Noncommutative birational rowmotion on a rectangle

A case study in noncommutative dynamics

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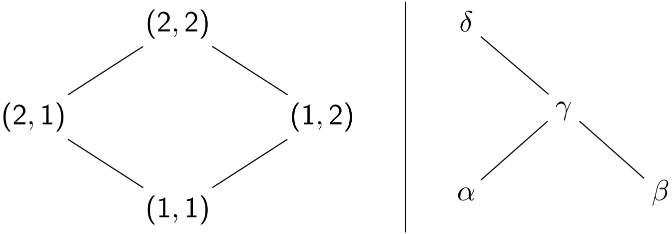
slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/cap2021.pdf

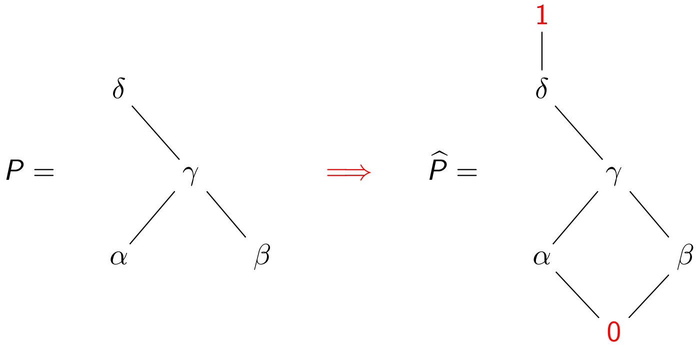
- A **poset** (= partially ordered set) is a set P with a reflexive, transitive and antisymmetric relation.
- We use the symbols $<$, $\leq$, $>$ and $\geq$ accordingly.
- We draw posets as Hasse diagrams:



- We only care about finite posets here.
- We say that $u \in P$ **is covered by** $v \in P$ (written $u \lessdot v$) if we have $u < v$ and there is no $w \in P$ satisfying $u < w < v$.
- We say that $u \in P$ **covers** $v \in P$ (written $u \gtrdot v$) if we have $u > v$ and there is no $w \in P$ satisfying $u > w > v$.

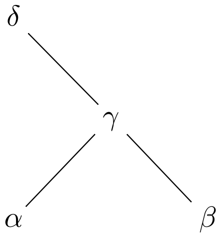
- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to P and forcing
  - 0 to be less than every other element, and
  - 1 to be greater than every other element.

Example:



More poset basics: linear extensions

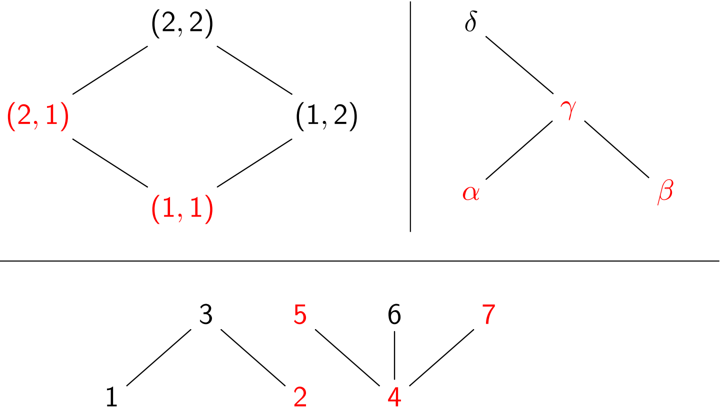
- A **linear extension** of P means a list $(v_1, v_2, \ldots, v_n)$ of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- For instance,



  has two linear extensions $(\alpha, \beta, \gamma, \delta)$ and $(\beta, \alpha, \gamma, \delta)$.
- Every finite poset has at least one linear extension.

More poset basics: order ideals

- An **order ideal** of a poset P is a subset S of P such that if $v \in S$ and $w \leq v$, then $w \in S$.
- Examples (the elements of the order ideal are marked in red):



- We let $J(P)$ denote the set of all order ideals of P.

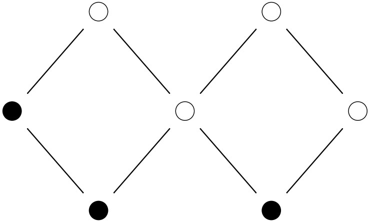
- **Classical rowmotion** is the rowmotion studied by Striker/Williams (arXiv:1108.1172). It has appeared many times before, under different guises:
  - Brouwer/Schrijver (1974) (as a permutation of the antichains),
  - Fon-der-Flaass (1993) (as a permutation of the antichains),
  - Cameron/Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
  - Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).

- Let P be a finite poset. **Classical rowmotion** is the map $\mathbf{r} : J(P) \to J(P)$ which sends every order ideal S to a new order ideal $\mathbf{r}(S)$ defined as follows:
  - **Invert colors** (i.e., take the complement $P \setminus S$).
  - **Boil down to generators** (i.e., take the set M of minimal elements of this complement).
  - **Complete downwards** (i.e., take the set J of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

  Then, $\mathbf{r}(S) = J$.

Example:

Let S be the following order ideal ($\bullet$ = inside order ideal):



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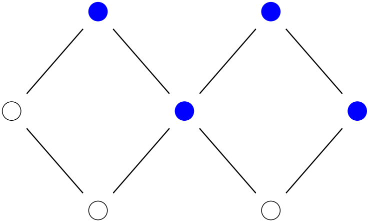
Example:

Mark the elements of the complement blue.



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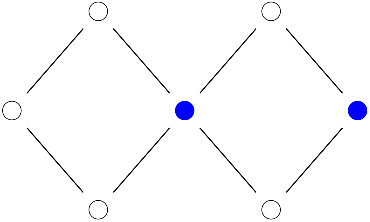
Example:

Leave only the minimal elements:



Classical rowmotion: the standard definition

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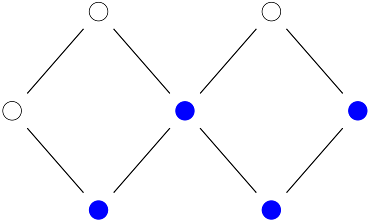
  Then, $\mathbf{r}(S) = J$.

