The Petrie symmetrie functions and Murnaghan–Nakayama rules

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/djursholm2020.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/petriesym.pdf
overview: http://www.cip.ifi.lmu.de/~grinberg/algebra/fps20pet.pdf

Manifest

- What you are going to see:
  - A new family $(G(k,m))_{m\geq 0}$ of symmetric functions for each $k > 0$. (So, a family of families.)
  - It "interpolates" between the e's and the h's in a sense.
  - Various nice properties if I do say so myself.
  - A proof (sketch) of a conjecture coming from algebraic groups.
  - A source of homework exercises for your symmetric functions class.

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  - Various nice properties if I do say so myself.
  - A proof (sketch) of a conjecture coming from algebraic groups.
  - A source of homework exercises for your symmetric functions class.
- What you are **not** going to see:
  - Meaning.
  - Theories.
  - (mostly) actual combinatorics (algorithms, bijections, etc.).

- We will use standard notations for symmetric functions, such as used in:
  - Richard Stanley, *Enumerative Combinatorics, volume 2*, CUP 2001.
  - D.G. and Victor Reiner, *Hopf algebras in Combinatorics*, 2012-2020+.

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- Let k be a commutative ring ($\mathbb{Z}$ and $\mathbb{Q}$ will suffice).
- Let $\mathbb{N} := \{0, 1, 2, \ldots\}$.

Symmetric functions: notation, 2

- A *weak composition* means a sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ such that all $i \gg 0$ satisfy $\alpha_i = 0$.
- We let WC be the set of all weak compositions.
- We write α_i for the i-th entry of a weak composition α.
- The *size* of a weak composition α is defined to be $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 + \cdots$.

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- We let Par denote the set of all partitions. For each $n \in \mathbb{Z}$, we let Par_n denote the set of all partitions of n.
- We often omit trailing zeroes from partitions: e.g., $(3, 2, 1, 0, 0, 0, \ldots) = (3, 2, 1) = (3, 2, 1, 0)$.
- The partition $(0, 0, 0, \ldots) = ()$ is called the *empty partition* and denoted by $\varnothing$.

Symmetric functions: notation, 3

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- The ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ consists of formal infinite $\mathbf{k}$-linear combinations of monomials $\mathbf{x}^\alpha$. These combinations are called *formal power series*.
- The *symmetric functions* are the formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that are
  - *of bounded degree* (i.e., all monomials in f have degrees $< N$ for some $N = N_f$);
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- We let

$$\Lambda = \{\text{symmetric functions } f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]\}.$$

  This is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, graded by the degree.

Symmetric functions: m

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For example:

$$m_{(2,2,1)} = \sum_{i<j<k} x_i^2 x_j^2 x_k + \sum_{i<j<k} x_i^2 x_j x_k^2 + \sum_{i<j<k} x_i x_j^2 x_k^2 .$$

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The family $(m_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the k-module Λ, called the *monomial basis*.

- The *complete basis* $(h_\lambda)_{\lambda \in \mathrm{Par}}$:
  For each $n \in \mathbb{Z}$, define the *complete homogeneous symmetric function h_n* by

$$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = n}} \mathrm{x}^\alpha = \sum_{\lambda \in \mathrm{Par}_n} m_\lambda .$$

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$$h_1 = x_1 + x_2 + x_3 + \cdots ;$$

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  We can make a basis out of (products of) p_n's when k is a $\mathbb{Q}$-algebra.

- The *Schur basis* $(s_\lambda)_{\lambda \in \mathrm{Par}}$:
  For each partition λ, we can define the *Schur function* s_λ in many equivalent ways, e.g.:
  - We have

    $$s_\lambda = \sum_{\substack{T \text{ is a semistandard} \\ \text{Young tableau of shape } \lambda}} \mathbf{x}_T,$$

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  - If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, then
    $$s_\lambda = \det\left(\left(h_{\lambda_i - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right)$$
    (the *first Jacobi–Trudi formula*).

  The family $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module Λ.

