# The Clifford algebra and the Chevalley map - a computational approach (detailed version[1])

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Version 0.6 (3 June 2016). Not proofread!

## 1. Introduction: the Clifford algebra

The theory of the Clifford algebra of a vector space with a given symmetric bilinear form is rather well-understood: One of the basic properties of the Clifford algebra gives an explicit basis for it in terms of a basis of the underlying vector space (Theorem 1 below), and another one provides a canonical vector space isomorphism between the Clifford algebra and the exterior algebra of the same vector space (the so-called Chevalley map, Theorem 2 below). While both of these properties appear in standard literature such as [1] and [2], sadly I have never seen them proven in the generality they deserve: first, the bilinear form needs not be symmetric[2]. Besides, the properties still hold over arbitrary commutative rings rather than just fields of characteristic 0. The proofs given in literature are usually not sufficient to cover these general cases. Here we are going to present a computational proof of both of these properties, giving integral[3] recursive formulas for the vector space isomorphism[4] between the Clifford algebra and the exterior algebra (in both directions).

> **Remark (added in 2016).** As I now know, most of what is done in this paper is not new. In particular, its main results already appear in §9 of Chapter IX of [7][5]; they also (essentially) appear in Chapter 2 of [8][6]; the main ideas also appear in (1.7) of Chapter IV of [9][7]. Moreover, the proofs given in [7], in [8] and in [9] are essentially the same as ours. (Moreover, similar ideas and a variant of our map $\alpha^f$ have been used for different purposes in [10].) The results in Sections 11–14 of this paper might still be new.

First, let us define everything in maximal generality:

> **Definition 1.** In this note, a *ring* will always mean a ring with 1. If $k$ is a ring, a $k$*-algebra* will mean a (not necessarily commutative) $k$-algebra with 1. Sometimes we will use the word "algebra" as an abbreviation for

[1] This is a version including all the proofs of the results given in [0]. While it is self-contained and detailed, I would recommend any reader to read the (much shorter) summary [0] and consult this detailed version only in case of unclarity.

[2] although this is not a substantial generalization as long as we are working over a field $k$ with characteristic $\neq 2$

[3] in the sense of: no division by $k$!

[4] or, respectively, module isomorphism if we are working over a commutative ring instead of a field

[5] More precisely: Our Theorem 33 is Proposition 3 in §9 of Chapter IX of [7] (and thus, our Theorem 1 is a consequence of said proposition); our Theorem 2 is a particular case (for $L = \{1, 2, \ldots, n\}$) of Théorème 1 in §9 of Chapter IX of [7].

[6] More precisely, Theorem (2.16) in Chapter 2 of [8] includes both our Theorem 1 and our Theorem 2 in the case when the $k$-module $L$ is finitely generated and projective. But the proof given in [8], as far as it concerns our Theorem (2.16), does not require the "finitely generated and projective" condition.

[7] Thanks to Rainer Schulze-Pillot for making me aware of [9].

"$k$-algebra". If $L$ is a $k$-algebra, then a *left $L$-module* is always supposed to be a left $L$-module on which the unity of $L$ acts as the identity. Whenever we use the tensor product sign $\otimes$ without an index, we mean $\otimes_k$.

**Definition 2.** Let $k$ be a commutative ring. Let $L$ be a $k$-module. A *bilinear form* on $L$ means a bilinear map $f : L \times L \to k$. A bilinear form $f$ on $L$ is said to be *symmetric* if it satisfies $(f(x,y) = f(y,x)$ for any $x \in L$ and $y \in L)$.

**Definition 3.** Let $k$ be a commutative ring. Let $L$ be a $k$-module, and $f : L \times L \to k$ be a bilinear form on $L$. For every $i \in \mathbb{N}$, we define the so-called *$i$-th tensor power* $L^{\otimes i}$ of $L$ to be the $k$-module $\underbrace{L \otimes L \otimes ... \otimes L}_{i \text{ times}}$. The *tensor algebra* $\otimes L$ of $L$ over $k$ is defined to be the algebra $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$, where the multiplication is given by the tensor product. Now, we define the *Clifford algebra* $\operatorname{Cl}(L, f)$ to be the factor algebra $(\otimes L) \diagup I_f$, where $I_f$ is the two-sided ideal

$$(\otimes L) \cdot \langle v \otimes v - f(v,v) \mid v \in L \rangle \cdot (\otimes L)$$

of the algebra $\otimes L$. [8]

*Remark.* We denote by $\mathbf{0}$ the symmetric bilinear form on $L$ defined by

$$(\mathbf{0}(x,y) = 0 \text{ for every } x \in L \text{ and } y \in L).$$

Then,

$$I_{\mathbf{0}} = (\otimes L) \cdot \left\langle v \otimes v - \underbrace{\mathbf{0}(v,v)}_{=0} \mid v \in L \right\rangle \cdot (\otimes L) = (\otimes L) \cdot \langle v \otimes v \mid v \in L \rangle \cdot (\otimes L),$$

and thus $\operatorname{Cl}(L, \mathbf{0}) = (\otimes L) \diagup I_{\mathbf{0}} = (\otimes L) \diagup ((\otimes L) \cdot \langle v \otimes v \mid v \in L \rangle \cdot (\otimes L))$ is the exterior algebra $\wedge L$ of the $k$-module $L$. Hence, the exterior algebra $\wedge L$ is a particular case of the Clifford algebra - namely, it is the Clifford algebra $\operatorname{Cl}(L, \mathbf{0})$.

In general, the Clifford algebra $\operatorname{Cl}(L, f)$ is not isomorphic to the exterior algebra $\wedge L$ as algebra. However, they are isomorphic as $k$-modules, as the following theorem states:

**Theorem 1 (Chevalley map theorem):** Let $k$ be a commutative ring. Let $L$ be a $k$-module, and $f : L \times L \to k$ be a bilinear form on $L$. Then, the $k$-modules $\wedge L$ and $\operatorname{Cl}(L, f)$ are isomorphic.

We are going to prove this theorem by explicitly constructing mutually inverse homomorphisms in both directions. This proof substantially differs from the proofs given in standard literature for the particular case of $k$ being a field of characteristic 0 and $L$ being a finite-dimensional $k$-vector space, which proceed by constructing the isomorphism in one direction and showing either its injectivity or its surjectivity, or

[8] Here, whenever $U$ is a set, and $P : U \to \otimes L$ is a map (not necessarily a linear map), we denote by $\langle P(v) \mid v \in U \rangle$ the $k$-submodule of $\otimes L$ generated by the elements $P(v)$ for all $v \in U$.

proving both using the basis theorem (Theorem 2 below).[9] Using Theorem 1 we will be able to construct a basis for $\mathrm{Cl}(L,f)$ in the case when $L$ has one:

> **Theorem 2 (Clifford basis theorem):** Let $k$ be a commutative ring. Let $L$ be a free $k$-module with a finite basis $(e_1, e_2, ..., e_n)$, and $f : L\times L \to k$ be a bilinear form on $L$. Let $\varphi_f : L \to \mathrm{Cl}(L,f)$ be the $k$-module homomorphism defined by $\varphi_f = \mathrm{proj}_f \circ \mathrm{inj}$, where $\mathrm{inj} : L \to \otimes L$ is the canonical injection of the $k$-module $L$ into its tensor algebra $\otimes L$, and where $\mathrm{proj}_f : \otimes L \to \mathrm{Cl}(L,f)$ is the canonical projection of the tensor algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \mathrm{Cl}(L,f)$.
>
> Then, $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is a basis of the $k$-module $\mathrm{Cl}(L,f)$, where $\mathcal{P}(\{1,2,...,n\})$ denotes the power set of the set $\{1,2,...,n\}$.

Here, we are using the following notation:

> **Definition 4.** Let $A$ be a ring, and let $I$ be a finite subset of $\mathbb{Z}$. Let $a_i$ be an element of $A$ for each $i \in I$. Then, we denote by $\overrightarrow{\prod\limits_{i\in I}}a_i$ the element of $A$ defined as follows: We write the set $I$ in the form $I = \{i_1, i_2, ..., i_\ell\}$ with $i_1 < i_2 < ... < i_\ell$ (in other words, we let $i_1$, $i_2$, ..., $i_\ell$ be the elements of $I$, written down in ascending order). Then, we define $\overrightarrow{\prod\limits_{i\in I}}a_i$ as the product $a_{i_1}a_{i_2}...a_{i_\ell}$. This product $\overrightarrow{\prod\limits_{i\in I}}a_i$ is called the *ascending product* of the elements $a_i$ of $A$.

One more theorem that is often (silently) used and will follow from our considerations:

> **Theorem 3.** Let $k$ be a commutative ring. Let $L$ be a $k$-module, and $f : L \times L \to k$ be a bilinear form on $L$. Let $\varphi_f : L \to \mathrm{Cl}(L,f)$ be the $k$-module homomorphism defined by $\varphi_f = \mathrm{proj}_f \circ \mathrm{inj}$, where $\mathrm{inj} : L \to \otimes L$ is the canonical injection of the $k$-module $L$ into its tensor algebra $\otimes L$, and where $\mathrm{proj}_f : \otimes L \to \mathrm{Cl}(L,f)$ is the canonical projection of the tensor algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \mathrm{Cl}(L,f)$. Then, the homomorphism $\varphi_f$ is injective.

Theorem 2 is known in the case of $k$ being a field and $L$ being a finite-dimensional $k$-vector space; in this case, it is often proved using orthogonal decomposition of $L$ into $f$-orthogonal subspaces - a tactic not available to us in the general case of $k$ being an arbitrary commutative ring. We will have to derive Theorem 2 from Theorem 1 to

[9] The proof of Theorem 1 in [2] (where Theorem 1 appears as Theorem 1.2, albeit only in the case of $k$ being a field) seems different, but I don't completely understand it; to me it seems that it has a flaw (it states that "the $r$-homogeneous part of $\varphi$ is then of the form $\varphi_r = \sum a_i \otimes v_i \otimes v_i \otimes b_i$ (where $\deg a_i + \deg b_i = r-2$ for each $i$)", which I am not sure about, because theoretically one could imagine that the representation of $\varphi$ in the form $\varphi = \sum a_i \otimes (v_i \otimes v_i + q(v_i)) \otimes b_i$ involves some $a_i$ and $b_i$ of extremely huge degree which cancel out in the sum).

prove it in this generality. Most proofs of Theorem 1 rely on Theorem 2, and Theorem 3 is usually proven using either Theorem 1 or Theorem 2.

The nature of our proof will be computational - we are going to define some $k$-module automorphisms of the tensor algebra $\otimes L$ by recursive formulae. During the course of the proof, we will show a lot of formulas, each of which has a more or less straightforward inductive proofs. The inductive proofs will always use one and the same tactic: a tactic I call *tensor induction*. Here is what it is about:

**Definition 5.** **(a)** Let $k$ be a commutative ring, and $L$ be a $k$-module. Let $p \in \mathbb{N}$. An element of $L^{\otimes p}$ is said to be *left-induced* if and only if it can be written in the form $u \otimes U$ for some $u \in L$ and some $U \in L^{\otimes (p-1)}$. Then, for every $p \in \mathbb{N}_+$, the $k$-module $L^{\otimes p}$ is generated by its left-induced elements (because $L^{\otimes p} = L \otimes L^{\otimes (p-1)}$, and therefore the $k$-module $L^{\otimes p}$ is generated by its elements of the form $u \otimes U$ for some $u \in L$ and some $U \in L^{\otimes (p-1)}$; in other words, the $k$-module $L^{\otimes p}$ is generated by its left-induced elements).

**(b)** Let $k$ be a commutative ring, and $L$ be a $k$-module. Let $p \in \mathbb{N}$. An element of $L^{\otimes p}$ is said to be *right-induced* if and only if it can be written in the form $U \otimes u$ for some $u \in L$ and some $U \in L^{\otimes (p-1)}$. Then, for every $p \in \mathbb{N}_+$, the $k$-module $L^{\otimes p}$ is generated by its right-induced elements (because $L^{\otimes p} = L^{\otimes (p-1)} \otimes L$, and therefore the $k$-module $L^{\otimes p}$ is generated by its elements of the form $U \otimes u$ for some $u \in L$ and some $U \in L^{\otimes (p-1)}$; in other words, the $k$-module $L^{\otimes p}$ is generated by its right-induced elements).

**The left tensor induction tactic.** Let $p \in \mathbb{N}_+$. Let $\eta$ and $\varepsilon$ be two $k$-linear maps from $L^{\otimes p}$ to some other $k$-module. Then, in order to prove that

$$\left(\eta\left(T\right) = \varepsilon\left(T\right) \qquad \text{for every } T \in L^{\otimes p}\right),$$

it is enough to prove that

$$\left(\eta\left(T\right) = \varepsilon\left(T\right) \qquad \text{for every left-induced } T \in L^{\otimes p}\right)$$

(because the $k$-module $L^{\otimes p}$ is generated by its left-induced elements).

In words: In order to prove that all elements of $L^{\otimes p}$ satisfy some given $k$-linear equation, it is enough to show that all left-induced elements of $L^{\otimes p}$ satisfy this equation.

**The right tensor induction tactic.** Let $p \in \mathbb{N}_+$. Let $\eta$ and $\varepsilon$ be two $k$-linear maps from $L^{\otimes p}$ to some other $k$-module. Then, in order to prove that

$$\left(\eta\left(T\right) = \varepsilon\left(T\right) \qquad \text{for every } T \in L^{\otimes p}\right),$$

it is enough to prove that

$$\left(\eta\left(T\right) = \varepsilon\left(T\right) \qquad \text{for every right-induced } T \in L^{\otimes p}\right)$$

(because the $k$-module $L^{\otimes p}$ is generated by its right-induced elements).

In words: In order to prove that all elements of $L^{\otimes p}$ satisfy some given $k$-linear equation, it is enough to show that all right-induced elements of $L^{\otimes p}$ satisfy this equation.

**The tensor algebra induction tactic.** Let $\eta$ and $\varepsilon$ be two $k$-linear maps from $\otimes L$ to some other $k$-module. Then, in order to prove that

$$(\eta (T) = \varepsilon (T) \qquad \text{for every } T \in \otimes L),$$

it is enough to prove that

$$\left(\eta (T) = \varepsilon (T) \qquad \text{for every } p \in \mathbb{N} \text{ and every } T \in L^{\otimes p}\right)$$

(because the $k$-module $\otimes L$ is the direct sum $L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$, and therefore is generated by its submodules $L^{\otimes p}$ for all $p \in \mathbb{N}$).

## 2. Left interior products on the tensor algebra

From now on, we fix a commutative ring $k$, and a $k$-module $L$. Let $f$ be some bilinear form on $L$.

First, we define some operations of $L$ on $\otimes L$ - the so-called *interior products*. Our definition will be rather dry - if you want a formula for these operations, scroll down to Theorem 5 below.

**Definition 6.** Let $f : L \times L \to k$ be a bilinear form. For every $p \in \mathbb{N}$ and every $v \in L$, we define a $k$-linear map $\delta^f_{v,p} : L^{\otimes p} \to L^{\otimes (p-1)}$ (where $L^{\otimes(-1)}$ means 0) by induction over $p$:

*Induction base:* For $p = 0$, we define the map $\delta^f_{v,p} : L^{\otimes 0} \to L^{\otimes(-1)}$ to be the zero map.

*Induction step:* For each $p \in \mathbb{N}_+$, we define a $k$-linear map $\delta^f_{v,p} : L^{\otimes p} \to L^{\otimes(p-1)}$ by

$$\left(\delta^f_{v,p} (u \otimes U) = f (v, u) U - u \otimes \delta^f_{v,p-1} (U) \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right), \tag{1}$$

assuming that we have already defined a $k$-linear map $\delta^f_{v,p-1} : L^{\otimes(p-1)} \to L^{\otimes(p-2)}$. (This definition is justified, because in order to define a $k$-linear map from $L^{\otimes p}$ to some other $k$-module, it is enough to define how it acts on tensors of the form $u \otimes U$ for every $u \in L$ and $U \in L^{\otimes(p-1)}$, as long as this action is bilinear with respect to $u$ and $U$. This is because $L^{\otimes p} = L \otimes L^{\otimes(p-1)}$.)

This way we have defined a $k$-linear map $\delta^f_{v,p} : L^{\otimes p} \to L^{\otimes(p-1)}$ for every $p \in \mathbb{N}$. We can combine these maps $\delta^f_{v,0}$, $\delta^f_{v,1}$, $\delta^f_{v,2}$, ... into one $k$-linear map $\delta^f_v : \otimes L \to \otimes L$ (since $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$), and the formula (1) rewrites as

$$\left(\delta^f_v (u \otimes U) = f (v, u) U - u \otimes \delta^f_v (U) \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right). \tag{2}$$

It is easily seen (by induction over $p \in \mathbb{N}$) that the map $\delta^f_{v,p}$ depends linearly on the vector $v \in L$. Hence, the combination $\delta^f_v$ of the maps $\delta^f_{v,0}$, $\delta^f_{v,1}$, $\delta^f_{v,2}$, ... must also depend linearly on $v \in L$. In other words, the map

$$L \times (\otimes L) \to \otimes L, \qquad (v, U) \mapsto \delta^f_v (U)$$

is $k$-bilinear. Hence, this map gives rise to a $k$-linear map

$$\delta^f : L \otimes (\otimes L) \to \otimes L, \qquad v \otimes U \mapsto \delta_v^f (U).$$

We are going to denote $\delta_v^f (U)$ by $v \overset{f}{\llcorner} U$ for each $v \in L$ and $U \in \otimes L$. Thus, the equality (2) takes the form

$$\left( v \overset{f}{\llcorner} (u \otimes U) = f(v,u) U - u \otimes \left( v \overset{f}{\llcorner} U \right) \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes (p-1)} \right). \tag{3}$$

The tensor $v \overset{f}{\llcorner} U$ is called the *left interior product of* $v$ *and* $U$ with respect to the bilinear form $f$.

Let us note that many authors omit the $f$ in the notation $\overset{f}{\llcorner}$; in other words, they simply write $\llcorner$ for $\overset{f}{\llcorner}$. However, we are going to avoid this abbreviation, as we aim at considering several bilinear forms at once, and omitting the name of the bilinear form could lead to confusion.

The above inductive definition of $\overset{f}{\llcorner}$ is not particularly vivid. Here is an explicit formula for $\overset{f}{\llcorner}$ (albeit we are mostly going to avoid using it in proofs):

**Theorem 5.** Let $f : L \times L \to k$ be a bilinear form.

**(a)** For every $\lambda \in k$ and every $v \in L$, we have $v \overset{f}{\llcorner} \lambda = 0$. [10]

**(b)** For every $u \in L$ and $v \in L$, we have $v \overset{f}{\llcorner} u = f(v,u)$. [11]

**(c)** Let $u_1$, $u_2$, ..., $u_p$ be $p$ elements of $L$. Let $v \in L$. Then,

$$v \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes u_p) = \sum_{i=1}^{p} (-1)^{i-1} f(v, u_i) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p. \tag{4}$$

Here, the hat over the vector $u_i$ means that the vector $u_i$ is being omitted from the tensor product; in other words, $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$ is just another way to write $\underbrace{u_1 \otimes u_2 \otimes ... \otimes u_{i-1}}_{\substack{\text{tensor product of the} \\ \text{first } i-1 \text{ vectors } u_\ell}} \otimes \underbrace{u_{i+1} \otimes u_{i+2} \otimes ... \otimes u_p}_{\substack{\text{tensor product of the} \\ \text{last } p-i \text{ vectors } u_\ell}}$.

*Proof of Theorem 5.* **(a)** We have $\lambda \in k = L^{\otimes 0}$ and thus $\delta_v^f (\lambda) = \underbrace{\delta_{v,0}^f}_{=0} (\lambda) = 0(\lambda) = 0$. Thus, $v \overset{f}{\llcorner} \lambda = \delta_v^f(\lambda) = 0$, and Theorem 5 **(a)** is proven.

**(b)** Applying (3) to $U = 1$, we see that

$$v \overset{f}{\llcorner} (u \otimes 1) = f(v,u) 1 - u \otimes \underbrace{\left( v \overset{f}{\llcorner} 1 \right)}_{=0 \text{ (by Theorem 5 } \textbf{(a)}\text{)}} = f(v,u) 1 - u \otimes 0 = f(v,u).$$

Since $u \otimes 1 = u$, this rewrites as $v \overset{f}{\llcorner} u = f(v,u)$. Thus, Theorem 5 **(b)** is proven.

[10] Here, $\lambda \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

[11] Here, $f(v,u) \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

**(c)** We are going to prove Theorem 5 **(c)** by induction over $p$:

The induction base is clear, since for $p = 0$, Theorem 5 **(c)** trivially follows from Theorem 5 **(a)**[12].

Now to the induction step: Let $p \in \mathbb{N}_+$. Let us prove Theorem 5 **(c)** for this $p$, assuming that we have already verified Theorem 5 **(c)** applied to $p-1$ instead of $p$.

In fact, we have assumed that we have already shown Theorem 5 **(c)** applied to $p-1$ instead of $p$. In other words, we have already shown the equality

$$v \llcorner^{f} (u_1 \otimes u_2 \otimes ... \otimes u_{p-1}) = \sum_{i=1}^{p-1} (-1)^{i-1} f(v, u_i) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{p-1} \qquad (5)$$

for any $p-1$ vectors $u_1$, $u_2$, ..., $u_{p-1}$ in $L$. Now, our goal is to prove the equality (4) for any $p$ vectors $u_1$, $u_2$, ..., $u_p$ in $L$.

In fact, substituting the vectors $u_2$, $u_3$, ..., $u_p$ instead of $u_1$, $u_2$, ..., $u_{p-1}$ into the (already proven) equality (5), we get

$$\begin{aligned}
v \llcorner^{f} (u_2 \otimes u_3 \otimes ... \otimes u_p) &= \sum_{i=1}^{p-1} (-1)^{i-1} f(v, u_{i+1}) \cdot u_2 \otimes u_3 \otimes ... \otimes \widehat{u_{i+1}} \otimes ... \otimes u_p \\
&= \sum_{i=2}^{p} \underbrace{(-1)^{i-2}}_{=-(-1)^{i-1}} f(v, u_i) \cdot u_2 \otimes u_3 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \\
&\qquad \text{(here, we have substituted } i \text{ for } i+1 \text{ in the sum)} \\
&= -\sum_{i=2}^{p} (-1)^{i-1} f(v, u_i) \cdot u_2 \otimes u_3 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p. \qquad (6)
\end{aligned}$$

[12]because for $p = 0$, we have $u_1 \otimes u_2 \otimes ... \otimes u_p = (\text{empty product}) = 1 \in k$ and $\sum_{i=1}^{p} (-1)^{i-1} f(v, u_i) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p = (\text{empty sum}) = 0$

Now, applying (3) to $u = u_1$ and $U = u_2 \otimes u_3 \otimes ... \otimes u_p$, we get

$$\begin{aligned}
& v \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes u_3 \otimes ... \otimes u_p) \\
&= f(v, u_1)\, u_2 \otimes u_3 \otimes ... \otimes u_p - u_1 \otimes \left( \underbrace{v \overset{f}{\llcorner} (u_2 \otimes u_3 \otimes ... \otimes u_p)}_{\substack{= -\sum_{i=2}^{p} (-1)^{i-1} f(v,u_i) \cdot u_2 \otimes u_3 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \\ \text{(by (6))}}} \right) \\
&= \underbrace{f(v, u_1)}_{=(-1)^{1-1} f(v,u_1)} \underbrace{u_2 \otimes u_3 \otimes ... \otimes u_p}_{=u_1 \otimes u_2 \otimes ... \otimes \widehat{u_1} \otimes ... \otimes u_p} \\
&\qquad + \underbrace{u_1 \otimes \left( \sum_{i=2}^{p} (-1)^{i-1} f(v, u_i) \cdot u_2 \otimes u_3 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \right)}_{\substack{= \sum_{i=2}^{p} (-1)^{i-1} f(v,u_i) \cdot u_1 \otimes (u_2 \otimes u_3 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \\ = \sum_{i=2}^{p} (-1)^{i-1} f(v,u_i) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p}} \\
&= (-1)^{1-1} f(v, u_1)\, u_1 \otimes u_2 \otimes ... \otimes \widehat{u_1} \otimes ... \otimes u_p \\
&\qquad + \sum_{i=2}^{p} (-1)^{i-1} f(v, u_i) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \\
&= \sum_{i=1}^{p} (-1)^{i-1} f(v, u_i) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p.
\end{aligned}$$

Thus, (4) is proven for our $p \in \mathbb{N}$. In other words, Theorem 5 **(c)** is proven for our $p \in \mathbb{N}$. This completes the induction step, and thus the proof of Theorem 5 **(c)** is complete.

We are now going to prove some properties of the interior product. The most important one is the *bilinearity of* $\overset{f}{\llcorner}$; this property states that the map

$$L \times (\otimes L) \to \otimes L, \qquad (v, U) \mapsto v \overset{f}{\llcorner} U$$

is $k$-bilinear[13], i. e. that $(\alpha v + \beta v') \overset{f}{\llcorner} U = \alpha v \overset{f}{\llcorner} U + \beta v' \overset{f}{\llcorner} U$ and that $v \overset{f}{\llcorner} (\alpha U + \beta U') = \alpha v \overset{f}{\llcorner} U + \beta v \overset{f}{\llcorner} U'$ for any $v \in L$, $v' \in L$, $U \in \otimes L$ and $U' \in \otimes L$.

**Theorem 6.** If $u \in L$, $U \in \otimes L$, and $v \in L$, then

$$v \overset{f}{\llcorner} (u \otimes U) = f(v, u)\, U - u \otimes \left( v \overset{f}{\llcorner} U \right). \tag{7}$$

[13] This is because $v \overset{f}{\llcorner} U = \delta_v^f (U)$, and because the map

$$L \times (\otimes L) \to \otimes L, \qquad (v, U) \mapsto \delta_v^f (U)$$

is $k$-bilinear.

*Proof of (7).* If $U$ is a homogeneous tensor (i. e. an element of $L^{\otimes r}$ for some $r \in \mathbb{N}$), then (7) follows directly from (3) (applied to $p = r + 1$). Otherwise, we can write $U$ as a $k$-linear combination of homogeneous tensors of various degrees, and then apply (3) to each of these tensors; summing up, we obtain (7). Thus, (7) is proven.

**Theorem 7.** If $v \in L$ and $U \in \otimes L$, then

$$v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right) = 0. \tag{8}$$

*Proof of Theorem 7.* Fix some $v \in L$. First we will prove that for every $p \in \mathbb{N}$ and every $U \in L^{\otimes p}$, the equation (8) holds. In fact, we will show this by induction over $p$: The induction base ($p = 0$) is clear (thanks to Theorem 5 **(a)**, which yields $v \overset{f}{\llcorner} U = 0$ for every $U \in L^{\otimes 0} = k$). Now for the induction step: Fix some $p \in \mathbb{N}_+$. Let us now prove (8) for all $U \in L^{\otimes p}$, assuming that (8) is already proven for all $U \in L^{\otimes (p-1)}$.

We want to prove (8) for all $U \in L^{\otimes p}$. But in order to achieve this, it is enough to prove (8) for all left-induced $U \in L^{\otimes p}$ (because of the left tensor induction tactic, since the equation (8) is linear in $U$). So let us prove (8) for all left-induced $U \in L^{\otimes p}$. In fact, let $U \in L^{\otimes p}$ be a left-induced tensor. Then, $U$ can be written in the form $U = u \otimes \ddot{U}$ for some $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ (since $U$ is left-induced). Then, $v \overset{f}{\llcorner} U = v \overset{f}{\llcorner} \left( u \otimes \ddot{U} \right) = f(v,u) \ddot{U} - u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right)$ (by (7), applied to $\ddot{U}$ instead of $U$) yields

$$\begin{aligned}
v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right) &= v \overset{f}{\llcorner} \left( f(v,u) \ddot{U} - u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) = f(v,u)\, v \overset{f}{\llcorner} \ddot{U} - v \overset{f}{\llcorner} \left( u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \\
&\qquad \left( \text{by the bilinearity of } \overset{f}{\llcorner} \right) \\
&= f(v,u)\, v \overset{f}{\llcorner} \ddot{U} - \left( f(v,u) \left( v \overset{f}{\llcorner} \ddot{U} \right) - u \otimes \left( v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \right) \\
&\qquad \left( \begin{array}{c} \text{since } v \overset{f}{\llcorner} \left( u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) = f(v,u) \left( v \overset{f}{\llcorner} \ddot{U} \right) - u \otimes \left( v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \\ \text{(by (7), applied to } v \overset{f}{\llcorner} \ddot{U} \text{ instead of } U) \end{array} \right) \\
&= u \otimes \left( v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) = 0
\end{aligned}$$

(because $v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \ddot{U} \right) = 0$ by (8) (applied to $\ddot{U}$ instead of $U$) [14]). Thus, we have proven that $v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right) = 0$ for all left-induced $U \in L^{\otimes p}$. Consequently, by the left tensor induction tactic (as we said above), we conclude that (8) holds for all $U \in L^{\otimes p}$. This completes the induction step. Therefore we have now proven that for every $p \in \mathbb{N}$, and every $U \in L^{\otimes p}$, the equation (8) holds.

This yields that the equation (8) holds for every $U \in \otimes L$ (since every element of $\otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and since the equation (8) is linear in $U$). This proves Theorem 7.

**Theorem 8.** If $v \in L$, $w \in L$ and $U \in \otimes L$, then

$$v \overset{f}{\llcorner} \left( w \overset{f}{\llcorner} U \right) = -w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right). \tag{9}$$

[14] In fact, we *are* allowed to apply (8) to $\ddot{U}$ instead of $U$, since $\ddot{U} \in L^{\otimes (p-1)}$ and since we have assumed that (8) is already proven for all $U \in L^{\otimes (p-1)}$.

*First proof of Theorem 8.* Theorem 7 yields $v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right) = 0$. Theorem 7, applied to $w$ instead of $v$, yields $w \overset{f}{\llcorner} \left( w \overset{f}{\llcorner} U \right) = 0$. Finally, Theorem 7, applied to $v + w$ instead of $v$, yields $(v+w) \overset{f}{\llcorner} \left( (v+w) \overset{f}{\llcorner} U \right) = 0$. Thus,

$$\begin{aligned}
0 &= (v+w) \overset{f}{\llcorner} \left( (v+w) \overset{f}{\llcorner} U \right) = (v+w) \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U + w \overset{f}{\llcorner} U \right) \qquad \left( \text{by the bilinearity of } \overset{f}{\llcorner} \right) \\
&= v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U + w \overset{f}{\llcorner} U \right) + w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U + w \overset{f}{\llcorner} U \right) \qquad \left( \text{by the bilinearity of } \overset{f}{\llcorner} \right) \\
&= \left( \underbrace{v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right)}_{=0} + v \overset{f}{\llcorner} \left( w \overset{f}{\llcorner} U \right) \right) + \left( w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right) + \underbrace{w \overset{f}{\llcorner} \left( w \overset{f}{\llcorner} U \right)}_{=0} \right) \\
&\qquad \left( \text{by the bilinearity of } \overset{f}{\llcorner} \right) \\
&= v \overset{f}{\llcorner} \left( w \overset{f}{\llcorner} U \right) + w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right).
\end{aligned}$$

This yields (9), and thus Theorem 8 is proven.

*Second proof of Theorem 8.* Fix some $v \in L$ and $w \in L$. First we will prove that for every $p \in \mathbb{N}$ and every $U \in L^{\otimes p}$, the equation (9) holds. In fact, we will show this by induction over $p$: The induction base ($p = 0$) is clear (since Theorem 5 **(a)** yields $w \overset{f}{\llcorner} U = 0$ and $v \overset{f}{\llcorner} U = 0$ in the case $p = 0$). Now for the induction step: Fix some $p \in \mathbb{N}_+$. Let us now prove (9) for all $U \in L^{\otimes p}$, assuming that (9) is already proven for all $U \in L^{\otimes (p-1)}$.

We want to prove (9) for all $U \in L^{\otimes p}$. But in order to achieve this, it is enough to prove (9) for all left-induced $U \in L^{\otimes p}$ (because of the left tensor induction tactic, since the equation (9) is linear in $U$). So let us prove (9) for all left-induced $U \in L^{\otimes p}$. In fact, let $U \in L^{\otimes p}$ be a left-induced tensor. Then, $U$ can be written in the form $U = u \otimes \ddot{U}$ for some $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ (since $U$ is left-induced). Therefore, $v \overset{f}{\llcorner} U = v \overset{f}{\llcorner} \left( u \otimes \ddot{U} \right) = f(v,u) \ddot{U} - u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right)$ (by (7), applied to $\ddot{U}$ instead of $U$) yields

$$\begin{aligned}
w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} U \right) &= w \overset{f}{\llcorner} \left( f(v,u) \ddot{U} - u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) = f(v,u)\, w \overset{f}{\llcorner} \ddot{U} - w \overset{f}{\llcorner} \left( u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \\
&\qquad \left( \text{by the bilinearity of } \overset{f}{\llcorner} \right) \\
&= f(v,u)\, w \overset{f}{\llcorner} \ddot{U} - \left( f(w,u)\, v \overset{f}{\llcorner} \ddot{U} - u \otimes \left( w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \right) \\
&\qquad \left( \begin{array}{c} \text{since } w \overset{f}{\llcorner} \left( u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) = f(w,u)\, v \overset{f}{\llcorner} \ddot{U} - u \otimes \left( w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \\ \text{(by (7), applied to } w \text{ and } v \overset{f}{\llcorner} \ddot{U} \text{ instead of } v \text{ and } U) \end{array} \right) \\
&= f(v,u)\, w \overset{f}{\llcorner} \ddot{U} - f(w,u)\, v \overset{f}{\llcorner} \ddot{U} + u \otimes \left( w \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \ddot{U} \right) \right). \qquad (10)
\end{aligned}$$

Applying this equality (10) to $w$ and $v$ instead of $v$ and $w$, we obtain

$$v \overset{f}{\llcorner} \left( w \overset{f}{\llcorner} U \right) = f(w,u)\, v \overset{f}{\llcorner} \ddot{U} - f(v,u)\, w \overset{f}{\llcorner} \ddot{U} + u \otimes \left( v \overset{f}{\llcorner} \left( w \overset{f}{\llcorner} \ddot{U} \right) \right).$$

Adding this equality to (10), we obtain

$$\begin{aligned}
& w \llcorner^{f} \left( v \llcorner^{f} U \right) + v \llcorner^{f} \left( w \llcorner^{f} U \right) \\
&= \left( f(v,u)\, w \llcorner^{f} \ddot{U} - f(w,u)\, v \llcorner^{f} \ddot{U} + u \otimes \left( w \llcorner^{f} \left( v \llcorner^{f} \ddot{U} \right) \right) \right) \\
&\qquad + \left( f(w,u)\, v \llcorner^{f} \ddot{U} - f(v,u)\, w \llcorner^{f} \ddot{U} + u \otimes \left( v \llcorner^{f} \left( w \llcorner^{f} \ddot{U} \right) \right) \right) \\
&= u \otimes \left( w \llcorner^{f} \left( v \llcorner^{f} \ddot{U} \right) \right) + u \otimes \left( v \llcorner^{f} \left( w \llcorner^{f} \ddot{U} \right) \right) = u \otimes \left( w \llcorner^{f} \left( v \llcorner^{f} \ddot{U} \right) + v \llcorner^{f} \left( w \llcorner^{f} \ddot{U} \right) \right) = 0
\end{aligned}$$

(because $w \llcorner^{f} \left( v \llcorner^{f} \ddot{U} \right) + v \llcorner^{f} \left( w \llcorner^{f} \ddot{U} \right) = 0$, since $v \llcorner^{f} \left( w \llcorner^{f} \ddot{U} \right) = -w \llcorner^{f} \left( v \llcorner^{f} \ddot{U} \right)$ by (9) (applied to $\ddot{U}$ instead of $U$) [15]), and therefore $v \llcorner^{f} \left( w \llcorner^{f} U \right) = -w \llcorner^{f} \left( v \llcorner^{f} U \right)$. Thus, we have proven that (9) holds for all left-induced $U \in L^{\otimes p}$. Consequently, by the left tensor induction tactic (as we said above), we conclude that (9) holds for all $U \in L^{\otimes p}$. This completes the induction step. Therefore we have now proven that for every $p \in \mathbb{N}$, and every $U \in L^{\otimes p}$, the equation (9) holds.

This yields that the equation (9) holds for every $U \in \otimes L$ (since every element of $\otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and since the equation (9) is linear in $U$). This proves Theorem 8.

**Theorem 9.** If $p \in \mathbb{N}$, $u \in L$, $U \in L^{\otimes p}$, and $v \in L$, then

$$v \llcorner^{f} (U \otimes u) = (-1)^{p} f(v,u)\, U + \left( v \llcorner^{f} U \right) \otimes u. \tag{11}$$

Instead of proving this directly, we show something more general:

**Theorem 10.** If $p \in \mathbb{N}$, $v \in L$, $U \in L^{\otimes p}$, and $V \in \otimes L$, then

$$v \llcorner^{f} (U \otimes V) = (-1)^{p}\, U \otimes \left( v \llcorner^{f} V \right) + \left( v \llcorner^{f} U \right) \otimes V. \tag{12}$$

*Proof of Theorem 10.* We are going to prove (12) by induction over $p$:

The induction base $p = 0$ is obvious[16]. Now let us come to the induction step: Fix some $p \in \mathbb{N}_{+}$ and some $V \in \otimes L$. Let us now prove (12) for all $U \in L^{\otimes p}$, assuming that (12) is already proven for all $U \in L^{\otimes (p-1)}$.

[15] In fact, we *are* allowed to apply (9) to $\ddot{U}$ instead of $U$, since $\ddot{U} \in L^{\otimes (p-1)}$ and since we have assumed that (9) is already proven for all $U \in L^{\otimes (p-1)}$.

[16] In fact, in the case $p = 0$, we have $U \in L^{\otimes p} = L^{\otimes 0} = k$ and thus

$$v \llcorner^{f} \Bigg( \underbrace{U \otimes V}_{=UV} \Bigg) = U \cdot v \llcorner^{f} V \qquad \text{and}$$

$$\underbrace{(-1)^{p}}_{=(-1)^{0}=1} \underbrace{U \otimes \left( v \llcorner^{f} V \right)}_{\substack{=U \cdot v \llcorner^{f} V \\ (\text{since } U \in k)}} + \underbrace{\left( v \llcorner^{f} U \right)}_{\substack{=0 \text{ (by Theorem 5 } \mathbf{(a)}, \\ \text{since } U \in k)}} \otimes V = 1U \cdot v \llcorner^{f} V + 0 \otimes V = U \cdot v \llcorner^{f} V,$$

and therefore (12) is valid in the case $p = 0$.

We want to prove (12) for all $U \in L^{\otimes p}$. But in order to achieve this, it is enough to prove (12) for all left-induced $U \in L^{\otimes p}$ (by the left tensor induction tactic, because the equation (12) is linear in $U$). Thus, let us prove (12) for all left-induced $U \in L^{\otimes p}$. In other words, let us prove (12) for all tensors $U \in L^{\otimes p}$ of the form $U = u \otimes \ddot{U}$ with $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ (because every left-induced tensor $U \in L^{\otimes p}$ has the form $U = u \otimes \ddot{U}$ for some $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$). In other words, let us prove that

$$v \llcorner^{f} \left( \left( u \otimes \ddot{U} \right) \otimes V \right) = (-1)^p \left( u \otimes \ddot{U} \right) \otimes \left( v \llcorner^{f} V \right) + \left( v \llcorner^{f} \left( u \otimes \ddot{U} \right) \right) \otimes V \tag{13}$$

for every $u \in L$, $\ddot{U} \in L^{\otimes (p-1)}$, $v \in L$ and $V \in \otimes L$.

In fact, $\left( u \otimes \ddot{U} \right) \otimes V = u \otimes \left( \ddot{U} \otimes V \right)$ yields

$$v \llcorner^{f} \left( \left( u \otimes \ddot{U} \right) \otimes V \right) = v \llcorner^{f} \left( u \otimes \left( \ddot{U} \otimes V \right) \right) = f(v,u) \left( \ddot{U} \otimes V \right) - u \otimes \left( v \llcorner^{f} \left( \ddot{U} \otimes V \right) \right).$$

(by (7), applied to $\ddot{U} \otimes V$ instead of $U$). But since

$$v \llcorner^{f} \left( \ddot{U} \otimes V \right) = (-1)^{p-1} \ddot{U} \otimes \left( v \llcorner^{f} V \right) + \left( v \llcorner^{f} \ddot{U} \right) \otimes V$$

(this follows from applying (12) to $p-1$ and $\ddot{U}$ instead of $p$ and $U$ [17]), this becomes

$$\begin{aligned} v \llcorner^{f} \left( \left( u \otimes \ddot{U} \right) \otimes V \right) &= f(v,u) \left( \ddot{U} \otimes V \right) - u \otimes \left( (-1)^{p-1} \ddot{U} \otimes \left( v \llcorner^{f} V \right) + \left( v \llcorner^{f} \ddot{U} \right) \otimes V \right) \\ &= f(v,u) \left( \ddot{U} \otimes V \right) - \underbrace{(-1)^{p-1}}_{=-(-1)^p} u \otimes \ddot{U} \otimes \left( v \llcorner^{f} V \right) - u \otimes \left( v \llcorner^{f} \ddot{U} \right) \otimes V \\ &= f(v,u) \left( \ddot{U} \otimes V \right) + (-1)^p u \otimes \ddot{U} \otimes \left( v \llcorner^{f} V \right) - u \otimes \left( v \llcorner^{f} \ddot{U} \right) \otimes V. \end{aligned}$$

Comparing this to

$$\begin{aligned} &(-1)^p \left( u \otimes \ddot{U} \right) \otimes \left( v \llcorner^{f} V \right) + \left( v \llcorner^{f} \left( u \otimes \ddot{U} \right) \right) \otimes V \\ &= (-1)^p \left( u \otimes \ddot{U} \right) \otimes \left( v \llcorner^{f} V \right) + \left( f(v,u) \ddot{U} - u \otimes \left( v \llcorner^{f} \ddot{U} \right) \right) \otimes V \\ &\qquad \left( \begin{array}{c} \text{because (7), applied to } \ddot{U} \text{ instead of } U, \\ \text{yields } v \llcorner^{f} \left( u \otimes \ddot{U} \right) = f(v,u) \ddot{U} - u \otimes \left( v \llcorner^{f} \ddot{U} \right) \end{array} \right) \\ &= (-1)^p \left( u \otimes \ddot{U} \right) \otimes \left( v \llcorner^{f} V \right) + f(v,u) \ddot{U} \otimes V - u \otimes \left( v \llcorner^{f} \ddot{U} \right) \otimes V \\ &= f(v,u) \ddot{U} \otimes V + (-1)^p \left( u \otimes \ddot{U} \right) \otimes \left( v \llcorner^{f} V \right) - u \otimes \left( v \llcorner^{f} \ddot{U} \right) \otimes V \\ &= f(v,u) \left( \ddot{U} \otimes V \right) + (-1)^p u \otimes \ddot{U} \otimes \left( v \llcorner^{f} V \right) - u \otimes \left( v \llcorner^{f} \ddot{U} \right) \otimes V, \end{aligned}$$

we obtain (13). Hence, we have proven (13). As already explained above, this completes the induction step. Thus, (12) is proven for all $p \in \mathbb{N}$. In other words, the proof of Theorem 10 is complete.

[17] In fact, we *are* allowed to apply (12) to $p-1$ and $\ddot{U}$ instead of $p$ and $U$, because we have assumed that (12) is already proven for all $U \in L^{\otimes (p-1)}$.

*Proof of Theorem 9.* Applying Theorem 10 to $V = u$, we obtain

$$v \llcorner^{f} (U \otimes u) = (-1)^p U \otimes \left(v \llcorner^{f} u\right) + \left(v \llcorner^{f} U\right) \otimes u.$$

Since $v \llcorner^{f} u = f(v, u)$ (by Theorem 5 **(b)**), this becomes

$$v \llcorner^{f} (U \otimes u) = (-1)^p \underbrace{U \otimes f(v, u)}_{= f(v,u) U} + \left(v \llcorner^{f} U\right) \otimes u = (-1)^p f(v, u) U + \left(v \llcorner^{f} U\right) \otimes u,$$

and therefore Theorem 9 is proven.

Note that we will often use a trivial generalization of Theorem 9 rather than Theorem 9 itself:

**Theorem $\mathbf{10\frac{1}{2}}$.** If $p \in \mathbb{N}$, $u \in L$, $U \in \bigoplus_{\substack{i \in \mathbb{N};\\ i \equiv p \bmod 2}} L^{\otimes i}$, and $v \in L$, then

$$v \llcorner^{f} (U \otimes u) = (-1)^p f(v, u) U + \left(v \llcorner^{f} U\right) \otimes u. \tag{14}$$

*Proof of Theorem $10\frac{1}{2}$.* Since $U \in \bigoplus_{\substack{i \in \mathbb{N};\\ i \equiv p \bmod 2}} L^{\otimes i}$, we can write $U$ in the form $U = \sum_{\substack{i \in \mathbb{N};\\ i \equiv p \bmod 2}} U_i$, where $U_i \in L^{\otimes i}$ for every $i \in \mathbb{N}$ satisfying $i \equiv p \bmod 2$. Now, (11) (applied to $i$ and $U_i$ instead of $p$ and $U$) yields

$$v \llcorner^{f} (U_i \otimes u) = (-1)^i f(v, u) U_i + \left(v \llcorner^{f} U_i\right) \otimes u$$

for every $i \in \mathbb{N}$ satisfying $i \equiv p \bmod 2$. Since $(-1)^i = (-1)^p$ (because $i \equiv p \bmod 2$), this becomes

$$v \llcorner^{f} (U_i \otimes u) = (-1)^p f(v, u) U_i + \left(v \llcorner^{f} U_i\right) \otimes u. \tag{15}$$

Now, $U = \sum\limits_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} U_i$ yields

$$\begin{aligned}
v \llcorner^f (U \otimes u) &= v \llcorner^f \left( \sum_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} U_i \otimes u \right) = \sum_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} v \llcorner^f (U_i \otimes u) \qquad \left(\text{by the bilinearity of } \llcorner^f\right) \\
&= \sum_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} \left( (-1)^p f(v,u) U_i + \left( v \llcorner^f U_i \right) \otimes u \right) \qquad (\text{by (15)}) \\
&= (-1)^p f(v,u) \sum_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} U_i + \underbrace{\sum_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} \left( v \llcorner^f U_i \right)}_{\substack{= v \llcorner^f \sum\limits_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} U_i \\ (\text{by the bilinearity of } \llcorner^f)}} \otimes u \\
&= (-1)^p f(v,u) \underbrace{\sum_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} U_i}_{=U} + \left( v \llcorner^f \underbrace{\sum_{\substack{i \in \mathbb{N};\\ i \equiv p \mod 2}} U_i}_{=U} \right) \otimes u \\
&= (-1)^p f(v,u) U + \left( v \llcorner^f U \right) \otimes u.
\end{aligned}$$

This proves Theorem $10\frac{1}{2}$.

