Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions

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Abstract

The dual immaculate functions are a basis of the ring QSym of quasisymmetric functions, and form one of the most natural analogues of the Schur functions. The dual immaculate function corresponding to a composition is a weighted generating function for immaculate tableaux in the same way as a Schur function is for semistandard Young tableaux; an "immaculate tableau" is defined similarly to a semistandard Young tableau, but the shape is a composition rather than a partition, and only the first column is required to strictly increase (whereas the other columns can be arbitrary; but each row has to weakly increase). Dual immaculate functions have been introduced by Berg, Bergeron, Saliola, Serrano and Zabrocki in arXiv:1208.5191, and have since been found to possess numerous nontrivial properties.

In this note, we prove a conjecture of Mike Zabrocki which provides an alternative construction for the dual immaculate functions in terms of certain "vertex operators". The proof uses a dendriform structure on the ring QSym; we discuss the relation of this structure to known dendriform structures on the combinatorial Hopf algebras FQSym and WQSym.

1. Introduction

The three most well-known combinatorial Hopf algebras that are defined over any commutative ring $\mathbf{k}$ are the Hopf algebra of symmetric functions, the Hopf algebra of quasisymmetric functions, and that of noncommutative symmetric functions. The first of these three Hopf algebras has been studied for several decades, while the latter two are newer (the quasisymmetric functions, for example, have been first defined by Ira M. Gessel in 1984); we refer to [HaGuKi10, Chapters 4 and 6] and [GriRei15, Chapters 2 and 5] for expositions of them[1]. All three of these Hopf algebras are known to carry multiple algebraic structures (such as additional products, skewing operators, pairings etc.) and have several bases of combinatorial and algebraic significance. The Schur functions – forming a basis of the symmetric functions – are probably the most important of these bases (certainly the most natural in terms of relations to representation theory and several other applications); a natural question is thus to seek similar bases for quasisymmetric and noncommutative symmetric functions.

Several answers to this question have been suggested, but the simplest one appears to be given in a 2013 paper by Berg, Bergeron, Saliola, Serrano and Zabrocki [BBSSZ13a]: They define the *immaculate (noncommutative symmetric) functions* (which form a basis of the noncommutative symmetric functions) and the *dual immaculate (quasi-symmetric) functions* (which form a basis of the quasisymmetric functions). These two bases are mutually dual and satisfy analogues of various properties of the Schur basis (i.e., the basis of the symmetric functions consisting of the Schur functions). Among these properties are a Littlewood-Richardson rule [BBSSZ13b], a Pieri rule [BSOZ13] (which is not a consequence of the Littlewood-Richardson rule), and a representation-theoretical interpretation [BBSSZ13c]. The immaculate functions can be defined by an analogue of the Jacobi-Trudi identity (see [BBSSZ13a, Remark 3.28] for details), whereas the dual immaculate functions can be defined as generating functions for "immaculate tableaux" in analogy to the Schur functions being generating functions for semistandard tableaux (see Proposition 4.4 below for details).

The original definition of the immaculate functions ([BBSSZ13a, Definition 3.2]) is by applying a sequence of so-called *noncommutative Bernstein operators* to the constant power series 1. Around 2013, Mike Zabrocki conjectured that the dual immaculate functions can be obtained by a similar use of "quasi-symmetric Bernstein operators". The purpose of this note is to prove this conjecture (Corollary 5.6 below). Along the way, we define certain new binary operations on QSym (the ring of quasisymmetric functions); two of them give rise to a structure of a dendriform algebra [EbrFar08], which seems to be interesting in its own right.

This note is organized as follows: In Section 2, we recall basic properties of quasisymmetric (and symmetric) functions and introduce the notations that we shall use. In Section 3, we define two binary operations $\prec$ and Φ on the power series ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and show that they restrict to operations on QSym which interact with the Hopf algebra structure of QSym in a useful way. In Section 4, we define the dual immaculate functions, and show that this definition agrees with the one given in [BBSSZ13a, Remark 3.28]; we then give a combinatorial interpretation of dual immaculate functions (which is not new, but

[1]Historically, the origin of the noncommutative symmetric functions is in [GKLLRT95], whereas the quasisymmetric functions have been introduced in [Gessel84]. See also [Stanle99, Section 7.19] specifically for the quasisymmetric functions and their enumerative applications (although the Hopf algebra structure does not appear in this source).

has apparently never been explicitly stated). In Section 5, we prove Zabrocki's conjecture. In Section 6, we discuss how our binary operations can be lifted to noncommutative power series and restrict to operations on WQSym, which are closely related to similar operations that have appeared in the literature. In the final Section 7, we ask some further questions.

This note is available in two versions: a short one and a long one (with more details, mainly in proofs). The former is available at `https://www.cip.ifi.lmu.de/~grinberg/algebra/dimcreation.pdf`, the latter at `https://www.cip.ifi.lmu.de/~grinberg/algebra/dimcreation-long.pdf`. The version you are currently reading is the long (detailed) one. Both versions are compiled from the same sourcecode (the short one compiles by default; see the comments at front of the TeX file for precise instructions to get the long one). Both versions appear on the arXiv as preprint arXiv:1410.0079 (the short version being the regular PDF download, while the long version is an ancillary file).

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The published version differs insignificantly from the above-mentioned short version of this note. (The former has editorial changes; the latter has some trivial corrections and updated references.)

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2. Quasisymmetric functions

We assume that the reader is familiar with the basics of the theory of symmetric and quasisymmetric functions (as presented, e.g., in [HaGuKi10, Chapters 4 and 6] and [GriRei15, Chapters 2 and 5]). However, let us define all the notations that we need (not least because they are not consistent across literature). We shall try to have our notations match those used in [BBSSZ13a, Section 2] as much as possible.

We use $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$.

A *composition* means a finite sequence of positive integers. For instance, $(2, 3)$ and $(1, 5, 1)$ are compositions. The *empty composition* (i.e., the empty sequence $()$) is denoted by $\varnothing$. We denote by Comp the set of all compositions. For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we denote by $|\alpha|$ the *size* of the composition α; this is the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$. If $n \in \mathbb{N}$, then a *composition of* n simply means a composition having size n. A *nonempty composition* means a composition that is not empty (or, equivalently, that has size > 0).

Let $\mathbf{k}$ be a commutative ring (which, for us, means a commutative ring with unity). This $\mathbf{k}$ will stay fixed throughout the paper. We shall define our symmetric and quasisymmetric functions over this commutative ring $\mathbf{k}$. [2] Every tensor sign $\otimes$ without a subscript should be understood to mean $\otimes_{\mathbf{k}}$.

Let $x_1, x_2, x_3, \ldots$ be countably many distinct indeterminates. We let Mon be the free abelian monoid on the set $\{x_1, x_2, x_3, \ldots\}$ (written multiplicatively); it consists of elements of the form $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ for finitely supported $(a_1, a_2, a_3, \ldots) \in \mathbb{N}^\infty$ (where "finitely supported" means that all but finitely many positive integers i satisfy $a_i = 0$). A *monomial* will mean an element of Mon. Thus, monomials are combinatorial objects (without coefficients), independent of $\mathbf{k}$.

We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. By abuse of notation, we shall identify every monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots \in \text{Mon}$ with the corresponding element $x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \cdot \cdots$ of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ when necessary (e.g., when we speak of the sum of two monomials or when we multiply a monomial with an element of $\mathbf{k}$); however, monomials don't live in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ per se[3].

The $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a topological $\mathbf{k}$-algebra; its topology is the product topology[4]. The polynomial ring $\mathbf{k}[x_1, x_2, x_3, \ldots]$ is a dense subset of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ with respect to this topology. This allows to prove certain identities in the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (such as the associativity of multiplication, just to give a stupid example) by first proving them in $\mathbf{k}[x_1, x_2, x_3, \ldots]$ (that

[2]We do not require anything from $\mathbf{k}$ other than being a commutative ring. Some authors prefer to work only over specific rings $\mathbf{k}$, such as $\mathbb{Z}$ or $\mathbb{Q}$ (for example, [BBSSZ13a] always works over $\mathbb{Q}$). Usually, their results (and often also their proofs) nevertheless are just as valid over arbitrary $\mathbf{k}$. We see no reason to restrict our generality here.

[3]This is a technicality. Indeed, the monomials 1 and x_1 are distinct, but the corresponding elements 1 and x_1 of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ are identical when $\mathbf{k} = 0$. So we could not regard the monomials as lying in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by default.

[4]More precisely, this topology is defined as follows (see also [GriRei15, Section 2.6]):

We endow the ring $\mathbf{k}$ with the discrete topology. To define a topology on the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, we (temporarily) regard every power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ as the family of its coefficients. Thus, $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ becomes a product of infinitely many copies of $\mathbf{k}$ (one for each monomial). This allows us to define a product topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This product topology is the topology that we will be using whenever we make statements about convergence in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ or write down infinite sums of power series. A sequence $(a_n)_{n \in \mathbb{N}}$ of power series converges to a power series a with respect to this topology if and only if for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the coefficient of } \mathfrak{m} \text{ in } a_n) = (\text{the coefficient of } \mathfrak{m} \text{ in } a).$$

Note that this is **not** the topology obtained by taking the completion of $\mathbf{k}[x_1, x_2, x_3, \ldots]$ with respect to the standard grading (in which all x_i have degree 1). Indeed, this completion is not even the whole $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

is, for polynomials), and then arguing that they follow by density in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

If $\mathfrak{m}$ is a monomial, then $\operatorname{Supp}\mathfrak{m}$ will denote the subset

$$\{i \in \{1,2,3,\ldots\} \mid \text{the exponent with which } x_i \text{ occurs in } \mathfrak{m} \text{ is } > 0\}$$

of $\{1,2,3,\ldots\}$; this subset is finite. The *degree* $\deg \mathfrak{m}$ of a monomial $\mathfrak{m} = x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ is defined to be $a_1 + a_2 + a_3 + \cdots \in \mathbb{N}$.

A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *bounded-degree* if there exists an $N \in \mathbb{N}$ such that every monomial of degree $> N$ appears with coefficient 0 in P. Let $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$ denote the $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ formed by the bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

The $\mathbf{k}$*-algebra of symmetric functions* over $\mathbf{k}$ is defined as the $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$ consisting of all bounded-degree power series which are invariant under any permutation of the indeterminates. This $\mathbf{k}$-subalgebra is denoted by Sym. (Notice that Sym is denoted Λ in [GriRei15].) As a $\mathbf{k}$-module, Sym is known to have several bases, such as the basis of complete homogeneous symmetric functions (h_λ) and that of the Schur functions (s_λ), both indexed by the integer partitions.

Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent* if they have the form $\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ and $\mathfrak{n} = x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_\ell}^{\alpha_\ell}$ for some $\ell \in \mathbb{N}$, some positive integers $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, some positive integers $i_1, i_2, \ldots, i_\ell$ satisfying $i_1 < i_2 < \cdots < i_\ell$, and some positive integers $j_1, j_2, \ldots, j_\ell$ satisfying $j_1 < j_2 < \cdots < j_\ell$ [5]. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if any two pack-equivalent monomials have equal coefficients in P. The $\mathbf{k}$*-algebra of quasisymmetric functions* over $\mathbf{k}$ is defined as the $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$ consisting of all bounded-degree power series which are quasisymmetric. It is clear that $\mathrm{Sym} \subseteq \mathrm{QSym}$.

For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, the *monomial quasisymmetric function M_α* is defined by

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}.$$

One easily sees that $M_\alpha \in \mathrm{QSym}$ for every $\alpha \in \mathrm{Comp}$. It is well-known that $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module QSym; this is the so-called *monomial basis* of QSym. Other bases of QSym exist as well, some of which we are going to encounter below.

It is well-known that the $\mathbf{k}$-algebras Sym and QSym can be canonically endowed with Hopf algebra structures such that Sym is a Hopf subalgebra of QSym. We refer to [HaGuKi10, Chapters 4 and 6] and [GriRei15, Chapters 2 and 5] for the definitions of these structures (and for a definition of the notion of a Hopf algebra); at this point, let us merely state a few properties. The comultipli-

[5] For instance, the monomial $x_1^4 x_2^2 x_3 x_7^6$ is pack-equivalent to $x_2^4 x_4^2 x_4 x_5^6$, but not to $x_2^2 x_1^4 x_3 x_7^6$.

cation $\Delta : \operatorname{QSym} \to \operatorname{QSym} \otimes \operatorname{QSym}$ of QSym satisfies

$$\Delta\left(M_\alpha\right) = \sum_{i=0}^{\ell} M_{(\alpha_1,\alpha_2,\ldots,\alpha_i)} \otimes M_{(\alpha_{i+1},\alpha_{i+2},\ldots,\alpha_\ell)}$$

for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}$. The counit $\varepsilon : \operatorname{QSym} \to \mathbf{k}$ of QSym satisfies $\varepsilon\left(M_\alpha\right) = \begin{cases} 1, & \text{if } \alpha = \varnothing; \\ 0, & \text{if } \alpha \neq \varnothing \end{cases}$ for every $\alpha \in \operatorname{Comp}$.

We shall always use the notation Δ for the comultiplication of a Hopf algebra, the notation ε for the counit of a Hopf algebra, and the notation S for the antipode of a Hopf algebra. Occasionally we shall use *Sweedler's notation* for working with coproducts of elements of a Hopf algebra[6].

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of an $n \in \mathbb{N}$, then we define a subset $D(\alpha)$ of $\{1, 2, \ldots, n-1\}$ by

$$D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}.$$

This subset $D(\alpha)$ is called the *set of partial sums* of the composition α; see [GriRei15, Definition 5.1.10] for its further properties. Most importantly, a composition α of size n can be uniquely reconstructed from n and $D(\alpha)$.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of an $n \in \mathbb{N}$, then the *fundamental quasisymmetric function* $F_\alpha \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$ can be defined by

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}. \qquad (1)$$

(This is only one of several possible definitions of F_α. In [GriRei15, Definition 5.2.4], the power series F_α is denoted by L_α and defined differently; but [GriRei15, Proposition 5.2.9] proves the equivalence of this definition with ours.[7]) One can easily see that $F_\alpha \in \operatorname{QSym}$ for every $\alpha \in \operatorname{Comp}$. The family $(F_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym as well; it is called the *fundamental basis* of QSym.

[6]In a nutshell, Sweedler's notation (or, more precisely, the special case of Sweedler's notation that we will use) consists in writing $\sum_{(c)} c_{(1)} \otimes c_{(2)}$ for the tensor $\Delta(c) \in C \otimes C$, where c is an element of a $\mathbf{k}$-coalgebra C. The sum $\sum_{(c)} c_{(1)} \otimes c_{(2)}$ symbolizes a representation of the tensor $\Delta(c)$ as a sum $\sum_{i=1}^{N} c_{1,i} \otimes c_{2,i}$ of pure tensors; it allows us to manipulate $\Delta(c)$ without having to explicitly introduce the N and the $c_{1,i}$ and the $c_{2,i}$. For instance, if $f : C \to \mathbf{k}$ is a $\mathbf{k}$-linear map, then we can write $\sum_{(c)} f\left(c_{(1)}\right) c_{(2)}$ for $\sum_{i=1}^{N} f(c_{1,i}) c_{2,i}$. Of course, we need to be careful not to use Sweedler's notation for terms which do depend on the specific choice of the N and the $c_{1,i}$ and the $c_{2,i}$; for instance, we must not write $\sum_{(c)} c_{(1)}^2 c_{(2)}$.

[7]In fact, [GriRei15, (5.2.3)] is exactly our equality (1).

3. Restricted-product operations

We shall now define two binary operations on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

Definition 3.1. We define a binary operation $\prec\ :\ \mathbf{k}[[x_1, x_2, x_3, \ldots]] \times \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (written in infix notation[8]) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and that it satisfy

$$\mathfrak{m} \prec \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases} \tag{2}$$

for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.

Some clarifications are in order. First, we are using $\prec$ as an operation symbol (rather than as a relation symbol as it is commonly used)[9]. Second, we consider $\min \varnothing$ to be ∞, and this symbol ∞ is understood to be greater than every integer[10]. Hence, $\mathfrak{m} \prec 1 = \mathfrak{m}$ for every nonconstant monomial $\mathfrak{m}$, and $1 \prec \mathfrak{m} = 0$ for every monomial $\mathfrak{m}$.

Let us first see why the operation $\prec$ in Definition 3.1 is well-defined. Recall that the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is the product topology. Hence, if $\prec$ is to be $\mathbf{k}$-bilinear and continuous with respect to it, we must have

$$\left(\sum_{\mathfrak{m} \in \operatorname{Mon}} \lambda_{\mathfrak{m}} \mathfrak{m}\right) \prec \left(\sum_{\mathfrak{n} \in \operatorname{Mon}} \mu_{\mathfrak{n}} \mathfrak{n}\right) = \sum_{\mathfrak{m} \in \operatorname{Mon}} \sum_{\mathfrak{n} \in \operatorname{Mon}} \lambda_{\mathfrak{m}} \mu_{\mathfrak{n}} \mathfrak{m} \prec \mathfrak{n}$$

for any families $(\lambda_{\mathfrak{m}})_{\mathfrak{m} \in \operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}}$ and $(\mu_{\mathfrak{n}})_{\mathfrak{n} \in \operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}}$ of scalars. Combined with (2), this uniquely determines $\prec$. Therefore, the binary operation $\prec$ satisfying the conditions of Definition 3.1 is unique (if it exists). But it also exists, because if we define a binary operation $\prec$ on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by the explicit formula

$$\left(\sum_{\mathfrak{m} \in \operatorname{Mon}} \lambda_{\mathfrak{m}} \mathfrak{m}\right) \prec \left(\sum_{\mathfrak{n} \in \operatorname{Mon}} \mu_{\mathfrak{n}} \mathfrak{n}\right) = \sum_{\substack{(\mathfrak{m},\mathfrak{n}) \in \operatorname{Mon} \times \operatorname{Mon}; \\ \min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n})}} \lambda_{\mathfrak{m}} \mu_{\mathfrak{n}} \mathfrak{m}\mathfrak{n}$$
$$\text{for all } (\lambda_{\mathfrak{m}})_{\mathfrak{m} \in \operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}} \text{ and } (\mu_{\mathfrak{n}})_{\mathfrak{n} \in \operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}},$$

then it clearly satisfies the conditions of Definition 3.1 (and is well-defined).

The operation $\prec$ is not associative; however, it is part of what is called a *dendriform algebra* structure on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (and on QSym, as we shall see

[8] By this we mean that we write $a \prec b$ instead of $\prec(a, b)$.

[9] Of course, the symbol has been chosen because it is reminiscent of the smaller symbol in "$\min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n})$".

[10] but not greater than itself

below). The following remark (which will not be used until Section 6, and thus can be skipped by a reader not familiar with dendriform algebras) provides some details:

Remark 3.2. Let us define another binary operation $\succeq$ on $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ similarly to $\prec$ except that we set

$$\mathfrak{m} \succeq \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min\left(\operatorname{Supp} \mathfrak{m}\right) \geq \min\left(\operatorname{Supp} \mathfrak{n}\right); \\ 0, & \text{if } \min\left(\operatorname{Supp} \mathfrak{m}\right) < \min\left(\operatorname{Supp} \mathfrak{n}\right) \end{cases}.$$

Then, the structure $\left(\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right], \prec, \succeq\right)$ is a dendriform algebra augmented to satisfy [EbrFar08, (15)]. In particular, any three elements a, b and c of $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ satisfy

$$\begin{aligned} a \prec b + a \succeq b &= ab; \\ (a \prec b) \prec c &= a \prec (bc); \\ (a \succeq b) \prec c &= a \succeq (b \prec c); \\ a \succeq (b \succeq c) &= (ab) \succeq c. \end{aligned}$$

Now, we introduce another binary operation.

Definition 3.3. We define a binary operation $\Phi : \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right] \times \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right] \to \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ and that it satisfy

$$\mathfrak{m} \Phi \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) \leq \min\left(\operatorname{Supp} \mathfrak{n}\right); \\ 0, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) > \min\left(\operatorname{Supp} \mathfrak{n}\right) \end{cases}$$

for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.

Here, $\max \varnothing$ is understood as 0. The welldefinedness of the operation Φ in Definition 3.3 is proven in the same way as that of the operation $\prec$.

Let us make a simple observation which will not be used until Section 6, but provides some context:

Proposition 3.4. The binary operation Φ is associative. It is also unital (with 1 serving as the unity).

Proof of Proposition 3.4. Let us first show that Φ is associative.

In order to show this, we must prove that

$$(a \Phi b) \Phi c = a \Phi (b \Phi c) \tag{3}$$

for any three elements a, b and c of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

But if $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{p}$ are three monomials, then the definition of Φ readily shows that

$$(\mathfrak{m} \Phi \mathfrak{n}) \Phi \mathfrak{p} = \begin{cases} \mathfrak{mnp}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{n}) \\ & \qquad \text{and } \max(\operatorname{Supp}(\mathfrak{mn})) \leq \min(\operatorname{Supp}\mathfrak{p}); \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \mathfrak{mnp}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{n}) \\ & \qquad \text{and } \max((\operatorname{Supp}\mathfrak{m}) \cup (\operatorname{Supp}\mathfrak{n})) \leq \min(\operatorname{Supp}\mathfrak{p}); \\ 0, & \text{otherwise} \end{cases}$$
$$(\text{since } \operatorname{Supp}(\mathfrak{mn}) = (\operatorname{Supp}\mathfrak{m}) \cup (\operatorname{Supp}\mathfrak{n}))$$

and

$$\mathfrak{m} \Phi (\mathfrak{n} \Phi \mathfrak{p}) = \begin{cases} \mathfrak{mnp}, & \text{if } \max(\operatorname{Supp}\mathfrak{n}) \leq \min(\operatorname{Supp}\mathfrak{p}) \\ & \qquad \text{and } \max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}(\mathfrak{np})); \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \mathfrak{mnp}, & \text{if } \max(\operatorname{Supp}\mathfrak{n}) \leq \min(\operatorname{Supp}\mathfrak{p}) \\ & \qquad \text{and } \max(\operatorname{Supp}\mathfrak{m}) \leq \min((\operatorname{Supp}\mathfrak{n}) \cup (\operatorname{Supp}\mathfrak{p})); \\ 0, & \text{otherwise} \end{cases}$$
$$(\text{since } \operatorname{Supp}(\mathfrak{np}) = (\operatorname{Supp}\mathfrak{n}) \cup (\operatorname{Supp}\mathfrak{p}));$$

thus, $(\mathfrak{m} \Phi \mathfrak{n}) \Phi \mathfrak{p} = \mathfrak{m} \Phi (\mathfrak{n} \Phi \mathfrak{p})$ (since it is straightforward to check that the condition
$(\max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{n}) \text{ and } \max((\operatorname{Supp}\mathfrak{m}) \cup (\operatorname{Supp}\mathfrak{n})) \leq \min(\operatorname{Supp}\mathfrak{p}))$
is equivalent to the condition
$(\max(\operatorname{Supp}\mathfrak{n}) \leq \min(\operatorname{Supp}\mathfrak{p}) \text{ and } \max(\operatorname{Supp}\mathfrak{m}) \leq \min((\operatorname{Supp}\mathfrak{n}) \cup (\operatorname{Supp}\mathfrak{p})))$
[11]). In other words, the equality (3) holds when a, b and c are monomials. Thus, this equality also holds whenever a, b and c are polynomials (since it is $\mathbf{k}$-linear in a, b and c), and consequently also holds whenever a, b and c are power series (since it is continuous in a, b and c). This proves that Φ is associative.

The proof of the fact that Φ is unital (with unity 1) is similar and left to the reader. Proposition 3.4 is thus shown. □

Here is another property of Φ that will not be used until Section 6:

Proposition 3.5. Every $a \in \operatorname{QSym}$ and $b \in \operatorname{QSym}$ satisfy $a \prec b \in \operatorname{QSym}$ and $a \Phi b \in \operatorname{QSym}$.

For example, we can explicitly describe the operation Φ on the monomial basis $(M_\gamma)_{\gamma \in \operatorname{Comp}}$ of QSym. Namely, any two nonempty compositions α and β

[11]Indeed, both conditions are equivalent to
$(\max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{n}) \text{ and } \max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{p}) \text{ and } \max(\operatorname{Supp}\mathfrak{n}) \leq \min(\operatorname{Supp}\mathfrak{p}))$.

satisfy $M_\alpha \mathbin{\Phi} M_\beta = M_{[\alpha,\beta]} + M_{\alpha\odot\beta}$, where $[\alpha, \beta]$ and $\alpha \odot \beta$ are two compositions defined by

$$[(\alpha_1, \alpha_2, \ldots, \alpha_\ell), (\beta_1, \beta_2, \ldots, \beta_m)] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m);$$
$$(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \odot (\beta_1, \beta_2, \ldots, \beta_m) = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m).$$

[12] If one of α and β is empty, then $M_\alpha \mathbin{\Phi} M_\beta = M_{[\alpha,\beta]}$.

