1]$. We must prove (178). We are in one of the following two cases:

Case 1: We have $i \in Q$.

Case 2: We have $i \notin Q$.

Let us first consider Case 1. In this case, we have $i \in Q$. Hence, $\{i\} \subseteq Q$, so that $Q \cap \{i\} = \{i\}$. But

$$(Q \cap [i-1]) \cap \{i\} = Q \cap \underbrace{[i-1] \cap \{i\}}_{\substack{=\varnothing \\ (\text{since } i \notin [i-1])}} = Q \cap \varnothing = \varnothing.$$

Also,

$$Q \cap \underbrace{[i]}_{=[i-1] \cup \{i\}} = Q \cap ([i-1] \cup \{i\}) = (Q \cap [i-1]) \cup \underbrace{(Q \cap \{i\})}_{=\{i\}} = (Q \cap [i-1]) \cup \{i\}.$$

Hence,

$$\begin{aligned} \left| \underbrace{Q \cap [i]}_{=(Q \cap [i-1]) \cup \{i\}} \right| &= |(Q \cap [i-1]) \cup \{i\}| \\ &= \underbrace{|Q \cap [i-1]|}_{\substack{=\rho(i) \\ (\text{since } \rho(i) = |Q \cap [i-1]|)}} + \underbrace{|\{i\}|}_{=1} \qquad (\text{since } (Q \cap [i-1]) \cap \{i\} = \varnothing) \\ &= \rho(i) + 1. \end{aligned}$$

[*Proof of (179):* Let $i \in Q$. Thus, $i \in Q \subseteq [n-1]$. Thus, $1 \leq i \leq n-1$, so that $n-1 \geq 1$. Also, $i \in [n-1]$, so that (178) yields $\rho(i+1) - \rho(i) = [i \in Q] = 1$ (since $i \in Q$). Now, the definition of ω yields $\omega(i+1) = -n\rho(i+1) + (i+1)$ and $\omega(i) = -n\rho(i) + i$. Hence,

$$\begin{aligned} \underbrace{\omega(i+1)}_{=-n\rho(i+1)+(i+1)} - \underbrace{\omega(i)}_{=-n\rho(i)+i} &= (-n\rho(i+1) + (i+1)) - (-n\rho(i) + i) \\ &= -n \underbrace{(\rho(i+1) - \rho(i))}_{=1} + \underbrace{(i+1) - i}_{=1} \\ &= -n + 1 = -\underbrace{(n-1)}_{\geq 1 > 0} < -0 = 0. \end{aligned}$$

In other words, $\omega(i+1) < \omega(i)$.

But $i+1 \prec i$ holds if and only if $\omega(i+1) < \omega(i)$ (by the definition of the relation $\prec$). Hence, $i+1 \prec i$ holds (since $\omega(i+1) < \omega(i)$). This proves (179).]

Furthermore, we claim that

$$(i \prec i+1 \qquad \text{for every } i \in \{1,2,\ldots,n-1\} \setminus Q). \tag{180}$$

[*Proof of (180):* Let $i \in \{1,2,\ldots,n-1\} \setminus Q$. Thus, $i \in \{1,2,\ldots,n-1\}$ but $i \notin Q$. We have $i \in \{1,2,\ldots,n-1\} = [n-1]$. Thus, (178) yields $\rho(i+1) - \rho(i) = [i \in Q] = 0$ (since $i \notin Q$). Now,

---

Now, the definition of ρ yields

$$\rho(i+1) = \left| Q \cap \left[\underbrace{(i+1) - 1}_{=i} \right] \right| = |Q \cap [i]| = \rho(i) + 1,$$

so that $\rho(i+1) - \rho(i) = 1 = [i \in Q]$ (since $[i \in Q] = 1$ (since $i \in Q$)). Thus, (178) is proven in Case 1.

Let us now consider Case 2. In this case, we have $i \notin Q$. Thus, $\{i\} \cap Q = \varnothing$. Now,

$$\begin{aligned} Q \cap \underbrace{[i]}_{=[i-1]\cup\{i\}} &= Q \cap ([i-1] \cup \{i\}) = (Q \cap [i-1]) \cup \underbrace{(Q \cap \{i\})}_{=\{i\}\cap Q = \varnothing} \\ &= (Q \cap [i-1]) \cup \varnothing = Q \cap [i-1]. \end{aligned}$$

Hence,

$$\left| \underbrace{Q \cap [i]}_{=Q\cap[i-1]} \right| = |Q \cap [i-1]| = \rho(i) \qquad (\text{since } \rho(i) = |Q \cap [i-1]|).$$

Now, the definition of ρ yields

$$\rho(i+1) = \left| Q \cap \left[\underbrace{(i+1) - 1}_{=i} \right] \right| = |Q \cap [i]| = \rho(i),$$

so that $\rho(i+1) - \rho(i) = 0 = [i \in Q]$ (since $[i \in Q] = 0$ (since $i \notin Q$)). Thus, (178) is proven in Case 2.

We have now proven (178) in each of the two Cases 1 and 2. Therefore, (178) always holds.

the definition of ω yields $\omega(i+1) = -n\rho(i+1) + (i+1)$ and $\omega(i) = -n\rho(i) + i$. Hence,

$$\begin{aligned}\underbrace{\omega(i+1)}_{=-n\rho(i+1)+(i+1)} - \underbrace{\omega(i)}_{=-n\rho(i)+i} &= (-n\rho(i+1) + (i+1)) - (-n\rho(i) + i) \\ &= -n\underbrace{(\rho(i+1) - \rho(i))}_{=0} + \underbrace{(i+1) - i}_{=1} \\ &= -0 + 1 = 1 > 0.\end{aligned}$$

In other words, $\omega(i+1) > \omega(i)$, so that $\omega(i) < \omega(i+1)$.

But $i \prec i+1$ holds if and only if $\omega(i) < \omega(i+1)$ (by the definition of the relation $\prec$). Hence, $i \prec i+1$ holds (since $\omega(i) < \omega(i+1)$). This proves (180).]

Now, we know that $\prec$ is a total order on the set E and satisfies (179) and (180). Hence, there exists a total order $<_2$ on the set E satisfying

$$(i+1 <_2 i \qquad \text{for every } i \in Q)$$

and

$$(i <_2 i+1 \qquad \text{for every } i \in \{1,2,\ldots,n-1\} \setminus Q)$$

(namely, $\prec$ is such a total order). This proves Proposition 10.63. $\square$

Proof of Proposition 10.62. We shall use the notation introduced in Definition 10.2 (that is, we shall write $[k]$ for $\{1,2,\ldots,k\}$ when $k \in \mathbb{Z}$). In particular, $[n] = \{1,2,\ldots,n\} = E$.

Lemma 10.4 shows that $D(\alpha) \subseteq [n-1] = \{1,2,\ldots,n-1\}$. In other words, $D(\alpha)$ is a subset of $\{1,2,\ldots,n-1\}$. Proposition 10.63 (applied to $Q = D(\alpha)$) thus shows that there exists a total order $<_2$ on the set E satisfying

$$(i+1 <_2 i \qquad \text{for every } i \in D(\alpha))$$

and

$$(i <_2 i+1 \qquad \text{for every } i \in \{1,2,\ldots,n-1\} \setminus D(\alpha)).$$

This proves Proposition 10.62. $\square$

Next, let us prove another claim made in Example 3.6 (c):

Proposition 10.65. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Define a set $D(\alpha)$ as in Definition 10.3. Let $E = \{1,2,\ldots,n\}$. Let $<_1$ be the total order on the set E inherited from $\mathbb{Z}$ (thus, two elements a and b of E satisfy $a <_1 b$ if and only if they satisfy $a < b$). Let $<_2$ be a strict partial order on the set E satisfying (174) and (175). Then,

$$\begin{aligned}\Gamma((E, <_1, <_2)) &= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ &= \sum_{\beta \text{ is a composition of } n;\ D(\beta) \supseteq D(\alpha)} M_\beta.\end{aligned}$$

Proof of Proposition 10.65. Let us first observe a simple fact: If u and v are two elements of $\{1,2,\ldots,n\}$ such that $u < v$ and $v <_2 u$, then

$$\{u, u+1, \ldots, v-1\} \cap D(\alpha) \neq \varnothing \tag{181}$$

[132].

Let Z be the set of all $(E, <_1, <_2)$-partitions. The definition of $\Gamma((E, <_1, <_2))$ yields

$$\Gamma((E, <_1, <_2)) = \sum_{\pi \text{ is an } (E,<_1,<_2)\text{-partition}} \mathbf{x}_\pi,$$

where $\mathbf{x}_\pi = \prod_{e \in E} x_{\pi(e)}$. Thus,

$$\Gamma((E, <_1, <_2)) = \underbrace{\sum_{\pi \text{ is an } (E,<_1,<_2)\text{-partition}}}_{\substack{=\sum_{\pi \in Z} \\ \text{(since } Z \text{ is the set of all} \\ (E,<_1,<_2)\text{-partitions)}}} \mathbf{x}_\pi = \sum_{\pi \in Z} \mathbf{x}_\pi. \tag{182}$$

On the other hand, define a set W by

$$W = \left\{ (i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \mid i_1 \leq i_2 \leq \cdots \leq i_n \right.$$
$$\left. \text{and } \left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha) \right) \right\}. \tag{183}$$

Thus, we have the following equality between summation signs:

$$\sum_{(i_1,i_2,\ldots,i_n) \in W} = \sum_{\substack{(i_1,i_2,\ldots,i_n) \in \{1,2,3,\ldots\}^n; \\ i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}}.$$

For every $\pi \in Z$, we have $(\pi(1), \pi(2), \ldots, \pi(n)) \in W$ [133]. Hence, we can define a map $\Phi : Z \to W$ by setting

$$(\Phi(\pi) = (\pi(1), \pi(2), \ldots, \pi(n)) \qquad \text{for every } \pi \in Z).$$

---

[132] *Proof of (181):* Let u and v be two elements of $\{1, 2, \ldots, n\}$ such that $u < v$ and $v <_2 u$. We must prove (181).

Indeed, assume (for the sake of contradiction) that $\{u, u+1, \ldots, v-1\} \cap D(\alpha) = \varnothing$.

We have $u \geq 1$ (since $u \in \{1, 2, \ldots, n\}$) and $v \leq n$ (since $v \in \{1, 2, \ldots, n\}$). Also, $u \neq v$ (since $u < v$).

Now, let $k \in \{u, u+1, \ldots, v-1\}$ be arbitrary. If we had $k \in D(\alpha)$, then we would have

$$k \in \{u, u+1, \ldots, v-1\} \cap D(\alpha) \qquad (\text{since } k \in \{u, u+1, \ldots, v-1\} \text{ and } k \in D(\alpha))$$
$$= \varnothing,$$

which would imply that the empty set $\varnothing$ has at least one element (namely, the element k); but this is clearly absurd. Thus, we cannot have $k \in D(\alpha)$. Therefore, we must have $k \notin D(\alpha)$. But $k \in \{u, u+1, \ldots, v-1\} \subseteq \{1, 2, \ldots, n-1\}$ (since $u \geq 1$ and $v \leq n$). Combining this with $k \notin D(\alpha)$, we obtain $k \in \{1, 2, \ldots, n-1\} \setminus D(\alpha)$. Hence, (175) (applied to $i = k$) shows that $k <_2 k+1$.

Now, forget that we fixed k. We thus have shown that $k <_2 k+1$ for each $k \in \{u, u+1, \ldots, v-1\}$. In other words, we have the relations $u <_2 u+1$, $u+1 <_2 u+2$, $\ldots$, $v-1 <_2 v$. Since $(E, <_2)$ is a poset (because $<_2$ is a strict partial order on E), we can combine these relations into a chain of inequalities:

$$u <_2 u+1 <_2 u+2 <_2 \cdots <_2 v.$$

Thus, $u <_2 v$ (since $u \neq v$). This contradicts $v <_2 u$ (since $<_2$ is a strict partial order on E).

This contradiction shows that our assumption (that $\{u, u+1, \ldots, v-1\} \cap D(\alpha) = \varnothing$) was false. Hence, we cannot have $\{u, u+1, \ldots, v-1\} \cap D(\alpha) = \varnothing$. We thus must have $\{u, u+1, \ldots, v-1\} \cap D(\alpha) \neq \varnothing$. This proves (181).

[133] *Proof.* Let $\pi \in Z$. We must show that $(\pi(1), \pi(2), \ldots, \pi(n)) \in W$.

Consider this map Φ. Thus, Φ is the map $Z \to W,\ \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(n))$.