Finally, another straightforward fact:

**Theorem $10\frac{3}{4}$.** Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. If $w \in L$ and $U \in \otimes L$, then

$$w \llcorner^f U + w \llcorner^g U = w \llcorner^{f+g} U. \tag{16}$$

This theorem is immediately trivial using Theorem 5 **(c)**, but as we want to avoid using Theorem 5 **(c)**, here is a straightforward proof of Theorem $10\frac{3}{4}$ using tensor induction:

*Proof of Theorem $10\frac{3}{4}$.* Fix some $w \in L$. We will first show that for every $p \in \mathbb{N}$, the equation (16) holds for every $U \in L^{\otimes p}$.

In fact, we will prove this by induction over $p$:

The induction base case $p = 0$ is obvious (because in this case, $U \in L^{\otimes p} = L^{\otimes 0} = k$ and thus Theorem 5 **(a)** yields $w \llcorner^f U = 0$, $w \llcorner^g U = 0$ and $w \llcorner^{f+g} U = 0$, making the equation (16) trivially true).

So let us now come to the induction step: Let $p \in \mathbb{N}_+$. We must prove (16) for every $U \in L^{\otimes p}$, assuming that (16) has already been proven for every $U \in L^{\otimes (p-1)}$.

We want to prove that (16) holds for every $U \in L^{\otimes p}$. In order to do this, it is enough to prove that (16) holds for every left-induced $U \in L^{\otimes p}$ (by the left tensor induction tactic, because the equation (16) is linear in $U$). So, let us prove this. Let $U \in L^{\otimes p}$ be a left-induced tensor. Then, we can write $U$ in the form $U = u \otimes \ddot{U}$ for some $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ (because $U$ is left-induced). Thus,

$$w \overset{f}{\llcorner} U = w \overset{f}{\llcorner} \left(u \otimes \ddot{U}\right) = f\left(w, u\right) \ddot{U} - u \otimes \left(w \overset{f}{\llcorner} \ddot{U}\right) \tag{17}$$

(by (7), applied to $w$ and $\ddot{U}$ instead of $v$ and $U$). Also,

$$w \overset{g}{\llcorner} U = g\left(w, u\right) \ddot{U} - u \otimes \left(w \overset{g}{\llcorner} \ddot{U}\right)$$

(by (17), applied to $g$ instead of $f$) and

$$w \overset{f+g}{\llcorner} U = \left(f+g\right)\left(w, u\right) \ddot{U} - u \otimes \left(w \overset{f+g}{\llcorner} \ddot{U}\right)$$

(by (17), applied to $f+g$ instead of $f$). Hence,

$$\begin{aligned}
& w \overset{f}{\llcorner} U + w \overset{g}{\llcorner} U \\
&= \left(f\left(w, u\right) \ddot{U} - u \otimes \left(w \overset{f}{\llcorner} \ddot{U}\right)\right) + \left(g\left(w, u\right) \ddot{U} - u \otimes \left(w \overset{g}{\llcorner} \ddot{U}\right)\right) \\
&= \underbrace{\left(f\left(w, u\right) \ddot{U} + g\left(w, u\right) \ddot{U}\right)}_{=\left(f(w,u)+g(w,u)\right)\ddot{U}} - \underbrace{\left(u \otimes \left(w \overset{f}{\llcorner} \ddot{U}\right) + u \otimes \left(w \overset{g}{\llcorner} \ddot{U}\right)\right)}_{=u \otimes \left(w \overset{f}{\llcorner} \ddot{U} + w \overset{g}{\llcorner} \ddot{U}\right)} \\
&= \left(f\left(w, u\right) + g\left(w, u\right)\right) \ddot{U} - u \otimes \left(w \overset{f}{\llcorner} \ddot{U} + w \overset{g}{\llcorner} \ddot{U}\right) \\
&= \underbrace{\left(f\left(w, u\right) + g\left(w, u\right)\right)}_{=(f+g)(w,u)} \ddot{U} - u \otimes \left(w \overset{f+g}{\llcorner} \ddot{U}\right) \\
&\qquad \left(\begin{array}{c}
\text{since (16) (applied to } \ddot{U} \text{ instead of } U\text{) yields } w \overset{f}{\llcorner} \ddot{U} + w \overset{g}{\llcorner} \ddot{U} = w \overset{f+g}{\llcorner} \ddot{U} \\
\text{(in fact, we \textit{are} allowed to apply (16) to } \ddot{U} \text{ instead of } U\text{, since } \ddot{U} \in L^{\otimes (p-1)} \\
\text{and since (16) has already been proven for every } U \in L^{\otimes (p-1)}\text{)}
\end{array}\right) \\
&= \left(f+g\right)\left(w, u\right) \ddot{U} - u \otimes \left(w \overset{f+g}{\llcorner} \ddot{U}\right) = w \overset{f+g}{\llcorner} U.
\end{aligned}$$

Hence, the equality (16) is proven for every left-induced tensor $U \in L^{\otimes p}$. As we already said above, this entails that (16) must also hold for every tensor $U \in L^{\otimes p}$, and thus the induction step is complete. Hence, (16) is proven for every $p \in \mathbb{N}$ and every $U \in L^{\otimes p}$.

Consequently, the equation (16) holds for every $U \in \otimes L$ (since every $U \in \otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and since the equation (16) is $k$-linear). In other words, Theorem $10\frac{3}{4}$ is proven.

### 3. Right interior products on the tensor algebra

We have proven a number of properties of the interior product $\overset{f}{\llcorner}$. We are now going to introduce a very analogous construction $\overset{f}{\lrcorner}$ which works "from the right" almost the same way as $\overset{f}{\llcorner}$ works "from the left":

**Definition 7.** Let $f : L \times L \to k$ be a bilinear form. For every $p \in \mathbb{N}$ and every $v \in L$, we define a $k$-linear map $\rho^f_{v,p} : L^{\otimes p} \to L^{\otimes(p-1)}$ (where $L^{\otimes(-1)}$ means 0) by induction over $p$:

*Induction base:* For $p = 0$, we define the map $\delta^f_{v,p} : L^{\otimes 0} \to L^{\otimes(-1)}$ to be the zero map.

*Induction step:* For each $p \in \mathbb{N}_+$, we define a $k$-linear map $\delta^f_{v,p} : L^{\otimes p} \to L^{\otimes(p-1)}$ by

$$\left(\rho^f_{v,p}(U \otimes u) = f(u,v)\,U - \rho^f_{v,p-1}(U) \otimes u \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right), \tag{18}$$

assuming that we have already defined a $k$-linear map $\rho^f_{v,p-1} : L^{\otimes(p-1)} \to L^{\otimes(p-2)}$. (This definition is justified, because in order to define a $k$-linear map from $L^{\otimes p}$ to some other $k$-module, it is enough to define how it acts on tensors of the form $U \otimes u$ for every $u \in L$ and $U \in L^{\otimes(p-1)}$, as long as this action is bilinear with respect to $u$ and $U$. This is because $L^{\otimes p} = L^{\otimes(p-1)} \otimes L$.)

This way we have defined a $k$-linear map $\rho^f_{v,p} : L^{\otimes p} \to L^{\otimes(p-1)}$ for every $p \in \mathbb{N}$. We can combine these maps $\rho^f_{v,0}$, $\rho^f_{v,1}$, $\rho^f_{v,2}$, ... into one $k$-linear map $\rho^f_v : \otimes L \to \otimes L$ (since $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$), and the formula (1) rewrites as

$$\left(\rho^f_v(u \otimes U) = f(u,v)\,U - \rho^f_v(U) \otimes u \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right). \tag{19}$$

It is easily seen (by induction over $p \in \mathbb{N}$) that the map $\rho^f_{v,p}$ depends linearly on the vector $v \in L$. Hence, the combination $\rho^f_v$ of the maps $\rho^f_{v,0}$, $\rho^f_{v,1}$, $\rho^f_{v,2}$, ... must also depend linearly on $v \in L$. In other words, the map

$$(\otimes L) \times L \to \otimes L, \qquad (U, v) \mapsto \rho^f_v(U)$$

is $k$-bilinear. Hence, this map gives rise to a $k$-linear map

$$\rho^f : (\otimes L) \otimes L \to \otimes L, \qquad U \otimes v \mapsto \rho^f_v(U).$$

We are going to denote $\rho^f_v(U)$ by $U \overset{f}{\llcorner} v$ for each $v \in L$ and $U \in \otimes L$. Thus, the equality (2) takes the form

$$\left((U \otimes u) \overset{f}{\llcorner} v = f(u,v)\,U - \left(U \overset{f}{\llcorner} v\right) \otimes u \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right). \tag{20}$$

The tensor $U \overset{f}{\llcorner} v$ is called the *right interior product of* $v$ *and* $U$ with respect to the bilinear form $f$.

Again, many authors omit the $f$ in the notation $\overset{f}{\llcorner}$; in other words, they simply write $\llcorner$ for $\overset{f}{\llcorner}$. However, we are going to avoid this abbreviation, as we aim at considering several bilinear forms at once, and omitting the name of the bilinear form could lead to confusion.

Everything that we have proven for $\overset{f}{\llcorner}$ has an analogue for $\overset{f}{\lrcorner}$. In fact, we can take any identity concerning $\overset{f}{\llcorner}$, and "read it from right to left" to obtain an analogous property of $\overset{f}{\lrcorner}$ [18] For instance, reading the property (3) of $\overset{f}{\llcorner}$ from right to left, we obtain (20), because

- "reading the term $v \overset{f}{\llcorner} (u \otimes U)$ from right to left" means replacing it by $(U \otimes u) \overset{f}{\lrcorner} v$;
- "reading the term $f(v, u) U$ from right to left" means replacing it by $U f(u, v) = f(u, v) U$ (since $f(u, v) \in k$ is a scalar);
- "reading the term $u \otimes \left( v \overset{f}{\llcorner} U \right)$ from right to left" means replacing it by $\left( U \overset{f}{\lrcorner} v \right) \otimes u$.

If we take a theorem about the left interior product $\overset{f}{\llcorner}$ (for example, one of the Theorems 5-10), and "read it from right to left", we obtain a new theorem about the right interior product $\overset{f}{\lrcorner}$, and this new theorem is valid because we can read not only the theorem, but also its proof from right to left. This way, we get the following new theorems:

**Theorem 11.** Let $f : L \times L \to k$ be a bilinear form.

**(a)** For every $\lambda \in k$ and every $v \in L$, we have $\lambda \overset{f}{\lrcorner} v = 0$. [19]

**(b)** For every $u \in L$ and $v \in L$, we have $u \overset{f}{\lrcorner} v = f(u, v)$. [20]

**(c)** Let $u_1$, $u_2$, ..., $u_p$ be $p$ elements of $L$. Let $v \in L$. Then,

$$(u_1 \otimes u_2 \otimes ... \otimes u_p) \overset{f}{\lrcorner} v = \sum_{i=1}^{p} (-1)^{p-i} f(u_i, v) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p. \quad (21)$$

Here, the hat over the vector $u_i$ means that the vector $u_i$ is being omitted from the tensor product; in other words, $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$ is just another way to write $\underbrace{u_1 \otimes u_2 \otimes ... \otimes u_{i-1}}_{\substack{\text{tensor product of the} \\ \text{first } i-1 \text{ vectors } u_\ell}} \otimes \underbrace{u_{i+1} \otimes u_{i+2} \otimes ... \otimes u_p}_{\substack{\text{tensor product of the} \\ \text{last } p-i \text{ vectors } u_\ell}}$.

---

[18] "Reading from right to left" means

- replacing every term of the form $v \overset{f}{\llcorner} U$ by $U \overset{f}{\lrcorner} v$ (where $v \in L$ and $U \in \otimes L$), and vice versa;
- reversing the order in every tensor product;
- replacing every $f(u, v)$ by $f(v, u)$.

However, some care must be taken here: when our identity is of the form

$$\left( \text{sum of some terms involving vectors, tensors and } \otimes \text{ and } \overset{f}{\llcorner} \text{ signs} \right) = (\text{another sum of terms of this kind}),$$

then we should *not* read each of the sums from right to left, but we should read *every of their terms* from right to left. (Thus, reading a term like $a \otimes b - c \overset{f}{\llcorner} d$ from right to left, we get $b \otimes a - d \overset{f}{\lrcorner} c$, and not $d \overset{f}{\lrcorner} c - b \otimes a$.)

[19] Here, $\lambda \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

[20] Here, $f(u, v) \in k$ is considered as an element of $\otimes L$ by means of the canonical inclusion $k = L^{\otimes 0} \subseteq \otimes L$.

**Theorem 12.** If $u \in L$, $U \in \otimes L$, and $v \in L$, then

$$(U \otimes u) \overset{f}{\lrcorner} v = f(u,v)\, U - \left(U \overset{f}{\lrcorner} v\right) \otimes u. \tag{22}$$

**Theorem 13.** If $v \in L$ and $U \in \otimes L$, then

$$\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} v = 0. \tag{23}$$

**Theorem 14.** If $v \in L$, $w \in L$ and $U \in \otimes L$, then

$$\left(U \overset{f}{\lrcorner} w\right) \overset{f}{\lrcorner} v = -\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} w. \tag{24}$$

**Theorem 15.** If $p \in \mathbb{N}$, $u \in L$, $U \in L^{\otimes p}$, and $v \in L$, then

$$(u \otimes U) \overset{f}{\lrcorner} v = (-1)^p f(u,v)\, U + u \otimes \left(U \overset{f}{\lrcorner} v\right). \tag{25}$$

**Theorem 16.** If $p \in \mathbb{N}$, $v \in L$, $U \in L^{\otimes p}$, and $V \in \otimes L$, then

$$(V \otimes U) \overset{f}{\lrcorner} v = (-1)^p \left(V \overset{f}{\lrcorner} v\right) \otimes U + V \otimes \left(U \overset{f}{\lrcorner} v\right). \tag{26}$$

**Theorem $16\frac{1}{2}$.** If $p \in \mathbb{N}$, $u \in L$, $U \in \bigoplus\limits_{\substack{i\in\mathbb{N};\\ i\equiv p \bmod 2}} L^{\otimes i}$, and $v \in L$, then

$$(u \otimes U) \overset{f}{\lrcorner} v = (-1)^p f(u,v)\, U + u \otimes \left(U \overset{f}{\lrcorner} v\right). \tag{27}$$

These Theorems 11-16 are simply the results of reading Theorems 5-10 from right to left, so as we said, we don't really need to give proofs for them (because one can simply read the proofs of Theorems 5-10 from right to left, and thus obtain proofs of Theorems 11-16). Yet, we are going to present the proof of Theorem 11 explicitly[21], and we will later reprove Theorems 12-16 in a different way.

*Proof of Theorem 11.* **(a)** We have $\lambda \in k = L^{\otimes 0}$ and thus $\rho_v^f(\lambda) = \underbrace{\rho_{v,0}^f}_{=0}(\lambda) = 0(\lambda) = 0$. Thus, $\lambda \overset{f}{\lrcorner} v = \delta_v^f(\lambda) = 0$, and Theorem 11 **(a)** is proven.

**(b)** Applying (20) to $U = 1$, we see that

$$(1 \otimes u) \overset{f}{\lrcorner} v = f(u,v)\, 1 - \underbrace{\left(1 \overset{f}{\lrcorner} v\right)}_{=0 \text{ (by Theorem 11 } \mathbf{(a)}\text{)}} \otimes u = f(u,v)\, 1 - 0 \otimes u = f(u,v).$$

Since $1 \otimes u = u$, this rewrites as $u \overset{f}{\lrcorner} v = f(u,v)$. Thus, Theorem 11 **(b)** is proven.

**(c)** We are going to prove Theorem 11 **(c)** by induction over $p$:

[21] This is mainly because Theorem 11 does not result *verbatim* from reading Theorem 5 from right to left, but instead requires some more changes (such as renaming $u_1 \otimes u_2 \otimes ... \otimes u_p$ by $u_p \otimes u_{p-1} \otimes ... \otimes u_1$, and renaming $i-1$ by $p-i$).

The induction base is clear, since for $p = 0$, Theorem 11 **(c)** trivially follows from Theorem 11 **(a)**[22].

Now to the induction step: Let $p \in \mathbb{N}_+$. Let us prove Theorem 11 **(c)** for this $p$, assuming that we have already shown Theorem 11 **(c)** applied to $p-1$ instead of $p$.

In fact, we have assumed that we have already shown Theorem 11 **(c)** applied to $p-1$ instead of $p$. In other words, we have already shown the equality

$$\left(u_1 \otimes u_2 \otimes ... \otimes u_{p-1}\right) \overset{f}{\lrcorner} v = \sum_{i=1}^{p-1} (-1)^{(p-1)-i} f\left(u_i, v\right) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{p-1}. \quad (28)$$

for any $p-1$ vectors $u_1$, $u_2$, ..., $u_{p-1}$ in $L$. Now, our goal is to prove the equality (21) for any $p$ vectors $u_1$, $u_2$, ..., $u_p$ in $L$.

[22] because for $p = 0$, we have $u_1 \otimes u_2 \otimes ... \otimes u_p = (\text{empty product}) = 1 \in k$ and $\sum\limits_{i=1}^{p} (-1)^{p-i} f\left(v, u_i\right) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p = (\text{empty sum}) = 0$

Applying (20) to $u = u_p$ and $U = u_1 \otimes u_2 \otimes ... \otimes u_{p-1}$, we get

$$\begin{aligned}
&\left(u_1 \otimes u_2 \otimes ... \otimes u_{p-1} \otimes u_p\right) \overset{f}{\lrcorner} v \\
&= f\left(u_p, v\right) u_1 \otimes u_2 \otimes ... \otimes u_{p-1} - \left( \underbrace{\left(u_1 \otimes u_2 \otimes ... \otimes u_{p-1}\right) \overset{f}{\lrcorner} v}_{\substack{= \sum\limits_{i=1}^{p-1} (-1)^{(p-1)-i} f(u_i, v) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{p-1} \\ \text{(by (28))}}} \right) \otimes u_p \\
&= \underbrace{f\left(u_p, v\right)}_{=(-1)^{p-p} f(u_p, v)} u_1 \otimes u_2 \otimes ... \otimes u_{p-1} \\
&\qquad - \left( \sum_{i=1}^{p-1} \underbrace{(-1)^{(p-1)-i}}_{=(-1)^{p-i-1} = -(-1)^{p-i}} f\left(u_i, v\right) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{p-1} \right) \otimes u_p \\
&= (-1)^{p-p} f\left(u_p, v\right) u_1 \otimes u_2 \otimes ... \otimes u_{p-1} \\
&\qquad - \left( \sum_{i=1}^{p-1} \left(-(-1)^{p-i}\right) f\left(u_i, v\right) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{p-1} \right) \otimes u_p \\
&= (-1)^{p-p} f\left(u_p, v\right) \underbrace{u_1 \otimes u_2 \otimes ... \otimes u_{p-1}}_{= u_1 \otimes u_2 \otimes ... \otimes \widehat{u_p} \otimes ... \otimes u_p} \\
&\qquad + \underbrace{\left( \sum_{i=1}^{p-1} (-1)^{p-i} f\left(u_i, v\right) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{p-1} \right) \otimes u_p}_{\substack{= \sum\limits_{i=1}^{p-1} (-1)^{p-i} f(u_i, v) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{p-1} \otimes u_p \\ = \sum\limits_{i=1}^{p-1} (-1)^{p-i} f(u_i, v) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p}} \\
&= (-1)^{p-p} f\left(u_p, v\right) u_1 \otimes u_2 \otimes ... \otimes \widehat{u_p} \otimes ... \otimes u_p \\
&\qquad + \sum_{i=1}^{p-1} (-1)^{p-i} f\left(u_i, v\right) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \\
&= \sum_{i=1}^{p} (-1)^{p-i} f\left(u_i, v\right) \cdot u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p .
\end{aligned}$$

Thus, (21) is proven for our $p \in \mathbb{N}$. In other words, we have proven Theorem 11 **(c)** for our $p \in \mathbb{N}$. This completes the induction step, and thus the proof of Theorem 11 **(c)** is complete.

Let us notice another property of $\overset{f}{\lrcorner}$: the *bilinearity of* $\overset{f}{\lrcorner}$. This property states that the map

$$(\otimes L) \times L \to \otimes L, \qquad (U, v) \mapsto U \overset{f}{\lrcorner} v$$

is $k$-bilinear[23], i. e. that $U\overset{f}{\lrcorner}(\alpha v+\beta v') = \alpha U\overset{f}{\lrcorner}v + \beta U\overset{f}{\lrcorner}v'$ and that $(\alpha U+\beta U')\overset{f}{\lrcorner}v = \alpha U\overset{f}{\lrcorner}v + \beta U'\overset{f}{\lrcorner}v$ for any $v \in L$, $v' \in L$, $U \in \otimes L$ and $U' \in \otimes L$.

As we said above, Theorems 12-16 don't need to be proven in details, because simply reading the proofs of Theorems 6-10 from right to left yields proofs of Theorems 12-16. However, there also is a different way to prove these theorems, namely by defining an automorphism of the $k$-module $\otimes L$:

**Definition 8.** For every $p \in \mathbb{N}$, we define an endomorphism $t_p : L^{\otimes p} \to L^{\otimes p}$ of the $k$-module $L^{\otimes p}$ by

$$\left(t_p\left(u_1 \otimes u_2 \otimes ... \otimes u_p\right) = u_p \otimes u_{p-1} \otimes ... \otimes u_1 \qquad \text{for any vectors } u_1,\ u_2,\ ...,\ u_p \text{ in } L\right). \tag{29}$$

[24] These endomorphisms $t_0$, $t_1$, $t_2$, ... can be combined together to an endomorphism $t : \otimes L \to \otimes L$ of the $k$-module $\otimes L$ (since $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$).

This map $t$ satisfies

$$t\left(u_1 \otimes u_2 \otimes ... \otimes u_p\right) = u_p \otimes u_{p-1} \otimes ... \otimes u_1 \tag{30}$$
$$\text{for any } p \in \mathbb{N} \text{ and any vectors } u_1,\ u_2,\ ...,\ u_p \text{ in } L.$$

(due to (29)). This obviously yields $t^2 = \mathrm{id}$ [25]. Hence, the map $t : \otimes L \to \otimes L$ is bijective. Besides,

$$t\left(U \otimes V\right) = t\left(V\right) \otimes t\left(U\right) \qquad \text{for every } U \in \otimes L \text{ and } V \in \otimes L. \tag{31}$$

[26] Also, obviously, $t(u) = u$ for every $u \in L$.

Our use for the map $t$ is now to reduce the right interior product $\overset{f}{\lrcorner}$ to the left interior product $\overset{f}{\llcorner}$. For this we need yet another definition:

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[23]This is because $U\overset{f}{\lrcorner}v = \rho_v^f(U)$, and because the map

$$(\otimes L) \times L \to \otimes L, \qquad (U, v) \mapsto \rho_v^f(U)$$

is $k$-bilinear.

[24]This definition is legitimate, because the map $\underbrace{L \times L \times ... \times L}_{p \text{ times}} \to L^{\otimes p}$ given by

$$(u_1, u_2, ..., u_p) \mapsto u_p \otimes u_{p-1} \otimes ... \otimes u_1 \qquad \text{for any vectors } u_1,\ u_2,\ ...,\ u_p \text{ in } L$$

is $k$-multilinear, and thus yields a map $t_p : L^{\otimes p} \to L^{\otimes p}$ satisfying (29).

[25]In fact,

$$t^2\left(u_1 \otimes u_2 \otimes ... \otimes u_p\right) = t\left(\underbrace{t\left(u_1 \otimes u_2 \otimes ... \otimes u_p\right)}_{\substack{=u_p \otimes u_{p-1} \otimes ... \otimes u_1 \\ \text{(due to (30))}}}\right) = t\left(u_p \otimes u_{p-1} \otimes ... \otimes u_1\right) = u_1 \otimes u_2 \otimes ... \otimes u_p$$
$$\text{(due to (30), applied to } u_p,\ u_{p-1},\ ...,\ u_1 \text{ instead of } u_1,\ u_2,\ ...,\ u_p\text{)}$$

for any $p \in \mathbb{N}$ and any vectors $u_1$, $u_2$, ..., $u_p$ in $L$. Thus, $t^2(U) = U$ for every $U \in \otimes L$ (because every $U \in \otimes L$ is a $k$-linear combination of tensors of the form $u_1 \otimes u_2 \otimes ... \otimes u_p$ (for $p \in \mathbb{N}$ and vectors $u_1$, $u_2$, ..., $u_p$ in $L$), and because the equation $t^2(U) = U$ is linear in $U$). In other words, $t^2 = \mathrm{id}$, qed.

[26]*Proof of (31).* We WLOG assume that $U = u_1 \otimes u_2 \otimes ... \otimes u_q$ for some $q \in \mathbb{N}$ and some vectors

**Definition 9.** Let $f : L \times L \to k$ be a bilinear form. Then, we define a new bilinear form $f^t : L \times L \to k$ by

$$\left( f^t(u,v) = f(v,u) \qquad \text{for every } u \in L \text{ and } v \in L \right).$$

This bilinear form $f^t$ is called the *transpose* of the bilinear form $f$.

It is clear that $(f^t)^t = f$ for any bilinear form $f$, and that a bilinear form $f$ is symmetric if and only if $f = f^t$.

Now, here is a way to write $\overset{f}{\lrcorner}$ in terms of $\overset{f^t}{\llcorner}$:

**Theorem 17.** Let $v \in L$ and $U \in \otimes L$. Then,

$$t\left( U \overset{f}{\lrcorner} v \right) = v \overset{f^t}{\llcorner} t(U) \tag{32}$$

and

$$t\left( v \overset{f^t}{\llcorner} U \right) = t(U) \overset{f}{\lrcorner} v. \tag{33}$$

*Proof of Theorem 17.* Fix some $v \in L$. We are first going to prove that $t\left( U \overset{f}{\lrcorner} v \right) = v \overset{f^t}{\llcorner} t(U)$ for every $p \in \mathbb{N}$ and every $U \in L^{\otimes p}$. In fact, we will prove this by induction over $p$: The induction base case $p = 0$ is trivial[27]. So let us come to the induction step: Let $p \in \mathbb{N}_+$. Assume that we have already proven

$$t\left( U \overset{f}{\lrcorner} v \right) = v \overset{f^t}{\llcorner} t(U) \qquad \text{for every} \qquad U \in L^{\otimes (p-1)}. \tag{34}$$

Now we must prove $t\left( U \overset{f}{\lrcorner} v \right) = v \overset{f^t}{\llcorner} t(U)$ for every $U \in L^{\otimes p}$. In order to do this, it is clearly enough to prove $t\left( U \overset{f}{\lrcorner} v \right) = v \overset{f^t}{\llcorner} t(U)$ for every right-induced $U \in L^{\otimes p}$ (by the right tensor induction tactic, because the equation $t\left( U \overset{f}{\lrcorner} v \right) = v \overset{f^t}{\llcorner} t(U)$ is linear in $U$).

---

$u_1$, $u_2$, ..., $u_q$ in $L$. (In fact, this assumption is legitimate, since every $U \in \otimes L$ can be written as a $k$-linear combination of tensors of the form $u_1 \otimes u_2 \otimes ... \otimes u_q$ for $q \in \mathbb{N}$ and vectors $u_1$, $u_2$, ..., $u_q$ in $L$, and since the equation (31) is linear in $U$.) We also WLOG assume that $V = u_{q+1} \otimes u_{q+2} \otimes ... \otimes u_p$ for some $p \in \mathbb{N}$ and for some vectors $u_{q+1}$, $u_{q+2}$, ..., $u_p$ in $L$. (In fact, this assumption is legitimate, since every $V \in \otimes L$ can be written as a $k$-linear combination of tensors of the form $u_{q+1} \otimes u_{q+2} \otimes ... \otimes u_p$ for $p \in \mathbb{N}$ and vectors $u_{q+1}$, $u_{q+2}$, ..., $u_p$ in $L$, and since the equation (31) is linear in $V$.) Then,

$$\begin{aligned} U \otimes V &= (u_1 \otimes u_2 \otimes ... \otimes u_q) \otimes (u_{q+1} \otimes u_{q+2} \otimes ... \otimes u_p) = u_1 \otimes u_2 \otimes ... \otimes u_p, \qquad \text{so that} \\ t(U \otimes V) &= t(u_1 \otimes u_2 \otimes ... \otimes u_p) = u_p \otimes u_{p-1} \otimes ... \otimes u_1 \\ &= \underbrace{(u_p \otimes u_{p-1} \otimes ... \otimes u_{q+1})}_{=t(u_{q+1} \otimes u_{q+2} \otimes ... \otimes u_p) = t(V)} \otimes \underbrace{(u_q \otimes u_{q-1} \otimes ... \otimes u_1)}_{=t(u_1 \otimes u_2 \otimes ... \otimes u_q) = t(U)} = t(V) \otimes t(U), \end{aligned}$$

so that (31) is proven.

[27]In fact, in the case $p = 0$, we have $U \in L^{\otimes p} = L^{\otimes 0} = k$ and thus $U \overset{f}{\lrcorner} v = 0$ (by Theorem 11 **(a)**) and $v \overset{f^t}{\llcorner} t(U) = 0$ (by Theorem 5 **(a)**, applied to $f^t$ instead of $f$), which makes the assertion $t\left( U \overset{f}{\lrcorner} v \right) = v \overset{f^t}{\llcorner} t(U)$ obvious.

But this is easy: If $U \in L^{\otimes p}$ is a right-induced tensor, then $U = \ddot{U} \otimes u$ for some $u \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$, and thus

$$\begin{aligned}
t\left(U \overset{f}{\lrcorner} v\right) &= t\left(\left(\ddot{U} \otimes u\right) \overset{f}{\lrcorner} v\right) = t\left(f(u,v)\,\ddot{U} - \left(\ddot{U} \overset{f}{\lrcorner} v\right) \otimes u\right) \\
&\qquad \left(\begin{array}{c} \text{since } \left(\ddot{U} \otimes u\right) \overset{f}{\lrcorner} v = f(u,v)\,\ddot{U} - \left(\ddot{U} \overset{f}{\lrcorner} v\right) \otimes u \\ \text{by (20) (applied to } \ddot{U} \text{ instead of } U) \end{array}\right) \\
&= f(u,v)\, t\left(\ddot{U}\right) - \underbrace{t\left(\left(\ddot{U} \overset{f}{\lrcorner} v\right) \otimes u\right)}_{\substack{= t(u) \otimes t\left(\ddot{U} \overset{f}{\lrcorner} v\right) \\ \text{(by (31), applied} \\ \text{to } \ddot{U} \overset{f}{\lrcorner} v \text{ and } u \text{ instead of } U \text{ and } V)}} \qquad \text{(since } t \text{ is } k\text{-linear)} \\
&= f(u,v)\, t\left(\ddot{U}\right) - \underbrace{t(u)}_{= u \text{ (since } u \in L)} \otimes t\left(\ddot{U} \overset{f}{\lrcorner} v\right) \\
&= f(u,v)\, t\left(\ddot{U}\right) - u \otimes t\left(\ddot{U} \overset{f}{\lrcorner} v\right) = f(u,v)\, t\left(\ddot{U}\right) - u \otimes \left(v \overset{f^t}{\llcorner} t\left(\ddot{U}\right)\right)
\end{aligned}$$

(since $\ddot{U} \in L^{\otimes(p-1)}$ yields $t\left(\ddot{U} \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} t\left(\ddot{U}\right)$, according to (34) (applied to $\ddot{U}$ instead of $U$)) and

$$\begin{aligned}
v \overset{f^t}{\llcorner} t(U) &= v \overset{f^t}{\llcorner} \left(\underbrace{t(u)}_{= u \text{ (since } u \in L)} \otimes t\left(\ddot{U}\right)\right) \\
&\qquad \left(\begin{array}{c} \text{since } U = \ddot{U} \otimes u \text{ yields } t(U) = t\left(\ddot{U} \otimes u\right) = t(u) \otimes t\left(\ddot{U}\right) \text{ by (31)} \\ \text{(applied to } \ddot{U} \text{ and } u \text{ instead of } U \text{ and } V) \end{array}\right) \\
&= v \overset{f^t}{\llcorner} \left(u \otimes t\left(\ddot{U}\right)\right) = \underbrace{f^t(v,u)}_{= f(u,v)}\, t\left(\ddot{U}\right) - u \otimes \left(v \overset{f^t}{\llcorner} f\left(\ddot{U}\right)\right) \\
&\qquad \left(\text{by (3), applied to } f^t \text{ and } t\left(\ddot{U}\right) \text{ instead of } f \text{ and } U\right) \\
&= f(u,v)\, t\left(\ddot{U}\right) - u \otimes \left(v \overset{f^t}{\llcorner} f\left(\ddot{U}\right)\right)
\end{aligned}$$

lead to $t\left(U \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} t(U)$. This completes our induction step, and thus we have proven that $t\left(U \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} t(U)$ for every $p \in \mathbb{N}$ and every $U \in L^{\otimes p}$. This immediately yields that $t\left(U \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} t(U)$ for every $U \in \otimes L$ (because every $U \in \otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for different $p \in \mathbb{N}$, and since the equation $t\left(U \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} t(U)$ is linear in $U$). Thus, (32) is proven. In order to prove Theorem 17, it now only remains to prove (33).

In fact, applying (32) to $t(U)$ instead of $U$, we obtain $t\left(t(U) \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} \underbrace{t(t(U))}_{\substack{= t^2(U) = U \\ \text{(since } t^2 = \mathrm{id})}} =$

$v \overset{f^t}{\llcorner} U$. Thus, $t\left(t\left(t\left(U\right) \overset{f}{\lrcorner} v\right)\right) = t\left(v \overset{f^t}{\llcorner} U\right)$. Since $t\left(t\left(t\left(U\right) \overset{f}{\lrcorner} v\right)\right) = t^2\left(t\left(U\right) \overset{f}{\lrcorner} v\right) = t\left(U\right) \overset{f}{\lrcorner} v$ (since $t^2 = \mathrm{id}$), this becomes $t\left(U\right) \overset{f}{\lrcorner} v = t\left(v \overset{f^t}{\llcorner} U\right)$, and thus (33) is proven.

So we have now proven both (32) and (33). Thus, Theorem 17 is proven.

Now, Theorem 17 enables us to prove Theorems 12-16 quickly:

*Proof of Theorem 12.* We have

$$\begin{aligned}
t\left(\left(U \overset{f}{\lrcorner} v\right) \otimes u\right) &= \underbrace{t\left(u\right)}_{=u \text{ (since } u \in L)} \otimes t\left(U \overset{f}{\lrcorner} v\right) \\
&\qquad \left(\text{by (31), applied to } U \overset{f}{\lrcorner} v \text{ and } u \text{ instead of } U \text{ and } V\right) \\
&= u \otimes t\left(U \overset{f}{\lrcorner} v\right) = u \otimes \left(v \overset{f^t}{\llcorner} t\left(U\right)\right) \qquad \text{(by (32))}
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
t\left(\left(U \otimes u\right) \overset{f}{\lrcorner} v\right) &= v \overset{f^t}{\llcorner} t\left(U \otimes u\right) \qquad \left(\text{by (32), applied to } U \otimes u \text{ instead of } U\right) \\
&= v \overset{f^t}{\llcorner} \left(\underbrace{t\left(u\right)}_{=u \text{ (since } u \in L)} \otimes t\left(U\right)\right) \\
&\qquad \left(\text{since (31) (applied to } V = u\text{) yields } t\left(U \otimes u\right) = t\left(u\right) \otimes t\left(U\right)\right) \\
&= v \overset{f^t}{\llcorner} \left(u \otimes t\left(U\right)\right) = \underbrace{f^t\left(v, u\right)}_{=f(u,v)} t\left(U\right) - u \otimes \left(v \overset{f^t}{\llcorner} t\left(U\right)\right) \\
&\qquad \left(\text{by (7), applied to } f^t \text{ and } t\left(U\right) \text{ instead of } f \text{ and } U\right) \\
&= f\left(u, v\right) t\left(U\right) - \underbrace{u \otimes \left(v \overset{f^t}{\llcorner} t\left(U\right)\right)}_{=t\left(\left(U \overset{f}{\lrcorner} v\right) \otimes u\right) \text{ (by (35))}} = f\left(u, v\right) t\left(U\right) - t\left(\left(U \overset{f}{\lrcorner} v\right) \otimes u\right) \\
&= t\left(f\left(u, v\right) U - \left(U \overset{f}{\lrcorner} v\right) \otimes u\right)
\end{aligned}$$

(since the map $t$ is $k$-linear). Since $t$ is injective (because $t$ is bijective), this yields

$$\left(U \otimes u\right) \overset{f}{\lrcorner} v = f\left(u, v\right) U - \left(U \overset{f}{\lrcorner} v\right) \otimes u.$$

This proves Theorem 12.

*Proof of Theorem 13.* The equation (32), applied to $U \overset{f}{\lrcorner} v$ instead of $U$, yields

$$\begin{aligned}
t\left(\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} v\right) &= v \overset{f^t}{\llcorner} t\left(U \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} \left(v \overset{f^t}{\llcorner} t\left(U\right)\right) \qquad \text{(by (32))} \\
&= 0
\end{aligned}$$

(by (8), applied to $f^t$ instead of $f$). This proves Theorem 13.

*Proof of Theorem 14.* The equation (32), applied to $U \overset{f}{\lrcorner} v$ and $w$ instead of $U$ and $v$, yields

$$t\left(\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} w\right) = w \overset{f^t}{\llcorner} t\left(U \overset{f}{\lrcorner} v\right) = w \overset{f^t}{\llcorner} \left(v \overset{f^t}{\llcorner} t\left(U\right)\right) \qquad \text{(by (32))}.$$

Similarly, $t\left(\left(U \overset{f}{\lrcorner} w\right) \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} \left(w \overset{f^t}{\llcorner} t\left(U\right)\right)$. Together with the equality $v \overset{f^t}{\llcorner} \left(w \overset{f^t}{\llcorner} U\right) = -w \overset{f^t}{\llcorner} \left(v \overset{f^t}{\llcorner} U\right)$ (which follows from (9), applied to $f^t$ instead of $f$), this yields

$$t\left(\left(U \overset{f}{\lrcorner} w\right) \overset{f}{\lrcorner} v\right) = v \overset{f^t}{\llcorner} \left(w \overset{f^t}{\llcorner} t\left(U\right)\right) = -\underbrace{w \overset{f^t}{\llcorner} \left(v \overset{f^t}{\llcorner} U\right)}_{=t\left(\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} w\right)} = -t\left(\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} w\right) = t\left(-\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} w\right).$$

Since $t$ is injective (because $t$ is bijective), this results in $\left(U \overset{f}{\lrcorner} w\right) \overset{f}{\lrcorner} v = -\left(U \overset{f}{\lrcorner} v\right) \overset{f}{\lrcorner} w$, which proves Theorem 14.

*Proof of Theorem 16.* Applying (32) to $V \otimes U$ instead of $U$, we get

$$\begin{aligned} t\left(\left(V \otimes U\right) \overset{f}{\lrcorner} v\right) &= v \overset{f^t}{\llcorner} t\left(V \otimes U\right) = v \overset{f^t}{\llcorner} \left(t\left(U\right) \otimes t\left(V\right)\right) \\ &\qquad \text{(due to (31), applied to } V \text{ and } U \text{ instead of } U \text{ and } V\text{)} \\ &= \left(-1\right)^p t\left(U\right) \otimes \left(v \overset{f^t}{\llcorner} t\left(V\right)\right) + \left(v \overset{f^t}{\llcorner} t\left(U\right)\right) \otimes t\left(V\right) \end{aligned} \tag{36}$$

(by (12), applied to $f^t$, $t\left(U\right)$ and $t\left(V\right)$ instead of $f$, $U$ and $V$). On the other hand, (31) (applied to $V \overset{f}{\lrcorner} v$ and $U$ instead of $U$ and $V$) leads to

$$t\left(\left(V \overset{f}{\lrcorner} v\right) \otimes U\right) = t\left(U\right) \otimes \underbrace{t\left(V \overset{f}{\lrcorner} v\right)}_{\substack{=v \overset{f^t}{\llcorner} t(V) \text{ (by (32), applied} \\ \text{to } V \text{ instead of } U\text{)}}} = t\left(U\right) \otimes \left(v \overset{f^t}{\llcorner} t\left(V\right)\right).$$

Also, (31) (applied to $V$ and $U \overset{f}{\lrcorner} v$ instead of $U$ and $V$) leads to

$$t\left(V \otimes \left(U \overset{f}{\lrcorner} v\right)\right) = \underbrace{t\left(U \overset{f}{\lrcorner} v\right)}_{\substack{=v \overset{f^t}{\llcorner} t(U) \\ \text{(by (32))}}} \otimes t\left(V\right) = \left(v \overset{f^t}{\llcorner} t\left(U\right)\right) \otimes t\left(V\right).$$

Thus,

$$\begin{aligned} &t\left(\left(-1\right)^p \left(V \overset{f}{\lrcorner} v\right) \otimes U + V \otimes \left(U \overset{f}{\lrcorner} v\right)\right) \\ &= \left(-1\right)^p \underbrace{t\left(\left(V \overset{f}{\lrcorner} v\right) \otimes U\right)}_{=t(U) \otimes \left(v \overset{f^t}{\llcorner} t(V)\right)} + \underbrace{t\left(V \otimes \left(U \overset{f}{\lrcorner} v\right)\right)}_{=\left(v \overset{f^t}{\llcorner} t(U)\right) \otimes t(V)} \\ &= \left(-1\right)^p t\left(U\right) \otimes \left(v \overset{f^t}{\llcorner} t\left(V\right)\right) + \left(v \overset{f^t}{\llcorner} t\left(U\right)\right) \otimes t\left(V\right) = t\left(\left(V \otimes U\right) \overset{f}{\lrcorner} v\right) \end{aligned}$$

(by (36)). Since $t$ is injective (because $t$ is bijective), this entails

$$(-1)^p \left(V \overset{f}{\lrcorner} v\right) \otimes U + V \otimes \left(U \overset{f}{\lrcorner} v\right) = (V \otimes U) \overset{f}{\lrcorner} v.$$

Thus, Theorem 16 is proven.

*Proof of Theorem 15.* Applying Theorem 16 to $V = u$, we obtain

$$(u \otimes U) \overset{f}{\lrcorner} v = (-1)^p \left(u \overset{f}{\lrcorner} v\right) \otimes U + u \otimes \left(U \overset{f}{\lrcorner} v\right).$$

Since $u \overset{f}{\lrcorner} v = f(u, v)$ (by Theorem 11 **(b)**), this becomes

$$(u \otimes U) \overset{f}{\lrcorner} v = (-1)^p \underbrace{f(u, v) \otimes U}_{= f(u,v) U} + u \otimes \left(U \overset{f}{\lrcorner} v\right) = (-1)^p f(u, v) U + u \otimes \left(U \overset{f}{\lrcorner} v\right),$$

and therefore Theorem 15 is proven.

*Proof of Theorem 16$\frac{1}{2}$.* Applying (32) to $u \otimes U$ instead of $U$, we get

$$\begin{aligned}
t\left((u \otimes U) \overset{f}{\lrcorner} v\right) &= v \overset{f^t}{\llcorner} t(u \otimes U) = v \overset{f^t}{\llcorner} (t(U) \otimes t(u)) \\
&\qquad \text{(due to (31), applied to } u \text{ and } U \text{ instead of } U \text{ and } V\text{)} \\
&= v \overset{f^t}{\llcorner} (t(U) \otimes u) \qquad \text{(since } t(u) = u \text{, because } u \in L\text{)} \\
&= (-1)^p f^t(v, u) t(U) + \left(v \overset{f^t}{\llcorner} t(U)\right) \otimes u \qquad (37)
\end{aligned}$$

(by (14) (applied to $f^t$ and $t(U)$ instead of $f$ and $U$), since $U \in \bigoplus\limits_{\substack{i \in \mathbb{N};\\ i \equiv p \bmod 2}} L^{\otimes i}$ yields $t(U) \in \bigoplus\limits_{\substack{i \in \mathbb{N};\\ i \equiv p \bmod 2}} L^{\otimes i}$ (because the map $t$ is composed of the maps $t_i : L^{\otimes i} \to L^{\otimes i}$ for all $i \in \mathbb{N}$, and thus maps $L^{\otimes i}$ into $L^{\otimes i}$ for all $i \in \mathbb{N}$)). On the other hand, (31) (applied to $u$ and $U \overset{f}{\lrcorner} v$ instead of $U$ and $V$) leads to

$$t\left(u \otimes \left(U \overset{f}{\lrcorner} v\right)\right) = \underbrace{t\left(U \overset{f}{\lrcorner} v\right)}_{\substack{= v \overset{f^t}{\llcorner} t(U) \\ \text{(by (32))}}} \otimes \underbrace{t(u)}_{= u \text{ (since } u \in L)} = \left(v \overset{f^t}{\llcorner} t(U)\right) \otimes u.$$

Since the map $t$ is $k$-linear, we now have

$$\begin{aligned}
t\left((-1)^p f(u, v) U + u \otimes \left(U \overset{f}{\lrcorner} v\right)\right) &= (-1)^p \underbrace{f(u, v)}_{= f^t(v,u)} t(U) + \underbrace{t\left(u \otimes \left(U \overset{f}{\lrcorner} v\right)\right)}_{= \left(v \overset{f^t}{\llcorner} t(U)\right) \otimes u} \\
&= (-1)^p f^t(v, u) t(U) + \left(v \overset{f^t}{\llcorner} t(U)\right) \otimes u \\
&= t\left((u \otimes U) \overset{f}{\lrcorner} v\right)
\end{aligned}$$

(by (37)). Since $t$ is injective (because $t$ is bijective), this entails

$$(-1)^p f(u,v)\,U + u \otimes \left(U \overset{f}{\lrcorner} v\right) = (u \otimes U) \overset{f}{\lrcorner} v.$$

Thus, Theorem $16\frac{1}{2}$ is proven.

(This proof of Theorem $16\frac{1}{2}$ yields a new proof of Theorem 15.)