Proposition 3.5 can reasonably be called obvious; the below proof owes its length mainly to the difficulty of formalizing the intuition.

Proof of Proposition 3.5. We shall first introduce a few more notations.

If $\mathfrak{m}$ is a monomial, then the *Parikh composition* of $\mathfrak{m}$ is defined as follows: Write $\mathfrak{m}$ in the form $\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ for some $\ell \in \mathbb{N}$, some positive integers α_1, α_2, …, α_ℓ, and some positive integers i_1, i_2, …, i_ℓ satisfying $i_1 < i_2 < \cdots < i_\ell$. (Notice that this way of writing $\mathfrak{m}$ is unique.) Then, the Parikh composition of $\mathfrak{m}$ is defined to be the composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$.

We denote by $\operatorname{Parikh} \mathfrak{m}$ the Parikh composition of a monomial $\mathfrak{m}$. Now, it is easy to see that the definition of a monomial quasisymmetric function M_α can be rewritten as follows: For every $\alpha \in \operatorname{Comp}$, we have

$$M_\alpha = \sum_{\substack{\mathfrak{m} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{m} = \alpha}} \mathfrak{m}. \tag{4}$$

(Indeed, for any given composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, the monomials $\mathfrak{m}$ satisfying $\operatorname{Parikh} \mathfrak{m} = \alpha$ are precisely the monomials of the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ with i_1, i_2, …, i_ℓ being positive integers satisfying $i_1 < i_2 < \cdots < i_\ell$.)

Now, pack-equivalent monomials can be characterized as follows: Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are pack-equivalent if and only if they have the same Parikh composition.

Now, we come to the proof of Proposition 3.5.

Let us first fix two compositions α and β. We shall prove that $M_\alpha \prec M_\beta \in \operatorname{QSym}$.

Write the compositions α and β as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Let $\mathcal{S}_0$ denote the ℓ-element set $\{0\} \times \{1, 2, \ldots, \ell\}$. Let $\mathcal{S}_1$ denote the m-element set $\{1\} \times \{1, 2, \ldots, m\}$. Let $\mathcal{S}$ denote the $(\ell + m)$-element set $\mathcal{S}_0 \cup \mathcal{S}_1$. Let $\operatorname{inc}_0 : \{1, 2, \ldots, \ell\} \to \mathcal{S}$ be the map which sends every $p \in \{1, 2, \ldots, \ell\}$ to $(0, p) \in \mathcal{S}_0 \subseteq \mathcal{S}$. Let $\operatorname{inc}_1 : \{1, 2, \ldots, m\} \to \mathcal{S}$ be the map which sends every $q \in \{1, 2, \ldots, m\}$ to $(1, q) \in \mathcal{S}_1 \subseteq \mathcal{S}$. Define a map $\rho : \mathcal{S} \to \{1, 2, 3, \ldots\}$ by setting

$$\rho(0, p) = \alpha_p \qquad \text{for all } p \in \{1, 2, \ldots, \ell\};$$
$$\rho(1, q) = \beta_q \qquad \text{for all } q \in \{1, 2, \ldots, m\}.$$

For every composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, we define a *$\gamma$-smap* to be a map $f : \mathcal{S} \to \{1, 2, \ldots, n\}$ satisfying the following three properties:

[12] What we call $[\alpha, \beta]$ is denoted by $\alpha \cdot \beta$ in [GriRei15, before Proposition 5.1.7].

- The maps $f \circ \operatorname{inc}_0$ and $f \circ \operatorname{inc}_1$ are strictly increasing.
- We have[13] $\min(f(\mathcal{S}_0)) < \min(f(\mathcal{S}_1))$.
- Every $u \in \{1, 2, \ldots, n\}$ satisfies

$$\sum_{s \in f^{-1}(u)} \rho(s) = \gamma_u.$$

These three properties will be called the three *defining properties* of a γ-smap.

Now, we make the following claim:

Claim 1: Let $\mathfrak{q}$ be any monomial. Let γ be the Parikh composition of $\mathfrak{q}$. The coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all γ-smaps.

Proof of Claim 1: Write the composition γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. Write the monomial $\mathfrak{q}$ in the form $\mathfrak{q} = x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_n}^{\gamma_n}$ for some positive integers k_1, k_2, ..., k_n satisfying $k_1 < k_2 < \cdots < k_n$. (This is possible because $(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma$ is the Parikh composition of $\mathfrak{q}$.) Then, $\operatorname{Supp} \mathfrak{q} = \{k_1, k_2, \ldots, k_n\}$.

From (4), we get $M_\alpha = \sum\limits_{\substack{\mathfrak{m} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{m} = \alpha}} \mathfrak{m}$. Similarly, $M_\beta = \sum\limits_{\substack{\mathfrak{n} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{n} = \beta}} \mathfrak{n}$. Hence,

$$\begin{aligned}
& M_\alpha \prec M_\beta \\
&= \left(\sum_{\substack{\mathfrak{m} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{m} = \alpha}} \mathfrak{m} \right) \prec \left(\sum_{\substack{\mathfrak{n} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{n} = \beta}} \mathfrak{n} \right) \\
&= \sum_{\substack{\mathfrak{m} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{m} = \alpha}} \sum_{\substack{\mathfrak{n} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{n} = \beta}} \underbrace{\mathfrak{m} \prec \mathfrak{n}}_{\substack{= \begin{cases} \mathfrak{m}\mathfrak{n}, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases} \\ \text{(by the definition of } \prec \text{ on monomials)}}} \\
& \qquad \left(\text{since the operation } \prec \text{ is } \mathbf{k}\text{-bilinear and continuous}\right) \\
&= \sum_{\substack{\mathfrak{m} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{m} = \alpha}} \sum_{\substack{\mathfrak{n} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{n} = \beta}} \begin{cases} \mathfrak{m}\mathfrak{n}, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases} \\
&= \sum_{\substack{(\mathfrak{m}, \mathfrak{n}) \in \operatorname{Mon} \times \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{m} = \alpha; \\ \operatorname{Parikh} \mathfrak{n} = \beta; \\ \min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n})}} \mathfrak{m}\mathfrak{n}.
\end{aligned}$$

Thus, the coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all pairs $(\mathfrak{m}, \mathfrak{n}) \in \operatorname{Mon} \times \operatorname{Mon}$ such that $\operatorname{Parikh} \mathfrak{m} = \alpha$, $\operatorname{Parikh} \mathfrak{n} = \beta$, $\min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n})$ and $\mathfrak{m}\mathfrak{n} = \mathfrak{q}$. These pairs shall be called *spairs*. (The concept of a spair depends on $\mathfrak{q}$; we nevertheless omit $\mathfrak{q}$ from the notation, since we regard $\mathfrak{q}$ as fixed.)

[13] Keep in mind that we set $\min \varnothing = \infty$.

Now, we shall construct a bijection between the γ-smaps and the spairs.

Indeed, we first define a map Φ from the set of γ-smaps to the set of spairs as follows: Let $f : \mathcal{S} \to \{1, 2, \ldots, n\}$ be a γ-smap. Then, $\Phi(f)$ is defined to be the spair

$$\left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p}, \prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right).$$

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Conversely, we define a map Ψ from the set of spairs to the set of γ-smaps as follows: Let $(\mathfrak{m}, \mathfrak{n})$ be a spair. Then, we write the monomial $\mathfrak{m}$ in the form

[14]This is a well-defined spair, for the following reasons:

- The first defining property of a γ-smap can be rewritten as "$f(0,1) < f(0,2) < \cdots < f(0,\ell)$ and $f(1,1) < f(1,2) < \cdots < f(1,m)$". Combined with $k_1 < k_2 < \cdots < k_n$, this shows that $k_{f(0,1)} < k_{f(0,2)} < \cdots < k_{f(0,\ell)}$ and $k_{f(1,1)} < k_{f(1,2)} < \cdots < k_{f(1,m)}$. Hence, $\operatorname{Parikh}\left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) = \alpha$ and $\operatorname{Parikh}\left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) = \beta$.
- The second defining property of a γ-smap shows that $\min(f(\mathcal{S}_0)) < \min(f(\mathcal{S}_1))$, so that $k_{\min(f(\mathcal{S}_0))} < k_{\min(f(\mathcal{S}_1))}$ (since $k_1 < k_2 < \cdots < k_n$). But $\operatorname{Supp}\left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) = \left\{ k_{f(s)} \mid s \in \mathcal{S}_0 \right\}$ and thus $\min\left(\operatorname{Supp}\left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) \right) = \min\left\{ k_{f(s)} \mid s \in \mathcal{S}_0 \right\} = k_{\min(f(\mathcal{S}_0))}$ (since $k_1 < k_2 < \cdots < k_n$). Similarly, $\min\left(\operatorname{Supp}\left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) \right) = k_{\min(f(\mathcal{S}_1))}$. Hence,
$$\min\left(\operatorname{Supp}\left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) \right) = k_{\min(f(\mathcal{S}_0))} < k_{\min(f(\mathcal{S}_1))} = \min\left(\operatorname{Supp}\left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) \right).$$
- The third defining property of a γ-smap shows that $\sum_{s \in f^{-1}(u)} \rho(s) = \gamma_u$ for every $u \in \{1, 2, \ldots, n\}$. Now, every $p \in \{1, 2, \ldots, \ell\}$ satisfies $\alpha_p = \rho(0,p)$. Hence, $\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} = \prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\rho(0,p)} = \prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)}$. Similarly, $\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} = \prod_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)}$. Multiplying these two identities, we obtain
$$\begin{aligned} \left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) \left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) &= \left(\prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)} \right) \left(\prod_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)} \right) = \prod_{s \in \mathcal{S}} x_{k_{f(s)}}^{\rho(s)} = \prod_{u=1}^{n} \prod_{s \in f^{-1}(u)} \underbrace{x_{k_{f(s)}}^{\rho(s)}}_{\substack{= x_{k_u}^{\rho(s)} \\ (\text{since } f(s) = u)}} \\ &= \prod_{u=1}^{n} \underbrace{\prod_{s \in f^{-1}(u)} x_{k_u}^{\rho(s)}}_{\substack{= x_{k_u}^{\gamma_u} \\ (\text{since } \sum_{s \in f^{-1}(u)} \rho(s) = \gamma_u)}} = \prod_{u=1}^{n} x_{k_u}^{\gamma_u} = x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_n}^{\gamma_n} = \mathfrak{q}. \end{aligned}$$

$\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ for some positive integers $i_1, i_2, \ldots, i_\ell$ satisfying $i_1 < i_2 < \cdots < i_\ell$ (this is possible since $\operatorname{Parikh} \mathfrak{m} = \alpha$), and we write the monomial $\mathfrak{n}$ in the form $\mathfrak{n} = x_{j_1}^{\beta_1} x_{j_2}^{\beta_2} \cdots x_{j_m}^{\beta_m}$ for some positive integers $j_1, j_2, \ldots, j_m$ satisfying $j_1 < j_2 < \cdots < j_m$ (this is possible since $\operatorname{Parikh} \mathfrak{n} = \beta$). Of course, $\operatorname{Supp} \mathfrak{m} = \{i_1, i_2, \ldots, i_\ell\}$ and $\operatorname{Supp} \mathfrak{n} = \{j_1, j_2, \ldots, j_m\}$, so that $\min \{i_1, i_2, \ldots, i_\ell\} < \min \{j_1, j_2, \ldots, j_m\}$ (since $\min (\operatorname{Supp} \mathfrak{m}) < \min (\operatorname{Supp} \mathfrak{n})$).

Now, we define a map $f : \mathcal{S} \to \{1, 2, \ldots, n\}$ as follows:

- For every $p \in \{1, 2, \ldots, \ell\}$, we let $f(0, p)$ be the unique $r \in \{1, 2, \ldots, n\}$ such that $i_p = k_r$. [15]
- For every $q \in \{1, 2, \ldots, m\}$, we let $f(1, q)$ be the unique $r \in \{1, 2, \ldots, n\}$ such that $j_q = k_r$. [16]

It is now straightforward to show that f is a γ-smap.[17] We define $\Psi(\mathfrak{m}, \mathfrak{n})$ to be this γ-smap f.

[15] To prove that this is well-defined, we need to show that this r exists and is unique. The uniqueness of r is obvious (since $k_1 < k_2 < \cdots < k_n$). To prove its existence, we notice that $i_p \in \operatorname{Supp} \mathfrak{m}$ (since $\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ and $\alpha_p > 0$) and thus $i_p \in \operatorname{Supp} \mathfrak{m} \subseteq \operatorname{Supp} \underbrace{(\mathfrak{m}\mathfrak{n})}_{=\mathfrak{q}} = \operatorname{Supp} \mathfrak{q} = \{k_1, k_2, \ldots, k_n\}$.

[16] This is again well-defined, for similar reasons as the r in the definition of $f(0, p)$.

[17] Indeed:

- The first defining property of a γ-smap holds. (*Proof:* Let us show that $f \circ \operatorname{inc}_0$ is strictly increasing (the proof for $f \circ \operatorname{inc}_1$ is similar). Assume it is not. Then there exist some $p, p' \in \{1, 2, \ldots, \ell\}$ satisfying $p < p'$ and $(f \circ \operatorname{inc}_0)(p) \geq (f \circ \operatorname{inc}_0)(p')$. Consider these p, p'. We have $p < p'$, and therefore $i_p < i_{p'}$ (since $i_1 < i_2 < \cdots < i_\ell$). But $(f \circ \operatorname{inc}_0)(p) \geq (f \circ \operatorname{inc}_0)(p')$, and thus $k_{(f \circ \operatorname{inc}_0)(p)} \geq k_{(f \circ \operatorname{inc}_0)(p')}$ (since $k_1 < k_2 < \cdots < k_n$). Since $k_{(f \circ \operatorname{inc}_0)(p)} = k_{f(0,p)} = i_p$ (by the definition of $f(0, p)$) and similarly $k_{(f \circ \operatorname{inc}_0)(p')} = i_{p'}$, this rewrites as $i_p \geq i_{p'}$. This contradicts $i_p < i_{p'}$. This contradiction completes the proof.)
- The second defining property of a γ-smap holds. (*Proof:* We WLOG assume that ℓ and m are positive, since the other case is straightforward. We have $i_1 < i_2 < \cdots < i_\ell$. In other words, $k_{f(0,1)} < k_{f(0,2)} < \cdots < k_{f(0,\ell)}$ (since $k_{f(0,p)} = i_p$ for every $p \in \{1, 2, \ldots, \ell\}$). Hence, $f(0, 1) < f(0, 2) < \cdots < f(0, \ell)$ (since $k_1 < k_2 < \cdots < k_n$). Hence, $\min(f(\mathcal{S}_0)) = f(0, 1)$. Similarly, $\min(f(\mathcal{S}_1)) = f(1, 1)$. But from $i_1 < i_2 < \cdots < i_\ell$, we obtain $i_1 = \min\{i_1, i_2, \ldots, i_\ell\}$; similarly, $j_1 = \min\{j_1, j_2, \ldots, j_m\}$. Hence, $k_{f(0,1)} = i_1 = \min\{i_1, i_2, \ldots, i_\ell\} < \min\{j_1, j_2, \ldots, j_m\} = j_1 = k_{f(1,1)}$, so that $f(0, 1) < f(1, 1)$ (since $k_1 < k_2 < \cdots < k_n$). Hence, $\min(f(\mathcal{S}_0)) = f(0, 1) < f(1, 1) = \min(f(\mathcal{S}_1))$, qed.)
- The third defining property of a γ-smap holds. (*Proof:* We have

$$\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \prod_{p=1}^{\ell} \underbrace{x_{i_p}^{\alpha_p}}_{\substack{= x_{k_{f(0,p)}}^{\rho(0,p)} \\ (\text{since } \alpha_p = \rho(0,p) \\ \text{and } i_p = k_{f(0,p)})}} = \prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\rho(0,p)} = \prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)}$$

We thus have defined a map Φ from the set of γ-smaps to the set of spairs, and a map Ψ from the set of spairs to the set of γ-smaps. It is straightforward to see that these two maps Φ and Ψ are mutually inverse, and thus Φ is a bijection. We thus have found a bijection between the set of γ-smaps and the set of spairs. Consequently, the number of all γ-smaps equals the number of all spairs.

Now, recall that the coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all spairs. Hence, the coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all γ-smaps (since the number of all γ-smaps equals the number of all spairs). In other words, Claim 1 is proven.

Claim 1 shows that the coefficient of a monomial $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ depends not on $\mathfrak{q}$ but only on the Parikh composition of $\mathfrak{q}$. Thus, any two pack-equivalent monomials have equal coefficients in $M_\alpha \prec M_\beta$ (since any two pack-equivalent monomials have the same Parikh composition). In other words, the power series $M_\alpha \prec M_\beta$ is quasisymmetric. Since $M_\alpha \prec M_\beta \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]_{\mathrm{bdd}}$, this yields that $M_\alpha \prec M_\beta \in \operatorname{QSym}$.

[At this point, let us remark that we can give an explicit formula for $M_\alpha \prec M_\beta$: Namely,

$$M_\alpha \prec M_\beta = \sum_{\gamma \in \operatorname{Comp}} \mathfrak{s}^{\gamma}_{\alpha,\beta} M_\gamma, \tag{5}$$

where $\mathfrak{s}^{\gamma}_{\alpha,\beta}$ is the number of all γ-smaps. Indeed, for every monomial $\mathfrak{q}$, the coefficient of $\mathfrak{q}$ on the left-hand side of (5) equals $\mathfrak{s}^{\gamma}_{\alpha,\beta}$ where γ is the Parikh composition of $\mathfrak{q}$ (because of Claim 1), whereas the coefficient of $\mathfrak{q}$ on the right-hand side of (5) also equals $\mathfrak{s}^{\gamma}_{\alpha,\beta}$ (for obvious reasons). Hence, every monomial has equal coefficients on the two sides of (5), and so (5) holds. Of course, (5) again proves that $M_\alpha \prec M_\beta \in \operatorname{QSym}$, since the sum $\sum\limits_{\gamma \in \operatorname{Comp}} \mathfrak{s}^{\gamma}_{\alpha,\beta} M_\gamma$ has only finitely many nonzero addends (indeed, γ-smaps can only exist if $|\gamma| \leq |\alpha| + |\beta|$).]

Now, let us forget that we fixed α and β. We thus have shown that every two compositions α and β satisfy $M_\alpha \prec M_\beta \in \operatorname{QSym}$.

Now, let $a \in \operatorname{QSym}$ and $b \in \operatorname{QSym}$. We shall only prove that $a \prec b \in \operatorname{QSym}$

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and similarly $\mathfrak{n} = \prod\limits_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)}$. Hence,

$$\mathfrak{m}\mathfrak{n} = \left(\prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)}\right) \left(\prod_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)}\right) = \prod_{s \in \mathcal{S}} x_{k_{f(s)}}^{\rho(s)} \qquad \left(\text{since } \mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \text{ and } \mathcal{S}_0 \cap \mathcal{S}_1 = \varnothing\right).$$

Thus, $\prod\limits_{s \in \mathcal{S}} x_{k_{f(s)}}^{\rho(s)} = \mathfrak{m}\mathfrak{n} = \mathfrak{q} = x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_n}^{\gamma_n}$. Now, for any $u \in \{1, 2, \ldots, n\}$, the exponent of x_{k_u} on the left hand side of this equality is $\sum\limits_{s \in f^{-1}(u)} \rho(s)$ (since $k_1 < k_2 < \cdots < k_n$), whereas the exponent of x_{k_u} on the right hand side is γ_u. Comparing these coefficients, we find $\sum\limits_{s \in f^{-1}(u)} \rho(s) = \gamma_u$.)

(since the proof of $a \varphi b \in \operatorname{QSym}$ is very similar[18]).

The statement that we need to prove ($a \prec b \in \operatorname{QSym}$) is $\mathbf{k}$-linear in each of a and b. Hence, we can WLOG assume that both a and b are elements of the monomial basis of QSym. Assume this. Thus, $a = M_\alpha$ and $b = M_\beta$ for some compositions α and β. Consider these α and β. Now, as we know, $M_\alpha \prec M_\beta \in \operatorname{QSym}$, so that $\underbrace{a}_{=M_\alpha} \prec \underbrace{b}_{=M_\beta} = M_\alpha \prec M_\beta \in \operatorname{QSym}$. This completes our proof of Proposition 3.5. $\square$

Remark 3.6. The proof of Proposition 3.5 given above actually yields a combinatorial formula for $M_\alpha \prec M_\beta$ whenever α and β are two compositions. Namely, let α and β be two compositions. Then,

$$M_\alpha \prec M_\beta = \sum_{\gamma \in \operatorname{Comp}} \mathfrak{s}^{\gamma}_{\alpha,\beta} M_\gamma, \tag{6}$$

where $\mathfrak{s}^{\gamma}_{\alpha,\beta}$ is the number of all smaps $(\alpha, \beta) \to \gamma$. Here a *smap* $(\alpha, \beta) \to \gamma$ means what was called a γ-smap in the above proof of Proposition 3.5.

This is similar to the well-known formula for $M_\alpha M_\beta$ (see, for example, [GriRei15, Proposition 5.1.3]) which (translated into our language) states that

$$M_\alpha M_\beta = \sum_{\gamma \in \operatorname{Comp}} \mathfrak{t}^{\gamma}_{\alpha,\beta} M_\gamma, \tag{7}$$

where $\mathfrak{t}^{\gamma}_{\alpha,\beta}$ is the number of all overlapping shuffles $(\alpha, \beta) \to \gamma$. Here, the *overlapping shuffles* $(\alpha, \beta) \to \gamma$ are defined in the same way as the γ-smaps, with the only difference that the second of the three properties that define a γ-smap (namely, the property $\min (f(\mathcal{S}_0)) < \min (f(\mathcal{S}_1))$) is omitted. Needless to say, (7) can be proven similarly to our proof of (6) above.

Here is a somewhat nontrivial property of φ and $\prec$:

Theorem 3.7. Let S denote the antipode of the Hopf algebra QSym. Let us use Sweedler's notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$, where b is any element of QSym. Then,

$$\sum_{(b)} \left(S\left(b_{(1)}\right) \varphi a \right) b_{(2)} = a \prec b$$

for any $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b \in \operatorname{QSym}$.

Proof of Theorem 3.7. Let $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. We can WLOG assume that a is a monomial (because all operations in sight are $\mathbf{k}$-linear and continuous).

[18] Alternatively, of course, $a \varphi b \in \operatorname{QSym}$ can be checked using the formula $M_\alpha \varphi M_\beta = M_{[\alpha,\beta]} + M_{\alpha \odot \beta}$ (which is easily proven). However, there is no such simple proof for $a \prec b \in \operatorname{QSym}$.

So assume this. That is, $a = \mathfrak{n}$ for some monomial $\mathfrak{n}$. Consider this $\mathfrak{n}$. Let $k = \min(\operatorname{Supp}\mathfrak{n})$. Notice that $k \in \{1, 2, 3, \ldots\} \cup \{\infty\}$.

(Some remarks about ∞ are in order. We use ∞ as an object which is greater than every integer. We will use summation signs like $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k}$ and $\sum\limits_{k < i_1 < i_2 < \cdots < i_\ell}$ in the following. Both of these summation signs range over $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell$ satisfying certain conditions ($1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$ in the first case, and $k < i_1 < i_2 < \cdots < i_\ell$ in the second case). In particular, none of the $i_1, i_2, \ldots, i_\ell$ is allowed to be ∞ (unlike k). So the summation $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k}$ is identical to $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell}$ when $k = \infty$, whereas the summation $\sum\limits_{k < i_1 < i_2 < \cdots < i_\ell}$ is empty when $k = \infty$ unless $\ell = 0$. (If $\ell = 0$, then the summation $\sum\limits_{k < i_1 < i_2 < \cdots < i_\ell}$ ranges over the empty 0-tuple, no matter what k is.)

We shall also use an additional symbol $\infty + 1$, which is understood to be greater than every element of $\{1, 2, 3, \ldots\} \cup \{\infty\}$.)

Every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$a \prec M_\alpha = \left(\sum_{k < i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \cdot a \tag{8}$$

[19].