Example:

$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):



Classical rowmotion: properties

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

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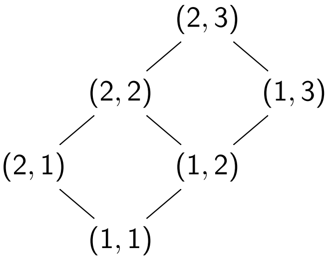
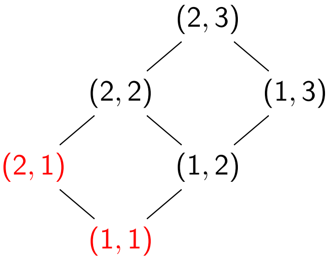
However, **for some types of** P, the order can be explicitly computed or bounded from above.
See Striker/Williams (arXiv:1108.1172) for an exposition of known results.

- If P is a $p \times q$-rectangle:



(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

Example:
Let S be the order ideal of the 2×3-rectangle given by:



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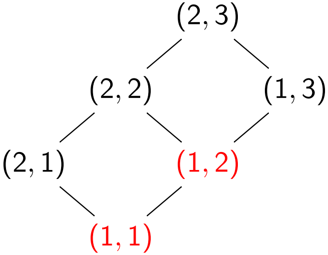

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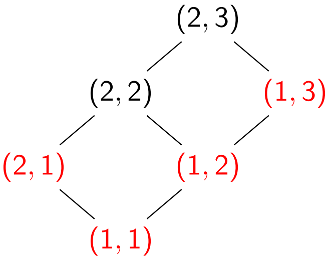
$\mathbf{r}^2(S)$ is



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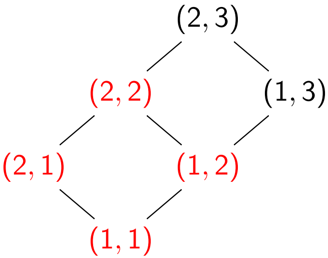

Classical rowmotion: properties

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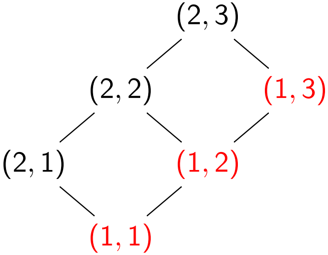
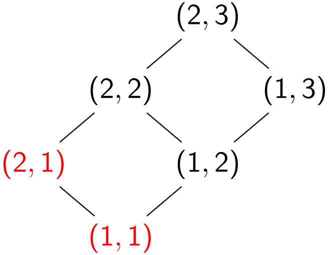


Example:

$\mathbf{r}^5(S)$ is

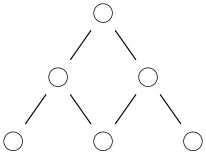


which is precisely the S we started with.

$\text{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$

Further posets for which classical rowmotion has small order:

- If P is a Δ-shaped triangle with sidelength $p-1$:



(shown here for $p=4$), then $\operatorname{ord}(\mathbf{r}) = 2p$ (if $p>2$).

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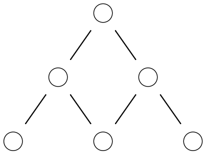

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- In this case, $\mathbf{r}^p$ is "reflection in the y-axis" (i.e., the central vertical axis).
- More general examples come from finite Weyl groups (Armstrong/Stump/Thomas, arXiv:1101.1277) and from minuscule weights of classical groups (Rush/Shi, arXiv:1108.5245; Okada, arXiv:2004.05364).

There is an alternative definition of classical rowmotion, which splits it into many little steps.

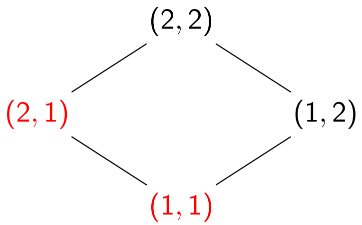
- If P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Simpler way to state this: $\mathbf{t}_v(S)$ is:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - S otherwise.

  ("Try to add or remove v from S; if this breaks the order ideal axiom, leave S fixed.")

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

Example:
Start with this order ideal S:



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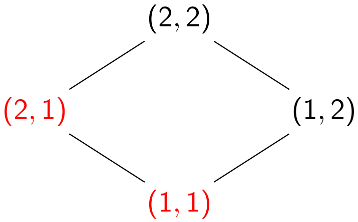
Example:
First apply $\mathbf{t}_{(2,2)}$, which changes nothing:



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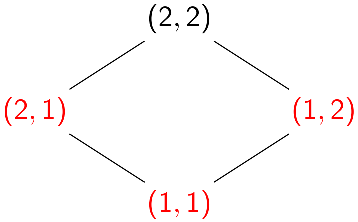
Then apply $\mathbf{t}_{(1,2)}$, which adds $(1,2)$ to the order ideal:



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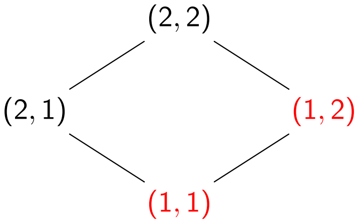
Then apply $\mathbf{t}_{(2,1)}$, which removes $(2,1)$ from the order ideal:



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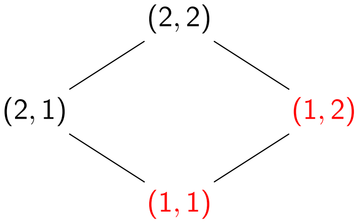

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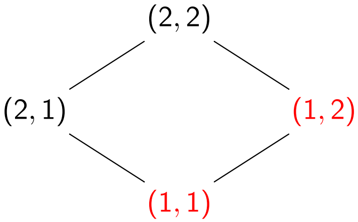
Example:
So this is $\mathbf{r}(S)$:

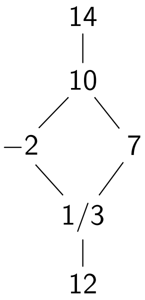


Goals of the talk

- define **noncommutative birational rowmotion**: a generalization of classical rowmotion on several levels, due to David Einstein, James Propp, Tom Roby and myself, based on ideas of Anatol Kirillov and Arkady Berenstein.
- discuss in detail how the "order $p + q$" theorem for rectangles generalizes to it.
- ask some questions.

- Let $\mathbb{K}$ be a ring (not necessarily commutative).
- A $\mathbb{K}$-**labelling of** P will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

Example: This is a $\mathbb{Q}$-labelling of the 2×2-rectangle:



- For any $v \in P$, define the **birational v-toggle** as the partial map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

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- This is a **partial** map. If any of the inverses does not exist in $\mathbb{K}$, then $T_v f$ is undefined!