Of course, Theorem $10\frac{3}{4}$ has its right counterpart as well:

**Theorem $16\frac{3}{4}$.** Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. If $w \in L$ and $U \in \otimes L$, then

$$U \overset{f}{\lrcorner} w + U \overset{g}{\lrcorner} w = U \overset{f+g}{\lrcorner} w.$$

We won't prove this theorem, since we won't ever use it and since it should now be absolutely clear how to derive it from Theorem $10\frac{3}{4}$ with the help of (32) (or how to prove it analogously to Theorem $10\frac{3}{4}$).

## 4. The two operations commute

Now that we know quite a lot about each of the operations $\overset{f}{\llcorner}$ and $\overset{f}{\lrcorner}$, let us show a relation between them:

**Theorem 18.** Let $v \in L$, $w \in L$ and $U \in \otimes L$. Then

$$v \overset{f}{\llcorner} \left(U \overset{f}{\lrcorner} w\right) = \left(v \overset{f}{\llcorner} U\right) \overset{f}{\lrcorner} w. \tag{38}$$

More generally, if $f : L \times L \to k$ and $g : L \times L \to k$ are two bilinear forms, then

$$v \overset{f}{\llcorner} \left(U \overset{g}{\lrcorner} w\right) = \left(v \overset{f}{\llcorner} U\right) \overset{g}{\lrcorner} w. \tag{39}$$

*Proof of Theorem 18.* In order to prove Theorem 18, it is enough to prove the equality (39) only (because the equality (38) directly follows from the equality (39), applied to $g = f$). So let us prove the equality (39).

First, let us prove that for every $p \in \mathbb{N}$, every $U \in L^{\otimes p}$ satisfies the equation (39). In fact, we are going to prove this by induction: The base case of our induction - the case $p = 0$ - is evident[28]. Now the induction step: Let $p \in \mathbb{N}_+$. Assume that we have proven that every $U \in L^{\otimes (p-1)}$ satisfies the equation (39). Now, in order to complete the induction step, we have to show that every $U \in L^{\otimes p}$ satisfies the equation (39). In order to achieve this goal, it will be enough to show that every left-induced $U \in L^{\otimes p}$

[28]In fact, in the case $p = 0$, every $U \in L^{\otimes p}$ is a scalar (since $L^{\otimes p} = L^{\otimes 0} = k$), and thus $U \overset{g}{\lrcorner} w = 0$ (by Theorem 11 **(a)**, applied to $g$ instead of $f$) and $v \overset{f}{\llcorner} U = 0$ (by Theorem 5 **(a)**), so that the equation (39) trivially holds.

satisfies the equation (39).[29] So let us prove this: For every left-induced $U \in L^{\otimes p}$, we can write $U$ in the form $U = u \otimes \ddot{U}$ for some $u \in L$ and some $\ddot{U} \in L^{\otimes (p-1)}$, and therefore $U$ satisfies

$$\begin{aligned}
& v \overset{f}{\llcorner} \left( U \overset{g}{\lrcorner} w \right) \\
&= v \overset{f}{\llcorner} \left( \left( u \otimes \ddot{U} \right) \overset{g}{\lrcorner} w \right) \qquad \left( \text{since } U = u \otimes \ddot{U} \right) \\
&= v \overset{f}{\llcorner} \left( (-1)^{p-1} g(u,w) \ddot{U} + u \otimes \left( \ddot{U} \overset{g}{\lrcorner} w \right) \right) \\
&\qquad \left( \begin{array}{c} \text{since (25) (applied to } p-1,\ g,\ \ddot{U} \text{ and } w \text{ instead of } p,\ f,\ U \text{ and } v\text{) yields} \\ \left( u \otimes \ddot{U} \right) \overset{g}{\lrcorner} w = (-1)^{p-1} g(u,w) \ddot{U} + u \otimes \left( \ddot{U} \overset{g}{\lrcorner} w \right) \ \text{(because } \ddot{U} \in L^{\otimes (p-1)}\text{)} \end{array} \right) \\
&= (-1)^{p-1} g(u,w) \cdot v \overset{f}{\llcorner} \ddot{U} + v \overset{f}{\llcorner} \left( u \otimes \left( \ddot{U} \overset{g}{\lrcorner} w \right) \right) \\
&= (-1)^{p-1} g(u,w) \cdot v \overset{f}{\llcorner} \ddot{U} + f(v,u) \cdot \ddot{U} \overset{g}{\lrcorner} w - u \otimes \left( v \overset{f}{\llcorner} \left( \ddot{U} \overset{g}{\lrcorner} w \right) \right) \\
&\qquad \left( \begin{array}{c} \text{since (7), applied to } \ddot{U} \overset{g}{\lrcorner} w \text{ instead of } U\text{, yields} \\ v \overset{f}{\llcorner} \left( u \otimes \left( \ddot{U} \overset{g}{\lrcorner} w \right) \right) = f(v,u) \cdot \ddot{U} \overset{g}{\lrcorner} w - u \otimes \left( v \overset{f}{\llcorner} \left( \ddot{U} \overset{g}{\lrcorner} w \right) \right) \end{array} \right) \\
&= (-1)^{p-1} g(u,w) \cdot v \overset{f}{\llcorner} \ddot{U} + f(v,u) \cdot \ddot{U} \overset{g}{\lrcorner} w - u \otimes \left( \left( v \overset{f}{\llcorner} \ddot{U} \right) \overset{g}{\lrcorner} w \right) \qquad (40)
\end{aligned}$$

(because $v \overset{f}{\llcorner} \left( \ddot{U} \overset{g}{\lrcorner} w \right) = \left( v \overset{f}{\llcorner} \ddot{U} \right) \overset{g}{\lrcorner} w$, which follows from applying the equality (39) to $\ddot{U}$ instead of $U$ [30]) and

$$\begin{aligned}
& \left( v \overset{f}{\llcorner} U \right) \overset{g}{\lrcorner} w \\
&= \left( v \overset{f}{\llcorner} \left( u \otimes \ddot{U} \right) \right) \overset{g}{\lrcorner} w \qquad \left( \text{since } U = u \otimes \ddot{U} \right) \\
&= \left( f(v,u) \ddot{U} - u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \overset{g}{\lrcorner} w \\
&\qquad \left( \begin{array}{c} \text{since (7), applied to } \ddot{U} \text{ instead of } U\text{, yields} \\ v \overset{f}{\llcorner} \left( u \otimes \ddot{U} \right) = f(v,u) \ddot{U} - u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \end{array} \right) \\
&= f(v,u) \ddot{U} \overset{g}{\lrcorner} w - \left( u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \overset{g}{\lrcorner} w \\
&= f(v,u) \ddot{U} \overset{g}{\lrcorner} w - \left( (-1)^{p-2} g(u,w) \cdot v \overset{f}{\llcorner} \ddot{U} + u \otimes \left( \left( v \overset{f}{\llcorner} \ddot{U} \right) \overset{g}{\lrcorner} w \right) \right) \\
&\qquad \left( \begin{array}{c} \text{since (25) (applied to } p-2,\ g,\ v \overset{f}{\llcorner} \ddot{U} \text{ and } w \text{ instead of } p,\ f,\ U \text{ and } v\text{)} \\ \text{yields } \left( u \otimes \left( v \overset{f}{\llcorner} \ddot{U} \right) \right) \overset{g}{\lrcorner} w = (-1)^{p-2} g(u,w) \cdot v \overset{f}{\llcorner} \ddot{U} + u \otimes \left( \left( v \overset{f}{\llcorner} \ddot{U} \right) \overset{g}{\lrcorner} w \right) \\ \text{(because } v \overset{f}{\llcorner} \ddot{U} \in L^{\otimes (p-2)}\text{)} \end{array} \right) \\
&= \underbrace{-(-1)^{p-2}}_{=(-1)^{p-1}} g(u,w) \cdot v \overset{f}{\llcorner} \ddot{U} + f(v,u) \ddot{U} \overset{g}{\lrcorner} w - u \otimes \left( \left( v \overset{f}{\llcorner} \ddot{U} \right) \overset{g}{\lrcorner} w \right) \\
&= (-1)^{p-1} g(u,w) \cdot v \overset{f}{\llcorner} \ddot{U} + f(v,u) \cdot \ddot{U} \overset{g}{\lrcorner} w - u \otimes \left( \left( v \overset{f}{\llcorner} \ddot{U} \right) \overset{g}{\lrcorner} w \right) = v \overset{f}{\llcorner} \left( U \overset{g}{\lrcorner} w \right)
\end{aligned}$$

[29] This follows from the left tensor induction tactic, because the equation (39) is linear in $U$.

[30] In fact, we are allowed to apply the equality (39) to $\ddot{U}$ instead of $U$, since $\ddot{U} \in L^{\otimes (p-1)}$ and since we have proven that every $U \in L^{\otimes (p-1)}$ satisfies the equation (39) (by our assumption).

(by (40)). Thus, we have proven the equality (39). Hence, every left-induced $U \in L^{\otimes p}$ satisfies the equation (39). As we said above, this yields that every $U \in L^{\otimes p}$ satisfies the equation (39). This completes our induction step, and thus we have shown that for every $p \in \mathbb{N}$, every $U \in L^{\otimes p}$ satisfies the equation (39). Consequently, every $U \in \otimes L$ satisfies the equation (39) (because every $U \in \otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and because the equation (39) is linear in $U$). This proves Theorem 18.

## 5. The endomorphism $\alpha^f$

We are now going to define an endomorphism $\alpha^f : \otimes L \to \otimes L$ which depends on the bilinear form $f$:

**Definition 10.** Let $f : L \times L \to k$ be a bilinear form. For every $p \in \mathbb{N}$, we define a $k$-linear map $\alpha_p^f : L^{\otimes p} \to \otimes L$ by induction over $p$:

*Induction base:* For $p = 0$, we define the map $\alpha_p^f : L^{\otimes 0} \to \otimes L$ to be the canonical inclusion of $L^{\otimes 0}$ into the tensor algebra $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$. (In other words, we define the map $\alpha_0^f : k \to \otimes L$ by $\alpha_p^f (\lambda) = \lambda$ for every $\lambda \in k = L^{\otimes 0}$.)

*Induction step:* For each $p \in \mathbb{N}_+$, we define a $k$-linear map $\alpha_p^f : L^{\otimes p} \to \otimes L$ by

$$\left(\alpha_p^f (u \otimes U) = u \otimes \alpha_{p-1}^f (U) - u \overset{f}{\llcorner} \alpha_{p-1}^f (U) \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes (p-1)}\right), \tag{41}$$

assuming that we have already defined a $k$-linear map $\alpha_{p-1}^f : L^{\otimes (p-1)} \to \otimes L$. (This definition is justified, because in order to define a $k$-linear map from $L^{\otimes p}$ to some other $k$-module, it is enough to define how it acts on tensors of the form $u \otimes U$ for every $u \in L$ and $U \in L^{\otimes (p-1)}$, as long as this action is bilinear with respect to $u$ and $U$. This is because $L^{\otimes p} = L \otimes L^{\otimes (p-1)}$.)

This way we have defined a $k$-linear map $\alpha_p^f : L^{\otimes p} \to \otimes L$ for every $p \in \mathbb{N}$. We can combine these maps $\alpha_0^f$, $\alpha_1^f$, $\alpha_2^f$, ... into one $k$-linear map $\alpha^f : \otimes L \to \otimes L$ (since $\otimes L = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ...$), and the formula (41) rewrites as

$$\left(\alpha^f (u \otimes U) = u \otimes \alpha^f (U) - u \overset{f}{\llcorner} \alpha^f (U) \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes (p-1)}\right). \tag{42}$$

We note that, in contrast to the map $\delta_v^f$ (which maps every homogeneous tensor from $L^{\otimes p}$ to $L^{\otimes (p-1)}$), the map $\alpha^f$ can map homogeneous tensors to inhomogeneous tensors.

This endomorphism $\alpha^f$ now turns out to have plenty of properties. But first let us first evaluate it on pure tensors of low rank (0, 1, 2, 3, 4): [31]

*Action of $\alpha^f$ on tensors of rank 0:* For any $\lambda \in k$, we have $\alpha^f (\lambda) = \lambda$, where we consider $\lambda$ as an element of $\otimes L$ through the canonical injection $k = L^{\otimes 0} \to \otimes L$. (In fact, $\lambda \in k = L^{\otimes 0}$ yields $\alpha^f (\lambda) = \alpha_0^f (\lambda) = \lambda$ by the definition of $\alpha_0^f$).

[31] Note that these evaluations will never be used in future (except of (43) and (44), which are pretty much trivial), so there is no need to read them. But I think they provide a good intuition for what the map $\alpha^f$ does to tensors of low degrees.

*Action of* $\alpha^f$ *on tensors of rank* 1*:* For any $u \in L$, we have

$$\alpha^f(u) = \alpha^f(u \otimes 1) = u \otimes \underbrace{\alpha^f(1)}_{=1 \text{ (since } 1 \in k)} - \underbrace{u \overset{f}{\llcorner} \alpha^f(1)}_{\substack{=0 \text{ (by Theorem 5 } \textbf{(a)}, \\ \text{since } \alpha^f(1) = 1 \in k)}} \qquad \text{(by (42), applied to } U = 1)$$

$$= u \otimes 1 - 0 = u \otimes 1 = u. \tag{43}$$

*Action of* $\alpha^f$ *on tensors of rank* 2*:* For any $u \in L$ and $v \in L$, we have

$$\alpha^f(u \otimes v) = u \otimes \underbrace{\alpha^f(v)}_{\substack{=v \text{ (by (43), applied to} \\ v \text{ instead of } u)}} - u \overset{f}{\llcorner} \underbrace{\alpha^f(v)}_{\substack{=v \text{ (by (43), applied to} \\ v \text{ instead of } u)}} \qquad \text{(by (42), applied to } U = v)$$

$$= u \otimes v - \underbrace{u \overset{f}{\llcorner} v}_{\substack{=f(u,v) \text{ (according to Theorem 5 } \textbf{(b)}, \\ \text{applied to } v \text{ and } u \text{ instead of } u \text{ and } v)}} = u \otimes v - f(u,v). \tag{44}$$

*Action of* $\alpha^f$ *on tensors of rank* 3*:* For any $u \in L$, $v \in L$ and $w \in L$, we have

$$\begin{aligned}
&\alpha^f(u \otimes v \otimes w) \\
&= u \otimes \alpha^f(v \otimes w) - u \overset{f}{\llcorner} \alpha^f(v \otimes w) \qquad \text{(by (42), applied to } U = v \otimes w) \\
&= u \otimes (v \otimes w - f(v,w)) - u \overset{f}{\llcorner} (v \otimes w - f(v,w)) \\
&\qquad \left( \begin{array}{c} \text{since (44) (applied to } v \text{ and } w \text{ instead of } u \text{ and } v) \text{ yields} \\ \alpha^f(v \otimes w) = v \otimes w - f(v,w) \end{array} \right) \\
&= \left( u \otimes v \otimes w - \underbrace{u \otimes f(v,w)}_{=f(v,w)u} \right) - \left( \underbrace{u \overset{f}{\llcorner} (v \otimes w)}_{\substack{=f(u,v)w - v \otimes \left( u \overset{f}{\llcorner} w \right) \\ \text{(by (7), applied} \\ \text{to } u,\, v \text{ and } w \text{ instead of } v,\, u \text{ and } U)}} - \underbrace{u \overset{f}{\llcorner} f(v,w)}_{\substack{=0 \text{ (by Theorem 5 } \textbf{(a)}, \\ \text{since } f(v,w) \in k)}} \right) \\
&= (u \otimes v \otimes w - f(v,w)u) - \left( f(u,v)w - v \otimes \underbrace{\left( u \overset{f}{\llcorner} w \right)}_{\substack{=f(u,w) \text{ (by Theorem 5 } \textbf{(b)}, \\ \text{applied to } u \text{ and } w \text{ instead of } v \text{ and } u)}} - 0 \right) \\
&= (u \otimes v \otimes w - f(v,w)u) - \left( f(u,v)w - \underbrace{v \otimes f(u,w)}_{=f(u,w)v} - 0 \right) \\
&= (u \otimes v \otimes w - f(v,w)u) - (f(u,v)w - f(u,w)v - 0) \\
&= u \otimes v \otimes w - f(v,w)u + f(u,w)v - f(u,v)w.
\end{aligned} \tag{45}$$

*Action of* $\alpha^f$ *on tensors of rank* 4*:* For any $u \in L$, $v \in L$, $w \in L$ and $x \in L$, we have

$$
\begin{aligned}
&\alpha^f(u\otimes v\otimes w\otimes x)\\
&=u\otimes\alpha^f(v\otimes w\otimes x)-u\overset{f}{\llcorner}\alpha^f(v\otimes w\otimes x) \qquad \text{(by (42), applied to } U=v\otimes w\otimes x\text{)}\\
&=u\otimes(v\otimes w\otimes x-f(w,x)v+f(v,x)w-f(v,w)x)\\
&\qquad -u\overset{f}{\llcorner}(v\otimes w\otimes x-f(w,x)v+f(v,x)w-f(v,w)x)\\
&\qquad \left(\begin{array}{c}\text{since (45) (applied to } v\text{, } w \text{ and } x \text{ instead of } u\text{, } v \text{ and } w\text{) yields}\\ \alpha^f(v\otimes w\otimes x)=v\otimes w\otimes x-f(w,x)v+f(v,x)w-f(v,w)x\end{array}\right)\\
&=(u\otimes v\otimes w\otimes x-f(w,x)u\otimes v+f(v,x)u\otimes w-f(v,w)u\otimes x)\\
&\qquad -\left(\underbrace{u\overset{f}{\llcorner}(v\otimes w\otimes x)}_{\substack{=f(u,v)w\otimes x-v\otimes\left(u\overset{f}{\llcorner}(w\otimes x)\right)\\ \text{(by (7), applied to } u\text{, } v \text{ and}\\ w\otimes x \text{ instead of } v\text{, } u \text{ and } U\text{)}}}-f(w,x)\underbrace{u\overset{f}{\llcorner}v}_{\substack{=f(u,v) \text{ (by}\\ \text{Theorem 5 } \textbf{(b)}\text{,}\\ \text{applied to } u \text{ and } v\\ \text{instead of } v \text{ and } u\text{)}}}\right.\\
&\qquad\qquad \left.+f(v,x)\underbrace{u\overset{f}{\llcorner}w}_{\substack{=f(u,w) \text{ (by}\\ \text{Theorem 5 } \textbf{(b)}\text{,}\\ \text{applied to } u \text{ and } w\\ \text{instead of } v \text{ and } u\text{)}}}-f(v,w)\underbrace{u\overset{f}{\llcorner}x}_{\substack{=f(u,x) \text{ (by}\\ \text{Theorem 5 } \textbf{(b)}\text{,}\\ \text{applied to } u \text{ and } x\\ \text{instead of } v \text{ and } u\text{)}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(u\otimes v\otimes w\otimes x - f\left(w,x\right)u\otimes v + f\left(v,x\right)u\otimes w - f\left(v,w\right)u\otimes x\right)\\
&\qquad - \left( f\left(u,v\right)w\otimes x - v\otimes \left( \underbrace{u \llcorner^{f} \left(w\otimes x\right)}_{\substack{=f(u,w)x - w\otimes\left(u\llcorner^{f} x\right)\\ \text{(by (7), applied}\\ \text{to } u,\ w \text{ and } x \text{ instead of } v,\ u \text{ and } U)}} \right) \right.\\
&\qquad\qquad \left. - f\left(w,x\right)f\left(u,v\right) + f\left(v,x\right)f\left(u,w\right) - f\left(v,w\right)f\left(u,x\right)\right)\\
&= \left(u\otimes v\otimes w\otimes x - f\left(w,x\right)u\otimes v + f\left(v,x\right)u\otimes w - f\left(v,w\right)u\otimes x\right)\\
&\qquad - \left( f\left(u,v\right)w\otimes x - v\otimes \left( f\left(u,w\right)x - w\otimes\left(u\llcorner^{f} x\right)\right)\right.\\
&\qquad\qquad \left. - f\left(w,x\right)f\left(u,v\right) + f\left(v,x\right)f\left(u,w\right) - f\left(v,w\right)f\left(u,x\right)\right)\\
&= u\otimes v\otimes w\otimes x - f\left(w,x\right)u\otimes v + f\left(v,x\right)u\otimes w - f\left(v,w\right)u\otimes x\\
&\qquad - f\left(u,v\right)w\otimes x + v\otimes\left( f\left(u,w\right)x - w\otimes\left(\underbrace{u\llcorner^{f} x}_{\substack{=f(u,x) \text{ (by Theorem 5 } \textbf{(b)},\\ \text{applied to } u \text{ and } x \text{ instead of } v \text{ and } u)}}\right)\right)\\
&\qquad + f\left(w,x\right)f\left(u,v\right) - f\left(v,x\right)f\left(u,w\right) + f\left(v,w\right)f\left(u,x\right)\\
&= u\otimes v\otimes w\otimes x - f\left(w,x\right)u\otimes v + f\left(v,x\right)u\otimes w - f\left(v,w\right)u\otimes x\\
&\qquad - f\left(u,v\right)w\otimes x + v\otimes\left(f\left(u,w\right)x - w\otimes f\left(u,x\right)\right)\\
&\qquad + f\left(w,x\right)f\left(u,v\right) - f\left(v,x\right)f\left(u,w\right) + f\left(v,w\right)f\left(u,x\right)\\
&= u\otimes v\otimes w\otimes x - f\left(w,x\right)u\otimes v + f\left(v,x\right)u\otimes w - f\left(v,w\right)u\otimes x\\
&\qquad - f\left(u,v\right)w\otimes x + \left(\underbrace{v\otimes f\left(u,w\right)x}_{=f(u,w)v\otimes x} - \underbrace{v\otimes w\otimes f\left(u,x\right)}_{=f(u,x)v\otimes w}\right)\\
&\qquad + f\left(w,x\right)f\left(u,v\right) - f\left(v,x\right)f\left(u,w\right) + f\left(v,w\right)f\left(u,x\right).\\
&= u\otimes v\otimes w\otimes x - f\left(w,x\right)u\otimes v + f\left(v,x\right)u\otimes w - f\left(v,w\right)u\otimes x\\
&\qquad - f\left(u,v\right)w\otimes x + f\left(u,w\right)v\otimes x - f\left(u,x\right)v\otimes w\\
&\qquad + f\left(w,x\right)f\left(u,v\right) - f\left(v,x\right)f\left(u,w\right) + f\left(v,w\right)f\left(u,x\right).
\end{aligned}
$$

These computations can be generalized to $\alpha^f\left(u_1\otimes u_2\otimes ...\otimes u_p\right)$ for general $p\in\mathbb{N}$. As a result, we get the formula

$$
\begin{aligned}
&\alpha^f\left(u_1\otimes u_2\otimes ...\otimes u_p\right)\\
&= \sum \left(-1\right)^{\text{(number of all bad pairs)}} f\left(u_{i_1},u_{j_1}\right)f\left(u_{i_2},u_{j_2}\right)...f\left(u_{i_k},u_{j_k}\right)u_{r_1}\otimes u_{r_2}\otimes ...\otimes u_{r_{p-2k}}
\end{aligned}
$$

for any $p$ vectors $u_1$, $u_2$, ..., $u_p$ in $L$, where the sum is over all partitions of the set $\{1,2,...,p\}$ into three subsets $\{i_1,i_2,...,i_k\}$, $\{j_1,j_2,...,j_k\}$ and $\{r_1,r_2,...,r_{p-2k}\}$ (for various $k$) which satisfy $i_1 < i_2 < ... < i_k$, $j_1 < j_2 < ... < j_k$, $r_1 < r_2 < ... < r_{p-2k}$ and ($i_\ell < j_\ell$ for every $\ell\in\{1,2,...,k\}$). Here, a "bad pair" means a pair $(\ell,\ell')\in\{1,2,...,k\}^2$ satisfying $\ell\geq\ell'$ and $i_\ell < j_{\ell'}$ (so, in particular, for every $\ell\in\{1,2,...,k\}$, the pair $(\ell, \ell)$ is bad, since $i_\ell < j_\ell$). [32] Thus we have an explicit formula for $\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_p)$, but it is extremely hard to deal with; this is the reason why I defined $\alpha^f$ by induction rather than by a direct formula.

We remark that the formula (42) can be slightly generalized, in the sense that $U$ doesn't have to be a homogeneous tensor:

**Theorem 19.** Let $u \in L$ and $U \in \otimes L$. Then,

$$\alpha^f (u \otimes U) = u \otimes \alpha^f (U) - u \overset{f}{\llcorner} \alpha^f (U) . \tag{46}$$

*Proof of Theorem 19.* Fix $u \in L$. We have to prove the equation (46) for every $U \in \otimes L$. We can WLOG assume that $U \in L^{\otimes p}$ for some $p \in \mathbb{N}$ (since every tensor $U \in \otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and since the equation (46) is linear in $U$). But then, (46) follows from (42) (applied to $p+1$ instead of $p$). Thus, the equation (46) is proven for every $U \in \otimes L$, and therefore Theorem 19 is proven.

Another fact is, while $\alpha^f$ is not necessarily homogeneous, the degrees of all the terms it spits out have the same parity as that of the original tensor:

**Theorem 20.** Let $U \in L^{\otimes p}$ for some $p \in \mathbb{N}$. Then,

$$\alpha^f (U) \in \bigoplus_{\substack{i \in \mathbb{N}; \\ i \equiv p \bmod 2}} L^{\otimes i}. \tag{47}$$

Even a stronger assertion holds:

$$\alpha^f (U) \in \bigoplus_{\substack{i \in \{0,1,...,p\}; \\ i \equiv p \bmod 2}} L^{\otimes i}. \tag{48}$$

*Proof of Theorem 20.* We are going to prove (48) by induction over $p$.

The induction base case $p = 0$ is obvious[33].

So let us pass on to the induction step: Let $p \in \mathbb{N}_+$. Assume that we have proven (48) for $p - 1$ instead of $p$; that is, we have shown that

$$\alpha^f (U) \in \bigoplus_{\substack{i \in \{0,1,...,p-1\}; \\ i \equiv p-1 \bmod 2}} L^{\otimes i} \qquad \text{for every } U \in L^{\otimes (p-1)}. \tag{49}$$

Now we have to establish (48) for our value of $p$ as well, i. e. we have to prove that

$$\alpha^f (U) \in \bigoplus_{\substack{i \in \{0,1,...,p\}; \\ i \equiv p \bmod 2}} L^{\otimes i} \qquad \text{for every } U \in L^{\otimes p}. \tag{50}$$

[32] I hope I haven't made a mistake in the formula.

[33] In fact, in this case, $U \in L^{\otimes p} = L^{\otimes 0} = k$, and thus $\alpha^f (U) = U \in \bigoplus_{\substack{i \in \{0,1,...,p\}; \\ i \equiv p \bmod 2}} L^{\otimes i}$.

So let us prove (50). First, we notice that every $u \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$ satisfy[34]

$$\begin{aligned}
&\alpha^f\left(u\otimes\ddot{U}\right)\\
&= u\otimes \underbrace{\alpha^f\left(\ddot{U}\right)}_{\substack{\in \bigoplus_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes i}\\ \text{(by (49), applied to }\ddot{U}\\ \text{instead of }U\text{)}}} - u\overset{f}{\llcorner} \underbrace{\alpha^f\left(\ddot{U}\right)}_{\substack{\in \bigoplus_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes i}\\ \text{(by (49), applied to }\ddot{U}\\ \text{instead of }U\text{)}}} \qquad \left(\text{by (46), applied to }\ddot{U}\text{ instead of }U\right)\\
&\in u\otimes \bigoplus_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes i} + u\overset{f}{\llcorner} \bigoplus_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes i}\\
&= u\otimes \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes i} + u\overset{f}{\llcorner} \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes i} \qquad (\text{since direct sums are sums})\\
&= \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} \underbrace{u\otimes L^{\otimes i}}_{\subseteq L^{\otimes(i+1)}\text{ (since }u\in L)} + \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} \underbrace{u\overset{f}{\llcorner} L^{\otimes i}}_{\substack{\subseteq L^{\otimes(i-1)}\text{ (since}\\ u\overset{f}{\llcorner}P\in L^{\otimes(i-1)}\text{ for every}\\ P\in L^{\otimes i})}}\\
&\qquad \left(\text{since both the tensor product and the operation }\overset{f}{\llcorner}\text{ are bilinear}\right)\\
&\subseteq \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes(i+1)} + \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i\equiv p-1 \bmod 2}} L^{\otimes(i-1)} = \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i+1\equiv p \bmod 2}} L^{\otimes(i+1)} + \sum_{\substack{i\in\{0,1,\ldots,p-1\};\\ i-1\equiv p \bmod 2}} L^{\otimes(i-1)}\\
&\qquad \left(\begin{array}{c}\text{since } i\equiv p-1 \bmod 2 \text{ is equivalent to } i+1\equiv p \bmod 2\text{, and}\\ \text{since } i\equiv p-1 \bmod 2 \text{ is equivalent to } i-1\equiv p \bmod 2 \text{ (the latter}\\ \text{is because } i\equiv p-1 \bmod 2 \text{ is equivalent}\\ \text{to } i+1\equiv p \bmod 2\text{, and because } i+1\equiv i-1 \bmod 2)\end{array}\right)\\
&= \underbrace{\sum_{\substack{i\in\{1,2,\ldots,p\};\\ i\equiv p \bmod 2}} L^{\otimes i}}_{\subseteq \sum_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \bmod 2}} L^{\otimes i}} + \underbrace{\sum_{\substack{i\in\{-1,0,\ldots,p\};\\ i\equiv p \bmod 2}} L^{\otimes i}}_{\subseteq L^{\otimes(-1)} + \sum_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \bmod 2}} L^{\otimes i} = \sum_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \bmod 2}} L^{\otimes i}\text{ (since }L^{\otimes(-1)}=0)}\\
&\qquad \left(\begin{array}{c}\text{here, we substituted } i \text{ for } i+1 \text{ in the first sum, and we}\\ \text{substituted } i \text{ for } i-1 \text{ in the second sum}\end{array}\right)
\end{aligned}$$

[34]In the following, whenever $P$ is a $k$-submodule of $\otimes L$, we denote by $u\overset{f}{\llcorner}P$ the $k$-submodule $\left\{u\overset{f}{\llcorner}p \mid p\in P\right\}$ of $\otimes L$. This is indeed a submodule, since $u\overset{f}{\llcorner}p$ is $k$-linear in $p$ (because of the bilinearity of $\overset{f}{\llcorner}$).

$$\subseteq \sum_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} + \sum_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \subseteq \sum_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \qquad \left(\text{since } \sum_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \text{ is a } k\text{-module}\right)$$

$$= \bigoplus_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \tag{51}$$

(since the sum $\sum\limits_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i}$ is a direct sum). Consequently, (50) is true for each tensor $U \in L^{\otimes p}$ (because every tensor $U \in L^{\otimes p}$ can be written in the form $U = \sum\limits_{i\in I} \alpha_i u_i \otimes \ddot{U}_i$ for a finite set $I$, a family $(\alpha_i)_{i\in I}$ of scalars in $k$, a family $(u_i)_{i\in I}$ of vectors in $L$ and a family $\left(\ddot{U}_i\right)_{i\in I}$ of tensors in $L^{\otimes(p-1)}$ [35], and thus it satisfies

$$\alpha^f(U) = \alpha^f\left(\sum_{i\in I} \alpha_i u_i \otimes \ddot{U}_i\right) = \sum_{i\in I} \alpha_i \underbrace{\alpha^f\left(u_i \otimes \ddot{U}_i\right)}_{\substack{\in \bigoplus_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \\ \text{(due to (51), applied} \\ \text{to } u_i \text{ and } \ddot{U}_i \text{ instead of } u \text{ and } \ddot{U})}}$$

$$\left(\text{since the map } \alpha^f \text{ is } k\text{-linear}\right)$$

$$\in \sum_{i\in I} \alpha_i \bigoplus_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \subseteq \bigoplus_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \qquad \left(\text{since } \bigoplus_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \text{ is a } k\text{-module}\right)$$

). Thus, the induction is complete, and (48) is proven. Clearly, (48) yields (47) (since $\bigoplus\limits_{\substack{i\in\{0,1,\ldots,p\};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i} \subseteq \bigoplus\limits_{\substack{i\in\mathbb{N};\\ i\equiv p \operatorname{mod} 2}} L^{\otimes i}$). Thus, Theorem 20 is proven.

Now let us show some more interesting properties of $\alpha^f$. The proofs will be again by induction akin to the proofs of Theorems 6-10 and 12-16.

First, we notice that the definition of $\alpha^f$ had a bias towards left tensoring: we defined the value of $\alpha_p^f$ on a tensor of rank $p$ by writing this tensor as a linear combination of tensors of the form $u \otimes U$ with $u \in L$ and $U \in L^{\otimes(p-1)}$, and then by setting the value of $\alpha_p^f$ on each such $u \otimes U$ tensor according (41). But what if we would try to define a "right analogue" $\widetilde{\alpha}^f$ of $\alpha^f$, which would be (inductively) defined by

$$\left(\widetilde{\alpha}_p^f(U \otimes u) = \widetilde{\alpha}_{p-1}^f(U) \otimes u - \widetilde{\alpha}_{p-1}^f(U) \overset{f}{\lrcorner} u \qquad \text{for every } u \in L \text{ and } U \in L^{\otimes(p-1)}\right)$$

instead of (41) ? It turns out that this wouldn't give us anything new: This "right analogue" $\widetilde{\alpha}^f$ would be the same as $\alpha^f$. This is explained by the following theorem:

**Theorem 21.** Let $u \in L$ and $U \in \otimes L$. Then,

$$\alpha^f(U \otimes u) = \alpha^f(U) \otimes u - \alpha^f(U) \overset{f}{\lrcorner} u. \tag{52}$$

[35] This is because $U \in L^{\otimes p} = L \otimes L^{\otimes(p-1)}$.

*Proof of Theorem 21.* Fix $u \in L$. We have to prove the equation (46) for every $U \in \otimes L$. In order to do this, it is enough to prove the equation (46) for every $U \in L^{\otimes p}$ for every $p \in \mathbb{N}$ (since every tensor $U \in \otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and since the equation (46) is linear in $U$). So let us prove the equation (46) for every $U \in L^{\otimes p}$ for every $p \in \mathbb{N}$. We are going to prove this by induction over $p$:

The induction base case $p = 0$ is trivial (because in this case, $U \in L^{\otimes p} = L^{\otimes 0}$ yields $\alpha^f(U) = \alpha_0^f(U) = U$ by the definition of $\alpha_0^f$, and $\underbrace{U}_{\in L^{\otimes 0} = k} \otimes \underbrace{u}_{\in L} \in k \otimes L = L$ yields $\alpha^f(U \otimes u) = U \otimes u$ due to (43)).

Hence, let us pass on to the induction step. Let $p \in \mathbb{N}_+$. Assume that (52) has already been proven for every $U \in L^{\otimes(p-1)}$. Now we must prove (52) for every $U \in L^{\otimes p}$. In order to do this, it is enough to prove (52) for every left-induced $U \in L^{\otimes p}$ (by the left tensor induction tactic, since the equation (52) is linear in $U$). So, let $U \in L^{\otimes p}$ be some left-induced tensor. Then, $U = \ddot{u} \otimes \ddot{U}$ for some $\ddot{u} \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$ (because $U$ is left-induced). Therefore,

$$\alpha^f(U \otimes u) = \alpha^f\left(\ddot{u} \otimes \ddot{U} \otimes u\right) = \ddot{u} \otimes \alpha^f\left(\ddot{U} \otimes u\right) - \ddot{u} \llcorner^{f} \alpha^f\left(\ddot{U} \otimes u\right)$$

(by (46), applied to $\ddot{u}$ and $\ddot{U} \otimes u$ instead of $u$ and $U$). But since $\alpha^f\left(\ddot{U} \otimes u\right) = \alpha^f\left(\ddot{U}\right) \otimes u - \alpha^f\left(\ddot{U}\right) \lrcorner^{f} u$ (this follows from (52), applied to $\ddot{U}$ instead of $U$ [36]), this rewrites as

$$\begin{aligned}
&\alpha^f(U \otimes u) \\
&= \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \otimes u - \alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right) - \ddot{u} \llcorner^{f} \left(\alpha^f\left(\ddot{U}\right) \otimes u - \alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right) \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) \otimes u - \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right)\right) - \left(\ddot{u} \llcorner^{f} \left(\alpha^f\left(\ddot{U}\right) \otimes u\right) - \ddot{u} \llcorner^{f} \left(\alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right)\right) \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) \otimes u - \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right)\right) \\
&\qquad - \left((-1)^{p-1} f(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \left(\ddot{u} \llcorner^{f} \alpha^f\left(\ddot{U}\right)\right) \otimes u - \ddot{u} \llcorner^{f} \left(\alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right)\right) \\
&\qquad \left(\begin{array}{c} \text{since (14) (applied to } p-1\text{, } \ddot{u} \text{ and } \alpha^f\left(\ddot{U}\right) \text{ instead of } p\text{, } v \text{ and } U\text{) yields} \\ \ddot{u} \llcorner^{f} \left(\alpha^f\left(\ddot{U}\right) \otimes u\right) = (-1)^{p-1} f(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \left(\ddot{u} \llcorner^{f} \alpha^f\left(\ddot{U}\right)\right) \otimes u \\ \text{(because } \ddot{U} \in L^{\otimes(p-1)} \text{ yields } \alpha^f\left(\ddot{U}\right) \in \bigoplus\limits_{\substack{i \in \mathbb{N};\\ i \equiv p-1 \bmod 2}} L^{\otimes i} \\ \text{(by Theorem 20, applied to } p-1 \text{ instead of } p\text{))} \end{array}\right) \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) \otimes u - \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right)\right) \\
&\qquad - \left((-1)^{p-1} f(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \left(\ddot{u} \llcorner^{f} \alpha^f\left(\ddot{U}\right)\right) \otimes u - \left(\ddot{u} \llcorner^{f} \alpha^f\left(\ddot{U}\right)\right) \lrcorner^{f} u\right) \qquad (53) \\
&\qquad \left(\begin{array}{c} \text{since } \ddot{u} \llcorner^{f} \left(\alpha^f\left(\ddot{U}\right) \lrcorner^{f} u\right) = \left(\ddot{u} \llcorner^{f} \alpha^f\left(\ddot{U}\right)\right) \lrcorner^{f} u\text{, according to (38)} \\ \text{(applied to } \alpha^f\left(\ddot{U}\right)\text{, } \ddot{u} \text{ and } u \text{ instead of } U\text{, } v \text{ and } w\text{)} \end{array}\right).
\end{aligned}$$

[36] In fact, we *are* allowed to apply (52) to $\ddot{U}$ instead of $U$, because $\ddot{U} \in L^{\otimes(p-1)}$ and because we have assumed that (52) has already been proven for every $U \in L^{\otimes(p-1)}$.

On the other hand, $U = \ddot{u} \otimes \ddot{U}$ yields

$$\alpha^f(U) = \alpha^f\left(\ddot{u} \otimes \ddot{U}\right) = \ddot{u} \otimes \alpha^f\left(\ddot{U}\right) - \ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)$$
$$\left(\text{by (46), applied to } \ddot{u} \text{ and } \ddot{U} \text{ instead of } u \text{ and } U\right)$$

and therefore

$$\begin{aligned}
&\alpha^f(U) \otimes u - \alpha^f(U) \overset{f}{\lrcorner} u \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) - \ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \otimes u - \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) - \ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \overset{f}{\lrcorner} u \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) \otimes u - \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \otimes u\right) - \left(\left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right)\right) \overset{f}{\lrcorner} u - \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \overset{f}{\lrcorner} u\right) \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) \otimes u - \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \otimes u\right) \\
&\qquad - \left((-1)^{p-1} f(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \overset{f}{\lrcorner} u\right) - \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \overset{f}{\lrcorner} u\right) \\
&\qquad \left(\begin{array}{c}
\text{since (27) (applied to } p-1,\ \ddot{u},\ u \text{ and } \alpha^f\left(\ddot{U}\right) \text{ instead of } p,\ u,\ v \text{ and } U) \\
\text{yields } \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right)\right) \overset{f}{\lrcorner} u = (-1)^{p-1} f(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \overset{f}{\lrcorner} u\right) \\
\text{(because } \ddot{U} \in L^{\otimes(p-1)} \text{ yields } \alpha^f\left(\ddot{U}\right) \in \bigoplus\limits_{\substack{i \in \mathbb{N}; \\ i \equiv p-1 \bmod 2}} L^{\otimes i} \\
\text{(by Theorem 20, applied to } p-1 \text{ instead of } p))
\end{array}\right) \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) \otimes u - \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \overset{f}{\lrcorner} u\right)\right) \\
&\qquad - \left((-1)^{p-1} f(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \otimes u - \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \overset{f}{\lrcorner} u\right).
\end{aligned}$$

Comparing this to (53), we obtain

$$\alpha^f(U \otimes u) = \alpha^f(U) \otimes u - \alpha^f(U) \overset{f}{\lrcorner} u.$$

In other words, (52) holds for our tensor $U$. Thus, we have proven (52) for every $U \in L^{\otimes p}$. This completes the induction step, and therefore the proof of Theorem 21 is completed.

Time for more invariancy properties of $\alpha^f$:

**Theorem 22.** Let $u \in L$ and $U \in \otimes L$. Let $g : L \times L \to k$ be a bilinear form. Then,

$$\alpha^f\left(U \overset{g}{\lrcorner} u\right) = \alpha^f(U) \overset{g}{\lrcorner} u. \tag{54}$$

**Theorem 23.** Let $u \in L$ and $U \in \otimes L$. Let $g : L \times L \to k$ be a bilinear form. Then,

$$\alpha^f\left(u \overset{g}{\llcorner} U\right) = u \overset{g}{\llcorner} \alpha^f(U). \tag{55}$$

**Theorem 24.** We have $\alpha^f \circ t = t \circ \alpha^{f^t}$.

*Proof of Theorem 22.* Let $u \in L$. We will now show that for every $p \in \mathbb{N}$, the equality (54) holds for every $U \in L^{\otimes p}$.

In fact, we will prove this by induction:

The induction base case $p = 0$ is trivial[37].

So let us now move on to the induction step: Let $p \in \mathbb{N}_+$. Assume that the equality (54) is proven for every $U \in L^{\otimes (p-1)}$. Our goal is now to prove the equality (54) for every $U \in L^{\otimes p}$. In order to achieve this goal, it is enough to prove the equality (54) for every left-induced $U \in L^{\otimes p}$ (by the left tensor induction tactic, because the equality (54) is linear in $U$). However, this is easy, because every left-induced $U \in L^{\otimes p}$ can be written in the form $U = \ddot{u} \otimes \ddot{U}$ for some $\ddot{u} \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ (by the definition of "left induced"), and therefore satisfies $\alpha^f \left( U \overset{g}{\lrcorner} u \right) = \alpha^f (U) \overset{g}{\lrcorner} u$ (since

$$
\begin{aligned}
&\alpha^f \left( U \overset{g}{\lrcorner} u \right) \\
&= \alpha^f \left( \left( \ddot{u} \otimes \ddot{U} \right) \overset{g}{\lrcorner} u \right) = \alpha^f \left( (-1)^{p-1} g(\ddot{u}, u) \ddot{U} + \ddot{u} \otimes \left( \ddot{U} \overset{g}{\lrcorner} u \right) \right) \\
&\qquad \left( \begin{array}{c} \text{since (25) (applied to } g, \ddot{U}, \ddot{u}, u \text{ and } p-1 \text{ instead of } f, U, u, v \text{ and } p\text{) yields} \\ \left( \ddot{u} \otimes \ddot{U} \right) \overset{g}{\lrcorner} u = (-1)^{p-1} g(\ddot{u}, u) \ddot{U} + \ddot{u} \otimes \left( \ddot{U} \overset{g}{\lrcorner} u \right) \end{array} \right) \\
&= (-1)^{p-1} g(\ddot{u}, u) \alpha^f \left( \ddot{U} \right) + \alpha^f \left( \ddot{u} \otimes \left( \ddot{U} \overset{g}{\lrcorner} u \right) \right) \qquad \left( \text{since } \alpha^f \text{ is a } k\text{-linear map} \right) \\
&= (-1)^{p-1} g(\ddot{u}, u) \alpha^f \left( \ddot{U} \right) + \ddot{u} \otimes \alpha^f \left( \ddot{U} \overset{g}{\lrcorner} u \right) - \ddot{u} \overset{f}{\llcorner} \alpha^f \left( \ddot{U} \overset{g}{\lrcorner} u \right) \\
&\qquad \left( \begin{array}{c} \text{since (46) (applied to } \ddot{u} \text{ and } \ddot{U} \overset{g}{\lrcorner} u \text{ instead of } u \text{ and } U\text{) yields} \\ \alpha^f \left( \ddot{u} \otimes \left( \ddot{U} \overset{g}{\lrcorner} u \right) \right) = \ddot{u} \otimes \alpha^f \left( \ddot{U} \overset{g}{\lrcorner} u \right) - \ddot{u} \overset{f}{\llcorner} \alpha^f \left( \ddot{U} \overset{g}{\lrcorner} u \right) \end{array} \right) \\
&= (-1)^{p-1} g(\ddot{u}, u) \alpha^f \left( \ddot{U} \right) + \ddot{u} \otimes \left( \alpha^f \left( \ddot{U} \right) \overset{g}{\lrcorner} u \right) - \ddot{u} \overset{f}{\llcorner} \left( \alpha^f \left( \ddot{U} \right) \overset{g}{\lrcorner} u \right) \\
&\qquad \left( \begin{array}{c} \text{since (54), applied to } \ddot{U} \text{ instead of } U, \text{ yields } \alpha^f \left( \ddot{U} \overset{g}{\lrcorner} u \right) = \alpha^f \left( \ddot{U} \right) \overset{g}{\lrcorner} u \\ \text{(in fact, we \textit{are} allowed to apply (54) to } \ddot{U} \text{ instead of } U, \text{ because} \\ \ddot{U} \in L^{\otimes (p-1)} \text{ and because we assumed that the equality (54) is proven} \\ \text{for every } U \in L^{\otimes (p-1)}) \end{array} \right) \\
&= (-1)^{p-1} g(\ddot{u}, u) \alpha^f \left( \ddot{U} \right) + \ddot{u} \otimes \left( \alpha^f \left( \ddot{U} \right) \overset{g}{\lrcorner} u \right) - \left( \ddot{u} \overset{f}{\llcorner} \alpha^f \left( \ddot{U} \right) \right) \overset{g}{\lrcorner} u \\
&\qquad \left( \begin{array}{c} \text{since (39) (applied to } \ddot{u}, \alpha^f \left( \ddot{U} \right) \text{ and } u \text{ instead of } v, U \text{ and } w\text{) yields} \\ \ddot{u} \overset{f}{\llcorner} \left( \alpha^f \left( \ddot{U} \right) \overset{g}{\lrcorner} u \right) = \left( \ddot{u} \overset{f}{\llcorner} \alpha^f \left( \ddot{U} \right) \right) \overset{g}{\lrcorner} u \end{array} \right)
\end{aligned}
$$

[37] In fact, in this case, $U \in L^{\otimes p} = L^{\otimes 0} = k$ yields $U \overset{g}{\lrcorner} u = 0$ (by Theorem 11 **(a)** (applied to $g$, $U$ and $u$ instead of $f$, $\lambda$ and $v$)) and $\alpha^f (U) \overset{g}{\lrcorner} u = 0$ (by Theorem 11 **(a)** (applied to $g$, $\alpha^f (U)$ and $u$ instead of $f$, $\lambda$ and $v$), since $U \in k$ yields $\alpha^f (U) = U \in k$), and therefore the equation (54) rewrites as $\alpha^f (0) = 0$, which is trivially true.

and

$$\begin{aligned}
&\alpha^f(U) \overset{g}{\lrcorner} u \\
&= \alpha^f\left(\ddot{u} \otimes \ddot{U}\right) \overset{g}{\lrcorner} u \qquad \left(\text{since } U = \ddot{u} \otimes \ddot{U}\right) \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right) - \ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \overset{g}{\lrcorner} u \\
&\qquad \left(\begin{array}{c} \text{since (46) (applied to } \ddot{u} \text{ and } \ddot{U} \text{ instead of } u \text{ and } U\text{) yields} \\ \alpha^f\left(\ddot{u} \otimes \ddot{U}\right) = \ddot{u} \otimes \alpha^f\left(\ddot{U}\right) - \ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right) \end{array}\right) \\
&= \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right)\right) \overset{g}{\lrcorner} u - \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \overset{g}{\lrcorner} u \\
&= (-1)^{p-1} g(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \overset{g}{\lrcorner} u\right) - \left(\ddot{u} \overset{f}{\llcorner} \alpha^f\left(\ddot{U}\right)\right) \overset{g}{\lrcorner} u \\
&\qquad \left(\begin{array}{c} \text{since (27) (applied to } g,\ \alpha^f\left(\ddot{U}\right),\ \ddot{u},\ u \text{ and } p-1 \text{ instead of } f,\ U,\ u,\ v \text{ and } p\text{)} \\ \text{yields } \left(\ddot{u} \otimes \alpha^f\left(\ddot{U}\right)\right) \overset{g}{\lrcorner} u = (-1)^{p-1} g(\ddot{u}, u)\, \alpha^f\left(\ddot{U}\right) + \ddot{u} \otimes \left(\alpha^f\left(\ddot{U}\right) \overset{g}{\lrcorner} u\right) \\ \text{(because Theorem 20 (applied to } p-1 \text{ instead of } p\text{) yields} \\ \alpha^f\left(\ddot{U}\right) \in \bigoplus\limits_{\substack{i \in \mathbb{N}; \\ i \equiv p-1 \bmod 2}} L^{\otimes i} \text{ (since } \ddot{U} \in L^{\otimes(p-1)}\text{))} \end{array}\right)
\end{aligned}$$

). Thus, we have proven the equality (54) for every $U \in L^{\otimes p}$. This completes the induction step, and thus we have successfully shown that for every $p \in \mathbb{N}$, the equality (54) holds for every $U \in L^{\otimes p}$. Hence, the equality (54) holds for every $U \in \otimes L$ (because every $U \in \otimes L$ is a $k$-linear combinations of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and because the equality (54) is linear in $U$). This proves Theorem 22.