Let us define a map $\mathfrak{B}_k : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$\mathfrak{B}_k(p) = p(x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots) \qquad \text{for every } p \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$$

(where $p(x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots)$ has to be understood as $p(x_1, x_2, x_3, \ldots) = p$ when $k = \infty$). Then, $\mathfrak{B}_k$ is an evaluation map (in an appropriate sense) and thus a continuous $\mathbf{k}$-algebra homomorphism. Any monomial $\mathfrak{m}$ satisfies

$$\mathfrak{B}_k(\mathfrak{m}) = \begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq k; \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) > k \end{cases} \tag{9}$$

[19] *Proof of (8):* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. The definition of M_α yields $M_\alpha =$

[20]. Any $p \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ satisfies

$$p \Phi a = a \cdot \mathfrak{B}_k(p) \tag{10}$$

$\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$. Combined with $a = \mathfrak{n}$, this yields

$$\begin{aligned} a \prec M_\alpha &= \mathfrak{n} \prec \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\mathfrak{n} \prec \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right)}_{\substack{= \begin{cases} \mathfrak{n} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \min(\operatorname{Supp} \mathfrak{n}) < \min\{i_1, i_2, \ldots, i_\ell\}; \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{n}) \geq \min\{i_1, i_2, \ldots, i_\ell\} \end{cases} \\ \text{(by the definition of } \prec \text{ on monomials)}}} \\ &\qquad \text{(since } \prec \text{ is } \mathbf{k}\text{-bilinear and continuous)} \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \begin{cases} \mathfrak{n} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \min(\operatorname{Supp} \mathfrak{n}) < \min\{i_1, i_2, \ldots, i_\ell\}; \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{n}) \geq \min\{i_1, i_2, \ldots, i_\ell\} \end{cases} \\ &= \underbrace{\sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_\ell; \\ \min(\operatorname{Supp} \mathfrak{n}) < \min\{i_1, i_2, \ldots, i_\ell\}}}}_{\substack{= \sum\limits_{\min(\operatorname{Supp} \mathfrak{n}) < i_1 < i_2 < \cdots < i_\ell} \\ = \sum\limits_{k < i_1 < i_2 < \cdots < i_\ell} \\ \text{(since } \min(\operatorname{Supp} \mathfrak{n}) = k)}} \underbrace{\mathfrak{n}}_{=a} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{k < i_1 < i_2 < \cdots < i_\ell} a \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \\ &= \left(\sum_{k < i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \cdot a. \end{aligned}$$

This proves (8).

[20]*Proof.* Let $\mathfrak{m}$ be a monomial. Then,

$$\begin{aligned} \mathfrak{B}_k(\mathfrak{m}) &= \mathfrak{m}(x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots) \qquad \text{(by the definition of } \mathfrak{B}_k) \\ &= (\text{the result of replacing the indeterminates } x_{k+1}, x_{k+2}, x_{k+3}, \ldots \text{ by } 0 \text{ in } \mathfrak{m}) \\ &= \begin{cases} \mathfrak{m}, & \text{if none of the indeterminates } x_{k+1}, x_{k+2}, x_{k+3}, \ldots \text{ appears in } \mathfrak{m}; \\ 0, & \text{if some of the indeterminates } x_{k+1}, x_{k+2}, x_{k+3}, \ldots \text{ appear in } \mathfrak{m} \end{cases} \\ &= \begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \leq k; \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) > k \end{cases} \end{aligned}$$

(because none of the indeterminates $x_{k+1}, x_{k+2}, x_{k+3}, \ldots$ appears in $\mathfrak{m}$ if and only if $\max(\operatorname{Supp} \mathfrak{m}) \leq k$). This proves (9).

[21]. Also, every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$\mathfrak{B}_k \left(M_\alpha\right) = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \tag{12}$$

[22].

We shall use one further obvious observation: If $i_1, i_2, \ldots, i_\ell$ are some positive integers satisfying $i_1 < i_2 < \cdots < i_\ell$, then

$$\text{there exists exactly one } j \in \{0, 1, \ldots, \ell\} \text{ satisfying } i_j \leq k < i_{j+1} \tag{13}$$

(where i_0 is to be understood as 1, and $i_{\ell+1}$ as $\infty + 1$).

Let us now notice that every $f \in \operatorname{QSym}$ satisfies

$$af = \sum_{(f)} \mathfrak{B}_k \left(f_{(1)}\right) \left(a \prec f_{(2)}\right). \tag{14}$$

Proof of (14): Both sides of the equality (14) are $\mathbf{k}$-linear in f. Hence, it is enough to check (14) on the basis $\left(M_\gamma\right)_{\gamma \in \operatorname{Comp}}$ of QSym, that is, to prove that (14) holds whenever $f = M_\gamma$ for some $\gamma \in \operatorname{Comp}$. In other words, it is enough to show that

$$aM_\gamma = \sum_{(M_\gamma)} \mathfrak{B}_k \left(\left(M_\gamma\right)_{(1)}\right) \cdot \left(a \prec \left(M_\gamma\right)_{(2)}\right) \qquad \text{for every } \gamma \in \operatorname{Comp}.$$

[21]*Proof of (10):* Fix $p \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$. Since the equality (10) is $\mathbf{k}$-linear and continuous in p, we can WLOG assume that p is a monomial. Assume this. Hence, $p = \mathfrak{m}$ for some monomial $\mathfrak{m}$. Consider this $\mathfrak{m}$. We have

$$\mathfrak{B}_k \left(\underbrace{p}_{=\mathfrak{m}}\right) = \mathfrak{B}_k \left(\mathfrak{m}\right) = \begin{cases} \mathfrak{m}, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) \leq k; \\ 0, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) > k \end{cases} \tag{11}$$

(by (9)). Now,

$$\begin{aligned} \underbrace{p}_{=\mathfrak{m}} \Phi \underbrace{a}_{=\mathfrak{n}} = \mathfrak{m} \Phi \mathfrak{n} &= \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) \leq \min\left(\operatorname{Supp} \mathfrak{n}\right); \\ 0, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) > \min\left(\operatorname{Supp} \mathfrak{n}\right) \end{cases} \\ &\qquad \left(\text{by the definition of } \Phi\right) \\ &= \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) \leq k; \\ 0, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) > k \end{cases} \qquad \left(\text{since } \min\left(\operatorname{Supp} \mathfrak{n}\right) = k\right) \\ &= \underbrace{\mathfrak{n}}_{=a} \cdot \underbrace{\begin{cases} \mathfrak{m}, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) \leq k; \\ 0, & \text{if } \max\left(\operatorname{Supp} \mathfrak{m}\right) > k \end{cases}}_{\substack{=\mathfrak{B}_k(p) \\ \text{(by (11))}}} \\ &= a \cdot \mathfrak{B}_k\left(p\right). \end{aligned}$$

This proves (10).

[22]*Proof of (12):* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. The definition of M_α yields $M_\alpha =$

But this is easily done: Let $\gamma \in \operatorname{Comp}$. Write γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$.

$\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$. Applying the map $\mathfrak{B}_k$ to both sides of this equality, we obtain

$$\begin{aligned}
\mathfrak{B}_k\left(M_\alpha\right) &= \mathfrak{B}_k\left(\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right) \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\mathfrak{B}_k\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)}_{\substack{= \begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) \leq k; \\ 0, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) > k \end{cases} \\ \text{(by (9), applied to } \mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\text{)}}} \\
&\qquad \left(\text{since } \mathfrak{B}_k \text{ is } \mathbf{k}\text{-linear and continuous}\right) \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) \leq k; \\ 0, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) > k \end{cases}}_{\substack{= \begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} \leq k; \\ 0, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} > k \end{cases} \\ \left(\text{since } \operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right) = \{i_1, i_2, \ldots, i_\ell\}\right)}} \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} \leq k; \\ 0, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} > k \end{cases} \\
&= \underbrace{\sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_\ell; \\ \max\{i_1, i_2, \ldots, i_\ell\} \leq k}}}_{= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}.
\end{aligned}$$

This proves (12).

Then,

$$\sum_{(M_\gamma)} \mathfrak{B}_k\left(\left(M_\gamma\right)_{(1)}\right) \cdot \left(a \prec \left(M_\gamma\right)_{(2)}\right)$$

$$= \sum_{j=0}^{\ell} \underbrace{\mathfrak{B}_k\left(M_{(\gamma_1,\gamma_2,\ldots,\gamma_j)}\right)}_{\substack{=\sum\limits_{1\leq i_1<i_2<\cdots<i_j\leq k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \\ \text{(by (12))}}} \cdot \underbrace{\left(a \prec M_{(\gamma_{j+1},\gamma_{j+2},\ldots,\gamma_\ell)}\right)}_{\substack{=\left(\sum\limits_{k<i_1<i_2<\cdots<i_{\ell-j}} x_{i_1}^{\gamma_{j+1}} x_{i_2}^{\gamma_{j+2}} \cdots x_{i_{\ell-j}}^{\gamma_\ell}\right) \cdot a \\ \text{(by (8))}}}$$

$$\left(\text{since } \sum_{(M_\gamma)} \left(M_\gamma\right)_{(1)} \otimes \left(M_\gamma\right)_{(2)} = \Delta\left(M_\gamma\right) = \sum_{j=0}^{\ell} M_{(\gamma_1,\gamma_2,\ldots,\gamma_j)} \otimes M_{(\gamma_{j+1},\gamma_{j+2},\ldots,\gamma_\ell)}\right)$$

$$= \sum_{j=0}^{\ell} \left(\sum_{1\leq i_1<i_2<\cdots<i_j\leq k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j}\right) \underbrace{\left(\sum_{k<i_1<i_2<\cdots<i_{\ell-j}} x_{i_1}^{\gamma_{j+1}} x_{i_2}^{\gamma_{j+2}} \cdots x_{i_{\ell-j}}^{\gamma_\ell}\right)}_{\substack{=\sum\limits_{k<i_{j+1}<i_{j+2}<\cdots<i_\ell} x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell} \\ \text{(here, we have renamed the summation index} \\ \left(i_1,i_2,\ldots,i_{\ell-j}\right) \text{ as } \left(i_{j+1},i_{j+2},\ldots,i_\ell\right))}} \cdot a$$

$$= \sum_{j=0}^{\ell} \left(\sum_{1\leq i_1<i_2<\cdots<i_j\leq k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j}\right) \left(\sum_{k<i_{j+1}<i_{j+2}<\cdots<i_\ell} x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell}\right) \cdot a$$

$$= \underbrace{\sum_{j=0}^{\ell} \sum_{1\leq i_1<i_2<\cdots<i_j\leq k} \sum_{k<i_{j+1}<i_{j+2}<\cdots<i_\ell}}_{\substack{=\sum\limits_{1\leq i_1<i_2<\cdots<i_\ell} \sum\limits_{\substack{j\in\{0,1,\ldots,\ell\}; \\ i_j\leq k<i_{j+1}}} \\ \text{(where } i_0 \text{ is to be understood as 1, and } i_{\ell+1} \text{ as } \infty+1)}} \underbrace{\left(x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j}\right) \left(x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell}\right)}_{=x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}} \cdot a$$

$$= \sum_{1\leq i_1<i_2<\cdots<i_\ell} \underbrace{\sum_{\substack{j\in\{0,1,\ldots,\ell\}; \\ i_j\leq k<i_{j+1}}} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}_{\substack{\text{this sum has precisely one addend,} \\ \text{(because of (13)),} \\ \text{and thus equals } x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}} \cdot a = \underbrace{\sum_{1\leq i_1<i_2<\cdots<i_\ell} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}_{=M_\gamma} \cdot a$$

$$= M_\gamma \cdot a = aM_\gamma,$$

qed. Thus, (14) is proven.

Now, every $b \in \operatorname{QSym}$ satisfies

$$\begin{aligned}
&\sum_{(b)} \underbrace{\left(S\left(b_{(1)}\right) \Phi\, a\right)}_{\substack{=a\cdot\mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \\ \text{(by (10), applied to } p=S\left(b_{(1)}\right))}} b_{(2)} \\
&= \sum_{(b)} a\cdot\mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) b_{(2)} = \sum_{(b)} \mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \cdot \underbrace{ab_{(2)}}_{\substack{=\sum_{\left(b_{(2)}\right)} \mathfrak{B}_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a\prec\left(b_{(2)}\right)_{(2)}\right) \\ \text{(by (14), applied to } f=b_{(2)})}} \\
&= \sum_{(b)} \mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \left(\sum_{\left(b_{(2)}\right)} \mathfrak{B}_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a\prec\left(b_{(2)}\right)_{(2)}\right)\right) \\
&= \sum_{(b)} \sum_{\left(b_{(2)}\right)} \mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \mathfrak{B}_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a\prec\left(b_{(2)}\right)_{(2)}\right) \\
&= \sum_{(b)} \underbrace{\sum_{\left(b_{(1)}\right)} \mathfrak{B}_k\left(S\left(\left(b_{(1)}\right)_{(1)}\right)\right) \mathfrak{B}_k\left(\left(b_{(1)}\right)_{(2)}\right)}_{\substack{=\mathfrak{B}_k\left(\sum_{\left(b_{(1)}\right)} S\left(\left(b_{(1)}\right)_{(1)}\right)\cdot\left(b_{(1)}\right)_{(2)}\right) \\ \text{(since } \mathfrak{B}_k \text{ is a } \mathbf{k}\text{-algebra homomorphism)}}} \left(a\prec b_{(2)}\right) \\
&\qquad \left(\begin{array}{c}\text{since the coassociativity of } \Delta \text{ yields} \\ \sum_{(b)} \sum_{\left(b_{(2)}\right)} b_{(1)} \otimes \left(b_{(2)}\right)_{(1)} \otimes \left(b_{(2)}\right)_{(2)} = \sum_{(b)} \sum_{\left(b_{(1)}\right)} \left(b_{(1)}\right)_{(1)} \otimes \left(b_{(1)}\right)_{(2)} \otimes b_{(2)}\end{array}\right) \\
&= \sum_{(b)} \mathfrak{B}_k\left(\underbrace{\sum_{\left(b_{(1)}\right)} S\left(\left(b_{(1)}\right)_{(1)}\right)\left(b_{(1)}\right)_{(2)}}_{\substack{=\varepsilon\left(b_{(1)}\right) \\ \text{(by one of the defining equations of the antipode)}}}\right) \left(a\prec b_{(2)}\right) \\
&= \sum_{(b)} \underbrace{\mathfrak{B}_k\left(\varepsilon\left(b_{(1)}\right)\right)}_{\substack{=\varepsilon\left(b_{(1)}\right) \\ \text{(since } \mathfrak{B}_k \text{ is a } \mathbf{k}\text{-algebra} \\ \text{homomorphism, and} \\ \varepsilon\left(b_{(1)}\right)\in\mathbf{k} \text{ is a scalar)}}} \left(a\prec b_{(2)}\right) = \sum_{(b)} \varepsilon\left(b_{(1)}\right)\cdot\left(a\prec b_{(2)}\right) \\
&= \sum_{(b)} a\prec\left(\varepsilon\left(b_{(1)}\right) b_{(2)}\right) = a\prec\underbrace{\left(\sum_{(b)} \varepsilon\left(b_{(1)}\right) b_{(2)}\right)}_{=b} = a\prec b.
\end{aligned}$$

This proves Theorem 3.7. $\square$

Let us connect the Φ operation with the fundamental basis of QSym:

Proposition 3.8. For any two compositions α and β, define a composition $\alpha \odot \beta$ as follows:

– If α is empty, then set $\alpha \odot \beta = \beta$.

– Otherwise, if β is empty, then set $\alpha \odot \beta = \alpha$.

– Otherwise, define $\alpha \odot \beta$ as $(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m)$, where α is written as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and where β is written as $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$.

Then, any two compositions α and β satisfy

$$F_\alpha \Phi F_\beta = F_{\alpha \odot \beta}.$$

Our proof of this proposition will rely on the following lemma:

Lemma 3.9. If G is a set of integers and r is an integer, then we let $G + r$ denote the set $\{g + r \mid g \in G\}$ of integers.

Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Let α be a composition of p. Let β be a composition of q. Consider the composition $\alpha \odot \beta$ defined in Proposition 3.8.

(a) Then, $\alpha \odot \beta$ is a composition of $p + q$ satisfying $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$.

(b) Also, define a composition $[\alpha, \beta]$ by $[\alpha, \beta] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$, where α and β are written in the forms $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Assume that $p > 0$ and $q > 0$. Then, $[\alpha, \beta]$ is a composition of $p + q$ satisfying $D([\alpha, \beta]) = D(\alpha) \cup \{p\} \cup (D(\beta) + p)$.

(Actually, part **(b)** of this lemma will not be used until much later, but part **(a)** will be used soon.)

Proof of Lemma 3.9. Write α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Thus, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$, so that $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = |\alpha| = p$ (since α is a composition of p).

Write β in the form $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Thus, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_m$, so that $\beta_1 + \beta_2 + \cdots + \beta_m = |\beta| = q$ (since β is a composition of q).

We have $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$, and thus

$$\begin{aligned} D(\beta) &= \{\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{m-1}\} \\ &\qquad \text{(by the definition of } D(\beta)) \\ &= \left\{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m-1\}\right\}. \end{aligned}$$

Also, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, and thus

$$\begin{aligned} D(\alpha) &= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\} \\ &\qquad \text{(by the definition of } D(\alpha)) \\ &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in \{1, 2, \ldots, \ell-1\}\}. \end{aligned}$$

(a) If α or β is empty, then Lemma 3.9 **(a)** holds for obvious reasons (because of the definition of $\alpha \odot \beta$ in this case). Thus, we WLOG assume that neither α nor β is empty.

We have $\alpha \odot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m)$ (by the definition of $\alpha \odot \beta$) and thus

$$\begin{aligned} |\alpha \odot \beta| &= \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3 + \cdots + \beta_m \\ &= \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{=p} + \underbrace{(\beta_1 + \beta_2 + \cdots + \beta_m)}_{=q} = p + q. \end{aligned}$$

Thus, $\alpha \odot \beta$ is a composition of $p+q$. Hence, it remains to show that $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$.

Now, $\alpha \odot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m)$, so that

$$\begin{aligned} & D(\alpha \odot \beta) \\ &= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}, \\ &\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1), \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2, \\ &\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3, \ldots, \\ &\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3 + \cdots + \beta_{m-1}\} \\ &\qquad \text{(by the definition of } D(\alpha \odot \beta)) \\ &= \underbrace{\{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in \{1, 2, \ldots, \ell - 1\}\}}_{=D(\alpha)} \\ &\qquad \cup \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3 + \cdots + \beta_j}_{\substack{=(\alpha_1+\alpha_2+\cdots+\alpha_\ell)+(\beta_1+\beta_2+\cdots+\beta_j) \\ =(\beta_1+\beta_2+\cdots+\beta_j)+(\alpha_1+\alpha_2+\cdots+\alpha_\ell)}} \mid j \in \{1, 2, \ldots, m-1\} \right\} \\ &= D(\alpha) \cup \left\{ (\beta_1 + \beta_2 + \cdots + \beta_j) + \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{=p} \mid j \in \{1, 2, \ldots, m-1\} \right\} \\ &= D(\alpha) \cup \underbrace{\{(\beta_1 + \beta_2 + \cdots + \beta_j) + p \mid j \in \{1, 2, \ldots, m-1\}\}}_{=\{\beta_1+\beta_2+\cdots+\beta_j \mid j \in \{1,2,\ldots,m-1\}\}+p} \\ &= D(\alpha) \cup \left(\underbrace{\{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m-1\}\}}_{=D(\beta)} + p \right) \\ &= D(\alpha) \cup (D(\beta) + p). \end{aligned}$$

This completes the proof of Lemma 3.9 **(a)**.

(b) We have $p > 0$. Thus, the composition α is nonempty (since α is a composition of p). In other words, the composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is nonempty (since $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$). Hence, $\ell > 0$.

We have $q > 0$. Thus, the composition β is nonempty (since β is a composition of q). In other words, the composition $(\beta_1, \beta_2, \ldots, \beta_m)$ is nonempty (since $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$). Hence, $m > 0$.

We have $[\alpha, \beta] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$ (by the definition of $[\alpha, \beta]$) and thus

$$\begin{aligned} |\alpha \odot \beta| &= \alpha_1 + \alpha_2 + \cdots + \alpha_\ell + \beta_1 + \beta_2 + \cdots + \beta_m \\ &= \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{=p} + \underbrace{(\beta_1 + \beta_2 + \cdots + \beta_m)}_{=q} = p + q. \end{aligned}$$

Thus, $[\alpha, \beta]$ is a composition of $p + q$. Hence, it remains to show that $D([\alpha, \beta]) = D(\alpha) \cup (D(\beta) + p)$.

Now, $[\alpha, \beta] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$, so that

$$\begin{aligned}
& D([\alpha, \beta]) \\
&= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}, \\
&\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1, \\
&\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1 + \beta_2, \ldots, \\
&\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1 + \beta_2 + \cdots + \beta_{m-1}\} \\
&\qquad \text{(by the definition of } D([\alpha, \beta])) \\
&= \underbrace{\{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in \{1, 2, \ldots, \ell - 1\}\}}_{= D(\alpha)} \\
&\qquad \cup \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_\ell}_{= p} \right\} \\
&\qquad \cup \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1 + \beta_2 + \cdots + \beta_j}_{\substack{= (\alpha_1 + \alpha_2 + \cdots + \alpha_\ell) + (\beta_1 + \beta_2 + \cdots + \beta_j) \\ = (\beta_1 + \beta_2 + \cdots + \beta_j) + (\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}} \mid j \in \{1, 2, \ldots, m - 1\} \right\} \\
&= D(\alpha) \cup \{p\} \cup \left\{ (\beta_1 + \beta_2 + \cdots + \beta_j) + \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{= p} \mid j \in \{1, 2, \ldots, m - 1\} \right\} \\
&= D(\alpha) \cup \{p\} \cup \underbrace{\{(\beta_1 + \beta_2 + \cdots + \beta_j) + p \mid j \in \{1, 2, \ldots, m - 1\}\}}_{= \{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m-1\}\} + p} \\
&= D(\alpha) \cup \{p\} \cup \left(\underbrace{\{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m - 1\}\}}_{= D(\beta)} + p \right) \\
&= D(\alpha) \cup \{p\} \cup (D(\beta) + p).
\end{aligned}$$

This completes the proof of Lemma 3.9 **(b)**. □

Proof of Proposition 3.8. If either α or β is empty, then this is obvious (since Φ is unital with 1 as its unity, and since $F_\varnothing = 1$). So let us WLOG assume that neither is. Write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, and write β as $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Thus, ℓ and m are positive (since α and β are nonempty).

Let $p = |\alpha|$ and $q = |\beta|$. Thus, p and q are positive (since α and β are nonempty). Recall that we use the notation $D(\alpha)$ for the set of partial sums of a composition α. If G is a set of integers and r is an integer, then we let $G + r$ denote the set $\{g + r \mid g \in G\}$ of integers.

Lemma 3.9 **(a)** shows that $\alpha \odot \beta$ is a composition of $p+q$ satisfying $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$.

Applying (1) to p instead of n, we obtain

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_p}. \tag{15}$$

Applying (1) to q and β instead of n and α, we obtain

$$F_\beta = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_q; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)}} x_{i_1} x_{i_2} \cdots x_{i_q} = \sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}$$

(here, we renamed the summation index $(i_1, i_2, \ldots, i_q)$ as $(i_{p+1}, i_{p+2}, \ldots, i_{p+q})$). This, together with (15), yields

$$\begin{aligned}
& F_\alpha \Phi F_\beta \\
&= \left(\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_p} \right) \Phi \left(\sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}} \right) \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} \sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} \underbrace{\left(x_{i_1} x_{i_2} \cdots x_{i_p} \right) \Phi \left(x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}} \right)}_{\substack{= \begin{cases} x_{i_1} x_{i_2} \cdots x_{i_p} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}, & \text{if } i_p \leq i_{p+1}; \\ 0, & \text{if } i_p > i_{p+1} \end{cases} \\ \text{(by the definition of } \Phi \text{ on monomials)}}} \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} \sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} \begin{cases} x_{i_1} x_{i_2} \cdots x_{i_p} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}, & \text{if } i_p \leq i_{p+1}; \\ 0, & \text{if } i_p > i_{p+1} \end{cases} \\
&= \underbrace{\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha); \\ i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p; \\ i_p \leq i_{p+1}}}}_{= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha) \cup (D(\beta)+p)}}} \underbrace{x_{i_1} x_{i_2} \cdots x_{i_p} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}}_{= x_{i_1} x_{i_2} \cdots x_{i_{p+q}}} \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha) \cup (D(\beta)+p)}} x_{i_1} x_{i_2} \cdots x_{i_{p+q}}.
\end{aligned} \tag{16}$$

On the other hand, $\alpha \odot \beta$ is a composition of $p+q$ satisfying $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$. Thus, (1) (applied to $\alpha \odot \beta$ and $p+q$ instead of α and n)

yields

$$F_{\alpha\odot\beta} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha\odot\beta)}} x_{i_1} x_{i_2} \cdots x_{i_{p+q}} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)\cup(D(\beta)+p)}} x_{i_1} x_{i_2} \cdots x_{i_{p+q}}$$

(since $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$). Compared with (16), this yields $F_\alpha \Phi F_\beta = F_{\alpha\odot\beta}$. This proves Proposition 3.8. □

For our goals, we need a certain particular case of Proposition 3.8. Namely, let us recall that for every $m \in \mathbb{N}$, the *m-th complete homogeneous symmetric function* h_m is defined as the element $\sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$ of Sym. It is easy to see that $h_m = F_{(m)}$ for every positive integer m. From this, we obtain:

Corollary 3.10. For any two compositions α and β, define a composition $\alpha \odot \beta$ as in Proposition 3.8. Then, every composition α and every positive integer m satisfy

$$F_{\alpha\odot(m)} = F_\alpha \Phi h_m. \tag{17}$$

Proof of Corollary 3.10. Let α be a composition. Let m be a positive integer. Recall that $h_m = F_{(m)}$. Proposition 3.8 yields $F_\alpha \Phi F_{(m)} = F_{\alpha\odot(m)}$. Hence, $F_{\alpha\odot(m)} = F_\alpha \Phi \underbrace{F_{(m)}}_{=h_m} = F_\alpha \Phi h_m$. This proves Corollary 3.10. □

For the sake of completeness (or, rather, in order not to lose old writing), let me write down the definitions of some more operations on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

Definition 3.11. We define a binary operation $\preceq : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \times \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and that it satisfy

$$\mathfrak{m} \preceq \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) \leq \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) > \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$

for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.

