Two interacting Hopf algebras of trees
Damien Calaque, Kurusch Ebrahimi-Fard, and Dominique Manchon
arXiv:0806.2238v4 [math.CO] 17 Dec 2011
Errata and questions

A. Corrections

- **Page 3, two lines above (2):** You write: "the group $G_0 = e + \mathfrak{g}_0$ of linear maps γ that send the bialgebra unit to the algebra unit, $\alpha(1) = 1_{\mathcal{A}}$". You mean $\gamma(1)$, not $\alpha(1)$ here.
- **Page 3, equation (3):** Add a condition $\rho(1) = 1_{\mathcal{A}}$ here (otherwise, the constant zero map 0 would be a character...).
- **Page 3, equation (5):** Remove the comma at the end of this equation.
- **Page 3, section 3:** You write: "Recall, that a –non-planar– *rooted tree* is either the empty set, or [...]". I doubt that you really want the empty set to count as a tree. If you do, then the definition of a rooted forest (page 4) allows for "invisible children" and infinitely many different trees with one vertex only, which is in conflict with all your examples.
- **Page 4, first line of the page:** Typo: "egde" should be "edge".
- **Page 4, second paragraph of the page:** You write: "A *rooted forest* is a finite collection $s = (t_1, t_2, ..., t_n)$ of rooted trees". There is nothing really wrong with this, but I find the notation $(t_1, t_2, ..., t_n)$ not really appropriate for a collection where the order of the elements doesn't matter (i.e., for a multiset): it conflicts with the standard notation for the n-tuple $(t_1, t_2, ..., t_n)$ (in which the order of the elements does matter). There doesn't seem to be a (universally agreed upon) standard notation for multisets, though.
- **Page 4, §4.1:** As I said above, the empty set does not really belong into the class of trees. Thus you don't need to say "*excluding the empty tree*". (The empty **forest** makes sense, but this is just the empty collection () of rooted trees.)
- **Page 4, equation (7):** It is worth saying that this formula can be rewritten as

$$\Delta(t) = \sum_{F \subseteq E(t)} (t \mid F) \otimes (t \diagup F), \tag{7'}$$

where

  - $E(t)$ is the set of all edges of t;
  - $t \mid F$ means the result of removing all edges $e \notin F$ from t;

  – t / F means the result of contracting each edge $f \in F$ to a point in t.

- **Page 4, §4.1:** It is worth pointing out that the formula (7) holds not only when t is a tree, but also when t is a forest (since the right hand side is clearly multiplicative in t). The same applies to the formula (7′). This is used tacitly in the proof of Theorem 8. (More precisely, the proof of Theorem 8 uses the analogous formulas for the map Φ, which lifts the map Δ to an algebra morphism from $\mathcal{H}_{\text{CK}}$ to $\mathcal{H} \otimes \mathcal{H}_{\text{CK}}$.)

- **Page 5, §4.2:** On the first line of §4.2, you write "$\widetilde{\mathcal{H}} = S(T)$". The S here should be a calligraphic $\mathcal{S}$.

- **Page 5, §4.3:** You write: "For any tree t the corresponding Z_t is an infinitesimal character of $\mathcal{H}$". This holds only for trees $t \neq \bullet$.

- **Page 5, §4.3:** In the formula

$$(Z_t \star Z_u - Z_u \star Z_t)(v) = \sum_s Z_t(s) Z_u(v/s) - \sum_s Z_u(s) Z_t(v/s),$$

  it is worth explaining that the sums range over all subforests of v.

- **Page 5, §4.3:** The formula

$$t \rhd u = \sum_{v,\ t \subset v \text{ and } v/t = u} N(t,u,v)\, v$$

  needs some explanations to be understood correctly (I believe).

  First of all, the sum $\sum_{v,\ t \subset v \text{ and } v/t = u}$ is a sum over all *trees* v satisfying $t \subset v$ and $v / t = u$.

  Second, "$t \subset v$ and $v / t = u$" is actually a shorthand notation for "there is a subtree s of v isomorphic to t and satisfying $v / s = u$ (with the $=$ sign denoting isomorphism of trees!)".