Now we could give a proof of Theorem 23 which is totally analogous to the above proof of Theorem 22, but instead we prefer to go another way: First we show Theorem 24, and then we conclude Theorem 23 from Theorem 22 using Theorem 24.

*Proof of Theorem 24.* Let us first prove that for every $p \in \mathbb{N}$, we have

$$\alpha^f(t(U)) = t\left(\alpha^{f^t}(U)\right) \tag{56}$$

for every $U \in L^{\otimes p}$.

In fact, we are going to prove this by induction over $p$:

The induction base case $p = 0$ is trivial[38].

Now, we must perform the induction step: Let $p \in \mathbb{N}_+$. Assume that (56) holds for every $U \in L^{\otimes(p-1)}$. Then, we must prove that (56) also holds for every $U \in L^{\otimes p}$. In fact, in order to achieve this, it is enough to prove that (56) holds for every left-induced $U \in L^{\otimes p}$ (due to the left tensor induction tactic, because the equality (56) is linear in $U$). So let $U \in L^{\otimes p}$ be a left-induced tensor. Then, $U$ can be written in the form $U = \ddot{u} \otimes \ddot{U}$ for some $\ddot{u} \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$ (since $U$ is left-induced). Therefore, $t(U) = t\left(\ddot{u} \otimes \ddot{U}\right) = t\left(\ddot{U}\right) \otimes t(\ddot{u})$ (by (31), applied to $\ddot{u}$ and $\ddot{U}$ instead of $U$ and $V$),

[38] In fact, in this case, we have $\left(\alpha^f \circ t\right)(U) = U = \left(t \circ \alpha^{f^t}\right)(U)$, since all three maps $\alpha^f$, $t$ and $\alpha^{f^t}$ leave elements of $L^{\otimes p} = L^{\otimes 0} = k$ fixed.

and thus

$$\alpha^f (t (U))$$

$$= \alpha^f \left( t \left( \ddot{U} \right) \otimes \underbrace{t (\ddot{u})}_{= \ddot{u} \text{ (since } \ddot{u} \in L)} \right) = \alpha^f \left( t \left( \ddot{U} \right) \otimes \ddot{u} \right) = \alpha^f \left( t \left( \ddot{U} \right) \right) \otimes \ddot{u} - \alpha^f \left( t \left( \ddot{U} \right) \right) \overset{f}{\lrcorner} \ddot{u}$$

$$= t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \otimes \ddot{u} - t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \overset{f}{\lrcorner} \ddot{u}$$

$$\left( \begin{array}{c} \text{since (56) (applied to } \ddot{U} \text{ instead of } U \text{) yields } \alpha^f \left( t \left( \ddot{U} \right) \right) = t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \\ \text{(in fact, we } \textit{are} \text{ allowed to apply (56) to } \ddot{U} \text{ instead of } U \text{, because} \\ \ddot{U} \in L^{\otimes (p-1)} \text{ and because we have assumed that} \\ \text{(56) holds for every } U \in L^{\otimes (p-1)} \text{)} \end{array} \right)$$

$$= t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \otimes \underbrace{\ddot{u}}_{= t(\ddot{u}) \text{ (since } \ddot{u} \in L)} - t \left( \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) \right)$$

$$\left( \begin{array}{c} \text{since (33) (applied to } t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \text{ and } \ddot{u} \text{ instead of } U \text{ and } v \text{) yields} \\ t \left( \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) \right) = t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \overset{f}{\lrcorner} \ddot{u} \text{, which rewrites as} \\ t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \overset{f}{\lrcorner} \ddot{u} = t \left( \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) \right) \end{array} \right)$$

$$= t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \otimes t (\ddot{u}) - t \left( \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) \right) = t \left( \ddot{u} \otimes \alpha^{f^t} \left( \ddot{U} \right) \right) - t \left( \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) \right)$$

$$\left( \begin{array}{c} \text{since (31) (applied to } \ddot{u} \text{ and } \alpha^{f^t} \left( \ddot{U} \right) \text{ instead of } U \text{ and } V \text{) yields} \\ t \left( \ddot{u} \otimes \alpha^{f^t} \left( \ddot{U} \right) \right) = t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \otimes t (\ddot{u}) \text{, so that} \\ t \left( \alpha^{f^t} \left( \ddot{U} \right) \right) \otimes t (\ddot{u}) = t \left( \ddot{u} \otimes \alpha^{f^t} \left( \ddot{U} \right) \right) \end{array} \right)$$

$$= t \left( \ddot{u} \otimes \alpha^{f^t} \left( \ddot{U} \right) - \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) \right) \qquad \text{(since the map } t \text{ is } k\text{-linear)}$$

$$= t \left( \alpha^{f^t} \left( \underbrace{\ddot{u} \otimes \ddot{U}}_{= U} \right) \right)$$

$$\left( \begin{array}{c} \text{since (46) (applied to } f^t \text{, } \ddot{u} \text{ and } \ddot{U} \text{ instead of } f \text{, } u \text{ and } U \text{) yields} \\ \alpha^{f^t} \left( \ddot{u} \otimes \ddot{U} \right) = \ddot{u} \otimes \alpha^{f^t} \left( \ddot{U} \right) - \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) \text{, so that} \\ \ddot{u} \otimes \alpha^{f^t} \left( \ddot{U} \right) - \ddot{u} \overset{f^t}{\llcorner} \alpha^{f^t} \left( \ddot{U} \right) = \alpha^{f^t} \left( \ddot{u} \otimes \ddot{U} \right) \end{array} \right)$$

$$= t \left( \alpha^{f^t} (U) \right).$$

Thus, we have proven that (56) holds for every left-induced $U \in L^{\otimes p}$. As we said, this is sufficient in order to complete the induction step, and therefore the induction step is completed, and we have successfully proven that for every $p \in \mathbb{N}$, we have $\alpha^f (t (U)) = t \left( \alpha^{f^t} (U) \right)$ for every $U \in L^{\otimes p}$. This immediately yields that $\alpha^f (t (U)) = t \left( \alpha^{f^t} (U) \right)$ for every $U \in \otimes L$ (because the equation $\alpha^f (t (U)) = t \left( \alpha^{f^t} (U) \right)$ is linear in $U$, and because every element of $\otimes L$ is a $k$-linear combinations of elements of $L^{\otimes p}$ for various

$p \in \mathbb{N}$). Hence, for every $U \in \otimes L$, we have

$$\left(\alpha^f \circ t\right)(U) = \alpha^f\left(t(U)\right) = t\left(\alpha^{f^t}(U)\right) = \left(t \circ \alpha^f\right)(U).$$

Thus, $\alpha^f \circ t = t \circ \alpha^{f^t}$. This proves Theorem 24.

*Proof of Theorem 23.* Applying (54) to $f^t$, $g^t$ and $t(U)$ instead of $f$, $g$ and $U$, we obtain

$$\alpha^{f^t}\left(t(U) \overset{g^t}{\lrcorner} u\right) = \alpha^{f^t}\left(t(U)\right) \overset{g^t}{\lrcorner} u.$$

Thus,

$$t\left(\alpha^{f^t}\left(t(U) \overset{g^t}{\lrcorner} u\right)\right) = t\left(\alpha^{f^t}\left(t(U)\right) \overset{g^t}{\lrcorner} u\right).$$

But since

$$t\left(\alpha^{f^t}\left(t(U) \overset{g^t}{\lrcorner} u\right)\right) = \left(\underbrace{t \circ \alpha^{f^t}}_{=\alpha^f \circ t \text{ (by Theorem 24)}}\right)\left(t(U) \overset{g^t}{\lrcorner} u\right) = \left(\alpha^f \circ t\right)\left(t(U) \overset{g^t}{\lrcorner} u\right)$$

$$= \alpha^f\left(t\left(t(U) \overset{g^t}{\lrcorner} u\right)\right) = \alpha^f\left(u \overset{g}{\llcorner} \underbrace{t\left(t(U)\right)}_{\substack{=t^2(U)=U \\ (\text{since } t^2=\mathrm{id})}}\right)$$

$$\left(\begin{array}{c} \text{since (32) (applied to } g^t,\ t(U) \text{ and } u \text{ instead of } f,\ U \text{ and } v) \\ \text{yields } t\left(t(U) \overset{g^t}{\lrcorner} u\right) = u \overset{\left(g^t\right)^t}{\llcorner} t\left(t(U)\right) = u \overset{g}{\llcorner} t\left(t(U)\right) \text{ (since } \left(g^t\right)^t = g) \end{array}\right)$$

$$= \alpha^f\left(u \overset{g}{\llcorner} U\right)$$

and

$$t\left(\alpha^{f^t}\left(t(U)\right) \overset{g^t}{\lrcorner} u\right)$$

$$= u \overset{\left(g^t\right)^t}{\llcorner} t\left(\alpha^{f^t}\left(t(U)\right)\right)$$

$$\left(\text{due to (32) (applied to } g^t,\ \alpha^f\left(t(U)\right) \text{ and } u \text{ instead of } f,\ U \text{ and } v)\right)$$

$$= u \overset{g}{\llcorner} \underbrace{t\left(\alpha^{f^t}\left(t(U)\right)\right)}_{=\left(t \circ \alpha^{f^t} \circ t\right)(U)} \qquad \left(\text{since } \left(g^t\right)^t = g\right)$$

$$= u \overset{g}{\llcorner} \left(t \circ \alpha^{f^t} \circ t\right)(U) = u \overset{g}{\llcorner} \left(\underbrace{t \circ t}_{=t^2=\mathrm{id}} \circ \alpha^f\right)(U)$$

$$\left(\begin{array}{c} \text{since Theorem 24 (applied to } f^t \text{ instead of } f) \\ \text{yields } \alpha^{f^t} \circ t = t \circ \alpha^{\left(f^t\right)^t} = t \circ \alpha^f \text{ (because } \left(f^t\right)^t = f) \end{array}\right)$$

$$= u \overset{g}{\llcorner} \alpha^f(U),$$

this becomes $\alpha^f\left(u \overset{g}{\llcorner} U\right) = u \overset{g}{\llcorner} \alpha^f(U)$. Thus, Theorem 23 is proven.

## 6. The endomorphism $\alpha^g$ and the ideals $I_f^{(v)}$

In Definition 3, we have introduced the two-sided ideal $I_f$ of the algebra $\otimes L$. It was defined as

$$(\otimes L) \cdot \langle v \otimes v - f(v,v) \mid v \in L \rangle \cdot (\otimes L).$$

We will now write this ideal $I_f$ as a sum (not a direct sum, however) of certain smaller $k$-modules, which we denote by $I_f^{(v)}$ and $I_f^{(v;p;q)}$ (the $I_f^{(v;p;q)}$ are an even finer subdivision of the $I_f^{(v)}$). These ideals are not really necessary for our further goals, but they help keeping our proof a bit more organized:

**Definition 11. (a)** For any vector $v \in L$, let $I_f^{(v)}$ be the $k$-submodule

$$(\otimes L) \cdot (v \otimes v - f(v,v)) \cdot (\otimes L)$$

of the $k$-module $\otimes L$.

**(b)** For any vector $v \in L$, and any $p \in \mathbb{N}$ and $q \in \mathbb{N}$, let $I_f^{(v;p;q)}$ be the $k$-submodule

$$L^{\otimes p} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q}$$

of the $k$-module $\otimes L$.

Note that the dot sign (the sign $\cdot$) in this definition stands for multiplication in the algebra $\otimes L$; in other words, it is synonymous to the tensor product sign (the sign $\otimes$).

We then have

$$I_f = \sum_{v \in L} I_f^{(v)} \tag{57}$$

(where the $\sum$ sign means a sum of $k$-modules), since Definition 3 yields

$$\begin{aligned} I_f &= (\otimes L) \cdot \underbrace{\langle v \otimes v - f(v,v) \mid v \in L \rangle}_{= \sum_{v \in L} (v \otimes v - f(v,v)) \cdot k} \cdot (\otimes L) \\ &= (\otimes L) \cdot \left( \sum_{v \in L} (v \otimes v - f(v,v)) \cdot k \right) \cdot (\otimes L) = \sum_{v \in L} (\otimes L) \cdot \underbrace{((v \otimes v - f(v,v)) \cdot k) \cdot (\otimes L)}_{= (v \otimes v - f(v,v)) \cdot k \cdot (\otimes L)} \\ &= \sum_{v \in L} (\otimes L) \cdot (v \otimes v - f(v,v)) \cdot \underbrace{k \cdot (\otimes L)}_{= \otimes L} = \sum_{v \in L} \underbrace{(\otimes L) \cdot (v \otimes v - f(v,v)) \cdot (\otimes L)}_{= I_f^{(v)}} = \sum_{v \in L} I_f^{(v)}. \end{aligned}$$

Besides, every $v \in L$ satisfies

$$I_f^{(v)} = \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{N}} I_f^{(v;p;q)} \tag{58}$$

(where the $\sum$ signs mean sums of $k$-modules), since Definition 11 **(a)** yields

$$\begin{aligned} I_f^{(v)} &= \underbrace{(\otimes L)}_{= \sum_{p \in \mathbb{N}} L^{\otimes p}} \cdot (v \otimes v - f(v,v)) \cdot \underbrace{(\otimes L)}_{= \sum_{q \in \mathbb{N}} L^{\otimes q}} = \left( \sum_{p \in \mathbb{N}} L^{\otimes p} \right) \cdot (v \otimes v - f(v,v)) \cdot \left( \sum_{q \in \mathbb{N}} L^{\otimes q} \right) \\ &= \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{N}} \underbrace{L^{\otimes p} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q}}_{= I_f^{(v;p;q)}} = \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{N}} I_f^{(v;p;q)}. \end{aligned}$$

Consequently, $I_f^{(v;p;q)} \subseteq I_f^{(v)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{N}$.

On the other hand, note that every vector $v \in L$, and any $p \in \mathbb{N}$ and $q \in \mathbb{N}$ satisfy

$$I_f^{(v;p;q)} = \langle (u_p \otimes u_{p-1} \otimes ... \otimes u_1) \otimes (v \otimes v - f(v,v)) \otimes (w_1 \otimes w_2 \otimes ... \otimes w_q) \\ \mid \ ((u_1, u_2, ..., u_p), (w_1, w_2, ..., w_q)) \in L^p \times L^q \rangle , \tag{59}$$

because

$$\begin{aligned} I_f^{(v;p;q)} &= \underbrace{L^{\otimes p}}_{=\langle u_p \otimes u_{p-1} \otimes ... \otimes u_1 \ \mid \ (u_1,u_2,...,u_p) \in L^p \rangle} \cdot (v \otimes v - f(v,v)) \cdot \underbrace{L^{\otimes q}}_{=\langle w_1 \otimes w_2 \otimes ... \otimes w_q \ \mid \ (w_1,w_2,...,w_q) \in L^q \rangle} \\ &= \langle u_p \otimes u_{p-1} \otimes ... \otimes u_1 \ \mid \ (u_1, u_2, ..., u_p) \in L^p \rangle \cdot (v \otimes v - f(v,v)) \\ &\qquad \cdot \langle w_1 \otimes w_2 \otimes ... \otimes w_q \ \mid \ (w_1, w_2, ..., w_q) \in L^q \rangle \\ &= \langle (u_p \otimes u_{p-1} \otimes ... \otimes u_1) \cdot (v \otimes v - f(v,v)) \cdot (w_1 \otimes w_2 \otimes ... \otimes w_q) \\ &\qquad \mid \ ((u_1, u_2, ..., u_p), (w_1, w_2, ..., w_q)) \in L^p \times L^q \rangle \\ &= \langle (u_p \otimes u_{p-1} \otimes ... \otimes u_1) \otimes (v \otimes v - f(v,v)) \otimes (w_1 \otimes w_2 \otimes ... \otimes w_q) \\ &\qquad \mid \ ((u_1, u_2, ..., u_p), (w_1, w_2, ..., w_q)) \in L^p \times L^q \rangle \end{aligned}$$

[39].

Our main goal in this section is to prove the following result:

**Theorem 25.** Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $\alpha^g (I_f) \subseteq I_{f+g}$.

In order to prove this theorem, we first start with an easy fact:

**Proposition 26.** If $w \in L$, $U \in \otimes L$, and $v \in L$, then

$$w \llcorner^f (v \otimes v \otimes U) = v \otimes v \otimes \left( w \llcorner^f U \right) \tag{60}$$

and

$$\alpha^f (v \otimes v \otimes U) = (v \otimes v - f(v,v)) \otimes \alpha^f (U) \tag{61}$$

*Proof of Proposition 26.* The formula (7) (applied to $w$, $v$ and $v \otimes U$ instead of $v$, $u$ and $U$) yields

$$\begin{aligned} w \llcorner^f (v \otimes v \otimes U) &= f(w,v)\, v \otimes U - v \otimes \left( w \llcorner^f (v \otimes U) \right) \\ &= f(w,v)\, v \otimes U - v \otimes \left( f(w,v)\, U - v \otimes \left( w \llcorner^f U \right) \right) \\ &\qquad \left( \begin{array}{c} \text{since (7) (applied to } w \text{ and } v \text{ instead of } v \text{ and } u\text{)} \\ \text{yields } w \llcorner^f (v \otimes U) = f(w,v)\, U - v \otimes \left( w \llcorner^f U \right) \end{array} \right) \\ &= f(w,v)\, v \otimes U - \left( \underbrace{v \otimes f(w,v)\, U}_{=f(w,v) v \otimes U} - v \otimes v \otimes \left( w \llcorner^f U \right) \right) = v \otimes v \otimes \left( w \llcorner^f U \right). \end{aligned}$$

[39] Here, we have replaced the dot signs (the $\cdot$ signs) by tensor product signs (the $\otimes$ signs), because the multiplication in the algebra $\otimes L$ is the tensor product.

This proves (60). Besides, the formula (46) (applied to $v$ and $v \otimes U$ instead of $u$ and $U$) yields

$$\alpha^f (v \otimes v \otimes U) = v \otimes \alpha^f (v \otimes U) - v \overset{f}{\llcorner} \alpha^f (v \otimes U) .$$

Since $\alpha^f (v \otimes U) = v \otimes \alpha^f (U) - v \overset{f}{\llcorner} \alpha^f (U)$ (as follows from (46), applied to $v$ instead of $u$), this rewrites as

$$\begin{aligned}
&\alpha^f (v \otimes v \otimes U) \\
&= v \otimes \left( v \otimes \alpha^f (U) - v \overset{f}{\llcorner} \alpha^f (U) \right) - v \overset{f}{\llcorner} \left( v \otimes \alpha^f (U) - v \overset{f}{\llcorner} \alpha^f (U) \right) \\
&= \left( v \otimes v \otimes \alpha^f (U) - v \otimes \left( v \overset{f}{\llcorner} \alpha^f (U) \right) \right) - \left( v \overset{f}{\llcorner} \left( v \otimes \alpha^f (U) \right) - v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \alpha^f (U) \right) \right) \\
&\qquad \left( \text{by the bilinearity of } \otimes \text{ and the bilinearity of } \overset{f}{\llcorner} \right) \\
&= v \otimes v \otimes \alpha^f (U) - v \otimes \left( v \overset{f}{\llcorner} \alpha^f (U) \right) - \underbrace{v \overset{f}{\llcorner} \left( v \otimes \alpha^f (U) \right)}_{\substack{= f(v,v) \alpha^f (U) - v \otimes \left( v \overset{f}{\llcorner} \alpha^f (U) \right) \\ \text{(by (7), applied} \\ \text{to } v \text{ and } \alpha^f (U) \text{ instead of } u \text{ and } U)}} + \underbrace{v \overset{f}{\llcorner} \left( v \overset{f}{\llcorner} \alpha^f (U) \right)}_{\substack{=0 \text{ (by (8), applied} \\ \text{to } \alpha^f (U) \text{ instead of } U)}} \\
&= v \otimes v \otimes \alpha^f (U) - v \otimes \left( v \overset{f}{\llcorner} \alpha^f (U) \right) - \left( f (v,v) \alpha^f (U) - v \otimes \left( v \overset{f}{\llcorner} \alpha^f (U) \right) \right) + 0 \\
&= v \otimes v \otimes \alpha^f (U) - \underbrace{f (v,v) \alpha^f (U)}_{= f(v,v) \otimes \alpha^f (U)} = (v \otimes v - f (v,v)) \otimes \alpha^f (U) ,
\end{aligned}$$

which proves (61). Thus, both (60) and (61) are verified, and therefore, we have proved Proposition 26.

Now we are going to prove that the ideal $I_f$ is stable under the map $w \overset{g}{\llcorner}$ for any two bilinear forms $f$ and $g$ and any vector $w$:

**Theorem 27.** Let $w \in L$. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $w \overset{g}{\llcorner} I_f \subseteq I_f$. (Here, whenever $P$ is a $k$-submodule of $\otimes L$, we denote by $w \overset{g}{\llcorner} P$ the $k$-submodule $\left\{ w \overset{g}{\llcorner} p \mid p \in P \right\}$ of $\otimes L$. This is indeed a $k$-submodule, as follows from the bilinearity of $\overset{g}{\llcorner}$.)

*Proof of Theorem 27.* Let us first show that

$$w \overset{g}{\llcorner} I_f^{(v;p;q)} \subseteq I_f^{(v)} \tag{62}$$

for every $v \in L$, $p \in \mathbb{N}$ and $q \in \mathbb{N}$.

*Proof of (62).* In fact, we are going to prove (62) by induction over $p \in \mathbb{N}$.

The induction base case - the case $p = 0$ - is trivial, because (62) is easily seen to hold for $p = 0$ [40].

[40] *Proof.* Let $T \in I_f^{(v;0;q)}$. By the definition of $I_f^{(v;p;q)}$, we have

$$\begin{aligned}
I_f^{(v;0;q)} &= \underbrace{L^{\otimes 0}}_{=k} \cdot (v \otimes v - f (v,v)) \cdot L^{\otimes q} \\
&= \underbrace{k \cdot (v \otimes v - f (v,v))}_{=(v \otimes v - f(v,v)) \cdot k} \cdot L^{\otimes q} = (v \otimes v - f (v,v)) \cdot \underbrace{k \cdot L^{\otimes q}}_{= L^{\otimes q}} = (v \otimes v - f (v,v)) \cdot L^{\otimes q}.
\end{aligned}$$

Now to the induction step: Let $p \in \mathbb{N}_+$. We must now prove (62) for this $p$, assuming that (62) is already proven for $p-1$ instead of $p$.

In fact, let us prove (62). Let $T \in I_f^{(v;p;q)}$. Then, $p > 0$ (because $p \in \mathbb{N}_+$) yields $L^{\otimes p} = L \otimes L^{\otimes(p-1)}$, and thus

$$\begin{aligned} T \in I_f^{(v;p;q)} &= \underbrace{L^{\otimes p}}_{\substack{=L\otimes L^{\otimes(p-1)}=L\cdot L^{\otimes(p-1)} \\ \text{(since the tensor product is the} \\ \text{multiplication in the algebra } \otimes L)}} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q} \\ &= L \cdot \underbrace{L^{\otimes(p-1)} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q}}_{=I_f^{(v;p-1;q)}} = L \cdot I_f^{(v;p-1;q)}. \end{aligned}$$

Hence, there exist a finite set $I$, a family $(\ell_i)_{i\in I}$ of elements of $L$ and a family $(U_i)_{i\in I}$ of elements of $I_f^{(v;p-1;q)}$ such that $T = \sum_{i\in I} \ell_i U_i$. Since the multiplication in the algebra $\otimes L$ is the tensor product, this rewrites as $T = \sum_{i\in I} \ell_i \otimes U_i$. Every $i \in I$ satisfies $U_i \in I_f^{(v;p-1;q)}$ and thus

$$w_{\llcorner}^{g} U_i \in w_{\llcorner}^{g} I_f^{(v;p-1;q)} \subseteq I_f^{(v)} \tag{63}$$

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Thus, $T \in I_f^{(v;0;q)} = (v \otimes v - f(v,v)) \cdot L^{\otimes q}$, so that there exists some $U \in L^{\otimes q}$ such that $T = (v \otimes v - f(v,v)) \cdot U$. Since the multiplication in the algebra $\otimes L$ is the tensor product, we can rewrite this as $T = (v \otimes v - f(v,v)) \otimes U$. Thus,

$$\begin{aligned} w_{\llcorner}^{g} T &= w_{\llcorner}^{g} \left( \underbrace{(v \otimes v - f(v,v)) \otimes U}_{=v\otimes v\otimes U - f(v,v)\otimes U} \right) = w_{\llcorner}^{g} \left( v \otimes v \otimes U - \underbrace{f(v,v) \otimes U}_{=f(v,v)U} \right) = w_{\llcorner}^{g} (v \otimes v \otimes U - f(v,v) U) \\ &= w_{\llcorner}^{g} (v \otimes v \otimes U) - f(v,v)\, w_{\llcorner}^{g} U \qquad \left(\text{since } \llcorner^{g} \text{ is bilinear}\right) \\ &= \underbrace{v \otimes v \otimes \left(w_{\llcorner}^{g} U\right)}_{\substack{=(v\otimes v)\cdot\left(w_{\llcorner}^{g}U\right) \\ \text{(since the tensor product} \\ \text{is the multiplication in the} \\ \text{algebra } \otimes L)}} - f(v,v)\, w_{\llcorner}^{g} U \qquad (\text{by (60), applied to } g \text{ instead of } f) \\ &= (v \otimes v) \cdot \left(w_{\llcorner}^{g} U\right) - f(v,v)\, w_{\llcorner}^{g} U = (v \otimes v - f(v,v)) \cdot \left(w_{\llcorner}^{g} U\right) \\ &= \underbrace{1}_{\in \otimes L} \cdot (v \otimes v - f(v,v)) \cdot \underbrace{\left(w_{\llcorner}^{g} U\right)}_{\in \otimes L} \in (\otimes L) \cdot (v \otimes v - f(v,v)) \cdot (\otimes L) = I_f^{(v)}. \end{aligned}$$

Thus, we have shown that $w_{\llcorner}^{g} T \in I_f^{(v)}$ for every $T \in I_f^{(v;0;q)}$. In other words, $w_{\llcorner}^{g} I_f^{(v;0;q)} \subseteq I_f^{(v)}$. This proves (62) for $p = 0$.

(by (62), applied to $p-1$ instead of $p$ [41]). On the other hand, every $i \in I$ satisfies

$$\begin{aligned}
\ell_i \otimes I_f^{(v)} &= \ell_i \cdot \underbrace{I_f^{(v)}}_{=(\otimes L)\cdot(v\otimes v - f(v,v))\cdot(\otimes L)} \qquad \left(\begin{array}{c}\text{since the tensor product is the}\\ \text{multiplication in the algebra } \otimes L\end{array}\right) \\
&= \underbrace{\ell_i \cdot (\otimes L)}_{\subseteq \otimes L} \cdot (v \otimes v - f(v,v)) \cdot (\otimes L) \\
&\subseteq (\otimes L) \cdot (v \otimes v - f(v,v)) \cdot (\otimes L) = I_f^{(v)}. 
\end{aligned} \tag{64}$$

Now, $T = \sum_{i \in I} \ell_i \otimes U_i$ yields

$$\begin{aligned}
w \llcorner^{g} T &= w \llcorner^{g} \left( \sum_{i \in I} \ell_i \otimes U_i \right) = \sum_{i \in I} w \llcorner^{g} (\ell_i \otimes U_i) \qquad \left(\text{by the bilinearity of } \llcorner^{g}\right) \\
&= \sum_{i \in I} \left( g(w, \ell_i) U_i - \ell_i \otimes \left( w \llcorner^{g} U_i \right) \right) \\
&\qquad \left(\begin{array}{c}\text{since (7) (applied to } g,\ w,\ \ell_i \text{ and } U_i \text{ instead of } f,\ v,\ u \text{ and } U)\\ \text{yields } w \llcorner^{g} (\ell_i \otimes U_i) = g(w,\ell_i) U_i - \ell_i \otimes \left( w \llcorner^{g} U_i \right)\end{array}\right) \\
&= \sum_{i \in I} g(w, \ell_i) \underbrace{U_i}_{\in I_f^{(v;p-1;q)} \subseteq I_f^{(v)}} - \sum_{i \in I} \ell_i \otimes \underbrace{\left( w \llcorner^{g} U_i \right)}_{\in I_f^{(v)} \text{ (by (63))}} \\
&\in \sum_{i \in I} g(w, \ell_i) I_f^{(v)} - \sum_{i \in I} \underbrace{\ell_i \otimes I_f^{(v)}}_{\substack{\subseteq I_f^{(v)} \\ \text{(by (64))}}} \subseteq \underbrace{\sum_{i \in I} g(w, \ell_i) I_f^{(v)}}_{\subseteq I_f^{(v)} \text{ (since } I_f^{(v)} \text{ is a } k\text{-module)}} - \underbrace{\sum_{i \in I} I_f^{(v)}}_{\subseteq I_f^{(v)} \text{ (since } I_f^{(v)} \text{ is a } k\text{-module)}} \\
&\subseteq I_f^{(v)} - I_f^{(v)} \subseteq I_f^{(v)} \qquad \left(\text{since } I_f^{(v)} \text{ is a } k\text{-module}\right).
\end{aligned}$$

Hence, we have proven that $w \llcorner^{g} T \in I_f^{(v)}$ for every $T \in I_f^{(v;p;q)}$. In other words, we have proven (62). Thus, the induction step is complete, and we have successfully shown that (62) holds for every $v \in L$, $p \in \mathbb{N}$ and $q \in \mathbb{N}$.

Now, every $v \in L$ satisfies

$$\begin{aligned}
w \llcorner^{g} I_f^{(v)} &= w \llcorner^{g} \left( \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{N}} I_f^{(v;p;q)} \right) \qquad \text{(by (58))} \\
&= \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{N}} \underbrace{\left( w \llcorner^{g} I_f^{(v;p;q)} \right)}_{\subseteq I_f^{(v)} \text{ (by (62))}} \qquad \left(\text{by the bilinearity of } \llcorner^{g}\right) \\
&\subseteq \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{N}} I_f^{(v)} \subseteq I_f^{(v)} \qquad \left(\text{since } I_f^{(v)} \text{ is a } k\text{-module}\right).
\end{aligned} \tag{65}$$

[41] In fact, we *are* allowed to apply (62) to $p-1$ instead of $p$, since we have assumed that (62) is already proven for $p-1$ instead of $p$.

Consequently, (57) yields

$$w \overset{g}{\llcorner} I_f = w \overset{g}{\llcorner} \left( \sum_{v \in L} I_f^{(v)} \right) = \sum_{v \in L} \left( \underbrace{w \overset{g}{\llcorner} I_f^{(v)}}_{\subseteq I_f^{(v)}} \right) \qquad \left( \text{by the bilinearity of } \overset{g}{\llcorner} \right)$$
$$\subseteq \sum_{v \in L} I_f^{(v)} = I_f \qquad (\text{by (57)}).$$

This proves Theorem 27.

As an analogue of Theorem 27, we can show:

**Theorem 28.** Let $w \in L$. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $I_f \overset{g}{\lrcorner} w \subseteq I_f$. (Here, whenever $P$ is a $k$-submodule of $\otimes L$, we denote by $P \overset{g}{\lrcorner} w$ the $k$-submodule $\left\{ p \overset{g}{\lrcorner} w \mid p \in P \right\}$ of $\otimes L$. This is indeed a $k$-submodule, as follows from the bilinearity of $\overset{g}{\lrcorner}$.)

We can either prove this in complete analogy to Theorem 27, or use Theorem 27 and the following fact:

**Theorem 29.** We have $t(I_f) = I_f$.

*Proof of Theorem 29.* We will prove more: We will prove that every $v \in L$, $p \in \mathbb{N}$ and $q \in \mathbb{N}$ satisfy $t\left( I_f^{(v;p;q)} \right) = I_f^{(v;q;p)}$.

In fact, the definition of $I_f^{(v;p;q)}$ yields

$$\begin{aligned} I_f^{(v;p;q)} &= L^{\otimes p} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q} = \left\langle U \cdot (v \otimes v - f(v,v)) \cdot V \mid (U,V) \in L^{\otimes p} \times L^{\otimes q} \right\rangle \\ &= \left\langle U \otimes (v \otimes v - f(v,v)) \otimes V \mid (U,V) \in L^{\otimes p} \times L^{\otimes q} \right\rangle \qquad (66) \\ &\qquad (\text{since the multiplication in } \otimes L \text{ is the tensor product}). \end{aligned}$$

But every $(U,V) \in L^{\otimes p} \times L^{\otimes q}$ satisfies

$$\begin{aligned} & t(U \otimes (v \otimes v - f(v,v)) \otimes V) \\ &= t((v \otimes v - f(v,v)) \otimes V) \otimes t(U) \\ &\qquad (\text{by (31), applied to } (v \otimes v - f(v,v)) \otimes V \text{ instead of } V) \\ &= t(V) \otimes t(v \otimes v - f(v,v)) \otimes t(U) \\ &\qquad \left( \begin{array}{c} \text{since (31) (applied to } v \otimes v - f(v,v) \text{ instead of } U\text{) yields} \\ t((v \otimes v - f(v,v)) \otimes V) = t(V) \otimes t(v \otimes v - f(v,v)) \end{array} \right) \\ &= \underbrace{t(V)}_{\in L^{\otimes q} \text{ (since } V \in L^{\otimes q})} \otimes (v \otimes v - f(v,v)) \otimes \underbrace{t(U)}_{\in L^{\otimes p} \text{ (since } U \in L^{\otimes p})} \\ &\qquad \left( \text{since } t(v \otimes v - f(v,v)) = \underbrace{t(v \otimes v)}_{= v \otimes v} - \underbrace{t(f(v,v))}_{f(v,v)} = v \otimes v - f(v,v) \right) \\ &\in L^{\otimes q} \otimes (v \otimes v - f(v,v)) \otimes L^{\otimes p} = I_f^{(v;q;p)} \end{aligned}$$

(since the definition of $I_f^{(v;q;p)}$ is $I_f^{(v;q;p)} = L^{\otimes q} \otimes (v \otimes v - f(v,v)) \otimes L^{\otimes p}$). Now, (66) yields

$$\begin{aligned}
t\left(I_f^{(v;p;q)}\right) &= t\left(\left\langle U \otimes (v \otimes v - f(v,v)) \otimes V \ \mid\ (U,V) \in L^{\otimes p} \times L^{\otimes q}\right\rangle\right) \\
&= \left\langle t\left(U \otimes (v \otimes v - f(v,v)) \otimes V\right) \ \mid\ (U,V) \in L^{\otimes p} \times L^{\otimes q}\right\rangle \\
&\qquad \text{(since } t \text{ is a } k\text{-linear map)} \\
&\subseteq I_f^{(v;q;p)}
\end{aligned}$$

(since $t(U \otimes (v \otimes v - f(v,v)) \otimes V) \in I_f^{(v;q;p)}$ for every $(U,V) \in L^{\otimes p} \times L^{\otimes q}$). Thus we have proven that $t\left(I_f^{(v;p;q)}\right) \subseteq I_f^{(v;q;p)}$. Upon transposing $p$ and $q$, this becomes $t\left(I_f^{(v;q;p)}\right) \subseteq I_f^{(v;p;q)}$. Applying $t$ to both sides of this relation, we get $t\left(t\left(I_f^{(v;q;p)}\right)\right) \subseteq t\left(I_f^{(v;p;q)}\right)$. Since $t\left(t\left(I_f^{(v;q;p)}\right)\right) = \underbrace{t^2}_{=\mathrm{id}}\left(I_f^{(v;q;p)}\right) = I_f^{(v;q;p)}$, this becomes $I_f^{(v;q;p)} \subseteq t\left(I_f^{(v;p;q)}\right)$. Combined with $t\left(I_f^{(v;p;q)}\right) \subseteq I_f^{(v;q;p)}$, this yields $t\left(I_f^{(v;p;q)}\right) = I_f^{(v;q;p)}$.

Thus, every $v \in L$ satisfies

$$\begin{aligned}
t\left(I_f^{(v)}\right) &= t\left(\sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}} I_f^{(v;p;q)}\right) \qquad \text{(by (58))} \\
&= \sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}} \underbrace{t\left(I_f^{(v;p;q)}\right)}_{=I_f^{(v;q;p)}} \qquad \text{(since the map } t \text{ is linear)} \\
&= \sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}} I_f^{(v;q;p)} = \sum_{q\in\mathbb{N}}\sum_{p\in\mathbb{N}} I_f^{(v;q;p)} \\
&= \sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}} I_f^{(v;p;q)} \qquad \text{(here we renamed } p \text{ and } q \text{ into } q \text{ and } p \text{ in the sum)} \\
&= I_f^{(v)} \qquad \text{(by (58))}.
\end{aligned}$$

But now, (57) yields

$$t(I_f) = t\left(\sum_{v\in L} I_f^{(v)}\right) = \sum_{v\in L} \underbrace{t\left(I_f^{(v)}\right)}_{=I_f^{(v)}} = \sum_{v\in L} I_f^{(v)} = I_f \qquad \text{(by (57))}.$$

This proves Theorem 29.

*Proof of Theorem 28.* Every $U \in I_f$ satisfies

$$\begin{aligned}
t\left(U \lrcorner^{g} w\right) &= w \llcorner^{g^t} \underbrace{t(U)}_{\substack{\in t(I_f) = I_f \\ \text{(by Theorem 29)}}} \qquad \text{(by (32), applied to } g \text{ and } w \text{ instead of } f \text{ and } v) \\
&\in w \llcorner^{g^t} I_f \subseteq I_f \qquad \left(\text{by Theorem 27, applied to } g^t \text{ instead of } g\right).
\end{aligned}$$

Applying $t$ to both sides of this relation, we get

$$t\left(t\left(U \lrcorner^{g} w\right)\right) \in t(I_f).$$

Since $t\left(t\left(U \overset{g}{\lrcorner} w\right)\right) = \underbrace{t^2}_{=\mathrm{id}}\left(U \overset{g}{\lrcorner} w\right) = U \overset{g}{\lrcorner} w$ and $t(I_f) = I_f$ (by Theorem 29), this rewrites as $U \overset{g}{\lrcorner} w \in I_f$. Since this holds for all $U \in I_f$, we can conclude from this that $I_f \overset{g}{\lrcorner} w \subseteq I_f$. This proves Theorem 28.

Now, something more interesting: The map $\alpha^g$ doesn't (in general) leave $I_f$ stable, but instead maps it to $I_{f+g}$:

**Theorem 30.** Let $w \in L$. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $\alpha^g(I_f) \subseteq I_{f+g}$.

*Proof of Theorem 30.* Let us first show that

$$\alpha^g\left(I_f^{(v;p;q)}\right) \subseteq I_{f+g}^{(v)} \tag{67}$$

for every $v \in L$, $p \in \mathbb{N}$ and $q \in \mathbb{N}$.

*Proof of (67).* In fact, we are going to prove (67) by induction over $p \in \mathbb{N}$.

The induction base case - the case $p = 0$ - is trivial, because (67) is easily seen to hold for $p = 0$ [42].

[42] *Proof.* Let $T \in I_f^{(v;0;q)}$. By the definition of $I_f^{(v;p;q)}$, we have

$$I_f^{(v;0;q)} = \underbrace{L^{\otimes 0}}_{=k} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q} = \underbrace{k \cdot (v \otimes v - f(v,v))}_{=(v\otimes v - f(v,v))\cdot k} \cdot L^{\otimes q}$$
$$= (v \otimes v - f(v,v)) \cdot \underbrace{k \cdot L^{\otimes q}}_{=L^{\otimes q}} = (v \otimes v - f(v,v)) \cdot L^{\otimes q}.$$

Thus, $T \in I_f^{(v;0;q)} = (v \otimes v - f(v,v)) \cdot L^{\otimes q}$, so that there exists some $U \in L^{\otimes q}$ such that $T = (v \otimes v - f(v,v)) \cdot U$. Since the multiplication in the algebra $\otimes L$ is the tensor product, we can rewrite this as $T = (v \otimes v - f(v,v)) \otimes U$. Thus,

$$\alpha^g(T) = \alpha^g\left(\underbrace{(v \otimes v - f(v,v)) \otimes U}_{=v\otimes v\otimes U - f(v,v)\otimes U}\right) = \alpha^g\left(v \otimes v \otimes U - \underbrace{f(v,v) \otimes U}_{=f(v,v)U}\right) = \alpha^g(v \otimes v \otimes U - f(v,v)U)$$
$$= \alpha^g(v \otimes v \otimes U) - f(v,v)\,\alpha^g(U) \qquad \text{(since } \alpha^g \text{ is linear)}$$
$$= \underbrace{(v \otimes v - g(v,v)) \otimes \alpha^g(U)}_{\substack{=(v\otimes v - g(v,v))\cdot \alpha^g(U) \\ \text{(since the tensor product} \\ \text{is the multiplication in the} \\ \text{algebra } \otimes L)}} - f(v,v)\,\alpha^g(U) \qquad \text{(by (61), applied to } g \text{ instead of } f)$$
$$= (v \otimes v - g(v,v)) \cdot \alpha^g(U) - f(v,v)\,\alpha^g(U)$$
$$= \left(\underbrace{v \otimes v - g(v,v) - f(v,v)}_{\substack{=v\otimes v - (f(v,v)+g(v,v)) \\ =v\otimes v - (f+g)(v,v)}}\right) \cdot \underbrace{\alpha^g(U)}_{\in \otimes L} \in (v \otimes v - (f+g)(v,v)) \cdot (\otimes L)$$
$$= \underbrace{1}_{\in \otimes L} \cdot (v \otimes v - (f+g)(v,v)) \cdot (\otimes L) \subseteq (\otimes L) \cdot (v \otimes v - (f+g)(v,v)) \cdot (\otimes L) = I_{f+g}^{(v)}.$$

Thus, we have shown that $\alpha^g(T) \in I_{f+g}^{(v)}$ for every $T \in I_f^{(v;0;q)}$. In other words, $\alpha^g\left(I_f^{(v;0;q)}\right) \subseteq I_{f+g}^{(v)}$. This proves (67) for $p = 0$.

Now to the induction step: Let $p \in \mathbb{N}_+$. We must now prove (67) for this $p$, assuming that (67) is already proven for $p-1$ instead of $p$.