Here, all the remarks we made after Definition 3.1 apply. In particular, $\min \varnothing = \infty$, and we are using $\preceq$ as an operation symbol.

We have $\mathfrak{m} \preceq 1 = \mathfrak{m}$ for every monomial $\mathfrak{m}$, and $1 \preceq \mathfrak{m} = 0$ for every nonconstant monomial $\mathfrak{m}$.

Remark 3.12. The operation $\preceq$ is part of a dendriform algebra structure on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (and on QSym). More precisely, if we define another binary operation $\succ$ on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ similarly to $\preceq$ except that we set

$$\mathfrak{m} \succ \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) > \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) \leq \min(\operatorname{Supp} \mathfrak{n}) \end{cases},$$

then the structure $(\mathbf{k}[[x_1, x_2, x_3, \ldots]], \preceq, \succ)$ is a dendriform algebra augmented to satisfy [EbrFar08, (15)]. In particular, any three elements a, b and c of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ satisfy

$$\begin{aligned} a \preceq b + a \succ b &= ab; \\ (a \preceq b) \preceq c &= a \preceq (bc); \\ (a \succ b) \preceq c &= a \succ (b \preceq c); \\ a \succ (b \succ c) &= (ab) \succ c. \end{aligned}$$

And here is an analogue of Φ which has *mostly* similar properties:

Definition 3.13. We define a binary operation $\between$: $\mathbf{k}[[x_1, x_2, x_3, \ldots]] \times \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and that it satisfy

$$\mathfrak{m} \between \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$

for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.

Here, again, $\max \varnothing$ is understood as 0. The binary operation $\between$ is associative. It is also unital (with 1 serving as the unity).

Proposition 3.14. Every $a \in \operatorname{QSym}$ and $b \in \operatorname{QSym}$ satisfy $a \preceq b \in \operatorname{QSym}$ and $a \between b \in \operatorname{QSym}$.

For example, any two compositions α and β satisfy $M_\alpha \between M_\beta = M_{[\alpha,\beta]}$, where $[\alpha, \beta]$ denotes the concatenation of α and β (defined by $[(\alpha_1, \alpha_2, \ldots, \alpha_\ell), (\beta_1, \beta_2, \ldots, \beta_m)] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$). (Recall that $(M_\gamma)_{\gamma \in \operatorname{Comp}}$ is the monomial basis of QSym.)

Theorem 3.15. Let S denote the antipode of the Hopf algebra QSym. Let us use Sweedler's notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$, where b is any element of QSym. Then,

$$\sum_{(b)} \left(S\left(b_{(1)}\right) \between a \right) b_{(2)} = a \preceq b$$

▌ for any $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b \in \mathrm{QSym}$.

Proof of Theorem 3.15. The following proof is mostly analogous to the proof of Theorem 3.7.

Let $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. We can WLOG assume that a is a monomial (because all operations in sight are $\mathbf{k}$-linear and continuous). So assume this. That is, $a = \mathfrak{n}$ for some monomial $\mathfrak{n}$. Consider this $\mathfrak{n}$. Let $k = \min(\operatorname{Supp}\mathfrak{n})$. Notice that $k \in \{1, 2, 3, \ldots\} \cup \{\infty\}$.

(Some remarks about ∞ are in order. We use ∞ as an object which is greater than every integer. We will use summation signs like $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell < k}$ and $\sum\limits_{k \leq i_1 < i_2 < \cdots < i_\ell}$ in the following. Both of these summation signs range over $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell$ satisfying certain conditions ($1 \leq i_1 < i_2 < \cdots < i_\ell < k$ in the first case, and $k \leq i_1 < i_2 < \cdots < i_\ell$ in the second case). In particular, none of the $i_1, i_2, \ldots, i_\ell$ is allowed to be ∞ (unlike k). So the summation $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell < k}$ is identical to $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell}$ when $k = \infty$, whereas the summation $\sum\limits_{k \leq i_1 < i_2 < \cdots < i_\ell}$ is empty when $k = \infty$ unless $\ell = 0$. (If $\ell = 0$, then the summation $\sum\limits_{k \leq i_1 < i_2 < \cdots < i_\ell}$ ranges over the empty 0-tuple, no matter what k is.))

Every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$a \preceq M_\alpha = \left(\sum_{k \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \cdot a \tag{18}$$

[23].

Let us define a map $\mathfrak{B}'_k : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$\mathfrak{B}'_k(p) = p(x_1, x_2, \ldots, x_{k-1}, 0, 0, 0, \ldots) \qquad \text{for every } p \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$$

(where $p(x_1, x_2, \ldots, x_{k-1}, 0, 0, 0, \ldots)$ has to be understood as $p(x_1, x_2, x_3, \ldots) = p$ when $k = \infty$). Then, $\mathfrak{B}'_k$ is an evaluation map (in an appropriate sense) and thus a continuous $\mathbf{k}$-algebra homomorphism. Any monomial $\mathfrak{m}$ satisfies

$$\mathfrak{B}'_k(\mathfrak{m}) = \begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) < k; \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \geq k \end{cases} \tag{19}$$

[23] *Proof of (18):* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. The definition of M_α yields $M_\alpha =$

[24]. Any $p \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ satisfies

$$p \divideontimes a = a \cdot \mathfrak{B}'_k\left(p\right) \tag{20}$$

$\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$. Combined with $a = \mathfrak{n}$, this yields

$$\begin{aligned}
a \preceq M_\alpha &= \mathfrak{n} \preceq \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\mathfrak{n} \preceq \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right)}_{\substack{= \begin{cases} \mathfrak{n} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \min\left(\operatorname{Supp}\mathfrak{n}\right) \leq \min\left\{i_1, i_2, \ldots, i_\ell\right\}; \\ 0, & \text{if } \min\left(\operatorname{Supp}\mathfrak{n}\right) > \min\left\{i_1, i_2, \ldots, i_\ell\right\} \end{cases} \\ \text{(by the definition of } \preceq \text{ on monomials)}}} \\
&\qquad \left(\text{since } \preceq \text{ is } \mathbf{k}\text{-bilinear and continuous}\right) \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \begin{cases} \mathfrak{n} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \min\left(\operatorname{Supp}\mathfrak{n}\right) \leq \min\left\{i_1, i_2, \ldots, i_\ell\right\}; \\ 0, & \text{if } \min\left(\operatorname{Supp}\mathfrak{n}\right) > \min\left\{i_1, i_2, \ldots, i_\ell\right\} \end{cases} \\
&= \underbrace{\sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_\ell; \\ \min\left(\operatorname{Supp}\mathfrak{n}\right) \leq \min\left\{i_1, i_2, \ldots, i_\ell\right\}}}}_{\substack{= \sum_{\min\left(\operatorname{Supp}\mathfrak{n}\right) \leq i_1 < i_2 < \cdots < i_\ell} \\ = \sum_{k \leq i_1 < i_2 < \cdots < i_\ell} \\ \text{(since } \min\left(\operatorname{Supp}\mathfrak{n}\right) = k\text{)}}} \underbrace{\mathfrak{n}}_{=a} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{k \leq i_1 < i_2 < \cdots < i_\ell} a \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \\
&= \left(\sum_{k \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \cdot a.
\end{aligned}$$

This proves (18).

[24]*Proof.* Let $\mathfrak{m}$ be a monomial. Then,

$$\begin{aligned}
\mathfrak{B}'_k\left(\mathfrak{m}\right) &= \mathfrak{m}\left(x_1, x_2, \ldots, x_{k-1}, 0, 0, 0, \ldots\right) \qquad \left(\text{by the definition of } \mathfrak{B}'_k\right) \\
&= \left(\text{the result of replacing the indeterminates } x_k, x_{k+1}, x_{k+2}, \ldots \text{ by } 0 \text{ in } \mathfrak{m}\right) \\
&= \begin{cases} \mathfrak{m}, & \text{if none of the indeterminates } x_k, x_{k+1}, x_{k+2}, \ldots \text{ appears in } \mathfrak{m}; \\ 0, & \text{if some of the indeterminates } x_k, x_{k+1}, x_{k+2}, \ldots \text{ appear in } \mathfrak{m} \end{cases} \\
&= \begin{cases} \mathfrak{m}, & \text{if } \max\left(\operatorname{Supp}\mathfrak{m}\right) < k; \\ 0, & \text{if } \max\left(\operatorname{Supp}\mathfrak{m}\right) \geq k \end{cases}
\end{aligned}$$

(because none of the indeterminates $x_k, x_{k+1}, x_{k+2}, \ldots$ appears in $\mathfrak{m}$ if and only if $\max\left(\operatorname{Supp}\mathfrak{m}\right) < k$). This proves (19).

[25]. Also, every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$\mathfrak{B}'_k (M_\alpha) = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell < k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \tag{22}$$

(where the sum $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_\ell < k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ has to be interpreted as being equal to 1, rather than being empty, when $\ell = 0$) [26].

We shall use one further obvious observation: If $i_1, i_2, \ldots, i_\ell$ are some positive integers satisfying $i_1 < i_2 < \cdots < i_\ell$, then

$$\text{there exists exactly one } j \in \{0, 1, \ldots, \ell\} \text{ satisfying } i_j < k \leq i_{j+1} \tag{23}$$

(where i_0 is to be understood as 0, and $i_{\ell+1}$ as ∞).

Let us now notice that every $f \in \operatorname{QSym}$ satisfies

$$af = \sum_{(f)} \mathfrak{B}'_k \left(f_{(1)} \right) \left(a \preceq f_{(2)} \right). \tag{24}$$

Proof of (24): Both sides of the equality (24) are $\mathbf{k}$-linear in f. Hence, it is enough to check (24) on the basis $(M_\gamma)_{\gamma \in \operatorname{Comp}}$ of QSym, that is, to prove that (24) holds whenever $f = M_\gamma$ for some $\gamma \in \operatorname{Comp}$. In other words, it is enough to show that

$$aM_\gamma = \sum_{(M_\gamma)} \mathfrak{B}'_k \left((M_\gamma)_{(1)} \right) \cdot \left(a \preceq (M_\gamma)_{(2)} \right) \qquad \text{for every } \gamma \in \operatorname{Comp}.$$

[25]*Proof of (20):* Fix $p \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. Since the equality (20) is $\mathbf{k}$-linear and continuous in p, we can WLOG assume that p is a monomial. Assume this. Hence, $p = \mathfrak{m}$ for some monomial $\mathfrak{m}$. Consider this $\mathfrak{m}$. We have

$$\mathfrak{B}'_k \left(\underbrace{p}_{=\mathfrak{m}} \right) = \mathfrak{B}'_k (\mathfrak{m}) = \begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < k; \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq k \end{cases} \tag{21}$$

(by (19)). Now,

$$\begin{aligned}
\underbrace{p}_{=\mathfrak{m}} \mathbin{\divideontimes} \underbrace{a}_{=\mathfrak{n}} = \mathfrak{m} \mathbin{\divideontimes} \mathfrak{n} &= \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases} \\
&\qquad (\text{by the definition of } \divideontimes) \\
&= \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < k; \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq k \end{cases} \qquad (\text{since } \min(\operatorname{Supp} \mathfrak{n}) = k) \\
&= \underbrace{\mathfrak{n}}_{=a} \cdot \underbrace{\begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < k; \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq k \end{cases}}_{\substack{=\mathfrak{B}'_k(p) \\ (\text{by } (21))}} \\
&= a \cdot \mathfrak{B}'_k (p).
\end{aligned}$$

This proves (20).

[26]*Proof of (22):* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. The definition of M_α yields $M_\alpha =$

But this is easily done: Let $\gamma \in \operatorname{Comp}$. Write γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$.

$\sum\limits_{1\leq i_1<i_2<\cdots<i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$. Applying the map $\mathfrak{B}'_k$ to both sides of this equality, we obtain

$$\begin{aligned}
\mathfrak{B}'_k(M_\alpha) &= \mathfrak{B}'_k\left(\sum_{1\leq i_1<i_2<\cdots<i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right) \\
&= \sum_{1\leq i_1<i_2<\cdots<i_\ell} \underbrace{\mathfrak{B}'_k\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)}_{\substack{=\begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) < k; \\ 0, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) \geq k \end{cases} \\ \text{(by (19), applied to } \mathfrak{m}=x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\text{)}}} \\
&\qquad \left(\text{since } \mathfrak{B}'_k \text{ is } \mathbf{k}\text{-linear and continuous}\right) \\
&= \sum_{1\leq i_1<i_2<\cdots<i_\ell} \underbrace{\begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) < k; \\ 0, & \text{if } \max\left(\operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)\right) \geq k \end{cases}}_{\substack{=\begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} < k; \\ 0, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} \geq k \end{cases} \\ \text{(since } \operatorname{Supp}\left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}\right)=\{i_1,i_2,\ldots,i_\ell\}\text{)}}} \\
&= \sum_{1\leq i_1<i_2<\cdots<i_\ell} \begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} < k; \\ 0, & \text{if } \max\{i_1, i_2, \ldots, i_\ell\} \geq k \end{cases} \\
&= \underbrace{\sum_{\substack{1\leq i_1<i_2<\cdots<i_\ell; \\ \max\{i_1,i_2,\ldots,i_\ell\}<k}}}_{=\sum\limits_{1\leq i_1<i_2<\cdots<i_\ell<k}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{1\leq i_1<i_2<\cdots<i_\ell<k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}.
\end{aligned}$$

This proves (22).

Then,

$$\begin{aligned}
&\sum_{(M_\gamma)} \mathfrak{B}'_k \left((M_\gamma)_{(1)} \right) \cdot \left(a \preceq (M_\gamma)_{(2)} \right) \\
&= \sum_{j=0}^{\ell} \underbrace{\mathfrak{B}'_k \left(M_{(\gamma_1,\gamma_2,\ldots,\gamma_j)} \right)}_{\substack{= \sum_{1 \le i_1 < i_2 < \cdots < i_j < k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \\ \text{(by (22))}}} \cdot \underbrace{\left(a \preceq M_{(\gamma_{j+1},\gamma_{j+2},\ldots,\gamma_\ell)} \right)}_{\substack{= \left(\sum_{k \le i_1 < i_2 < \cdots < i_{\ell-j}} x_{i_1}^{\gamma_{j+1}} x_{i_2}^{\gamma_{j+2}} \cdots x_{i_{\ell-j}}^{\gamma_\ell} \right) \cdot a \\ \text{(by (18))}}} \\
&\qquad \left(\text{since } \sum_{(M_\gamma)} (M_\gamma)_{(1)} \otimes (M_\gamma)_{(2)} = \Delta(M_\gamma) = \sum_{j=0}^{\ell} M_{(\gamma_1,\gamma_2,\ldots,\gamma_j)} \otimes M_{(\gamma_{j+1},\gamma_{j+2},\ldots,\gamma_\ell)} \right) \\
&= \sum_{j=0}^{\ell} \left(\sum_{1 \le i_1 < i_2 < \cdots < i_j < k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \right) \underbrace{\left(\sum_{k \le i_1 < i_2 < \cdots < i_{\ell-j}} x_{i_1}^{\gamma_{j+1}} x_{i_2}^{\gamma_{j+2}} \cdots x_{i_{\ell-j}}^{\gamma_\ell} \right)}_{\substack{= \sum_{k \le i_{j+1} < i_{j+2} < \cdots < i_\ell} x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell} \\ \text{(here, we have renamed the summation index} \\ (i_1, i_2, \ldots, i_{\ell-j}) \text{ as } (i_{j+1}, i_{j+2}, \ldots, i_\ell))}} \cdot a \\
&= \sum_{j=0}^{\ell} \left(\sum_{1 \le i_1 < i_2 < \cdots < i_j < k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \right) \left(\sum_{k \le i_{j+1} < i_{j+2} < \cdots < i_\ell} x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell} \right) \cdot a \\
&= \underbrace{\sum_{j=0}^{\ell} \sum_{1 \le i_1 < i_2 < \cdots < i_j < k} \sum_{k \le i_{j+1} < i_{j+2} < \cdots < i_\ell}}_{\substack{= \sum_{1 \le i_1 < i_2 < \cdots < i_\ell} \sum_{j \in \{0,1,\ldots,\ell\};\ i_j < k \le i_{j+1}} \\ \text{(where } i_0 \text{ is to be understood as } 0 \text{, and } i_{\ell+1} \text{ as } \infty)}} \underbrace{\left(x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \right) \left(x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell} \right)}_{= x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}} \cdot a \\
&= \sum_{1 \le i_1 < i_2 < \cdots < i_\ell} \underbrace{\sum_{\substack{j \in \{0,1,\ldots,\ell\}; \\ i_j < k \le i_{j+1}}} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}_{\substack{\text{this sum has precisely one addend} \\ \text{(because of (23)),} \\ \text{and thus equals } x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}} \cdot a = \underbrace{\sum_{1 \le i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}_{= M_\gamma} \cdot a \\
&= M_\gamma \cdot a = a M_\gamma,
\end{aligned}$$

qed. Thus, (24) is proven.

Now, every $b \in \operatorname{QSym}$ satisfies

$$\sum_{(b)} \underbrace{\left(S\left(b_{(1)}\right) \divideontimes a\right)}_{\substack{=a\cdot\mathfrak{B}'_k\left(S\left(b_{(1)}\right)\right) \\ \text{(by (20), applied to } p=S\left(b_{(1)}\right))}} b_{(2)}$$

$$= \sum_{(b)} a \cdot \mathfrak{B}'_k\left(S\left(b_{(1)}\right)\right) b_{(2)} = \sum_{(b)} \mathfrak{B}'_k\left(S\left(b_{(1)}\right)\right) \cdot \underbrace{ab_{(2)}}_{\substack{=\sum_{\left(b_{(2)}\right)} \mathfrak{B}'_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a\preceq\left(b_{(2)}\right)_{(2)}\right) \\ \text{(by (24), applied to } f=b_{(2)})}}$$

$$= \sum_{(b)} \mathfrak{B}'_k\left(S\left(b_{(1)}\right)\right)\left(\sum_{\left(b_{(2)}\right)} \mathfrak{B}'_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a \preceq \left(b_{(2)}\right)_{(2)}\right)\right)$$

$$= \sum_{(b)} \sum_{\left(b_{(2)}\right)} \mathfrak{B}'_k\left(S\left(b_{(1)}\right)\right) \mathfrak{B}'_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a \preceq \left(b_{(2)}\right)_{(2)}\right)$$

$$= \sum_{(b)} \underbrace{\sum_{\left(b_{(1)}\right)} \mathfrak{B}'_k\left(S\left(\left(b_{(1)}\right)_{(1)}\right)\right) \mathfrak{B}'_k\left(\left(b_{(1)}\right)_{(2)}\right)}_{\substack{=\mathfrak{B}'_k\left(\sum_{\left(b_{(1)}\right)} S\left(\left(b_{(1)}\right)_{(1)}\right)\cdot\left(b_{(1)}\right)_{(2)}\right) \\ \text{(since } \mathfrak{B}'_k \text{ is a } \mathbf{k}\text{-algebra homomorphism)}}} \left(a \preceq b_{(2)}\right)$$

$$\left(\begin{array}{c}\text{since the coassociativity of } \Delta \text{ yields} \\ \sum_{(b)} \sum_{\left(b_{(2)}\right)} b_{(1)} \otimes \left(b_{(2)}\right)_{(1)} \otimes \left(b_{(2)}\right)_{(2)} = \sum_{(b)} \sum_{\left(b_{(1)}\right)} \left(b_{(1)}\right)_{(1)} \otimes \left(b_{(1)}\right)_{(2)} \otimes b_{(2)}\end{array}\right)$$

$$= \sum_{(b)} \mathfrak{B}'_k\left(\underbrace{\sum_{\left(b_{(1)}\right)} S\left(\left(b_{(1)}\right)_{(1)}\right)\left(b_{(1)}\right)_{(2)}}_{\substack{=\varepsilon\left(b_{(1)}\right) \\ \text{(by one of the defining equations of the antipode)}}}\right)\left(a \preceq b_{(2)}\right)$$

$$= \sum_{(b)} \underbrace{\mathfrak{B}'_k\left(\varepsilon\left(b_{(1)}\right)\right)}_{\substack{=\varepsilon\left(b_{(1)}\right) \\ \text{(since } \mathfrak{B}'_k \text{ is a } \mathbf{k}\text{-algebra} \\ \text{homomorphism, and} \\ \varepsilon\left(b_{(1)}\right)\in\mathbf{k} \text{ is a scalar)}}} \left(a \preceq b_{(2)}\right) = \sum_{(b)} \varepsilon\left(b_{(1)}\right) \cdot \left(a \preceq b_{(2)}\right)$$

$$= \sum_{(b)} a \preceq \left(\varepsilon\left(b_{(1)}\right) b_{(2)}\right) = a \preceq \underbrace{\left(\sum_{(b)} \varepsilon\left(b_{(1)}\right) b_{(2)}\right)}_{=b} = a \preceq b.$$

This proves Theorem 3.15. $\square$

4. Dual immaculate functions and the operation $\prec$

We will now study the dual immaculate functions defined in [BBSSZ13a]. However, instead of defining them as was done in [BBSSZ13a, Section 3.7], we shall give a different (but equivalent) definition. First, we introduce immaculate tableaux (which we define as in [BBSSZ13a, Definition 3.9]), which are an analogue of the well-known semistandard Young tableaux (also known as "column-strict tableaux")[27]:

Definition 4.1. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition.

(a) The *Young diagram* of α will mean the subset $\left\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq \ell;\ 1 \leq j \leq \alpha_i\right\}$ of $\mathbb{Z}^2$. It is denoted by $Y(\alpha)$.

(b) An *immaculate tableau of shape* α will mean a map $T : Y(\alpha) \to \{1,2,3,\ldots\}$ which satisfies the following two axioms:

1. We have $T(i,1) < T(j,1)$ for any integers i and j satisfying $1 \leq i < j \leq \ell$.

2. We have $T(i,u) \leq T(i,v)$ for any integers i, u and v satisfying $1 \leq i \leq \ell$ and $1 \leq u < v \leq \alpha_i$.

The *entries* of an immaculate tableau T mean the images of elements of $Y(\alpha)$ under T.

We will use the same graphical representation of immaculate tableaux (analogous to the "English notation" for semistandard Young tableaux) that was used in [BBSSZ13a]: An immaculate tableau T of shape $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is represented as a table whose rows are left-aligned (but can have different lengths), and whose i-th row (counted from top) has α_i boxes, which are respectively filled with the entries $T(i,1)$, $T(i,2)$, ..., $T(i,\alpha_i)$ (from left to right). For example, an immaculate tableau T of shape $(3,1,2)$ is represented by the picture

$a_{1,1}$	$a_{1,2}$	$a_{1,3}$
$a_{2,1}$		
$a_{3,1}$	$a_{3,2}$	

,

where $a_{i,j} = T(i,j)$ for every $(i,j) \in Y((3,1,2))$. Thus, the first of the above two axioms for an immaculate tableau T says that the entries of T are strictly increasing down the first column of $Y(\alpha)$, whereas the second of the above

[27] See, e.g., [Stanle99, Chapter 7] for a study of semistandard Young tableaux. We will not use them in this note; however, our terminology for immaculate tableaux will imitate some of the classical terminology defined for semistandard Young tableaux.

two axioms says that the entries of T are weakly increasing along each row of $Y(\alpha)$.