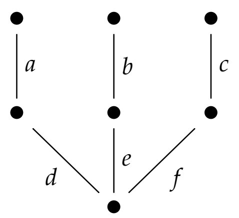
  It would be better to rewrite the above formula in the following less confusing way:

$$t \rhd u = \sum_{v \text{ is a tree}} N(t,u,v)\, v, \qquad \text{(Nab)}$$

  where $N(t,u,v)$ is the number of subforests s of v that are isomorphic to t and satisfy $v / s \cong u$ (note that I don't like speaking of v/t, since t is itself not a subtree of v but merely isomorphic to a such, but the isomorphism class of v/t does not depend on v and t alone). Of course, the sum can be restricted to range only over those v for which $N(t,u,v) \neq 0$ (in particular, we only need to sum over the trees v with at most $e(t) + e(u)$ many edges).

- **Page 5, §4.3:** You write: "where $\mathfrak{g} = \operatorname{Prim} \mathcal{H}^\circ$ is the Lie algebra spanned by the Z_t's for rooted trees t". You mean "[...] for rooted trees $t \neq \bullet$" here, since $Z_\bullet$ is not primitive.

- **Page 5, §4.3:** You write: "The product $\triangleright$ satisfies the left pre-Lie relation (5)." This is true, but not at all obvious. In the appendix below (Appendix B), I give a proof (actually, this is the wonderful proof that you sent me in an email).
- **Page 6, §4:** "See [24] for more on the combinatorics of rooted trees and Hopf algebras.": The reference [24] says nothing about trees. Probably a mis-reference.
- **Page 6, Corollary 2:** It should be said that the summation indices $k_1, k_2, \ldots, k_r$ are required to be positive.
- **Page 6, proof of Proposition 3:** I don't understand what "any class of subforests contains only one element" means and why this is needed.
- **Page 7, §5:** "perharps" $\to$ "perhaps".
- **Page 7, (13):** Replace "$\sum_{j\geq 0} p_{2j}$" by "$\sum_{j\geq 1} p_{2j}$", since there is no p_0.
- **Page 8, Proposition 5:** I'm not sure this is true. For example, let t be the tree

  .

  Then, $\Delta(t)$ contains an addend corresponding to the contraction of $s = \{a, c\}$. But there are (at least) three "floor" functions $\mathbf{fl} : E(t) \to \mathbb{N}$ that correspond to this addend:

  - one that sends d, e, f to 0 and a to 1 and b to 2 and c to 3;
  - one that sends d, e, f to 0 and a to 3 and b to 2 and c to 1;
  - one that sends d, e, f, b to 0 and a, c to 1.

  This can be fixed by changing Definition 1 as follows: Instead of requiring the image of $\mathbf{fl}$ to be an interval, we require that any two edges e, f with a common endpoint satisfy $|\mathbf{fl}(e) - \mathbf{fl}(f)| \leq 1$. Then, it is easy to see that the map

  $$\{\text{floored tree structures on } t\} \to \{\text{subsets of } E(t)\}, \qquad \mathbf{fl} \mapsto \{e \in E(t) \mid \mathbf{fl}(e) \text{ is even}\}$$

is a bijection[1]. This bijection furthermore has the property that if it sends a floored tree structure **fl** (with corresponding floored tree $\widetilde{t}$) to a subset F of $E(t)$, then $\prod_{j\geq 1} s_{2j-1}\left(\widetilde{t}\right) = t \mid F$ and $t / \prod_{j\geq 1} s_{2j-1}\left(\widetilde{t}\right) = t/F$. Hence, the claim of Proposition 5 follows from the formula (7') by reindexing the sum using this bijection. (I hope this was correct...)

Also, in (14), the summation sign "$\sum\limits_{r=1}^{h(t)}$" should be "$\sum\limits_{r=1}^{h(t)+1}$" or just "$\sum\limits_{r\geq 1}$".

- **Page 8, §8:** In the first displayed equation of §8, replace "E_σ" by "$E_\sigma(t)$".

  While at that, it is worth saying that $E_\sigma(t)$ also equals $\dfrac{1}{t!\sigma(t)}$ (by the definition of $\mathrm{CM}(t)$).

- **Page 8, §8:** In all "$\sum\limits_{k_1+\cdots+k_r=n}$" sums, the summation indices $k_1, k_2, \ldots, k_r$ must be required to be positive.

- **Page 9, §9.1:** "spanned by nonempty rooted trees" $\to$ "spanned by rooted trees", since the empty graph is not a tree.

- **Page 9, §9.1:** After (17), replace "Here the notation $V < W$ means that $x < y$ for any vertex x of v and any vertex y of w such that x and y are comparable" by "Here the notation $W < V$ means that $x < y$ for any vertex x of w and any vertex y of v such that x and y are comparable". This way, the order of W and V matches the one in (17) and also in the next sentence.