In fact, let us prove (67). Let $T \in I_f^{(v;p;q)}$. Then, $p > 0$ (because $p \in \mathbb{N}_+$) yields $L^{\otimes p} = L \otimes L^{\otimes (p-1)}$, and thus

$$\begin{aligned} T \in I_f^{(v;p;q)} &= \underbrace{L^{\otimes p}}_{\substack{= L \otimes L^{\otimes (p-1)} = L \cdot L^{\otimes (p-1)} \\ \text{(since the tensor product is the} \\ \text{multiplication in the algebra } \otimes L)}} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q} \\ &= L \cdot \underbrace{L^{\otimes (p-1)} \cdot (v \otimes v - f(v,v)) \cdot L^{\otimes q}}_{= I_f^{(v;p-1;q)}} = L \cdot I_f^{(v;p-1;q)}. \end{aligned}$$

Hence, there exist a finite set $I$, a family $(\ell_i)_{i \in I}$ of elements of $L$ and a family $(U_i)_{i \in I}$ of elements of $I_f^{(v;p-1;q)}$ such that $T = \sum_{i \in I} \ell_i U_i$. Since the multiplication in the algebra $\otimes L$ is the tensor product, this rewrites as $T = \sum_{i \in I} \ell_i \otimes U_i$. Every $i \in I$ satisfies $U_i \in I_f^{(v;p-1;q)}$ and thus

$$\alpha^g(U_i) \in \alpha^g\left(I_f^{(v;p-1;q)}\right) \subseteq I_{f+g}^{(v)}$$

(by (67), applied to $p-1$ instead of $p$ [43]) and

$$\begin{aligned} \ell_i \otimes I_{f+g}^{(v)} &= \underbrace{\ell_i \otimes (\otimes L)}_{\subseteq \otimes L} \cdot (v \otimes v - (f+g)(v,v)) \cdot (\otimes L) \\ &\qquad \left(\text{since } I_{f+g}^{(v)} = (\otimes L) \cdot (v \otimes v - (f+g)(v,v)) \cdot (\otimes L)\right) \\ &\subseteq (\otimes L) \cdot (v \otimes v - (f+g)(v,v)) \cdot (\otimes L) = I_{f+g}^{(v)}. \end{aligned}$$

[43]In fact, we *are* allowed to apply (67) to $p-1$ instead of $p$, since we have assumed that (67) is already proven for $p-1$ instead of $p$.

Now, $T=\sum\limits_{i\in I}\ell_i\otimes U_i$ yields

$$\begin{aligned}
\alpha^g\left(T\right) &= \alpha^g\left(\sum_{i\in I}\ell_i\otimes U_i\right)=\sum_{i\in I}\alpha^g\left(\ell_i\otimes U_i\right) && \left(\text{since } \alpha^g \text{ is a linear map}\right)\\
&= \sum_{i\in I}\left(\ell_i\otimes\underbrace{\alpha^g\left(U_i\right)}_{\in I_{f+g}^{(v)}}-\ell_i\overset{g}{\llcorner}\underbrace{\alpha^g\left(U_i\right)}_{\in I_{f+g}^{(v)}}\right)\\
&\qquad \left(\begin{array}{c}\text{since (46) (applied to } g\text{, } \ell_i \text{ and } U_i \text{ instead of } f\text{, } u \text{ and } U\text{)}\\ \text{yields } \alpha^g\left(\ell_i\otimes U_i\right)=\ell_i\otimes\alpha^g\left(U_i\right)-\ell_i\overset{g}{\llcorner}\alpha^g\left(U_i\right)\end{array}\right)\\
&\in \sum_{i\in I}\left(\underbrace{\ell_i\otimes I_{f+g}^{(v)}}_{\subseteq I_{f+g}^{(v)}}-\underbrace{\ell_i\overset{g}{\llcorner}I_{f+g}^{(v)}}_{\substack{\subseteq I_{f+g}^{(v)} \text{ (by (65),}\\ \text{applied to } f+g \text{ and } \ell_i \text{ instead of } f \text{ and } w)}}\right)\\
&\subseteq \sum_{i\in I}\left(\underbrace{I_{f+g}^{(v)}-I_{f+g}^{(v)}}_{\subseteq I_{f+g}^{(v)} \text{ (since } I_{f+g}^{(v)} \text{ is a } k\text{-module)}}\right)=\sum_{i\in I}I_{f+g}^{(v)}\subseteq I_{f+g}^{(v)}\\
&\qquad \left(\text{since } I_{f+g}^{(v)} \text{ is a } k\text{-module}\right).
\end{aligned}$$

Hence, we have proven that $\alpha^g\left(T\right)\in I_{f+g}^{(v)}$ for every $T\in I_f^{(v;p;q)}$. In other words, we have proven (67). Thus, the induction step is complete, and we have successfully shown that (67) holds for every $v\in L$, $p\in\mathbb{N}$ and $q\in\mathbb{N}$.

Now, every $v\in L$ satisfies

$$\begin{aligned}
\alpha^g\left(I_f^{(v)}\right) &= \alpha^g\left(\sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}}I_f^{(v;p;q)}\right) && \left(\text{by (58)}\right)\\
&= \sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}}\underbrace{\alpha^g\left(I_f^{(v;p;q)}\right)}_{\subseteq I_{f+g}^{(v)} \text{ (by (67))}} && \left(\text{by the linearity of } \alpha^g\right)\\
&\subseteq \sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}}I_{f+g}^{(v)}\subseteq I_{f+g}^{(v)} && \left(\text{since } I_{f+g}^{(v)} \text{ is a } k\text{-module}\right).
\end{aligned}$$

Consequently, (57) yields

$$\begin{aligned}
\alpha^g\left(I_f\right) &= \alpha^g\left(\sum_{v\in L}I_f^{(v)}\right)=\sum_{v\in L}\left(\underbrace{\alpha^g\left(I_f^{(v)}\right)}_{\subseteq I_{f+g}^{(v)}}\right) && \left(\text{by the linearity of } \alpha^g\right)\\
&\subseteq \sum_{v\in L}I_{f+g}^{(v)}=I_{f+g} && \left(\text{by (57), applied to } f+g \text{ instead of } f\right).
\end{aligned}$$

This proves Theorem 30.

Actually a stronger fact holds:

**Theorem 31.** Let $w \in L$. Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $\alpha^g (I_f) = I_{f+g}$.

We will prove this in the next section, using the inverse of $\alpha^g$.

## 7. $\alpha^f \circ \alpha^g = \alpha^{f+g}$

Until now, each of our results involved $\alpha^f$ only for one bilinear form $f$. Though we sometimes called it $g$ instead of $f$, never did we consider the maps $\alpha^f$ for two *different* forms $f$ together in one and the same theorem. Let us change this now:

**Theorem 32.** **(a)** Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, $\alpha^f \circ \alpha^g = \alpha^{f+g}$.

**(b)** The bilinear form $\mathbf{0} : L \times L \to k$ defined by ($\mathbf{0}(x, y) = 0$ for every $x \in L$ and $y \in L$) satisfies $\alpha^{\mathbf{0}} = \mathrm{id}$.

**(c)** Let $f : L \times L \to k$ be a bilinear form. Then, the map $\alpha^f$ is invertible, and its inverse is $\alpha^{-f}$.

*Proof of Theorem 32.* **(a)** We will first show that for every $p \in \mathbb{N}$, we have

$$\alpha^f (\alpha^g (U)) = \alpha^{f+g} (U) \tag{68}$$

for every $U \in L^{\otimes p}$.

In fact, we will prove (68) by induction over $p$:

The induction base case $p = 0$ is obvious (because in this case, $U \in L^{\otimes p} = L^{\otimes 0} = k$ and thus $\alpha^f \left( \underbrace{\alpha^g (U)}_{=U} \right) = \alpha^f (U) = U$ and $\alpha^{f+g} (U) = U$, rending the equation (68) trivially true).

So let us now come to the induction step: Let $p \in \mathbb{N}_+$. We must prove (68), assuming that (68) has already been proven for $p - 1$ instead of $p$.

We want to prove (68). In other words, we want to prove that (68) holds for every $U \in L^{\otimes p}$. In order to do this, it is enough to prove that (68) holds for every left-induced $U \in L^{\otimes p}$ (by the left tensor induction tactic, because the equation (68) is linear in $U$). So, let us prove this. Let $U \in L^{\otimes p}$ be a left-induced tensor. Then, we can write $U$ in the form $U = u \otimes \ddot{U}$ for some $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ (because $U$ is left-induced).

Thus,

$$\begin{aligned}
&\alpha^f\left(\alpha^g\left(U\right)\right)\\
&=\alpha^f\left(\alpha^g\left(u\otimes\ddot{U}\right)\right)=\alpha^f\left(u\otimes\alpha^g\left(\ddot{U}\right)-u\llcorner^g\alpha^g\left(\ddot{U}\right)\right)\\
&\qquad\left(\begin{array}{c}\text{since (46) (applied to } g \text{ and } \ddot{U} \text{ instead of } f \text{ and } U\text{)}\\ \text{yields } \alpha^g\left(u\otimes\ddot{U}\right)=u\otimes\alpha^g\left(\ddot{U}\right)-u\llcorner^g\alpha^g\left(\ddot{U}\right)\end{array}\right)\\
&=\underbrace{\alpha^f\left(u\otimes\alpha^g\left(\ddot{U}\right)\right)}_{\substack{=u\otimes\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)-u\llcorner^f\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)\\ \text{(by (46), applied to}\\ \alpha^g\left(\ddot{U}\right)\text{ instead of }U\text{)}}}-\underbrace{\alpha^f\left(u\llcorner^g\alpha^g\left(\ddot{U}\right)\right)}_{\substack{=u\llcorner^g\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)\text{ (by (55), applied}\\ \text{to }\alpha^g\left(\ddot{U}\right)\text{ instead of }U\text{)}}}\\
&=\left(u\otimes\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)-u\llcorner^f\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)\right)-\left(u\llcorner^g\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)\right)\\
&=u\otimes\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)-\left(\underbrace{u\llcorner^f\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)+u\llcorner^g\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)}_{\substack{=u\llcorner^{f+g}\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)\\ \text{(by (16), applied to}\\ u\text{ and }\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)\text{ instead of }w\text{ and }U\text{)}}}\right)\\
&=u\otimes\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)-u\llcorner^{f+g}\alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)=u\otimes\alpha^{f+g}\left(\ddot{U}\right)-u\llcorner^{f+g}\alpha^{f+g}\left(\ddot{U}\right)\\
&\qquad\left(\begin{array}{c}\text{because (68) (applied to } \ddot{U} \text{ instead of } U\text{) yields } \alpha^f\left(\alpha^g\left(\ddot{U}\right)\right)=\alpha^{f+g}\left(\ddot{U}\right)\\ \text{(in fact, we \textit{are} allowed to apply (68) to } \ddot{U} \text{ instead of } U\text{, because}\\ \ddot{U}\in L^{\otimes(p-1)} \text{ and because (68) has already been proven for } p-1\\ \text{instead of } p\text{)}\end{array}\right)\\
&=\alpha^{f+g}\left(\underbrace{u\otimes\ddot{U}}_{=U}\right)\qquad\left(\begin{array}{c}\text{since (46) (applied to } f+g \text{ and } \ddot{U} \text{ instead of } f \text{ and } U\text{)}\\ \text{yields } \alpha^{f+g}\left(u\otimes\ddot{U}\right)=u\otimes\alpha^{f+g}\left(\ddot{U}\right)-u\llcorner^{f+g}\alpha^{f+g}\left(\ddot{U}\right)\end{array}\right)\\
&=\alpha^{f+g}\left(U\right).
\end{aligned}$$

Hence, the equality (68) is proven for every left-induced tensor $U\in L^{\otimes p}$. As we already said above, this entails that (68) must also hold for every tensor $U\in L^{\otimes p}$, and thus the induction step is complete. Hence, (68) is proven for every $p\in\mathbb{N}$ and every $U\in L^{\otimes p}$.

Consequently, the equation (68) holds for every $U\in\otimes L$ (since every $U\in\otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p\in\mathbb{N}$, and since the equation (68) is $k$-linear). In other words, Theorem 32 **(a)** is proven.

**(b)** We will first show that for every $p\in\mathbb{N}$, we have

$$\alpha^{\mathbf{0}}\left(U\right)=U \tag{69}$$

for every $U\in L^{\otimes p}$.

In fact, we will prove (69) by induction over $p$:

The induction base case $p=0$ is obvious (because in this case, $U\in L^{\otimes p}=L^{\otimes 0}=k$ and thus $\alpha^{\mathbf{0}}\left(U\right)=U$, and thus the equation (69) holds in this case).

So let us now come to the induction step: Let $p \in \mathbb{N}_+$. We must prove (69), assuming that (69) has already been proven for $p-1$ instead of $p$.

We want to prove (69). In other words, we want to prove that (69) holds for every $U \in L^{\otimes p}$. In order to do this, it is enough to prove that (69) holds for every left-induced $U \in L^{\otimes p}$ (by the left tensor induction tactic, because the equation (69) is linear in $U$). So, let us prove this. Let $U \in L^{\otimes p}$ be a left-induced tensor. Then, we can write $U$ in the form $U = u \otimes \ddot{U}$ for some $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ (because $U$ is left-induced). Thus,

$$\begin{aligned}
&\alpha^{\mathbf{0}}(U) \\
&= \alpha^{\mathbf{0}}\left(u \otimes \ddot{U}\right) = u \otimes \alpha^{\mathbf{0}}\left(\ddot{U}\right) - u \llcorner^{\mathbf{0}} \alpha^{\mathbf{0}}\left(\ddot{U}\right) \\
&\qquad \left(\begin{array}{c} \text{since (46) (applied to } \mathbf{0} \text{ and } \ddot{U} \text{ instead of } f \text{ and } U) \\ \text{yields } \alpha^{g}\left(u \otimes \ddot{U}\right) = u \otimes \alpha^{g}\left(\ddot{U}\right) - u \llcorner^{g} \alpha^{g}\left(\ddot{U}\right) \end{array}\right) \\
&= u \otimes \ddot{U} - u \llcorner^{\mathbf{0}} \ddot{U} \\
&\qquad \left(\begin{array}{c} \text{because (69) (applied to } \ddot{U} \text{ instead of } U) \text{ yields } \alpha^{\mathbf{0}}\left(\ddot{U}\right) = \ddot{U} \\ \text{(in fact, we \textit{are} allowed to apply (69) to } \ddot{U} \text{ instead of } U\text{, because} \\ \ddot{U} \in L^{\otimes (p-1)} \text{ and because (69) has already been proven for } p-1 \\ \text{instead of } p) \end{array}\right) \\
&= u \otimes \ddot{U} - 0 \qquad \left(\begin{array}{c} \text{since (16) (applied to } u, \ddot{U}, 0 \text{ and } 0 \text{ instead of } w, U, f \text{ and } g) \\ \text{yields } u \llcorner^{\mathbf{0}} \ddot{U} + u \llcorner^{\mathbf{0}} \ddot{U} = u \llcorner^{\mathbf{0}+\mathbf{0}} \ddot{U} = u \llcorner^{\mathbf{0}} \ddot{U}\text{, so that } u \llcorner^{\mathbf{0}} \ddot{U} = 0 \end{array}\right) \\
&= u \otimes \ddot{U} = U.
\end{aligned}$$

Hence, the equality (69) is proven for every left-induced tensor $U \in L^{\otimes p}$. As we already said above, this entails that (69) must also hold for every tensor $U \in L^{\otimes p}$, and thus the induction step is complete. Hence, (69) is proven for every $p \in \mathbb{N}$ and every $U \in L^{\otimes p}$.

Consequently, the equation (69) holds for every $U \in \otimes L$ (since every $U \in \otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and since the equation (69) is $k$-linear). In other words, Theorem 32 **(b)** is proven.

**(c)** Applying Theorem 32 **(a)** to $-f$ instead of $g$, we obtain $\alpha^{f} \circ \alpha^{-f} = \alpha^{f+(-f)} = \alpha^{\mathbf{0}} = \mathrm{id}$ (by Theorem 32 **(b)**). On the other hand, applying Theorem 32 **(a)** to $-f$ and $f$ instead of $f$ and $g$, we obtain $\alpha^{-f} \circ \alpha^{f} = \alpha^{(-f)+f} = \alpha^{\mathbf{0}} = \mathrm{id}$ (by Theorem 32 **(b)**). Combining $\alpha^{f} \circ \alpha^{-f} = \mathrm{id}$ with $\alpha^{-f} \circ \alpha^{f} = \mathrm{id}$, we see that the map $\alpha^{f}$ is invertible, and its inverse is $\alpha^{-f}$. This proves Theorem 32 **(c)**.

Now, we can prove Theorem 31:

*Proof of Theorem 31.* Theorem 30 yields $\alpha^{g}(I_f) \subseteq I_{f+g}$. On the other hand, Theorem 30, applied to $f+g$ and $-g$ instead of $f$ and $g$, yields $\alpha^{-g}(I_{f+g}) \subseteq I_{(f+g)+(-g)} = I_f$. Applying $\alpha^{g}$ to both sides of this relation, we obtain $\alpha^{g}(\alpha^{-g}(I_{f+g})) \subseteq \alpha^{g}(I_f)$. But Theorem 31 **(c)** (applied to $g$ instead of $f$) yields that the map $\alpha^{g}$ is invertible, and its inverse is $\alpha^{-g}$; thus, $\alpha^{g} \circ \alpha^{-g} = \mathrm{id}$. So we have $\alpha^{g}(\alpha^{-g}(I_{f+g})) = \underbrace{\left(\alpha^{g} \circ \alpha^{-g}\right)}_{=\mathrm{id}}(I_{f+g}) = I_{f+g}$, and therefore $\alpha^{g}(\alpha^{-g}(I_{f+g})) \subseteq \alpha^{g}(I_f)$ becomes $I_{f+g} \subseteq \alpha^{g}(I_f)$. When combined with $\alpha^{g}(I_f) \subseteq I_{f+g}$, this leads to $\alpha^{g}(I_f) = I_{f+g}$, and therefore Theorem 31 is proven.

Now we are able to give a *proof of Theorem 1.* First a definition:

**Definition 12.** Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Theorem 25 yields $\alpha^g(I_f) \subseteq I_{f+g}$. Therefore, the $k$-module homomorphism $\alpha^g : \otimes L \to \otimes L$ induces a $k$-module homomorphism $(\otimes L) \diagup I_f \to (\otimes L) \diagup I_{f+g}$. We denote this homomorphism by $\overline{\alpha}_f^g$. Since $(\otimes L) \diagup I_f = \mathrm{Cl}(L, f)$ and $(\otimes L) \diagup I_{f+g} = \mathrm{Cl}(L, f+g)$, this homomorphism $\overline{\alpha}_f^g$ is a homomorphism $\overline{\alpha}_f^g : \mathrm{Cl}(L, f) \to \mathrm{Cl}(L, f+g)$.

Now consider two bilinear forms $f$ and $g$. According to Theorem 32 **(c)** (applied to $g$ instead of $f$), the map $\alpha^g$ is invertible, and its inverse is $\alpha^{-g}$. Thus, $\alpha^g \circ \alpha^{-g} = \mathrm{id}$ and $\alpha^{-g} \circ \alpha^g = \mathrm{id}$. Now, the homomorphism $\overline{\alpha}_{f+g}^{-g}$ is a homomorphism from $\mathrm{Cl}(L, f+g)$ to $\mathrm{Cl}\left(L, \underbrace{(f+g) + (-g)}_{=f}\right) = \mathrm{Cl}(L, f)$, while the homomorphism $\overline{\alpha}_f^g$ is a homomorphism from $\mathrm{Cl}(L, f)$ to $\mathrm{Cl}(L, f+g)$. Therefore, $\alpha^g \circ \alpha^{-g} = \mathrm{id}$ becomes $\overline{\alpha}_f^g \circ \overline{\alpha}_{f+g}^{-g} = \mathrm{id}$, and for the same reason $\alpha^{-g} \circ \alpha^g = \mathrm{id}$ becomes $\overline{\alpha}_{f+g}^{-g} \circ \overline{\alpha}_f^g = \mathrm{id}$. Thus, the homomorphism $\overline{\alpha}_f^g$ has an inverse - namely, the homomorphism $\overline{\alpha}_{f+g}^{-g}$. Therefore, $\overline{\alpha}_f^g$ and $\overline{\alpha}_{f+g}^{-g}$ are isomorphisms. We have thus proven the following fact:

**Theorem 33.** Let $f : L \times L \to k$ and $g : L \times L \to k$ be two bilinear forms. Then, the $k$-modules $\mathrm{Cl}(L, f)$ and $\mathrm{Cl}(L, f+g)$ are isomorphic, and the maps $\overline{\alpha}_f^g : \mathrm{Cl}(L, f) \to \mathrm{Cl}(L, f+g)$ and $\overline{\alpha}_{f+g}^{-g} : \mathrm{Cl}(L, f+g) \to \mathrm{Cl}(L, f)$ are two mutually inverse isomorphisms between them.

In particular, this generalizes the following fact:

**Theorem 34.** Let $f : L \times L \to k$ be a bilinear form. Then, the $k$-modules $\mathrm{Cl}(L, f)$ and $\wedge L$ are isomorphic, and the maps $\overline{\alpha}_f^{-f} : \mathrm{Cl}(L, f) \to \wedge L$ and $\overline{\alpha}_{\mathbf{0}}^f : \wedge L \to \mathrm{Cl}(L, f)$ are two mutually inverse isomorphisms between them.

In fact, Theorem 34 follows from applying Theorem 33 to $g = -f$ (because if we set $g = -f$, then $f + g = \mathbf{0}$, $\mathrm{Cl}\left(L, \underbrace{f+g}_{=f+(-f)=\mathbf{0}}\right) = \mathrm{Cl}(L, \mathbf{0}) = \wedge L$ and $-g = -(-f) = f$).

Clearly, Theorem 34 immediately yields Theorem 1. Theorem 3 is a simple consequence, as well:

*Proof of Theorem 3.* Let $\mathrm{proj}_f : \otimes L \to \mathrm{Cl}(L, f)$ denote the canonical projection of the $k$-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \mathrm{Cl}(L, f)$, and let $\mathrm{proj}_{\mathbf{0}} : \otimes L \to \wedge L$ denote the canonical projection of the $k$-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_{\mathbf{0}} = \wedge L$. The isomorphism $\overline{\alpha}_{\mathbf{0}}^f$ is the map from $\wedge L$ to $\mathrm{Cl}(L, f)$ induced by the homomorphism $\alpha^f : \otimes L \to \otimes L$; in other words, $\overline{\alpha}_{\mathbf{0}}^f \circ \mathrm{proj}_{\mathbf{0}} = \mathrm{proj}_f \circ \alpha^f$.

We identify any vector $v \in L$ with the 1-tensor $\mathrm{inj}(v)$ in the tensor algebra $\otimes L$. In other words, we write $\mathrm{inj}(v) = v$ for every vector $v \in L$. This makes $L$ a subspace of $\otimes L$. It is known that the map $\mathrm{proj}_{\mathbf{0}} |_L : L \to \wedge L$ (this is the canonical map from the $k$-module $L$ to the exterior algebra of $L$) is injective. Also, the map $\overline{\alpha}_{\mathbf{0}}^f : \wedge L \to \mathrm{Cl}(L, f)$ is injective (since it is an isomorphism, according to Theorem 34). Thus, the composition

$\overline{\alpha}_{\mathbf{0}}^{f} \circ (\operatorname{proj}_{\mathbf{0}} \mid_L)$ is also an injective map (because the two maps $\operatorname{proj}_{\mathbf{0}} \mid_L$ and $\overline{\alpha}_{\mathbf{0}}^{f}$ are injective). But every $v \in L$ satisfies

$$\left(\overline{\alpha}_{\mathbf{0}}^{f} \circ (\operatorname{proj}_{\mathbf{0}} \mid_L)\right)(v) = \overline{\alpha}_{\mathbf{0}}^{f}\left(\underbrace{\operatorname{proj}_{\mathbf{0}} \mid_L (v)}_{=\operatorname{proj}_{\mathbf{0}}(v)}\right) = \overline{\alpha}_{\mathbf{0}}^{f}(\operatorname{proj}_{\mathbf{0}}(v)) = \underbrace{\left(\overline{\alpha}_{\mathbf{0}}^{f} \circ \operatorname{proj}_{\mathbf{0}}\right)}_{=\operatorname{proj}_f \circ \alpha^f}(v)$$
$$= \left(\operatorname{proj}_f \circ \alpha^f\right)(v) = \operatorname{proj}_f\left(\underbrace{\alpha^f(v)}_{=v \text{ (by (43))}}\right) = \operatorname{proj}_f(v) = \varphi_f(v)$$

(since we identify any vector $v \in L$ with its image $\operatorname{inj}(v)$ in the tensor algebra $\otimes L$, and thus $\operatorname{proj}_f(v) = \operatorname{proj}_f(\operatorname{inj}(v)) = \underbrace{\left(\operatorname{proj}_f \circ \operatorname{inj}\right)}_{=\varphi_f}(v) = \varphi_f(v)$). In other words, $\overline{\alpha}_{\mathbf{0}}^{f} \circ (\operatorname{proj}_{\mathbf{0}} \mid_L) = \varphi_f$. Since the map $\overline{\alpha}_{\mathbf{0}}^{f} \circ (\operatorname{proj}_{\mathbf{0}} \mid_L)$ is injective, this yields that the map $\varphi_f$ is injective, and Theorem 3 is proven.

## 8. A simple formula for $\alpha^f$ on special pure tensors

We record the following simple formula to compute $\alpha^f$ of certain kinds of pure tensors. It doesn't help us to compute $\alpha^f$ generally, but can be used to compute $\overline{\alpha}_{\mathbf{0}}^{f}$ and $\overline{\alpha}_{f}^{-f}$.

**Theorem 35.** Let $p \in \mathbb{N}$. Let $u_1, u_2, ..., u_p$ be $p$ elements of $L$ such that

$$(f(u_i, u_j) = 0 \text{ for every } i \in \{1, 2, ..., p\} \text{ and } j \in \{1, 2, ..., p\} \text{ satisfying } i < j). \tag{70}$$

Then,

$$\alpha^f(u_1 \otimes u_2 \otimes ... \otimes u_p) = u_1 \otimes u_2 \otimes ... \otimes u_p.$$

Before we prove this, a lemma about the right interior product:

**Theorem 36.** Let $p \in \mathbb{N}$. Let $u_1, u_2, ..., u_p$ be $p$ elements of $L$, and let $v$ be another element of $L$ such that

$$(f(u_i, v) = 0 \text{ for every } i \in \{1, 2, ..., p\}). \tag{71}$$

Then,

$$(u_1 \otimes u_2 \otimes ... \otimes u_p) \overset{f}{\llcorner} v = 0.$$

While this theorem is trivial using Theorem 5 **(c)**, let us give here a proof avoiding Theorem 5 **(c)** here:

*Proof of Theorem 36.* Let us prove that every $i \in \{0, 1, ..., p\}$ satisfies

$$(u_1 \otimes u_2 \otimes ... \otimes u_i) \overset{f}{\llcorner} v = 0. \tag{72}$$

In fact, we are going to prove (72) by induction over $i$.

The induction base case $i = 0$ is obvious[44].

Now, let us come to the induction step: Let $j \in \{1, 2, ..., p\}$. We assume that (72) holds for $i = j - 1$, and we want to prove that (72) also holds for $i = j$.

Since (72) holds for $i = j-1$, we have $(u_1 \otimes u_2 \otimes ... \otimes u_{j-1}) \overset{f}{\lrcorner} v = 0$. In other words, $U \overset{f}{\lrcorner} v = 0$, where we denote the tensor $u_1 \otimes u_2 \otimes ... \otimes u_{j-1}$ by $U$. Now, $u_1 \otimes u_2 \otimes ... \otimes u_j = \underbrace{(u_1 \otimes u_2 \otimes ... \otimes u_{j-1})}_{=U} \otimes u_j = U \otimes u_j$ and thus

$$\begin{aligned}
&(u_1 \otimes u_2 \otimes ... \otimes u_j) \overset{f}{\lrcorner} v \\
&= (U \otimes u_j) \overset{f}{\lrcorner} v \\
&= \underbrace{f(u_j, v)}_{=0 \text{ (by (71))}} U - \underbrace{\left(U \overset{f}{\lrcorner} v\right)}_{=0} \otimes u_j \qquad \text{(by (22) (applied to } u_j \text{ instead of } u\text{))} \\
&= 0U - 0 \otimes u_j = 0.
\end{aligned}$$

In other words, (72) holds for $i = j$. This completes the induction step. Hence, (72) is proved for every $i \in \{0, 1, ..., p\}$. In particular, we can therefore apply (72) to $i = p$, and thus obtain $(u_1 \otimes u_2 \otimes ... \otimes u_p) \overset{f}{\lrcorner} v = 0$. Thus, Theorem 36 is proven.

*Proof of Theorem 35.* Let us prove that every $i \in \{0, 1, ..., p\}$ satisfies

$$\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_i) = u_1 \otimes u_2 \otimes ... \otimes u_i. \tag{73}$$

In fact, we are going to prove (73) by induction over $i$.

The induction base case $i = 0$ is obvious[45].

Now, let us come to the induction step: Let $j \in \{1, 2, ..., p\}$. We assume that (73) holds for $i = j - 1$, and we want to prove that (73) also holds for $i = j$.

Since (73) holds for $i = j-1$, we have $\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_{j-1}) = u_1 \otimes u_2 \otimes ... \otimes u_{j-1}$. In other words, $\alpha^f (U) = U$, where we denote the tensor $u_1 \otimes u_2 \otimes ... \otimes u_{j-1}$ by $U$. Now, $u_1 \otimes u_2 \otimes ... \otimes u_j = \underbrace{(u_1 \otimes u_2 \otimes ... \otimes u_{j-1})}_{=U} \otimes u_j = U \otimes u_j$ and thus

$$\begin{aligned}
&\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_j) \\
&= \alpha^f (U \otimes u_j) \\
&= \alpha^f (U) \otimes u_j - \alpha^f (U) \overset{f}{\lrcorner} u_j \qquad \text{(by (52) (applied to } u_j \text{ instead of } u\text{))} \\
&= U \otimes u_j - U \overset{f}{\lrcorner} u_j \qquad \left(\text{since } \alpha^f (U) = U\right).
\end{aligned} \tag{74}$$

But on the other hand, we have

$$(f (u_i, u_j) = 0 \text{ for every } i \in \{1, 2, ..., j-1\})$$

[44] In fact, in the case $i = 0$, we have $\underbrace{(u_1 \otimes u_2 \otimes ... \otimes u_i)}_{=(\text{empty tensor product})=1} \overset{f}{\lrcorner} v = 1 \overset{f}{\lrcorner} v = 0$ (by Theorem 11 **(a)**, applied to $\lambda = 1$), and thus (72) holds for $i = 0$.

[45] In fact, in the case $i = 0$, we have $\alpha^f \underbrace{(u_1 \otimes u_2 \otimes ... \otimes u_i)}_{=(\text{empty tensor product})=1} = \alpha^f (1) = 1 =$ (empty tensor product) $= (u_1 \otimes u_2 \otimes ... \otimes u_i)$, and thus (73) holds for $i = 0$.

(by (70), since $i \in \{1, 2, ..., j-1\}$ yields $i < j$), and therefore Theorem 36 (applied to $j-1$ and $u_j$ instead of $p$ and $v$) yields $(u_1 \otimes u_2 \otimes ... \otimes u_{j-1}) \overset{f}{\lrcorner} u_j = 0$. Since $U = u_1 \otimes u_2 \otimes ... \otimes u_{j-1}$, this becomes $U \overset{f}{\lrcorner} u_j = 0$. Hence, (74) becomes

$$\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_j) = U \otimes u_j - \underbrace{U \overset{f}{\lrcorner} u_j}_{=0} = U \otimes u_j = u_1 \otimes u_2 \otimes ... \otimes u_j.$$

In other words, (73) holds for $i = j$. This completes the induction step. Hence, (73) is proved for every $i \in \{0, 1, ..., p\}$. In particular, we can therefore apply (73) to $i = p$, and thus obtain $\alpha^f (u_1 \otimes u_2 \otimes ... \otimes u_p) = u_1 \otimes u_2 \otimes ... \otimes u_p$. Thus, Theorem 35 is proven.

## 9. The Clifford basis theorem

We now come closer to proving Theorem 2 - the Clifford basis theorem. First let us make Theorem 20 a bit more precise:

**Theorem 37.** Let $U \in L^{\otimes p}$ for some $p \in \mathbb{N}$. Then,

$$\alpha^f (U) - U \in \bigoplus_{\substack{i \in \{0,1,...,p-2\}; \\ i \equiv p \bmod 2}} L^{\otimes i}. \tag{75}$$

Our proof of this fact will be more or less a copy of the proof of Theorem 20, with the only difference that we take a closer look at the highest-degree terms:

*Proof of Theorem 37.* We are going to prove (75) by induction over $p$.

The induction base case $p = 0$ is obvious[46].

So let us pass on to the induction step: Let $p \in \mathbb{N}_+$. Assume that we have proven (75) for $p-1$ instead of $p$; that is, we have shown that

$$\alpha^f (U) - U \in \bigoplus_{\substack{i \in \{0,1,...,(p-1)-2\}; \\ i \equiv p-1 \bmod 2}} L^{\otimes i} \qquad \text{for every } U \in L^{\otimes (p-1)}. \tag{76}$$

Now we have to establish (75) for our value of $p$ as well, i. e. we have to prove that

$$\alpha^f (U) - U \in \bigoplus_{\substack{i \in \{0,1,...,p-2\}; \\ i \equiv p \bmod 2}} L^{\otimes i} \qquad \text{for every } U \in L^{\otimes p}. \tag{77}$$

So let us prove (77). First, we notice that every $u \in L$ and $\ddot{U} \in L^{\otimes (p-1)}$ satisfy

$$\begin{aligned}
&\alpha^f \left(u \otimes \ddot{U}\right) \\
&= u \otimes \underbrace{\alpha^f \left(\ddot{U}\right)}_{=\left(\alpha^f\left(\ddot{U}\right) - \ddot{U}\right) + \ddot{U}} - u \overset{f}{\llcorner} \underbrace{\alpha^f \left(\ddot{U}\right)}_{=\left(\alpha^f\left(\ddot{U}\right) - \ddot{U}\right) + \ddot{U}} \qquad \left(\text{by (46), applied to } \ddot{U} \text{ instead of } U\right) \\
&= u \otimes \left(\left(\alpha^f \left(\ddot{U}\right) - \ddot{U}\right) + \ddot{U}\right) - u \overset{f}{\llcorner} \left(\left(\alpha^f \left(\ddot{U}\right) - \ddot{U}\right) + \ddot{U}\right) \\
&= \left(u \otimes \left(\alpha^f \left(\ddot{U}\right) - \ddot{U}\right) + u \otimes \ddot{U}\right) - \left(u \overset{f}{\llcorner} \left(\alpha^f \left(\ddot{U}\right) - \ddot{U}\right) + u \overset{f}{\llcorner} \ddot{U}\right) \\
&= \left(u \otimes \left(\alpha^f \left(\ddot{U}\right) - \ddot{U}\right) - u \overset{f}{\llcorner} \left(\alpha^f \left(\ddot{U}\right) - \ddot{U}\right) - u \overset{f}{\llcorner} \ddot{U}\right) + u \otimes \ddot{U}
\end{aligned}$$

[46] In fact, in this case, $U \in L^{\otimes p} = L^{\otimes 0} = k$, and thus $\alpha^f (U) = U$, which yields $\alpha^f (U) - U = 0 \in \bigoplus_{\substack{i \in \{0,1,...,p-2\}; \\ i \equiv p \bmod 2}} L^{\otimes i}$.

and therefore[47]

$$
\begin{aligned}
&\alpha^f\left(u\otimes\ddot{U}\right)-u\otimes\ddot{U}\\
&= u\otimes\underbrace{\left(\alpha^f\left(\ddot{U}\right)-\ddot{U}\right)}_{\substack{\in\bigoplus_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes i}\\ \text{(by (76), applied to }\ddot{U}\\ \text{instead of }U)}} - u\overset{f}{\llcorner}\underbrace{\left(\alpha^f\left(\ddot{U}\right)-\ddot{U}\right)}_{\substack{\in\bigoplus_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes i}\\ \text{(by (76), applied to }\ddot{U}\\ \text{instead of }U)}} - \underbrace{u\overset{f}{\llcorner}\ddot{U}}_{\substack{\in L^{\otimes(p-2)}\\ \text{(since }\ddot{U}\in L^{\otimes(p-1)},\\ \text{and since }u\overset{f}{\llcorner}P\in L^{\otimes(p-2)}\\ \text{for every }P\in L^{\otimes(p-1)})}}\\
&\in u\otimes\bigoplus_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes i}+u\overset{f}{\llcorner}\bigoplus_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes i}-L^{\otimes(p-2)}\\
&= u\otimes\sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes i}+u\overset{f}{\llcorner}\sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes i}-L^{\otimes(p-2)}\\
&\qquad\text{(since direct sums are sums)}\\
&= \sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} \underbrace{u\otimes L^{\otimes i}}_{\subseteq L^{\otimes(i+1)}\text{ (since }u\in L)} + \sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} \underbrace{u\overset{f}{\llcorner}L^{\otimes i}}_{\substack{\subseteq L^{\otimes(i-1)}\text{ (since}\\ u\overset{f}{\llcorner}P\in L^{\otimes(i-1)}\text{ for every}\\ P\in L^{\otimes i})}} - L^{\otimes(p-2)}\\
&\qquad\left(\text{since both the tensor product and the operation }\overset{f}{\llcorner}\text{ are bilinear}\right)\\
&\subseteq \sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes(i+1)}+\sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i\equiv p-1\bmod 2}} L^{\otimes(i-1)}-L^{\otimes(p-2)}\\
&= \sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i+1\equiv p\bmod 2}} L^{\otimes(i+1)}+\sum_{\substack{i\in\{0,1,\ldots,(p-1)-2\};\\ i-1\equiv p\bmod 2}} L^{\otimes(i-1)}-L^{\otimes(p-2)}\\
&\qquad\left(\begin{array}{c}\text{since }i\equiv p-1\bmod 2\text{ is equivalent to }i+1\equiv p\bmod 2\text{, and}\\ \text{since }i\equiv p-1\bmod 2\text{ is equivalent to }i-1\equiv p\bmod 2\\ \text{(the latter is because }i\equiv p-1\bmod 2\text{ is equivalent}\\ \text{to }i+1\equiv p\bmod 2\text{, and because }i+1\equiv i-1\bmod 2)\end{array}\right)\\
&= \sum_{\substack{i\in\{0,1,\ldots,p-3\};\\ i+1\equiv p\bmod 2}} L^{\otimes(i+1)}+\sum_{\substack{i\in\{0,1,\ldots,p-3\};\\ i-1\equiv p\bmod 2}} L^{\otimes(i-1)}-L^{\otimes(p-2)}\qquad\text{(since }(p-1)-2=p-3)
\end{aligned}
$$

[47] In the following, whenever $P$ is a $k$-submodule of $\otimes L$, we denote by $u\overset{f}{\llcorner}P$ the $k$-submodule $\left\{u\overset{f}{\llcorner}p\ \mid\ p\in P\right\}$ of $\otimes L$. This is indeed a submodule, since $u\overset{f}{\llcorner}p$ is $k$-linear in $p$ (because of the bilinearity of $\overset{f}{\llcorner}$).

$$= \underbrace{\sum_{\substack{i\in\{1,2,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i}}_{\subseteq \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i}} + \underbrace{\sum_{\substack{i\in\{-1,0,\ldots,p-4\};\\ i\equiv p \bmod 2}} L^{\otimes i}}_{\substack{\subseteq L^{\otimes(-1)} + \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \\ = \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \text{ (since } L^{\otimes(-1)}=0)}} - \underbrace{L^{\otimes(p-2)}}_{\substack{\subseteq \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \\ \text{(because } p-2\in\{0,1,\ldots,p-2\} \\ \text{and } p-2\equiv p \bmod 2)}}$$

$$\left(\begin{array}{c}\text{here, we substituted } i \text{ for } i+1 \text{ in the first sum, and we} \\ \text{substituted } i \text{ for } i-1 \text{ in the second sum}\end{array}\right)$$

$$\subseteq \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} + \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} + \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i}$$

$$\subseteq \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \qquad \left(\text{since } \sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \text{ is a } k\text{-module}\right)$$

$$= \bigoplus_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \tag{78}$$

(since the sum $\sum_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i}$ is a direct sum). Consequently, (77) is true for each tensor $U \in L^{\otimes p}$ (because every tensor $U \in L^{\otimes p}$ can be written in the form $U = \sum_{i\in I} \alpha_i u_i \otimes \ddot{U}_i$ for a finite set $I$, a family $(\alpha_i)_{i\in I}$ of scalars in $k$, a family $(u_i)_{i\in I}$ of vectors in $L$ and a family $\left(\ddot{U}_i\right)_{i\in I}$ of tensors in $L^{\otimes(p-1)}$ [48], and thus it satisfies

$$\begin{aligned}
&\alpha^f(U) - U \\
&= \underbrace{\alpha^f\left(\sum_{i\in I} \alpha_i u_i \otimes \ddot{U}_i\right)}_{\substack{=\sum_{i\in I} \alpha_i \alpha^f\left(u_i\otimes \ddot{U}_i\right) \\ \text{(since the map } \alpha^f \text{ is } k\text{-linear)}}} - \sum_{i\in I} \alpha_i u_i \otimes \ddot{U}_i = \sum_{i\in I} \alpha_i \alpha^f\left(u_i \otimes \ddot{U}_i\right) - \sum_{i\in I} \alpha_i u_i \otimes \ddot{U}_i \\
&= \sum_{i\in I} \alpha_i \underbrace{\left(\alpha^f\left(u_i \otimes \ddot{U}_i\right) - u_i \otimes \ddot{U}_i\right)}_{\substack{\in \bigoplus_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \\ \text{(due to (78), applied} \\ \text{to } u_i \text{ and } \ddot{U}_i \text{ instead of } u \text{ and } \ddot{U})}} \in \sum_{i\in I} \alpha_i \bigoplus_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \\
&\subseteq \bigoplus_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \qquad \left(\text{since } \bigoplus_{\substack{i\in\{0,1,\ldots,p-2\};\\ i\equiv p \bmod 2}} L^{\otimes i} \text{ is a } k\text{-module}\right)
\end{aligned}$$

). Thus, the induction is complete, and (75) is proven. Thus, Theorem 37 is proven.

[48] This is because $U \in L^{\otimes p} = L \otimes L^{\otimes(p-1)}$.

Before we can finally prove Theorem 2, some preliminary work is needed. First, we define some notations:

In Definition 4, we defined the ascending product $\overrightarrow{\prod\limits_{i\in I}} a_i$ of a finite family $(a_i)_{i\in I}$ of elements of a ring $A$. However, this notation can turn out to be ambiguous if $a_i$ are elements of two different rings with different multiplications. For instance, we consider every vector in $L$ both as an element of the tensor algebra $\otimes L$ and as an element of the exterior algebra $\wedge L$. So, if $a_i$ is a vector in $L$ for each $i \in I$, then what exactly does the product $\overrightarrow{\prod\limits_{i\in I}} a_i$ mean: does it mean the ascending product of the vectors $a_i$ seen as elements of $\otimes L$, or does it mean the ascending product of the vectors $a_i$ seen as elements of $\wedge L$ ? In order to avoid this ambiguity, we shall rename the ascending product $\overrightarrow{\prod\limits_{i\in I}} a_i$ in the algebra $\otimes L$ as $\overrightarrow{\bigotimes\limits_{i\in I}} a_i$, and we shall rename the ascending product $\overrightarrow{\prod\limits_{i\in I}} a_i$ in the algebra $\wedge L$ as $\overrightarrow{\bigwedge\limits_{i\in I}} a_i$. In other words, we declare the following notation:

**Definition 13. (a)** Let $I$ be a finite subset of $\mathbb{Z}$. Let $a_i$ be an element of $\otimes L$ for each $i \in I$. Then, we will denote by $\overrightarrow{\bigotimes\limits_{i\in I}} a_i$ the ascending product of the elements $a_i$ of $\otimes L$ (this product is built using the multiplication in the ring $\otimes L$, i. e., using the tensor product multiplication).

**(b)** Let $I$ be a finite subset of $\mathbb{Z}$. Let $a_i$ be an element of $\wedge L$ for each $i \in I$. Then, we will denote by $\overrightarrow{\bigwedge\limits_{i\in I}} a_i$ the ascending product of the elements $a_i$ of $\wedge L$ (this product is built using the multiplication in the ring $\wedge L$, i. e., using the exterior product multiplication).

One more definition:

**Definition 14.** If $N$ is a set, and $\ell \in \mathbb{N}$, then we denote by $\mathcal{P}_\ell(N)$ the set of all $\ell$-element subsets of the set $N$.

It is known that if $(e_1, e_2, ..., e_n)$ is a basis of the $k$-module $L$, then

$$\left(\overrightarrow{\bigwedge\limits_{i\in I}} e_i\right)_{I\in\mathcal{P}_\ell(\{1,2,...,n\})} \quad \text{is a basis of the } k\text{-module } \wedge^\ell L. \tag{79}$$

(This is one of the many classical properties of the exterior algebra.)

*Proof of Theorem 2.* We want to prove that the family $\left(\overrightarrow{\prod\limits_{i\in I}} \varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is a basis of the $k$-module $\mathrm{Cl}(L,f)$. In order to prove this, we must show that this family is linearly independent, and that it generates the $k$-module $\mathrm{Cl}(L,f)$. Let us first prove that it is linearly independent:

*Proof of the linear independence of the family* $\left(\overrightarrow{\prod\limits_{i\in I}} \varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$*:*

Let $(\lambda_I)_{I\in\mathcal{P}(\{1,2,...,n\})}$ be a family of elements of $k$ such that

$$\sum_{I\in\mathcal{P}(\{1,2,...,n\})} \lambda_I \cdot \overrightarrow{\prod\limits_{i\in I}} \varphi_f(e_i) = 0. \tag{80}$$

We are now going to prove that this family $(\lambda_I)_{I\in\mathcal{P}(\{1,2,...,n\})}$ satisfies $\lambda_I = 0$ for all $I \in \mathcal{P}(\{1,2,...,n\})$. In order to prove this, we will show that for every $j \in \{0,1,...,n+1\}$, we have

$$\left(\lambda_I = 0 \text{ for all } I \in \mathcal{P}(\{1,2,...,n\}) \text{ satisfying } |I| > n-j\right). \tag{81}$$

In fact, we will prove (81) by induction over $j$:

The induction base case $j=0$ is trivial[49].

Now we begin with the induction step: Let $\mathbf{i} \in \{0,1,...,n\}$. Assume that (81) has already been proven for $j = \mathbf{i}$. Now, we must prove (81) for $j = \mathbf{i}+1$.

We have assumed that (81) has already been proven for $j = \mathbf{i}$. In other words, we have assumed that

$$\left(\lambda_I = 0 \text{ for all } I \in \mathcal{P}(\{1,2,...,n\}) \text{ satisfying } |I| > n-\mathbf{i}\right) \tag{82}$$

has already been proven.

