A quotient of the ring of symmetric functions generalizing quantum cohomology

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/cap2020.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf
overview: http://www.cip.ifi.lmu.de/~grinberg/algebra/fpsac19.pdf

What is this about?

- From a modern point of view, **Schubert calculus** (a.k.a. classical enumerative geometry, or Hilbert's 15th problem) is about two cohomology rings:

$$\mathrm{H}^*\left(\underbrace{\mathrm{Gr}\left(k,n\right)}_{\text{Grassmannian}}\right) \quad \text{and} \quad \mathrm{H}^*\left(\underbrace{\mathrm{Fl}\left(n\right)}_{\text{flag variety}}\right)$$

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- In this talk, we are concerned with the first.
- Classical result: as rings,

$$\begin{aligned} &\mathrm{H}^*\left(\mathrm{Gr}\left(k,n\right)\right) \\ &\cong \left(\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}\right) \\ &\qquad \diagup \left(h_{n-k+1}, h_{n-k+2}, \ldots, h_n\right)_{\text{ideal}}, \end{aligned}$$

where the h_i are complete homogeneous symmetric polynomials (to be defined soon).

- (Small) **Quantum cohomology** is a deformation of cohomology from the 1980–90s. For the Grassmannian, it is

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$$\begin{aligned}&\operatorname{QH}^*(\operatorname{Gr}(k,n))\\ &\cong (\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}[q])\\ &\qquad / \left(h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n + (-1)^k q\right)_{\text{ideal}}.\end{aligned}$$

- Many properties of classical cohomology still hold here. In particular: $\operatorname{QH}^*(\operatorname{Gr}(k,n))$ has a $\mathbb{Z}[q]$-module basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of (projected) Schur polynomials (to be defined soon), with λ ranging over all partitions with $\leq k$ parts and each part $\leq n-k$. The structure constants are the **Gromov–Witten invariants**. References:
  - Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, 1999.
  - Alexander Postnikov, *Affine approach to quantum Schubert calculus*, 2005.

Where are we going?

- **Goal:** Deform $\mathrm{H}^*(\mathrm{Gr}(k,n))$ using k parameters instead of one, generalizing $\mathrm{QH}^*(\mathrm{Gr}(k,n))$.

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- I will now start from scratch and define standard notations around symmetric polynomials, then introduce the deformed cohomology ring algebraically.
- There is a number of open questions and things to explore.

A more general setting: $\mathcal{P}$ and $\mathcal{S}$

- Let $\mathbf{k}$ be a commutative ring.
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- Let $\mathcal{S}$ denote the ring of *symmetric* polynomials in $\mathcal{P}$.
  These are the polynomials $f \in \mathcal{P}$ satisfying

  $$f(x_1, x_2, \ldots, x_k) = f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}\right)$$

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- **Theorem (Artin $\leq$1944):** The $\mathcal{S}$-module $\mathcal{P}$ is free with basis

$$\left(x^\alpha\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i} \qquad \text{(or, informally: "}\left(x_1^{<1} x_2^{<2} \cdots x_k^{<k}\right)\text{")}.$$

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  Example: For $k = 3$, this basis is $\left(1, x_3, x_3^2, x_2, x_2 x_3, x_2 x_3^2\right)$.

- The ring $\mathcal{S}$ of symmetric polynomials in $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$ has several bases, usually indexed by certain sets of (integer) partitions.
  First, let us recall what partitions are:

- A *partition* means a **weakly decreasing** sequence of nonnegative integers that has only finitely many nonzero entries.

k-partitions: definition

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  Examples: $(4, 2, 2, 0, 0, 0, \ldots)$ and $(3, 2, 0, 0, 0, 0, \ldots)$ and $(5, 0, 0, 0, 0, 0, \ldots)$ are three partitions.
  $(2, 3, 2, 0, 0, 0, \ldots)$ and $(2, 1, 1, 1, \ldots)$ are not.

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- Thus there is a bijection

$$\{k\text{-partitions}\} \to \{\text{partitions with at most } k \text{ nonzero entries}\},$$
$$\lambda \mapsto (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, 0, \ldots).$$

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  $(2, 3, 2)$ is not.
- If $\lambda \in \mathbb{N}^k$ is a k-partition, then its *Young diagram $Y(\lambda)$* is defined as a table made out of k left-aligned rows, where the i-th row has λ_i boxes.
  Example: If $k = 6$ and $\lambda = (5, 5, 3, 2, 0, 0)$, then

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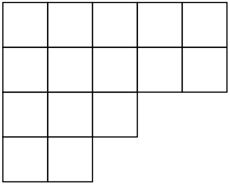

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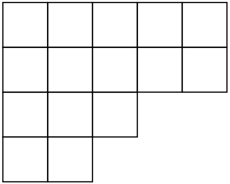
  $Y(\lambda) =$  .