First, let us see that $(\pi(1), \pi(2), \ldots, \pi(n))$ is well-defined. We have $\pi \in Z$. In other words, π is an $(E, <_1, <_2)$-partition (since Z is the set of all $(E, <_1, <_2)$-partitions). Hence, π is a map $E \to \{1, 2, 3, \ldots\}$. In other words, π is a map $\{1, 2, \ldots, n\} \to \{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, n\}$). Hence, $(\pi(1), \pi(2), \ldots, \pi(n))$ is well-defined and an element of $\{1, 2, 3, \ldots\}^n$.

Recall the definition of an $(E, <_1, <_2)$-partition. This definition shows that π is an $(E, <_1, <_2)$-partition if and only if it satisfies the following two assertions:

Assertion $\mathcal{A}_1$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\pi(e) \leq \pi(f)$.

Assertion $\mathcal{A}_2$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\pi(e) < \pi(f)$.

Thus, π satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ (since π is an $(E, <_1, <_2)$-partition).

We now shall show that

$$\pi(k) \leq \pi(k+1) \qquad \text{for every } k \in \{1, 2, \ldots, n-1\}. \tag{184}$$

[*Proof of (184):* Let $k \in \{1, 2, \ldots, n-1\}$ be arbitrary. We shall show that $\pi(k) \leq \pi(k+1)$.

We have $k \in \{1, 2, \ldots, n-1\}$. Thus, both k and $k+1$ belong to the set $\{1, 2, \ldots, n\}$. In other words, both k and $k+1$ belong to the set E (since $E = \{1, 2, \ldots, n\}$).

Recall that $<_1$ is the total order on the set E inherited from $\mathbb{Z}$. Hence, $k <_1 k+1$ (since $k < k+1$). Therefore, Assertion $\mathcal{A}_1$ (applied to $e = k$ and $f = k+1$) yields $\pi(k) \leq \pi(k+1)$. This proves (184).]

Now, (184) shows that $\pi(1) \leq \pi(2) \leq \cdots \leq \pi(n)$.

Next, let us show that

$$\pi(j) < \pi(j+1) \qquad \text{whenever } j \in D(\alpha). \tag{185}$$

[*Proof of (185):* Let $j \in D(\alpha)$.

We shall use the notation introduced in Definition 10.2 (that is, we shall write $[k]$ for $\{1, 2, \ldots, k\}$ when $k \in \mathbb{Z}$). In particular, $[n-1] = \{1, 2, \ldots, n-1\}$. Lemma 10.4 shows that $D(\alpha) \subseteq [n-1] = \{1, 2, \ldots, n-1\}$. Hence, $j \in D(\alpha) \subseteq \{1, 2, \ldots, n-1\}$. Thus, both j and $j+1$ belong to the set $\{1, 2, \ldots, n\}$. In other words, both j and $j+1$ belong to the set E (since $E = \{1, 2, \ldots, n\}$).

Recall that $<_1$ is the total order on the set E inherited from $\mathbb{Z}$. Hence, $j <_1 j+1$ (since $j < j+1$). Also, $j+1 <_2 j$ (by (174), applied to $i = j$). Therefore, Assertion $\mathcal{A}_2$ (applied to $e = j$ and $f = j+1$) yields $\pi(k) < \pi(k+1)$. This proves (185).]

Now, we know that $(\pi(1), \pi(2), \ldots, \pi(n))$ is an element of $\{1, 2, 3, \ldots\}^n$ and satisfies $\pi(1) \leq \pi(2) \leq \cdots \leq \pi(n)$ and $(\pi(j) < \pi(j+1) \text{ whenever } j \in D(\alpha))$. Thus,

$$\begin{aligned}
(\pi(1), \pi(2), \ldots, \pi(n)) \in \Big\{ & (i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \ \mid \ i_1 \leq i_2 \leq \cdots \leq i_n \\
& \text{and } \left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha)\right) \Big\} \\
= W. &
\end{aligned}$$

The map Φ is surjective[134] and injective[135]. In other words, the map Φ is bijective, i.e., a bijection.

---

[134] *Proof.* Let $\mathbf{g} \in W$. We shall show that $\mathbf{g} \in \Phi(Z)$.

We have

$$\mathbf{g} \in W = \left\{ (i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \ \mid \ i_1 \leq i_2 \leq \cdots \leq i_n \right. \\ \left. \text{and } \left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha)\right) \right\}.$$

In other words, $\mathbf{g}$ is an $(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$ and $\left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha)\right)$. Consider this $(i_1, i_2, \ldots, i_n)$. Thus, $\mathbf{g} = (i_1, i_2, \ldots, i_n)$.

We have $i_1 \leq i_2 \leq \cdots \leq i_n$. In other words, for any $u \in \{1, 2, \ldots, n\}$ and $v \in \{1, 2, \ldots, n\}$ satisfying $u \leq v$, we have

$$i_u \leq i_v. \tag{186}$$

Notice also that we have

$$\left(i_j < i_{j+1} \text{ whenever } j \in D(\alpha)\right). \tag{187}$$

Define a map $\pi : \{1, 2, \ldots, n\} \to \{1, 2, 3, \ldots\}$ by setting

$$(\pi(k) = i_k \qquad \text{for every } k \in \{1, 2, \ldots, n\}).$$

Thus, π is a map from E to $\{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, n\}$).

Recall the definition of an $(E, <_1, <_2)$-partition. This definition shows that π is an $(E, <_1, <_2)$-partition if and only if it satisfies the following two assertions:

Assertion $\mathcal{A}_1$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\pi(e) \leq \pi(f)$.

Assertion $\mathcal{A}_2$: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\pi(e) < \pi(f)$.

We shall now show that π satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proof of Assertion $\mathcal{A}_1$: Let $e \in E$ and $f \in E$ be such that $e <_1 f$. Recall that $<_1$ is the total order on the set E inherited from $\mathbb{Z}$. Hence, $e <_1 f$ holds if and only if $e < f$. Thus, $e < f$ (since $e <_1 f$ holds). Thus, $e \leq f$. Hence, (186) (applied to $u = e$ and $v = f$) yields $i_e \leq i_f$. Now, the definition of π yields $\pi(e) = i_e$ and $\pi(f) = i_f$. Hence, $\pi(e) = i_e \leq i_f = \pi(f)$. This proves Assertion $\mathcal{A}_1$.

Proof of Assertion $\mathcal{A}_2$: Let $e \in E$ and $f \in E$ be such that $e <_1 f$ and $f <_2 e$. Recall that $<_1$ is the total order on the set E inherited from $\mathbb{Z}$. Hence, $e <_1 f$ holds if and only if $e < f$. Thus, $e < f$ (since $e <_1 f$ holds). Thus, $e \leq f$.

Both e and f are elements of E. In other words, both e and f are elements of $\{1, 2, \ldots, n\}$ (since $E = \{1, 2, \ldots, n\}$). We have $e < f$ and $f <_2 e$. Thus, (181) (applied to $u = e$ and $v = f$) yields $\{e, e+1, \ldots, f-1\} \cap D(\alpha) \neq \varnothing$. In other words, the set $\{e, e+1, \ldots, f-1\} \cap D(\alpha)$ is nonempty; hence, this set contains an element. In other words, there exists some $j \in \{e, e+1, \ldots, f-1\} \cap D(\alpha)$. Consider such j.

We have $j \in \{e, e+1, \ldots, f-1\} \cap D(\alpha) \subseteq \{e, e+1, \ldots, f-1\}$. Thus, $e \leq j \leq f-1$. From $j \leq f-1$, we obtain $j+1 \leq f$.

From $e \in \{1, 2, \ldots, n\}$, we obtain $e \geq 1$. From $f \in \{1, 2, \ldots, n\}$, we obtain $f \leq n$. Thus, $j \in \{e, e+1, \ldots, f-1\} \subseteq \{1, 2, \ldots, n-1\}$ (since $e \geq 1$ and $f \leq n$). Hence, both j and $j+1$ are elements of $\{1, 2, \ldots, n\}$.

We have $j \in \{e, e+1, \ldots, f-1\} \cap D(\alpha) \subseteq D(\alpha)$. Thus, $i_j < i_{j+1}$ (by (187)). Applying (186) to $u = e$ and $v = j$, we obtain $i_e \leq i_j$ (since $e \leq j$). Applying (186) to $u = j+1$ and $v = f$, we obtain $i_{j+1} \leq i_f$ (since $j+1 \leq f$). Hence, $i_e \leq i_j < i_{j+1} \leq i_f$.

Now, the definition of π yields $\pi(e) = i_e$ and $\pi(f) = i_f$. Hence, $\pi(e) = i_e < i_f = \pi(f)$. This proves Assertion $\mathcal{A}_2$.

Now, we have shown that π satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus, π is an $(E, <_1, <_2)$-partition (since we know that π is an $(E, <_1, <_2)$-partition if and only if it satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$). In other words, $\pi \in Z$ (since Z is the set of all $(E, <_1, <_2)$-partitions).

In other words, the map $Z \to W,\ \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(n))$ is a bijection (since Φ is the map $Z \to W,\ \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(n))$).

Every $\pi \in Z$ satisfies

$$\mathbf{x}_\pi = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)} \tag{188}$$

Now, the definition of Φ yields

$$\begin{aligned}\Phi(\pi) &= (\pi(1), \pi(2), \ldots, \pi(n)) = (i_1, i_2, \ldots, i_n) \\ &\qquad (\text{since } \pi(k) = i_k \text{ for every } k \in \{1, 2, \ldots, n\}) \\ &= \mathbf{g}.\end{aligned}$$

Hence, $\mathbf{g} = \Phi\left(\underbrace{\pi}_{\in Z}\right) \in \Phi(Z)$.

Let us now forget that we fixed $\mathbf{g}$. We thus have shown that $\mathbf{g} \in \Phi(Z)$ for every $\mathbf{g} \in W$. In other words, $W \subseteq \Phi(Z)$. In other words, the map Φ is surjective, qed.

[135]*Proof.* Let π_1 and π_2 be two elements of Z such that $\Phi(\pi_1) = \Phi(\pi_2)$. We shall prove that $\pi_1 = \pi_2$.

The definition of Φ yields $\Phi(\pi_1) = (\pi_1(1), \pi_1(2), \ldots, \pi_1(n))$ and $\Phi(\pi_2) = (\pi_2(1), \pi_2(2), \ldots, \pi_2(n))$. Thus,

$$(\pi_1(1), \pi_1(2), \ldots, \pi_1(n)) = \Phi(\pi_1) = \Phi(\pi_2) = (\pi_2(1), \pi_2(2), \ldots, \pi_2(n)).$$

In other words,

$$\pi_1(i) = \pi_2(i) \qquad \text{for every } i \in \{1, 2, \ldots, n\}.$$

In other words,

$$\pi_1(i) = \pi_2(i) \qquad \text{for every } i \in E$$

(since $E = \{1, 2, \ldots, n\}$). In other words, $\pi_1 = \pi_2$.

Now, forget that we fixed π_1 and π_2. We thus have shown that if π_1 and π_2 are two elements of Z such that $\Phi(\pi_1) = \Phi(\pi_2)$, then $\pi_1 = \pi_2$. In other words, the map Φ is injective, qed.

[136]. Now, (182) becomes

$$\begin{aligned}
\Gamma\left((E,<_1,<_2)\right) &= \sum_{\pi\in Z} \underbrace{\mathbf{x}_\pi}_{\substack{=x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)} \\ \text{(by (188))}}} = \sum_{\pi\in Z} x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)} \\
&= \underbrace{\sum_{(i_1,i_2,\ldots,i_n)\in W}}_{\substack{=\sum\limits_{\substack{i_1\leq i_2\leq\cdots\leq i_n; \\ i_j<i_{j+1} \text{ whenever } j\in D(\alpha)}}}} x_{i_1}x_{i_2}\cdots x_{i_n} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } (i_1,i_2,\ldots,i_n) \text{ for } (\pi(1),\pi(2),\ldots,\pi(n)), \\ \text{since the map } Z\to W,\ \pi\mapsto(\pi(1),\pi(2),\ldots,\pi(n)) \\ \text{is a bijection}\end{array}\right) \\
&= \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n; \\ i_j<i_{j+1} \text{ whenever } j\in D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n} \\
&= \sum_{\beta \text{ is a composition of } n;\ D(\beta)\supseteq D(\alpha)} M_\beta
\end{aligned}$$

(by Corollary 10.18). This proves Proposition 10.65. □

Definition 10.66. Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ be a composition of a nonnegative integer n. Define a set $D(\alpha)$ as in Definition 10.3. Define a formal power series $F_\alpha \in \mathbf{k}[[x_1,x_2,x_3,\ldots]]$ by

$$F_\alpha = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n; \\ i_j<i_{j+1} \text{ whenever } j\in D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n}. \tag{189}$$

This formal power series F_α is called the *fundamental quasisymmetric function* corresponding to the composition α. We have

$$F_\alpha = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n; \\ i_j<i_{j+1} \text{ whenever } j\in D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n} = \sum_{\beta \text{ is a composition of } n;\ D(\beta)\supseteq D(\alpha)} M_\beta$$

(by Proposition 10.65), so that $F_\alpha \in \mathrm{QSym}$.