Petrie functions: definition of $G(k)$

- For any positive integer k, set

$$G(k) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < k \text{ for all } i}} \mathrm{x}^\alpha = \sum (\text{all monomials whose exponents are all } < k) \in \mathrm{k}[[x_1, x_2, x_3, \ldots]] \qquad (\text{not } \in \Lambda \text{ in general}).$$

- For any positive integer k and any $m \in \mathbb{N}$, we let

$$G(k,m) = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m;\\ \alpha_i < k \text{ for all } i}} \mathrm{x}^\alpha = \sum (\text{all degree-}m\text{ monomials whose exponents are all } < k) \in \Lambda.$$

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  $$G(k,m) = \sum_{\substack{\alpha \in WC; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} x^\alpha$$
  $$= \sum (\text{all degree-}m \text{ monomials whose exponents are all } < k)$$
  $$\in \Lambda.$$

  For example,

  $$G(3,4) = \sum_{i<j<k<\ell} x_i x_j x_k x_\ell + \sum_{i<j<k} x_i^2 x_j x_k + \sum_{i<j<k} x_i x_j^2 x_k + \sum_{i<j<k} x_i x_j x_k^2 + \sum_{i<j} x_i^2 x_j^2$$
  $$= m_{(1,1,1,1)} + m_{(2,1,1)} + m_{(2,2)}.$$

Petrie functions: basic properties

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  - $$G(k) = \sum_{\substack{\lambda \in \text{Par};\\ \lambda_i < k \text{ for all } i}} m_\lambda = \prod_{i=1}^{\infty} \left(x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right).$$

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  - $G(m,m) = h_m - p_m$.

Petrie functions and the coproduct of Λ

- This is for the friends of Hopf algebras:

$$\Delta\left(G\left(k,m\right)\right)=\sum_{i=0}^{m}G\left(k,i\right)\otimes G\left(k,m-i\right)$$

for each $k>0$ and $m\in\mathbb{N}$.

Here, Δ is the *comultiplication* of Λ, defined to be the k-algebra homomorphism

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- In terms of alphabets, this says

$$\left(G\left(k,m\right)\right)\left(x_1,x_2,x_3,\ldots,y_1,y_2,y_3,\ldots\right)$$
$$=\sum_{i=0}^{m}\left(G\left(k,i\right)\right)\left(x_1,x_2,x_3,\ldots\right)\cdot\left(G\left(k,m-i\right)\right)\left(y_1,y_2,y_3,\ldots\right).$$

- We can expand the $G(k, m)$ in the Schur basis $(s_\lambda)_{\lambda \in \mathrm{Par}}$: e.g.,

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Expanding Petries in the Schur basis

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- Better yet: Any product $G(k, m) \cdot s_\mu$ expands in the Schur basis with coefficients in $\{0, 1, -1\}$.
- Let us see what the coefficients are.

Petrie numbers

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- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \operatorname{Par}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \operatorname{Par}$, and let k be a positive integer. Then, the *k-Petrie number* $\operatorname{pet}_k(\lambda, \mu)$ of λ and μ is the integer defined by

$$\operatorname{pet}_k(\lambda, \mu) = \det\left(\left([0 \leq \lambda_i - \mu_j - i + j < k]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right).$$

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For example, for $\ell = 3$, we have

$$\begin{aligned} &\mathrm{pet}_k(\lambda, \mu) \\ &= \det\begin{pmatrix} [0 \leq \lambda_1 - \mu_1 < k] & [0 \leq \lambda_1 - \mu_2 + 1 < k] & [0 \leq \lambda_1 - \mu_3 + 2 < k] \\ [0 \leq \lambda_2 - \mu_1 - 1 < k] & [0 \leq \lambda_2 - \mu_2 < k] & [0 \leq \lambda_2 - \mu_3 + 1 < k] \\ [0 \leq \lambda_3 - \mu_1 - 2 < k] & [0 \leq \lambda_3 - \mu_2 - 1 < k] & [0 \leq \lambda_3 - \mu_3 < k] \end{pmatrix}. \end{aligned}$$

For example,

$$\mathrm{pet}_4((3,1,1),(2,1)) = \det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

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- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathsf{Par}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathsf{Par}$, and let k be a positive integer. Then, the *k-Petrie number* $\operatorname{pet}_k(\lambda, \mu)$ of λ and μ is the integer defined by

$$\operatorname{pet}_k(\lambda, \mu) = \det \left(\left([0 \leq \lambda_i - \mu_j - i + j < k] \right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell} \right).$$

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- **Proposition:** We have $\mathrm{pet}_k(\lambda, \mu) \in \{0, 1, -1\}$ for all λ and μ.
- *Proof idea.* Each row of the matrix $\left([0 \leq \lambda_i - \mu_j - i + j < k]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}$ has the form

$$(\underbrace{0, 0, \ldots, 0}_{a \text{ zeroes}}, \underbrace{1, 1, \ldots, 1}_{b \text{ ones}}, \underbrace{0, 0, \ldots, 0}_{c \text{ zeroes}}) \qquad \text{for some } a, b, c \in \mathbb{N}.$$

Thus, this matrix is the transpose of a *Petrie matrix*. Hence, its determinant is $\in \{-1, 0, 1\}$ (by Gordon and Wilkinson 1974).