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- The same convention applies to partitions.

Symmetric polynomials: the e-basis

- For each $m \in \mathbb{Z}$, we let e_m denote the m-th *elementary symmetric polynomial*:

$$e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

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- Note that $e_m = 0$ when $m > k$.

- For each $m \in \mathbb{Z}$, we let h_m denote the m-th *complete homogeneous symmetric polynomial*:

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- **Equivalent restatement:** $(h_\lambda)_{\lambda \text{ is a partition whose entries are } \leq k}$ is a basis of the **k**-module $\mathcal{S}$.
- **Theorem:** $(h_\lambda)_{\lambda \text{ is a } k\text{-partition}}$ is a basis of the **k**-module $\mathcal{S}$. (Another basis!)

- For each k-partition λ, we let s_λ be the *λ-th Schur polynomial*:

$$s_\lambda = \frac{\det\left(\left(x_i^{\lambda_j+k-j}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right)}{\det\left(\left(x_i^{k-j}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right)} \qquad \text{(alternant formula)}$$

$$= \det\left(\left(h_{\lambda_i - i + j}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right) \qquad \text{(Jacobi-Trudi)}\ .$$

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- **Theorem:** The equality above holds, and s_λ is a symmetric polynomial with nonnegative coefficients. Explicitly,

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard } \lambda\text{-tableau} \\ \text{with entries } 1,2,\ldots,k}} \prod_{i=1}^{k} x_i^{(\text{number of } i\text{'s in } T)},$$

where a *semistandard λ-tableau with entries $1, 2, \ldots, k$* is a way of putting an integer $i \in \{1, 2, \ldots, k\}$ into each box of $Y(\lambda)$ such that the entries **weakly** increase along rows and **strictly** increase along columns.

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- **Theorem:** $(s_\lambda)_{\lambda \text{ is a } k\text{-partition}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$.

- If λ and μ are two k-partitions, then the product $s_\lambda s_\mu$ can be again written as a **k**-linear combination of Schur polynomials (since these form a basis):

$$s_\lambda s_\mu = \sum_{\nu \text{ is a } k\text{-partition}} c_{\lambda,\mu}^{\nu} s_\nu,$$

where the $c_{\lambda,\mu}^{\nu}$ lie in **k**. These $c_{\lambda,\mu}^{\nu}$ are called the *Littlewood-Richardson coefficients*.

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  where the $c_{\lambda,\mu}^{\nu}$ lie in **k**. These $c_{\lambda,\mu}^{\nu}$ are called the *Littlewood-Richardson coefficients*.
- **Theorem:** These Littlewood-Richardson coefficients $c_{\lambda,\mu}^{\nu}$ are nonnegative integers (and count something).

- We have defined
$$s_\lambda = \det\left(\left(h_{\lambda_i - i + j}\right)_{1 \leq i \leq k,\ 1 \leq j \leq k}\right)$$
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Apply the same definition to arbitrary $\lambda \in \mathbb{Z}^k$.

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  (So we get nothing really new.)

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  - If β has a negative entry, then $s_\alpha = 0$.
  - If β has two equal entries, then $s_\alpha = 0$.
  - Otherwise, let γ be the k-tuple obtained by sorting β in decreasing order, and let σ be the permutation of the indices that causes this sorting. Let λ be the k-partition $(\gamma_1 - (k-1), \gamma_2 - (k-2), \ldots, \gamma_k - (k-k))$. Then, $s_\alpha = (-1)^\sigma s_\lambda$.

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- Also, the alternant formula still holds if all $\lambda_i + (k-i)$ are ≥ 0.

A more general setting: $a_1, a_2, \ldots, a_k$ and J

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$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

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- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{P} / J$ is free with basis

$$\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$$
$$\left(\text{informally: } \text{``}\left(\overline{x_1^{<n-k+1} x_2^{<n-k+2} \cdots x_n^{<n}}\right)\text{''}\right)$$

where the overline $\overline{}$ means "projection" onto whatever quotient we need (here: from $\mathcal{P}$ onto $\mathcal{P} / J$).
(This basis has $n(n-1)\cdots(n-k+1)$ elements.)