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- Notice that this is a **local change** to the label at v; all other labels stay the same.
- If $\mathbb{K}$ is commutative, then $T_v^2 = \mathrm{id}$ (on the range of T_v).

- We define **(noncommutative) birational rowmotion** as the partial map

$$R := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent on the linear extension, because:

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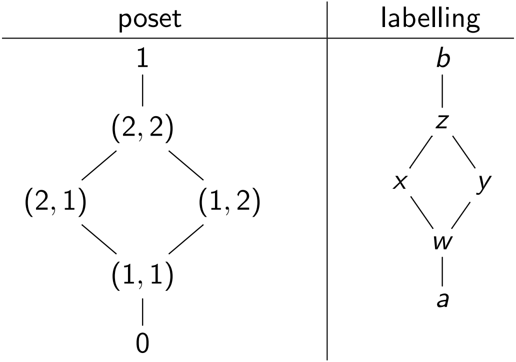
- This is indeed independent on the linear extension, because:
  - T_v and T_w commute whenever v and w are incomparable (or just don't cover each other);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



Birational rowmotion: example

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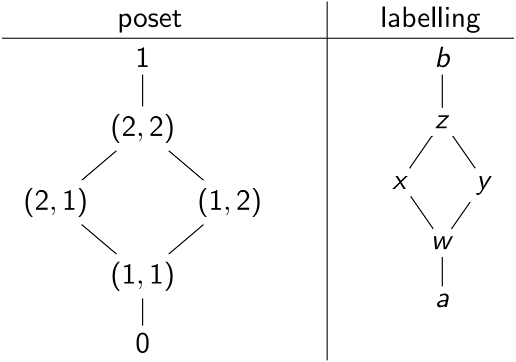

We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1),(1,2),(2,1),(2,2))$).
That is, toggle in the order "top, left, right, bottom".

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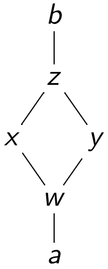
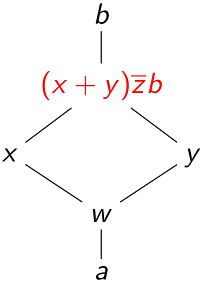
original labelling f	labelling $T_{(2,2)} f$
	

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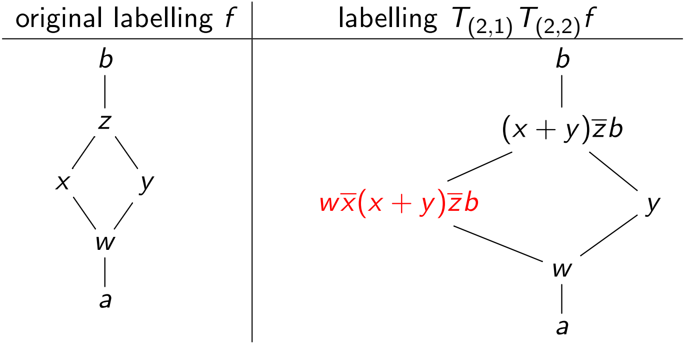

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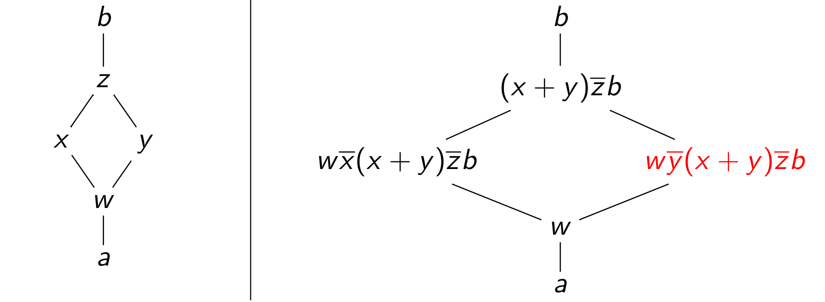

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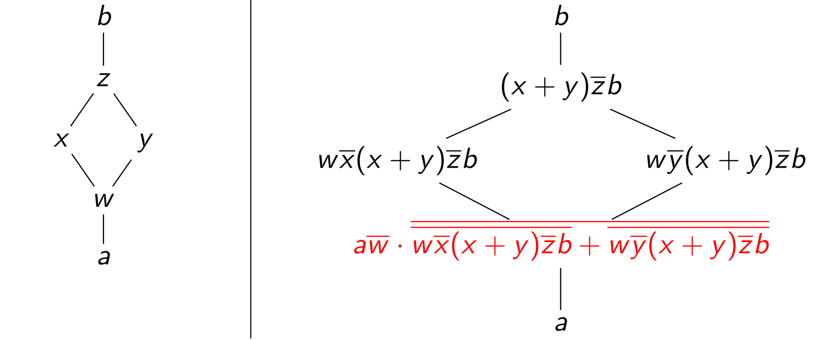

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Birational rowmotion: example

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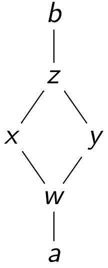	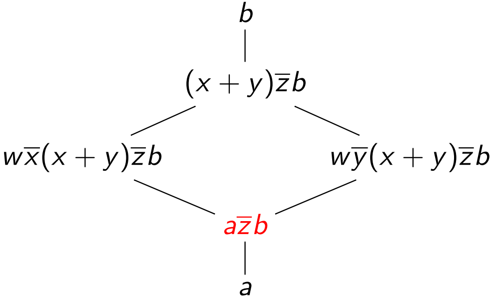

We have used $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ and simplified the result.

- Why is this called birational rowmotion?
- Indeed, it generalizes classical rowmotion of order ideals:
  - Let $\operatorname{Trop}\mathbb{Z}$ be the **tropical semiring** over $\mathbb{Z}$. This is the set $\mathbb{Z} \cup \{-\infty\}$ with "addition" $(a, b) \mapsto \max\{a, b\}$ and "multiplication" $(a, b) \mapsto a + b$. This is a semifield.

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  - To every order ideal $S\in J(P)$, assign a $\operatorname{Trop}\mathbb{Z}$-labelling $\operatorname{tlab} S$ defined by

$$(\operatorname{tlab} S)(v)=\begin{cases}1, & \text{if } v\notin S\cup\{0\}\,;\\ 0, & \text{if } v\in S\cup\{0\}\,.\end{cases}$$

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  - Don't like semifields? Use $\mathbb{Q}$ and take the "tropical limit".