Expanding Petries in the Schur basis: the formula

- **Theorem:** Let k be a positive integer. Let $\mu \in \operatorname{Par}$. Then,

$$G(k) \cdot s_{\mu} = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu) s_{\lambda}.$$

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- **Corollary:** Let k be a positive integer. Then,

$$G(k) = \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda, \varnothing) s_\lambda.$$

Thus, for each $m \in \mathbb{N}$, we have

$$G(k, m) = \sum_{\lambda \in \text{Par}_m} \text{pet}_k(\lambda, \varnothing) s_\lambda.$$

- **Theorem:** Let k be a positive integer. Let $\mu \in \operatorname{Par}$. Then,
$$G(k) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu)\, s_\lambda.$$
Thus, for each $m \in \mathbb{N}$, we have
$$G(k, m) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}_{m+|\mu|}} \operatorname{pet}_k(\lambda, \mu)\, s_\lambda.$$

- One proof of the Theorem uses alternants; the other uses the "semi-skew Cauchy identity"
$$\sum_{\lambda \in \operatorname{Par}} s_\lambda(\mathrm{x})\, s_{\lambda/\mu}(\mathrm{y}) = s_\mu(\mathrm{x}) \cdot \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1}$$
$$= s_\mu(\mathrm{x}) \cdot \sum_{\lambda \in \operatorname{Par}} h_\lambda(\mathrm{x})\, m_\lambda(\mathrm{y})$$
(for any $\mu \in \operatorname{Par}$ and for two sets of indeterminates $\mathrm{x} = (x_1, x_2, x_3, \ldots)$ and $\mathrm{y} = (y_1, y_2, y_3, \ldots)$).

What are the Petrie numbers?

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What are the Petrie numbers?

- We have shown that $\text{pet}_k(\lambda, \mu) \in \{0, 1, -1\}$, but what exactly is it?
- Gordon and Wilkinson 1974 prove that Petrie matrices have determinants $\in \{0, 1, -1\}$ by induction. This is little help to us.

- **Proposition:** Let $\lambda \in \mathrm{Par}$ and $k > 0$ be such that $\lambda_1 \geq k$. Then, $\mathrm{pet}_k(\lambda, \varnothing) = 0$.

- **Proposition:** Let $\lambda \in \operatorname{Par}$ and $k > 0$ be such that $\lambda_1 \geq k$. Then, $\operatorname{pet}_k(\lambda, \varnothing) = 0$.
- To get a description in all other cases, recall the definition of *transpose (aka conjugate) partitions*:
  Given a partition $\lambda \in \operatorname{Par}$, we define the *transpose partition λ^t* of λ to be the partition μ given by

  $$\mu_i = |\{j \in \{1, 2, 3, \ldots\} \mid \lambda_j \geq i\}| \qquad \text{for all } i \geq 1.$$

  In terms of Young diagrams, this is just flipping the diagram of λ across the diagonal.

- **Theorem:** Let $\lambda \in \operatorname{Par}$ and $k > 0$ be such that $\lambda_1 < k$. Let $\mu = \lambda^t$ (the transpose partition of λ). Thus, $\mu_k = 0$.
  For each $i \in \{1, 2, \ldots, k-1\}$, set

$$\beta_i = \mu_i - i \qquad \text{and} \qquad \gamma_i = 1 + \underbrace{(\beta_i - 1)\,\%k}_{\text{remainder of } \beta_i - 1 \text{ modulo } k} .$$

  (a) If the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are not distinct, then $\operatorname{pet}_k(\lambda, \varnothing) = 0$.
  (b) If the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are distinct, then

$$\operatorname{pet}_k(\lambda, \varnothing) = (-1)^{(\beta_1+\beta_2+\cdots+\beta_{k-1})+g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})},$$

  where

$$g = \left| \left\{ (i,j) \in \{1, 2, \ldots, k-1\}^2 \ \mid \ i < j \text{ and } \gamma_i < \gamma_j \right\} \right| .$$

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- **Question:** Is there such a description for $\operatorname{pet}_k(\lambda, \mu)$?