10.11. The antipode of M_α

Next, we shall prove a fact which is (in a sense) similar to Corollary 10.18:

Corollary 10.67. Let α be a composition of a nonnegative integer n. Then,

$$\sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n; \\ \{j\in[n-1] \mid i_j<i_{j+1}\}\subseteq D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n} = \sum_{\substack{\beta \text{ is a composition of } n; \\ D(\beta)\subseteq D(\alpha)}} M_\beta.$$

(Here, we are using the notations of Definition 10.2 and Definition 10.3.)

[136]*Proof of (188):* Let $\pi \in Z$. The definition of $\mathbf{x}_\pi$ yields

$$\begin{aligned}
\mathbf{x}_\pi = \prod_{e\in E} x_{\pi(e)} &= \prod_{e\in\{1,2,\ldots,n\}} x_{\pi(e)} \qquad (\text{since } E=\{1,2,\ldots,n\}) \\
&= x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)}.
\end{aligned}$$

This proves (188).

Proof of Corollary 10.67. Proposition 10.17 yields that the maps D and comp are mutually inverse. Thus, the map D is invertible, i.e., a bijection.

Now,

$$\begin{aligned}
&\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} \subseteq D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
&= \underbrace{\sum_{\substack{G \subseteq [n-1]; \\ G \subseteq D(\alpha)}}}_{= \sum_{\substack{G \in \mathcal{P}([n-1]); \\ G \subseteq D(\alpha)}}} \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} = G}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
&\qquad \left(\text{since } \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} \subseteq [n-1] \text{ for every } \left(i_1 \leq i_2 \leq \cdots \leq i_n\right)\right) \\
&= \sum_{\substack{G \in \mathcal{P}([n-1]); \\ G \subseteq D(\alpha)}} \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} = G}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
&= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} = D(\beta)}} x_{i_1} x_{i_2} \cdots x_{i_n}
\end{aligned}$$

(here, we have substituted $D(\beta)$ for G in the outer sum, since the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection). Comparing this with

$$\begin{aligned}
&\underbrace{\sum_{\substack{\beta \text{ is a composition of } n; \\ D(\beta) \subseteq D(\alpha)}}}_{\substack{= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \\ \text{(since } \operatorname{Comp}_n \text{ is the set} \\ \text{of all compositions of } n)}} \overbrace{M_\beta}^{\substack{= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} = D(\beta)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ \text{(by Proposition 10.10 (applied to } \beta \text{ instead of } \alpha))}} \\
&= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} = D(\beta)}} x_{i_1} x_{i_2} \cdots x_{i_n},
\end{aligned}$$

we obtain

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \left\{j \in [n-1] \ \mid \ i_j < i_{j+1}\right\} \subseteq D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\beta \text{ is a composition of } n; \\ D(\beta) \subseteq D(\alpha)}} M_\beta.$$

This proves Corollary 10.67. □

Next, we shall prove an analogue of Proposition 10.26:

Proposition 10.68. Let $\ell \in \mathbb{N}$. Let $E = \{1, 2, \ldots, \ell\}$. Let $<_1$ be the restriction of the standard relation $<$ on $\mathbb{Z}$ to the subset E. (Thus, two elements e and f of E satisfy $e <_1 f$ if and only if $e < f$.) Let $>_1$ be the opposite relation of $<_1$. (Thus, two elements e and f of E satisfy $e >_1 f$ if and only if $f <_1 e$.) Let $\mathbf{E}' = (E, >_1, >_1)$.

(a) Then, $\mathbf{E}'$ is a special double poset.

(b) Let $w : E \to \{1, 2, 3, \ldots\}$ be any map. Set $\alpha = (w(1), w(2), \ldots, w(\ell))$. Then, α is a composition. Write α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. We have

$$\Gamma\left(\mathbf{E}', w\right) = \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}.$$

Proof of Proposition 10.68. (a) The relation $<_1$ is a total order (since it is a restriction of the relation $<$ on $\mathbb{Z}$, which is a total order). Hence, the relation $>_1$ is a total order as well (since it is the opposite relation of the total order $<_1$). Thus, $(E, >_1, >_1)$ is a special double poset (by the definition of "special"). In other words, $\mathbf{E}'$ is a special double poset (since $\mathbf{E}' = (E, >_1, >_1)$). This proves Proposition 10.68 (a).

(b) The map w is a map $E \to \{1, 2, 3, \ldots\}$. In other words, the map w is a map $\{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, \ell\}$). Hence, for every $i \in \{1, 2, \ldots, \ell\}$, we have $w(i) \in \{1, 2, 3, \ldots\}$. Thus, $(w(1), w(2), \ldots, w(\ell))$ is a sequence of positive integers, i.e., a composition. In other words, α is a composition (since $\alpha = (w(1), w(2), \ldots, w(\ell))$).

We have $\alpha = (w(1), w(2), \ldots, w(\ell))$. Thus, $(\alpha_1, \alpha_2, \ldots, \alpha_\ell) = \alpha = (w(1), w(2), \ldots, w(\ell))$. In other words, $\alpha_k = w(k)$ for each $k \in \{1, 2, \ldots, \ell\}$. Hence,

$$\sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} \underbrace{x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}}_{\substack{= x_{i_1}^{w(1)} x_{i_2}^{w(2)} \cdots x_{i_\ell}^{w(\ell)} \\ (\text{since } \alpha_k = w(k) \text{ for each } k \in \{1,2,\ldots,\ell\})}} = \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{w(1)} x_{i_2}^{w(2)} \cdots x_{i_\ell}^{w(\ell)}. \tag{190}$$

It remains to prove that $\Gamma(\mathbf{E}', w) = \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$. The order $>_1$ is an extension of the order $>_1$ (obviously). Thus, Proposition 10.20 (applied to $\mathbf{E}'$, $>_1$ and $>_1$ instead of $\mathbf{E}$, $<_1$ and $<_2$) shows that the $\mathbf{E}'$-partitions are precisely the weakly increasing maps from the poset $(E, >_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

On the other hand, let $\mathcal{J}$ denote the set of all length-ℓ weakly decreasing sequences of positive integers. In other words,

$$\mathcal{J} = \left\{ (i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid i_1 \geq i_2 \geq \cdots \geq i_\ell \right\}.$$

Thus,

$$\sum_{(i_1, i_2, \ldots, i_\ell) \in \mathcal{J}} = \sum_{\substack{(i_1, i_2, \ldots, i_\ell) \in \{1,2,3,\ldots\}^\ell; \\ i_1 \geq i_2 \geq \cdots \geq i_\ell}} = \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell}$$

(an equality between summation signs).

Let Z denote the set of all $\mathbf{E}'$-partitions.

For every $\phi \in Z$, we have $(\phi(1), \phi(2), \ldots, \phi(\ell)) \in \mathcal{J}$ [137]. Hence, we can define a map

[137] *Proof.* Let $\phi \in Z$. Thus, ϕ is an $\mathbf{E}'$-partition (since Z is the set of all $\mathbf{E}'$-partitions). In other words, ϕ is a weakly increasing map from the poset $(E, >_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$ (since the $\mathbf{E}'$-partitions are precisely the weakly increasing maps from the poset $(E, >_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$). In other words, ϕ is a map $E \to \{1, 2, 3, \ldots\}$ which has the property that if e and f are two elements of E satisfying $e >_1 f$, then

$$\phi(e) \leq \phi(f) \tag{191}$$

(by the definition of a "weakly increasing map").

The map ϕ is a map $E \to \{1, 2, 3, \ldots\}$. In other words, the map ϕ is a map $\{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, \ell\}$). Thus, $(\phi(1), \phi(2), \ldots, \phi(\ell))$ is an element of $\{1, 2, 3, \ldots\}^\ell$.

Now, let i and j be two elements of $\{1, 2, \ldots, \ell\}$ satisfying $i < j$. We have $i \in \{1, 2, \ldots, \ell\} = E$; thus, $\phi(i)$ is well-defined. Similarly, $\phi(j)$ is well-defined. The definition of the relation $<_1$ shows that the relation $<_1$ is the restriction of the standard relation $<$ on $\mathbb{Z}$ to the subset E. Thus, $i <_1 j$ if and only if $i < j$. Hence, $i <_1 j$ (since $i < j$). Thus, $j >_1 i$ (since $>_1$ is the opposite relation of $<_1$). Hence, (191) (applied to $e = j$ and $f = i$) shows that $\phi(j) \leq \phi(i)$. In other words, $\phi(i) \geq \phi(j)$.

Now, forget that we fixed i and j. We thus have shown that if i and j are two elements of $\{1, 2, \ldots, \ell\}$ satisfying $i < j$, then $\phi(i) \geq \phi(j)$. In other words, $\phi(1) \geq \phi(2) \geq \cdots \geq \phi(\ell)$.

$\Phi : Z \to \mathcal{J}$ by

$$(\Phi(\phi) = (\phi(1), \phi(2), \ldots, \phi(\ell)) \qquad \text{for every } \phi \in Z).$$

Consider this map Φ. This map Φ is injective[138] and surjective[139]. In other words, the map Φ is

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Now, $(\phi(1), \phi(2), \ldots, \phi(\ell))$ is an element of $\{1, 2, 3, \ldots\}^{\ell}$ and satisfies $\phi(1) \geq \phi(2) \geq \cdots \geq \phi(\ell)$. In other words,

$$\begin{aligned}&(\phi(1), \phi(2), \ldots, \phi(\ell)) \\ &\in \left\{(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^{\ell} \mid i_1 \geq i_2 \geq \cdots \geq i_\ell\right\} = \mathcal{J},\end{aligned}$$

qed.

[138] *Proof.* Let ϕ_1 and ϕ_2 be two elements of Z such that $\Phi(\phi_1) = \Phi(\phi_2)$. We shall show that $\phi_1 = \phi_2$.

The definition of Φ shows that $\Phi(\phi_1) = (\phi_1(1), \phi_1(2), \ldots, \phi_1(\ell))$. The definition of Φ shows that $\Phi(\phi_2) = (\phi_2(1), \phi_2(2), \ldots, \phi_2(\ell))$. Hence,

$$(\phi_1(1), \phi_1(2), \ldots, \phi_1(\ell)) = \Phi(\phi_1) = \Phi(\phi_2) = (\phi_2(1), \phi_2(2), \ldots, \phi_2(\ell)).$$

In other words, $\phi_1(i) = \phi_2(i)$ for each $i \in \{1, 2, \ldots, \ell\}$. In other words, $\phi_1(i) = \phi_2(i)$ for each $i \in E$ (since $E = \{1, 2, \ldots, \ell\}$). In other words, $\phi_1 = \phi_2$.

Now, forget that we fixed ϕ_1 and ϕ_2. We thus have shown that if ϕ_1 and ϕ_2 are two elements of Z such that $\Phi(\phi_1) = \Phi(\phi_2)$, then $\phi_1 = \phi_2$. In other words, the map Φ is injective, qed.

[139] *Proof.* Let $\mathbf{j} \in \mathcal{J}$. We shall show that $\mathbf{j} \in \Phi(Z)$.

We have $\mathbf{j} \in \mathcal{J} = \left\{(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^{\ell} \mid i_1 \geq i_2 \geq \cdots \geq i_\ell\right\}$. In other words, $\mathbf{j}$ has the form $(i_1, i_2, \ldots, i_\ell)$ for some $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^{\ell}$ satisfying $i_1 \geq i_2 \geq \cdots \geq i_\ell$. Consider this $(i_1, i_2, \ldots, i_\ell)$. Thus, $\mathbf{j} = (i_1, i_2, \ldots, i_\ell)$.