We consider the map $\overline{\alpha}_f^{-f} : \mathrm{Cl}(L,f) \to \wedge L$. We have defined this map $\overline{\alpha}_f^{-f}$ as the map from $(\otimes L) \diagup I_f = \mathrm{Cl}(L,f)$ to $(\otimes L) \diagup I_{\mathbf{0}} = \mathrm{Cl}(L,0) = \wedge L$ canonically induced by the map $\alpha^{-f} : \otimes L \to \otimes L$. In other words, if we denote by $\mathrm{proj}_f : \otimes L \to \mathrm{Cl}(L,f)$ the canonical projection of the $k$-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \mathrm{Cl}(L,f)$, and if we denote by $\mathrm{proj}_{\mathbf{0}} : \otimes L \to \wedge L$ the canonical projection of the $k$-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_{\mathbf{0}} = \wedge L$, then we have $\overline{\alpha}_f^{-f} \circ \mathrm{proj}_f = \mathrm{proj}_{\mathbf{0}} \circ \alpha^{-f}$. Note that

$$\wedge^{\ell} L = \mathrm{proj}_{\mathbf{0}}\left(L^{\otimes \ell}\right) \qquad \text{for every } \ell \in \mathbb{N}. \tag{83}$$

Clearly, for every subset $I$ of $\{1,2,...,n\}$, we have

$$\overrightarrow{\prod_{i\in I}} \underbrace{\varphi_f(e_i)}_{=\mathrm{proj}_f(e_i)} = \overrightarrow{\prod_{i\in I}} \mathrm{proj}_f(e_i) = \mathrm{proj}_f\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right)$$

(because $\overrightarrow{\prod\limits_{i\in I}}$ denotes an ascending product in the algebra $\mathrm{Cl}(L,f)$, whereas $\overrightarrow{\bigotimes\limits_{i\in I}}$ denotes an ascending product in the algebra $\otimes L$, and because taking products commutes with $\mathrm{proj}_f$ since $\mathrm{proj}_f$ is a $k$-algebra homomorphism). Therefore,

$$\begin{aligned}
\overline{\alpha}_f^{-f}\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right) &= \overline{\alpha}_f^{-f}\left(\mathrm{proj}_f\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right)\right) = \underbrace{\left(\overline{\alpha}_f^{-f} \circ \mathrm{proj}_f\right)}_{=\mathrm{proj}_{\mathbf{0}} \circ \alpha^{-f}}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) \\
&= \left(\mathrm{proj}_{\mathbf{0}} \circ \alpha^{-f}\right)\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) \\
&= \mathrm{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right)\right).
\end{aligned} \tag{84}$$

But (80) yields

$$\overline{\alpha}_f^{-f}\left(\sum_{I\in\mathcal{P}(\{1,2,...,n\})} \lambda_I \cdot \overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right) = \overline{\alpha}_f^{-f}(0) = 0.$$

[49]In fact, (81) is vacuously true in the case $j=0$, since there is no $I \in \mathcal{P}(\{1,2,...,n\})$ satisfying $|I| > n-j$ in the case $j=0$.

This, in view of

$$\begin{aligned}
&\overline{\alpha}_f^{-f}\left(\sum_{I\in\mathcal{P}(\{1,2,\ldots,n\})}\lambda_I\cdot\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)\\
&=\sum_{I\in\mathcal{P}(\{1,2,\ldots,n\})}\lambda_I\cdot\overline{\alpha}_f^{-f}\left(\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)\qquad\qquad\left(\text{since }\overline{\alpha}_f^{-f}\text{ is }k\text{-linear}\right)\\
&=\sum_{I\in\mathcal{P}(\{1,2,\ldots,n\})}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)\qquad\qquad(\text{by }(84))\,,
\end{aligned}$$

becomes

$$\sum_{I\in\mathcal{P}(\{1,2,\ldots,n\})}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)=0. \tag{85}$$

But we have

$$\begin{aligned}
&\sum_{I\in\mathcal{P}(\{1,2,\ldots,n\})}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)\\
&=\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|\leq n-\mathbf{i}}}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)+\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|>n-\mathbf{i}}}\underbrace{\lambda_I}_{\substack{=0\text{ (by (82),}\\\text{since }|I|>n-\mathbf{i})}}\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)\\
&=\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|\leq n-\mathbf{i}}}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)+\underbrace{\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|>n-\mathbf{i}}}0\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)}_{=0}\\
&=\sum_{\substack{I\in\mathcal{P}(\{1,2,\ldots,n\});\\|I|\leq n-\mathbf{i}}}\lambda_I\cdot\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right).
\end{aligned}\tag{86}$$

Now, every $I \in \mathcal{P}(\{1,2,...,n\})$ satisfies $\overrightarrow{\bigotimes_{i\in I}} e_i \in L^{\otimes |I|}$ and therefore

$$\begin{aligned}
&\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) - \overrightarrow{\bigotimes_{i\in I}} e_i \\
&\in \bigoplus_{\substack{i\in\{0,1,...,|I|-2\};\\ i\equiv |I| \bmod 2}} L^{\otimes i} \\
&\qquad \left(\text{due to Theorem 37, applied to } \overrightarrow{\bigotimes_{i\in I}} e_i,\ |I| \text{ and } -f \text{ instead of } U,\ p \text{ and } f\right) \\
&\subseteq \bigoplus_{i\in\{0,1,...,|I|-2\}} L^{\otimes i} \subseteq \bigoplus_{i\in\{0,1,...,|I|-1\}} L^{\otimes i} = \bigoplus_{i=0}^{|I|-1} L^{\otimes i} \\
&= \bigoplus_{\ell=0}^{|I|-1} L^{\otimes \ell} \qquad \text{(here, we renamed } i \text{ into } \ell \text{ in the direct sum)} \\
&= \sum_{\ell=0}^{|I|-1} L^{\otimes \ell} \qquad \text{(since direct sums are sums)}
\end{aligned}$$

and therefore

$$\begin{aligned}
\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right)\right) - \operatorname{proj}_{\mathbf{0}}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) &= \operatorname{proj}_{\mathbf{0}}\left(\underbrace{\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}} e_i\right) - \overrightarrow{\bigotimes_{i\in I}} e_i}_{\in \sum_{\ell=0}^{|I|-1} L^{\otimes \ell}}\right) \\
&\in \operatorname{proj}_{\mathbf{0}}\left(\sum_{\ell=0}^{|I|-1} L^{\otimes \ell}\right) = \sum_{\ell=0}^{|I|-1} \underbrace{\operatorname{proj}_{\mathbf{0}}\left(L^{\otimes \ell}\right)}_{=\wedge^\ell L \text{ (by (83))}} \\
&= \sum_{\ell=0}^{|I|-1} \wedge^\ell L.
\end{aligned}$$

Since

$$\begin{aligned}
&\operatorname{proj}_{\mathbf{0}}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\\
&=\overrightarrow{\bigwedge_{i\in I}}\operatorname{proj}_{\mathbf{0}}\left(e_i\right)\\
&\qquad\left(\begin{array}{c}\text{since } \overrightarrow{\bigotimes_{i\in I}} \text{ denotes the ascending product in the algebra } \otimes L\text{, while } \overrightarrow{\bigwedge_{i\in I}}\\ \text{denotes the ascending product in the algebra } \wedge L\text{, and since the map}\\ \operatorname{proj}_{\mathbf{0}} \text{ commutes with taking products (because } \operatorname{proj}_{\mathbf{0}} \text{ is a } k\text{-algebra}\\ \text{homomorphism)}\end{array}\right)\\
&=\overrightarrow{\bigwedge_{i\in I}}e_i \qquad \left(\begin{array}{c}\text{since } \operatorname{proj}_{\mathbf{0}}\left(e_i\right)=e_i\text{, because we identify any}\\ \text{vector } v\in L \text{ with its images in } \otimes L \text{ and in } \wedge L\end{array}\right),
\end{aligned}$$

this rewrites as

$$\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)-\overrightarrow{\bigwedge_{i\in I}}e_i\in\sum_{\ell=0}^{|I|-1}\wedge^{\ell}L. \tag{87}$$

Thus, if $|I|\leq n-\mathbf{i}$, then

$$\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)-\overrightarrow{\bigwedge_{i\in I}}e_i\in\sum_{\ell=0}^{|I|-1}\wedge^{\ell}L\subseteq\sum_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L$$

(since $|I|\leq n-\mathbf{i}$ yields $|I|-1\leq n-\mathbf{i}-1$). In other words, if $|I|\leq n-\mathbf{i}$, then

$$\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)\equiv\overrightarrow{\bigwedge_{i\in I}}e_i \operatorname{mod}\sum_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L.$$

Hence, (86) becomes

$$\begin{aligned}
&\sum_{I\in\mathcal{P}(\{1,2,...,n\})} \lambda_I \cdot \operatorname{proj}_{\mathbf{0}} \left( \alpha^{-f} \left( \overrightarrow{\bigotimes_{i\in I}} e_i \right) \right) \\
&= \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|\leq n-\mathbf{i}}} \lambda_I \cdot \underbrace{\operatorname{proj}_{\mathbf{0}} \left( \alpha^{-f} \left( \overrightarrow{\bigotimes_{i\in I}} e_i \right) \right)}_{\equiv \overrightarrow{\bigwedge_{i\in I}} e_i \bmod \sum_{\ell=0}^{n-\mathbf{i}-1} \wedge^\ell L} \equiv \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|\leq n-\mathbf{i}}} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \\
&= \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|< n-\mathbf{i}}} \lambda_I \cdot \underbrace{\overrightarrow{\bigwedge_{i\in I}} e_i}_{\substack{\equiv 0 \bmod \sum_{\ell=0}^{n-\mathbf{i}-1} \wedge^\ell L \\ \text{(because for every } I\in\mathcal{P}(\{1,2,...,n\}) \\ \text{satisfying } |I|<n-\mathbf{i}\text{, we have } 0\leq|I|\leq n-\mathbf{i}-1 \\ \text{and thus } \overrightarrow{\bigwedge_{i\in I}} e_i \in \wedge^{|I|} L \subseteq \sum_{\ell=0}^{n-\mathbf{i}-1} \wedge^\ell L)}} + \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|= n-\mathbf{i}}} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \\
&\equiv \underbrace{\sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|< n-\mathbf{i}}} \lambda_I \cdot 0}_{=0} + \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|= n-\mathbf{i}}} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i = \sum_{\substack{I\in\mathcal{P}(\{1,2,...,n\});\\ |I|= n-\mathbf{i}}} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \\
&= \sum_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \bmod \sum_{\ell=0}^{n-\mathbf{i}-1} \wedge^\ell L
\end{aligned}$$

(since $\{I \in \mathcal{P}(\{1,2,...,n\}) \mid |I| = n-\mathbf{i}\} = \mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})$). Combined with (85), this yields

$$\sum_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \equiv 0 \bmod \sum_{\ell=0}^{n-\mathbf{i}-1} \wedge^\ell L,$$

which is equivalent to

$$\sum_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \in \sum_{\ell=0}^{n-\mathbf{i}-1} \wedge^\ell L. \tag{88}$$

But on the other hand,

$$\sum_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \in \wedge^{n-\mathbf{i}} L$$

(since every $I \in \mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})$ satisfies $\overrightarrow{\bigwedge_{i\in I}} e_i \in \wedge^{n-\mathbf{i}} L$ [50], and since $\wedge^{n-\mathbf{i}} L$ is a $k$-module). Combining this with (88), we obtain

$$\sum_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})} \lambda_I \cdot \overrightarrow{\bigwedge_{i\in I}} e_i \in \left( \sum_{\ell\in\{0,1,...,n-\mathbf{i}-1\}} \wedge^\ell L \right) \cap \left( \wedge^{n-\mathbf{i}} L \right).$$

[50] because $I \in \mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})$ yields $|I| = n - \mathbf{i}$

But since $\left(\sum\limits_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L\right)\cap\left(\wedge^{n-\mathbf{i}}L\right)=0$ [51], this becomes

$$\sum_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})}\lambda_I\cdot\overrightarrow{\bigwedge_{i\in I}}e_i\in 0,$$

so that

$$\sum_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})}\lambda_I\cdot\overrightarrow{\bigwedge_{i\in I}}e_i=0.$$

But since $\left(\overrightarrow{\bigwedge_{i\in I}}e_i\right)_{I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})}$ is a basis of the $k$-module $\wedge^{\ell}L$ (this follows from (79), applied to $\ell=n-\mathbf{i}$), this yields that

$$\left(\lambda_I=0\text{ for every }I\in\mathcal{P}_{n-\mathbf{i}}\left(\{1,2,...,n\}\right)\right).\tag{89}$$

Consequently,

$$\left(\lambda_I=0\text{ for all }I\in\mathcal{P}\left(\{1,2,...,n\}\right)\text{ satisfying }|I|>n-(\mathbf{i}+1)\right)$$

[52]. In other words, (81) is true for $j=\mathbf{i}+1$. This completes the induction step, and thus we have proven (81) for every $j\in\{0,1,...,n+1\}$.

Now, we conclude that $\lambda_I=0$ for all $I\in\mathcal{P}(\{1,2,...,n\})$ (because every $I\in\mathcal{P}(\{1,2,...,n\})$ satisfies $|I|>-1=n-(n+1)$, and thus (81) (applied to $j=n+1$) yields $\lambda_I=0$). Hence, we have shown that if some family $(\lambda_I)_{I\in\mathcal{P}(\{1,2,...,n\})}$ of elements of $k$ satisfies (80), then $\lambda_I=0$ for all $I\in\mathcal{P}(\{1,2,...,n\})$. In other words, the family $\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is linearly independent.

*Proof that the family* $\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ *generates the whole $k$-module* $\operatorname{Cl}(L,f)$*:*

Next we are going to prove that the family $\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ generates the $k$-module $\operatorname{Cl}(L,f)$. In order to verify this, we denote by $S$ the sub-$k$-module of $\operatorname{Cl}(L,f)$

[51] In fact, it is known that $\wedge L=\bigoplus\limits_{\ell=0}^{n}\wedge^{\ell}L$, so that

$$\wedge L=\bigoplus_{\ell=0}^{n}\wedge^{\ell}L=\left(\bigoplus_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L\right)\oplus\left(\wedge^{n-\mathbf{i}}L\right)\oplus\left(\bigoplus_{\ell=n-\mathbf{i}+1}^{n}\wedge^{\ell}L\right)$$

and therefore $\left(\bigoplus\limits_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L\right)\cap\left(\wedge^{n-\mathbf{i}}L\right)=0$, which rewrites as $\left(\sum\limits_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L\right)\cap\left(\wedge^{n-\mathbf{i}}L\right)=0$ (because $\bigoplus\limits_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L=\sum\limits_{\ell=0}^{n-\mathbf{i}-1}\wedge^{\ell}L$, since every direct sum is a sum).

[52] *Proof.* We have $|I|\in\mathbb{Z}$ and $|I|>n-(\mathbf{i}+1)$. Therefore, only the following two cases are possible:

*Case 1:* We have $|I|=n-(\mathbf{i}+1)+1$.

*Case 2:* We have $|I|>n-(\mathbf{i}+1)+1$.

In Case 1, we have $|I|=n-(\mathbf{i}+1)+1=n-\mathbf{i}$, so that $I\in\mathcal{P}_{n-\mathbf{i}}(\{1,2,...,n\})$ and therefore $\lambda_I=0$ according to (89).

In Case 2, we have $|I|>n-(\mathbf{i}+1)+1=n-\mathbf{i}$ and therefore $\lambda_I=0$ according to (82).

Thus, in both cases 1 and 2, we have $\lambda_I=0$. Hence, $\lambda_I=0$ is proven.

generated by the family $\left(\overrightarrow{\prod\limits_{i\in I}}\varphi_f\left(e_i\right)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$. In other words, we denote by $S$ the sub-$k$-module

$$\left\langle \overrightarrow{\prod\limits_{i\in I}}\varphi_f\left(e_i\right) \mid I\in\mathcal{P}\left(\{1,2,...,n\}\right)\right\rangle$$

of $\operatorname{Cl}(L,f)$. Now we are going to prove that for any $j\in\{0,1,...,n+1\}$, we have

$$\sum_{\ell=0}^{j-1}\wedge^\ell L\subseteq\overline{\alpha}_f^{-f}\left(S\right). \tag{90}$$

In fact, we are going to prove (90) by induction over $j$:

The base case - the case $j=0$ - is obvious[53].

So let us now come to the induction step: Let $\rho\in\{0,1,...,n\}$. Assume that (90) has already been proven for $j=i$. We must now show that (90) also holds for $j=\rho+1$.

We have assumed that (90) has already been proven for $j=\rho$. In other words, we have assumed that

$$\sum_{\ell=0}^{\rho-1}\wedge^\ell L\subseteq\overline{\alpha}_f^{-f}\left(S\right). \tag{91}$$

Now, we will show that $\wedge^\rho L\subseteq\overline{\alpha}_f^{-f}\left(S\right)$.

In fact, according to (79) (applied to $\rho$ instead of $\ell$), the family $\left(\overrightarrow{\bigwedge\limits_{i\in I}}e_i\right)_{I\in\mathcal{P}_\rho(\{1,2,...,n\})}$ is a basis of the $k$-module $\wedge^\rho L$. In particular, this yields that this family generates the $k$-module $\wedge^\rho L$; in other words, $\wedge^\rho L=\left\langle\overrightarrow{\bigwedge\limits_{i\in I}}e_i \mid I\in\mathcal{P}_\rho\left(\{1,2,...,n\}\right)\right\rangle$. Hence, in order to prove that $\wedge^\rho L\subseteq\overline{\alpha}_f^{-f}\left(S\right)$, it will be enough to show that $\overrightarrow{\bigwedge\limits_{i\in I}}e_i\in\overline{\alpha}_f^{-f}\left(S\right)$ for each $I\in\mathcal{P}_\rho\left(\{1,2,...,n\}\right)$ (because $\overline{\alpha}_f^{-f}\left(S\right)$ is a $k$-module).

Now, let $I\in\mathcal{P}_\rho\left(\{1,2,...,n\}\right)$ be arbitrary. We are going to prove that $\overrightarrow{\bigwedge\limits_{i\in I}}e_i\in\overline{\alpha}_f^{-f}\left(S\right)$.

According to (87), we have

$$\begin{aligned}\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)-\overrightarrow{\bigwedge_{i\in I}}e_i&\in\sum_{\ell=0}^{|I|-1}\wedge^\ell L=\sum_{\ell=0}^{\rho-1}\wedge^\ell L\\ &\qquad\left(\text{since } I\in\mathcal{P}_\rho\left(\{1,2,...,n\}\right)\text{ yields } |I|=\rho\right)\\ &\subseteq\overline{\alpha}_f^{-f}\left(S\right)\qquad\left(\text{by (91)}\right).\end{aligned}$$

In other words,

$$\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right)\equiv\overrightarrow{\bigwedge_{i\in I}}e_i\operatorname{mod}\overline{\alpha}_f^{-f}\left(S\right).$$

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[53]In fact, in the case $j=0$, the assertion (90) is trivial (because $j=0$ yields $\sum\limits_{\ell=0}^{j-1}\wedge^\ell L=$ (empty sum) $=0$).

But since (84) yields $\operatorname{proj}_{\mathbf{0}}\left(\alpha^{-f}\left(\overrightarrow{\bigotimes_{i\in I}}e_i\right)\right) = \overline{\alpha}_f^{-f}\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right) \equiv 0 \operatorname{mod} \overline{\alpha}_f^{-f}(S)$ [54], this rewrites as $0 \equiv \overrightarrow{\bigwedge_{i\in I}}e_i \operatorname{mod} \overline{\alpha}_f^{-f}(S)$. Hence, $\overrightarrow{\bigwedge_{i\in I}}e_i \in \overline{\alpha}_f^{-f}(S)$.

So we have now proven that $\overrightarrow{\bigwedge_{i\in I}}e_i \in \overline{\alpha}_f^{-f}(S)$ for every $I \in \mathcal{P}_\rho(\{1,2,...,n\})$. Now,

$$\wedge^\rho L = \left\langle \underbrace{\overrightarrow{\bigwedge_{i\in I}}e_i}_{\in \overline{\alpha}_f^{-f}(S)} \mid I \in \mathcal{P}_\rho(\{1,2,...,n\}) \right\rangle \subseteq \overline{\alpha}_f^{-f}(S) \tag{92}$$

(since $\overline{\alpha}_f^{-f}(S)$ is a $k$-module). Now,

$$\sum_{\ell=0}^{(\rho+1)-1} \wedge^\ell L = \sum_{\ell=0}^{\rho} \wedge^\ell L = \underbrace{\left(\sum_{\ell=0}^{\rho-1} \wedge^\ell L\right)}_{\substack{\subseteq \overline{\alpha}_f^{-f}(S) \\ \text{(by (91))}}} + \underbrace{\wedge^\rho L}_{\substack{\subseteq \overline{\alpha}_f^{-f}(S) \\ \text{(by (92))}}} \subseteq \overline{\alpha}_f^{-f}(S) + \overline{\alpha}_f^{-f}(S) = \overline{\alpha}_f^{-f}(S)$$

(since $\overline{\alpha}_f^{-f}(S)$ is a $k$-module). In other words, (90) holds for $j = \rho + 1$. This completes the induction step, and thus we have proven (90) for every $j \in \{0, 1, ..., n+1\}$.

Now, applying (90) to $j = n+1$, we get

$$\sum_{\ell=0}^{(n+1)-1} \wedge^\ell L \subseteq \overline{\alpha}_f^{-f}(S).$$

Since

$$\sum_{\ell=0}^{(n+1)-1} \wedge^\ell L = \sum_{\ell=0}^{n} \wedge^\ell L = \wedge L \qquad \left(\text{because } \wedge L = \bigoplus_{\ell=0}^{n} \wedge^\ell L = \sum_{\ell=0}^{n} \wedge^\ell L\right),$$

this rewrites as $\wedge L \subseteq \overline{\alpha}_f^{-f}(S)$. Thus, $\overline{\alpha}_{\mathbf{0}}^f(\wedge L) \subseteq \overline{\alpha}_{\mathbf{0}}^f\left(\overline{\alpha}_f^{-f}(S)\right)$. But $\overline{\alpha}_{\mathbf{0}}^f(\wedge L) = \operatorname{Cl}(L,f)$ (because $\overline{\alpha}_{\mathbf{0}}^f : \wedge L \to \operatorname{Cl}(L,f)$ is an isomorphism) and $\overline{\alpha}_{\mathbf{0}}^f\left(\overline{\alpha}_f^{-f}(S)\right) = \left(\overline{\alpha}_{\mathbf{0}}^f \circ \overline{\alpha}_f^{-f}\right)(S) = S$ (since the maps $\overline{\alpha}_f^{-f}$ and $\overline{\alpha}_{\mathbf{0}}^f$ are mutually inverse). Hence, $\overline{\alpha}_{\mathbf{0}}^f(\wedge L) \subseteq \overline{\alpha}_{\mathbf{0}}^f\left(\overline{\alpha}_f^{-f}(S)\right)$ becomes $\operatorname{Cl}(L,f) \subseteq S$. Since $S \subseteq \operatorname{Cl}(L,f)$ (because $S$ is a sub-$k$-module of $\operatorname{Cl}(L,f)$), this yields $S = \operatorname{Cl}(L,f)$. But since $S$ is the sub-$k$-module of $\operatorname{Cl}(L,f)$ generated by the family $\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$, this yields that the $k$-module $\operatorname{Cl}(L,f)$ is generated by the family $\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$.

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[54] In fact, $\overrightarrow{\prod_{i\in I}}\varphi_f(e_i) \in S$ (since we defined $S$ as the sub-$k$-module of $\operatorname{Cl}(L,f)$ generated by the family $\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$) yields $\overline{\alpha}_f^{-f}\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right) \in \overline{\alpha}_f^{-f}(S)$ and thus $\overline{\alpha}_f^{-f}\left(\overrightarrow{\prod_{i\in I}}\varphi_f(e_i)\right) \equiv 0 \operatorname{mod} \overline{\alpha}_f^{-f}(S)$.

We have now proven that the family $\left(\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is linearly independent, and that the $k$-module $\mathrm{Cl}\left(L,f\right)$ is generated by this family. In other words, the family $\left(\overrightarrow{\prod_{i\in I}}\varphi_f\left(e_i\right)\right)_{I\in\mathcal{P}(\{1,2,...,n\})}$ is a basis of the $k$-module $\mathrm{Cl}\left(L,f\right)$. This proves Theorem 2.

## 10. The antisymmetrizer formula

We have constructed the Chevalley map $\overline{\alpha}_{\mathbf{0}}^f : \wedge L \to \mathrm{Cl}\left(L,f\right)$ through a canonical, inductively defined map $\alpha^f : \otimes L \to \otimes L$. This, however, is not the most common definition of the Chevalley map. The purpose of this section is to prove a different formula for $\overline{\alpha}_{\mathbf{0}}^f$ (although the word "formula" is not to be taken too seriously here, since it gives a unique value for $\overline{\alpha}_{\mathbf{0}}^f$ only if $k$ is a $\mathbb{Q}$-algebra), at least in the case when the form $f$ is symmetric:

**Theorem 38.** Let $f : L\times L\to k$ be a symmetric bilinear form. Let $p\in\mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$p!\cdot\overline{\alpha}_{\mathbf{0}}^f\left(u_1\wedge u_2\wedge...\wedge u_p\right)=\sum_{\sigma\in S_p}\left(-1\right)^\sigma\varphi^f\left(u_{\sigma(1)}\right)\varphi^f\left(u_{\sigma(2)}\right)...\varphi^f\left(u_{\sigma(p)}\right).$$

Here and in the following, we denote by $S_p$ the group of all permutations of the set $\{1,2,...,p\}$, and we denote by $\left(-1\right)^\sigma$ the sign of the permutation $\sigma$ for every $\sigma\in S_p$.

Theorem 38 is often used as a definition of the map $\overline{\alpha}_{\mathbf{0}}^f$ in the case when $k$ is a $\mathbb{Q}$-algebra (because in this case, we can divide by $p!$). However, it does not yield a unique value of $\overline{\alpha}_{\mathbf{0}}^f\left(u_1\wedge u_2\wedge...\wedge u_p\right)$ if the characteristic of $k$ is too small, and therefore I believe my definition of $\overline{\alpha}_{\mathbf{0}}^f$ (through the map $\alpha^f$ introduced in Definition 10 above) to be a better one.

Theorem 38 is an equality in the Clifford algebra $\mathrm{Cl}\left(L,f\right)$. However, it can be "lifted" into $\otimes L$:

**Theorem 39.** Let $f : L\times L\to k$ be a symmetric bilinear form. Let $p\in\mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$\alpha^f\left(\sum_{\sigma\in S_p}\left(-1\right)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p)}\right)=\sum_{\sigma\in S_p}\left(-1\right)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p)}.$$

We will prove this... you guessed right, by induction. In the induction step we will use a lemma which is interesting for its own merit:

**Theorem 40.** Let $f : L\times L\to k$ be a symmetric bilinear form. Let $p\in\mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$\alpha^f\left(\sum_{\sigma\in S_p}\left(-1\right)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p)}\right)$$
$$=\sum_{\sigma\in S_p}\left(-1\right)^\sigma\alpha^f\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p-1)}\right)\otimes u_{\sigma(p)}.$$

This, in turn, will be concluded from the following result:

**Theorem 41.** Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let $u_1, u_2, ..., u_p$ be $p$ vectors in $L$. Then,

$$\sum_{\sigma \in S_p} (-1)^\sigma \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \overset{f}{\lrcorner} u_{\sigma(p)} = 0.$$

*Proof of Theorem 41.* Let $\sigma \in S_p$. Applying Theorem 11 **(c)** to the $p-1$ vectors $u_{\sigma(1)}, u_{\sigma(2)}, ..., u_{\sigma(p-1)}$ instead of the $p$ vectors $u_1, u_2, ..., u_p$, we obtain

$$\begin{aligned}
&\left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \overset{f}{\lrcorner} v \\
&= \sum_{i=1}^{p-1} (-1)^{(p-1)-i} f\left(u_{\sigma(i)}, v\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)}
\end{aligned}$$

[55] for any vector $v \in L$. Applying this to $v = u_{\sigma(p)}$, we get

$$\begin{aligned}
&\left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \overset{f}{\lrcorner} u_{\sigma(p)} \\
&= \sum_{i=1}^{p-1} (-1)^{(p-1)-i} f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{\sigma \in S_p} (-1)^\sigma \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \overset{f}{\lrcorner} u_{\sigma(p)} \\
&= \sum_{\sigma \in S_p} (-1)^\sigma \sum_{i=1}^{p-1} (-1)^{(p-1)-i} f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)} \\
&= \sum_{i=1}^{p-1} (-1)^{(p-1)-i} \sum_{\sigma \in S_p} (-1)^\sigma f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)}.
\end{aligned} \tag{93}$$

Now, fix some $i \in \{1, 2, ..., p-1\}$. Consider the transposition $\tau \in S_p$ defined by

$$\left(\tau(j) = \begin{cases} p, \text{ if } j = i; \\ i, \text{ if } j = p; \\ j, \text{ if } j \notin \{p, i\} \end{cases} \qquad \text{for any } j \in \{1, 2, ..., p\}\right). \tag{94}$$

(This transposition $\tau$ is usually denoted $(p, i)$ or $(pi)$ in the notation of group theorists.) It is known that $(-1)^\tau = -1$ and $\tau\tau = \mathrm{id}$ [56]. Let us consider the normal subgroup $A_p = \{\eta \in S_p \mid (-1)^\eta = 1\}$ of $S_p$. Define a map $Z : A_p \to S_p \diagdown A_p$ by

$$\left(Z(\sigma) = \sigma\tau \qquad \text{for every } \sigma \in A_p\right).$$

[55] Here, the hat over the vector $u_{\sigma(i)}$ means that the vector $u_{\sigma(i)}$ is being omitted from the tensor product; in other words, $u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)}$ is just another way to write $\underbrace{u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(i-1)}}_{\substack{\text{tensor product of the} \\ \text{first } i-1 \text{ vectors } u_{\sigma(\ell)}}} \otimes \underbrace{u_{\sigma(i+1)} \otimes u_{\sigma(i+2)} \otimes ... \otimes u_{\sigma(p-1)}}_{\substack{\text{tensor product of the} \\ \text{last } (p-1)-i \text{ vectors } u_{\sigma(\ell)} \text{ with } \ell \leq p-1}}$.

[56] Here and in the following, whenever $a$ and $b$ are two elements of $S_p$, we denote by $ab$ the product of $a$ and $b$ in the group $S_p$ (in other words, the composition of the permutations $a$ and $b$).

(This map is indeed well-defined, since $\sigma\tau \in S_p \diagdown A_p$ for every $\sigma \in A_p$ [57].) Also, define a map $W : S_p \diagdown A_p \to A_p$ by

$$\left(W(\sigma) = \sigma\tau \qquad \text{for every } \sigma \in S_p \diagdown A_p\right).$$

(This map is indeed well-defined, since $\sigma\tau \in A_p$ for every $\sigma \in S_p \diagdown A_p$ [58].) The two maps $Z$ and $W$ are mutually inverse[59]. Thus, the map $Z$ is a bijection.

Clearly, (94) yields

$$\tau(p) = \begin{cases} p, \text{ if } p = i; \\ i, \text{ if } p = p; \\ j, \text{ if } p \notin \{p, i\} \end{cases} = i$$

(since $p = p$) and

$$\tau(i) = \begin{cases} p, \text{ if } i = i; \\ i, \text{ if } i = p; \\ j, \text{ if } i \notin \{p, i\} \end{cases} = p$$

(since $i = i$).

We note that every permutation $\sigma \in S_p$ satisfies

$$u_{(\sigma\tau)(1)} \otimes u_{(\sigma\tau)(2)} \otimes ... \otimes u_{(\sigma\tau)(i-1)} = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(i-1)}$$

(since for every $j \in \{1, 2, ..., i-1\}$, the equation (94) yields $\tau(j) = \begin{cases} p, \text{ if } j = i; \\ i, \text{ if } j = p; \\ j, \text{ if } j \notin \{p, i\} \end{cases} = j$ (since $j \in \{1, 2, ..., i-1\}$ yields $j \notin \{p, i\}$) and thus $(\sigma\tau)(j) = \sigma\left(\underbrace{\tau(j)}_{=j}\right) = \sigma(j)$)

and

$$u_{(\sigma\tau)(i+1)} \otimes u_{(\sigma\tau)(i+2)} \otimes ... \otimes u_{(\sigma\tau)(p-1)} = u_{\sigma(i+1)} \otimes u_{\sigma(i+2)} \otimes ... \otimes u_{\sigma(p-1)}$$

(since for every $j \in \{i+1, i+2, ..., p-1\}$, the equation (94) yields $\tau(j) = \begin{cases} p, \text{ if } j = i; \\ i, \text{ if } j = p; \\ j, \text{ if } j \notin \{p, i\} \end{cases} = j$ (since $j \in \{i+1, i+2, ..., p-1\}$ yields $j \notin \{p, i\}$) and

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[57] In fact, $\sigma \in A_p = \{\eta \in S_p \mid (-1)^\eta = 1\}$ yields $(-1)^\sigma = 1$ and thus $(-1)^{\sigma\tau} = \underbrace{(-1)^\sigma}_{=1} \underbrace{(-1)^\tau}_{=-1} = -1 \neq 1$, so that $\sigma\tau \notin \{\eta \in S_p \mid (-1)^\eta = 1\} = A_p$ and therefore $\sigma\tau \in S_p \diagdown A_p$.

[58] In fact, $\sigma \in S_p \diagdown A_p$ yields $\sigma \notin A_p = \{\eta \in S_p \mid (-1)^\eta = 1\}$ and thus $(-1)^\sigma \neq 1$, so that $(-1)^\sigma = -1$ (since the term $(-1)^\sigma$ can only take the values 1 and $-1$) and thus $(-1)^{\sigma\tau} = \underbrace{(-1)^\sigma}_{=-1} \underbrace{(-1)^\tau}_{=-1} = 1$, so that $\sigma\tau \in \{\eta \in S_p \mid (-1)^\eta = 1\} = A_p$.

[59] In fact, $Z \circ W = \mathrm{id}$ (since every $\sigma \in A_p$ satisfies $(Z \circ W)(\sigma) = Z(W(\sigma)) = \tau \underbrace{W(\sigma)}_{=\tau\sigma} = \underbrace{\tau\tau}_{=\mathrm{id}} \sigma = \sigma$) and $W \circ Z = \mathrm{id}$ (since every $\sigma \in S_p \diagdown A_p$ satisfies $(W \circ Z)(\sigma) = W(Z(\sigma)) = \tau \underbrace{Z(\sigma)}_{=\tau\sigma} = \underbrace{\tau\tau}_{=\mathrm{id}} \sigma = \sigma$).

thus $(\sigma\tau)(j) = \sigma\left(\underbrace{\tau(j)}_{=j}\right) = \sigma(j)$), and therefore

$$\begin{aligned}
& u_{(\sigma\tau)(1)} \otimes u_{(\sigma\tau)(2)} \otimes ... \otimes \widehat{u_{(\sigma\tau)(i)}} \otimes ... \otimes u_{(\sigma\tau)(p-1)} \\
&= \underbrace{\left(u_{(\sigma\tau)(1)} \otimes u_{(\sigma\tau)(2)} \otimes ... \otimes u_{(\sigma\tau)(i-1)}\right)}_{=u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes ...\otimes u_{\sigma(i-1)}} \otimes \underbrace{\left(u_{(\sigma\tau)(i+1)} \otimes u_{(\sigma\tau)(i+2)} \otimes ... \otimes u_{(\sigma\tau)(p-1)}\right)}_{=u_{\sigma(i+1)}\otimes u_{\sigma(i+2)}\otimes ...\otimes u_{\sigma(p-1)}} \\
&= \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(i-1)}\right) \otimes \left(u_{\sigma(i+1)} \otimes u_{\sigma(i+2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \\
&= u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)}.
\end{aligned} \tag{95}$$

Also, every permutation $\sigma$ satisfies

$$f\left(u_{(\sigma\tau)(i)}, u_{(\sigma\tau)(p)}\right) = f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \tag{96}$$

(since $(\sigma\tau)(i) = \sigma\left(\underbrace{\tau(i)}_{=p}\right) = \sigma(p)$ and $(\sigma\tau)(p) = \sigma\left(\underbrace{\tau(p)}_{=i}\right) = \sigma(i)$ yield $f\left(u_{(\sigma\tau)(i)}, u_{(\sigma\tau)(p)}\right) = f\left(u_{\sigma(p)}, u_{\sigma(i)}\right) = f\left(u_{\sigma(i)}, u_{\sigma(p)}\right)$ (since $f$ is symmetric)) and

$$(-1)^{\sigma\tau} = (-1)^{\sigma} \cdot \underbrace{(-1)^{\tau}}_{=-1} = -(-1)^{\sigma}. \tag{97}$$

Now,

$$\begin{aligned}
& \sum_{\sigma\in S_p} (-1)^{\sigma} f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)} \\
&= \sum_{\sigma\in A_p} (-1)^{\sigma} f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)} \\
&\qquad + \sum_{\sigma\in S_p\setminus A_p} (-1)^{\sigma} f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)} \\
&\qquad \left(\text{since the set } S_p \text{ is the union of the two disjoint sets } A_p \text{ and } S_p\setminus A_p\right) \\
&= 0,
\end{aligned}$$

since

$$\begin{aligned}
& \sum_{\sigma\in S_p\setminus A_p} (-1)^{\sigma} f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)} \\
&= \sum_{\sigma\in A_p} (-1)^{Z(\sigma)} f\left(u_{(Z(\sigma))(i)}, u_{(Z(\sigma))(p)}\right) \cdot u_{(Z(\sigma))(1)} \otimes u_{(Z(\sigma))(2)} \otimes ... \otimes \widehat{u_{(Z(\sigma))(i)}} \otimes ... \otimes u_{(Z(\sigma))(p-1)} \\
&\qquad \left(\begin{array}{c}\text{here, we substituted } Z(\sigma) \text{ for } \sigma \text{ in the sum,} \\ \text{because the map } Z : A_p \to S_p\setminus A_p \text{ is a bijection}\end{array}\right) \\
&= \sum_{\sigma\in A_p} \underbrace{(-1)^{\sigma\tau}}_{\substack{=-(-1)^{\sigma} \\ (\text{by (97)})}} \underbrace{f\left(u_{(\sigma\tau)(i)}, u_{(\sigma\tau)(p)}\right)}_{\substack{=f\left(u_{\sigma(i)},u_{\sigma(p)}\right) \\ (\text{by (96)})}} \cdot \underbrace{u_{(\sigma\tau)(1)} \otimes u_{(\sigma\tau)(2)} \otimes ... \otimes \widehat{u_{(\sigma\tau)(i)}} \otimes ... \otimes u_{(\sigma\tau)(p-1)}}_{\substack{=u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes ...\otimes\widehat{u_{\sigma(i)}}\otimes ...\otimes u_{\sigma(p-1)} \\ (\text{by (95)})}} \\
&\qquad \left(\text{since } Z(\sigma) = \sigma\tau\right) \\
&= -\sum_{\sigma\in A_p} (-1)^{\sigma} f\left(u_{\sigma(i)}, u_{\sigma(p)}\right) \cdot u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes \widehat{u_{\sigma(i)}} \otimes ... \otimes u_{\sigma(p-1)}.
\end{aligned}$$

Hence, (93) becomes

$$\sum_{\sigma\in S_p}(-1)^\sigma\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p-1)}\right)\overset{f}{\lrcorner}u_{\sigma(p)}$$
$$=\sum_{i=1}^{p-1}(-1)^{(p-1)-i}\underbrace{\sum_{\sigma\in S_p}(-1)^\sigma f\left(u_{\sigma(i)},u_{\sigma(p)}\right)\cdot u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes\widehat{u_{\sigma(i)}}\otimes...\otimes u_{\sigma(p-1)}}_{=0}$$
$$=\sum_{i=1}^{p-1}(-1)^{(p-1)-i}0=0.$$

This proves Theorem 41.

Theorem 41 has a "left" analogue:

**Theorem 42.** Let $f:L\times L\to k$ be a symmetric bilinear form. Let $p\in\mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$\sum_{\sigma\in S_p}(-1)^\sigma u_{\sigma(1)}\overset{f}{\llcorner}\left(u_{\sigma(2)}\otimes u_{\sigma(3)}\otimes...\otimes u_{\sigma(p)}\right)=0.$$

*Proof of Theorem 42.* Let $\zeta\in S_p$ be the permutation defined by

$$\left(\zeta(j)=p+1-j\qquad\text{for every } j\in\{1,2,...,p\}\right).$$

Then, $\zeta\zeta=\mathrm{id}$ (since any $j\in\{1,2,...,p\}$ satisfies

$$(\zeta\zeta)(j)=\zeta\left(\underbrace{\zeta(j)}_{=p+1-j}\right)=\zeta(p+1-j)=p+1-(p+1-j)\qquad\text{(by the definition of }\zeta)$$
$$=j$$

). Define a map $U:S_p\to S_p$ by

$$\left(U(\sigma)=\sigma\zeta\qquad\text{for every }\sigma\in S_p\right).$$

Then, $U^2=\mathrm{id}$ (since every $\sigma\in S_p$ satisfies

$$U^2(\sigma)=U\left(\underbrace{U(\sigma)}_{=\sigma\zeta}\right)=U(\sigma\zeta)=\sigma\underbrace{\zeta\zeta}_{=\mathrm{id}}\qquad\text{(by the definition of }U)$$
$$=\sigma$$

), and thus the map $U$ is a bijection.

Now, Theorem 41 yields

$$\begin{aligned}
0 &= \sum_{\sigma\in S_p} (-1)^{\sigma} \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \overset{f}{\lrcorner} u_{\sigma(p)} \\
&= \sum_{\sigma\in S_p} (-1)^{U(\sigma)} \left(u_{(U(\sigma))(1)} \otimes u_{(U(\sigma))(2)} \otimes ... \otimes u_{(U(\sigma))(p-1)}\right) \overset{f}{\lrcorner} u_{(U(\sigma))(p)} \\
&\qquad \text{(here, we substituted } U(\sigma) \text{ for } \sigma \text{ in the sum, since the map } U \text{ is a bijection)} \\
&= \sum_{\sigma\in S_p} (-1)^{\sigma\zeta} \left(u_{(\sigma\zeta)(1)} \otimes u_{(\sigma\zeta)(2)} \otimes ... \otimes u_{(\sigma\zeta)(p-1)}\right) \overset{f}{\lrcorner} u_{(\sigma\zeta)(p)} \\
&\qquad \left(\text{since } U(\sigma) = \sigma\zeta \text{ for every } \sigma \in S_p\right) \\
&= \sum_{\sigma\in S_p} (-1)^{\sigma\zeta} \left(u_{\sigma(p+1-1)} \otimes u_{\sigma(p+1-2)} \otimes ... \otimes u_{\sigma(p+1-(p-1))}\right) \overset{f}{\lrcorner} u_{\sigma(p+1-p)} \\
&\qquad \left(\text{since every } j \in \{1,2,...,p\} \text{ satisfies } (\sigma\zeta)(j) = \sigma\left(\underbrace{\zeta(j)}_{=p+1-j}\right) = \sigma(p+1-j)\right) \\
&= \sum_{\sigma\in S_p} \underbrace{(-1)^{\sigma\zeta}}_{=(-1)^{\sigma}(-1)^{\zeta}} \left(u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}\right) \overset{f}{\lrcorner} u_{\sigma(1)} \\
&= (-1)^{\zeta} \sum_{\sigma\in S_p} (-1)^{\sigma} \left(u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}\right) \overset{f}{\lrcorner} u_{\sigma(1)}.
\end{aligned}$$

Dividing this equality by $(-1)^{\zeta}$, we obtain

$$0 = \sum_{\sigma\in S_p} (-1)^{\sigma} \left(u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}\right) \overset{f}{\lrcorner} u_{\sigma(1)}. \tag{98}$$

Since the bilinear form $f$ is symmetric, it satisfies $f^t = f$ (since any two $x \in L$ and $y \in L$ satisfy $f^t(x,y) = f(y,x) = f(x,y)$ because the form $f$ is symmetric). Now, for every $\sigma \in S_p$, the identity (32) (applied to $U = u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}$ and $v = u_{\sigma(1)}$) yields

$$\begin{aligned}
t\left(\left(u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}\right) \overset{f}{\lrcorner} u_{\sigma(1)}\right) &= u_{\sigma(1)} \overset{f^t}{\llcorner} \underbrace{t\left(u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}\right)}_{=u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes ... \otimes u_{\sigma(p)}} \\
&= u_{\sigma(1)} \overset{f^t}{\llcorner} \left(u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes ... \otimes u_{\sigma(p)}\right) \\
&= u_{\sigma(1)} \overset{f}{\llcorner} \left(u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes ... \otimes u_{\sigma(p)}\right)
\end{aligned}$$

(since $f^t = f$). Now,

$$\begin{aligned}
0 = t(0) &= t\left(\sum_{\sigma\in S_p} (-1)^{\sigma} \left(u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}\right) \overset{f}{\lrcorner} u_{\sigma(1)}\right) \qquad \text{(by (98))} \\
&= \sum_{\sigma\in S_p} (-1)^{\sigma} \underbrace{t\left(\left(u_{\sigma(p)} \otimes u_{\sigma(p-1)} \otimes ... \otimes u_{\sigma(2)}\right) \overset{f}{\lrcorner} u_{\sigma(1)}\right)}_{=u_{\sigma(1)} \overset{f}{\llcorner} \left(u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes ... \otimes u_{\sigma(p)}\right)} \qquad \text{(since the map } t \text{ is linear)} \\
&= \sum_{\sigma\in S_p} (-1)^{\sigma} u_{\sigma(1)} \overset{f}{\llcorner} \left(u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes ... \otimes u_{\sigma(p)}\right).
\end{aligned}$$

This proves Theorem 42.