- **Page 9, (17):** The "$t \otimes \mathbf{1}$" and "$\mathbf{1} \otimes t$" addends should be removed from the right hand side. Indeed, as you explain two sentences further on, you do count the empty cut and the total cut as admissible cuts; thus, these two addends are already included in the sum $\sum\limits_{c\in\mathrm{Adm}(t)} P^c(t) \otimes R^c(t)$.

- **Page 10:** On the first line of this page, "Here we denote by $\mathrm{Adm}(t)$ the set of admissible cuts of a forest, i.e. the set of collections of edges such that any path from the root to a leaf contains at most one edge of the collection.". Even with the clarification given in the next sentence, this is only true when t is a tree, not when t is a forest. For a forest, admissible cuts need to be defined as sets of vertices, not of edges.

[1] Indeed, the inverse of this map sends any subset F of $E(t)$ to the floored tree structure **fl** defined as follows: For any $e \in E(t)$, we consider the path from the top vertex of e down to the root of t. Color each edge of this path black if it belongs to F and white if it does not. Also append an extra black edge at the end of this path (after the root of t). Then, $\mathbf{fl}(e)$ is the number of times that the color of the edge changes as we walk this path from one end to the other.

- **Page 10:** "the subforest formed by the edges above the cut $c \in \operatorname{Adm}(t)$ (resp. the subforest formed by the edges under the cut)": In both cases, you want not just the edges but also the vertices.

- **Page 10:** After "Note that the trunk of a tree is a tree", add "or empty".

- **Page 10, equation (19):** The forest u must be assumed to be nonempty here.

- **Page 10, equation (20):** The "$V_n < \cdots < V_1$" under the summation sign should be understood as saying that $V_j < V_i$ for all $i < j$ (not only that $V_{i+1} < V_i$ for all i, which would be a weaker requirement).

- **Page 10, definition of pre-Lie algebra structure:** Replace "$\mathcal{H}^\circ$" by "$\mathcal{H}^\circ_{\mathrm{CK}}$" twice (in "it is a primitive element of $\mathcal{H}^\circ$" and in "the (convolution) product of $\mathcal{H}^\circ$").

- **Page 10, definition of pre-Lie algebra structure:** The formula "$t \to u = \sum\limits_v N'(t,u,v)\, v$" seems to be meant in the sense that all of t, u, v are assumed to be trees. Otherwise, it would be pretty clear that $\delta_t * \delta_u = \delta_{t \to u}$, which makes the formula $\delta_t * \delta_u - \delta_u * \delta_t = \delta_{t \to u - u \to t}$ a trivial corollary.

- **Page 10, definition of pre-Lie algebra structure:** In "where $N'(t,u,v)$ is the number of partitions $V(t) = V \amalg W$", replace "$V(t)$" by "$V(v)$".

- **Page 11, proof of Theorem 8:** This proof uses the formula
$$\Phi(t) = \sum_{\substack{s \text{ is a subforest} \\ \text{of } t}} s \otimes t/s \qquad (\text{in } \mathcal{H} \otimes \mathcal{H}_{\mathrm{CK}})$$
not just for all trees t but also for all forests t.

- **Page 11, proof of Theorem 8:** In the first display, the parentheses around the "t" in "$\Phi(t)$" have different sizes.

- **Page 12, proof of Theorem 8:** In the first display, the $\Big/$ signs are misleadingly tall, looking as if they span the $\otimes$ signs. It would probably be better to make them normally-sized but put parentheses around "$s \cap P^c(t)$" and around "$s \cap R^c(t)$".

- **Page 13, Corollary 12:** "le α" should be "let α".

- **Page 13, Corollary 12:** "linear maps form" should be "linear maps from".

- **Page 15, (31):** Replace the comma in "$B(\alpha * \beta, a)$" by a semicolon.

- **Page 17, §11.1:** "Let $\diamond$ the product" $\to$ "Let $\diamond$ be the product".