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  - More generally, if P is the minuscule poset associated to a minuscule weight λ of a finite-dimensional simple Lie algebra $\mathfrak{g}$, then $R^{h} = \text{id}$, where h is the Coxeter number of $\mathfrak{g}$. (Soichi Okada, doi:10.37236/9557 .)

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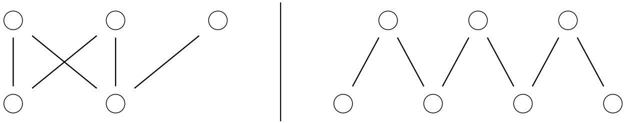
  - If P is an "n-graded forest" (a forest with all leaves having rank n), then $R^{\ell} = \text{id}$ for $\ell = \text{lcm}\,(1, 2, \ldots, n+1)$.

- In general, even if $\mathbb{K}$ is commutative, R can have infinite order – e.g., for the following two posets:



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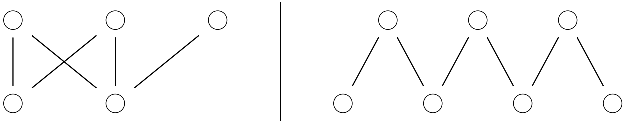

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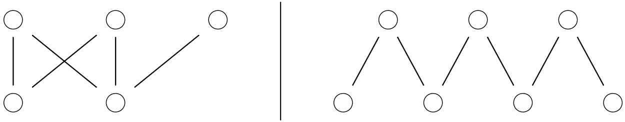

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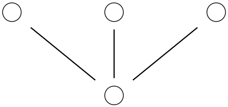

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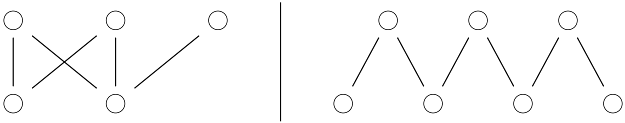

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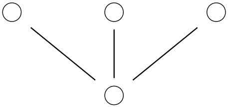

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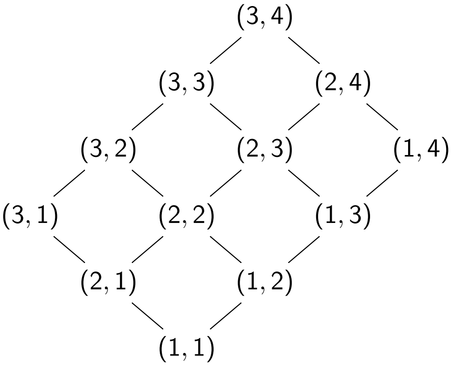
- However, not all is lost!

- Let p and q be two positive integers. Let $\mathbb{K}$ be a ring. Let P be the $p \times q$-rectangle poset: i.e.,

$$P := [p] \times [q], \qquad \text{where } [m] := \{1, 2, \ldots, m\}.$$

(The order on P is entrywise.)
Example: For $p = 3$ and $q = 4$, this is



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Periodicity theorem (* 2015, † 2021+ G & Roby):

If a and b are invertible and $R^{p+q} f$ is well-defined, then

$$\left(R^{p+q} f\right)(x) = a\overline{b} \cdot f(x) \cdot \overline{a}b \qquad \text{for each } x \in \widehat{P}.$$

Note that $a\overline{b} \cdot f(x) \cdot \overline{a}b$ is **not** generally conjugate to $f(x)$.

Birational rowmotion: the rectangle case

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Reciprocity theorem (* 2015, † 2021+ G & Roby):

Let $\ell \in \mathbb{N}$. If $R^\ell f$ is well-defined and $\ell \geq i + j - 1$, then

$$\left(R^\ell f\right)(i,j) = a \cdot \overline{\left(R^{\ell - i - j + 1} f\right) \underbrace{(p + 1 - i, q + 1 - j)}_{= \text{antipode of } (i,j) \text{ in } P}} \cdot b$$

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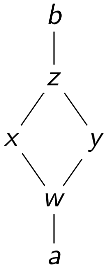
- **Example:** Iteratively apply R to a labelling of the 2×2-rectangle.
  Here is $R^0 f$:



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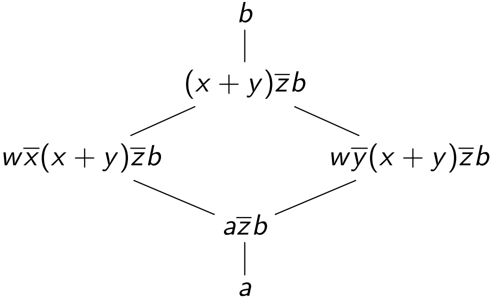

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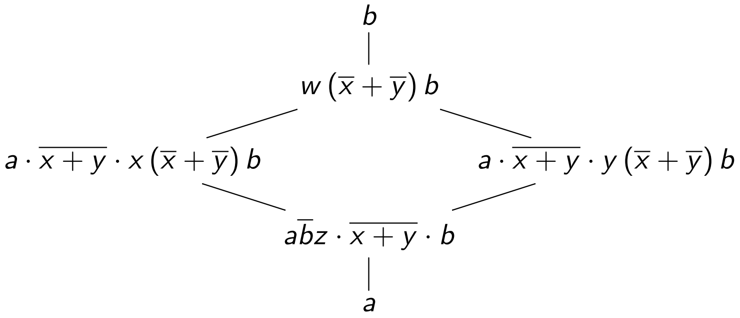

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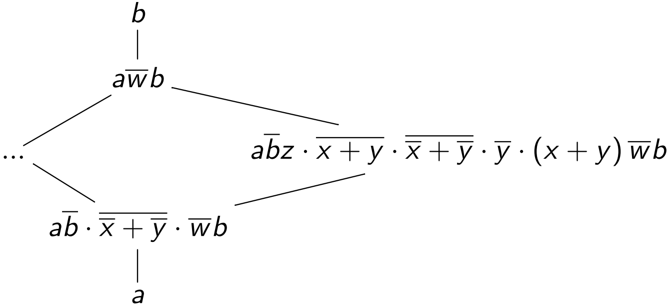

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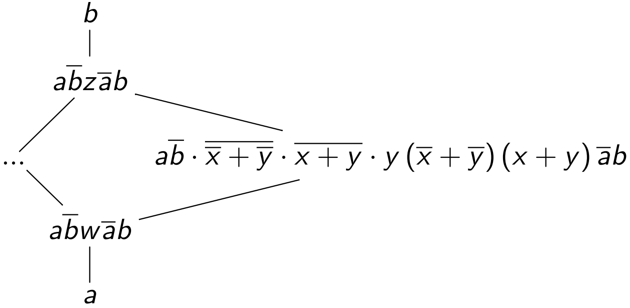

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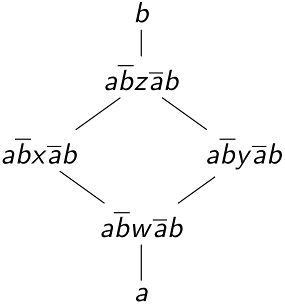

(after nontrivial simplifications).