Other properties

- For any $k > 0$, we define a map $\mathbf{f}_k : \Lambda \to \Lambda$ by setting

$$\mathbf{f}_k(a) = a\left(x_1^k, x_2^k, x_3^k, \ldots\right) \qquad \text{for each } a \in \Lambda.$$

  This map $\mathbf{f}_k$ is called the *k-th Frobenius endomorphism* of Λ. (Also known as plethysm by p_k. Perhaps the nicest plethysm!)

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- **Theorem:** Let k be a positive integer. Let $m \in \mathbb{N}$. Then,

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- **Theorem:** Fix a positive integer k. Assume that $1 - k$ is invertible in $\mathbf{k}$. Then, the family $(G(k, m))_{m \geq 1} = (G(k, 1), G(k, 2), G(k, 3), \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ.
- Thus, products of several elements of this family form a basis of Λ (if $1 - k$ is invertible in $\mathbf{k}$). These bases remain to be studied.

- This all begin with the following conjecture (Liu and Polo, arXiv:1908.08432):

$$\sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ (n-1,n-1,1) \triangleright \lambda}} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})} \qquad \text{for any } n > 1.$$

Here, the symbol $\triangleright$ stands for *dominance* of partitions (also known as majorization); i.e., for two partitions λ and μ, we have

$$\lambda \triangleright \mu \quad \text{if and only if}$$
$$(\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \text{ for all } i).$$

- Let me briefly outline how this conjecture can be proved.

The Liu–Polo conjecture, proof: 1

- The partitions $\lambda \in \operatorname{Par}_{2n-1}$ satisfying $(n-1, n-1, 1) \triangleright \lambda$ are precisely the partitions $\lambda \in \operatorname{Par}_{2n-1}$ satisfying $\lambda_i < n$ for all i.

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- So it remains to show that

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- The formula for $\operatorname{pet}_k(\lambda, \varnothing)$ should be useful here, but the combinatorics is tortuous.
  Instead, we can work algebraically:

- We can easily see that

$$G(n, n+k) = h_{n+k} - h_k p_n \qquad \text{for each } k \in \{0, 1, \ldots, n-1\}.$$

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- By the way, this is also a particular case of the

  $$G(k, m) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot \mathrm{f}_k(e_i)$$

  formula.

- Recall the *skewing operations* $f^{\perp} : \Lambda \to \Lambda$ for all $f \in \Lambda$.

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- For any $m \in \mathbb{N}$, we define a map $\mathrm{B}_m : \Lambda \to \Lambda$ (known as a *m-th Bernstein operator* in Zelevinsky's language, or as a *Schur row-adder* in Garsia's) by setting

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- **Theorem** (implicit in Zelevinsky's book; solved exercise in G./Reiner)**:** If $\lambda \in \mathrm{Par}$ and $m \in \mathbb{Z}$ satisfy $m \geq \lambda_1$, then

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- On the other hand, it is not hard to see that

$$\mathrm{B}_m(h_n) = h_m h_n - h_{m+1} h_{n-1} \qquad \text{and}$$
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for each $n > 0$ and each $m \in \{0, 1, \ldots, n\}$.

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  for each $n > 0$ and each $m \in \{0, 1, \ldots, n\}$.
  Hence,

$$\mathrm{B}_{n-1}(h_n - p_n) = h_{2n-1} - h_{n-1} p_n = G(n, 2n-1).$$

- The Murnaghan–Nakayama rule yields

$$p_n = \sum_{i=0}^{n-1} (-1)^i \, s_{(n-i, 1^i)}.$$

Subtracting this from $h_n = s_{(n)} = s_{(n-0,1^0)}$, we find

$$h_n - p_n = \sum_{i=0}^{n-2} (-1)^i \, s_{(n-1-i, 1^{i+1})}.$$

Hence,

$$\mathrm{B}_{n-1}(h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i \, \mathrm{B}_{n-1}\left(s_{(n-1-i, 1^{i+1})}\right)$$
$$= \sum_{i=0}^{n-2} (-1)^i \, s_{(n-1, n-1-i, 1^{i+1})}$$

(by $\mathrm{B}_m(s_\lambda) = s_{(m, \lambda_1, \lambda_2, \lambda_3, \ldots)}$).

- Since $\mathrm{B}_{n-1}(h_n - p_n) = G(n, 2n-1)$, we now get

$$G(n, 2n-1) = \mathrm{B}_{n-1}(h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i \, s_{(n-1, n-1-i, 1^{i+1})}.$$

This proves the conjecture from Liu/Polo.