(c) Let $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ be a composition of $|\alpha|$. An immaculate tableau T of shape α is said to have *content* β if every $j \in \{1, 2, 3, \ldots\}$ satisfies

$$\left|T^{-1}(j)\right| = \begin{cases} \beta_j, & \text{if } j \leq k; \\ 0, & \text{if } j > k \end{cases}.$$

Notice that not every immaculate tableau has a content (with this definition), because we only allow compositions as contents. More precisely, if T is an immaculate tableau of shape α, then there exists a composition β such that T has content β if and only if there exists a $k \in \mathbb{N}$ such that $T(Y(\alpha)) = \{1, 2, \ldots, k\}$.

(d) Let β be a composition of $|\alpha|$. Then, $K_{\alpha,\beta}$ denotes the number of immaculate tableaux of shape α and content β.

For future reference, let us notice that if α is a nonempty composition and if T is an immaculate tableau of shape α, then

$$\text{the smallest entry of } T \text{ is } T(1,1) \tag{25}$$

(because every $(i,j) \in Y(\alpha)$ satisfies $T(1,1) \leq T(i,1) \leq T(i,j)$). Moreover, if α is a composition, if T is an immaculate tableau of shape α, and if $(i,j) \in Y(\alpha)$ is such that $i > 1$, then

$$T(1,1) < T(i,1) \leq T(i,j). \tag{26}$$

Definition 4.2. Let α be a composition. The *dual immaculate function* $\mathfrak{S}^*_\alpha$ corresponding to α is defined as the quasisymmetric function

$$\sum_{\beta \models |\alpha|} K_{\alpha,\beta} M_\beta.$$

This definition is not identical to the definition of $\mathfrak{S}^*_\alpha$ used in [BBSSZ13a], but it is equivalent to it, as the following proposition shows.

Proposition 4.3. Definition 4.2 is equivalent to the definition of $\mathfrak{S}^*_\alpha$ used in [BBSSZ13a].

Proof of Proposition 4.3. Let $\leq_\ell$ denote the lexicographic order on compositions.

Let α be a composition. Then, [BBSSZ13a, Proposition 3.36] yields the following:

$$\left(\text{the dual immaculate function } \mathfrak{S}^*_\alpha \text{ as defined in [BBSSZ13a]}\right) = \sum_{\substack{\beta \models |\alpha|; \\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta.$$

Compared with

$$\begin{aligned}
&\left(\text{the dual immaculate function } \mathfrak{S}_\alpha^* \text{ as defined in Definition 4.2}\right)\\
&= \sum_{\beta \models |\alpha|} K_{\alpha,\beta} M_\beta = \sum_{\substack{\beta \models |\alpha|;\\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta + \sum_{\substack{\beta \models |\alpha|;\\ \text{not } \beta \leq_\ell \alpha}} \underbrace{K_{\alpha,\beta}}_{\substack{=0\\ \text{(by [BBSSZ13a, Proposition 3.15 (2)])}}} M_\beta\\
&= \sum_{\substack{\beta \models |\alpha|;\\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta + \underbrace{\sum_{\substack{\beta \models |\alpha|;\\ \text{not } \beta \leq_\ell \alpha}} 0 M_\beta}_{=0} = \sum_{\substack{\beta \models |\alpha|;\\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta,
\end{aligned}$$

this yields

$$\begin{aligned}
&\left(\text{the dual immaculate function } \mathfrak{S}_\alpha^* \text{ as defined in [BBSSZ13a]}\right)\\
&= \left(\text{the dual immaculate function } \mathfrak{S}_\alpha^* \text{ as defined in Definition 4.2}\right).
\end{aligned}$$

Hence, Definition 4.2 is equivalent to the definition of $\mathfrak{S}_\alpha^*$ used in [BBSSZ13a]. This proves Proposition 4.3. □

It is helpful to think of dual immaculate functions as analogues of Schur functions obtained by replacing semistandard Young tableaux by immaculate tableaux. Definition 4.2 is the analogue of the well-known formula $s_\lambda = \sum_{\mu \vdash |\lambda|} k_{\lambda,\mu} m_\mu$ for any partition λ, where s_λ denotes the Schur function corresponding to λ, where m_μ denotes the monomial symmetric function corresponding to the partition μ, and where $k_{\lambda,\mu}$ is the (λ, μ)-th Kostka number (i.e., the number of semistandard Young tableaux of shape λ and content μ). The following formula for the $\mathfrak{S}_\alpha^*$ (known to the authors of [BBSSZ13a] but not explicitly stated in their work) should not come as a surprise:

Proposition 4.4. Let α be a composition. Then,

$$\mathfrak{S}_\alpha^* = \sum_{\substack{T \text{ is an immaculate}\\ \text{tableau of shape } \alpha}} \mathbf{x}_T.$$

Here, $\mathbf{x}_T$ is defined as $\prod_{(i,j) \in Y(\alpha)} x_{T(i,j)}$ when T is an immaculate tableau of shape α.

Before we prove this proposition, let us state a fundamental and simple lemma:

Lemma 4.5. (a) If I is a finite subset of $\{1, 2, 3, \ldots\}$, then there exists a unique strictly increasing bijection $\{1, 2, \ldots, |I|\} \to I$. Let us denote this bijection by r_I. Its inverse r_I^{-1} is obviously again a strictly increasing bijection.

Now, let α be a composition.

(b) If T is an immaculate tableau of shape α, then $r_{T(Y(\alpha))}^{-1} \circ T$ (remember that immaculate tableaux are maps from $Y(\alpha)$ to $\{1, 2, 3, \ldots\}$) is an immaculate tableau of shape α as well, and has the additional property that there exists a unique composition β of $|\alpha|$ such that $r^{-1}_{T(Y(\alpha))} \circ T$ has content β.

(c) Let Q be an immaculate tableau of shape α. Let β be a composition of $|\alpha|$ such that Q has content β. Then,

$$M_\beta = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha; \\ r^{-1}_{T(Y(\alpha))} \circ T = Q}} \mathbf{x}_T. \tag{27}$$

Proof of Lemma 4.5. **(a)** Lemma 4.5 **(a)** is obvious.

(b) Let T be an immaculate tableau of shape α. Then, $r^{-1}_{T(Y(\alpha))} \circ T$ is an immaculate tableau of shape α as well[28]. Let $R = r^{-1}_{T(Y(\alpha))} \circ T : Y(\alpha) \to \{1, 2, \ldots, |T(Y(\alpha))|\}$. Then,

$$\begin{aligned} \underbrace{R}_{= r^{-1}_{T(Y(\alpha))} \circ T} (Y(\alpha)) &= \left(r^{-1}_{T(Y(\alpha))} \circ T \right) (Y(\alpha)) \\ &= r^{-1}_{T(Y(\alpha))} (T(Y(\alpha))) = \{1, 2, \ldots, |T(Y(\alpha))|\}. \end{aligned}$$

Hence, $\left(\left| R^{-1}(1) \right|, \left| R^{-1}(2) \right|, \ldots, \left| R^{-1}(|T(Y(\alpha))|) \right| \right)$ is a composition. Therefore, there exists a unique composition β of $|\alpha|$ such that R has content β (namely, $\beta = \left(\left| R^{-1}(1) \right|, \left| R^{-1}(2) \right|, \ldots, \left| R^{-1}(|T(Y(\alpha))|) \right| \right)$). In other words, there exists a unique composition β of $|\alpha|$ such that $r^{-1}_{T(Y(\alpha))} \circ T$ has content β (since $R = r^{-1}_{T(Y(\alpha))} \circ T$). This completes the proof of Lemma 4.5 **(b)**.

(c) If T is a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r^{-1}_{T(Y(\alpha))} \circ T = Q$, then T is automatically an immaculate tableau of shape α [29]. Hence, the summation sign "$\sum\limits_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha; \\ r^{-1}_{T(Y(\alpha))} \circ T = Q}}$" on the right hand side of (27) can be replaced by

[28] This is because the map $r^{-1}_{T(Y(\alpha))}$ is strictly increasing, and the inequality conditions which decide whether a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ is an immaculate tableau of shape α are preserved under composition with a strictly increasing map.

[29] *Proof.* Let T be a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r^{-1}_{T(Y(\alpha))} \circ T = Q$. Thus, $T = r_{T(Y(\alpha))} \circ Q$. Since Q is an immaculate tableau of shape α, this shows that T is an immaculate tableau of shape α (since the map $r_{T(Y(\alpha))}$ is strictly increasing, and the inequality conditions which decide whether a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ is an immaculate tableau of shape α are preserved under composition with a strictly increasing map).

"$\sum\limits_{\substack{T:Y(\alpha)\to\{1,2,3,\ldots\};\\ r_{T(Y(\alpha))}^{-1}\circ T=Q}}$". Hence,

$$\sum_{\substack{T \text{ is an immaculate}\\ \text{tableau of shape } \alpha;\\ r_{T(Y(\alpha))}^{-1}\circ T=Q}} \mathbf{x}_T = \sum_{\substack{T:Y(\alpha)\to\{1,2,3,\ldots\};\\ r_{T(Y(\alpha))}^{-1}\circ T=Q}} \mathbf{x}_T .$$

Now, let us write the composition β in the form $(\beta_1, \beta_2, \ldots, \beta_\ell)$. Then, we have

$$\left|Q^{-1}(k)\right| = \begin{cases} \beta_k, & \text{if } k \leq \ell; \\ 0, & \text{if } k > \ell \end{cases} \qquad \text{for every positive integer } k \tag{28}$$

(since Q has content β). Hence, $Q(Y(\alpha)) = \{1, 2, \ldots, \ell\}$. As a consequence, the maps $T : Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r_{T(Y(\alpha))}^{-1} \circ T = Q$ are in 1-to-1 correspondence with the ℓ-element subsets of $\{1, 2, 3, \ldots\}$ (the correspondence sends a map T to the ℓ-element subset $T(Y(\alpha))$, and the inverse correspondence sends an ℓ-element subset I to the map $r_I \circ Q$). But these latter subsets, in turn, are in 1-to-1 correspondence with the strictly increasing length-ℓ sequences $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers (the correspondence sends a subset G to the sequence $(r_G(1), r_G(2), \ldots, r_G(\ell))$; of course, this latter sequence is just the list of all elements of G in increasing order). Composing these two 1-to-1 correspondences, we conclude that the maps $T : Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r_{T(Y(\alpha))}^{-1} \circ T = Q$ are in 1-to-1 correspondence with the strictly increasing length-ℓ sequences $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers (the correspondence sends a map T to the sequence $\left(r_{T(Y(\alpha))}(1), r_{T(Y(\alpha))}(2), \ldots, r_{T(Y(\alpha))}(\ell)\right)$), and this correspondence has the property that $\mathbf{x}_T = x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$ whenever some map T gets sent to some sequence $(i_1 < i_2 < \cdots < i_\ell)$ (because if some map T gets sent to some sequence $(i_1 < i_2 < \cdots < i_\ell)$, then $(i_1, i_2, \ldots, i_\ell) = \left(r_{T(Y(\alpha))}(1), r_{T(Y(\alpha))}(2), \ldots, r_{T(Y(\alpha))}(\ell)\right)$, so that every $k \in \{1, 2, \ldots, \ell\}$ satisfies

$i_k = r_{T(Y(\alpha))}(k)$, and now we have

$$\mathbf{x}_T = \prod_{(i,j)\in Y(\alpha)} x_{T(i,j)} = \prod_{k=1}^{\ell} \prod_{\substack{(i,j)\in Y(\alpha);\\ Q(i,j)=k}} \underbrace{x_{T(i,j)}}_{\substack{=x_{r_{T(Y(\alpha))}(Q(i,j))} \\ \text{(since } T(i,j)=r_{T(Y(\alpha))}(Q(i,j)) \\ \text{(because } r^{-1}_{T(Y(\alpha))}\circ T=Q \\ \text{and thus } T=r_{T(Y(\alpha))}\circ Q))}}$$

$$\left(\text{since } Q\left(Y\left(\alpha\right)\right) = \{1,2,\ldots,\ell\}\right)$$

$$= \prod_{k=1}^{\ell} \underbrace{\prod_{\substack{(i,j)\in Y(\alpha);\\ Q(i,j)=k}}}_{=\prod_{(i,j)\in Q^{-1}(k)}} \underbrace{x_{r_{T(Y(\alpha))}(Q(i,j))}}_{\substack{=x_{r_{T(Y(\alpha))}(k)} \\ \text{(since } Q(i,j)=k)}}$$

$$= \prod_{k=1}^{\ell} \underbrace{\prod_{(i,j)\in Q^{-1}(k)} x_{r_{T(Y(\alpha))}(k)}}_{\substack{=x_{r_{T(Y(\alpha))}(k)}^{\left|Q^{-1}(k)\right|}=x_{i_k}^{\left|Q^{-1}(k)\right|} \\ \text{(since } r_{T(Y(\alpha))}(k)=i_k)}} = \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\left|Q^{-1}(k)\right|}}_{\substack{=x_{i_k}^{\beta_k} \\ \text{(since } \left|Q^{-1}(k)\right|=\beta_k \\ \text{(by (28)))}}} = \prod_{k=1}^{\ell} x_{i_k}^{\beta_k} = x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$$

). Hence,

$$\sum_{\substack{T:Y(\alpha)\to\{1,2,3,\ldots\};\\ r^{-1}_{T(Y(\alpha))}\circ T=Q}} \mathbf{x}_T = \sum_{1\le i_1<i_2<\cdots<i_\ell} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell} = M_\beta$$

(by the definition of M_β). Altogether, we thus have

$$\sum_{\substack{T \text{ is an immaculate}\\ \text{tableau of shape } \alpha;\\ r^{-1}_{T(Y(\alpha))}\circ T=Q}} \mathbf{x}_T = \sum_{\substack{T:Y(\alpha)\to\{1,2,3,\ldots\};\\ r^{-1}_{T(Y(\alpha))}\circ T=Q}} \mathbf{x}_T = M_\beta.$$

This proves Lemma 4.5 **(c)**. ☐

Proof of Proposition 4.4. For every finite subset I of $\{1,2,3,\ldots\}$, we shall use the notation r_I introduced in Lemma 4.5 **(a)**. Recall Lemma 4.5 **(b)**; it says that if T is an immaculate tableau of shape α, then $r^{-1}_{T(Y(\alpha))}\circ T$ is an immaculate tableau of shape α as well, and has the additional property that there exists a unique composition β of $|\alpha|$ such that $r^{-1}_{T(Y(\alpha))}\circ T$ has content β.

Now,

$$\mathfrak{S}^*_\alpha = \sum_{\beta\models|\alpha|} \underbrace{K_{\alpha,\beta} M_\beta}_{\substack{=\sum_{\substack{Q \text{ is an immaculate}\\ \text{tableau of shape } \alpha\\ \text{and content } \beta}} M_\beta \\ \text{(by the definition of } K_{\alpha,\beta})}} = \sum_{\beta\models|\alpha|} \sum_{\substack{Q \text{ is an immaculate}\\ \text{tableau of shape } \alpha\\ \text{and content } \beta}} M_\beta. \qquad (29)$$

But (27) shows that every composition β of $|\alpha|$ satisfies

$$\sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{and content } \beta}} M_\beta = \sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{and content } \beta}} \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha; \\ r^{-1}_{T(Y(\alpha))} \circ T = Q}} \mathbf{x}_T = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{such that } r^{-1}_{T(Y(\alpha))} \circ T \\ \text{has content } \beta}} \mathbf{x}_T$$

(because for every immaculate tableau T of shape α, the map $r^{-1}_{T(Y(\alpha))} \circ T$ is an immaculate tableau of shape α as well). Substituting this into (29), we obtain

$$\begin{aligned}
\mathfrak{S}^*_\alpha &= \sum_{\beta \models |\alpha|} \underbrace{\sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{and content } \beta}} M_\beta}_{= \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{such that } r^{-1}_{T(Y(\alpha))} \circ T \\ \text{has content } \beta}} \mathbf{x}_T} = \sum_{\beta \models |\alpha|} \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{such that } r^{-1}_{T(Y(\alpha))} \circ T \\ \text{has content } \beta}} \mathbf{x}_T \\
&= \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_T
\end{aligned}$$

(because for every immaculate tableau T of shape α, there exists a unique composition β of $|\alpha|$ such that $r^{-1}_{T(Y(\alpha))} \circ T$ has content β), whence Proposition 4.4 follows. ☐

Corollary 4.6. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition with $\ell > 0$. Let $\overline{\alpha}$ denote the composition $(\alpha_2, \alpha_3, \ldots, \alpha_\ell)$ of $|\alpha| - \alpha_1$. Then,

$$\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}}.$$

Here, h_n denotes the n-th complete homogeneous symmetric function for every $n \in \mathbb{N}$.

Proof of Corollary 4.6. Proposition 4.4 shows that

$$\mathfrak{S}^*_\alpha = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_T = \sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_Q \tag{30}$$

(here, we have renamed the summation index T as Q).

Let $n = \alpha_1$. If $i_1, i_2, \ldots, i_n$ are positive integers satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$, and if T is an immaculate tableau of shape $\overline{\alpha}$, then

$$\begin{aligned}
&\left(x_{i_1}x_{i_2}\cdots x_{i_n}\right) \prec \mathbf{x}_T \\
&= \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } \min\left(\operatorname{Supp}\left(x_{i_1}x_{i_2}\cdots x_{i_n}\right)\right) < \min\left(\operatorname{Supp}\left(\mathbf{x}_T\right)\right); \\ 0, & \text{if } \min\left(\operatorname{Supp}\left(x_{i_1}x_{i_2}\cdots x_{i_n}\right)\right) \geq \min\left(\operatorname{Supp}\left(\mathbf{x}_T\right)\right) \end{cases} \\
&\qquad \left(\text{by the definition of } \prec \text{ on monomials}\right) \\
&= \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } i_1 < \min\left(T\left(Y\left(\overline{\alpha}\right)\right)\right); \\ 0, & \text{if } i_1 \geq \min\left(T\left(Y\left(\overline{\alpha}\right)\right)\right) \end{cases} \qquad (31) \\
&\qquad \left(\text{since } \min\left(\operatorname{Supp}\left(x_{i_1}x_{i_2}\cdots x_{i_n}\right)\right) = i_1 \text{ and } \operatorname{Supp}\left(\mathbf{x}_T\right) = T\left(Y\left(\overline{\alpha}\right)\right)\right).
\end{aligned}$$

But from $n = \alpha_1$, we obtain $h_n = h_{\alpha_1}$, so that $h_{\alpha_1} = h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$ and $\mathfrak{S}^*_{\overline{\alpha}} = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \mathbf{x}_T$ (by Proposition 4.4). Hence,

$$\begin{aligned}
&h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}} \\
&= \left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2}\cdots x_{i_n}\right) \prec \left(\sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \mathbf{x}_T\right) \\
&= \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \underbrace{\left(x_{i_1}x_{i_2}\cdots x_{i_n}\right) \prec \mathbf{x}_T}_{\substack{= \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } i_1 < \min\left(T\left(Y\left(\overline{\alpha}\right)\right)\right); \\ 0, & \text{if } i_1 \geq \min\left(T\left(Y\left(\overline{\alpha}\right)\right)\right) \end{cases} \\ \text{(by (31))}}} \\
&= \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } i_1 < \min\left(T\left(Y\left(\overline{\alpha}\right)\right)\right); \\ 0, & \text{if } i_1 \geq \min\left(T\left(Y\left(\overline{\alpha}\right)\right)\right) \end{cases} \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}; \\ i_1 < \min(T(Y(\overline{\alpha})))}} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T. \qquad (32)
\end{aligned}$$

We need to check that this equals $\mathfrak{S}^*_{\alpha} = \sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_Q$.

Now, let us define a map Φ from:

- the set of all pairs $\left(\left(i_1, i_2, \ldots, i_n\right), T\right)$, where $i_1, i_2, \ldots, i_n$ are positive integers satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$, and where T is an immaculate tableau of shape $\overline{\alpha}$ satisfying $i_1 < \min\left(T\left(Y\left(\overline{\alpha}\right)\right)\right)$

to:

- the set of all immaculate tableaux of shape α.

Namely, we define the image of a pair $((i_1, i_2, \ldots, i_n), T)$ under Φ to be the immaculate tableau obtained by adding a new row, filled with the entries $i_1, i_2, \ldots, i_n$ (from left to right), to the top[30] of the tableau T [31].

This map Φ is a bijection[32], and has the property that if Q denotes the image of a pair $((i_1, i_2, \ldots, i_n), T)$ under the bijection Φ, then $\mathbf{x}_Q = x_{i_1} x_{i_2} \cdots x_{i_n} \mathbf{x}_T$. Hence,

$$\sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_Q = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}; \\ i_1 < \min(T(Y(\overline{\alpha})))}} x_{i_1} x_{i_2} \cdots x_{i_n} \mathbf{x}_T.$$

In light of (30) and (32), this rewrites as $\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}}$. So Corollary 4.6 is proven. □

Corollary 4.7. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. Then,

$$\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots)).$$

Proof of Corollary 4.7. We prove Corollary 4.7 by induction over ℓ:

Induction base: If $\ell = 0$, then $\alpha = \varnothing$ and thus $\mathfrak{S}^*_\alpha = \mathfrak{S}^*_\varnothing = 1$. But if $\ell = 0$, then we also have $h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots)) = 1$. Hence, if $\ell = 0$, then $\mathfrak{S}^*_\alpha = 1 = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))$. Thus, Corollary 4.7 is proven when $\ell = 0$. The induction base is complete.

Induction step: Let L be a positive integer. Assume that Corollary 4.7 holds for $\ell = L - 1$. We now need to prove that Corollary 4.7 by holds for $\ell = L$.

So let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition with $\ell = L$. Then, $\ell = L > 0$. Now, let $\overline{\alpha}$ denote the composition $(\alpha_2, \alpha_3, \ldots, \alpha_\ell)$ of $|\alpha| - \alpha_1$. Then, Corollary 4.6 yields $\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}}$. But by our induction hypothesis, we can apply Corollary 4.7 to $\overline{\alpha} = (\alpha_2, \alpha_3, \ldots, \alpha_\ell)$ instead of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ (since $\ell - 1 = L - 1$). As a result, we obtain $\mathfrak{S}^*_{\overline{\alpha}} = h_{\alpha_2} \prec (h_{\alpha_3} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))$. Hence,

$$\begin{aligned} \mathfrak{S}^*_\alpha &= h_{\alpha_1} \prec \underbrace{\mathfrak{S}^*_{\overline{\alpha}}}_{= h_{\alpha_2} \prec (h_{\alpha_3} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))} = h_{\alpha_1} \prec (h_{\alpha_2} \prec (h_{\alpha_3} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))) \\ &= h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots)). \end{aligned}$$

[30] Here, we are using the graphical representation of immaculate tableaux introduced in Definition 4.1.

[31] Formally speaking, this means that the image of $((i_1, i_2, \ldots, i_n), T)$ is the map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ which sends every $(u, v) \in Y(\alpha)$ to $\begin{cases} i_v, & \text{if } u = 1; \\ T(u-1, v), & \text{if } u \neq 1 \end{cases}$. Proving that this map is an immaculate tableau is easy.

[32] *Proof.* The injectivity of the map Φ is obvious. Its surjectivity follows from the observation that if Q is an immaculate tableau of shape α, then the first entry of its top row is smaller than the smallest entry of the immaculate tableau formed by all other rows of Q. (This is a consequence of (26), applied to Q instead of T.)

Now, let us forget that we fixed α. We thus have shown that $\mathfrak{S}_\alpha^* = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))$ for every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ which satisfies $\ell = L$. In other words, Corollary 4.7 holds for $\ell = L$. This completes the induction step. The induction proof of Corollary 4.7 is thus complete. $\square$

5. An alternative description of $h_m \prec$

In this section, we shall also use the Hopf algebra of *noncommutative symmetric functions*. This Hopf algebra (a noncommutative one, for a change) is denoted by NSym and has been discussed in [GriRei15, Section 5.4] and [HaGuKi10, Chapter 6]; all we need to know about it are the following properties:

- There is a nondegenerate pairing between NSym and QSym, that is, a nondegenerate $\mathbf{k}$-bilinear form $\operatorname{NSym} \times \operatorname{QSym} \to \mathbf{k}$. We shall denote this bilinear form by $(\cdot, \cdot)$. This $\mathbf{k}$-bilinear form is a Hopf algebra pairing, i.e., it satisfies

$$(ab, c) = \sum_{(c)} \left(a, c_{(1)}\right) \left(b, c_{(2)}\right) \qquad \text{for all } a \in \operatorname{NSym},\ b \in \operatorname{NSym} \text{ and } c \in \operatorname{QSym}; \tag{33}$$

$$(1, c) = \varepsilon(c) \qquad \text{for all } c \in \operatorname{QSym};$$

$$\sum_{(a)} \left(a_{(1)}, b\right) \left(a_{(2)}, c\right) = (a, bc) \qquad \text{for all } a \in \operatorname{NSym},\ b \in \operatorname{QSym} \text{ and } c \in \operatorname{QSym};$$

$$(a, 1) = \varepsilon(a) \qquad \text{for all } a \in \operatorname{NSym};$$

$$(S(a), b) = (a, S(b)) \qquad \text{for all } a \in \operatorname{NSym} \text{ and } b \in \operatorname{QSym}$$

  (where we use Sweedler's notation).