A slightly less general setting: symmetric $a_1, a_2, \ldots, a_k$ and J

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  (Same differences as for J, but we are generating an ideal of $\mathcal{S}$ now.)
- Let $\omega = \underbrace{(n-k, n-k, \ldots, n-k)}_{k \text{ entries}}$ and

$$P_{k,n} = \{\lambda \text{ is a } k\text{-partition} \mid \lambda_1 \leq n-k\} = \{k\text{-partitions } \lambda \subseteq \omega\}.$$

- Here, for two k-partitions α and β, we say that $\alpha \subseteq \beta$ if and only if $\alpha_i \leq \beta_i$ for all i.
- **Theorem (G.):** The **k**-module $\mathcal{S} / I$ is free with basis

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An even less general setting: constant $a_1, a_2, \ldots, a_k$

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- This setting still is general enough to encompass ...
  - **classical cohomology:** If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S} / I$ becomes the cohomology ring $\mathrm{H}^* (\mathrm{Gr}(k, n))$; the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ corresponds to the Schubert classes.
  - **quantum cohomology:** If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then $\mathcal{S} / I$ becomes the quantum cohomology ring $\mathrm{QH}^* (\mathrm{Gr}(k, n))$.

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- The above theorem lets us work in these rings (and more generally) without relying on geometry.

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- For every k-partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we define

  $$\nu^\vee := (n - k - \nu_k, n - k - \nu_{k-1}, \ldots, n - k - \nu_1) \in P_{k,n}.$$

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- For any three k-partitions $\alpha, \beta, \gamma \in P_{k,n}$, let

$$g_{\alpha,\beta,\gamma} := \text{coeff}_{\gamma^\vee}\left(\overline{s_\alpha s_\beta}\right) \in \mathbf{k}.$$

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- **Equivalent restatement:** Each $\nu \in P_{k,n}$ and $f \in \mathcal{S} / I$ satisfy $\text{coeff}_\omega \left(\overline{s_\nu} f \right) = \text{coeff}_{\nu^\vee} (f)$.

The h-basis

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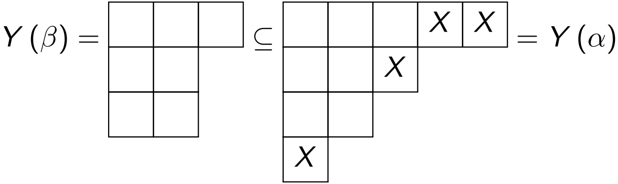
- **Proposition (G.):** Let m be a positive integer. Then,

$$\overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j \, a_{k-j} \overline{s_{(m,1^j)}},$$

where $\left(m, 1^j\right) := (m, \underbrace{1, 1, \ldots, 1}_{j \text{ ones}}, 0, 0, 0, \ldots)$ (a hook-shaped k-partition).

- If α and β are two k-partitions, then we say that *α / β is a horizontal strip* if and only if the Young diagram $Y(\alpha)$ is obtained from $Y(\beta)$ by adding some (possibly none) extra boxes with no two of these new boxes lying in the same column.

  Example: If $k = 4$ and $\alpha = (5, 3, 2, 1)$ and $\beta = (3, 2, 2, 0)$, then α / β is a horizontal strip, since



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- Furthermore, given $j \in \mathbb{N}$, we say that *α/β is a horizontal j-strip* if α/β is a horizontal strip and $|\alpha| - |\beta| = j$.
- **Theorem (Pieri).** Let λ be a k-partition. Let $j \in \mathbb{N}$. Then,

$$s_\lambda h_j = \sum_{\substack{\mu \text{ is a } k\text{-partition;} \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} s_\mu .$$

- **Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu / \lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{(n-k-j+1, 1^{i-1}), \nu} \overline{s_\nu}.$$

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- This generalizes the h-Pieri rule from Bertram, Ciocan-Fontanine and Fulton, but note that $c^{\lambda}_{(n-k-j+1,1^{i-1}),\nu}$ may be > 1.

A Pieri rule for $\mathcal{S}/I$: example

- **Example:** For $n = 7$ and $k = 3$, we have

$$\overline{s_{(4,3,2)} h_2} = \overline{s_{(4,4,3)}} + a_1 \left(\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}} \right)$$
$$- a_2 \left(\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}} \right)$$
$$+ a_3 \left(\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}} \right).$$

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- Multiplying by e_j appears harder: For $n = 5$ and $k = 3$, we have

$$\overline{s_{(2,2,1)} e_2} = a_1 \overline{s_{(2,2)}} - 2a_2 \overline{s_{(2,1)}} + a_3 \left(\overline{s_{(2)}} + \overline{s_{(1,1)}} \right) + a_1^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{()}}.$$

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- Multiplying by e_j appears harder: For $n = 5$ and $k = 3$, we have

$$\overline{s_{(2,2,1)} e_3} = -a_1 \overline{s_{(2,2)}} + a_2 \overline{s_{(2,1)}} + a_0^2 \overline{s_{(2)}} - 2a_0 a_1 \overline{s_{(1)}} + a_1^2 \overline{s_{()}}.$$

So, even multiplying by e_k can give a mess...