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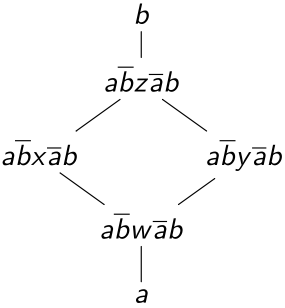

  This confirms the periodicity theorem for $p = q = 2$.
- Note that this is similar to Kontsevich's periodicity conjecture, proved by Iyudu/Shkarin (arXiv:1305.1965).

Birational rowmotion: the rectangle case, example

- Here are $R^0 f, R^1 f, \ldots, R^4 f$ for a generic $f \in \mathbb{K}^{\widehat{[2]\times[2]}}$ again, this time fully simplified and with the $f(0) = a$ and $f(1) = b$ labels removed:

$$R^0 f = \begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array} \quad ; \quad R^1 f = \begin{array}{ccc} & (x+y)\,\overline{z}b & \\ w\overline{x}\,(x+y)\,\overline{z}b & & w\overline{y}\,(x+y)\,\overline{z}b \\ & a\overline{z}b & \end{array}$$

$$R^2 f = \begin{array}{ccc} & w\,(\overline{x}+\overline{y})\,b & \\ a\overline{y}b & & a\overline{x}b \\ & a\overline{b}z\overline{x+y}b & \end{array} \quad ; \quad R^3 f = \begin{array}{ccc} & a\overline{w}b & \\ a\overline{b}z\overline{x+y}y\overline{w}b & & a\overline{b}z\overline{x+y}x\overline{w}b \\ & a\overline{b}\cdot\overline{\overline{x}+\overline{y}}\cdot\overline{w}b & \end{array}$$

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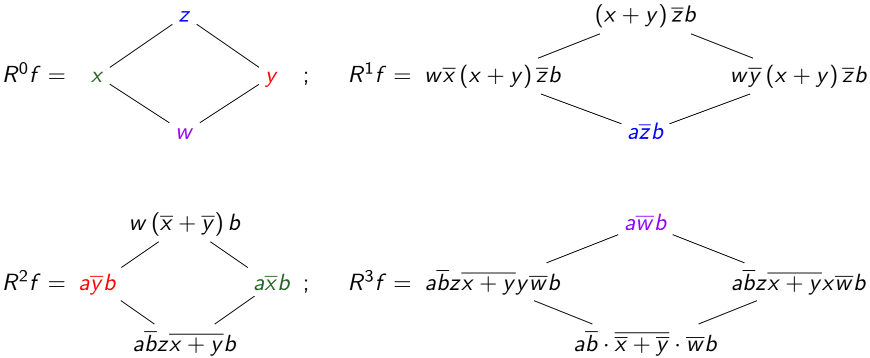

Equally colored labels are related by reciprocity. Can you spot some more?

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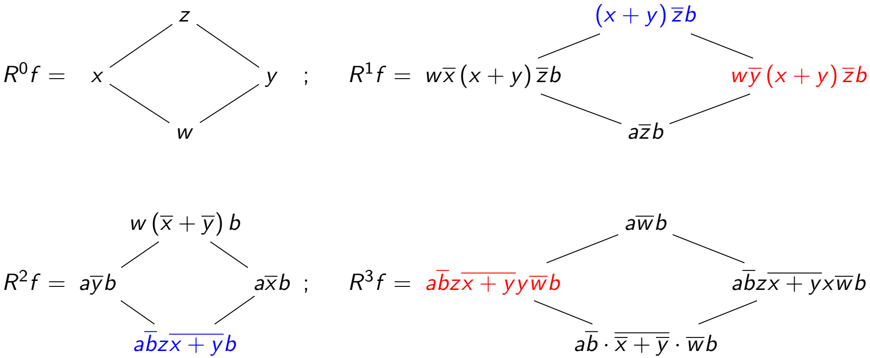

Here are some more instances of reciprocity. (There are more.)

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    Explicitly, if $A \in \mathbb{K}^{p\times(p+q)}$ is any matrix, then $(\operatorname{Grasp}_0 A)(0) = (\operatorname{Grasp}_0 A)(1) = 1$ and

$$(\operatorname{Grasp}_0 A)(i,j) = \frac{\det(A[1:i \mid i+j-1:p+j])}{\det(A[0:i \mid i+j:p+j])}$$

    for all $(i,j) \in P$, where the $A[a:b \mid c:d]$s are certain submatrices of A. (Note that this map Grasp_0 actually factors through the Grassmannian.)

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  - Reciprocity also easy using Grasp_0.

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  The "algebra" works!
- Unfortunately, the technical parts no longer work:
  - What does "almost all" mean for noncommutative $\mathbb{K}$?

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- **In some sense, yes:** Replace determinants by quasideterminants (Gelfand/Retakh, arXiv:q-alg/9705026; see also arXiv:math/0208146).

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  $$(\operatorname{Grasp}_0 A)(i,j) = (-1)^i \, q_{0,i+j-1}^{\{1:i|i+j:p+j\}}(A).$$

  The "algebra" works!
- Unfortunately, the technical parts no longer work:
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    No: e.g., the identity $x\overline{y}\overline{x}y = 1$ holds in all skew fields but not in all rings.
- We now believe this approach is a dead end.