*Proof of Theorem 40.* For every $\sigma \in S_p$, let us denote the tensor $u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}$ by $U_\sigma$. Then, every $\sigma \in S_p$ satisfies

$$\begin{aligned}
&\alpha^f \left( \underbrace{u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}}_{=\left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \otimes u_{\sigma(p)}} \right) \\
&= \alpha^f \left( \underbrace{\left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right)}_{=U_\sigma} \otimes u_{\sigma(p)} \right) \\
&= \alpha^f \left(U_\sigma \otimes u_{\sigma(p)}\right) = \alpha^f \left(U_\sigma\right) \otimes u_{\sigma(p)} - \alpha^f \left(U_\sigma\right) \overset{f}{\lrcorner} u_{\sigma(p)} \\
&\qquad \left(\text{by (52), applied to } U_\sigma \text{ and } u_{\sigma(p)} \text{ instead of } U \text{ and } u\right) \\
&= \alpha^f \left(U_\sigma\right) \otimes u_{\sigma(p)} - \alpha^f \left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right)
\end{aligned}$$

(since (54) (applied to $f$, $U_\sigma$ and $u_{\sigma(p)}$ instead of $g$, $U$ and $u$) yields $\alpha^f \left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right) = \alpha^f \left(U_\sigma\right) \overset{f}{\lrcorner} u_{\sigma(p)}$, so that $\alpha^f \left(U_\sigma\right) \overset{f}{\lrcorner} u_{\sigma(p)} = \alpha^f \left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right)$). Thus,

$$\begin{aligned}
&\alpha^f \left( \sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)} \right) \\
&= \sum_{\sigma \in S_p} (-1)^\sigma \underbrace{\alpha^f \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right)}_{=\alpha^f(U_\sigma) \otimes u_{\sigma(p)} - \alpha^f\left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right)} \qquad \left(\text{since the map } \alpha^f \text{ is linear}\right) \\
&= \sum_{\sigma \in S_p} (-1)^\sigma \left(\alpha^f \left(U_\sigma\right) \otimes u_{\sigma(p)} - \alpha^f \left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right)\right) \\
&= \sum_{\sigma \in S_p} (-1)^\sigma \alpha^f \left(U_\sigma\right) \otimes u_{\sigma(p)} - \sum_{\sigma \in S_p} (-1)^\sigma \alpha^f \left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{\sigma \in S_p} (-1)^\sigma \alpha^f \left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right) &= \alpha^f \left( \sum_{\sigma \in S_p} (-1)^\sigma U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)} \right) \qquad \left(\text{since } \alpha^f \text{ is a linear map}\right) \\
&= \alpha^f \left( \underbrace{\sum_{\sigma \in S_p} (-1)^\sigma \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \overset{f}{\lrcorner} u_{\sigma(p)}}_{=0 \text{ (by Theorem 41)}} \right) \\
&\qquad \left(\text{because } U_\sigma = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p-1)}\right) \\
&= \alpha^f (0) = 0,
\end{aligned}$$

this simplifies to

$$\alpha^f\left(\sum_{\sigma\in S_p}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p)}\right)$$
$$=\sum_{\sigma\in S_p}(-1)^\sigma \alpha^f\left(U_\sigma\right)\otimes u_{\sigma(p)}-\underbrace{\sum_{\sigma\in S_p}(-1)^\sigma \alpha^f\left(U_\sigma \overset{f}{\lrcorner} u_{\sigma(p)}\right)}_{=0}=\sum_{\sigma\in S_p}(-1)^\sigma \alpha^f\left(U_\sigma\right)\otimes u_{\sigma(p)}$$
$$=\sum_{\sigma\in S_p}(-1)^\sigma \alpha^f\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p-1)}\right)\otimes u_{\sigma(p)}$$
$$\left(\text{because } U_\sigma=u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p-1)}\right).$$

This proves Theorem 40.

*Proof of Theorem 39.* We are going to prove Theorem 39 by induction over $p$.

The induction base case $p=0$ is trivial[60].

Now let us handle the induction step: Let $q\in\mathbb{N}_+$. Assume that Theorem 39 holds for $p=q-1$. We are now going to show that Theorem 39 holds for $p=q$ as well.

We assumed that Theorem 39 holds for $p=q-1$. In other words, we assumed that

$$\alpha^f\left(\sum_{\sigma\in S_{q-1}}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)}\right)=\sum_{\sigma\in S_{q-1}}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)} \tag{99}$$

for any $q-1$ vectors $u_1$, $u_2$, ..., $u_{q-1}$ in $L$.

Now, for every $i\in\{1,2,...,q\}$, let us denote by $S_{q,(i)}$ the set of all permutations $\sigma\in S_q$ satisfying $\sigma(q)=i$. Then, the sets $S_{q,(1)}$, $S_{q,(2)}$, ..., $S_{q,(q)}$ are pairwise disjoint[61]. Besides, the set $S_q$ is the union of these sets $S_{q,(1)}$, $S_{q,(2)}$, ..., $S_{q,(q)}$ (since for every $\sigma\in S_q$, there exists one and only one $i\in\{1,2,...,q\}$ such that $\sigma(q)=i$, and thus this $i$ satisfies $\sigma\in S_{q,(i)}$).

[60] In fact, in the case $p=0$, the sum $\sum\limits_{\sigma\in S_p}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(p)}$ equals 1 (since it consists of one summand only, and this summand is 1). Hence, in the case $p=0$, Theorem 39 claims that $\alpha^f(1)=1$, which is trivial.

[61] In fact, if two elements $i$ and $j$ of $\{1,2,...,q\}$ are distinct, then $S_{q,(i)}\cap S_{q,(j)}=\varnothing$ (because if the sets $S_{q,(i)}$ and $S_{q,(j)}$ had a common element $\sigma$, then this element $\sigma$ would satisfy $\sigma(q)=i$ (since $\sigma\in S_{q,(i)}$) and $\sigma(q)=j$ (since $\sigma\in S_{q,(j)}$), and thus $i=\sigma(q)=j$ in contradiction to $i\neq j$).

Now, let $u_1, u_2, ..., u_q$ be $q$ vectors in $L$. Then,

$$\sum_{\sigma\in S_q} (-1)^\sigma \underbrace{u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(q)}}_{=\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes ...\otimes u_{\sigma(q-1)}\right)\otimes u_{\sigma(q)}}$$
$$= \sum_{\sigma\in S_q} (-1)^\sigma \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(q-1)}\right) \otimes u_{\sigma(q)}$$
$$= \sum_{i=1}^{q} \sum_{\sigma\in S_{q,(i)}} (-1)^\sigma \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(q-1)}\right) \otimes \underbrace{u_{\sigma(q)}}_{\substack{=u_i \text{ (since } \sigma\in S_{q,(i)} \\ \text{yields } \sigma(q)=i)}}$$
$$\left(\text{since the set } S_q \text{ is the union of the pairwise disjoint sets } S_{q,(1)},\ S_{q,(2)},\ ...,\ S_{q,(q)}\right)$$
$$= \sum_{i=1}^{q} \sum_{\sigma\in S_{q,(i)}} (-1)^\sigma \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(q-1)}\right) \otimes u_i$$
$$= \sum_{i=1}^{q} \left(\sum_{\sigma\in S_{q,(i)}} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(q-1)}\right) \otimes u_i. \tag{100}$$

Now, for every $i \in \{1, 2, ..., q\}$, let $\kappa_i$ denote the $(q-i+1)$-cycle $(i, i+1, ..., q) \in S_q$. Then,

$$\kappa_i(j) = j \qquad \text{for all } j \in \{1, 2, ..., q\} \text{ satisfying } j < i; \tag{101}$$
$$\kappa_i(j) = j+1 \qquad \text{for all } j \in \{1, 2, ..., q\} \text{ satisfying } i \le j < q; \tag{102}$$
$$\kappa_i(q) = i. \tag{103}$$

Now, we are going to prove that

$$\sum_{\sigma\in S_{q,(i)}} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(q-1)} = \sum_{\sigma\in S_{q-1}} (-1)^\sigma u_{\kappa_i(\sigma(1))} \otimes u_{\kappa_i(\sigma(2))} \otimes ... \otimes u_{\kappa_i(\sigma(q-1))} \tag{104}$$

for every $i \in \{1, 2, ..., q\}$.

In fact, fix some $i \in \{1, 2, ..., q\}$. Define a map $P : S_{q,(i)} \to S_{q,(q)}$ by

$$\left(P(\sigma) = \kappa_i^{-1}\sigma \text{ for every } \sigma \in S_{q,(i)}\right).$$

[62] (This map is well-defined, since every $\sigma \in S_{q,(i)}$ satisfies $\kappa_i^{-1}\sigma \in S_{q,(q)}$ [63].) Also, define a map $Q : S_{q,(q)} \to S_{q,(i)}$ by

$$\left(Q(\sigma) = \kappa_i\sigma \text{ for every } \sigma \in S_{q,(q)}\right).$$

[62] Here, whenever $a$ and $b$ are two elements of $S_q$, we denote by $ab$ the product of $a$ and $b$ in the group $S_q$ (in other words, the composition of the permutations $a$ and $b$).

[63] In fact, $\sigma \in S_{q,(i)}$ yields $\sigma(q) = i$ and thus $\left(\kappa_i^{-1}\sigma\right)(q) = \kappa_i^{-1}\left(\underbrace{\sigma(q)}_{=i}\right) = \kappa_i^{-1}(i) = q$ (since $\kappa_i(q) = i$), so that $\kappa_i^{-1}\sigma \in S_{q,(q)}$.

(This map is well-defined, since every $\sigma \in S_{q,(q)}$ satisfies $\kappa_i \sigma \in S_{q,(i)}$ [64].) The two maps $P$ and $Q$ are mutually inverse[65]. Thus, the map $P$ is a bijection.

Now, define a map $R : S_{q,(q)} \to S_{q-1}$ as follows: For every $\sigma \in S_{q,(q)}$, let $R(\sigma) \in S_{q-1}$ be the permutation of $\{1,2,...,q-1\}$ defined by

$$((R(\sigma))(j) = \sigma(j) \text{ for every } j \in \{1,2,...,q-1\}).$$

(This map $R(\sigma)$ is indeed well-defined[66] and is indeed a permutation of $\{1,2,...,q-1\}$ [67].)

On the other hand, let us define a map $T : S_{q-1} \to S_{q,(q)}$ as follows: For every $\sigma \in S_{q-1}$, let $T(\sigma) \in S_{q,(q)}$ be the permutation of $\{1,2,...,q\}$ defined by

$$\left( (T(\sigma))(j) = \begin{cases} \sigma(j), \text{ if } j \in \{1,2,...,q-1\}; \\ q, \text{ if } j = q \end{cases} \text{ for every } j \in \{1,2,...,q\} \right).$$

(This map $T(\sigma)$ is indeed a permutation of $\{1,2,...,q\}$ [68] and indeed lies in $S_{q,(q)}$ (since $(T(\sigma))(q) = q$).)

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[64] In fact, $\sigma \in S_{q,(q)}$ yields $\sigma(q) = q$ and thus $(\kappa_i \sigma)(q) = \kappa_i \left( \underbrace{\sigma(q)}_{=q} \right) = \kappa_i(q) = i$, so that $\kappa_i \sigma \in S_{q,(i)}$.

[65] In fact, $P \circ Q = \mathrm{id}$ (since every $\sigma \in S_{q,(q)}$ satisfies $(P \circ Q)(\sigma) = P(Q(\sigma)) = \kappa_i^{-1} \underbrace{Q(\sigma)}_{=\kappa_i \sigma} = \kappa_i^{-1} \kappa_i \sigma = \sigma$) and $Q \circ P = \mathrm{id}$ (since every $\sigma \in S_{q,(i)}$ satisfies $(Q \circ P)(\sigma) = Q(P(\sigma)) = \kappa_i \underbrace{P(\sigma)}_{=\kappa_i^{-1}\sigma} = \kappa_i \kappa_i^{-1} \sigma = \sigma$).

[66] because $\sigma(j) \in \{1,2,...,q-1\}$ for every $j \in \{1,2,...,q-1\}$ (since $j \in \{1,2,...,q-1\}$ yields $j \neq q$, and thus $\sigma(j) \neq \sigma(q)$ (since $\sigma$ is a permutation), so that $\sigma(j) \neq q$ (since $\sigma(q) = q$ because $\sigma \in S_{q,(q)}$) and therefore $\sigma(j) \in \{1,2,...,q\} \setminus \{q\} = \{1,2,...,q-1\}$)

[67] since any two distinct elements $j_1$ and $j_2$ of $\{1,2,...,q-1\}$ satisfy

$$\begin{aligned} (R(\sigma))(j_1) = \sigma(j_1) \neq \sigma(j_2) \qquad & \text{(since } j_1 \neq j_2 \text{ and since } \sigma \text{ is a permutation)} \\ \neq (R(\sigma))(j_2), & \end{aligned}$$

and therefore the map $R(\sigma)$ is injective, so that it is a permutation of $\{1,2,...,q-1\}$ (because any injective map from a finite set to itself must be a permutation of this set)

[68] because it is a surjective map from the set $\{1,2,...,q\}$ to itself (since

$$\begin{aligned} & (T(\sigma)) \left( \underbrace{\{1,2,...,q\}}_{=\{1,2,...,q-1\} \cup \{q\}} \right) \\ & = (T(\sigma))(\{1,2,...,q-1\} \cup \{q\}) = \underbrace{(T(\sigma))(\{1,2,...,q-1\})}_{\substack{=\sigma(\{1,2,...,q-1\}) \\ (\text{since } (T(\sigma))(j)=j \text{ for every } j \in \{1,2,...,q-1\})}} \cup \underbrace{(T(\sigma))(\{q\})}_{\substack{=\{(T(\sigma))(q)\}=\{q\} \\ (\text{since } (T(\sigma))(q)=q)}} \\ & = \underbrace{\sigma(\{1,2,...,q-1\})}_{\substack{=\{1,2,...,q-1\} \text{ (since } \sigma \in S_{q-1} \text{ is a} \\ \text{permutation of } \{1,2,...,q-1\})}} \cup \{q\} = \{1,2,...,q-1\} \cup \{q\} = \{1,2,...,q\} \end{aligned}$$

), and because a surjective map from a finite set to itself must always be a permutation of this set

The maps $R$ and $T$ are mutually inverse[69]. Thus, $R$ is a bijection.

Now,

$$\begin{aligned}
&\sum_{\sigma\in S_{q,(i)}} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(q-1)} \\
&= \sum_{\sigma\in S_{q,(i)}} (-1)^\sigma u_{\kappa_i((P(\sigma))(1))} \otimes u_{\kappa_i((P(\sigma))(2))} \otimes ... \otimes u_{\kappa_i((P(\sigma))(q-1))} \\
&\qquad \left(\begin{array}{c}\text{since every } j \in \{1,2,...,q-1\} \text{ satisfies } \sigma(j) = \kappa_i((P(\sigma))(j)), \\ \text{because } \kappa_i\left(\underbrace{(P(\sigma))}_{=\kappa_i^{-1}\sigma}(j)\right) = \kappa_i\left(\left(\kappa_i^{-1}\sigma\right)(j)\right) = \left(\underbrace{\kappa_i\kappa_i^{-1}}_{=\mathrm{id}}\sigma\right)(j) = \sigma(j)\end{array}\right) \\
&= \sum_{\sigma\in S_{q,(q)}} (-1)^\sigma u_{\kappa_i(\sigma(1))} \otimes u_{\kappa_i(\sigma(2))} \otimes ... \otimes u_{\kappa_i(\sigma(q-1))} \\
&\qquad \left(\text{here, we substituted } \sigma \text{ for } P(\sigma), \text{ since the map } P : S_{q,(i)} \to S_{q,(q)} \text{ is a bijection}\right) \\
&= \sum_{\sigma\in S_{q-1}} (-1)^\sigma u_{\kappa_i((R(\sigma))(1))} \otimes u_{\kappa_i((R(\sigma))(2))} \otimes ... \otimes u_{\kappa_i((R(\sigma))(q-1))} \\
&\qquad \left(\begin{array}{c}\text{since every } j \in \{1,2,...,q-1\} \text{ satisfies } \sigma(j) = (R(\sigma))(j) \text{ (because the map} \\ R(\sigma) \text{ was defined through the equation } (R(\sigma))(j) = \sigma(j)\text{)}\end{array}\right) \\
&= \sum_{\sigma\in S_{q-1}} (-1)^\sigma u_{\kappa_i(\sigma(1))} \otimes u_{\kappa_i(\sigma(2))} \otimes ... \otimes u_{\kappa_i(\sigma(q-1))} \\
&\qquad \left(\text{here, we substituted } \sigma \text{ for } R(\sigma), \text{ since the map } R : S_{q,(q)} \to S_{q-1} \text{ is a bijection}\right),
\end{aligned}$$

[69] In fact, every $\sigma \in S_{q-1}$ satisfies $(R\circ T)(\sigma) = \sigma$ (because for every $j \in \{1,2,...,q-1\}$, we have

$$\begin{aligned}
&\left(\underbrace{(R\circ T)(\sigma)}_{=R(T(\sigma))}\right)(j) = (R(T(\sigma)))(j) = (T(\sigma))(j) \qquad \text{(by the definition of } R\text{)} \\
&= \begin{cases} \sigma(j), & \text{if } j \in \{1,2,...,q-1\}; \\ q, & \text{if } j = q \end{cases} \qquad \text{(by the definition of } T\text{)} \\
&= \sigma(j) \qquad \text{(since } j \in \{1,2,...,q-1\}\text{)}
\end{aligned}$$

). Thus, $R\circ T = \mathrm{id}$. Also, every $\sigma \in S_{q,(q)}$ satisfies $(T\circ R)(\sigma) = \sigma$ (because for every $j \in \{1,2,...,q\}$, we have

$$\begin{aligned}
&\left(\underbrace{(T\circ R)(\sigma)}_{=T(R(\sigma))}\right)(j) \\
&= (T(R(\sigma)))(j) = \begin{cases} (R(\sigma))(j), & \text{if } j \in \{1,2,...,q-1\}; \\ q, & \text{if } j = q \end{cases} \qquad \text{(by the definition of } T\text{)} \\
&= \begin{cases} \sigma(j), & \text{if } j \in \{1,2,...,q-1\}; \\ q, & \text{if } j = q \end{cases} \qquad \text{(since } (R(\sigma))(j) = \sigma(j) \text{ by the definition of } R\text{)} \\
&= \begin{cases} \sigma(j), & \text{if } j \in \{1,2,...,q-1\}; \\ \sigma(j), & \text{if } j = q \end{cases} \qquad \left(\begin{array}{c}\text{since } \sigma \in S_{q,(q)} \text{ yields } \sigma(q) = q, \text{ and thus} \\ q = \sigma(q) = \sigma(j) \text{ in the case } j = q\end{array}\right) \\
&= \sigma(j)
\end{aligned}$$

). Thus, $T\circ R = \mathrm{id}$. Together with $R\circ T = \mathrm{id}$, this yields that the maps $R$ and $T$ are mutually inverse.

and thus (104) is proven. But applying (99) to the $q-1$ vectors $u_{\kappa_i(1)}$, $u_{\kappa_i(2)}$, ..., $u_{\kappa_i(q-1)}$ instead of $u_1$, $u_2$, ..., $u_{q-1}$, we obtain

$$\alpha^f\left(\sum_{\sigma\in S_{q-1}}(-1)^\sigma u_{\kappa_i(\sigma(1))}\otimes u_{\kappa_i(\sigma(2))}\otimes...\otimes u_{\kappa_i(\sigma(q-1))}\right)$$
$$=\sum_{\sigma\in S_{q-1}}(-1)^\sigma u_{\kappa_i(\sigma(1))}\otimes u_{\kappa_i(\sigma(2))}\otimes...\otimes u_{\kappa_i(\sigma(q-1))}. \tag{105}$$

But the $k$-linearity of the map $\alpha^f$ yields

$$\sum_{\sigma\in S_{q,(i)}}(-1)^\sigma \alpha^f\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)}\right)$$
$$=\alpha^f\left(\sum_{\sigma\in S_{q,(i)}}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)}\right)$$
$$=\alpha^f\left(\sum_{\sigma\in S_{q-1}}(-1)^\sigma u_{\kappa_i(\sigma(1))}\otimes u_{\kappa_i(\sigma(2))}\otimes...\otimes u_{\kappa_i(\sigma(q-1))}\right) \qquad \text{(by (104))}$$
$$=\sum_{\sigma\in S_{q-1}}(-1)^\sigma u_{\kappa_i(\sigma(1))}\otimes u_{\kappa_i(\sigma(2))}\otimes...\otimes u_{\kappa_i(\sigma(q-1))} \qquad \text{(by (105))}$$
$$=\sum_{\sigma\in S_{q,(i)}}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)} \qquad \text{(by (104) again)}. \tag{106}$$

Now, Theorem 40 (applied to $q$ instead of $p$) yields

$$\alpha^f\left(\sum_{\sigma\in S_q}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q)}\right)$$
$$=\sum_{\sigma\in S_q}(-1)^\sigma \alpha^f\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)}\right)\otimes u_{\sigma(q)}$$
$$=\sum_{i=1}^{q}\sum_{\sigma\in S_{q,(i)}}(-1)^\sigma \alpha^f\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)}\right)\otimes \underbrace{u_{\sigma(q)}}_{\substack{=u_i \text{ (since } \sigma\in S_{q,(i)} \\ \text{yields } \sigma(q)=i)}}$$
$$\left(\text{since the set } S_q \text{ is the union of the pairwise disjoint sets } S_{q,(1)},\ S_{q,(2)},\ ...,\ S_{q,(q)}\right)$$
$$=\sum_{i=1}^{q}\underbrace{\sum_{\sigma\in S_{q,(i)}}(-1)^\sigma \alpha^f\left(u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)}\right)}_{\substack{=\sum\limits_{\sigma\in S_{q,(i)}}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)} \\ \text{(by (106))}}}\otimes u_i$$
$$=\sum_{i=1}^{q}\left(\sum_{\sigma\in S_{q,(i)}}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q-1)}\right)\otimes u_i$$
$$=\sum_{\sigma\in S_q}(-1)^\sigma u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes...\otimes u_{\sigma(q)}$$

(by (100)). In other words, Theorem 39 is valid for $p = q$. This completes the induction step, and thus Theorem 39 is proven.

*Proof of Theorem 38.* Let us denote by $\operatorname{proj}_f : \otimes L \to \operatorname{Cl}(L, f)$ the canonical projection of the $k$-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_f = \operatorname{Cl}(L, f)$. Let us also denote by $\operatorname{proj}_{\mathbf{0}} : \otimes L \to \wedge L$ the canonical projection of the $k$-algebra $\otimes L$ onto its factor algebra $(\otimes L) \diagup I_{\mathbf{0}} = \wedge L$. Then, the homomorphism $\overline{\alpha}_{\mathbf{0}}^f$ was defined as the homomorphism $(\otimes L) \diagup I_{\mathbf{0}} \to (\otimes L) \diagup I_f$ induced by the homomorphism $\alpha^f : \otimes L \to \otimes L$; in other words, $\operatorname{proj}_f \circ \alpha^f = \overline{\alpha}_{\mathbf{0}}^f \circ \operatorname{proj}_{\mathbf{0}}$. Thus,

$$\begin{aligned}
&\left(\operatorname{proj}_f \circ \alpha^f\right) \left(\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) \\
&= \left(\overline{\alpha}_{\mathbf{0}}^f \circ \operatorname{proj}_{\mathbf{0}}\right) \left(\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) \\
&= \overline{\alpha}_{\mathbf{0}}^f \left(\operatorname{proj}_{\mathbf{0}} \left(\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right)\right). \qquad (107)
\end{aligned}$$

Now, the multiplication in the algebra $\otimes L$ is the tensor multiplication (denoted by $\otimes$), and the multiplication in the algebra $\wedge L$ is the exterior multiplication (denoted by $\wedge$). Since the map $\operatorname{proj}_{\mathbf{0}}$ commutes with multiplication (since $\operatorname{proj}_{\mathbf{0}}$ is an algebra homomorphism), we thus have

$$\begin{aligned}
&\operatorname{proj}_{\mathbf{0}} \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) \\
&= \operatorname{proj}_{\mathbf{0}} \left(u_{\sigma(1)}\right) \wedge \operatorname{proj}_{\mathbf{0}} \left(u_{\sigma(2)}\right) \wedge ... \wedge \operatorname{proj}_{\mathbf{0}} \left(u_{\sigma(p)}\right) \\
&= u_{\sigma(1)} \wedge u_{\sigma(2)} \wedge ... \wedge u_{\sigma(p)} \qquad \left(\begin{array}{c} \text{since we identify the element } \operatorname{proj}_{\mathbf{0}}(v) \in \wedge L \\ \text{with } v \text{ for every vector } v \in L \end{array}\right) \\
&= (-1)^\sigma u_1 \wedge u_2 \wedge ... \wedge u_p
\end{aligned}$$

(because if we interchange the factors in an exterior product of vectors, then the product becomes multiplied with $(-1)^\sigma$ where $\sigma$ is the permutation we used to interchange the factors) for every $\sigma \in S_p$. Now, since $\operatorname{proj}_{\mathbf{0}}$ is a linear map, we have

$$\begin{aligned}
&\operatorname{proj}_{\mathbf{0}} \left(\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) \\
&= \sum_{\sigma \in S_p} (-1)^\sigma \underbrace{\operatorname{proj}_{\mathbf{0}} \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right)}_{=(-1)^\sigma u_1 \wedge u_2 \wedge ... \wedge u_p} \\
&= \sum_{\sigma \in S_p} \underbrace{(-1)^\sigma (-1)^\sigma}_{\substack{=((-1)^\sigma)^2 = 1 \\ (\text{since } (-1)^\sigma \in \{1, -1\})}} u_1 \wedge u_2 \wedge ... \wedge u_p = \sum_{\sigma \in S_p} u_1 \wedge u_2 \wedge ... \wedge u_p \\
&= \underbrace{|S_p|}_{=p!} \cdot u_1 \wedge u_2 \wedge ... \wedge u_p = p! \cdot u_1 \wedge u_2 \wedge ... \wedge u_p,
\end{aligned}$$

and therefore (107) becomes

$$\begin{aligned}
&\left(\operatorname{proj}_f \circ \alpha^f\right)\left(\sum_{\sigma\in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) \\
&= \overline{\alpha}_{\mathbf{0}}^f \left(\underbrace{\operatorname{proj}_{\mathbf{0}} \left(\sum_{\sigma\in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right)}_{=p!\cdot u_1\wedge u_2\wedge...\wedge u_p}\right) \\
&= \overline{\alpha}_{\mathbf{0}}^f \left(p! \cdot u_1 \wedge u_2 \wedge ... \wedge u_p\right) = p! \cdot \overline{\alpha}_{\mathbf{0}}^f \left(u_1 \wedge u_2 \wedge ... \wedge u_p\right)
\end{aligned} \tag{108}$$

(since $\overline{\alpha}_{\mathbf{0}}^f$ is a linear map).

On the other hand, the multiplication in the algebra $\otimes L$ is the tensor multiplication (denoted by $\otimes$), and the multiplication in the algebra $\operatorname{Cl}(L, f)$ is simply written as product. Since the map $\operatorname{proj}_f$ commutes with multiplication (since $\operatorname{proj}_f$ is an algebra homomorphism), we thus have

$$\begin{aligned}
\operatorname{proj}_f \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) &= \operatorname{proj}_f \left(u_{\sigma(1)}\right) \operatorname{proj}_f \left(u_{\sigma(2)}\right) ... \operatorname{proj}_f \left(u_{\sigma(p)}\right) \\
&= \varphi^f \left(u_{\sigma(1)}\right) \varphi^f \left(u_{\sigma(2)}\right) ... \varphi^f \left(u_{\sigma(p)}\right)
\end{aligned}$$

(because $\operatorname{proj}_f (v) = \varphi^f (v)$ for every $v \in L$) for every $\sigma \in S_p$.

Comparing the equation

$$\begin{aligned}
&\left(\operatorname{proj}_f \circ \alpha^f\right)\left(\sum_{\sigma\in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) \\
&= \operatorname{proj}_f \left(\alpha^f \left(\sum_{\sigma\in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right)\right) \\
&= \operatorname{proj}_f \left(\sum_{\sigma\in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right) \qquad \text{(by Theorem 39)} \\
&= \sum_{\sigma\in S_p} (-1)^\sigma \underbrace{\operatorname{proj}_f \left(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}\right)}_{=\varphi^f\left(u_{\sigma(1)}\right)\varphi^f\left(u_{\sigma(2)}\right)...\varphi^f\left(u_{\sigma(p)}\right)} \qquad \left(\text{since } \operatorname{proj}_f \text{ is a linear map}\right) \\
&= \sum_{\sigma\in S_p} (-1)^\sigma \varphi^f \left(u_{\sigma(1)}\right) \varphi^f \left(u_{\sigma(2)}\right) ... \varphi^f \left(u_{\sigma(p)}\right)
\end{aligned}$$

with (108), we obtain

$$p! \cdot \overline{\alpha}_{\mathbf{0}}^f \left(u_1 \wedge u_2 \wedge ... \wedge u_p\right) = \sum_{\sigma\in S_p} (-1)^\sigma \varphi^f \left(u_{\sigma(1)}\right) \varphi^f \left(u_{\sigma(2)}\right) ... \varphi^f \left(u_{\sigma(p)}\right).$$

This proves Theorem 38.

## 11. Some more identities

Let us prove some more curious properties of $\overset{f}{\llcorner}$, $\overset{f}{\lrcorner}$ and $\alpha^f$ for a symmetric bilinear form $f$. The following theorems 43-45 bear a certain similarity to theorems 40-42 (and can actually be used to give an alternative proof of Theorem 39, although we are not going to elaborate on this proof).

**Theorem 43.** Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$\begin{aligned}
&\sum_{i=1}^{p} (-1)^{i-1} \alpha^f \left( (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \otimes u_i \right) \\
&= \sum_{i=1}^{p} (-1)^{i-1} \alpha^f \left( u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \right) \otimes u_i .
\end{aligned}$$

Here, the hat over the vector $u_i$ means that the vector $u_i$ is being omitted from the tensor product; in other words, $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$ is just another way to write $\underbrace{u_1 \otimes u_2 \otimes ... \otimes u_{i-1}}_{\substack{\text{tensor product of the} \\ \text{first } i-1 \text{ vectors } u_\ell}} \otimes \underbrace{u_{i+1} \otimes u_{i+2} \otimes ... \otimes u_p}_{\substack{\text{tensor product of the} \\ \text{last } p-i \text{ vectors } u_\ell}}$.

This, in turn, will be concluded from the following result:

**Theorem 44.** Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$\sum_{i=1}^{p} (-1)^{i-1} \left( u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p \right) \overset{f}{\lrcorner} u_i = 0.$$

(For the meaning of the term $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$, see Theorem 43.)

*Proof of Theorem 44.* We are going to prove that every $j \in \{0, 1, ..., p\}$ satisfies

$$\sum_{i=1}^{j} (-1)^{i-1} \left( u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_j \right) \overset{f}{\lrcorner} u_i = 0. \qquad (109)$$

In fact, we will prove this by induction over $j$:

The base case $j = 0$ is trivial[70].

Now, let us come to the induction step: Let $q \in \{1, 2, ..., p\}$. Assume that we have proven (109) for $j = q - 1$. Let us now prove (109) for $j = q$.

For any $i \in \{1, 2, ..., q-1\}$, let us denote the tensor $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_{q-1}$ by $U_i$.

[70] because in the case $j = 0$, we have $\sum\limits_{i=1}^{j} (-1)^{i-1} \left( u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_j \right) \overset{f}{\lrcorner} u_i = (\text{empty sum}) = 0$, and thus the equation (109) is trivially true in the case $j = 0$

We have

$$\sum_{i=1}^{q}(-1)^{i-1}\left(u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_q\right)\overset{f}{\lrcorner}u_i$$

$$=\sum_{i=1}^{q-1}(-1)^{i-1}\left(\underbrace{u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_q}_{\substack{=(u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_{q-1})\otimes u_q=U_i\otimes u_q\\ \text{(since } u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_{q-1}=U_i)}}\right)\overset{f}{\lrcorner}u_i$$

$$+(-1)^{q-1}\underbrace{\left(u_1\otimes u_2\otimes...\otimes\widehat{u_q}\otimes...\otimes u_q\right)}_{=u_1\otimes u_2\otimes...\otimes u_{q-1}}\overset{f}{\lrcorner}u_q$$

$$=\sum_{i=1}^{q-1}(-1)^{i-1}\underbrace{\left(U_i\otimes u_q\right)\overset{f}{\lrcorner}u_i}_{\substack{=f(u_q,u_i)U_i-\left(U_i\overset{f}{\lrcorner}u_i\right)\otimes u_q\\ \text{(by (22), applied to } U_i\text{,}\\ u_q\text{ and }u_i\text{ instead of }U\text{, }u\text{ and }v}}+(-1)^{q-1}\underbrace{\left(u_1\otimes u_2\otimes...\otimes u_{q-1}\right)\overset{f}{\lrcorner}u_q}_{\substack{=\sum_{i=1}^{q-1}(-1)^{q-1-i}f(u_i,u_q)\cdot u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_{q-1}\\ \text{(by (21), applied to }q-1\text{ and }u_q\text{ instead of }p\text{ and }v)}}$$

$$=\sum_{i=1}^{q-1}(-1)^{i-1}\left(\underbrace{f\left(u_q,u_i\right)}_{\substack{=f(u_i,u_q)\text{ (since the form}\\ f\text{ is symmetric)}}}U_i-\left(U_i\overset{f}{\lrcorner}u_i\right)\otimes u_q\right)$$

$$+(-1)^{q-1}\sum_{i=1}^{q-1}(-1)^{q-1-i}f\left(u_i,u_q\right)\cdot\underbrace{u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_{q-1}}_{=U_i}$$

$$=\sum_{i=1}^{q-1}(-1)^{i-1}\left(f\left(u_i,u_q\right)U_i-\left(U_i\overset{f}{\lrcorner}u_i\right)\otimes u_q\right)+(-1)^{q-1}\sum_{i=1}^{q-1}(-1)^{q-1-i}f\left(u_i,u_q\right)U_i$$

$$=\left(\sum_{i=1}^{q-1}(-1)^{i-1}f\left(u_i,u_q\right)U_i-\sum_{i=1}^{q-1}(-1)^{i-1}\left(U_i\overset{f}{\lrcorner}u_i\right)\otimes u_q\right)$$

$$+(-1)^{q-1}\sum_{i=1}^{q-1}(-1)^{q-1-i}f\left(u_i,u_q\right)U_i.$$

Since $U_i=u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_{q-1}$ yields

$$\sum_{i=1}^{q-1}(-1)^{i-1}\left(U_i\overset{f}{\lrcorner}u_i\right)=\sum_{i=1}^{q-1}(-1)^{i-1}\left(\left(u_1\otimes u_2\otimes...\otimes\widehat{u_i}\otimes...\otimes u_{q-1}\right)\overset{f}{\lrcorner}u_i\right)=0$$

(according to (109), applied to $j = q - 1$ [71]), this simplifies to

$$\sum_{i=1}^{q} (-1)^{i-1} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_q) \overset{f}{\lrcorner} u_i$$
$$= \left( \sum_{i=1}^{q-1} (-1)^{i-1} f(u_i, u_q) U_i - \underbrace{\sum_{i=1}^{q-1} (-1)^{i-1} \left( U_i \overset{f}{\lrcorner} u_i \right)}_{=0} \otimes u_q \right)$$
$$+ \underbrace{(-1)^{q-1} \sum_{i=1}^{q-1} (-1)^{q-1-i} f(u_i, u_q) U_i}_{= \sum_{i=1}^{q-1} (-1)^{q-1} (-1)^{q-1-i} f(u_i, u_q) U_i}$$
$$= \left( \sum_{i=1}^{q-1} (-1)^{i-1} f(u_i, u_q) U_i - \underbrace{0 \otimes u_q}_{=0} \right) + \sum_{i=1}^{q-1} (-1)^{q-1} (-1)^{q-1-i} f(u_i, u_q) U_i$$
$$= \sum_{i=1}^{q-1} (-1)^{i-1} f(u_i, u_q) U_i + \sum_{i=1}^{q-1} (-1)^{q-1} (-1)^{q-1-i} f(u_i, u_q) U_i$$
$$= \sum_{i=1}^{q-1} \left( (-1)^{i-1} + (-1)^{q-1} (-1)^{q-1-i} \right) f(u_i, u_q) U_i.$$

Since

$$\underbrace{(-1)^{i-1}}_{=-(-1)^i} + \underbrace{(-1)^{q-1} (-1)^{q-1-i}}_{\substack{=(-1)^{(q-1)+(q-1-i)}=(-1)^{2(q-1)-i} \\ =(-1)^{2(q-1-i)+i}=(-1)^{2(q-1-i)}(-1)^i}} = -(-1)^i + \underbrace{(-1)^{2(q-1-i)}}_{=\left((-1)^2\right)^{q-1-i}=1^{q-1-i}=1} (-1)^i$$
$$= (-1)^i + 1 \cdot (-1)^i = 0,$$

this simplifies to

$$\sum_{i=1}^{q} (-1)^{i-1} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_q) \overset{f}{\lrcorner} u_i$$
$$= \sum_{i=1}^{q-1} \underbrace{\left( (-1)^{i-1} + (-1)^{q-1} (-1)^{q-1-i} \right)}_{=0} f(u_i, u_q) U_i = \sum_{i=1}^{q-1} 0 f(u_i, u_q) U_i = 0.$$

In other words, the equation (109) holds for $j = q$. Thus we have completed the induction step. Consequently, we have proven (109) for every $j \in \{0, 1, ..., p\}$. Thus, applying (109) to $j = p$, we obtain

$$\sum_{i=1}^{p} (-1)^{i-1} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \overset{f}{\lrcorner} u_i = 0.$$

[71] In fact, we *are* allowed to apply (109) to $j = q - 1$, because we assumed that we have proven (109) for $j = q - 1$.

This proves Theorem 44.

Theorem 44 has a "left" analogue:

**Theorem 45.** Let $f : L \times L \to k$ be a symmetric bilinear form. Let $p \in \mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$\sum_{i=1}^{p} (-1)^{i-1} u_i \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) = 0.$$

(For the meaning of the term $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$, see Theorem 43.)

*Proof of Theorem 45.* Applying Theorem 44 to the $p$ vectors $u_p$, $u_{p-1}$, ..., $u_1$ instead of $u_1$, $u_2$, ..., $u_p$, we obtain

$$\sum_{i=1}^{p} (-1)^{i-1} \left(u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_{p-i+1}} \otimes ... \otimes u_1\right) \overset{f}{\lrcorner} u_{p-i+1} = 0. \qquad (110)$$

Since the bilinear form $f$ is symmetric, it satisfies $f^t = f$ (since any two $x \in L$ and $y \in L$ satisfy $f^t(x, y) = f(y, x) = f(x, y)$ because the form $f$ is symmetric). Now, for every $i \in \{1, 2, ..., p\}$, the identity (32) (applied to $U = u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_1$ and $v = u_i$) yields

$$\begin{aligned} t\left(\left(u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_1\right) \overset{f}{\lrcorner} u_i\right) &= u_i \overset{f^t}{\llcorner} \underbrace{t\left(u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_1\right)}_{=u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p} \\ &= u_i \overset{f^t}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \\ &= u_i \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \end{aligned}$$

(since $f^t = f$). Now, (110) yields

$$\begin{aligned} 0 &= \sum_{i=1}^{p} (-1)^{i-1} \left(u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_{p-i+1}} \otimes ... \otimes u_1\right) \overset{f}{\lrcorner} u_{p-i+1} \\ &= \sum_{i=1}^{p} (-1)^{p-i} \left(u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_1\right) \overset{f}{\lrcorner} u_i \\ &\qquad \left(\text{here we substituted } i \text{ for } p-i+1 \text{ in the sum}\right), \end{aligned}$$

so that

$$\begin{aligned} t(0) &= t\left(\sum_{i=1}^{p} (-1)^{p-i} \left(u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_1\right) \overset{f}{\lrcorner} u_i\right) \\ &= \sum_{i=1}^{p} (-1)^{p-i} \underbrace{t\left(\left(u_p \otimes u_{p-1} \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_1\right) \overset{f}{\lrcorner} u_i\right)}_{=u_i \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p)} \\ &= \sum_{i=1}^{p} \underbrace{(-1)^{p-i}}_{\substack{=(-1)^{(p+1)+(i-1)} \\ =(-1)^{p+1}(-1)^{i-1}}} u_i \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \\ &= (-1)^{p+1} \sum_{i=1}^{p} (-1)^{i-1} u_i \overset{f}{\llcorner} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p). \end{aligned}$$

Since $t(0) = 0$, this rewrites as

$$0 = (-1)^{p+1} \sum_{i=1}^{p} (-1)^{i-1} u_i \llcorner^{f} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p).$$

Dividing this equation by $(-1)^{p+1}$, we get

$$0 = \sum_{i=1}^{p} (-1)^{i-1} u_i \llcorner^{f} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p).$$

This proves Theorem 45.

*Proof of Theorem 43.* For every $i \in \{1, 2, ..., p\}$, denote the tensor $u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$ by $U_i$. Then,

$$\begin{aligned}
&\alpha^f \left( \underbrace{(u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p)}_{=U_i} \otimes u_i \right) \\
&= \alpha^f (U_i \otimes u_i) = \alpha^f (U_i) \otimes u_i - \alpha^f (U_i) \lrcorner^{f} u_i \\
&\qquad \text{(by (52), applied to } U_i \text{ and } u_i \text{ instead of } U \text{ and } u) \\
&= \alpha^f (U_i) \otimes u_i - \alpha^f \left( U_i \lrcorner^{f} u_i \right) \\
&\qquad \left( \begin{array}{c} \text{since (54) (applied to } f, U_i \text{ and } u_i \text{ instead of } g, U \text{ and } u) \\ \text{yields } \alpha^f \left( U_i \lrcorner^{f} u_i \right) = \alpha^f (U_i) \lrcorner^{f} u_i \text{, so that } \alpha^f (U_i) \lrcorner^{f} u_i = \alpha^f \left( U_i \lrcorner^{f} u_i \right) \end{array} \right)
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_{i=1}^{p} (-1)^{i-1} \alpha^f ((u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \otimes u_i) \\
&= \sum_{i=1}^{p} (-1)^{i-1} \left( \alpha^f (U_i) \otimes u_i - \alpha^f \left( U_i \lrcorner^{f} u_i \right) \right) \\
&= \sum_{i=1}^{p} (-1)^{i-1} \alpha^f (U_i) \otimes u_i - \sum_{i=1}^{p} (-1)^{i-1} \alpha^f \left( U_i \lrcorner^{f} u_i \right).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{i=1}^{p} (-1)^{i-1} \alpha^f \left( U_i \lrcorner^{f} u_i \right) &= \alpha^f \left( \sum_{i=1}^{p} (-1)^{i-1} U_i \lrcorner^{f} u_i \right) \qquad \left(\text{since } \alpha^f \text{ is a linear map}\right) \\
&= \alpha^f \left( \underbrace{\sum_{i=1}^{p} (-1)^{i-1} (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \lrcorner^{f} u_i}_{=0 \text{ (by Theorem 44)}} \right) \\
&\qquad \text{(since } U_i = u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \\
&= \alpha^f (0) = 0,
\end{aligned}$$

this rewrites as

$$\sum_{i=1}^{p} (-1)^{i-1} \alpha^f ((u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \otimes u_i)$$
$$= \sum_{i=1}^{p} (-1)^{i-1} \alpha^f (U_i) \otimes u_i - 0 = \sum_{i=1}^{p} (-1)^{i-1} \alpha^f (U_i) \otimes u_i$$
$$= \sum_{i=1}^{p} (-1)^{i-1} \alpha^f (u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p) \otimes u_i$$

(since $U_i = u_1 \otimes u_2 \otimes ... \otimes \widehat{u_i} \otimes ... \otimes u_p$). This proves Theorem 43.

## 12. The invariant module of the $\alpha^f$ maps for all symmetric bilinear $f$

Let us consider a fixed commutative ring $k$, and a fixed $k$-module $L$. However, in this section, we are *not* going to fix a bilinear form $f$ on $L$, but we will consider all bilinear forms $f$ at once. Each bilinear form $f$ gives rise to an endomorphism $\alpha^f : \otimes L \to \otimes L$, and we are going to study the module $\operatorname{Fix} \alpha^{\text{symm}}$ of all tensors in $\otimes L$ that are fixed under $\alpha^f$ for *all symmetric* bilinear forms $f$. [72]

**Definition 15.** Let $k$ be a commutative ring, and $L$ be a $k$-module. We denote by $\operatorname{Fix} \alpha^{\text{symm}}$ the subset

$$\left\{U \in \otimes L \ \mid \ \text{every symmetric bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f (U) = U\right\}$$
$$= \bigcap_{\substack{f:L\times L\to k \text{ is a} \\ \text{symmetric bilinear form}}} \left\{ U \in \otimes L \ \mid \ \underbrace{\alpha^f (U) = U}_{\substack{\text{this equation is equivalent} \\ \text{to } \alpha^f(U)-U=0}} \right\}$$
$$= \bigcap_{\substack{f:L\times L\to k \text{ is a} \\ \text{symmetric bilinear form}}} \left\{ U \in \otimes L \ \mid \ \underbrace{\alpha^f (U) - U}_{=\left(\alpha^f-\mathrm{id}\right)(U)} = 0 \right\}$$
$$= \bigcap_{\substack{f:L\times L\to k \text{ is a} \\ \text{symmetric bilinear form}}} \underbrace{\left\{U \in \otimes L \ \mid \ \left(\alpha^f - \mathrm{id}\right)(U) = 0\right\}}_{=\operatorname{Ker}\left(\alpha^f-\mathrm{id}\right)}$$
$$= \bigcap_{\substack{f:L\times L\to k \text{ is a} \\ \text{symmetric bilinear form}}} \operatorname{Ker}\left(\alpha^f - \mathrm{id}\right)$$

of $\otimes L$. Clearly, this subset $\operatorname{Fix} \alpha^{\text{symm}}$ is a sub-$k$-module of $\otimes L$ (since $\operatorname{Fix} \alpha^{\text{symm}} = \bigcap_{\substack{f:L\times L\to k \text{ is a} \\ \text{symmetric bilinear form}}} \operatorname{Ker}\left(\alpha^f - \mathrm{id}\right)$, and since $\operatorname{Ker}\left(\alpha^f - \mathrm{id}\right)$ is a sub-$k$-module of $\otimes L$ for each $f$).

It seems to be a nontrivial question to further characterize $\operatorname{Fix} \alpha^{\text{symm}}$. First we note that antisymmetrizers always lie in $\operatorname{Fix} \alpha^{\text{symm}}$:

[72] As for the space $\operatorname{Fix} \alpha$ of all tensors in $\otimes L$ that are fixed under $\alpha^f$ for *all* (not only symmetric) bilinear forms $f$, we are planning to study this space later.

**Corollary 46.** Let $p \in \mathbb{N}$, and let $u_1$, $u_2$, ..., $u_p$ be $p$ vectors in $L$. Then,

$$\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)} \in \operatorname{Fix} \alpha^{\text{symm}}.$$

*Proof of Corollary 46.* Every symmetric bilinear form $f$ satisfies

$$\alpha^f \left( \sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)} \right) = \sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}$$

(by Theorem 39). Thus,

$$\begin{aligned} &\sum_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)} \\ &\in \left\{ U \in \otimes L \ \mid \ \text{every symmetric bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f (U) = U \right\} \\ &= \operatorname{Fix} \alpha^{\text{symm}}. \end{aligned}$$

This proves Corollary 46.