- For $\mathrm{QH}^*(\mathrm{Gr}(k,n))$, Bertram, Ciocan-Fontanine and Fulton give a “rim hook algorithm” that rewrites an arbitrary $\overline{s_\mu}$ as $(-1)^{\text{something}} q^{\text{something}} \overline{s_\lambda}$ with $\lambda \in P_{k,n}$.
  Is there such a thing for $\mathcal{S} \diagup I$?

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- **Theorem (G.):** Let μ be a k-partition with $\mu_1 > n - k$. Let
  $$W = \Big\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k \ \mid \ \lambda_1 = \mu_1 - n \text{ and } \lambda_i - \mu_i \in \{0,1\} \text{ for all } i \in \{2,3,\ldots,k\} \Big\}.$$
  (Not all elements of W are k-partitions, but all belong to $\mathbb{Z}^k$, so we know how to define s_λ for them.)

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  Then,

$$\overline{s_\mu} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\lambda \in W; \\ |\lambda| = |\mu| - (n-k+j)}} \overline{s_\lambda}.$$

Positivity?

- **Conjecture:** Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1, 2, \ldots, k\}$. Let $\lambda, \mu, \nu \in P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|} \operatorname{coeff}_\nu \left(\overline{s_\lambda s_\mu}\right)$ is a polynomial in $b_1, b_2, \ldots, b_k$ with coefficients in $\mathbb{N}$.
- Verified for all $n \leq 8$ using SageMath.
- This would generalize positivity of Gromov–Witten invariants.

- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{S} / I$ is free with basis

$$\left(\overline{m_\lambda}\right)_{\lambda \in P_{k,n}},$$

where

$$m_\lambda = \left(\text{the sum of all distinct permutations of the monomial } x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}\right)$$

is a *monomial symmetric polynomial*.

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- What about other bases? Forgotten symmetric functions?

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S_k-module structure

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- What is the S_k-module structure on $\mathcal{P} / J$?
- **Almost-theorem (G., needs to be checked):** Assume that **k** is a $\mathbb{Q}$-algebra. Then, as S_k-modules,

$$\mathcal{P} / J \cong \left(\mathcal{P} / \mathcal{P}\mathcal{S}^+\right)^{\times \binom{n}{k}} \cong \left(\underbrace{\mathbf{k} S_k}_{\text{regular rep}} \right)^{\times \binom{n}{k}},$$

where $\mathcal{P}\mathcal{S}^+$ is the ideal of $\mathcal{P}$ generated by symmetric polynomials with constant term 0.

Deforming symmetric functions, 1

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

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$$\mathbf{e}_{k+1}, \quad \mathbf{e}_{k+2}, \quad \mathbf{e}_{k+3}, \quad \ldots)_{\text{ideal}}.$$

- So why not replace the $\mathbf{e}_j$ by $\mathbf{e}_j - b_j$ too?

- **Theorem (G.):** Assume that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ as well as $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots$ are elements of Λ such that

$$\deg \mathbf{a}_i < n - k + i \qquad \text{and} \quad \deg \mathbf{b}_i < k + i.$$

Then,

$$\Lambda \diagup (\mathbf{h}_{n-k+1} - \mathbf{a}_1, \quad \mathbf{h}_{n-k+2} - \mathbf{a}_2, \quad \ldots, \quad \mathbf{h}_n - \mathbf{a}_k, \\ \mathbf{e}_{k+1} - \mathbf{b}_1, \quad \mathbf{e}_{k+2} - \mathbf{b}_2, \quad \mathbf{e}_{k+3} - \mathbf{b}_3, \quad \ldots)_{\text{ideal}}$$

is a free $\mathbf{k}$-module with basis $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda \in P_{k,n}}$.

On the proofs, 1

- Proofs of all the above (except for the S_k-action) can be found in
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  - Using that + Jacobi–Trudi, show that $\mathcal{S} / I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$.
  - As for the rest, compute in Λ... a lot.

Gröbner bases, 1: the degree-lexicographic order

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- Gröbner bases are "particularly uncomplicated" generating sets for ideals in polynomial rings.
  (But take the word "basis" with a grain of salt – they can have redundant elements, for example.)