We have $i_e \in \{1, 2, 3, \ldots\}$ for every $e \in \{1, 2, \ldots, \ell\}$ (since $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^{\ell}$). In other words, $i_e \in \{1, 2, 3, \ldots\}$ for every $e \in E$ (since $E = \{1, 2, \ldots, \ell\}$). Thus, we can define a map $\phi : E \to \{1, 2, 3, \ldots\}$ by ($\phi(e) = i_e$ for every $e \in E$). Consider this map ϕ.

We have $i_1 \geq i_2 \geq \cdots \geq i_\ell$. In other words, if e and f are two elements of $\{1, 2, \ldots, \ell\}$ such that $e < f$, then

$$i_e \geq i_f. \tag{192}$$

Let e and f be two elements of E satisfying $e >_1 f$. We have $e >_1 f$. In other words, $f <_1 e$ (since $>_1$ is the opposite relation of $<_1$). In other words, $f < e$ (since $<_1$ is the restriction of the standard relation $<$ on $\mathbb{Z}$ to the subset E). Thus, (192) (applied to f and e instead of e and f) shows that $i_f \geq i_e$. In other words, $i_e \leq i_f$. But the definition of ϕ shows that $\phi(e) = i_e$ and $\phi(f) = i_f$. Hence, $\phi(e) = i_e \leq i_f = \phi(f)$.

Now, forget that we fixed e and f. We thus have shown that if e and f are two elements of E satisfying $e >_1 f$, then $\phi(e) \leq \phi(f)$. In other words, ϕ is a weakly increasing map from the poset $(E, >_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$ (by the definition of a "weakly increasing map"). In other words, ϕ is an $\mathbf{E}'$-partition (since the $\mathbf{E}'$-partitions are precisely the weakly increasing maps from the poset $(E, >_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$). In other words, $\phi \in Z$ (since Z is the set of all $\mathbf{E}'$-partitions).

We have $\phi(e) = i_e$ for every $e \in E$ (by the definition of ϕ). In other words, $\phi(e) = i_e$ for every $e \in \{1, 2, \ldots, \ell\}$ (since $E = \{1, 2, \ldots, \ell\}$). In other words, $(\phi(1), \phi(2), \ldots, \phi(\ell)) = (i_1, i_2, \ldots, i_\ell)$.

Now, the definition of Φ yields

$$\Phi(\phi) = (\phi(1), \phi(2), \ldots, \phi(\ell)) = (i_1, i_2, \ldots, i_\ell) = \mathbf{j}.$$

Thus, $\mathbf{j} = \Phi\left(\underbrace{\phi}_{\in Z}\right) \in \Phi(Z)$.

bijective. Thus, Φ is a bijection. In other words, the map

$$Z \to \mathcal{J}, \qquad \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(\ell)) \tag{193}$$

is a bijection (since the map (193) is the map Φ).

For every $\pi \in Z$, we have

$$\mathbf{x}_{\pi,w} = x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)} \tag{194}$$

[140].

Now, the definition of $\Gamma(\mathbf{E}', w)$ yields

$$\begin{aligned}
\Gamma(\mathbf{E}', w) &= \underbrace{\sum_{\pi \text{ is an } \mathbf{E}'\text{-partition}}}_{\substack{=\sum_{\pi \in Z} \\ \text{(since } Z \text{ is the set of} \\ \text{all } \mathbf{E}'\text{-partitions)}}} \mathbf{x}_{\pi,w} = \sum_{\pi \in Z} \underbrace{\mathbf{x}_{\pi,w}}_{\substack{= x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)} \\ \text{(by (194))}}} = \sum_{\pi \in Z} x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)} \\
&= \underbrace{\sum_{(i_1, i_2, \ldots, i_\ell) \in \mathcal{J}}}_{= \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell}} x_{i_1}^{w(1)} x_{i_2}^{w(2)} \cdots x_{i_\ell}^{w(\ell)} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } (i_1, i_2, \ldots, i_\ell) \text{ for } (\phi(1), \phi(2), \ldots, \phi(\ell)), \\ \text{since the map } Z \to \mathcal{J},\ \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(\ell)) \\ \text{is a bijection} \end{array} \right) \\
&= \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{w(1)} x_{i_2}^{w(2)} \cdots x_{i_\ell}^{w(\ell)} = \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \qquad \text{(by (190))}.
\end{aligned}$$

This completes the proof of Proposition 10.68 (b). ☐

Our next claim is an analogue of Proposition 10.10:

Proposition 10.69. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Then,

$$\sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \{j \in [n-1] \mid i_j < i_{j+1}\} \subseteq D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Proof of Proposition 10.69. Let $\mathcal{J}$ denote the set of all length-ℓ weakly increasing sequences of positive integers. In other words,

$$\mathcal{J} = \left\{ (i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid i_1 \leq i_2 \leq \cdots \leq i_\ell \right\}. \tag{195}$$

Now, forget that we fixed $\mathbf{j}$. We thus have shown that $\mathbf{j} \in \Phi(Z)$ for every $\mathbf{j} \in \mathcal{J}$. In other words, $\mathcal{J} \subseteq \Phi(Z)$. In other words, the map Φ is surjective, qed.

[140] *Proof of (194):* Let $\pi \in Z$. Then, the definition of $\mathbf{x}_{\pi,w}$ yields

$$\begin{aligned}
\mathbf{x}_{\pi,w} &= \prod_{e \in E} x_{\pi(e)}^{w(e)} = \underbrace{\prod_{e \in \{1, 2, \ldots, \ell\}}}_{= \prod_{e=1}^{\ell}} x_{\pi(e)}^{w(e)} \qquad (\text{since } E = \{1, 2, \ldots, \ell\}) \\
&= \prod_{e=1}^{\ell} x_{\pi(e)}^{w(e)} = x_{\pi(1)}^{w(1)} x_{\pi(2)}^{w(2)} \cdots x_{\pi(\ell)}^{w(\ell)}.
\end{aligned}$$

This proves (194).

Renaming the index $(i_1, i_2, \ldots, i_\ell)$ as $(j_1, j_2, \ldots, j_\ell)$ in this formula, we obtain

$$\mathcal{J} = \left\{ (j_1, j_2, \ldots, j_\ell) \in \{1, 2, 3, \ldots\}^\ell \ \mid \ j_1 \leq j_2 \leq \cdots \leq j_\ell \right\}.$$

Now,

$$\begin{aligned} &\underbrace{\sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell}}_{\substack{= \sum_{\substack{(i_1,i_2,\ldots,i_\ell) \in \{1,2,3,\ldots\}^\ell; \\ i_1 \leq i_2 \leq \cdots \leq i_\ell}} = \sum_{(i_1,i_2,\ldots,i_\ell) \in \mathcal{J}} \\ \left(\text{since } \left\{(i_1,i_2,\ldots,i_\ell) \in \{1,2,3,\ldots\}^\ell \ \mid \ i_1 \leq i_2 \leq \cdots \leq i_\ell\right\} = \mathcal{J}\right)}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \\ &= \sum_{(i_1,i_2,\ldots,i_\ell) \in \mathcal{J}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{(j_1,j_2,\ldots,j_\ell) \in \mathcal{J}} x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_\ell}^{\alpha_\ell} \end{aligned} \tag{196}$$

(here, we have renamed $(i_1, i_2, \ldots, i_\ell)$ as $(j_1, j_2, \ldots, j_\ell)$ in the sum).

Define a set $\mathcal{I}$ by

$$\begin{aligned} \mathcal{I} = \{ (i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \ \mid \ & i_1 \leq i_2 \leq \cdots \leq i_n \\ & \text{and } \{ j \in [n-1] \ \mid \ i_j < i_{j+1} \} \subseteq D(\alpha) \}. \end{aligned} \tag{197}$$

Thus, $\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \{j \in [n-1] \ \mid \ i_j < i_{j+1}\} \subseteq D(\alpha)}} = \sum_{(i_1,i_2,\ldots,i_n) \in \mathcal{I}}$ (an equality between summation signs). Hence,

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \{j \in [n-1] \ \mid \ i_j < i_{j+1}\} \subseteq D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{(i_1,i_2,\ldots,i_n) \in \mathcal{I}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{198}$$

The definition of $\mathcal{I}$ shows that

$$\begin{aligned} \mathcal{I} &= \{ (i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \ \mid \ i_1 \leq i_2 \leq \cdots \leq i_n \\ &\qquad \text{and } \{ j \in [n-1] \ \mid \ i_j < i_{j+1} \} \subseteq D(\alpha) \} \\ &= \{ (k_1, k_2, \ldots, k_n) \in \{1, 2, 3, \ldots\}^n \ \mid \ k_1 \leq k_2 \leq \cdots \leq k_n \\ &\qquad \text{and } \{ j \in [n-1] \ \mid \ k_j < k_{j+1} \} \subseteq D(\alpha) \} \end{aligned}$$

(here, we have renamed the index $(i_1, i_2, \ldots, i_n)$ as $(k_1, k_2, \ldots, k_n)$).

Now, for every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

Define a map $f : [n] \to [\ell]$ as in Lemma 10.7.

Now, for every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \mathcal{J}$ [141]. Hence, we can define a map $\Phi : \mathcal{I} \to \mathcal{J}$ by setting

$$(\Phi(i_1, i_2, \ldots, i_n) = (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \qquad \text{for every } (i_1, i_2, \ldots, i_n) \in \mathcal{I}).$$

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[141] *Proof.* Let $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$. Thus,

$$\begin{aligned} &(i_1, i_2, \ldots, i_n) \in \mathcal{I} \\ &\quad = \{ (k_1, k_2, \ldots, k_n) \in \{1, 2, 3, \ldots\}^n \ \mid \ k_1 \leq k_2 \leq \cdots \leq k_n \\ &\qquad\qquad \text{and } \{ j \in [n-1] \ \mid \ k_j < k_{j+1} \} \subseteq D(\alpha) \}. \end{aligned}$$

In other words, $(i_1, i_2, \ldots, i_n)$ is an element of $\{1, 2, 3, \ldots\}^n$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$ and $\{ j \in [n-1] \ \mid \ i_j < i_{j+1} \} \subseteq D(\alpha)$.

Lemma 10.8 (b) shows that we have $(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \{1, 2, 3, \ldots\}^\ell$ and $i_{s_1} \leq i_{s_2} \leq \cdots \leq i_{s_\ell}$. In

Consider this Φ.

For every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have

$$i_{s_{f(k)}} = i_k \qquad \text{for every } k \in [n] \tag{199}$$

[142].

Now, for every $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$, we have $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \mathcal{I}$ [143]. Hence, we can define a map $\Psi : \mathcal{J} \to \mathcal{I}$ by setting

$$\left(\Psi\left(h_1, h_2, \ldots, h_\ell\right) = \left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \qquad \text{for every } (h_1, h_2, \ldots, h_\ell) \in \mathcal{J}\right).$$

Consider this Ψ.

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other words,

$$(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \in \left\{(j_1, j_2, \ldots, j_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid j_1 \leq j_2 \leq \cdots \leq j_\ell\right\} = \mathcal{J},$$

qed.

[142] *Proof of (199):* Fix $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$. We need to prove the equality (199).

We have

$$\begin{aligned}(i_1, i_2, \ldots, i_n) &\in \mathcal{I} \\ &= \left\{(k_1, k_2, \ldots, k_n) \in \{1, 2, 3, \ldots\}^n \mid k_1 \leq k_2 \leq \cdots \leq k_n \right. \\ &\qquad \left. \text{and } \left\{j \in [n-1] \mid k_j < k_{j+1}\right\} \subseteq D(\alpha)\right\}.\end{aligned}$$

In other words, $(i_1, i_2, \ldots, i_n)$ is an element of $\{1, 2, 3, \ldots\}^n$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$ and $\left\{j \in [n-1] \mid i_j < i_{j+1}\right\} \subseteq D(\alpha)$. Hence, Lemma 10.8 (a) shows that we have $i_{s_{f(k)}} = i_k$ for every $k \in [n]$. Hence, (199) is proven.

[143] *Proof.* Let $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$. Thus, $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J} = \left\{(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell \mid i_1 \leq i_2 \leq \cdots \leq i_\ell\right\}$. In other words, $(h_1, h_2, \ldots, h_\ell)$ is an ℓ-tuple in $\{1, 2, 3, \ldots\}^\ell$ and satisfies $h_1 \leq h_2 \leq \cdots \leq h_\ell$. Hence, Lemma 10.9 (a) shows that we have $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \{1, 2, 3, \ldots\}^n$ and $h_{f(1)} \leq h_{f(2)} \leq \cdots \leq h_{f(n)}$. Furthermore, Lemma 10.9 (b) yields $\left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\} \subseteq D(\alpha)$.