- Now to something different.
  Recall our formula

$$G(k, m) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}_{m+|\mu|}} \underbrace{\operatorname{pet}_k(\lambda, \mu)}_{\in\{0,1,-1\}} s_\lambda .$$

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- **Problem:** What other functions can we replace $G\left(k,m\right)$ by and still get such a formula?
  In other words, what other $f\in\Lambda$ satisfy

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- Let us restate this more formally.

- We recall the *Hall inner product* $(\cdot, \cdot) : \Lambda \times \Lambda \to \mathbf{k}$; it is the unique k-bilinear form on Λ that satisfies

$$(s_\lambda, s_\mu) = \delta_{\lambda,\mu} \qquad \text{for all } \lambda, \mu \in \operatorname{Par}.$$

It also is symmetric and nondegenerate and satisfies

$$(h_\lambda, m_\mu) = \delta_{\lambda,\mu} \qquad \text{for all } \lambda, \mu \in \operatorname{Par}.$$

- **Definition:** Let $k = \mathbb{Z}$ from now on.
  - A symmetric function $f \in \Lambda$ will be called *signed multiplicity-free* if f can be expanded as a linear combination of distinct Schur functions with all coefficients in $\{-1, 0, 1\}$. (That is, if the Hall inner product (f, s_μ) is -1 or 0 or 1 for each partition μ.)

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MNable symmetric functions: definition

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- For example, $h_3 p_2$ is signed multiplicity-free, since

$$h_3 p_2 = s_{(5)} + s_{(3,2)} - s_{(3,1,1)};$$

but it is not MNable, since the product

$$h_3 p_2 s_{(2)} = -s_{(3,2,1,1)} + s_{(3,2,2)} - s_{(4,1,1,1)} + s_{(4,3)} - s_{(5,1,1)} + 2s_{(5,2)} + s_{(6,1)} + s_{(7)}$$

is not signed multiplicity-free (due to the coefficient of $s_{(5,2)}$ being 2).

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  - A symmetric function $f \in \Lambda$ will be called *MNable* if for each partition μ, the product fs_μ is signed multiplicity-free.
- **First Pieri rule:** Each $\mu \in \mathrm{Par}$ and $i \in \mathbb{N}$ satisfy

$$h_i s_\mu = \sum_{\substack{\lambda \in \mathrm{Par}; \\ \lambda/\mu \text{ is a horizontal } i\text{-strip}}} s_\lambda .$$

The right hand side is signed multiplicity-free (without any -1's). Thus, h_i is MNable.

MNable symmetric functions: examples

- **Definition:** Let $k = \mathbb{Z}$ from now on.
  - A symmetric function $f \in \Lambda$ will be called *signed multiplicity-free* if f can be expanded as a linear combination of distinct Schur functions with all coefficients in $\{-1, 0, 1\}$. (That is, if the Hall inner product (f, s_μ) is -1 or 0 or 1 for each partition μ.)
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- **Second Pieri rule:** Each $\mu \in \mathrm{Par}$ and $i \in \mathbb{N}$ satisfy

$$e_i s_\mu = \sum_{\substack{\lambda \in \mathrm{Par}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} s_\lambda .$$

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- **Definition:** Let $\mathbf{k} = \mathbb{Z}$ from now on.
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  - A symmetric function $f \in \Lambda$ will be called *MNable* if for each partition μ, the product fs_μ is signed multiplicity-free.
- **Murnaghan–Nakayama rule:** Each $\mu \in \mathrm{Par}$ and $i > 0$ satisfy

$$p_i s_\mu = \sum_{\substack{\lambda \in \mathrm{Par}; \\ \lambda/\mu \text{ is a rim hook of size } i}} \pm s_\lambda .$$

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  - A symmetric function $f \in \Lambda$ will be called *MNable* if for each partition μ, the product fs_μ is signed multiplicity-free.
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$$p_i s_\mu = \sum_{\substack{\lambda \in \mathrm{Par}; \\ \lambda/\mu \text{ is a rim hook of size } i}} \pm s_\lambda .$$

  The right hand side is signed multiplicity-free. Thus, p_i is MNable.
- Roughly speaking, an $f \in \Lambda$ is MNable if and only if there is a **M**urnaghan-**N**akayama-like rule for fs_μ. Thus, the name "**MN**able".