- There is a basis of the $\mathbf{k}$-module NSym which is dual to the fundamental basis $(F_\alpha)_{\alpha \in \operatorname{Comp}}$ of QSym with respect to the bilinear form $(\cdot, \cdot)$. This basis is called the *ribbon basis* and will be denoted by $(R_\alpha)_{\alpha \in \operatorname{Comp}}$.

Both of these properties are immediate consequences of the definitions of NSym and of $(R_\alpha)_{\alpha \in \operatorname{Comp}}$ given in [GriRei15, Section 5.5] (although other sources define these objects differently, and then the properties no longer are immediate). The notations we are using here are the same as the ones used in [GriRei15, Section 5.5] (except that [GriRei15, Section 5.5] calls L_α what we denote by F_α), and only slightly differ from those in [BBSSZ13a] (namely, [BBSSZ13a] denotes the pairing $(\cdot, \cdot)$ by $\langle \cdot, \cdot \rangle$ instead).

We need some more definitions. For any $g \in \operatorname{NSym}$, let $\mathrm{L}_g : \operatorname{NSym} \to \operatorname{NSym}$ denote the left multiplication by g on NSym (that is, the $\mathbf{k}$-linear map $\operatorname{NSym} \to \operatorname{NSym}$, $f \mapsto gf$). For any $g \in \operatorname{NSym}$, let $g^{\perp} : \operatorname{QSym} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map adjoint to $\mathrm{L}_g : \operatorname{NSym} \to \operatorname{NSym}$ with respect to the pairing $(\cdot, \cdot)$ between NSym and QSym. Thus, for any $g \in \operatorname{NSym}$, $a \in \operatorname{NSym}$ and $c \in \operatorname{QSym}$, we have

$$\left(a, g^{\perp} c\right) = \left(\underbrace{\mathrm{L}_g\, a}_{=ga}, c\right) = (ga, c). \tag{34}$$

The following fact is well-known (and also is an easy formal consequence of the definition of $g^{\perp}$ and of (33)):

Lemma 5.1. Every $g \in \operatorname{NSym}$ and $f \in \operatorname{QSym}$ satisfy

$$g^{\perp} f = \sum_{(f)} \left(g, f_{(1)}\right) f_{(2)}. \tag{35}$$

Proof of Lemma 5.1. Let $g \in \operatorname{NSym}$ and $f \in \operatorname{QSym}$. For every $a \in \operatorname{NSym}$, we have

$$\begin{aligned} \left(a, g^{\perp} f\right) &= \left(\underbrace{\mathrm{L}_g\, a}_{\substack{=ga \\ (\text{by the definition of } \mathrm{L}_g)}}, f\right) && \left(\begin{array}{c}\text{since the map } g^{\perp} \text{ is adjoint to } \mathrm{L}_g \\ \text{with respect to the pairing } (\cdot,\cdot)\end{array}\right) \\ &= (ga, f) = \sum_{(f)} \left(g, f_{(1)}\right) \left(a, f_{(2)}\right) && \left(\begin{array}{c}\text{by (33), applied to } g,\ a \text{ and } f \\ \text{instead of } a,\ b \text{ and } c\end{array}\right) \\ &= \left(a, \sum_{(f)} \left(g, f_{(1)}\right) f_{(2)}\right) && (\text{since the pairing } (\cdot,\cdot) \text{ is } \mathbf{k}\text{-bilinear}). \end{aligned}$$

Since the pairing $(\cdot, \cdot)$ is nondegenerate, this entails that $g^{\perp} f = \sum_{(f)} \left(g, f_{(1)}\right) f_{(2)}$. This proves Lemma 5.1. ☐

For any composition α, we define a composition $\omega(\alpha)$ as follows: Let $n = |\alpha|$, and write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Let $\operatorname{rev}\alpha$ denote the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ of n. Then, $\omega(\alpha)$ shall be the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\operatorname{rev}\alpha)$. (This definition is identical with that in [GriRei15, Definition 5.2.14]. Some authors denote $\omega(\alpha)$ by α' instead.) We notice that $\omega(\omega(\alpha)) = \alpha$ for any composition α.

Here is a simple property of the composition $\omega(\alpha)$ that will later be used:

Proposition 5.2. **(a)** We have $\omega([\alpha, \beta]) = \omega(\beta) \odot \omega(\alpha)$ for any two compositions α and β.

(b) We have $\omega(\alpha \odot \beta) = [\omega(\beta), \omega(\alpha)]$ for any two compositions α and β.

(c) We have $\omega(\omega(\gamma)) = \gamma$ for every composition γ.

Proof of Proposition 5.2. For any composition α, we define a composition $\operatorname{rev}\alpha$ as follows: Let $n = |\alpha|$, and write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Let $\operatorname{rev}\alpha$ denote the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ of n. (This definition of $\operatorname{rev}\alpha$ is the same as the one we gave above during the definition of $\omega(\alpha)$.) Clearly,

$$|\operatorname{rev}\gamma| = |\gamma| \qquad \text{for any composition } \gamma. \tag{36}$$

It is easy to see that

$$\operatorname{rev}([\alpha, \beta]) = [\operatorname{rev}\beta, \operatorname{rev}\alpha] \qquad \text{and} \tag{37}$$

$$\operatorname{rev}(\alpha \odot \beta) = (\operatorname{rev}\beta) \odot (\operatorname{rev}\alpha) \tag{38}$$

for any two compositions α and β.

Recall that a composition γ of a nonnegative integer n is uniquely determined by the set $D(\gamma)$ and the number n. Thus, if γ_1 and γ_2 are two compositions of one and the same nonnegative integer n satisfying $D(\gamma_1) = D(\gamma_2)$, then

$$\gamma_1 = \gamma_2. \tag{39}$$

For every composition γ, we define a composition $\rho(\gamma)$ as follows: Let $n = |\gamma|$. Let $\rho(\gamma)$ be the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$. (This is well-defined, because for every subset T of $\{1, 2, \ldots, n-1\}$, there exists a unique composition τ of n which satisfies $D(\tau) = T$.) Notice that

$$|\rho(\gamma)| = |\gamma| \qquad \text{for any composition } \gamma. \tag{40}$$

Also, if $n \in \mathbb{N}$, and if γ is a composition of n, then

$$D(\rho(\gamma)) = \{1, 2, \ldots, n-1\} \setminus D(\gamma) \tag{41}$$

[33].

Notice also that

$$\omega(\alpha) = \rho(\operatorname{rev}\alpha) \qquad \text{for any composition } \alpha \tag{42}$$

[33]*Proof of (41):* Let $n \in \mathbb{N}$. Let γ be a composition of n. Then, $\rho(\gamma)$ is the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$ (because this is how $\rho(\gamma)$ was defined). Thus, $\rho(\gamma)$ is a composition of n which satisfies $D(\rho(\gamma)) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$. This proves (41).

[34].

Now, we shall prove that

$$\rho([\alpha,\beta]) = \rho(\alpha) \odot \rho(\beta) \tag{43}$$

for any two compositions α and β.

Proof of (43): Let α and β be two compositions. Let $p = |\alpha|$ and $q = |\beta|$; thus, α and β are compositions of p and q, respectively. We WLOG assume that both compositions α and β are nonempty (since otherwise, (43) is fairly obvious). The composition α is a composition of p. Thus, $p > 0$ (since α is nonempty). Similarly, $q > 0$.

Hence, $[\alpha,\beta]$ is a composition of $p+q$ satisfying $D([\alpha,\beta]) = D(\alpha) \cup \{p\} \cup (D(\beta)+p)$ (by Lemma 3.9 **(b)**). The definition of $\rho([\alpha,\beta])$ thus yields

$$\begin{aligned} D(\rho([\alpha,\beta])) &= \{1,2,\ldots,p+q-1\} \setminus \underbrace{D([\alpha,\beta])}_{=D(\alpha)\cup\{p\}\cup(D(\beta)+p)} \\ &= \{1,2,\ldots,p+q-1\} \setminus (\{p\} \cup D(\alpha) \cup (D(\beta)+p)). \end{aligned} \tag{44}$$

Applying (40) to $\gamma = \alpha$, we obtain $|\rho(\alpha)| = |\alpha| = p$. Thus, $\rho(\alpha)$ is a composition of p. Similarly, $\rho(\beta)$ is a composition of q. Thus, Lemma 3.9 **(a)** (applied to $\rho(\alpha)$ and $\rho(\beta)$ instead of α and β) shows that $\rho(\alpha) \odot \rho(\beta)$ is a composition of $p+q$ satisfying $D(\rho(\alpha) \odot \rho(\beta)) = D(\rho(\alpha)) \cup (D(\rho(\beta))+p)$. Also, applying (40) to $\gamma = [\alpha,\beta]$, we obtain $|\rho([\alpha,\beta])| = |[\alpha,\beta]| = p+q$ (since $[\alpha,\beta]$ is a composition of $p+q$). In other words, $\rho([\alpha,\beta])$ is a composition of $p+q$.

But the definition of $\rho(\alpha)$ shows that $D(\rho(\alpha)) = \{1,2,\ldots,p-1\} \setminus D(\alpha)$. Also, the definition of $\rho(\beta)$ shows that $D(\rho(\beta)) = \{1,2,\ldots,q-1\} \setminus D(\beta)$. Hence,

$$\begin{aligned} &\underbrace{D(\rho(\beta))}_{=\{1,2,\ldots,q-1\}\setminus D(\beta)} + p \\ &= (\{1,2,\ldots,q-1\} \setminus D(\beta)) + p \\ &= \underbrace{(\{1,2,\ldots,q-1\}+p)}_{=\{p+1,p+2,\ldots,p+q-1\}} \setminus (D(\beta)+p) \\ &= \{p+1,p+2,\ldots,p+q-1\} \setminus (D(\beta)+p). \end{aligned}$$

[34] *Proof of (42):* Let α be a composition. Let $n = |\alpha|$. Thus, α is a composition of n. Hence, $\omega(\alpha)$ is a composition of n as well. Also, $\operatorname{rev}\alpha$ is a composition of n. Now, the definition of $\rho(\operatorname{rev}\alpha)$ shows that $\rho(\operatorname{rev}\alpha)$ is the unique composition β of n which satisfies $D(\beta) = \{1,2,\ldots,n-1\} \setminus D(\operatorname{rev}\alpha)$. Hence, $\rho(\operatorname{rev}\alpha)$ is a composition of n and satisfies $D(\rho(\operatorname{rev}\alpha)) = \{1,2,\ldots,n-1\} \setminus D(\operatorname{rev}\alpha)$.

On the other hand, $\omega(\alpha)$ is the unique composition β of n which satisfies $D(\beta) = \{1,2,\ldots,n-1\} \setminus D(\operatorname{rev}\alpha)$ (by the definition of $\omega(\alpha)$). Thus, $\omega(\alpha)$ is a composition of n and satisfies $D(\omega(\alpha)) = \{1,2,\ldots,n-1\} \setminus D(\operatorname{rev}\alpha)$.

Hence,

$$D(\rho(\operatorname{rev}\alpha)) = \{1,2,\ldots,n-1\} \setminus D(\operatorname{rev}\alpha) = D(\omega(\alpha)).$$

Applying (39) to $\gamma_1 = \rho(\operatorname{rev}\alpha)$ and $\gamma_2 = \omega(\alpha)$, we therefore obtain $\rho(\operatorname{rev}\alpha) = \omega(\alpha)$. Qed.

Also, $D(\beta) \subseteq \{1,2,\ldots,q-1\}$, so that $D(\beta)+p \subseteq \{1,2,\ldots,q-1\}+p = \{p+1,p+2,\ldots,p+q-1\}$.

Now, it is well-known that if X, Y, X' and Y' are four sets such that $X' \subseteq X$, $Y' \subseteq Y$ and $X \cap Y = \varnothing$, then

$$\left(X \setminus X'\right) \cup \left(Y \setminus Y'\right) = (X \cup Y) \setminus \left(X' \cup Y'\right). \tag{45}$$

Now,

$$\begin{aligned}
&D(\rho(\alpha) \odot \rho(\beta)) \\
&= \underbrace{D(\rho(\alpha))}_{=\{1,2,\ldots,p-1\}\setminus D(\alpha)} \cup \underbrace{(D(\rho(\beta))+p)}_{=\{p+1,p+2,\ldots,p+q-1\}\setminus(D(\beta)+p)} \\
&= (\{1,2,\ldots,p-1\} \setminus D(\alpha)) \cup (\{p+1,p+2,\ldots,p+q-1\} \setminus (D(\beta)+p)) \\
&= \underbrace{(\{1,2,\ldots,p-1\} \cup \{p+1,p+2,\ldots,p+q-1\})}_{=\{1,2,\ldots,p+q-1\}\setminus\{p\}} \\
&\qquad \setminus (D(\alpha) \cup (D(\beta)+p)) \\
&\qquad \left(\begin{array}{c} \text{by (45), applied to } X = \{1,2,\ldots,p-1\}, \\ Y = \{p+1,p+2,\ldots,p+q-1\},\ X' = D(\alpha) \text{ and } Y' = D(\beta)+p \end{array}\right) \\
&= (\{1,2,\ldots,p+q-1\} \setminus \{p\}) \setminus (D(\alpha) \cup (D(\beta)+p)) \\
&= \{1,2,\ldots,p+q-1\} \setminus \underbrace{(\{p\} \cup D(\alpha) \cup (D(\beta)+p))}_{=D(\alpha)\cup\{p\}\cup(D(\beta)+p)} \\
&= \{1,2,\ldots,p+q-1\} \setminus (\{p\} \cup D(\alpha) \cup (D(\beta)+p)) \\
&= D(\rho([\alpha,\beta])) \qquad \text{(by (44))}.
\end{aligned}$$

Thus, (39) (applied to $n = p+q$, $\gamma_1 = \rho(\alpha) \odot \rho(\beta)$ and $\gamma_2 = \rho([\alpha,\beta])$) shows that $\rho(\alpha) \odot \rho(\beta) = \rho([\alpha,\beta])$. This proves (43).

(a) Let α and β be two compositions. Then, (42) yields $\omega(\alpha) = \rho(\operatorname{rev}\alpha)$. Also, (42) (applied to β instead of α) yields $\omega(\beta) = \rho(\operatorname{rev}\beta)$.

From (42) (applied to $[\alpha,\beta]$ instead of α), we obtain

$$\begin{aligned}
\omega([\alpha,\beta]) &= \rho\left(\underbrace{\operatorname{rev}([\alpha,\beta])}_{\substack{=[\operatorname{rev}\beta,\operatorname{rev}\alpha] \\ \text{(by (37))}}}\right) = \rho([\operatorname{rev}\beta,\operatorname{rev}\alpha]) \\
&= \underbrace{\rho(\operatorname{rev}\beta)}_{=\omega(\beta)} \odot \underbrace{\rho(\operatorname{rev}\alpha)}_{=\omega(\alpha)} \qquad \left(\begin{array}{c} \text{by (43), applied to } \operatorname{rev}\beta \\ \text{and } \operatorname{rev}\alpha \text{ instead of } \alpha \text{ and } \beta \end{array}\right) \\
&= \omega(\beta) \odot \omega(\alpha).
\end{aligned}$$

This proves Proposition 5.2 **(a)**.

(c) First of all, it is clear that

$$\operatorname{rev}(\operatorname{rev}\gamma) = \gamma \qquad \text{for every composition } \gamma. \tag{46}$$

Furthermore,

$$\rho(\rho(\gamma)) = \gamma \qquad \text{for every composition } \gamma \tag{47}$$

[35].

On the other hand, if G is a set of integers and r is an integer, then we let $r - G$ denote the set $\{r - g \mid g \in G\}$ of integers. Then, for any $n \in \mathbb{N}$ and any composition γ of n, we have

$$D(\operatorname{rev}\gamma) = n - D(\gamma) \tag{48}$$

[36].

Now,

$$\rho(\operatorname{rev}\gamma) = \operatorname{rev}(\rho(\gamma)) \qquad \text{for every composition } \gamma \tag{51}$$

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[35] *Proof of (47):* Let γ be a composition. Let $n = |\gamma|$. Thus, γ is a composition of n. The definition of $\rho(\gamma)$ shows that $\rho(\gamma)$ is the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$. Thus, $\rho(\gamma)$ is a composition of n and satisfies $D(\rho(\gamma)) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$.

Therefore, the definition of $\rho(\rho(\gamma))$ shows that $\rho(\rho(\gamma))$ is the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\rho(\gamma))$. Thus, $\rho(\rho(\gamma))$ is a composition of n and satisfies $D(\rho(\rho(\gamma))) = \{1, 2, \ldots, n-1\} \setminus D(\rho(\gamma))$. Hence,

$$\begin{aligned} D(\rho(\rho(\gamma))) &= \{1, 2, \ldots, n-1\} \setminus \underbrace{D(\rho(\gamma))}_{=\{1,2,\ldots,n-1\}\setminus D(\gamma)} \\ &= \{1, 2, \ldots, n-1\} \setminus (\{1, 2, \ldots, n-1\} \setminus D(\gamma)) \\ &= D(\gamma) \qquad (\text{since } D(\gamma) \subseteq \{1, 2, \ldots, n-1\}). \end{aligned}$$

Hence, (39) (applied to $\gamma_1 = \rho(\rho(\gamma))$ and $\gamma_2 = \gamma$) shows that $\rho(\rho(\gamma)) = \gamma$. This proves (47).

[36] *Proof of (48):* Let $n \in \mathbb{N}$. Let γ be a composition of n. Thus, γ is a composition satisfying $|\gamma| = n$.

Write γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$. Then, $\operatorname{rev}\gamma = (\gamma_\ell, \gamma_{\ell-1}, \ldots, \gamma_1)$ (by the definition of $\operatorname{rev}\gamma$). Also, from $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$, we obtain $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_\ell$, whence $\gamma_1 + \gamma_2 + \cdots + \gamma_\ell = |\gamma| = n$. Hence, every $i \in \{1, 2, \ldots, \ell - 1\}$ satisfies

$$\begin{aligned} n &= \gamma_1 + \gamma_2 + \cdots + \gamma_\ell = (\gamma_1 + \gamma_2 + \cdots + \gamma_i) + \underbrace{(\gamma_{i+1} + \gamma_{i+2} + \cdots + \gamma_\ell)}_{=\gamma_\ell + \gamma_{\ell-1} + \cdots + \gamma_{i+1}} \\ &= (\gamma_1 + \gamma_2 + \cdots + \gamma_i) + (\gamma_\ell + \gamma_{\ell-1} + \cdots + \gamma_{i+1}). \end{aligned} \tag{49}$$

Also, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$, so that the definition of $D(\gamma)$ yields

$$\begin{aligned} D(\gamma) &= \{\gamma_1, \gamma_1 + \gamma_2, \gamma_1 + \gamma_2 + \gamma_3, \ldots, \gamma_1 + \gamma_2 + \cdots + \gamma_{\ell-1}\} \\ &= \{\gamma_1 + \gamma_2 + \cdots + \gamma_i \mid i \in \{1, 2, \ldots, \ell - 1\}\}. \end{aligned} \tag{50}$$

[37].

Now, let γ be a composition. Then, (42) (applied to $\alpha = \gamma$) yields $\omega(\gamma) = \rho(\operatorname{rev}\gamma) = \operatorname{rev}(\rho(\gamma))$ (by (51)). But (42) (applied to $\alpha = \omega(\gamma)$) yields

$$\omega(\omega(\gamma)) = \rho\left(\operatorname{rev}\left(\underbrace{\omega(\gamma)}_{=\operatorname{rev}(\rho(\gamma))}\right)\right) = \rho\left(\underbrace{\operatorname{rev}(\operatorname{rev}(\rho(\gamma)))}_{\substack{=\rho(\gamma) \\ \text{(by (46), applied to} \\ \rho(\gamma) \text{ instead of } \gamma)}}\right)$$
$$= \rho(\rho(\gamma)) = \gamma \qquad \text{(by (47))}.$$

But $\operatorname{rev}\gamma = (\gamma_\ell, \gamma_{\ell-1}, \ldots, \gamma_1)$. Hence, the definition of $D(\operatorname{rev}\gamma)$ yields

$$\begin{aligned} D(\operatorname{rev}\gamma) &= \{\gamma_\ell, \gamma_\ell + \gamma_{\ell-1}, \gamma_\ell + \gamma_{\ell-1} + \gamma_{\ell-2}, \ldots, \gamma_\ell + \gamma_{\ell-1} + \gamma_{\ell-2} + \cdots + \gamma_2\} \\ &= \left\{ \underbrace{\gamma_\ell + \gamma_{\ell-1} + \cdots + \gamma_{i+1}}_{\substack{=n-(\gamma_1+\gamma_2+\cdots+\gamma_i) \\ \text{(by (49))}}} \mid i \in \{1, 2, \ldots, \ell-1\} \right\} \\ &= \{n - (\gamma_1 + \gamma_2 + \cdots + \gamma_i) \mid i \in \{1, 2, \ldots, \ell-1\}\} \\ &= n - \underbrace{\{\gamma_1 + \gamma_2 + \cdots + \gamma_i \mid i \in \{1, 2, \ldots, \ell-1\}\}}_{\substack{=D(\gamma) \\ \text{(by (50))}}} \\ &= n - D(\gamma). \end{aligned}$$

This proves (48).

[37]*Proof of (51):* Let γ be a composition. Let $n = |\gamma|$. Thus, γ is a composition of n.

Now, (40) (applied to $\operatorname{rev}\gamma$ instead of γ) yields $|\rho(\operatorname{rev}\gamma)| = |\operatorname{rev}\gamma| = |\gamma|$ (by (36)). Also, (36) (applied to $\rho(\gamma)$ instead of γ) yields $|\operatorname{rev}(\rho(\gamma))| = |\rho(\gamma)| = |\gamma|$ (by (40)). Now, $|\rho(\operatorname{rev}\gamma)| = |\gamma| = n$, $|\operatorname{rev}\gamma| = |\gamma| = n$, $|\rho(\gamma)| = |\gamma| = n$ and $|\operatorname{rev}(\rho(\gamma))| = |\gamma| = n$. Hence, all of $\rho(\operatorname{rev}\gamma)$, $\operatorname{rev}\gamma$, $\rho(\gamma)$ and $\operatorname{rev}(\rho(\gamma))$ are compositions of n.

Applying (41) to $\operatorname{rev}\gamma$ instead of γ, we obtain

$$\begin{aligned} D(\rho(\operatorname{rev}\gamma)) &= \underbrace{\{1, 2, \ldots, n-1\}}_{=n-\{1,2,\ldots,n-1\}} \setminus \underbrace{D(\operatorname{rev}\gamma)}_{\substack{=n-D(\gamma) \\ \text{(by (48))}}} \\ &= (n - \{1, 2, \ldots, n-1\}) \setminus (n - D(\gamma)) \\ &= n - \underbrace{(\{1, 2, \ldots, n-1\} \setminus D(\gamma))}_{\substack{=D(\rho(\gamma)) \\ \text{(by (41))}}} \\ &= n - D(\rho(\gamma)). \end{aligned}$$

Comparing this with

$$D(\operatorname{rev}(\rho(\gamma))) = n - D(\rho(\gamma)) \qquad \text{(by (48), applied to } \rho(\gamma) \text{ instead of } \gamma\text{)},$$

we obtain $D(\rho(\operatorname{rev}\gamma)) = D(\operatorname{rev}(\rho(\gamma)))$. Hence, (39) (applied to $\gamma_1 = \rho(\operatorname{rev}\gamma)$ and $\gamma_2 = \operatorname{rev}(\rho(\gamma))$) yields $\rho(\operatorname{rev}\gamma) = \operatorname{rev}(\rho(\gamma))$. This proves (51).

This proves Proposition 5.2 **(c)**.