Thus, $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right)$ is an n-tuple in $\{1, 2, 3, \ldots\}^n$ which satisfies $h_{f(1)} \leq h_{f(2)} \leq \cdots \leq h_{f(n)}$ and $\left\{j \in [n-1] \mid h_{f(j)} < h_{f(j+1)}\right\} \subseteq D(\alpha)$. In other words,

$$\begin{aligned}&\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \\ &\in \left\{(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \mid i_1 \leq i_2 \leq \cdots \leq i_n \right. \\ &\qquad \left. \text{and } \left\{j \in [n-1] \mid i_j < i_{j+1}\right\} \subseteq D(\alpha)\right\}.\end{aligned}$$

In light of (197), this rewrites as $\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \in \mathcal{I}$. Qed.

Now, $\Phi \circ \Psi = \mathrm{id}$ [144] and $\Psi \circ \Phi = \mathrm{id}$ [145]. Hence, the maps Φ and Ψ are mutually inverse. Thus, the map Φ is a bijection. In other words, the map

$$\mathcal{I} \to \mathcal{J}, \qquad (i_1, i_2, \ldots, i_n) \mapsto (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell})$$

is a bijection[146].

Now, for every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have

$$x_{i_1} x_{i_2} \cdots x_{i_n} = x_{i_{s_1}}^{\alpha_1} x_{i_{s_2}}^{\alpha_2} \cdots x_{i_{s_\ell}}^{\alpha_\ell} \tag{200}$$

[144] *Proof.* Let $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$. For every $i \in [\ell]$, we have $f(s_i) = i$ (by (83)) and thus $h_{f(s_i)} = h_i$. Now,

$$\begin{aligned}(\Phi \circ \Psi)(h_1, h_2, \ldots, h_\ell) &= \Phi\left(\underbrace{\Psi(h_1, h_2, \ldots, h_\ell)}_{=(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)})}\right) = \Phi\left(h_{f(1)}, h_{f(2)}, \ldots, h_{f(n)}\right) \\ &= \left(h_{f(s_1)}, h_{f(s_2)}, \ldots, h_{f(s_\ell)}\right) \qquad \text{(by the definition of } \Phi) \\ &= (h_1, h_2, \ldots, h_\ell)\end{aligned}$$

(since $h_{f(s_i)} = h_i$ for every $i \in [\ell]$).

Now, forget that we fixed $(h_1, h_2, \ldots, h_\ell)$. We thus have shown that $(\Phi \circ \Psi)(h_1, h_2, \ldots, h_\ell) = (h_1, h_2, \ldots, h_\ell)$ for every $(h_1, h_2, \ldots, h_\ell) \in \mathcal{J}$. In other words, $\Phi \circ \Psi = \mathrm{id}$, qed.

[145] *Proof.* For every $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$, we have

$$\begin{aligned}(\Psi \circ \Phi)(i_1, i_2, \ldots, i_n) &= \Psi\left(\underbrace{\Phi(i_1, i_2, \ldots, i_n)}_{\substack{=(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \\ \text{(by the definition of } \Phi)}}\right) = \Psi(i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \\ &= \left(i_{s_{f(1)}}, i_{s_{f(2)}}, \ldots, i_{s_{f(n)}}\right) \qquad \text{(by the definition of } \Psi) \\ &= (i_1, i_2, \ldots, i_n) \qquad \text{(by (199))}.\end{aligned}$$

In other words, $\Psi \circ \Phi = \mathrm{id}$, qed.

[146] since Φ is the map

$$\mathcal{I} \to \mathcal{J}, \qquad (i_1, i_2, \ldots, i_n) \mapsto (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell})$$

(by the definition of Φ)

[147]. But (196) becomes

$$M_\alpha = \sum_{(j_1,j_2,\ldots,j_\ell)\in\mathcal{J}} x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_\ell}^{\alpha_\ell} = \sum_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}} \underbrace{x_{i_{s_1}}^{\alpha_1} x_{i_{s_2}}^{\alpha_2} \cdots x_{i_{s_\ell}}^{\alpha_\ell}}_{\substack{=x_{i_1}x_{i_2}\cdots x_{i_n} \\ \text{(by (200))}}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \text{ for } (j_1, j_2, \ldots, j_\ell) \text{ in the} \\ \text{sum, since the map } \mathcal{I} \to \mathcal{J},\ (i_1, i_2, \ldots, i_n) \mapsto (i_{s_1}, i_{s_2}, \ldots, i_{s_\ell}) \\ \text{is a bijection}\end{array}\right)$$

$$= \sum_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \{j\in[n-1] \mid i_j < i_{j+1}\} \subseteq D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

(by (198)). This proves Proposition 10.69. □

Let us now prove the main statements made in Example 4.8 (b):

Proposition 10.70. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Then,

$$S(M_\alpha) = (-1)^\ell \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = (-1)^\ell \sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1}$$
$$= (-1)^\ell \sum_{\substack{\gamma \text{ is a composition of } n; \\ D(\gamma)\subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\gamma.$$

Proof of Proposition 10.70. Let $E = \{1, 2, \ldots, \ell\}$. Thus, $|E| = \ell$.

Let $<_1$ be the restriction of the standard relation $<$ on $\mathbb{Z}$ to the subset E. (Thus, two elements e and f of E satisfy $e <_1 f$ if and only if $e < f$.) Let $>_1$ be the opposite relation of $<_1$. (Thus, two elements e and f of E satisfy $e >_1 f$ if and only if $f <_1 e$.) Let $\mathbf{E} = (E, <_1, >_1)$. Then, Proposition 10.26 (a) shows that $\mathbf{E}$ is a special double poset. The double poset $\mathbf{E}$ is special, thus semispecial[148], and therefore tertispecial[149]. Hence, Theorem 4.2 (applied to $>_1$ instead of $<_2$) yields

$$S(\Gamma((E, <_1, >_1), w)) = \underbrace{(-1)^{|E|}}_{\substack{=(-1)^\ell \\ \text{(since } |E|=\ell)}} \Gamma((E, >_1, >_1), w)$$
$$= (-1)^\ell \Gamma((E, >_1, >_1), w). \tag{201}$$

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[147]*Proof of (200):* Let $(i_1, i_2, \ldots, i_n) \in \mathcal{I}$. Then,

$$x_{i_1} x_{i_2} \cdots x_{i_n} = \prod_{k\in[n]} x_{i_k} = \prod_{j\in[\ell]} \underbrace{\prod_{\substack{k\in[n]; \\ f(k)=j}}}_{=\prod_{k\in f^{-1}(j)}} \underbrace{x_{i_k}}_{\substack{=x_{i_{s_j}} \\ \text{(because (199) shows that} \\ i_k = i_{s_{f(k)}} = i_{s_j} \text{ (since } f(k)=j))}} \qquad (\text{since } f(k) \in [\ell] \text{ for every } k \in [n])$$

$$= \prod_{j\in[\ell]} \underbrace{\prod_{k\in f^{-1}(j)} x_{i_{s_j}}}_{\substack{=x_{i_{s_j}}^{|f^{-1}(j)|} = x_{i_{s_j}}^{\alpha_j} \\ \text{(by (88))}}} = \prod_{j\in[\ell]} x_{i_{s_j}}^{\alpha_j} = x_{i_{s_1}}^{\alpha_1} x_{i_{s_2}}^{\alpha_2} \cdots x_{i_{s_\ell}}^{\alpha_\ell}.$$

This proves (200).

[148]since every special double poset is semispecial

[149]since every semispecial double poset is tertispecial

Let $w : \{1,2,\ldots,\ell\} \to \{1,2,3,\ldots\}$ be the map sending every i to α_i. Thus, w is a map from $\{1,2,\ldots,\ell\}$ to $\{1,2,3,\ldots\}$. In other words, w is a map from E to $\{1,2,3,\ldots\}$ (since $E = \{1,2,\ldots,\ell\}$). Now, the definition of w yields $w(i) = \alpha_i$ for every $i \in \{1,2,\ldots,\ell\}$. In other words, $\alpha_i = w(i)$ for every $i \in \{1,2,\ldots,\ell\}$. We have

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) = (w(1), w(2), \ldots, w(\ell))$$

(since $\alpha_i = w(i)$ for every $i \in \{1,2,\ldots,\ell\}$). Hence, Proposition 10.26 (b) shows that α is a composition and satisfies $\Gamma(\mathbf{E}, w) = M_\alpha$. Thus, $M_\alpha = \Gamma\left(\underbrace{\mathbf{E}}_{=(E,<_1,>_1)}, w\right) = \Gamma((E,<_1,>_1), w)$. Applying the map S to both sides of this equality, we obtain

$$S(M_\alpha) = S(\Gamma((E,<_1,>_1), w)) = (-1)^\ell \Gamma((E,>_1,>_1), w) \tag{202}$$

(by (201)).

Let $\mathbf{E}' = (E,>_1,>_1)$. Then, Proposition 10.68 (a) shows that $\mathbf{E}'$ is a special double poset. Proposition 10.68 (b) furthermore shows that

$$\Gamma(\mathbf{E}', w) = \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}.$$

Now, (202) becomes

$$\begin{aligned} S(M_\alpha) &= (-1)^\ell \Gamma\left(\underbrace{(E,>_1,>_1)}_{=\mathbf{E}'}, w\right) = (-1)^\ell \underbrace{\Gamma(\mathbf{E}', w)}_{=\sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}} \\ &= (-1)^\ell \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \qquad (203) \\ &= (-1)^\ell \underbrace{\sum_{i_\ell \geq i_{\ell-1} \geq \cdots \geq i_1}}_{=\sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell}} \underbrace{x_{i_\ell}^{\alpha_1} x_{i_{\ell-1}}^{\alpha_2} \cdots x_{i_1}^{\alpha_\ell}}_{=x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1}} \\ &\qquad \left(\begin{array}{c}\text{here, we have substituted } (i_\ell, i_{\ell-1}, \ldots, i_1) \\ \text{for } (i_1, i_2, \ldots, i_\ell) \text{ in the sum}\end{array}\right) \\ &= (-1)^\ell \sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1}. \qquad (204) \end{aligned}$$

Now, recall that $(\alpha_1, \alpha_2, \ldots, \alpha_\ell) = \alpha$ is a composition of n. Thus, $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. Hence, $\alpha_\ell + \alpha_{\ell-1} + \cdots + \alpha_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. Thus, $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ is a composition of n. Therefore, Corollary 10.67 (applied to $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ instead of α) yields

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \{j \in [n-1] \mid i_j < i_{j+1}\} \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\beta \text{ is a composition of } n; \\ D(\beta) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\beta. \tag{205}$$

(Here, we are using the notations of Definition 10.2 and Definition 10.3.) On the other hand, Proposition 10.69 (applied to $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ and $\alpha_{\ell+1-k}$ instead of α and α_k) yields

$$\begin{aligned} \sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1} &= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ \{j \in [n-1] \mid i_j < i_{j+1}\} \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ &= \sum_{\substack{\beta \text{ is a composition of } n; \\ D(\beta) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\beta \qquad (\text{by } (205)). \end{aligned}$$

Hence, (204) becomes

$$S(M_\alpha) = (-1)^\ell \underbrace{\sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1}}_{= \sum_{\substack{\beta \text{ is a composition of } n; \\ D(\beta) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\beta} = (-1)^\ell \sum_{\substack{\beta \text{ is a composition of } n; \\ D(\beta) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\beta.$$

This equality, combined with (203) and (204), proves Proposition 10.70. □

10.12. The antipode of F_α

Recall how we defined a power series $F_\alpha \in \operatorname{QSym}$ for every composition α in Definition 10.66. We shall now study the antipode of this F_α. First, let us introduce an operation on compositions:

Definition 10.71. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Thus, $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of n. In other words, $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers, and their sum is $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. Hence, $\alpha_\ell + \alpha_{\ell-1} + \cdots + \alpha_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. Thus, $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ is a composition of n as well. We denote this composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ by $\operatorname{rev} \alpha$, and we call it the *reversal* of α.