However, elements of the form $\sum\limits_{\sigma \in S_p} (-1)^\sigma u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(p)}$ as in Corollary 46 are not the only inhabitants of $\operatorname{Fix} \alpha^{\text{symm}}$. There are more. I do not claim that I know all of them, but here is a result that construct at least a part:

**Theorem 47.** Let $k$ be a commutative ring. Let $L$ be a $k$-module.

**(a)** We have $k \subseteq \operatorname{Fix} \alpha^{\text{symm}}$ (where $k$ is regarded as a sub-$k$-module of $\otimes L$ because $k = L^{\otimes 0} \subseteq L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ... = \otimes L$) and $L \subseteq \operatorname{Fix} \alpha^{\text{symm}}$ (where $L$ is regarded as a sub-$k$-module of $\otimes L$ because $L = L^{\otimes 1} \subseteq L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ... = \otimes L$).

**(b)** Let $m \in \mathbb{N}$. Any two elements $u \in L$ and $V \in L^{\otimes m} \cap \operatorname{Fix} \alpha^{\text{symm}}$ satisfy $u \otimes V + (-1)^m V \otimes u \in \operatorname{Fix} \alpha^{\text{symm}}$.

The proof of Theorem 47 relies on the following result:

**Lemma 48.** Let $m \in \mathbb{N}$. Let $k$ be a commutative ring. Let $L$ be a $k$-module. Let $f : L \times L \to k$ be a symmetric bilinear form. Then, any $u \in L$ and $U \in L^{\otimes m}$ satisfy $\alpha^f (u \otimes U + (-1)^m U \otimes u) = u \otimes \alpha^f (U) + (-1)^m \alpha^f (U) \otimes u$ (where $u$ is regarded as an element of $\otimes L$ because $u \in L = L^{\otimes 1} \subseteq L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus ... = \otimes L$).

This lemma, in turn, will be proven using the following fact:

**Lemma 49.** Let $m \in \mathbb{N}$. Let $k$ be a commutative ring. Let $L$ be a $k$-module. Let $f : L \times L \to k$ be a bilinear form. Then, any $u \in L$ and $V \in L^{\otimes m}$ satisfy $u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u$.

*Proof of Lemma 49.* Fix some $u \in L$. We are now going to prove that for every $m \in \mathbb{N}$,

$$\left(\text{every } V \in L^{\otimes m} \text{ satisfies } u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u\right). \tag{111}$$

In fact, let us prove this by induction over $m$:

The induction base case, $m = 0$, is trivial (because in the case $m = 0$, we have $V \in L^{\otimes m} = L^{\otimes 0} = k$, so that $u \overset{f}{\llcorner} V = 0$ (by Theorem 5 **(a)**, applied to $u$ and $V$ instead of $v$ and $\lambda$) and $V \overset{f^t}{\lrcorner} u = 0$ (by Theorem 11 **(a)**, applied to $u$, $f^t$ and $V$ instead of $v$, $f$ and $\lambda$), and therefore clearly $u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u$, so that (111) is proven in the case $m = 0$).

Now let us come to the induction step. Fix some positive $\mu \in \mathbb{N}_+$. Now, let us prove (111) for $m = \mu$, assuming that (111) has already been proven for $m = \mu - 1$.

In fact, we have assumed that (111) has already been proven for $m = \mu - 1$. In other words, we have assumed that

$$\left(\text{every } V \in L^{\otimes(\mu-1)} \text{ satisfies } u \overset{f}{\llcorner} V = (-1)^{(\mu-1)-1} V \overset{f^t}{\lrcorner} u\right). \tag{112}$$

Now, our goal is to show (111) for $m = \mu$. That is, our goal is to show that every $V \in L^{\otimes \mu}$ satisfies $u \overset{f}{\llcorner} V = (-1)^{\mu-1} V \overset{f^t}{\lrcorner} u$. In order to achieve this goal, it is obviously enough to show that every left-induced $V \in L^{\otimes \mu}$ satisfies $u \overset{f}{\llcorner} V = (-1)^{\mu-1} V \overset{f^t}{\lrcorner} u$ (by the left tensor induction tactic, since the equation $u \overset{f}{\llcorner} V = (-1)^{\mu-1} V \overset{f^t}{\lrcorner} u$ is linear in $V$). So, let $V \in L^{\otimes \mu}$ be some left-induced tensor. Then, $V = v \otimes \widetilde{V}$ for some $v \in L$ and

$\widetilde{V} \in L^{\otimes(\mu-1)}$ (because $V$ is left-induced). Thus,

$$u \overset{f}{\llcorner} V = u \overset{f}{\llcorner} \left(v \otimes \widetilde{V}\right) = f\left(u,v\right) \widetilde{V} - v \otimes \underbrace{\left(u \overset{f}{\llcorner} \widetilde{V}\right)}_{\substack{=(-1)^{(\mu-1)-1} \widetilde{V} \overset{f^t}{\lrcorner} u \text{ (by (112),} \\ \text{applied to } \widetilde{V} \text{ instead of } V)}}$$

$$\left(\text{by (7), applied to } u,\ v \text{ and } \widetilde{V} \text{ instead of } v,\ u \text{ and } U\right)$$

$$= f\left(u,v\right) \widetilde{V} - \underbrace{v \otimes \left(\left(-1\right)^{(\mu-1)-1} \widetilde{V} \overset{f^t}{\lrcorner} u\right)}_{=(-1)^{(\mu-1)-1} v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right)} = f\left(u,v\right) \widetilde{V} - \underbrace{\left(-1\right)^{(\mu-1)-1}}_{=-(-1)^{\mu-1}} v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right)$$

$$= f\left(u,v\right) \widetilde{V} + \left(-1\right)^{\mu-1} v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right)$$

$$= \left(-1\right)^{\mu-1} \left(\underbrace{\frac{1}{\left(-1\right)^{\mu-1}}}_{\substack{=(-1)^{\mu-1} \text{ (since} \\ \left((-1)^{\mu-1}\right)^2 = (-1)^{2(\mu-1)} = 1, \\ \text{because } 2(\mu-1) \text{ is even)}}} f\left(u,v\right) \widetilde{V} + v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right)\right)$$

$$= \left(-1\right)^{\mu-1} \left(\left(-1\right)^{\mu-1} f\left(u,v\right) \widetilde{V} + v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right)\right). \tag{113}$$

But $V = v \otimes \widetilde{V}$ also yields

$$V \overset{f^t}{\lrcorner} u = \left(v \otimes \widetilde{V}\right) \overset{f^t}{\lrcorner} u = \left(-1\right)^{\mu-1} \underbrace{f^t\left(v,u\right)}_{=f(u,v)} \widetilde{V} + v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right)$$

$$\left(\text{by (25), applied to } v,\ \widetilde{V},\ f^t,\ u \text{ and } \mu-1 \text{ instead of } u,\ U,\ f,\ v \text{ and } p\right)$$

$$= \left(-1\right)^{\mu-1} f\left(u,v\right) \widetilde{V} + v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right).$$

Thus, (113) becomes

$$u \overset{f}{\llcorner} V = \left(-1\right)^{\mu-1} \left(\underbrace{\left(-1\right)^{\mu-1} f\left(u,v\right) \widetilde{V} + v \otimes \left(\widetilde{V} \overset{f^t}{\lrcorner} u\right)}_{=V \overset{f^t}{\lrcorner} u}\right) = \left(-1\right)^{\mu-1} V \overset{f^t}{\lrcorner} u.$$

Thus, we have proven that every left-induced $V \in L^{\otimes\mu}$ satisfies $u \overset{f}{\llcorner} V = \left(-1\right)^{\mu-1} V \overset{f^t}{\lrcorner} u$. As we already said above, this yields that every $V \in L^{\otimes\mu}$ satisfies $u \overset{f}{\llcorner} V = \left(-1\right)^{\mu-1} V \overset{f^t}{\lrcorner} u$. Therefore, (111) is proved for $m = \mu$, so that the induction step is complete. Consequently, we have proven that (111) holds for all $m \in \mathbb{N}$. In other words, we have verified Lemma 49.

Here is a little strengthening of Lemma 49:

**Lemma 50.** Let $m \in \mathbb{N}$. Let $k$ be a commutative ring. Let $L$ be a $k$-module. Let $f : L \times L \to k$ be a bilinear form. Then, any $u \in L$ and $V \in \bigoplus_{\substack{i \in \mathbb{N}; \\ i \equiv m \operatorname{mod} 2}} L^{\otimes i}$ satisfy $u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u$.

*Proof of Lemma 50.* Let $u \in L$ be fixed. Our goal is to show that every $V \in \bigoplus_{\substack{i \in \mathbb{N}; \\ i \equiv m \operatorname{mod} 2}} L^{\otimes i}$ satisfies $u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u$.

Define a map $\Phi_u : \otimes L \to \otimes L$ by

$$\left( \Phi_u (V) = u \overset{f}{\llcorner} V - (-1)^{m-1} V \overset{f^t}{\lrcorner} u \qquad \text{for every } V \in \otimes L \right).$$

This map $\Phi_u$ is $k$-linear (because $\overset{f}{\llcorner}$ and $\overset{f^t}{\lrcorner}$ are $k$-bilinear). Thus, $\operatorname{Ker} \Phi_u$ is a sub-$k$-module of $\otimes L$.

For every $i \in \mathbb{N}$ satisfying $i \equiv m \operatorname{mod} 2$, we have $L^{\otimes i} \subseteq \operatorname{Ker} \Phi_u$ (since every $V \in L^{\otimes i}$ satisfies

$$\begin{aligned} \Phi_u (V) &= \underbrace{u \overset{f}{\llcorner} V}_{\substack{= (-1)^{i-1} V \overset{f^t}{\lrcorner} u \\ \text{(by Lemma 49, applied} \\ \text{to } i \text{ instead of } m)}} - (-1)^{m-1} V \overset{f^t}{\lrcorner} u = \underbrace{(-1)^{i-1}}_{\substack{= (-1)^{m-1} \\ \text{(since } i \equiv m \operatorname{mod} 2 \text{ and} \\ \text{thus } i-1 \equiv m-1 \operatorname{mod} 2)}} V \overset{f^t}{\lrcorner} u - (-1)^{m-1} V \overset{f^t}{\lrcorner} u \\ &= (-1)^{m-1} V \overset{f^t}{\lrcorner} u - (-1)^{m-1} V \overset{f^t}{\lrcorner} u = 0 \end{aligned}$$

). Thus, $\bigoplus_{\substack{i \in \mathbb{N}; \\ i \equiv m \operatorname{mod} 2}} L^{\otimes i} \subseteq \bigoplus_{\substack{i \in \mathbb{N}; \\ i \equiv m \operatorname{mod} 2}} \operatorname{Ker} \Phi_u \subseteq \operatorname{Ker} \Phi_u$ (since $\operatorname{Ker} \Phi_u$ is a sub-$k$-module of $\otimes L$). Hence, every $V \in \bigoplus_{\substack{i \in \mathbb{N}; \\ i \equiv m \operatorname{mod} 2}} L^{\otimes i}$ satisfies $\Phi_u (V) = 0$. Since $\Phi_u (V) = u \overset{f}{\llcorner} V - (-1)^{m-1} V \overset{f^t}{\lrcorner} u$, this rewrites as $u \overset{f}{\llcorner} V - (-1)^{m-1} V \overset{f^t}{\lrcorner} u = 0$, so that $u \overset{f}{\llcorner} V = (-1)^{m-1} V \overset{f^t}{\lrcorner} u$. This proves Lemma 50.

*Proof of Lemma 48.* Applying (46), we get

$$\alpha^f (u \otimes U) = u \otimes \alpha^f (U) - u \overset{f}{\llcorner} \alpha^f (U).$$

Applying (52), we get

$$\alpha^f (U \otimes u) = \alpha^f (U) \otimes u - \alpha^f (U) \overset{f}{\lrcorner} u.$$

On the other hand, (47) (applied to $m$ instead of $p$) yields $\alpha^f (U) \in \bigoplus_{\substack{i \in \mathbb{N}; \\ i \equiv m \operatorname{mod} 2}} L^{\otimes i}$. Hence, we can apply Lemma 50 to $V = \alpha^f (U)$ and obtain $u \overset{f}{\llcorner} \alpha^f (U) = (-1)^{m-1} \alpha^f (U) \overset{f^t}{\lrcorner} u$. Since $f = f^t$ (because $f$ is symmetric), we can replace $f^t$ by $f$ in this equality, and thus obtain $u \overset{f}{\llcorner} \alpha^f (U) = (-1)^{m-1} \alpha^f (U) \overset{f}{\lrcorner} u$.

Since $\alpha^f$ is linear, we have

$$\begin{aligned}
&\alpha^f\left(u \otimes U + (-1)^m U \otimes u\right) \\
&= \underbrace{\alpha^f\left(u \otimes U\right)}_{=u\otimes\alpha^f(U) - u \overset{f}{\llcorner} \alpha^f(U)} + (-1)^m \underbrace{\alpha^f\left(U \otimes u\right)}_{=\alpha^f(U)\otimes u - \alpha^f(U) \overset{f}{\lrcorner} u} \\
&= \left(u \otimes \alpha^f(U) - u \overset{f}{\llcorner} \alpha^f(U)\right) + (-1)^m \left(\alpha^f(U) \otimes u - \alpha^f(U) \overset{f}{\lrcorner} u\right) \\
&= u \otimes \alpha^f(U) - \underbrace{u \overset{f}{\llcorner} \alpha^f(U)}_{=(-1)^{m-1}\alpha^f(U) \overset{f}{\lrcorner} u} + (-1)^m \alpha^f(U) \otimes u - \underbrace{(-1)^m}_{=-(-1)^{m-1}} \alpha^f(U) \overset{f}{\lrcorner} u \\
&= u \otimes \alpha^f(U) - (-1)^{m-1} \alpha^f(U) \overset{f}{\lrcorner} u + (-1)^m \alpha^f(U) \otimes u - \left(-(-1)^{m-1}\right) \alpha^f(U) \overset{f}{\lrcorner} u \\
&= u \otimes \alpha^f(U) + (-1)^m \alpha^f(U) \otimes u - \underbrace{\left((-1)^{m-1} \alpha^f(U) \overset{f}{\lrcorner} u + \left(-(-1)^{m-1}\right) \alpha^f(U) \overset{f}{\lrcorner} u\right)}_{=0} \\
&= u \otimes \alpha^f(U) + (-1)^m \alpha^f(U) \otimes u.
\end{aligned}$$

This proves Lemma 48.

*Proof of Theorem 47.* **(a)** For every $\lambda \in k$, we have $\lambda \in \operatorname{Fix} \alpha^{\text{symm}}$. [73] Thus, $k \subseteq \operatorname{Fix} \alpha^{\text{symm}}$.

For every $u \in L$, we have $u \in \operatorname{Fix} \alpha^{\text{symm}}$. [74] Thus, $L \subseteq \operatorname{Fix} \alpha^{\text{symm}}$.

Theorem 47 **(a)** is now proven.

**(b)** Let $u \in L$ and $V \in L^{\otimes m} \cap \operatorname{Fix} \alpha^{\text{symm}}$ be arbitrary. Then, clearly, $V \in L^{\otimes m}$ and

$$\begin{aligned}
&V \in \operatorname{Fix} \alpha^{\text{symm}} \\
&\quad = \left\{U \in \otimes L \ \mid \ \text{every symmetric bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f(U) = U\right\}.
\end{aligned}$$

Thus, every symmetric bilinear form $f : L \times L \to k$ satisfies $\alpha^f(V) = V$.

Let $f : L \times L \to k$ be a symmetric bilinear form. Then, Lemma 48 (applied to $U = V$) yields

$$\alpha^f\left(u \otimes V + (-1)^m V \otimes u\right) = u \otimes \underbrace{\alpha^f(V)}_{=V} + (-1)^m \underbrace{\alpha^f(V)}_{=V} \otimes u = u \otimes V + (-1)^m V \otimes u.$$

Thus,

$$\begin{aligned}
&u \otimes V + (-1)^m V \otimes u \\
&\in \left\{U \in \otimes L \ \mid \ \text{every symmetric bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f(U) = U\right\} \\
&= \operatorname{Fix} \alpha^{\text{symm}}.
\end{aligned}$$

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[73] *Proof.* Every symmetric bilinear form $f : L \times L \to k$ satisfies $\alpha^f(\lambda) = \lambda$. Thus,

$$\lambda \in \left\{U \in \otimes L \ \mid \ \text{every symmetric bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f(U) = U\right\} = \operatorname{Fix} \alpha^{\text{symm}}.$$

[74] *Proof.* Every symmetric bilinear form $f : L \times L \to k$ satisfies $\alpha^f(u) = u$ (according to (43)). Thus,

$$u \in \left\{U \in \otimes L \ \mid \ \text{every symmetric bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f(U) = U\right\} = \operatorname{Fix} \alpha^{\text{symm}}.$$

This proves Theorem 47 **(b)**.

Theorem 47 yields an inductive way to construct elements of $\operatorname{Fix} \alpha^{\text{symm}}$ beginning from elements of $L$. For example, for any two vectors $u \in L$ and $v \in L$, Theorem 47 shows that $u \otimes v - v \otimes u \in \operatorname{Fix} \alpha^{\text{symm}}$ (not surprisingly). For any three vectors $u \in L$, $v \in L$ and $w \in L$, Theorem 47 shows that $u \otimes (v \otimes w - w \otimes v) + (v \otimes w - w \otimes v) \otimes u \in \operatorname{Fix} \alpha^{\text{symm}}$. For any four vectors $u \in L$, $v \in L$, $w \in L$ and $x \in L$, Theorem 47 shows that

$$\begin{aligned} & u \otimes (v \otimes (w \otimes x - x \otimes w) + (w \otimes x - x \otimes w) \otimes v) \\ & \qquad - (v \otimes (w \otimes x - x \otimes w) + (w \otimes x - x \otimes w) \otimes v) \otimes u \end{aligned} \tag{114}$$

lies in $\operatorname{Fix} \alpha^{\text{symm}}$. And so on.

Do we get all elements of $\operatorname{Fix} \alpha^{\text{symm}}$ this way? No. For example, for any four vectors $a \in L$, $b \in L$, $c \in L$ and $d \in L$, the tensor

$$a \otimes b \otimes (c \otimes d + d \otimes c) - (c \otimes d + d \otimes c) \otimes a \otimes b$$

lies in $\operatorname{Fix} \alpha^{\text{symm}}$ (and is even fixed under $\alpha^f$ for *all* (not only symmetric) bilinear forms $f$). In general, this tensor cannot be written as a linear combination of elements of the form (114) (with $u \in L$, $v \in L$, $w \in L$ and $x \in L$), even if the underlying ring $k$ is a field of characteristic 0. (This was computed by Andrew Rupinski in [4].)

## 13. Further remarks

As we said, instead of $\operatorname{Fix} \alpha^{\text{symm}}$ we could consider the subset $\operatorname{Fix} \alpha$ of $\otimes L$ defined by

$$\begin{aligned} \operatorname{Fix} \alpha &= \left\{ U \in \otimes L \mid \text{every bilinear form } f : L \times L \to k \text{ satisfies } \alpha^f(U) = U \right\} \\ &= \bigcap_{\substack{f : L \times L \to k \text{ is a} \\ \text{bilinear form}}} \left\{ U \in \otimes L \mid \underbrace{\alpha^f(U) = U}_{\substack{\text{this equation is equivalent} \\ \text{to } \alpha^f(U) - U = 0}} \right\} \\ &= \bigcap_{\substack{f : L \times L \to k \text{ is a} \\ \text{bilinear form}}} \left\{ U \in \otimes L \mid \underbrace{\alpha^f(U) - U}_{= (\alpha^f - \operatorname{id})(U)} = 0 \right\} \\ &= \bigcap_{\substack{f : L \times L \to k \text{ is a} \\ \text{bilinear form}}} \underbrace{\left\{ U \in \otimes L \mid \left(\alpha^f - \operatorname{id}\right)(U) = 0 \right\}}_{= \operatorname{Ker}(\alpha^f - \operatorname{id})} = \bigcap_{\substack{f : L \times L \to k \text{ is a} \\ \text{bilinear form}}} \operatorname{Ker}\left(\alpha^f - \operatorname{id}\right). \end{aligned}$$

We can then prove that any three vectors $b \in L$, $c \in L$ and $d \in L$ satisfy

$$b \otimes (c \otimes d + d \otimes c) - (c \otimes d + d \otimes c) \otimes b \in \operatorname{Fix} \alpha.$$

We can also show that any four vectors $a \in L$, $b \in L$, $c \in L$ and $d \in L$ satisfy

$$a \otimes b \otimes (c \otimes d + d \otimes c) - (c \otimes d + d \otimes c) \otimes a \otimes b \in \operatorname{Fix} \alpha$$

(as we have already seen). Does this have a reasonable generalization?

Can invariant theory help us in understanding $\operatorname{Fix} \alpha^{\text{symm}}$ and $\operatorname{Fix} \alpha$ ? After all, Theorem 32 shows that $f \mapsto \alpha^f$ is a representation of the additive group of bilinear forms on $L$ (resp. symmetric bilinear forms on $L$) on $\otimes L$, and we are looking for the invariant space of this representation.

Another interesting question would be to generalize $\alpha^f$ to super-vector spaces, thus obtaining results about Weyl algebras rather than just Clifford algebras.

## 14. The $\alpha^f$ morphisms and direct sums

In this section we are going to deal with the behaviour of $\alpha^f$ morphisms when the $k$-module $L$ is a direct sum of two smaller $k$-modules.

First a relative triviality on submodules:

**Lemma 60.** Let $k$ be a commutative ring. Let $L$ be a $k$-module. Let $f : L \times L \to k$ be a bilinear form. Let $M$ be a $k$-submodule of $L$ such that $f(L \times M) = 0$. Then:

**(a)** Every $U \in \otimes L$ and every $m \in M$ satisfy $U \overset{f}{\lrcorner} m = 0$.

**(b)** Every $U \in \otimes L$ and every $m \in M$ satisfy $\alpha^f(U \otimes m) = \alpha^f(U) \otimes m$.

**(c)** We have $\alpha^f((\otimes L) \cdot M) = (\otimes L) \cdot M$.

*Proof of Lemma 60.* **(a)** Let $m \in M$ be fixed.

First we will prove that for every $p \in \mathbb{N}$ and every $U \in L^{\otimes p}$, the equation

$$U \overset{f}{\lrcorner} m = 0 \qquad (115)$$

holds. In fact, we will show this by induction over $p$: The induction base ($p = 0$) is clear (since Theorem 11 **(a)** yields $U \overset{f}{\lrcorner} m = 0$ in the case $p = 0$). Now for the induction step: Fix some $p \in \mathbb{N}_+$. Let us now prove (115) for all $U \in L^{\otimes p}$, assuming that (115) is already proven for all $U \in L^{\otimes(p-1)}$.

We want to prove (115) for all $U \in L^{\otimes p}$. But in order to achieve this, it is enough to prove (115) for all right-induced $U \in L^{\otimes p}$ (because of the right tensor induction tactic, since the equation (115) is linear in $U$). So let us prove (115) for all right-induced $U \in L^{\otimes p}$. In fact, let $U \in L^{\otimes p}$ be a right-induced tensor. Then, $U$ can be written in the form $U = \ddot{U} \otimes u$ for some $u \in L$ and $\ddot{U} \in L^{\otimes(p-1)}$ (since $U$ is right-induced).

Since we have assumed that (115) is already proven for all $U \in L^{\otimes(p-1)}$, we can apply (115) to $\ddot{U}$ instead of $U$. Thus we obtain $\ddot{U} \overset{f}{\lrcorner} m = 0$. On the other hand, $(u, m) \in L \times M$ yields $f(u, m) \in f(L \times M) = 0$, so that $f(u, m) = 0$.

Now, from $U = \ddot{U} \otimes u$ we get

$$\begin{aligned} U \overset{f}{\lrcorner} m &= \left(\ddot{U} \otimes u\right) \overset{f}{\lrcorner} m = \underbrace{f(u, m)}_{=0} \ddot{U} - \underbrace{\left(\ddot{U} \overset{f}{\lrcorner} m\right)}_{=0} \otimes u \\ &\qquad \left(\text{by (22), applied to } \ddot{U} \text{ and } m \text{ instead of } U \text{ and } v\right) \\ &= 0 \ddot{U} - 0 \otimes u = 0. \end{aligned}$$

Thus, we have proven that (115) holds for all right-induced $U \in L^{\otimes p}$. Consequently, by the right tensor induction tactic (as we said above), we conclude that (115) holds for all $U \in L^{\otimes p}$. This completes the induction step. Therefore we have now proven that for every $p \in \mathbb{N}$, and every $U \in L^{\otimes p}$, the equation (115) holds. Consequently, for every $U \in \otimes L$, the equation (115) holds (because every $U \in \otimes L$ is a $k$-linear combination of elements of $L^{\otimes p}$ for various $p \in \mathbb{N}$, and because the equation (115) is linear in $U$). This proves Lemma 60 **(a)**.

**(b)** Let $U \in \otimes L$ and $m \in M$ be arbitrary. Then, (52) (applied to $m$ instead of $u$) yields

$$\alpha^f (U \otimes m) = \alpha^f (U) \otimes m - \underbrace{\alpha^f (U) \overset{f}{\lrcorner} m}_{\substack{=0 \text{ (due to Lemma 60 } \mathbf{(a)}, \\ \text{applied to } \alpha^f(U) \text{ instead of } U)}} = \alpha^f (U) \otimes m.$$

This proves Lemma 60 **(b)**.

**(c)** We know that the map $\alpha^f : \otimes L \to \otimes L$ is invertible (by Theorem 32), so that the map $\alpha^f \times \mathrm{id} : (\otimes L) \times M \to (\otimes L) \times M$ is invertible as well. In other words, $\alpha^f \times \mathrm{id} : (\otimes L) \times M \to (\otimes L) \times M$ is a bijection.

We have

$$\begin{aligned}(\otimes L) \cdot M &= \left\langle \underbrace{U \cdot m}_{\substack{=U \otimes m \\ \text{(since the multiplication} \\ \text{in } \otimes L \text{ is the tensor product)}}} \mid (U, m) \in (\otimes L) \times M \right\rangle \\ &= \langle U \otimes m \mid (U, m) \in (\otimes L) \times M \rangle ,\end{aligned}$$

so that

$$\begin{aligned}&\alpha^f ((\otimes L) \cdot M) \\ &= \alpha^f (\langle U \otimes m \mid (U, m) \in (\otimes L) \times M \rangle) \\ &= \left\langle \underbrace{\alpha^f (U \otimes m)}_{\substack{=\alpha^f(U) \otimes m \\ \text{(by Lemma 60 } \mathbf{(b)}, \\ \text{since } U \in \otimes L \text{ and } m \in M)}} \mid (U, m) \in (\otimes L) \times M \right\rangle \qquad \left(\text{since } \alpha^f \text{ is a } k\text{-linear map}\right) \\ &= \left\langle \underbrace{\alpha^f (U) \otimes m}_{\substack{=\alpha^f(U) \cdot m \\ \text{(since the multiplication} \\ \text{in } \otimes L \text{ is the tensor product)}}} \mid (U, m) \in (\otimes L) \times M \right\rangle \\ &= \left\langle \alpha^f (U) \cdot m \mid (U, m) \in (\otimes L) \times M \right\rangle = \langle V \cdot m \mid (V, m) \in (\otimes L) \times M \rangle \\ &\qquad \left(\begin{array}{c}\text{here, we substituted } (V, m) \text{ for } \left(\alpha^f \times \mathrm{id}\right)(U, m) = \left(\alpha^f (U), m\right), \\ \text{because the map } \alpha^f \times \mathrm{id} \text{ is a bijection}\end{array}\right) \\ &= (\otimes L) \cdot M.\end{aligned}$$

This proves Lemma 60 **(c)**.

Note that we could also have derived Lemma 60 **(a)** from Theorem 11, but we prefer the inductive approach.

Now we can prove:

**Theorem 61.** Let $k$ be a commutative ring. Let $L$ be a $k$-module. Let $h : L \times L \to k$ be a bilinear form. Let $M$ and $N$ be two $k$-submodules of $L$ such that $h(M \times M) = 0$ and $L = M \oplus N$. Then:

**(a)** For every bilinear form $g : L \times L \to k$, there exists a $k$-module isomorphism $\operatorname{Cl}(L, g) \to \operatorname{Cl}(L, h+g)$ which sends the $k$-submodule $\operatorname{Cl}(L, g) \cdot \varphi_g(M)$ of $\operatorname{Cl}(L, g)$ to the $k$-submodule $\operatorname{Cl}(L, h+g) \cdot \varphi_{h+g}(M)$ of $\operatorname{Cl}(L, h+g)$.

**(b)** There exists a $k$-module isomorphism $\wedge L \to \operatorname{Cl}(L, h)$ which sends the $k$-submodule $(\wedge L) \cdot \varphi_{\mathbf{0}}(M)$ of $\wedge L$ to the $k$-submodule $\operatorname{Cl}(L, h) \cdot \varphi_h(M)$ of $\operatorname{Cl}(L, h)$. Therefore,

$$(\operatorname{Cl}(L,h)) \diagup (\operatorname{Cl}(L,h) \cdot \varphi_h(M)) \cong (\wedge L) \diagup ((\wedge L) \cdot \varphi_{\mathbf{0}}(M)) \cong \wedge (L \diagup M) \cong \wedge N$$

as $k$-modules.

**(c)** Let $\operatorname{proj}_M$ be the projection from $L$ on $M$ with kernel $N$, and let $\operatorname{proj}_N$ be the projection from $L$ on $N$ with kernel $M$. (These two projections are well-defined because $L = M \oplus N$). Define a map $f : L \times L \to k$ by

$$(f(u,v) = h(\operatorname{proj}_M u, v) + h(\operatorname{proj}_N v, u) \qquad \text{for every } (u,v) \in L \times L). \tag{116}$$

Then, $f$ is a bilinear form satisfying $f(L \times M) = 0$. Also,

$$f(v,v) = h(v,v) \qquad \text{for every } v \in L. \tag{117}$$

As a consequence, $I_f = I_h$ and $\operatorname{Cl}(L, f) = \operatorname{Cl}(L, h)$. Moreover, $I_{f+g} = I_{h+g}$, and $\operatorname{Cl}(L, f+g) = \operatorname{Cl}(L, h+g)$ for every bilinear form $g : L \times L \to k$. We also have $\alpha^f((\otimes L) \cdot M) = (\otimes L) \cdot M$. Finally, for every bilinear form $g : L \times L \to k$, the isomorphism $\overline{\alpha}_g^f : \operatorname{Cl}(L, g) \to \operatorname{Cl}(L, g+f)$ is a $k$-module isomorphism from $\operatorname{Cl}(L, g)$ to $\operatorname{Cl}(L, h+g)$ satisfying

$$\overline{\alpha}_g^f(\operatorname{Cl}(L,g) \cdot \varphi_g(M)) = \operatorname{Cl}(L, h+g) \cdot \varphi_{h+g}(M). \tag{118}$$

*Proof of Theorem 61.* **(c)** Clearly, both projections $\operatorname{proj}_M$ and $\operatorname{proj}_N$ are $k$-linear maps, and $h$ is bilinear (because $h$ is a bilinear form). This yields that the map $f$ is bilinear (because $f$ was defined by (116)).

Since $\operatorname{proj}_M$ is the projection from $L$ on $M$ with kernel $N$, we have $N = \operatorname{Ker} \operatorname{proj}_M$ and $M = \operatorname{proj}_M L$.

Since $\operatorname{proj}_N$ is the projection from $L$ on $N$ with kernel $M$, we have $M = \operatorname{Ker} \operatorname{proj}_N$ and $N = \operatorname{proj}_N L$.

Every $(u,v) \in L \times M$ satisfies $f(u,v) = 0$. [75] In other words, $f(L \times M) = 0$. According to Lemma 60 **(c)**, this yields $\alpha^f((\otimes L) \cdot M) = (\otimes L) \cdot M$.

[75] *Proof.* Let $(u,v) \in L \times M$ be arbitrary. Then, $u \in L$ and $v \in M$. Now, $u \in L$ leads to $\operatorname{proj}_M u \in \operatorname{proj}_M L = M$, so that $(\operatorname{proj}_M u, v) \in M \times M$ and thus $h(\operatorname{proj}_M u, v) \in h(M \times M) = 0$. In other words, $h(\operatorname{proj}_M u, v) = 0$. Now,

$$f(u,v) = \underbrace{h(\operatorname{proj}_M u, v)}_{=0} + h\left(\underbrace{\operatorname{proj}_N v}_{=0 \text{ (since } v \in M = \operatorname{Ker} \operatorname{proj}_N)}, u\right) = 0 + \underbrace{h(0,u)}_{=0 \text{ (since } h \text{ is bilinear)}} = 0 + 0 = 0,$$

qed.

Every $v \in L$ satisfies $v = \operatorname{proj}_M v + \operatorname{proj}_N v$. [76] But every $v \in L$ satisfies

$$\begin{aligned} f(v,v) &= h(\operatorname{proj}_M v, v) + h(\operatorname{proj}_N v, v) && \text{(by (116), applied to } u = v\text{)} \\ &= h\left(\underbrace{\operatorname{proj}_M v + \operatorname{proj}_N v}_{=v}, v\right) && \text{(since } h \text{ is bilinear)} \\ &= h(v,v). \end{aligned}$$

This proves (117).

Every bilinear form $g : L \times L \to k$ satisfies

$$(f+g)(v,v) = \underbrace{f(v,v)}_{=h(v,v) \text{ (due to (117))}} + g(v,v) = h(v,v) + g(v,v) = (h+g)(v,v)$$

for every $v \in L$. Therefore, every bilinear form $g : L \times L \to k$ satisfies

$$\begin{aligned} I_{f+g} &= (\otimes L) \cdot \left\langle v \otimes v - \underbrace{(f+g)(v,v)}_{=(h+g)(v,v)} \mid v \in L \right\rangle \cdot (\otimes L) && \text{(by the definition of } I_{f+g}\text{)} \\ &= (\otimes L) \cdot \langle v \otimes v - (h+g)(v,v) \mid v \in L \rangle \cdot (\otimes L) = I_{h+g} \\ &\qquad \left(\begin{array}{c} \text{since } I_{h+g} = (\otimes L) \cdot \langle v \otimes v - (h+g)(v,v) \mid v \in L \rangle \cdot (\otimes L) \\ \text{by the definition of } I_{h+g} \end{array}\right) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Cl}(L, f+g) &= (\otimes L) \diagup I_{f+g} && \text{(by the definition of } \operatorname{Cl}(L, f+g)\text{)} \\ &= (\otimes L) \diagup I_{h+g} && \text{(since } I_{f+g} = I_{h+g}\text{)} \\ &= \operatorname{Cl}(L, h+g) \\ &\qquad \left(\text{since } \operatorname{Cl}(L, h+g) = (\otimes L) \diagup I_{h+g} \text{ by the definition of } \operatorname{Cl}(L, h+g)\right). \end{aligned}$$

Applying this to $g = \mathbf{0}$, we obtain that $I_{f+\mathbf{0}} = I_{h+\mathbf{0}}$ and $\operatorname{Cl}(L, f+\mathbf{0}) = \operatorname{Cl}(L, h+\mathbf{0})$. In other words, $I_f = I_h$ and $\operatorname{Cl}(L, f) = \operatorname{Cl}(L, h)$.

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[76] *Proof.* Let $v \in L$ be arbitrary. Then, $v \in L = M \oplus N$, so that there exist two elements $m \in M$ and $n \in N$ such that $v = m + n$. Consider these $m$ and $n$. We have $\operatorname{proj}_M m = m$ (since $m \in M$, while $\operatorname{proj}_M$ is a projection on $M$) and $\operatorname{proj}_M n = 0$ (since $n \in N = \operatorname{Ker} \operatorname{proj}_M$). Thus,

$$\begin{aligned} \operatorname{proj}_M \underbrace{v}_{=m+n} &= \operatorname{proj}_M (m+n) = \underbrace{\operatorname{proj}_M m}_{=m} + \underbrace{\operatorname{proj}_M n}_{=0} && \text{(since } \operatorname{proj}_M \text{ is } k\text{-linear)} \\ &= m. \end{aligned}$$

Further, $\operatorname{proj}_N n = n$ (since $n \in N$, while $\operatorname{proj}_N$ is a projection on $N$) and $\operatorname{proj}_N m = 0$ (since $m \in M = \operatorname{Ker} \operatorname{proj}_N$). Thus,

$$\begin{aligned} \operatorname{proj}_N \underbrace{v}_{=m+n} &= \operatorname{proj}_N (m+n) = \underbrace{\operatorname{proj}_N m}_{=0} + \underbrace{\operatorname{proj}_N n}_{=n} && \text{(since } \operatorname{proj}_N \text{ is } k\text{-linear)} \\ &= n. \end{aligned}$$

Now, $v = \underbrace{m}_{=\operatorname{proj}_M v} + \underbrace{n}_{=\operatorname{proj}_N v} = \operatorname{proj}_M v + \operatorname{proj}_N v$, qed.

Now let $g : L \times L \to k$ be any bilinear form. We know from Theorem 33 (applied to $f$ and $g$ instead of $g$ and $f$) that $\overline{\alpha}_g^f : \operatorname{Cl}(L,g) \to \operatorname{Cl}(L,g+f)$ is a $k$-module isomorphism. Since $\operatorname{Cl}(L,g+f) = \operatorname{Cl}(L,f+g) = \operatorname{Cl}(L,h+g)$, this yields that $\overline{\alpha}_g^f$ is a $k$-module isomorphism from $\operatorname{Cl}(L,g)$ to $\operatorname{Cl}(L,h+g)$. Now let us prove (118).

By the definition of $\varphi_g$, we have $\varphi_g = \operatorname{proj}_g \circ \operatorname{inj}$. Thus, $\varphi_g(M) = \left(\operatorname{proj}_g \circ \operatorname{inj}\right)(M) = \operatorname{proj}_g(\operatorname{inj} M) = \operatorname{proj}_g M$ (since we identify $M$ with $\operatorname{inj} M$). On the other hand, $\operatorname{Cl}(L,g) = \operatorname{proj}_g(\otimes L)$ (since $\operatorname{proj}_g$ is the projection $\otimes L \to \operatorname{Cl}(L,g)$, and therefore surjective). Thus,

$$\underbrace{\operatorname{Cl}(L,g)}_{=\operatorname{proj}_g(\otimes L)} \cdot \underbrace{\varphi_g(M)}_{=\operatorname{proj}_g M} = \left(\operatorname{proj}_g(\otimes L)\right) \cdot \left(\operatorname{proj}_g M\right) = \operatorname{proj}_g((\otimes L)\cdot M)$$
$$\left(\text{since } \operatorname{proj}_g \text{ is a } k\text{-algebra homomorphism}\right),$$

so that

$$\overline{\alpha}_g^f(\operatorname{Cl}(L,g)\cdot\varphi_g(M)) = \overline{\alpha}_g^f\left(\operatorname{proj}_g((\otimes L)\cdot M)\right) = \left(\overline{\alpha}_g^f \circ \operatorname{proj}_g\right)((\otimes L)\cdot M). \tag{119}$$

Now let us recall how the homomorphism $\overline{\alpha}_g^f$ was defined: It was defined as the $k$-module homomorphism $(\otimes L)\diagup I_g \to (\otimes L)\diagup I_{g+f}$ induced by the $k$-module homomorphism $\alpha^f : \otimes L \to \otimes L$. Thus, $\overline{\alpha}_g^f \circ \operatorname{proj}_g = \operatorname{proj}_{g+f} \circ \alpha^f$. Hence, (119) becomes

$$\overline{\alpha}_g^f(\operatorname{Cl}(L,g)\cdot\varphi_g(M)) = \underbrace{\left(\overline{\alpha}_g^f \circ \operatorname{proj}_g\right)}_{=\operatorname{proj}_{g+f}\circ\alpha^f}((\otimes L)\cdot M) = \left(\operatorname{proj}_{g+f}\circ\alpha^f\right)((\otimes L)\cdot M)$$
$$= \operatorname{proj}_{g+f}\left(\underbrace{\alpha^f((\otimes L)\cdot M)}_{=(\otimes L)\cdot M}\right) = \operatorname{proj}_{g+f}((\otimes L)\cdot M)$$
$$= \left(\operatorname{proj}_{g+f}(\otimes L)\right)\cdot\left(\operatorname{proj}_{g+f} M\right) \tag{120}$$
$$\left(\text{since } \operatorname{proj}_{g+f} \text{ is a } k\text{-algebra homomorphism}\right).$$

By the definition of $\varphi_{g+f}$, we have $\varphi_{g+f} = \operatorname{proj}_{g+f} \circ \operatorname{inj}$. Thus, $\varphi_{g+f}(M) = \left(\operatorname{proj}_{g+f}\circ\operatorname{inj}\right)(M) = \operatorname{proj}_{g+f}(\operatorname{inj} M) = \operatorname{proj}_{g+f} M$ (since we identify $M$ with $\operatorname{inj} M$). On the other hand, $\operatorname{Cl}(L,g+f) = \operatorname{proj}_{g+f}(\otimes L)$ (since $\operatorname{proj}_{g+f}$ is the projection $\otimes L \to \operatorname{Cl}(L,g+f)$, and therefore surjective).

But $\operatorname{Cl}(L,g+f) = \operatorname{Cl}(L,h+g)$. By the definition of $\varphi_{g+f}$, we have

$$\varphi_{g+f} = \left(\text{the canonical projection } \otimes L \to \underbrace{\operatorname{Cl}(L,g+f)}_{=\operatorname{Cl}(L,h+g)}\right)$$
$$= (\text{the canonical projection } \otimes L \to \operatorname{Cl}(L,h+g)) = \varphi_{h+g}$$

(since $\varphi_{h+g}$ was defined as the canonical projection $\otimes L \to \operatorname{Cl}(L,h+g)$).

Thus, (120) becomes

$$\overline{\alpha}_g^f(\operatorname{Cl}(L,g)\cdot\varphi_g(M)) = \underbrace{\left(\operatorname{proj}_{g+f}(\otimes L)\right)}_{=\operatorname{Cl}(L,g+f)=\operatorname{Cl}(L,h+g)} \cdot \underbrace{\left(\operatorname{proj}_{g+f} M\right)}_{=\varphi_{g+f}(M)}$$
$$= \operatorname{Cl}(L,h+g)\cdot\underbrace{\varphi_{g+f}}_{=\varphi_{h+g}}(M) = \operatorname{Cl}(L,h+g)\cdot\varphi_{h+g}(M).$$

This proves Theorem 61 **(c)**.

**(a)** Let $g : L \times L \to k$ be a bilinear form. Define a bilinear form $f : L \times L \to k$ as in Theorem 61 **(c)**.

We know (from Theorem 61 **(c)**) that $\overline{\alpha}_g^f$ is a $k$-module isomorphism $\operatorname{Cl}(L, g) \to \operatorname{Cl}(L, h+g)$ which sends the $k$-submodule $\operatorname{Cl}(L, g) \cdot \varphi_g(M)$ of $\operatorname{Cl}(L, g)$ to the $k$-submodule $\operatorname{Cl}(L, h+g) \cdot \varphi_{h+g}(M)$ of $\operatorname{Cl}(L, h+g)$ (due to (118)). Thus, there exists a $k$-module isomorphism $\operatorname{Cl}(L, g) \to \operatorname{Cl}(L, h+g)$ which sends the $k$-submodule $\operatorname{Cl}(L, g) \cdot \varphi_g(M)$ of $\operatorname{Cl}(L, g)$ to the $k$-submodule $\operatorname{Cl}(L, h+g) \cdot \varphi_{h+g}(M)$ of $\operatorname{Cl}(L, h+g)$. This proves Theorem 61 **(a)**.

**(b)** Applying Theorem 61 **(a)** to $g = \mathbf{0}$, we conclude that there exists a $k$-module isomorphism $\operatorname{Cl}(L, \mathbf{0}) \to \operatorname{Cl}(L, h+\mathbf{0})$ which sends the $k$-submodule $\operatorname{Cl}(L, \mathbf{0}) \cdot \varphi_{\mathbf{0}}(M)$ of $\operatorname{Cl}(L, \mathbf{0})$ to the $k$-submodule $\operatorname{Cl}(L, h+\mathbf{0}) \cdot \varphi_{h+\mathbf{0}}(M)$ of $\operatorname{Cl}(L, h+\mathbf{0})$.

Since $\operatorname{Cl}(L, \mathbf{0}) = \wedge L$ and $h + \mathbf{0} = h$, this rewrites as follows: There exists a $k$-module isomorphism $\wedge L \to \operatorname{Cl}(L, h)$ which sends the $k$-submodule $(\wedge L) \cdot \varphi_{\mathbf{0}}(M)$ of $\operatorname{Cl}(L, \mathbf{0})$ to the $k$-submodule $\operatorname{Cl}(L, h) \cdot \varphi_h(M)$ of $\operatorname{Cl}(L, h)$.

This isomorphism therefore induces an isomorphism between the factor module $(\wedge L) \diagup ((\wedge L) \cdot \varphi_{\mathbf{0}}(M))$ and the factor module $(\operatorname{Cl}(L, h)) \diagup (\operatorname{Cl}(L, h) \cdot \varphi_h(M))$. We thus have

$$\begin{aligned}
&(\operatorname{Cl}(L, h)) \diagup (\operatorname{Cl}(L, h) \cdot \varphi_h(M)) \\
&\cong (\wedge L) \diagup ((\wedge L) \cdot \varphi_{\mathbf{0}}(M)) \\
&\cong \wedge (L \diagup M) \qquad \left( \begin{array}{c} \text{due to Corollary 80 } \textbf{(b)} \text{ from [5] (applied to} \\ L \text{ and } M \text{ instead of } V \text{ and } W) \end{array} \right) \\
&\cong \wedge N \qquad (\text{since } L = M \oplus N \text{ yields } L \diagup M \cong N)
\end{aligned}$$

as $k$-modules. This proves Theorem 61 **(b)**.

Note that Theorem 61 **(b)** was inspired by the results of the paper [6] by Calaque, Căldăraru and Tu. They considered, instead of a bilinear form $h$, a Lie bracket on $L$, and instead of $h(M \times M) = 0$ they required $[M, M] \subseteq M$. In this situation, analogues of Theorem 61 **(b)** for the universal enveloping algebra instead of the Clifford algebra were shown; however, these analogues are much harder and require some additional conditions.

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