- New proofs of periodicity and reciprocity in the commutative-$\mathbb{K}$ case were found by Gregg Musiker and Tom Roby in arXiv:1801.03877.
  They proceed by giving an explicit formula for $\left(R^k f\right)(i,j)$.
  For instance, $\left(R^3 f\right)(3,2)$

$$= \frac{1}{A_{02}+A_{11}+A_{20}} (A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}),$$

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- Lattice paths can be generalized to noncommutative $\mathbb{K}$, but NILPs? Unclear in what order to multiply different paths.

What now?

- We are back at square 1: no known theory available.

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- Let's play around with the setting.
  Step 1: Introduce notations...

A new beginning

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Thus, $x_0 = f(x)$ and $0_{\ell} = a$ and $1_{\ell} = b$.

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- In other words,

$$v_1 = \left(\sum_{u \lessdot v} u_0\right) \cdot \overline{v_0} \cdot \overline{\sum_{u \gtrdot v} \overline{u_1}} \qquad \text{for each } v \in P.$$

Transition equation

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- We haven't done anything serious yet, just rewritten the setup using the (more convenient) $x_\ell := \left(R^\ell f\right)(x)$ notation.

Simplifying the goal

- We must prove:

periodicity: $x_{p+q} = a\overline{b} \cdot x_0 \cdot \overline{a}b$;

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- Moreover, reciprocity in general follows from reciprocity for $\ell = i + j - 1$ (just apply it to $R^k f$ instead of f otherwise).

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Paths, Deltas and Nablas

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  Induction step: Given $v \in P$ such that $\nabla^{1\to u} = b\overline{u_\ell}$ for all $u \gtrdot v$. Since any path $1 \to v$ passes through a unique $u \gtrdot v$, we have
  $$\nabla^{1\to v} = \sum_{u \gtrdot v} \nabla^{1\to u}\nabla^v = \sum_{u \gtrdot v} b\overline{u_\ell}\nabla^v \qquad \text{(by induction hypothesis)}$$
  $$= b\overline{v_\ell} \qquad \text{(by definition of } \nabla^v\text{)}, \qquad \text{qed.}$$

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  (b) Analogous, but use upwards induction instead.

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  Similarly, **(d)** follows from **(b)**.

- **Transition equation in Δ-∇-form:**

$$\nabla_{\ell+1}^{v} = \Delta_{\ell}^{v} \qquad \text{for each } v \in \widehat{P} \text{ and } \ell \in \mathbb{N}.$$

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Take reciprocals on both sides, multiply by $\overline{\sum_{u \gtrdot v} \overline{u_{\ell+1}}}$ and rewrite using $\nabla^v_{\ell+1}$ and Δ^v_ℓ.

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- As a consequence of $\nabla^{v}_{\ell+1} = \Delta^{v}_{\ell}$, we have

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  Hence, $\nabla_{\ell+1}^{u \to v} = \Delta_{\ell}^{u \to v}$ for any $u, v \in \widehat{P}$.

- Now, for the bottommost element $(1, 1)$ of P, we have

$$\begin{aligned}
(1,1)_1 &= \overline{\nabla_1^{(p,q)\to(1,1)}} \cdot b && \text{(by path formula } \textbf{(c)}\text{)} \\
&= \overline{\Delta_0^{(p,q)\to(1,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\
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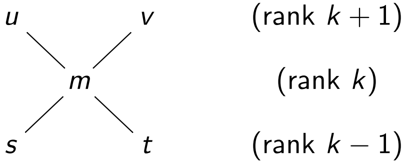
- What now?

The case $j = 1$ suffices: part 1

- We can simplify our goal one bit further. Consider the "neighborhood" of an element of our rectangle P:



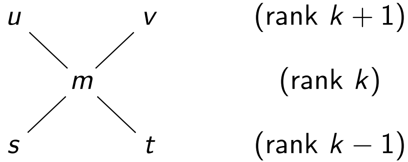
(where the **rank** of an $(i, j) \in P$ is defined to be $i + j - 1$). Say we have shown (our "induction hypotheses") that reciprocity holds for each of s, t, m, u; that is, we have

$$s_\ell = a \cdot \overline{s'_{\ell-(k-1)}} \cdot b, \qquad t_\ell = a \cdot \overline{t'_{\ell-(k-1)}} \cdot b,$$
$$m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b, \qquad u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b$$

for all sufficiently high ℓ, where x' denotes the antipode of x (that is, if $x = (i, j)$, then $x' = (p + 1 - i, q + 1 - j)$).

The case $j = 1$ suffices: part 1

- We can simplify our goal one bit further. Consider the "neighborhood" of an element of our rectangle P:



(where the **rank** of an $(i, j) \in P$ is defined to be $i + j - 1$).
Say we have shown (our "induction hypotheses") that reciprocity holds for each of s, t, m, u; that is, we have

$$s_\ell = a \cdot \overline{s'_{\ell-(k-1)}} \cdot b, \qquad t_\ell = a \cdot \overline{t'_{\ell-(k-1)}} \cdot b,$$
$$m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b, \qquad u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b$$

for all sufficiently high ℓ, where x' denotes the antipode of x (that is, if $x = (i, j)$, then $x' = (p + 1 - i, q + 1 - j)$).
Claim: Then, reciprocity also holds for v; that is, we have $v_\ell = a \cdot \overline{v'_{\ell-(k+1)}} \cdot b$ for all $\ell \geq k + 1$.

- *Proof idea.* Fix $\ell \geq k+1$, and compare the transition equations

$$m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}} \qquad \text{and}$$
$$m'_{\ell-k} = \left(u'_{\ell-k-1} + v'_{\ell-k-1}\right) \cdot \overline{m'_{\ell-k-1}} \cdot \overline{\overline{s'_{\ell-k}} + \overline{t'_{\ell-k}}}$$

using the induction hypotheses $m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b$,

$$s_{\ell-1} = a \cdot \overline{s'_{\ell-k}} \cdot b, \qquad t_{\ell-1} = a \cdot \overline{t'_{\ell-k}} \cdot b,$$
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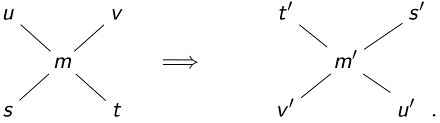

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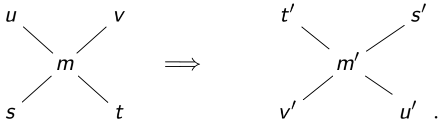

After subtracting $u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b$, out comes $v_\ell = a \cdot \overline{v'_{\ell-(k+1)}} \cdot b$.