(b) Let α and β be two compositions. Then, Proposition 5.2 **(a)** (applied to $\omega(\beta)$ and $\omega(\alpha)$ instead of α and β) yields

$$\omega([\omega(\beta), \omega(\alpha)]) = \underbrace{\omega(\omega(\alpha))}_{\substack{=\alpha \\ \text{(by Proposition 5.2 (c),} \\ \text{applied to } \gamma=\alpha)}} \odot \underbrace{\omega(\omega(\beta))}_{\substack{=\beta \\ \text{(by Proposition 5.2 (c),} \\ \text{applied to } \gamma=\beta)}} = \alpha \odot \beta.$$

Hence, $\alpha \odot \beta = \omega([\omega(\beta), \omega(\alpha)])$. Applying the map ω to both sides of this equality, we conclude that

$$\omega(\alpha \odot \beta) = \omega(\omega([\omega(\beta), \omega(\alpha)])) = [\omega(\beta), \omega(\alpha)]$$

(by Proposition 5.2 **(c)**, applied to $\gamma = [\omega(\beta), \omega(\alpha)]$). This proves Proposition 5.2 **(b)**. □

The notion of $\omega(\alpha)$ gives rise to a simple formula for the antipode S of the Hopf algebra QSym in terms of its fundamental basis:

Proposition 5.3. Let α be a composition. Then, $S(F_\alpha) = (-1)^{|\alpha|} F_{\omega(\alpha)}$.

This is proven in [GriRei15, Proposition 5.2.15].

We now state the main result of this note:

Theorem 5.4. Let $f \in \operatorname{QSym}$ and let m be a positive integer. For any two compositions α and β, define a composition $\alpha \odot \beta$ as in Proposition 3.8. Then,

$$h_m \prec f = \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_{\alpha \odot (m)} R^{\perp}_{\omega(\alpha)} f.$$

(Here, the sum on the right hand side converges, because all but finitely many compositions α satisfy $R^{\perp}_{\omega(\alpha)} f = 0$ for degree reasons.)

The proof is based on the following simple lemma:

Lemma 5.5. Let $a \in \operatorname{QSym}$ and $f \in \operatorname{QSym}$. Then,

$$\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} (F_\alpha \mathbin{\Phi} a) R^{\perp}_{\omega(\alpha)} f = a \prec f.$$

Proof of Lemma 5.5. The basis $(F_\alpha)_{\alpha \in \operatorname{Comp}}$ of QSym and the basis $(R_\alpha)_{\alpha \in \operatorname{Comp}}$ of NSym are dual bases. Thus,

$$\sum_{\alpha \in \operatorname{Comp}} F_\alpha (R_\alpha, g) = g \qquad \text{for every } g \in \operatorname{QSym}. \tag{52}$$

Let us use Sweedler's notation. The map $\operatorname{Comp} \to \operatorname{Comp}$, $\alpha \mapsto \omega(\alpha)$ is a bijection (since $\omega(\omega(\alpha)) = \alpha$ for any composition α). Hence, we can substitute $\omega(\alpha)$ for α in the sum $\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} (F_\alpha \mathbin{\Phi} a) R^\perp_{\omega(\alpha)} f$. We thus obtain

$$\begin{aligned}
&\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} (F_\alpha \mathbin{\Phi} a) R^\perp_{\omega(\alpha)} f \\
&= \sum_{\alpha \in \operatorname{Comp}} \underbrace{(-1)^{|\omega(\alpha)|}}_{\substack{=(-1)^{|\alpha|} \\ (\text{since } |\omega(\alpha)|=|\alpha|)}} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \underbrace{R^\perp_{\omega(\omega(\alpha))}}_{\substack{=R^\perp_\alpha \\ (\text{since } \omega(\omega(\alpha))=\alpha)}} f \\
&= \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \underbrace{R^\perp_\alpha f}_{\substack{=\sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} \\ (\text{by (35)})}} \\
&= \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} \\
&= \sum_{(f)} \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \left(R_\alpha, f_{(1)}\right) f_{(2)} \\
&= \sum_{(f)} \left(\left(\sum_{\alpha \in \operatorname{Comp}} \underbrace{(-1)^{|\alpha|} F_{\omega(\alpha)}}_{\substack{=S(F_\alpha) \\ (\text{by Proposition 5.3})}} \left(R_\alpha, f_{(1)}\right) \right) \mathbin{\Phi} a \right) f_{(2)} \\
&= \sum_{(f)} \left(\left(\sum_{\alpha \in \operatorname{Comp}} S(F_\alpha) \left(R_\alpha, f_{(1)}\right) \right) \mathbin{\Phi} a \right) f_{(2)} \\
&= \sum_{(f)} \left(S \left(\underbrace{\sum_{\alpha \in \operatorname{Comp}} F_\alpha \left(R_\alpha, f_{(1)}\right)}_{\substack{=f_{(1)} \\ (\text{by (52), applied to } g=f_{(1)})}} \right) \mathbin{\Phi} a \right) f_{(2)} = \sum_{(f)} \left(S\left(f_{(1)}\right) \mathbin{\Phi} a \right) f_{(2)} = a \prec f
\end{aligned}$$

(by Theorem 3.7, applied to $b = f$). This proves Lemma 5.5. □

Proof of Theorem 5.4. We have

$$\sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} \underbrace{F_{\alpha\odot(m)}}_{\substack{=F_\alpha \,\Phi\, h_m \\ \text{(by (17))}}} R_{\omega(\alpha)}^{\perp} f$$
$$= \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} (F_\alpha \,\Phi\, h_m) R_{\omega(\alpha)}^{\perp} f = h_m \prec f$$

(by Lemma 5.5, applied to $a = h_m$). This proves Theorem 5.4. □

As a consequence, we obtain the following result, conjectured by Mike Zabrocki (private correspondence):

Corollary 5.6. For every positive integer m, define a $\mathbf{k}$-linear operator $\mathbf{W}_m$: QSym $\to$ QSym by

$$\mathbf{W}_m = \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_{\alpha\odot(m)} R_{\omega(\alpha)}^{\perp}$$

(where $F_{\alpha\odot(m)}$ means left multiplication by $F_{\alpha\odot(m)}$). Then, every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$\mathfrak{S}_\alpha^* = (\mathbf{W}_{\alpha_1} \circ \mathbf{W}_{\alpha_2} \circ \cdots \circ \mathbf{W}_{\alpha_\ell})(1).$$

Proof of Corollary 5.6. For every positive integer m and every $f \in$ QSym, we have

$$\mathbf{W}_m f = \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_{\alpha\odot(m)} R_{\omega(\alpha)}^{\perp} f = h_m \prec f \qquad \text{(by Theorem 5.4)}.$$

Hence, by induction, for every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we have

$$\mathbf{W}_{\alpha_1}(\mathbf{W}_{\alpha_2}(\cdots(\mathbf{W}_{\alpha_\ell}(1))\cdots)) = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1)\cdots)) = \mathfrak{S}_\alpha^*$$

(by Corollary 4.7). In other words,

$$\mathfrak{S}_\alpha^* = \mathbf{W}_{\alpha_1}(\mathbf{W}_{\alpha_2}(\cdots(\mathbf{W}_{\alpha_\ell}(1))\cdots)) = (\mathbf{W}_{\alpha_1} \circ \mathbf{W}_{\alpha_2} \circ \cdots \circ \mathbf{W}_{\alpha_\ell})(1).$$

This proves Corollary 5.6. □

Let us finish this section with two curiosities: two analogues of Theorem 5.4, one of which can be viewed as an "$m = 0$ version" and the other as a "negative m version". We begin with the "$m = 0$ one", as it is the easier one to state:

Proposition 5.7. Let $f \in$ QSym. Then,

$$\varepsilon(f) = \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R_{\omega(\alpha)}^{\perp} f.$$

Proof of Proposition 5.7. Let us use Sweedler's notation. The map

$$\operatorname{Comp} \to \operatorname{Comp}, \ \alpha \mapsto \omega(\alpha)$$

is a bijection (since $\omega(\omega(\alpha)) = \alpha$ for any composition α). Hence, we can substitute $\omega(\alpha)$ for α in the sum $\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} f$. We thus obtain

$$\begin{aligned}
&\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} f \\
&= \sum_{\alpha \in \operatorname{Comp}} \underbrace{(-1)^{|\omega(\alpha)|}}_{\substack{=(-1)^{|\alpha|} \\ (\text{since } |\omega(\alpha)|=|\alpha|)}} F_{\omega(\alpha)} \underbrace{R^\perp_{\omega(\omega(\alpha))}}_{\substack{=R^\perp_\alpha \\ (\text{since } \omega(\omega(\alpha))=\alpha)}} f \\
&= \sum_{\alpha \in \operatorname{Comp}} \underbrace{(-1)^{|\alpha|} F_{\omega(\alpha)}}_{\substack{=S(F_\alpha) \\ (\text{by Proposition 5.3})}} \underbrace{R^\perp_\alpha f}_{\substack{=\sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} \\ (\text{by (35)})}} \\
&= \sum_{\alpha \in \operatorname{Comp}} S(F_\alpha) \sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} = \sum_{\alpha \in \operatorname{Comp}} \sum_{(f)} S(F_\alpha) \left(R_\alpha, f_{(1)}\right) f_{(2)} \\
&= \sum_{(f)} \left(\sum_{\alpha \in \operatorname{Comp}} S(F_\alpha) \left(R_\alpha, f_{(1)}\right) \right) f_{(2)} = \sum_{(f)} S \left(\underbrace{\sum_{\alpha \in \operatorname{Comp}} F_\alpha \left(R_\alpha, f_{(1)}\right)}_{\substack{=f_{(1)} \\ (\text{by (52), applied to } g=f_{(1)})}} \right) f_{(2)} \\
&= \sum_{(f)} S\left(f_{(1)}\right) f_{(2)} = \varepsilon(f)
\end{aligned}$$

(by one of the defining properties of the antipode). This proves Proposition 5.7. ☐

The "negative m" analogue is less obvious:[38]

Proposition 5.8. Let $f \in \operatorname{QSym}$ and let m be a positive integer. For any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we define an element $F_\alpha^{\backslash m}$ of QSym as follows:

- If $\ell = 0$ or $\alpha_\ell < m$, then $F_\alpha^{\backslash m} = 0$.
- If $\alpha_\ell = m$, then $F_\alpha^{\backslash m} = F_{(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})}$.

[38] Proposition 5.8 does not literally involve a negative m, but it involves an element $F_\alpha^{\backslash m}$ which can be viewed as "something like $F_{(\alpha) \odot (-m)}$".

- If $\alpha_\ell > m$, then $F_\alpha^{\backslash m} = F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell - m)}$.

(Here, any equality or inequality in which α_ℓ is mentioned is understood to include the statement that $\ell > 0$.)

Then,

$$(-1)^m \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_\alpha^{\backslash m} R^\perp_{\omega(\alpha)} f = \varepsilon \left(R^\perp_{(1^m)} f \right).$$

Here, (1^m) denotes the composition $\left(\underbrace{1, 1, \ldots, 1}_{m \text{ times}} \right)$.

Proof of Proposition 5.8. Let us first make some auxiliary observations.

Any two elements a and b of NSym satisfy

$$(ab)^\perp = b^\perp \circ a^\perp \tag{53}$$

[39].

For every two compositions α and β, we define a composition $[\alpha, \beta]$ by $[\alpha, \beta] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$, where α and β are written as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. We further define a composition $\alpha \odot \beta$ as in Proposition 3.8. Then, every two nonempty compositions α and β satisfy

$$R_\alpha R_\beta = R_{[\alpha,\beta]} + R_{\alpha \odot \beta}. \tag{54}$$

(This is part of [GriRei15, Theorem 5.4.10(c)].) Now it is easy to see that

$$R_{\omega([\alpha,(m)])} + R_{\omega(\alpha \odot (m))} = R_{(1^m)} R_{\omega(\alpha)} \tag{55}$$

---

[39] *Proof of (53):* Let a and b be two elements of NSym. Let $c \in \operatorname{QSym}$. Then,

$$\begin{aligned}
(ab)^\perp c &= \sum_{(c)} \underbrace{\left(ab, c_{(1)} \right)}_{\substack{= \sum_{(c_{(1)})} \left(a, (c_{(1)})_{(1)}\right) \left(b, (c_{(1)})_{(2)}\right) \\ \text{(by (33), applied to } c_{(1)} \text{ instead of } c)}} c_{(2)} \qquad \text{(by (35), applied to } g = ab \text{ and } f = c) \\
&= \sum_{(c)} \sum_{(c_{(1)})} \left(a, \left(c_{(1)}\right)_{(1)} \right) \left(b, \left(c_{(1)}\right)_{(2)} \right) c_{(2)} = \sum_{(c)} \sum_{(c_{(2)})} \left(a, c_{(1)} \right) \left(b, \left(c_{(2)}\right)_{(1)} \right) \left(c_{(2)}\right)_{(2)} \\
&\qquad \left(\begin{array}{c} \text{since the coassociativity of } \Delta \text{ yields} \\ \sum_{(c)} \sum_{(c_{(1)})} \left(c_{(1)}\right)_{(1)} \otimes \left(c_{(1)}\right)_{(2)} \otimes c_{(2)} = \sum_{(c)} \sum_{(c_{(2)})} c_{(1)} \otimes \left(c_{(2)}\right)_{(1)} \otimes \left(c_{(2)}\right)_{(2)} \end{array} \right) \\
&= \sum_{(c)} \left(a, c_{(1)} \right) \sum_{(c_{(2)})} \left(b, \left(c_{(2)}\right)_{(1)} \right) \left(c_{(2)}\right)_{(2)}.
\end{aligned}$$

for every nonempty composition α [40]. Hence, for every nonempty composition α, we have

$$\left(\underbrace{R_{\omega([\alpha,(m)])}+R_{\omega(\alpha\odot(m))}}_{=R_{(1^m)}R_{\omega(\alpha)}}\right)^{\perp}=\left(R_{(1^m)}R_{\omega(\alpha)}\right)^{\perp}=R_{\omega(\alpha)}^{\perp}\circ R_{(1^m)}^{\perp}\qquad(56)$$

(by (53), applied to $a=R_{(1^m)}$ and $b=R_{\omega(\alpha)}$).

We furthermore notice that $\omega(\varnothing)=\varnothing$ and thus $R_{\omega(\varnothing)}^{\perp}=R_{\varnothing}^{\perp}=\mathrm{id}$ (since $R_{\varnothing}=1$).

Compared with

$$\begin{aligned}\left(b^{\perp}\circ a^{\perp}\right)(c)&=b^{\perp}\left(\underbrace{a^{\perp}c}_{\substack{=\sum_{(c)}\left(a,c_{(1)}\right)c_{(2)}\\ \text{(by (35), applied to } g=a \text{ and } f=c)}}\right)=b^{\perp}\left(\sum_{(c)}\left(a,c_{(1)}\right)c_{(2)}\right)\\&=\sum_{(c)}\left(a,c_{(1)}\right)\underbrace{b^{\perp}\left(c_{(2)}\right)}_{\substack{=\sum_{\left(c_{(2)}\right)}\left(b,\left(c_{(2)}\right)_{(1)}\right)\left(c_{(2)}\right)_{(2)}\\ \text{(by (35), applied to } g=b \text{ and } f=c_{(2)})}}\qquad\left(\text{since the map } b^{\perp} \text{ is } \mathbf{k}\text{-linear}\right)\\&=\sum_{(c)}\left(a,c_{(1)}\right)\sum_{\left(c_{(2)}\right)}\left(b,\left(c_{(2)}\right)_{(1)}\right)\left(c_{(2)}\right)_{(2)},\end{aligned}$$

this yields $(ab)^{\perp}c=\left(b^{\perp}\circ a^{\perp}\right)(c)$.

Now, let us forget that we fixed c. We thus have shown that $(ab)^{\perp}c=\left(b^{\perp}\circ a^{\perp}\right)(c)$ for every $c\in\mathrm{QSym}$. In other words, $(ab)^{\perp}=b^{\perp}\circ a^{\perp}$. This proves (53).

[40] *Proof of (55):* Let α be a nonempty composition. Proposition 5.2 **(a)** shows that $\omega([\alpha,\beta])=\omega(\beta)\odot\omega(\alpha)$ for every nonempty composition β. Applying this to $\beta=(m)$, we obtain $\omega([\alpha,(m)])=\underbrace{\omega((m))}_{=(1^m)}\odot\omega(\alpha)=(1^m)\odot\omega(\alpha)$. But Proposition 5.2**(b)** shows that $\omega(\alpha\odot\beta)=[\omega(\beta),\omega(\alpha)]$ for every nonempty composition β. Applying this to $\beta=(m)$, we obtain $\omega(\alpha\odot(m))=\left[\underbrace{\omega((m))}_{=(1^m)},\omega(\alpha)\right]=[(1^m),\omega(\alpha)]$. Now,

$$\begin{aligned}R_{\omega([\alpha,(m)])}+R_{\omega(\alpha\odot(m))}&=R_{\omega(\alpha\odot(m))}+R_{\omega([\alpha,(m)])}=R_{[(1^m),\omega(\alpha)]}+R_{(1^m)\odot\omega(\alpha)}\\&\qquad\left(\text{since } \omega(\alpha\odot(m))=[(1^m),\omega(\alpha)] \text{ and } \omega([\alpha,(m)])=(1^m)\odot\omega(\alpha)\right)\\&=R_{(1^m)}R_{\omega(\alpha)}\end{aligned}$$

(since (54) (applied to (1^m) and $\omega(\alpha)$ instead of α and β) shows that $R_{(1^m)}R_{\omega(\alpha)}=R_{[(1^m),\omega(\alpha)]}+R_{(1^m)\odot\omega(\alpha)}$). This proves (55).

Now,

$$\begin{aligned}
&\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\mathrm{Comp};\\ \alpha_\ell=m}} \underbrace{(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|}}_{\substack{=(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m)|}\\ (\text{since } \alpha_\ell=m)}} \underbrace{F^{\backslash m}_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}}_{\substack{=F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})}\\ (\text{since } \alpha_\ell=m)}} \underbrace{R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}}_{\substack{=R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m))}\\ (\text{since } \alpha_\ell=m)}} f \\
&= \underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\mathrm{Comp};\\ \alpha_\ell=m}}}_{=\sum_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})\in\mathrm{Comp}}} \underbrace{(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m)|}}_{=(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})|+m}} F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})} \underbrace{R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m))}}_{\substack{=R^{\perp}_{\omega([(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}),(m)])}\\ (\text{since } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m)\\ =[(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}),(m)])}} f \\
&= \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})\in\mathrm{Comp}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})|+m} F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})} R^{\perp}_{\omega([(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}),(m)])} f \\
&= \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|+m} F_\alpha R^{\perp}_{\omega([\alpha,(m)])} f \\
&\qquad (\text{here, we have substituted } \alpha \text{ for } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}) \text{ in the sum}) \\
&= (-1)^{|\varnothing|+m} F_\varnothing R^{\perp}_{\omega([\varnothing,(m)])} f + \sum_{\substack{\alpha\in\mathrm{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^{\perp}_{\omega([\alpha,(m)])} f
\end{aligned} \tag{57}$$

(here, we have split off the addend for $\alpha = \varnothing$ from the sum). On the other hand,

$$\begin{aligned}
&\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\mathrm{Comp};\\ \alpha_\ell>m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} \underbrace{F^{\backslash m}_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}}_{\substack{=F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell-m)}\\ (\text{since } \alpha_\ell>m)}} R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))} f \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\mathrm{Comp};\\ \alpha_\ell>m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell-m)} R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))} f \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\mathrm{Comp};\\ \ell>0}} \underbrace{(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell+m)|}}_{=(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|+m}} F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)} \\
&\qquad \underbrace{R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell+m))}}_{\substack{=R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell)\odot(m))}\\ (\text{since } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell+m)=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\odot(m))}} f \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } (\alpha_1,\alpha_2,\ldots,\alpha_\ell) \\ \text{for } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell-m) \text{ in the sum}\end{array}\right) \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\mathrm{Comp};\\ \ell>0}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|+m} F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)} R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell)\odot(m))} f \\
&= \sum_{\substack{\alpha\in\mathrm{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^{\perp}_{\omega(\alpha\odot(m))} f
\end{aligned} \tag{58}$$

(here, we have substituted α for $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ in the sum).

But

$$\begin{aligned}
&\sum_{\alpha \in \mathrm{Comp}} (-1)^{|\alpha|} F_\alpha^{\backslash m} R_{\omega(\alpha)}^{\perp} f \\
&= \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \mathrm{Comp}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m} R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp} f \\
&\qquad \text{(here, we have renamed the summation index } \alpha \text{ as } (\alpha_1, \alpha_2, \ldots, \alpha_\ell)) \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \mathrm{Comp}; \\ \ell=0 \text{ or } \alpha_\ell < m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} \underbrace{F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m}}_{\substack{=0 \\ (\text{since } \ell=0 \text{ or } \alpha_\ell<m)}} R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp} f \\
&\qquad + \underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \mathrm{Comp}; \\ \alpha_\ell = m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m} R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp} f}_{\substack{=(-1)^{|\varnothing|+m} F_\varnothing R_{\omega([\varnothing,(m)])}^{\perp} f + \sum_{\substack{\alpha \in \mathrm{Comp}; \\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R_{\omega([\alpha,(m)])}^{\perp} f \\ (\text{by } (57))}} \\
&\qquad + \underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \mathrm{Comp}; \\ \alpha_\ell > m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m} R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp} f}_{\substack{= \sum_{\substack{\alpha \in \mathrm{Comp}; \\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R_{\omega(\alpha \odot (m))}^{\perp} f \\ (\text{by } (58))}} \\
&= \underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \mathrm{Comp}; \\ \ell=0 \text{ or } \alpha_\ell < m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} 0 R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp} f}_{=0} \\
&\qquad + (-1)^{|\varnothing|+m} F_\varnothing R_{\omega([\varnothing,(m)])}^{\perp} f + \sum_{\substack{\alpha \in \mathrm{Comp}; \\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R_{\omega([\alpha,(m)])}^{\perp} f \\
&\qquad + \sum_{\substack{\alpha \in \mathrm{Comp}; \\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R_{\omega(\alpha \odot (m))}^{\perp} f
\end{aligned}$$

$$= (-1)^{|\varnothing|+m} F_\varnothing R^\perp_{\omega([\varnothing,(m)])} f + \sum_{\substack{\alpha\in\mathrm{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^\perp_{\omega([\alpha,(m)])} f$$

$$+ \sum_{\substack{\alpha\in\mathrm{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^\perp_{\omega(\alpha\odot(m))} f$$

$$= (-1)^{|\varnothing|+m} F_\varnothing \underbrace{R^\perp_{\omega([\varnothing,(m)])}}_{\substack{=R^\perp_{(1^m)} \\ (\text{since } \omega([\varnothing,(m)])=\omega((m))=(1^m))}} f$$

$$+ \sum_{\substack{\alpha\in\mathrm{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha \underbrace{\left(R_{\omega([\alpha,(m)])} + R_{\omega(\alpha\odot(m))}\right)^\perp}_{\substack{=R^\perp_{\omega(\alpha)}\circ R^\perp_{(1^m)} \\ (\text{by (56)})}} f$$

$$= (-1)^{|\varnothing|+m} F_\varnothing \underbrace{R^\perp_{(1^m)} f}_{\substack{=R^\perp_{\omega(\varnothing)}\left(R^\perp_{(1^m)} f\right) \\ (\text{since } R^\perp_{\omega(\varnothing)}=\mathrm{id} \text{ and thus} \\ R^\perp_{\omega(\varnothing)}\left(R^\perp_{(1^m)} f\right)=R^\perp_{(1^m)} f)}} + \sum_{\substack{\alpha\in\mathrm{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha \underbrace{\left(R^\perp_{\omega(\alpha)} \circ R^\perp_{(1^m)}\right) f}_{=R^\perp_{\omega(\alpha)}\left(R^\perp_{(1^m)} f\right)}$$

$$= (-1)^{|\varnothing|+m} F_\varnothing R^\perp_{\omega(\varnothing)} \left(R^\perp_{(1^m)} f\right) + \sum_{\substack{\alpha\in\mathrm{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right)$$

$$= \sum_{\alpha\in\mathrm{Comp}} \underbrace{(-1)^{|\alpha|+m}}_{=(-1)^m(-1)^{|\alpha|}} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right) \qquad \left(\begin{array}{c}\text{here, we have incorporated the} \\ \alpha = \varnothing \text{ addend into the sum}\end{array}\right)$$

$$= (-1)^m \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right).$$

Multiplying both sides of this equality with $(-1)^m$, we obtain

$$(-1)^m \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} F_\alpha^{\backslash m} R^\perp_{\omega(\alpha)} f = \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right).$$

Comparing this with

$$\varepsilon\left(R^\perp_{(1^m)} f\right) = \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right)$$

(by Proposition 5.7, applied to $R^\perp_{(1^m)} f$ instead of f), we obtain

$$(-1)^m \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} F_\alpha^{\backslash m} R^\perp_{\omega(\alpha)} f = \varepsilon\left(R^\perp_{(1^m)} f\right).$$

This proves Proposition 5.8. □

6. Lifts to WQSym and FQSym

We have so far been studying the Hopf algebras Sym, QSym and NSym. These are merely the tip of an iceberg; dozens of combinatorial Hopf algebras are currently known, many of which are extensions of these. In this final section, we shall discuss how (and whether) our operations $\prec$ and Φ as well as some similar operations can be lifted to the bigger Hopf algebras WQSym and FQSym. We shall give no proofs, as these are not difficult and the whole discussion is tangential to this note.