Let us pause to see how this allows rewriting part of Proposition 10.70:

Proposition 10.72. Let α be a composition of a nonnegative integer n. Let $\ell \in \mathbb{N}$ be such that α is an ℓ-tuple. Consider the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ defined in Definition 10.13. Then,

$$S(M_\alpha) = (-1)^\ell \sum_{\substack{\gamma \text{ is a composition of } n; \\ D(\gamma) \subseteq D(\operatorname{rev} \alpha)}} M_\gamma.$$

Proof of Proposition 10.72. Write α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ (this is possible since α is an ℓ-tuple). Then, the definition of $\operatorname{rev} \alpha$ yields $\operatorname{rev} \alpha = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$. But Proposition 10.70 yields

$$S(M_\alpha) = (-1)^\ell \sum_{\substack{\gamma \text{ is a composition of } n; \\ D(\gamma) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\gamma = (-1)^\ell \sum_{\substack{\gamma \text{ is a composition of } n; \\ D(\gamma) \subseteq D(\operatorname{rev} \alpha)}} M_\gamma$$

(since $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1) = \operatorname{rev} \alpha$). This proves Proposition 10.72. □

Proposition 10.72 is easily seen to be equivalent to [GriRei14, Theorem 5.11].[150]

Next, let us define the conjugate of a composition (following [GriRei14, Definition 5.22]):

Definition 10.73. Let α be a composition of a nonnegative integer n. We shall use the notations that were used in Proposition 10.17.

We know that $\operatorname{rev} \alpha$ is a composition of n; in other words, $\operatorname{rev} \alpha \in \operatorname{Comp}_n$. Hence, $D(\operatorname{rev} \alpha) \in \mathcal{P}([n-1])$. Thus, $[n-1] \setminus D(\operatorname{rev} \alpha) \in \mathcal{P}([n-1])$. Hence, $\operatorname{comp}([n-1] \setminus D(\operatorname{rev} \alpha)) \in \operatorname{Comp}_n$. In other words, $\operatorname{comp}([n-1] \setminus D(\operatorname{rev} \alpha))$ is a composition of n.

We denote this composition $\operatorname{comp}([n-1] \setminus D(\operatorname{rev} \alpha))$ by $\omega(\alpha)$, and call it the *conjugate* of α.

[150] Indeed, the $\ell(\alpha)$ in [GriRei14, Theorem 5.11] is precisely what we call ℓ in Proposition 10.72, whereas the condition "γ coarsens $\operatorname{rev}(\alpha)$" in [GriRei14, Theorem 5.11] is equivalent to "$D(\gamma) \subseteq D(\operatorname{rev} \alpha)$".

Remark 10.74. Let α be a composition of a nonnegative integer n. Then,

$$D(\omega(\alpha)) = [n-1] \setminus D(\operatorname{rev}\alpha)$$

(where we are using the notations that were used in Proposition 10.17).

Proof of Remark 10.74. Proposition 10.17 shows that the maps D and comp are mutually inverse. Thus, $D \circ \operatorname{comp} = \operatorname{id}$. Now, the definition of $\omega(\alpha)$ shows that $\omega(\alpha) = \operatorname{comp}([n-1] \setminus D(\operatorname{rev}\alpha))$. Applying the map D to both sides of this equality, we obtain

$$\begin{aligned} D(\omega(\alpha)) &= D(\operatorname{comp}([n-1] \setminus D(\operatorname{rev}\alpha))) = \underbrace{(D \circ \operatorname{comp})}_{=\operatorname{id}}([n-1] \setminus D(\operatorname{rev}\alpha)) \\ &= [n-1] \setminus D(\operatorname{rev}\alpha). \end{aligned}$$

This proves Remark 10.74. □

Lemma 10.75. Let α be a composition of a nonnegative integer n. Consider the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ defined in Definition 10.13.

We have $D(\operatorname{rev}\alpha) = \{n-u \mid u \in D(\alpha)\}$.

Proof of Lemma 10.75. Write the composition α in the form $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Thus, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Hence, $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of n (since α is a composition of n). Therefore, $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers whose sum is n. Thus, $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$.

For every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer s_i by

$$s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

Lemma 10.5 (c) thus shows that $D(\alpha) = \{s_1, s_2, \ldots, s_{\ell-1}\}$.

For every $i \in \{0, 1, \ldots, \ell\}$, define a nonnegative integer t_i by

$$t_i = \alpha_\ell + \alpha_{\ell-1} + \cdots + \alpha_{(\ell+1)-i}.$$

Lemma 10.5 (c) (applied to $\operatorname{rev}\alpha$, $\alpha_{\ell+1-k}$ and t_i instead of α, α_k and s_i) thus shows that $D(\operatorname{rev}\alpha) = \{t_1, t_2, \ldots, t_{\ell-1}\}$ (since $\operatorname{rev}\alpha = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ (by the definition of $\operatorname{rev}\alpha$)). But each $i \in \{1, 2, \ldots, \ell-1\}$ satisfies $t_i = n - s_{\ell-i}$ [151]. Hence,

$$(t_1, t_2, \ldots, t_{\ell-1}) = \left(n - s_{\ell-1}, n - s_{\ell-2}, \ldots, n - s_{\ell-(\ell-1)}\right) = (n - s_{\ell-1}, n - s_{\ell-2}, \ldots, n - s_1),$$

so that $\{t_1, t_2, \ldots, t_{\ell-1}\} = \{n - s_{\ell-1}, n - s_{\ell-2}, \ldots, n - s_1\}$. Now,

$$\begin{aligned} D(\operatorname{rev}\alpha) &= \{t_1, t_2, \ldots, t_{\ell-1}\} = \{n - s_{\ell-1}, n - s_{\ell-2}, \ldots, n - s_1\} \\ &= \left\{ n - u \mid u \in \underbrace{\{s_{\ell-1}, s_{\ell-2}, \ldots, s_1\}}_{=\{s_1, s_2, \ldots, s_{\ell-1}\} = D(\alpha)} \right\} = \{n - u \mid u \in D(\alpha)\}. \end{aligned}$$

This proves Lemma 10.75. □

[151] *Proof.* Let $i \in \{1, 2, \ldots, \ell-1\}$. The definition of $s_{\ell-i}$ yields $s_{\ell-i} = \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-i}$. But from $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$, we obtain

$$n = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-i})}_{=s_{\ell-i}} + \underbrace{(\alpha_{\ell-i+1} + \alpha_{\ell-i+2} + \cdots + \alpha_\ell)}_{\substack{=\alpha_\ell + \alpha_{\ell-1} + \cdots + \alpha_{\ell-i+1} \\ =\alpha_\ell + \alpha_{\ell-1} + \cdots + \alpha_{(\ell+1)-i} = t_i}} = s_{\ell-i} + t_i.$$

Hence, $t_i = n - s_{\ell-i}$, qed.

Now, let us prove an analogue of Proposition 10.65:

Proposition 10.76. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Define a set $D(\alpha)$ as in Definition 10.3. Let $E = \{1, 2, \ldots, n\}$. Let $<_1$ be the total order on the set E inherited from $\mathbb{Z}$ (thus, two elements a and b of E satisfy $a <_1 b$ if and only if they satisfy $a < b$). Let $>_1$ be the opposite relation of $<_1$. (Thus, two elements e and f of E satisfy $e >_1 f$ if and only if $f <_1 e$.) Let $<_2$ be a strict partial order on the set E satisfying (174) and (175). We shall use the notation introduced in Definition 10.2 (that is, we shall write $[k]$ for $\{1, 2, \ldots, k\}$ when $k \in \mathbb{Z}$).

(a) We have
$$\Gamma\left((E, >_1, <_2)\right) = \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n; \\ i_j > i_{j+1} \text{ whenever } j \in [n-1] \setminus D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

(b) We have
$$\Gamma\left((E, >_1, <_2)\right) = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in [n-1] \setminus D(\operatorname{rev} \alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} = F_{\omega(\alpha)}.$$

Proof of Proposition 10.76. (a) Let us first observe a simple fact: If u and v are two elements of $\{1, 2, \ldots, n\}$ such that $u < v$ and $u <_2 v$, then
$$\{u, u+1, \ldots, v-1\} \not\subseteq D(\alpha) \qquad (206)$$
[152].

Let Z be the set of all $(E, >_1, <_2)$-partitions. The definition of $\Gamma\left((E, >_1, <_2)\right)$ yields
$$\Gamma\left((E, >_1, <_2)\right) = \sum_{\pi \text{ is an } (E, >_1, <_2)\text{-partition}} \mathbf{x}_\pi,$$
where $\mathbf{x}_\pi = \prod_{e \in E} x_{\pi(e)}$. Thus,
$$\Gamma\left((E, >_1, <_2)\right) = \underbrace{\sum_{\pi \text{ is an } (E, >_1, <_2)\text{-partition}}}_{\substack{= \sum_{\pi \in Z} \\ \text{(since } Z \text{ is the set of all} \\ (E, >_1, <_2)\text{-partitions)}}} \mathbf{x}_\pi = \sum_{\pi \in Z} \mathbf{x}_\pi. \qquad (207)$$

[152] *Proof of (206):* Let u and v be two elements of $\{1, 2, \ldots, n\}$ such that $u < v$ and $u <_2 v$. We must prove (206).

Indeed, assume (for the sake of contradiction) that $\{u, u+1, \ldots, v-1\} \subseteq D(\alpha)$.

We have $u \geq 1$ (since $u \in \{1, 2, \ldots, n\}$) and $v \leq n$ (since $v \in \{1, 2, \ldots, n\}$). Also, $u \neq v$ (since $u < v$), so that $v \neq u$.

Now, let $k \in \{u, u+1, \ldots, v-1\}$ be arbitrary. Thus, $k \in \{u, u+1, \ldots, v-1\} \subseteq D(\alpha)$. Hence, (174) (applied to $i = k$) shows that $k+1 <_2 k$.

Now, forget that we fixed k. We thus have shown that $k+1 <_2 k$ for each $k \in \{u, u+1, \ldots, v-1\}$. In other words, we have the relations $v <_2 v-1$, $v-1 <_2 v-2$, $\ldots$, $u+1 <_2 u$. Since $(E, <_2)$ is a poset (because $<_2$ is a strict partial order on E), we can combine these relations into a chain of inequalities:
$$v <_2 v-1 <_2 v-2 <_2 \cdots <_2 u.$$
Thus, $v <_2 u$ (since $v \neq u$). This contradicts $u <_2 v$ (since $<_2$ is a strict partial order on E).

This contradiction shows that our assumption (that $\{u, u+1, \ldots, v-1\} \subseteq D(\alpha)$) was false. Hence, we cannot have $\{u, u+1, \ldots, v-1\} \subseteq D(\alpha)$. We thus must have $\{u, u+1, \ldots, v-1\} \not\subseteq D(\alpha)$. This proves (206).

On the other hand, define a set W by

$$W = \left\{ (i_1, i_2, \ldots, i_n) \in \{1,2,3,\ldots\}^n \mid i_1 \geq i_2 \geq \cdots \geq i_n \right.$$
$$\left. \text{and } \left(i_j > i_{j+1} \text{ whenever } j \in [n-1] \setminus D(\alpha)\right) \right\}. \tag{208}$$

Thus, we have the following equality between summation signs:

$$\sum_{(i_1,i_2,\ldots,i_n)\in W} = \sum_{\substack{(i_1,i_2,\ldots,i_n)\in\{1,2,3,\ldots\}^n; \\ i_1\geq i_2\geq\cdots\geq i_n; \\ i_j>i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\alpha)}} = \sum_{\substack{i_1\geq i_2\geq\cdots\geq i_n; \\ i_j>i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\alpha)}} .$$

For every $\pi \in Z$, we have $(\pi(1), \pi(2), \ldots, \pi(n)) \in W$ [153]. Hence, we can define a map $\Phi : Z \to W$ by setting

$$(\Phi(\pi) = (\pi(1), \pi(2), \ldots, \pi(n)) \qquad \text{for every } \pi \in Z).$$

[153] *Proof.* Let $\pi \in Z$. We must show that $(\pi(1), \pi(2), \ldots, \pi(n)) \in W$.

First, let us see that $(\pi(1), \pi(2), \ldots, \pi(n))$ is well-defined. We have $\pi \in Z$. In other words, π is an $(E, >_1, <_2)$-partition (since Z is the set of all $(E, >_1, <_2)$-partitions). Hence, π is a map $E \to \{1,2,3,\ldots\}$. In other words, π is a map $\{1,2,\ldots,n\} \to \{1,2,3,\ldots\}$ (since $E = \{1,2,\ldots,n\}$). Thus, $(\pi(1), \pi(2), \ldots, \pi(n))$ is well-defined and an element of $\{1,2,3,\ldots\}^n$.

Recall the definition of an $(E, >_1, <_2)$-partition. This definition shows that π is an $(E, >_1, <_2)$-partition if and only if it satisfies the following two assertions:

Assertion $\mathcal{A}_1$: Every $e \in E$ and $f \in E$ satisfying $e >_1 f$ satisfy $\pi(e) \leq \pi(f)$.