Gröbner bases, 1: the degree-lexicographic order

- A *monomial order* is a total order on the monomials in $\mathcal{P}$ with the properties that
  - $1 \leq \mathfrak{m}$ for each monomial $\mathfrak{m}$;
  - $\mathfrak{a} \leq \mathfrak{b}$ implies $\mathfrak{a}\mathfrak{m} \leq \mathfrak{b}\mathfrak{m}$ for any monomials $\mathfrak{a}, \mathfrak{b}, \mathfrak{m}$;
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- The *degree-lexicographic order* is the monomial order defined as follows: Two monomials $\mathfrak{a} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $\mathfrak{b} = x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ satisfy $\mathfrak{a} > \mathfrak{b}$ if and only if
  - either $\deg \mathfrak{a} > \deg \mathfrak{b}$
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- Given a monomial order,
  - each nonzero polynomial $f \in \mathcal{P}$ has a well-defined *leading monomial* ($=$ the highest monomial appearing in f).

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- Given a monomial order,
  - each nonzero polynomial $f \in \mathcal{P}$ has a well-defined *leading monomial* (= the highest monomial appearing in f).
  - a polynomial f is called *quasi-monic* if the coefficient of its leading term in f is invertible.

Gröbner bases, 2: What is a Gröbner basis?

- If $\mathcal{I}$ is an ideal of $\mathcal{P}$, then a *Gröbner basis* of $\mathcal{I}$ (for a fixed monomial order) means a family $(f_i)_{i \in G}$ of quasi-monic polynomials that
  - generates $\mathcal{I}$, and
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- **Example:** Let $k = 3$, and rename x_1, x_2, x_3 as x, y, z. Use the degree-lexicographic order. Let $\mathcal{I}$ be the ideal generated by $x^2 - yz, y^2 - zx, z^2 - xy$. Then:

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  - The quadruple $(y^3 - z^3, x^2 - yz, xy - z^2, xz - y^2)$ is a Gröbner basis of $\mathcal{I}$. (Thanks SageMath, and whatever packages it uses for this.)

Gröbner bases, 3: Buchberger's first criterion

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- **Theorem (Buchberger's first criterion).** Let $\mathcal{I}$ be an ideal of $\mathcal{P}$.
  Let $(f_i)_{i \in G}$ be a family of quasi-monic polynomials that generates $\mathcal{I}$.
  Assume that the leading monomials of all the f_i are mutually coprime (i.e., each indeterminate appears in the leading monomial of f_i for at most one $i \in G$).
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- **Theorem (Macaulay's basis theorem).** Let $\mathcal{I}$ be an ideal of $\mathcal{P}$ that has a Gröbner basis $(f_i)_{i \in G}$. A monomial $\mathfrak{m}$ will be called *reduced* if it is not divisible by the leading term of any f_i. Then, the projections of the reduced monomials form a basis of the $\mathbf{k}$-module $\mathcal{P} / \mathcal{I}$.

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  Thus, $(\overline{x}) \cup \left(\overline{y^j z^\ell}\right)_{j < 3}$ is a basis of $\mathcal{P} / \mathcal{I}$.

- It is easy to prove the identity

$$h_p(x_{i..k}) = \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) h_{p-t}(x_{1..k})$$

for all $i \in \{1, 2, \ldots, k+1\}$ and $p \in \mathbb{N}$.
Here, $x_{a..b}$ means $x_a, x_{a+1}, \ldots, x_b$.

- Use this to show that

$$\left(h_{n-k+i}(x_{i..k}) - \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) a_{i-t} \right)_{i \in \{1,2,\ldots,k\}}$$

is a Gröbner basis of the ideal J wrt the degree-lexicographic order.

- Thus, Macaulay's basis theorem shows that $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$ is a basis of the **k**-module $\mathcal{P} / J$.

On the proofs, 3: the first basis of $\mathcal{S} / I$

- How to prove that $\mathcal{S} / I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$?

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- How to prove that $\mathcal{S} / I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$?
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- Thus, $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ is a basis of $\mathcal{P}/J$.
- $\Longrightarrow (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of $\mathcal{S}/I$.

On the proofs, 4: Bernstein's identity

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- Maybe the most important one:
  Bernstein's identity: Let λ be a partition. Let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Then,

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{m+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \mathbf{s}_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)}.$$

  Here, $\mathbf{f}^\perp \mathbf{g}$ means "$\mathbf{g}$ skewed by $\mathbf{f}$" (so that $(\mathbf{s}_\mu)^\perp \mathbf{s}_\lambda = \mathbf{s}_{\lambda / \mu}$).

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