Let us first define these two Hopf algebras (which are discussed, for example, in [FoiMal14]).

We start with WQSym. (Our definition of WQSym follows the papers of the Marne-la-Vallée school, such as [AFNT13, Section 5.1][41]; it will differ from that in [FoiMal14], but we will explain why it is equivalent.)

Let $X_1, X_2, X_3, \ldots$ be countably many distinct symbols. These symbols will be called *letters*. We define a *word* to be an ℓ-tuple of elements of $\{X_1, X_2, X_3, \ldots\}$ for some $\ell \in \mathbb{N}$. Thus, for example, (X_3, X_5, X_2) and (X_6) are words. We denote the empty word $()$ by 1, and we often identify the one-letter word (X_i) with the symbol X_i for every $i > 0$. For any two words $u = \left(X_{i_1}, X_{i_2}, \ldots, X_{i_n}\right)$ and $v = \left(X_{j_1}, X_{j_2}, \ldots, X_{j_m}\right)$, we define the concatenation uv as the word $\left(X_{i_1}, X_{i_2}, \ldots, X_{i_n}, X_{j_1}, X_{j_2}, \ldots, X_{j_m}\right)$. Concatenation is an associative operation and the empty word 1 is a neutral element for it; thus, the words form a monoid. We let Wrd denote this monoid. This monoid is the free monoid on the set $\{X_1, X_2, X_3, \ldots\}$. Concatenation allows us to rewrite any word $\left(X_{i_1}, X_{i_2}, \ldots, X_{i_n}\right)$ in the shorter form $X_{i_1} X_{i_2} \cdots X_{i_n}$.

Notice that Mon (the set of all monomials) is also a monoid under multiplication. We can thus define a monoid homomorphism $\pi : \mathrm{Wrd} \to \mathrm{Mon}$ by $\pi(X_i) = x_i$ for all $i \in \{1, 2, 3, \ldots\}$. This homomorphism π is surjective.

We define $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ to be the $\mathbf{k}$-module $\mathbf{k}^{\mathrm{Wrd}}$; its elements are all families $(\lambda_w)_{w \in \mathrm{Wrd}} \in \mathbf{k}^{\mathrm{Wrd}}$. We define a multiplication on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ by

$$(\lambda_w)_{w \in \mathrm{Wrd}} \cdot (\mu_w)_{w \in \mathrm{Wrd}} = \left(\sum_{(u,v) \in \mathrm{Wrd}^2;\ uv = w} \lambda_u \mu_v \right)_{w \in \mathrm{Wrd}} . \tag{59}$$

This makes $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ into a $\mathbf{k}$-algebra, with unity $(\delta_{w,1})_{w \in \mathrm{Wrd}}$. This $\mathbf{k}$-algebra is called the *$\mathbf{k}$-algebra of noncommutative power series in $X_1, X_2, X_3, \ldots$*. For every $u \in \mathrm{Wrd}$, we identify the word u with the element $(\delta_{w,u})_{w \in \mathrm{Wrd}}$ of $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ [42]. The $\mathbf{k}$-algebra $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ becomes a topological $\mathbf{k}$-algebra via the product topology (recalling that $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle = \mathbf{k}^{\mathrm{Wrd}}$ as sets). Thus, every element $(\lambda_w)_{w \in \mathrm{Wrd}}$ of $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ can be rewritten in the form $\sum_{w \in \mathrm{Wrd}} \lambda_w w$. This turns the equality (59) into

[41] where WQSym is denoted by **WQSym**

[42] This identification is harmless, since the map $\mathrm{Wrd} \to \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$, $u \mapsto (\delta_{w,u})_{w \in \mathrm{Wrd}}$ is a monoid homomorphism from Wrd to $(\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle, \cdot)$. (However, it fails to be injective if $\mathbf{k} = 0$.)

a distributive law (for infinite sums), and explains why we refer to elements of $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ as "noncommutative power series". We think of words as noncommutative analogues of monomials.

The *degree* of a word w will mean its length (i.e., the integer n for which w is an n-tuple). Let $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle_{\mathrm{bdd}}$ denote the $\mathbf{k}$-subalgebra of $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ formed by the *bounded-degree noncommutative power series*[43] in $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$. The surjective monoid homomorphism $\pi : \mathrm{Wrd} \to \mathrm{Mon}$ canonically gives rise to surjective $\mathbf{k}$-algebra homomorphisms $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle \to \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ and $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle_{\mathrm{bdd}} \to \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]_{\mathrm{bdd}}$, which we also denote by π. Notice that the $\mathbf{k}$-algebra $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle_{\mathrm{bdd}}$ is denoted $R\left\langle \mathbf{X}\right\rangle$ in [GriRei15, Section 8.1].

If w is a word, then we denote by $\operatorname{Supp} w$ the subset

$$\left\{i \in \left\{1, 2, 3, \ldots\right\} \mid \text{the symbol } X_i \text{ is an entry of } w\right\}$$

of $\left\{1, 2, 3, \ldots\right\}$. Notice that $\operatorname{Supp} w = \operatorname{Supp}\left(\pi\left(w\right)\right)$ is a finite set.

A word w is said to be *packed* if there exists an $\ell \in \mathbb{N}$ such that $\operatorname{Supp} w = \left\{1, 2, \ldots, \ell\right\}$.

For each word w, we define a packed word $\operatorname{pack} w$ as follows: Replace the smallest letter[44] that appears in w by X_1, the second-smallest letter by X_2, etc..[45] This word $\operatorname{pack} w$ is called the *packing* of w. For example, $\operatorname{pack}\left(X_3 X_1 X_6 X_1\right) = X_2 X_1 X_3 X_1$.

For every packed word u, we define an element $\mathbf{M}_u$ of $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle_{\mathrm{bdd}}$ by

$$\mathbf{M}_u = \sum_{\substack{w \in \mathrm{Wrd}; \\ \operatorname{pack} w = u}} w.$$

(This element $\mathbf{M}_u$ is denoted P_u in [AFNT13, Section 5.1].) We denote by WQSym the $\mathbf{k}$-submodule of $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle_{\mathrm{bdd}}$ spanned by the $\mathbf{M}_u$ for all packed words u. It is known that WQSym is a $\mathbf{k}$-subalgebra of $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle_{\mathrm{bdd}}$ which can furthermore be endowed with a Hopf algebra structure (the so-called *Hopf algebra of word quasisymmetric functions*) such that π restricts to a Hopf algebra surjection $\mathrm{WQSym} \to \mathrm{QSym}$. Notice that $\pi\left(\mathbf{M}_u\right) = M_{\operatorname{Parikh}(\pi(u))}$ for every packed word u, where the Parikh composition $\operatorname{Parikh} \mathfrak{m}$ of any monomial $\mathfrak{m}$ is defined as in the proof of Proposition 3.5.

The elements $\mathbf{M}_u$ with u ranging over all packed words form a basis of the $\mathbf{k}$-module WQSym, which is usually called the *monomial basis*[46]. Furthermore,

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[43] A noncommutative power series $\left(\lambda_w\right)_{w \in \mathrm{Wrd}} \in \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ is said to be *bounded-degree* if there is an $N \in \mathbb{N}$ such that every word w of length $> N$ satisfies $\lambda_w = 0$.

[44] We use the total ordering on the set $\left\{X_1, X_2, X_3, \ldots\right\}$ given by $X_1 < X_2 < X_3 < \cdots$.

[45] Here is a more pedantic way to restate this definition: Write w as $\left(X_{i_1}, X_{i_2}, \ldots, X_{i_\ell}\right)$, and let $I = \operatorname{Supp} w$ (so that $I = \left\{i_1, i_2, \ldots, i_\ell\right\}$). Let r_I be the unique increasing bijection $\left\{1, 2, \ldots, \left|I\right|\right\} \to I$. Then, $\operatorname{pack} w$ denotes the word $\left(X_{r_I^{-1}(i_1)}, X_{r_I^{-1}(i_2)}, \ldots, X_{r_I^{-1}(i_\ell)}\right)$.

[46] Sometimes it is parametrized not by packed words but instead by set compositions (i.e., ordered set partitions) of sets of the form $\left\{1, 2, \ldots, n\right\}$ with $n \in \mathbb{N}$. But the packed words of length n are in a 1-to-1 correspondence with set compositions of $\left\{1, 2, \ldots, n\right\}$, so this is merely a matter of relabelling.

the product of two such elements can be computed by the well-known formula[47]

$$\mathbf{M}_u \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell]) = u;\ \operatorname{pack}(w[\ell:]) = v}} \mathbf{M}_w, \tag{60}$$

where ℓ is the length of u, and where we use the notation $w\,[: \ell]$ for the word formed by the first ℓ letters of w and we use the notation $w\,[\ell :]$ for the word formed by the remaining letters of w. This equality (which should be considered a noncommutative analogue of (7), and can be proven similarly) makes it possible to give an alternative definition of WQSym, by defining WQSym as the free $\mathbf{k}$-module with basis $(\mathbf{M}_u)_{u \text{ is a packed word}}$ and defining multiplication using (60). This is precisely the approach taken in [FoiMal14, Section 1.1].

The Hopf algebra WQSym has also appeared under the name NCQSym ("quasisymmetric functions in noncommuting variables") in [BerZab05, Section 5.2] and other sources.

We now define five binary operations $\prec$, $\circ$, $\succ$, Φ, and $\between$ on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$.

Definition 6.1. (a) We define a binary operation $\prec\ : \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and that it satisfy

$$u \prec v = \begin{cases} uv, & \text{if } \min(\operatorname{Supp} u) < \min(\operatorname{Supp} v); \\ 0, & \text{if } \min(\operatorname{Supp} u) \geq \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

(b) We define a binary operation $\circ : \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and that it satisfy

$$u \circ v = \begin{cases} uv, & \text{if } \min(\operatorname{Supp} u) = \min(\operatorname{Supp} v); \\ 0, & \text{if } \min(\operatorname{Supp} u) \neq \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

(c) We define a binary operation $\succ\ : \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and that it satisfy

$$u \succ v = \begin{cases} uv, & \text{if } \min(\operatorname{Supp} u) > \min(\operatorname{Supp} v); \\ 0, & \text{if } \min(\operatorname{Supp} u) \leq \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

[47] This formula appears in [MeNoTh11, Proposition 4.1].

(d) We define a binary operation $\Phi : \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and that it satisfy

$$u \Phi v = \begin{cases} uv, & \text{if } \max(\operatorname{Supp} u) \leq \min(\operatorname{Supp} v); \\ 0, & \text{if } \max(\operatorname{Supp} u) > \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

(e) We define a binary operation $\divideontimes : \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and that it satisfy

$$u \divideontimes v = \begin{cases} uv, & \text{if } \max(\operatorname{Supp} u) < \min(\operatorname{Supp} v); \\ 0, & \text{if } \max(\operatorname{Supp} u) \geq \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

The first three of these five operations are closely related to those defined by Novelli and Thibon in [NovThi05a]; the main difference is the use of minima instead of maxima in our definitions.

The operations $\prec$, Φ and $\divideontimes$ on WQSym lift the operations $\prec$, Φ and $\divideontimes$ on QSym. More precisely, any $a \in \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and $b \in \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ satisfy

$$\pi(a) \prec \pi(b) = \pi(a \prec b) = \pi(b \succ a);$$
$$\pi(a) \Phi \pi(b) = \pi(a \Phi b);$$
$$\pi(a) \divideontimes \pi(b) = \pi(a \divideontimes b)$$

(and similar formulas would hold for $\circ$ and $\succ$ had we bothered to define such operations on QSym). Also, using the operation $\succeq$ defined in Remark 3.2, we have

$$\pi(a) \succeq \pi(b) = \pi(a \succ b + a \circ b) \qquad \text{for any } a \in \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \text{ and } b \in \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle.$$

We now have the following analogue of Proposition 3.5:

Proposition 6.2. Every $a \in$ WQSym and $b \in$ WQSym satisfy $a \prec b \in$ WQSym, $a \circ b \in$ WQSym, $a \succ b \in$ WQSym, $a \Phi b \in$ WQSym and $a \divideontimes b \in$ WQSym.

The proof of Proposition 6.2 is easier than that of Proposition 3.5; we omit it here. In analogy to Remark 3.6 and to (60), let us give explicit formulas for these five operations on the basis $(\mathbf{M}_u)_{u \text{ is a packed word}}$ of WQSym:

Remark 6.3. Let u and v be two packed words. Let ℓ be the length of u. Then:

(a) We have

$$\mathbf{M}_u \prec \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \min(\operatorname{Supp}(w[:\ell]))<\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

(b) We have

$$\mathbf{M}_u \circ \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \min(\operatorname{Supp}(w[:\ell]))=\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

(c) We have

$$\mathbf{M}_u \succ \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \min(\operatorname{Supp}(w[:\ell]))>\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

(d) We have

$$\mathbf{M}_u \mathbin{\phi} \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \max(\operatorname{Supp}(w[:\ell]))\leq\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

The sum on the right hand side consists of two addends (unless u or v is empty), namely $\mathbf{M}_{uv^{+h-1}}$ and $\mathbf{M}_{uv^{+h}}$, where $h = \max(\operatorname{Supp} u)$, and where v^{+j} denotes the word obtained by replacing every letter X_k in v by X_{k+j}.

(e) We have

$$\mathbf{M}_u \mathbin{\divideontimes} \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \max(\operatorname{Supp}(w[:\ell]))<\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

The sum on the right hand side consists of one addend only, namely $\mathbf{M}_{uv^{+h}}$.

Let us now move on to the combinatorial Hopf algebra FQSym, which is known as the *Malvenuto-Reutenauer Hopf algebra* or the *Hopf algebra of free quasi-symmetric functions*. We shall define it as a Hopf subalgebra of WQSym. This is not identical to the definition in [GriRei15, Section 8.1], but equivalent to it.

For every $n \in \mathbb{N}$, we let $\mathfrak{S}_n$ be the symmetric group on the set $\{1, 2, \ldots, n\}$. (This notation is identical with that in [GriRei15]. It has nothing to do with the $\mathfrak{S}_\alpha$ from [BBSSZ13a].) We let $\mathfrak{S}$ denote the disjoint union $\bigsqcup_{n \in \mathbb{N}} \mathfrak{S}_n$. We identify permutations in $\mathfrak{S}$ with certain words – namely, every permutation $\pi \in \mathfrak{S}$ is identified with the word $\left(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}\right)$, where n is such that $\pi \in \mathfrak{S}_n$. The words thus identified with permutations in $\mathfrak{S}$ are precisely the packed words which do not have repeated elements.

For every word w, we define a word $\operatorname{std} w \in \mathfrak{S}$ as follows: Write w in the form $\left(X_{i_1}, X_{i_2}, \ldots, X_{i_n}\right)$. Then, $\operatorname{std} w$ shall be the unique permutation $\pi \in \mathfrak{S}_n$ such that, whenever u and v are two elements of $\{1, 2, \ldots, n\}$ satisfying $u < v$, we have $(\pi(u) < \pi(v)$ if and only if $i_u \leq i_v)$. Equivalently (and less formally),

$\operatorname{std} w$ is the word which is obtained by

- replacing the leftmost smallest letter of w by X_1, and marking it as "processed";
- then replacing the leftmost smallest letter of w that is not yet processed by X_2, and marking it as "processed";
- then replacing the leftmost smallest letter of w that is not yet processed by X_3, and marking it as "processed";
- etc., until all letters of w are processed.

For instance, $\operatorname{std}(X_3X_5X_2X_3X_2X_3) = X_3X_6X_1X_4X_2X_5$ (which, regarded as permutation, is the permutation written in one-line notation as $(3,6,1,4,2,5)$).

We call $\operatorname{std} w$ the *standardization* of w.

Now, for every $\sigma \in \mathfrak{S}$, we define an element $\mathbf{G}_\sigma \in \mathrm{WQSym}$ by

$$\mathbf{G}_\sigma = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{std} w = \sigma}} \mathbf{M}_w = \sum_{\substack{w \in \mathrm{Wrd}; \\ \operatorname{std} w = \sigma}} w.$$

(The second equality sign can easily be checked.) Then, the $\mathbf{k}$-submodule of WQSym spanned by $(\mathbf{G}_\sigma)_{\sigma\in\mathfrak{S}}$ turns out to be a Hopf subalgebra, with basis $(\mathbf{G}_\sigma)_{\sigma\in\mathfrak{S}}$. This Hopf subalgebra is denoted by FQSym. This definition is not identical with the one given in [GriRei15, Section 8.1]; however, it gives an isomorphic Hopf algebra, as our $\mathbf{G}_\sigma$ correspond to the images of the G_σ introduced in [GriRei15, Section 8.1] under the embedding $\mathrm{FQSym} \to R\left\langle \{X_i\}_{i\in I}\right\rangle$ also defined therein.

Only two of the five operations $\prec$, $\circ$, $\succ$, Φ, and $⋇$ defined in Definition 6.1 can be restricted to binary operations on FQSym:

Proposition 6.4. Every $a \in \mathrm{FQSym}$ and $b \in \mathrm{FQSym}$ satisfy $a \succ b \in \mathrm{FQSym}$ and $a \Phi b \in \mathrm{FQSym}$.

Moreover, we have the following explicit formulas on the basis $(\mathbf{G}_\sigma)_{\sigma\in\mathfrak{S}}$:

Remark 6.5. Let $\sigma \in \mathfrak{S}$ and $\tau \in \mathfrak{S}$. Let ℓ be the length of σ (so that $\sigma \in \mathfrak{S}_\ell$).

(a) We have

$$\mathbf{G}_\sigma \succ \mathbf{G}_\tau = \sum_{\substack{\pi\in\mathfrak{S}; \\ \operatorname{std}(\pi[:\ell])=\sigma;\ \operatorname{std}(\pi[\ell:])=\tau; \\ \min(\operatorname{Supp}(\pi[:\ell]))>\min(\operatorname{Supp}(\pi[\ell:]))}} \mathbf{G}_\pi.$$

(b) We have

$$\mathbf{G}_\sigma \Phi \mathbf{G}_\tau = \sum_{\substack{\pi\in\mathfrak{S}; \\ \operatorname{std}(\pi[:\ell])=\sigma;\ \operatorname{std}(\pi[\ell:])=\tau; \\ \max(\operatorname{Supp}(\pi[:\ell]))\leq\min(\operatorname{Supp}(\pi[\ell:]))}} \mathbf{G}_\pi.$$

The sum on the right hand side consists of one addend only, namely $\mathbf{G}_{\sigma\tau^{+\ell}}$.

The statements of Remark 6.5 can be easily derived from Remark 6.3. The proof for **(a)** rests on the following simple observations:

- Every word w satisfies $\operatorname{std}(\operatorname{pack} w) = \operatorname{std} w$.
- Every $n \in \mathbb{N}$, every word w of length n and every $\ell \in \{0, 1, \ldots, n\}$ satisfy
$$\operatorname{std}((\operatorname{std} w)[: \ell]) = \operatorname{std}(w[: \ell]) \qquad \text{and} \qquad \operatorname{std}((\operatorname{std} w)[\ell :]) = \operatorname{std}(w[\ell :]).$$
- Every $n \in \mathbb{N}$, every word w of length n and every $\ell \in \{0, 1, \ldots, n\}$ satisfy the equivalence
$$\begin{aligned} &(\min(\operatorname{Supp}(w[: \ell])) > \min(\operatorname{Supp}(w[\ell :]))) \\ \Longleftrightarrow \ &(\min(\operatorname{Supp}((\operatorname{std} w)[: \ell])) > \min(\operatorname{Supp}((\operatorname{std} w)[\ell :]))). \end{aligned}$$

The third of these three observations would fail if the greater sign were to be replaced by a smaller sign; this is essentially why $\operatorname{FQSym} \subseteq \operatorname{WQSym}$ is not closed under $\prec$.

The operation $\succ$ on FQSym defined above is closely related to the operation $\succ$ on FQSym introduced by Foissy in [Foissy07, Section 4.2]. Indeed, the latter differs from the former in the use of max instead of min.

7. Epilogue

We have introduced five binary operations $\prec$, $\circ$, $\succ$, Φ, and $\divideontimes$ on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and their restrictions to QSym; we have further introduced five analogous operations on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and their restrictions to WQSym (as well as the restrictions of two of them to FQSym). We have used these operations (specifically, $\prec$ and Φ) to prove a formula (Corollary 5.6) for the dual immaculate functions $\mathfrak{S}^*_\alpha$. Along the way, we have found that the $\mathfrak{S}^*_\alpha$ can be obtained by repeated application of the operation $\prec$ (Corollary 4.7). A similar (but much more obvious) result can be obtained for the fundamental quasisymmetric functions: For every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}$, we have

$$F_\alpha = h_{\alpha_1} \divideontimes h_{\alpha_2} \divideontimes \cdots \divideontimes h_{\alpha_\ell} \divideontimes 1$$

(we do not use parentheses here, since $\divideontimes$ is associative). This shows that the $\mathbf{k}$-algebra $(\operatorname{QSym}, \divideontimes)$ is free. Moreover,

$$F_{\omega(\alpha)} = e_{\alpha_\ell} \Phi e_{\alpha_{\ell-1}} \Phi \cdots \Phi e_{\alpha_1} \Phi 1,$$

where e_m stands for the m-th elementary symmetric function; thus, the $\mathbf{k}$-algebra $(\operatorname{QSym}, \Phi)$ is also free.[48] (Incidentally, this shows that $S(a \divideontimes b) = S(b) \Phi S(a)$ for any $a, b \in \operatorname{QSym}$. But this does not hold for $a, b \in \operatorname{WQSym}$.)

[48] We owe these two observations to the referee.

One might wonder what "functions" can be similarly constructed using the operations $\prec$, $\circ$, $\succ$, Φ, and $\divideontimes$ in WQSym, using the noncommutative analogues $H_m = \sum_{i_1 \leq i_2 \leq \cdots \leq i_m} X_{i_1} X_{i_2} \cdots X_{i_m} = \mathbf{G}_{(1,2,\ldots,m)}$ and $E_m = \sum_{i_1 > i_2 > \cdots > i_m} X_{i_1} X_{i_2} \cdots X_{i_m} = \mathbf{G}_{(m,m-1,\ldots,1)}$ of h_m and e_m. (These analogues actually live in NSym, where NSym is embedded into FQSym as in [GriRei15, Corollary 8.1.14(b)]; but the operations do not preserve NSym, and only two of them preserve FQSym.) However, it seems somewhat tricky to ask the right questions here; for instance, the $\mathbf{k}$-linear span of the $\succ$-closure of $\{H_m \mid m \geq 0\}$ is not a $\mathbf{k}$-subalgebra of FQSym (since $H_2 H_1$ is not a $\mathbf{k}$-linear combination of H_3, $H_1 \succ (H_1 \succ H_1)$, $(H_1 \succ H_1) \succ H_1$, $H_1 \succ H_2$ and $H_2 \succ H_1$).

On the other hand, one might also try to write down the set of identities satisfied by the operations $\cdot$, $\prec$, $\circ$, $\succeq$, Φ and $\divideontimes$ on the various spaces ($\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, QSym, $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$, WQSym and FQSym), or by subsets of these operations; these identities could then be used to define new operads, i.e., algebraic structures comprising a $\mathbf{k}$-module and some operations on it that imitate (some of) the operations $\cdot$, $\prec$, $\circ$, $\succeq$, Φ and $\divideontimes$. For instance, apart from being associative, the operations Φ and $\divideontimes$ on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ satisfy the identity

$$(a \Phi b) \divideontimes c + (a \divideontimes b) \Phi c = a \Phi (b \divideontimes c) + a \divideontimes (b \Phi c) \tag{61}$$

for all $a, b, c \in \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$. This follows from the (easily verified) identities

$$(a \Phi b) \divideontimes c - a \Phi (b \divideontimes c) = \varepsilon(b)(a \divideontimes c - a \Phi c);$$
$$(a \divideontimes b) \Phi c - a \divideontimes (b \Phi c) = \varepsilon(b)(a \Phi c - a \divideontimes c),$$

where $\varepsilon : \mathbf{k}\langle\langle \mathbf{X} \rangle\rangle \to \mathbf{k}$ is the map which sends every noncommutative power series to its constant term. The equality (61) (along with the associativity of Φ and $\divideontimes$) makes $(\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle, \Phi, \divideontimes)$ into what is called an *$As^{\langle 2 \rangle}$-algebra* (see [Zinbie10, p. 39]). Is QSym or WQSym a free $As^{\langle 2 \rangle}$-algebra? What if we add the existence of a common neutral element for the operations Φ and $\divideontimes$ to the axioms of this operad?

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