- **Page 17, §11.1:** Replace "connected graded Hopf algebra" by "connected filtered Hopf algebra". (I am also not sure if [23] is the right reference for this.)
- **Page 17, §11.1:** In the displayed equation just above Remark 18, the sign "$\sum_{r=0}^{s}$" should be "$\sum_{r=0}^{k}$".
- **Page 19, proof of Theorem 20:** In the first display, replace "$\mathcal{H}_{\mathrm{CK}}^{(n)}$" by "$\mathcal{H}_{\mathrm{CK}}^{(k)}$".
- **Page 19, Lemma 21:** "Let $\widetilde{\delta} : \mathcal{A} \to k$ the linear map" $\to$ "Let $\widetilde{\delta} : \mathcal{A} \to k$ be the linear map".
- **Page 19, proof of Lemma 21:** "bigger than $\sup(l(u), l(v))$" should be "$\geq \sup(l(u), l(v))$".
- **Page 20, Corollary 23:** "the number" $\to$ "be the number".
- **Page 20, §11.4:** In the first displayed equation of §11.4, replace "$\sum_{k=0}^{n}$" by "$\sum_{r=0}^{k}$".
- **Page 20, §11.4:** "the choice of a subset E of $\{1, \ldots l\}$" should be "the choice of a subset E of $\{1, \ldots, k\}$".
- **Page 20, §11.4:** "mutinomial" $\to$ "multinomial".
- **Page 21, definition of the generalized corolla $\mathcal{C}_{k_1,\ldots,k_n}$:** What is called E_j here was previously denoted E_{j-1}.

B. Appendix: Proof of the pre-Lie relation for the product $\triangleright$ defined in §4.3

Proof that the product $\triangleright$ defined in §4.3 satisfies the left pre-Lie relation (5):

First, some preparations.

For any assertion $\mathcal{A}$, we are going to denote by $[\mathcal{A}]$ the truth value of this assertion $\mathcal{A}$ (defined by $[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$). Then, clearly, $[\mathcal{A}] \cdot [\mathcal{B}] = [\mathcal{A} \text{ and } \mathcal{B}]$ for any two assertions $\mathcal{A}$ and $\mathcal{B}$.

Lemma B.1. Any two trees $t \neq \bullet$ and $u \neq \bullet$ satisfy

$$Z_t \star Z_u = (1 + [t = u]) Z_{tu} + Z_{t \triangleright u}. \tag{A1}$$

Proof. Let $t \neq \bullet$ and $u \neq \bullet$ be two trees. For any forest w, we have

$$\begin{aligned}
&\left(Z_t \star Z_u\right)(w) \\
&= \sum_{s \subseteq w} \underbrace{Z_t(s)}_{=[s \cong t]} \underbrace{Z_u(w/s)}_{=[w/s \cong u]} \qquad \left(\text{since } \Delta(w) = \sum_{s \subseteq w} s \otimes (w/s) \text{ by (7)}\right) \\
&= \sum_{s \subseteq w} \underbrace{[s \cong t]\,[w/s \cong u]}_{\substack{=[s \cong t \text{ and } w/s \cong u] \\ (\text{since } [\mathcal{A}]\cdot[\mathcal{B}]=[\mathcal{A} \text{ and } \mathcal{B}] \\ \text{for any two assertions } \mathcal{A} \text{ and } \mathcal{B})}} = \sum_{s \subseteq w} [s \cong t \text{ and } w/s \cong u] \\
&= \sum_{\substack{s \subseteq w; \\ s \cong t \text{ and } w/s \cong u}} \underbrace{[s \cong t \text{ and } w/s \cong u]}_{\substack{=1 \\ (\text{since } (s \cong t \text{ and } w/s \cong u))}} + \sum_{\substack{s \subseteq w; \\ \text{not } (s \cong t \text{ and } w/s \cong u)}} \underbrace{[s \cong t \text{ and } w/s \cong u]}_{\substack{=0 \\ (\text{since not } (s \cong t \text{ and } w/s \cong u))}} \\
&= \underbrace{\sum_{\substack{s \subseteq w; \\ s \cong t \text{ and } w/s \cong u}} 1}_{\substack{=(\text{number of subforests } s \subseteq w \text{ such that } s \cong t \text{ and } w/s \cong u)\cdot 1 \\ =(\text{number of subforests } s \subseteq w \text{ such that } s \cong t \text{ and } w/s \cong u)}} + \underbrace{\sum_{\substack{s \subseteq w; \\ \text{not } (s \cong t \text{ and } w/s \cong u)}} 0}_{=0} \\
&= (\text{number of subforests } s \subseteq w \text{ such that } s \cong t \text{ and } w/s \cong u) \\
&= (\text{number of subtrees } s \subseteq w \text{ such that } s \cong t \text{ and } w/s \cong u) \\
&\qquad \left(\begin{array}{c} \text{since all subforests } s \subseteq w \text{ such that } s \cong t \text{ are actually subtrees of } w \\ (\text{because if } s \cong t, \text{ then } s \text{ is a tree (since } t \text{ is a tree))} \end{array}\right) \\
&= N(t,u,w) \qquad (\text{by the definition of } N(t,u,w)).
\end{aligned}$$