The case $j = 1$ suffices: part 2

- *Proof idea.* Fix $\ell \geq k+1$, and compare the transition equations

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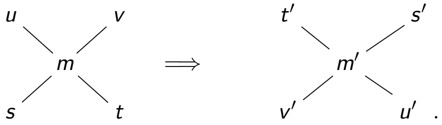

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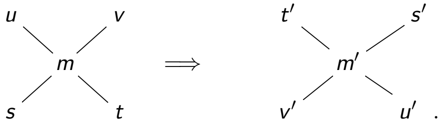

- This argument still works if s, t or u does not exist.
- Thus, in order to prove reciprocity for all (i, j), it suffices (by induction) to prove it in the case when $j = 1$.

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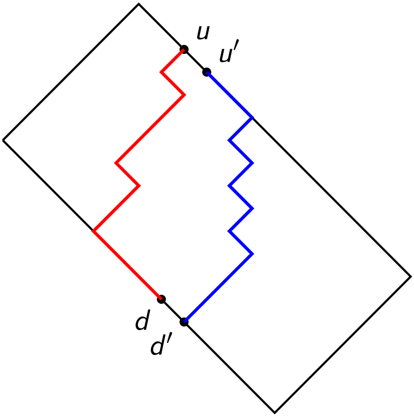
  Note the lack of rowmotion in this formula! The ℓ here is constantly 1, so it is a property of a single labeling. Thus, we drop the subscripts.
- **Our new goal:** Prove that

$$\Delta^{(p,q)\to(2,1)} = \nabla^{(p-1,q)\to(1,1)}.$$

- More generally:
- **Conversion lemma:** Let u and u' be two adjacent elements on the top-right edge of P (that is, $u = (k, q)$ and $u' = (k-1, q)$). Let d and d' be two adjacent elements on the bottom-left edge of P (that is, $d = (i, 1)$ and $d' = (i-1, 1)$). Then,



$$\Delta_\ell^{u \to d} = \nabla_\ell^{u' \to d'} \qquad \text{for each } \ell \in \mathbb{N}.$$

In short:

$$\Delta^{u \to d} = \nabla^{u' \to d'}.$$

- If we can prove the conversion lemma, we will obtain reciprocity not only for $(i, j) = (2, 1)$, but also for all (i, j) on the bottom-left edge of P (that is, for the entire case $j = 1$), because we can argue as follows:

$$\begin{aligned}
(i,1)_i &= \overline{\nabla_i^{(p,q)\to(i,1)}} \cdot b && \text{(by path formula } \mathbf{(c)}\text{)} \\
&= \overline{\Delta_{i-1}^{(p,q)\to(i,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\
&= \overline{\nabla_{i-1}^{(p-1,q)\to(i-1,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \overline{\Delta_{i-2}^{(p-1,q)\to(i-1,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\
&= \overline{\nabla_{i-2}^{(p-2,q)\to(i-2,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \cdots \\
&= \overline{\nabla_{1}^{(p-i+1,q)\to(1,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \overline{\Delta_{0}^{(p-i+1,q)\to(1,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\
&= a \cdot \overline{(p-i+1,q)_0} \cdot b && \text{(by path formula } \mathbf{(d)}\text{)}\,.
\end{aligned}$$

- This proves reciprocity

$$(i,1)_\ell = a \cdot \overline{(p-i+1,q)_{\ell-i}} \cdot b$$

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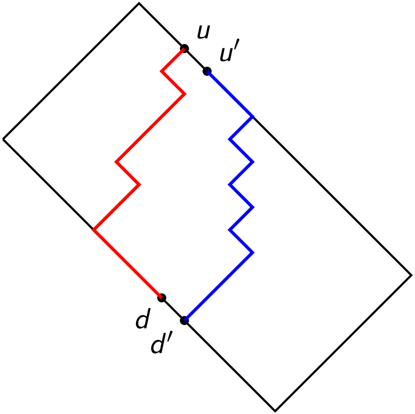
- Thus, we only need to prove the conversion lemma. We can now drop all subscripts forever!

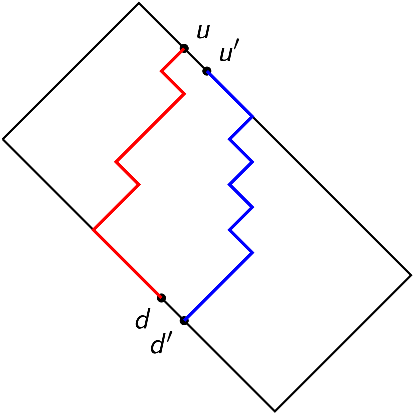
- Let us again look at the picture:



We must prove $\Delta^{u\to d} = \nabla^{u'\to d'}$.

Proving the conversion lemma: the intuition

- Let us again look at the picture:



We must prove $\Delta^{u \to d} = \nabla^{u' \to d'}$.

- How do we interpolate between paths $u \to d$ and paths $u' \to d'$?

- We define a **path-jump-path** to be a sequence

$$\mathbf{p} = \left(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k\right)$$

of elements of P, where the relation $x \blacktriangleright y$ means "y is one step down and some steps to the right of x" (that is, if $x = (r, s)$, then $y = (r - k, s + k - 1)$ for some $k > 0$). We say that this path-jump-path $\mathbf{p}$ has **jump at** i.

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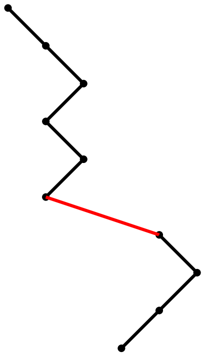
Example of a path-jump-path:



(The red edge is the jump.)

Proving the conversion lemma: path-jump-paths

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We say that this path-jump-path $\mathbf{p}$ has **jump at** i.
For any such path-jump-path $\mathbf{p}$, we set

$$E_{\mathbf{p}} := \Delta^{v_0} \Delta^{v_1} \cdots \Delta^{v_{i-1}} v_i \overline{v_{i+1}} \nabla^{v_{i+2}} \nabla^{v_{i+3}} \cdots \nabla^{v_k}.$$

(Here, we are omitting the ℓ subscripts – so v_i means $(v_i)_\ell$ and v_{i+1} means $(v_{i+1})_\ell$.)