Assertion $\mathcal{A}_2$: Every $e \in E$ and $f \in E$ satisfying $e >_1 f$ and $f <_2 e$ satisfy $\pi(e) < \pi(f)$.

Thus, π satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ (since π is an $(E, >_1, <_2)$-partition).

We now shall show that

$$\pi(k) \geq \pi(k+1) \qquad \text{for every } k \in \{1,2,\ldots,n-1\}. \tag{209}$$

[*Proof of (209):* Let $k \in \{1,2,\ldots,n-1\}$ be arbitrary. We shall show that $\pi(k) \geq \pi(k+1)$.

We have $k \in \{1,2,\ldots,n-1\}$. Thus, both k and $k+1$ belong to the set $\{1,2,\ldots,n\}$. In other words, both k and $k+1$ belong to the set E (since $E = \{1,2,\ldots,n\}$).

Recall that $<_1$ is the total order on the set E inherited from $\mathbb{Z}$. Hence, $k <_1 k+1$ (since $k < k+1$).

But $k+1 >_1 k$ holds if and only if $k <_1 k+1$ (since $>_1$ is the opposite relation of $<_1$). Thus, $k+1 >_1 k$ holds (since $k <_1 k+1$). Therefore, Assertion $\mathcal{A}_1$ (applied to $e = k+1$ and $f = k$) yields $\pi(k+1) \leq \pi(k)$. Hence, $\pi(k) \geq \pi(k+1)$. This proves (209).]

Now, (209) shows that $\pi(1) \geq \pi(2) \geq \cdots \geq \pi(n)$.

Next, let us show that

$$\pi(j) > \pi(j+1) \qquad \text{whenever } j \in [n-1] \setminus D(\alpha). \tag{210}$$

[*Proof of (210):* Let $j \in [n-1] \setminus D(\alpha)$. Thus, $j \in [n-1]$ and $j \notin D(\alpha)$.

From $j \in [n-1] = \{1,2,\ldots,n-1\}$, we conclude that both j and $j+1$ belong to the set $\{1,2,\ldots,n\}$. In other words, both j and $j+1$ belong to the set E (since $E = \{1,2,\ldots,n\}$). We have $j \in \underbrace{[n-1]}_{=\{1,2,\ldots,n-1\}} \setminus D(\alpha) = \{1,2,\ldots,n-1\} \setminus D(\alpha)$, so that $j <_2 j+1$ (by (175), applied to $i = j$).

Recall that $<_1$ is the total order on the set E inherited from $\mathbb{Z}$. Hence, $j <_1 j+1$ (since $j < j+1$). But $j+1 >_1 j$ holds if and only if $j <_1 j+1$ (since $>_1$ is the opposite relation of $<_1$). Thus, $j+1 >_1 j$ holds (since $j <_1 j+1$). Also, $j <_2 j+1$. Therefore, Assertion $\mathcal{A}_2$ (applied to $e = j+1$ and $f = j$) yields $\pi(j+1) < \pi(j)$. In other words, $\pi(j) > \pi(j+1)$. This proves (210).]

Now, we know that $(\pi(1), \pi(2), \ldots, \pi(n))$ is an element of $\{1,2,3,\ldots\}^n$ and satisfies $\pi(1) \geq$

Consider this map Φ. Thus, Φ is the map $Z \to W$, $\pi \mapsto (\pi(1), \pi(2), \ldots, \pi(n))$ (since $\Phi(\pi) = (\pi(1), \pi(2), \ldots, \pi(n))$ for every $\pi \in Z$).

$\pi(2) \geq \cdots \geq \pi(n)$ and $(\pi(j) > \pi(j+1)$ whenever $j \in [n-1] \setminus D(\alpha))$. In other words,

$$\begin{aligned}(\pi(1), \pi(2), \ldots, \pi(n)) &\in \{(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \mid i_1 \geq i_2 \geq \cdots \geq i_n \\ &\qquad \text{and } (i_j > i_{j+1} \text{ whenever } j \in [n-1] \setminus D(\alpha))\} \\ &= W.\end{aligned}$$

The map Φ is surjective[154] and injective[155]. In other words, the map Φ is bijective, i.e., a bijection.

[154] *Proof.* Let $\mathbf{g} \in W$. We shall show that $\mathbf{g} \in \Phi(Z)$.

We have

$$\mathbf{g} \in W = \{(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n \mid i_1 \geq i_2 \geq \cdots \geq i_n \text{ and } (i_j > i_{j+1} \text{ whenever } j \in [n-1] \setminus D(\alpha))\}.$$

In other words, $\mathbf{g}$ is an $(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n$ satisfying $i_1 \geq i_2 \geq \cdots \geq i_n$ and $(i_j > i_{j+1}$ whenever $j \in [n-1] \setminus D(\alpha))$. Consider this $(i_1, i_2, \ldots, i_n)$. Thus, $\mathbf{g} = (i_1, i_2, \ldots, i_n)$.

We have $i_1 \geq i_2 \geq \cdots \geq i_n$. In other words, for any $u \in \{1, 2, \ldots, n\}$ and $v \in \{1, 2, \ldots, n\}$ satisfying $u \leq v$, we have

$$i_u \geq i_v. \tag{211}$$

Notice also that we have

$$(i_j > i_{j+1} \text{ whenever } j \in [n-1] \setminus D(\alpha)). \tag{212}$$

Define a map $\pi : \{1, 2, \ldots, n\} \to \{1, 2, 3, \ldots\}$ by setting

$$(\pi(k) = i_k \qquad \text{for every } k \in \{1, 2, \ldots, n\}).$$

Thus, π is a map from E to $\{1, 2, 3, \ldots\}$ (since $E = \{1, 2, \ldots, n\}$).

Recall the definition of an $(E, >_1, <_2)$-partition. This definition shows that π is an $(E, >_1, <_2)$-partition if and only if it satisfies the following two assertions:

Assertion $\mathcal{A}_1$: Every $e \in E$ and $f \in E$ satisfying $e >_1 f$ satisfy $\pi(e) \leq \pi(f)$.

Assertion $\mathcal{A}_2$: Every $e \in E$ and $f \in E$ satisfying $e >_1 f$ and $f <_2 e$ satisfy $\pi(e) < \pi(f)$.

We shall now show that π satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proof of Assertion $\mathcal{A}_1$: Let $e \in E$ and $f \in E$ be such that $e >_1 f$. Recall that $>_1$ is the opposite relation of $<_1$. Hence, $e >_1 f$ holds if and only if $f <_1 e$. Thus, $f <_1 e$ (since $e >_1 f$ holds). Now, recall that $<_1$ is the total order on the set E inherited from $\mathbb{Z}$. Hence, $f <_1 e$ holds if and only if $f < e$. Thus, $f < e$ (since $f <_1 e$ holds). Thus, $f \leq e$. Hence, (211) (applied to $u = f$ and $v = e$) yields $i_f \geq i_e$. In other words, $i_e \leq i_f$. Now, the definition of π yields $\pi(e) = i_e$ and $\pi(f) = i_f$. Hence, $\pi(e) = i_e \leq i_f = \pi(f)$. This proves Assertion $\mathcal{A}_1$.

Proof of Assertion $\mathcal{A}_2$: Let $e \in E$ and $f \in E$ be such that $e >_1 f$ and $f <_2 e$. Thus, $f < e$ (indeed, this can be proven just as in our proof of Assertion $\mathcal{A}_1$).

Both e and f are elements of E. In other words, both e and f are elements of $\{1, 2, \ldots, n\}$ (since $E = \{1, 2, \ldots, n\}$). We have $f < e$ and $f <_2 e$. Thus, (206) (applied to $u = f$ and $v = e$) yields $\{f, f+1, \ldots, e-1\} \not\subseteq D(\alpha)$. In other words, there exists some $j \in \{f, f+1, \ldots, e-1\}$ such that $j \notin D(\alpha)$. Consider such j.

We have $j \in \{f, f+1, \ldots, e-1\}$. Thus, $f \leq j \leq e-1$. From $j \leq e-1$, we obtain $j+1 \leq e$.

From $f \in \{1, 2, \ldots, n\}$, we obtain $f \geq 1$. From $e \in \{1, 2, \ldots, n\}$, we obtain $e \leq n$. Thus, $j \in \{f, f+1, \ldots, e-1\} \subseteq \{1, 2, \ldots, n-1\}$ (since $f \geq 1$ and $e \leq n$). Hence, both j and $j+1$ are elements of $\{1, 2, \ldots, n\}$.

We have $j \in \{1, 2, \ldots, n-1\} = [n-1]$. Combining this with $j \notin D(\alpha)$, we obtain $j \in [n-1] \setminus D(\alpha)$. Thus, $i_j > i_{j+1}$ (by (212)). Applying (211) to $u = f$ and $v = j$, we obtain $i_f \geq i_j$ (since $f \leq j$). Applying (211) to $u = j+1$ and $v = e$, we obtain $i_{j+1} \geq i_e$ (since $j+1 \leq e$). Hence, $i_f \geq i_j > i_{j+1} \geq i_e$, so that $i_e < i_f$.

Now, the definition of π yields $\pi(e) = i_e$ and $\pi(f) = i_f$. Hence, $\pi(e) = i_e < i_f = \pi(f)$. This proves Assertion $\mathcal{A}_2$.

Now, we have shown that π satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus, π is an $(E, >_1, <_2)$-partition (since we know that π is an $(E, >_1, <_2)$-partition if and only if it satisfies Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$). In other words, $\pi \in Z$ (since Z is the set of all $(E, >_1, <_2)$-partitions).

In other words, the map $Z \to W,\ \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(n))$ is a bijection (since Φ is the map $Z \to W,\ \pi \mapsto (\pi(1), \pi(2), \ldots, \pi(n))$).

Every $\pi \in Z$ satisfies

$$\mathbf{x}_\pi = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)} \tag{213}$$

Now, the definition of Φ yields

$$\begin{aligned}\Phi(\pi) &= (\pi(1), \pi(2), \ldots, \pi(n)) = (i_1, i_2, \ldots, i_n) \\ &\qquad (\text{since } \pi(k) = i_k \text{ for every } k \in \{1, 2, \ldots, n\}) \\ &= \mathbf{g}.\end{aligned}$$

Hence, $\mathbf{g} = \Phi\left(\underbrace{\pi}_{\in Z}\right) \in \Phi(Z)$.

Let us now forget that we fixed $\mathbf{g}$. We thus have shown that $\mathbf{g} \in \Phi(Z)$ for every $\mathbf{g} \in W$. In other words, $W \subseteq \Phi(Z)$. In other words, the map Φ is surjective, qed.

[155] *Proof.* Let π_1 and π_2 be two elements of Z such that $\Phi(\pi_1) = \Phi(\pi_2)$. We shall prove that $\pi_1 = \pi_2$.

The definition of Φ yields $\Phi(\pi_1) = (\pi_1(1), \pi_1(2), \ldots, \pi_1(n))$ and $\Phi(\pi_2) = (\pi_2(1), \pi_2(2), \ldots, \pi_2(n))$. Thus,

$$(\pi_1(1), \pi_1(2), \ldots, \pi_1(n)) = \Phi(\pi_1) = \Phi(\pi_2) = (\pi_2(1), \pi_2(2), \ldots, \pi_2(n)).$$

In other words,

$$\pi_1(i) = \pi_2(i) \qquad \text{for every } i \in \{1, 2, \ldots, n\}.$$

In other words,

$$\pi_1(i) = \pi_2(i) \qquad \text{for every } i \in E$$

(since $E = \{1, 2, \ldots, n\}$). In other words, $\pi_1 = \pi_2$.

Now, forget that we fixed π_1 and π_2. We thus have shown that if π_1 and π_2 are two elements of Z such that $\Phi(\pi_1) = \Phi(\pi_2)$, then $\pi_1 = \pi_2$. In other words, the map Φ is injective, qed.