Since the forests form a basis of $\mathcal{H}$, this yields that

$$\begin{aligned}
&Z_t \star Z_u \\
&= \sum_{v \text{ forest}} N(t,u,v)\, Z_v \\
&= \underbrace{\sum_{\substack{v \text{ forest}; \\ v \text{ is a tree}}} N(t,u,v)\, Z_v}_{\substack{= \sum_{v \text{ tree}} N(t,u,v) Z_v \\ = \sum_{v = \bullet} N(t,u,v) Z_v + \sum_{v \text{ tree } \neq \bullet} N(t,u,v) Z_v}} + \underbrace{\sum_{\substack{v \text{ forest}; \\ v \text{ is the forest } tu}} N(t,u,v)\, Z_v}_{= N(t,u,tu) Z_{tu}} + \sum_{\substack{v \text{ forest}; \\ v \text{ is neither a tree} \\ \text{nor the forest } tu}} N(t,u,v)\, Z_v \\
&= \sum_{v=\bullet} N(t,u,v)\, Z_v + \sum_{v \text{ tree } \neq \bullet} N(t,u,v)\, Z_v + N(t,u,tu)\, Z_{tu} + \sum_{\substack{v \text{ forest}; \\ v \text{ is neither a tree} \\ \text{nor the forest } tu}} N(t,u,v)\, Z_v.
\end{aligned} \tag{A2}$$

Now, we recall that for every forest v, the number $N(t,u,v)$ is defined as the number of subtrees $s \subseteq v$ such that $s \cong t$ and $v/s \cong u$. As a consequence, we have:

- If v is the tree $\bullet$, then $N(t, u, v) = 0$ (since, in this case, there are no subtrees $s \subseteq v$ such that $s \cong t$ and $v \diagup s \cong u$ (because $t \neq \bullet$)).
- If v is neither a tree (not even $\bullet$) nor the forest tu, then $N(t, u, v) = 0$ (since, in this case, there are no subtrees $s \subseteq v$ such that $s \cong t$ and $v \diagup s \cong u$ [2]).
- If v is the forest tu, then $N(t, u, v) = 1 + [t = u]$ [3]. In other words, $N(t, u, tu) = 1 + [t = u]$.

[2] *Proof.* Let v be neither a tree (not even $\bullet$) nor the forest tu. Assume that there exists some subtree $s \subseteq v$ such that $s \cong t$ and $v \diagup s \cong u$. Consider this subtree s.

Since $v \diagup s \cong u$, it is clear that the forest $v \diagup s$ has only one connected component (since u is a tree and thus has only one connected component). Here, when we say "connected component", we are *not* counting isolated vertices as extra components (because the one-vertex tree $\bullet$ is the unity of our algebra $\mathcal{H}$, so we identify every forest f with the forest $f\bullet$).

Since v is not a tree, it is clear that v is a forest with at least two connected components. The subtree s must be a subset of one of them (since it is a tree, thus connected). If s were a *proper* subset of one of these connected components, then the forest $v \diagup s$ would still have at least two connected components (because when passing from v to $v \diagup s$, the connected component containing s would not completely disappear, and the other connected components would stay unchanged), which would contradict the fact that $v \diagup s$ has only one connected component. Hence, s is a subset but not a proper subset of one of these connected components. In other words, s is one of these connected components. Hence, $v \diagup s$ is the union of all the connected components other than s. Since $v \diagup s$ has only one connected component, this means that there is only one connected component other than s, and this connected component is isomorphic to $v \diagup s$. Hence, our forest v consists of two connected components, one of them being $s \cong t$ and the other being $\cong v \diagup s \cong u$. Hence, our forest v is tu. But this contradicts the assumption $v \neq tu$.

This contradiction shows that our assumption (that there exists some subtree $s \subseteq v$ such that $s \cong t$ and $v \diagup s \cong u$) was wrong. Hence, there are no subtrees $s \subseteq v$ such that $s \cong t$ and $v \diagup s \cong u$, qed.