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- Now, if $k = \operatorname{rank} u - \operatorname{rank}(d')$, then

$$\Delta^{u \to d} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } k-1}} E_{\mathbf{p}},$$

since $\Delta^d = d\overline{d'}$, and similarly

$$\nabla^{u' \to d'} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } 0}} E_{\mathbf{p}}.$$

Proving the conversion lemma: moving the jump

- So we need to show that

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for each $0 \leq i < k - 1$.

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- And yes, this is true and can be proved by a "local" argument (rewriting two consecutive steps of the path).
- This is similar to the "zipper argument" in lattice models. (Is there a Yang–Baxter equation lurking?)

Proving the conversion lemma: the civilized version, part 1

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- However, the path-jump-path argument is somewhat messy. We can make it slicker by rewriting it in matrix notation:
- Define three $P \times P$-matrices Δ, ∇ and U by

$$\Delta_{x,y} := \Delta^x [x \gtrdot y], \qquad \nabla_{x,y} := \nabla^y [x \gtrdot y],$$
$$U_{x,y} := x\overline{y} [x \blacktriangleright y] \qquad \text{for all } x, y \in P.$$

  Here, $[\mathcal{A}]$ is the Iverson bracket (i.e., truth value) of a statement $\mathcal{A}$; the relation $x \blacktriangleright y$ means "y is one step down and some steps to the right of x" as before. And again, we are omitting the ℓ subscripts, so $x\overline{y}$ actually means $x_\ell \overline{y_\ell}$.
- Now, we claim that

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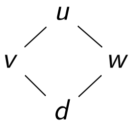
  Indeed, this follows easily from the following neat lemma: If



are four adjacent elements of P, then

$$\overline{w} \cdot \nabla^{d} \cdot d = \overline{u} \cdot \Delta^{u} \cdot v \qquad \text{and} \qquad \overline{v} \cdot \nabla^{d} \cdot d = \overline{u} \cdot \Delta^{u} \cdot w.$$

(The u and d here are unrelated to the u and d from the conversion lemma!)

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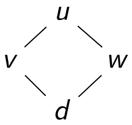

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- From $\Delta U = U \nabla$, we easily obtain

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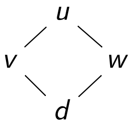

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- Setting $k = \operatorname{rank} u - \operatorname{rank} d$ and comparing the (u, d')-entries of both sides, we quickly obtain $\Delta^{u \to d} = \nabla^{u' \to d'}$ (since $x \blacktriangleright d'$ holds only for $x = d$, and since $u \blacktriangleright x$ holds only for $x = u'$). This proves the conversion lemma again.

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  In the **commutative** case, the theorems hold for semifields (and, more generally, commutative semirings) because they hold for fields and because they are "essentially" polynomial identities (once you clear denominators).
  This **fails** for noncommutative $\mathbb{K}$!
- **Scary example** (David Speyer, MathOverflow #401273): If x and y are two elements of a ring such that $x + y$ is invertible, then

$$x \cdot \overline{x+y} \cdot y = y \cdot \overline{x+y} \cdot x.$$

But this is not true if "ring" is replaced by "semiring"!

Is that all? Part 2: Questions

- Thus, we are left with a

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Are the periodicity and reciprocity theorems still true if "ring" is replaced by "semiring"? (I.e., we no longer require $\mathbb{K}$ to have a subtraction.)

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- We have partial results, e.g., for $p = q = 3$ and for $p = 2$.

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Are the periodicity and reciprocity theorems still true if "ring" is replaced by "semiring"? (I.e., we no longer require $\mathbb{K}$ to have a subtraction.)

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Question:

Are any other results like ours known in the noncommutative case?

Acknowledgments

- **Tom Roby**: collaboration
- **Mathematisches Forschungsinstitut Oberwolfach**: hospitality in July/August 2021
- **Gérard Duchamp, Maxim Kontsevich, Gleb Koshevoy, Hoang Ngoc Minh**: this conference
- **Sage and Sage-combinat**: computations
- **the birational combinatorics community**: keeping the subject exciting since 2013
- **you**: your patience

Some references

- David Einstein, James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains*, 2013. http://arxiv.org/abs/1310.5294
- David Einstein, James Propp, *Piecewise-linear and birational toggling*, 2014. https://arxiv.org/abs/1404.3455
- Darij Grinberg, Tom Roby, *Iterative properties of birational rowmotion*, 2014. http://arxiv.org/abs/1402.6178
- Michael Joseph, Tom Roby, *Birational and noncommutative lifts of antichain toggling and rowmotion*, 2019. https://arxiv.org/abs/1909.09658
- Michael Joseph, Tom Roby, *A birational lifting of the Stanley-Thomas word on products of two chains*, 2020. https://arxiv.org/abs/2001.03811
- Gregg Musiker, Tom Roby, *Paths to Understanding Birational Rowmotion on Products of Two Chains*, 2019. https://arxiv.org/abs/1801.03877

- **Zamolodchikov periodicity conjecture in type AA** (proved by A. Yu. Volkov, arXiv:hep-th/0606094v1): Let r and s be positive integers. Let $Y_{i,\ j,\ k}$ be elements of a commutative ring for $i \in [r]$ and $j \in [s]$ and $k \in \mathbb{Z}$. Assume that

$$Y_{i,\ j,\ k+1} Y_{i,\ j,\ k-1} = \frac{(1 + Y_{i+1,\ j,\ k})(1 + Y_{i-1,\ j,\ k})}{(1 + 1/Y_{i,\ j+1,\ k})(1 + 1/Y_{i,\ j-1,\ k})}$$

for all $i,\ j,\ k$, where sums involving "off-grid" points (e.g., $1 + Y_{0,\ j,\ k}$) are understood as 1.
Then, $Y_{i,\ j,\ k+2(r+s+2)} = Y_{i,\ j,\ k}$ for all $i,\ j,\ k$.

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- **Observation (Max Glick and others, ca. 2015?):** This is equivalent to periodicity of birational rowmotion ($R^{p+q} = 1$) for $[p] \times [q]$, where $p = r + 1$ and $q = s + 1$, when the ring is commutative. Explicitly,

$$Y_{i,\ j,\ i+j-2k} = (R^k f)(i, j+1) \ / \ (R^k f)(i+1, j).$$

  (Fine points omitted.)

The Y-system connection

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Question:

Can Zamolodchikov periodicity be generalized to noncommutative rings?