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MNable symmetric functions: results, 1

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  - The product $p_i p_j$ is MNable whenever $i > j > 0$.
  - The function $m_{(k^n)}$ as well as the differences $h_{nk} - m_{(k^n)}$ and $e_{nk} - (-1)^{(k-1)n} m_{(k^n)}$ are MNable for any positive integers n and k (where (k^n) denotes the n-tuple $(k, k, \ldots, k)$).

- **Theorem (continued):**
  - If some $f \in \Lambda$ is MNable, then so are $-f$ and $\omega(f)$, where $\omega : \Lambda \to \Lambda$ is the *fundamental involution* of Λ (that is, the k-algebra automorphism sending $e_n \mapsto h_n$).

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- The proofs use various techniques; the coefficients are not always easy to describe.

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- The MNability of a symmetric function can be tested in finite time using the last bullet point.

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- The families listed above cover all MNable homogeneous symmetric functions of degree < 4. In degree 4, we also have

$$s_{(1,1,1,1)} - s_{(3,1)} + s_{(4)} \qquad \text{and} \qquad s_{(4)} - s_{(2,2)}.$$

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- All MNable s_λ, m_λ, h_λ and e_λ appear in the list above. Not sure if all MNable p_λ.

MNable symmetric functions: question

- **Question:** What symmetric functions are MNable?
  - Any hope of a full classification?
  - Any more infinite families?

Bonus problem

Dual stable Grothendieck polynomials

Reminder on Schur functions

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  Recall that for any skew partition λ/μ, the *(skew) Schur function $s_{\lambda/\mu}$* is defined as the power series

$$\sum_{T \text{ is an SST of shape } \lambda/\mu} \mathbf{x}^{\operatorname{cont} T} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]],$$

  where "SST" is short for "semistandard Young tableau", and where

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- Let us generalize this by extending the sum and introducing extra parameters.

Dual stable Grothendieck polynomials, 1: RPPs

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2	4		

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(In detail: An RPP is a map T from a skew Young diagram to $\{\text{positive integers}\}$ such that $T(i,j) \leq T(i,j+1)$ and $T(i,j) \leq T(i+1,j)$ whenever these are defined.)

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- Let $\mathbf{k}$ be a commutative ring, and fix any elements $t_1, t_2, t_3, \ldots \in \mathbf{k}$.

Dual stable Grothendieck polynomials, 2: definition

- Given a skew partition λ/μ, we define the *refined dual stable Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}$ to be the formal power series

$$\sum_{T \text{ is an RPP of shape } \lambda/\mu} \mathbf{x}^{\operatorname{ircont} T} \mathbf{t}^{\operatorname{ceq} T} \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right],$$

where

$$\mathbf{x}^{\operatorname{ircont} T} = \prod_{k \geq 1} x_k^{\text{number of columns of } T \text{ containing entry } k}$$

and

$$\mathbf{t}^{\operatorname{ceq} T} = \prod_{i \geq 1} t_i^{\text{number of } j \text{ such that } T(i,j) = T(i+1,j)}$$

(where $T(i,j) = T(i+1,j)$ implies, in particular, that both (i,j) and $(i+1,j)$ are cells of T).

This is a formal power series in $x_1, x_2, x_3, \ldots$ (despite the name "polynomial").

Dual stable Grothendieck polynomials, 3: examples on $\mathbf{x}^{\operatorname{ircont} T}$

- Recall:

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- If T is an SST, then $\mathrm{t}^{\mathrm{ceq}\ T} = 1$.
- In general, $\mathrm{t}^{\mathrm{ceq}\ T}$ measures "how often" T breaks the SST condition.

Dual stable Grothendieck polynomials, 5

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- **Example 3:** If $\lambda = (2, 1)$ and $\mu = ()$, then $\widetilde{g}_{\lambda/\mu} = \sum_{a \leq b;\ a < c} x_a x_b x_c + t_1 \sum_{a \leq b} x_a x_b = s_{(2,1)} + t_1 s_{(2)}$.

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Here, $(\mathrm{x}, \mathrm{t}[k : \ell])$ denotes the alphabet $(x_1, x_2, x_3, \ldots, t_k, t_{k+1}, \ldots, t_{\ell-1})$.

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- The case $\mu = \varnothing$ has been proven by Damir Yeliussizov in arXiv:1601.01581.

Thank you

- **Linyuan Liu, Patrick Polo** for the original motivation.
- **Ira Gessel, Jim Haglund, Christopher Ryba, Richard Stanley and Mark Wildon** for interesting discussions.
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