[156]. Now, (207) becomes

$$\begin{aligned}
&\Gamma\left(\left(E,>_1,<_2\right)\right) \\
&= \sum_{\pi\in Z} \underbrace{\mathbf{x}_\pi}_{\substack{=x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)} \\ \text{(by (213))}}} = \sum_{\pi\in Z} x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)} \\
&= \underbrace{\sum_{(i_1,i_2,\ldots,i_n)\in W}}_{\substack{= \sum_{\substack{i_1\geq i_2\geq\cdots\geq i_n; \\ i_j>i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\alpha)}}}} x_{i_1}x_{i_2}\cdots x_{i_n} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \left(i_1,i_2,\ldots,i_n\right) \text{ for } \left(\pi\left(1\right),\pi\left(2\right),\ldots,\pi\left(n\right)\right), \\ \text{since the map } Z\to W,\ \pi\mapsto\left(\pi\left(1\right),\pi\left(2\right),\ldots,\pi\left(n\right)\right) \\ \text{is a bijection}\end{array}\right) \\
&= \sum_{\substack{i_1\geq i_2\geq\cdots\geq i_n; \\ i_j>i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n}.
\end{aligned}$$

This proves Proposition 10.76 (a).

(b) Write the composition α in the form $\left(\alpha_1,\alpha_2,\ldots,\alpha_\ell\right)$. Thus, $\alpha=\left(\alpha_1,\alpha_2,\ldots,\alpha_\ell\right)$. The definition of $\operatorname{rev}\alpha$ yields $\operatorname{rev}\alpha=\left(\alpha_\ell,\alpha_{\ell-1},\ldots,\alpha_1\right)$.

Proposition 10.76 (a) yields

$$\begin{aligned}
\Gamma\left(\left(E,>_1,<_2\right)\right) &= \sum_{\substack{i_1\geq i_2\geq\cdots\geq i_n; \\ i_j>i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n} \\
&= \underbrace{\sum_{\substack{i_n\geq i_{n-1}\geq\cdots\geq i_1; \\ i_{n+1-j}>i_{n+1-(j+1)} \text{ whenever } j\in[n-1]\setminus D(\alpha)}}}_{\substack{= \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n; \\ i_{n+1-j}>i_{n+1-(j+1)} \text{ whenever } j\in[n-1]\setminus D(\alpha)}} \\ \text{(since the condition } i_n\geq i_{n-1}\geq\cdots\geq i_1 \text{ is equivalent} \\ \text{to the condition } i_1\leq i_2\leq\cdots\leq i_n)}} \underbrace{x_{i_n}x_{i_{n-1}}\cdots x_{i_1}}_{=x_{i_1}x_{i_2}\cdots x_{i_n}} \\
&\qquad \left(\begin{array}{c}\text{here, we have renamed the summation index} \\ \left(i_1,i_2,\ldots,i_n\right) \text{ as } \left(i_n,i_{n-1},\ldots,i_1\right)\end{array}\right) \\
&= \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n; \\ i_{n+1-j}>i_{n+1-(j+1)} \text{ whenever } j\in[n-1]\setminus D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n}. \qquad (214)
\end{aligned}$$

But for every n-tuple $\left(i_1,i_2,\ldots,i_n\right)\in\left\{1,2,3,\ldots\right\}^n$, the condition

$$\left(i_{n+1-j}>i_{n+1-(j+1)} \text{ whenever } j\in[n-1]\setminus D\left(\alpha\right)\right) \qquad (215)$$

is equivalent to the condition

$$\left(i_j<i_{j+1} \text{ whenever } j\in[n-1]\setminus D\left(\operatorname{rev}\alpha\right)\right) \qquad (216)$$

[156]*Proof of (213):* Let $\pi\in Z$. The definition of $\mathbf{x}_\pi$ yields

$$\begin{aligned}
\mathbf{x}_\pi &= \prod_{e\in E} x_{\pi(e)} = \prod_{e\in\{1,2,\ldots,n\}} x_{\pi(e)} \qquad \left(\text{since } E=\left\{1,2,\ldots,n\right\}\right) \\
&= x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)}.
\end{aligned}$$

This proves (213).

[157].

[157]*Proof.* Let $(i_1, i_2, \ldots, i_n) \in \{1, 2, 3, \ldots\}^n$ be an n-tuple. We must show that the condition (215) is equivalent to the condition (216).

Lemma 10.75 yields $D(\operatorname{rev}\alpha) = \{n - u \mid u \in D(\alpha)\}$.

Let us first prove that (215) implies (216).

Proof that (215) implies (216): Assume that (215) holds. Now, let $j \in [n-1] \setminus D(\operatorname{rev}\alpha)$. Thus, $j \in [n-1]$ but $j \notin D(\operatorname{rev}\alpha)$.

Now, we shall prove that $n - j \notin D(\alpha)$. Indeed, assume the contrary. Thus, $n - j \in D(\alpha)$, so that $n - (n-j) \in \{n - u \mid u \in D(\alpha)\}$ (since $n - (n-j)$ has the form $n - u$ for some $u \in D(\alpha)$ (namely, for $u = n - j$)). Therefore, $n - (n-j) \in \{n - u \mid u \in D(\alpha)\} = D(\operatorname{rev}\alpha)$. Since $n - (n-j) = j$, this rewrites as $j \in D(\operatorname{rev}\alpha)$. This contradicts $j \notin D(\operatorname{rev}\alpha)$. This contradiction proves that our assumption was wrong. Hence, $n - j \notin D(\alpha)$ is proven.

Also, $j \in [n-1]$, so that $n - j \in [n-1]$. Combining this with $n - j \notin D(\alpha)$, we obtain $n - j \in [n-1] \setminus D(\alpha)$. Now, we assumed that (215) holds. Hence, (215) (applied to $n - j$ instead of j) shows that $i_{n+1-(n-j)} > i_{n+1-((n-j)+1)}$ (since $n - j \in [n-1] \setminus D(\alpha)$). In other words, $i_{j+1} > i_j$ (since $n + 1 - (n-j) = j+1$ and $n + 1 - ((n-j)+1) = j$). In other words, $i_j < i_{j+1}$.

Now, forget that we fixed j. We thus have shown that $i_j < i_{j+1}$ whenever $j \in [n-1] \setminus D(\operatorname{rev}\alpha)$. In other words, (216) holds.

Now, forget that we assumed that (215) holds. We thus have shown that if (215) holds, then (216) holds. In other words, we have proven that (215) implies (216).

Let us now prove that (216) implies (215).

Proof that (216) implies (215): Assume that (216) holds. Now, let $j \in [n-1] \setminus D(\alpha)$. Thus, $j \in [n-1]$ but $j \notin D(\alpha)$.

Now, we shall prove that $n - j \notin D(\operatorname{rev}\alpha)$. Indeed, assume the contrary. Thus, $n - j \in D(\operatorname{rev}\alpha)$. Hence, $n - j \in D(\operatorname{rev}\alpha) = \{n - u \mid u \in D(\alpha)\}$. In other words, $n - j$ has the form $n - u$ for some $u \in D(\alpha)$. Consider this u. Thus, $n - j = n - u$, so that $j = u$. Hence, $j = u \in D(\alpha)$. This contradicts $j \notin D(\alpha)$. This contradiction proves that our assumption was wrong. Hence, $n - j \notin D(\operatorname{rev}\alpha)$ is proven.

Also, $n - j \in [n-1]$ (since $j \in [n-1]$). Combining this with $n - j \notin D(\operatorname{rev}\alpha)$, we obtain $n - j \in [n-1] \setminus D(\operatorname{rev}\alpha)$. Now, we assumed that (216) holds. Hence, (216) (applied to $n - j$ instead of j) shows that $i_{n-j} < i_{(n-j)+1}$ (since $n - j \in [n-1] \setminus D(\operatorname{rev}\alpha)$). In other words, $i_{(n-j)+1} > i_{n-j}$. Hence,

$$\begin{aligned} i_{n+1-j} &= i_{(n-j)+1} \qquad (\text{since } n+1-j = (n-j)+1) \\ &> i_{n-j} = i_{n+1-(j+1)} \qquad (\text{since } n - j = n + 1 - (j+1)). \end{aligned}$$

Now, forget that we fixed j. We thus have shown that $i_{n+1-j} > i_{n+1-(j+1)}$ whenever $j \in [n-1] \setminus D(\alpha)$. In other words, (215) holds.

Now, forget that we assumed that (216) holds. We thus have shown that if (216) holds, then (215) holds. In other words, we have proven that (216) implies (215).

We have now shown the following two facts:

- (215) implies (216).
- (216) implies (215).

Combining these two facts, we conclude that the condition (215) is equivalent to the condition (216). Qed.

Now, (214) becomes

$$\Gamma\left(\left(E,>_1,<_2\right)\right) = \underbrace{\sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_{n+1-j}>i_{n+1-(j+1)} \text{ whenever } j\in[n-1]\setminus D(\alpha)}}}_{\substack{=\sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_j<i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\operatorname{rev}\alpha)}} \\ \text{(because for every } n\text{-tuple } (i_1,i_2,\ldots,i_n)\in\{1,2,3,\ldots\}^n, \\ \text{the condition } \left(i_{n+1-j}>i_{n+1-(j+1)} \text{ whenever } j\in[n-1]\setminus D(\alpha)\right) \\ \text{is equivalent to the condition } \left(i_j<i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\operatorname{rev}\alpha)\right))}} x_{i_1}x_{i_2}\cdots x_{i_n}$$

$$= \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_j<i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\operatorname{rev}\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n}. \tag{217}$$

Now, recall that $\omega\left(\alpha\right)$ is a composition of n (since α is a composition of n). Hence, the definition of $F_{\omega(\alpha)}$ yields

$$F_{\omega(\alpha)} = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_j<i_{j+1} \text{ whenever } j\in D(\omega(\alpha))}} x_{i_1}x_{i_2}\cdots x_{i_n} = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_j<i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\operatorname{rev}\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n}$$

(since $D\left(\omega\left(\alpha\right)\right) = [n-1]\setminus D\left(\operatorname{rev}\alpha\right)$ (by Remark 10.74)). Comparing this with (217), we obtain $\Gamma\left(\left(E,>_1,<_2\right)\right) = F_{\omega(\alpha)}$. This equality and (217) together complete the proof of Proposition 10.76 (b). $\square$

Finally, we can derive a formula for the antipode of F_α:

Proposition 10.77. Let α be a composition of a nonnegative integer n. Then, $S\left(F_\alpha\right) = \left(-1\right)^n F_{\omega(\alpha)}$.

Proposition 10.77 is [GriRei14, (5.9)] (although [GriRei14] uses the notation L_α for what we call F_α).

Proof of Proposition 10.77. Write the composition α in the form $\left(\alpha_1,\alpha_2,\ldots,\alpha_\ell\right)$. Thus, $\alpha = \left(\alpha_1,\alpha_2,\ldots,\alpha_\ell\right)$. Define a set $D\left(\alpha\right)$ as in Definition 10.3.

Let $E = \{1,2,\ldots,n\}$. Thus, $|E| = n$.

Let $<_1$ be the total order on the set E inherited from $\mathbb{Z}$ (thus, two elements a and b of E satisfy $a <_1 b$ if and only if they satisfy $a < b$). Let $>_1$ be the opposite relation of $<_1$. (Thus, two elements e and f of E satisfy $e >_1 f$ if and only if $f <_1 e$.)

Proposition 10.62 shows that there exists a total order $<_2$ on the set E satisfying (174) and (175). Consider such a $<_2$. Thus, $<_2$ is a total order, hence a strict partial order. Hence, Proposition 10.65 yields

$$\Gamma\left(\left(E,<_1,<_2\right)\right) = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_j<i_{j+1} \text{ whenever } j\in D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

Comparing this with (189), we obtain $\Gamma\left(\left(E,<_1,<_2\right)\right) = F_\alpha$.

We shall use the notation introduced in Definition 10.2 (that is, we shall write $[k]$ for $\{1,2,\ldots,k\}$ when $k\in\mathbb{Z}$). Proposition 10.76 (b) yields

$$\Gamma\left(\left(E,>_1,<_2\right)\right) = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n;\\ i_j<i_{j+1} \text{ whenever } j\in[n-1]\setminus D(\operatorname{rev}\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n} = F_{\omega(\alpha)}.$$

The double poset $\left(E,<_1,<_2\right)$ is special (since $<_2$ is a total order), thus semispecial[158], and therefore tertispecial[159]. Hence, Corollary 4.3 yields $S\left(\Gamma\left(\left(E,<_1,<_2\right)\right)\right) = \left(-1\right)^{|E|}\Gamma\left(\left(E,>_1,<_2\right)\right)$.

[158] since every special double poset is semispecial

[159] since every semispecial double poset is tertispecial

Since $\Gamma((E, <_1, <_2)) = F_\alpha$, $|E| = n$ and $\Gamma((E, >_1, <_2)) = F_{\omega(\alpha)}$, this rewrites as follows: $S(F_\alpha) = (-1)^n F_{\omega(\alpha)}$. This proves Proposition 10.77. □

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