[3] *Proof.* Let v be the forest tu. Let $s \subseteq v$ be a subtree such that $s \cong t$ and $v \diagup s \cong u$.

Since $v \diagup s \cong u$, it is clear that the forest $v \diagup s$ has only one connected component (since u is a tree and thus has only one connected component). Here, when we say "connected component", we are *not* counting isolated vertices as extra components (because the one-vertex tree $\bullet$ is the unity of our algebra $\mathcal{H}$, so we identify every forest f with the forest $f\bullet$).

Since $v = tu$, it is clear that v is a forest with exactly two connected components t and u. The subtree s must be a subset of one of them (since it is a tree, thus connected). If s were a *proper* subset of one of these connected components, then the forest $v \diagup s$ would still have at least two connected components (because when passing from v to $v \diagup s$, the connected component containing s would not completely disappear, and the other connected components would stay unchanged), which would contradict the fact that $v \diagup s$ has only one connected component. Hence, s is a subset but not a proper subset of one of these connected components. In other words, s is one of these connected components.

Now we recall that the two connected components of v are t and u. If $t \neq u$, then this forces s to be the component t (because s is one of these two connected components, but it cannot be u since $s \cong t \neq u$).

Now forget that we fixed s. We have thus shown that every subtree $s \subseteq v$ such that $s \cong t$ and $v \diagup s \cong u$ must be the connected component t of v. And conversely, $t \subseteq v$ is a subtree such that $t \cong t$ and $v \diagup t \cong u$ (since $v = tu$). Hence, if $t \neq u$, then there is exactly one subtree $s \subseteq v$ such that $s \cong t$ and $v \diagup s \cong u$. In other words, if $t \neq u$, then $N(t, u, v) = 1$ (since

Using these three observations, we see that (A2) becomes

$$\begin{aligned}
&Z_t \star Z_u \\
&= \sum_{v=\bullet} \underbrace{N(t,u,v)}_{\substack{=0 \\ \text{(since } v \text{ is} \\ \text{the tree } \bullet)}} Z_v + \sum_{v \text{ tree } \neq \bullet} N(t,u,v) Z_v + \underbrace{N(t,u,tu)}_{=1+[t=u]} Z_{tu} \\
&\qquad + \sum_{\substack{v \text{ forest;} \\ v \text{ is neither a tree} \\ \text{nor the forest } tu}} \underbrace{N(t,u,v)}_{\substack{=0 \\ \text{(since } v \text{ is neither a} \\ \text{tree nor the forest } tu)}} Z_v \\
&= \underbrace{\sum_{v=\bullet} 0 Z_v}_{=0} + \sum_{v \text{ tree } \neq \bullet} N(t,u,v) Z_v + (1+[t=u]) Z_{tu} + \underbrace{\sum_{\substack{v \text{ forest;} \\ v \text{ is neither a tree} \\ \text{nor the forest } tu}} 0 Z_v}_{=0} \\
&= \sum_{v \text{ tree } \neq \bullet} N(t,u,v) Z_v + (1+[t=u]) Z_{tu}. \qquad \text{(A3)}
\end{aligned}$$

On the other hand, the equality (Nab) above says that

$$\begin{aligned}
t \triangleright u &= \sum_{v \text{ tree}} N(t,u,v) v \\
&= \underbrace{N(t,u,\bullet)}_{\substack{=0 \\ \text{(since there is no subforest } s \text{ of } \bullet \\ \text{isomorphic to } t \text{ (because } t \neq \bullet))}} \bullet + \sum_{v \text{ tree } \neq \bullet} N(t,u,v) v \\
&= \sum_{v \text{ tree } \neq \bullet} N(t,u,v) v.
\end{aligned}$$

In other words,

$$\sum_{v \text{ tree } \neq \bullet} N(t,u,v) v = t \triangleright u. \qquad \text{(A4)}$$

$N(t,u,v)$ is defined as the number of subtrees $s \subseteq v$ such that $s \cong t$ and $v/s \cong u$).

By a similar argument, we see that if $t = u$, then $N(t,u,v) = 2$ (in fact, if $t = u$, then every subtree $s \subseteq v$ such that $s \cong t$ and $v/s \cong u$ must be one of the two isomorphic connected components t and u of v, and each of these two components can be chosen to be s).

So we know that $N(t,u,v)$ equals 1 if $t \neq u$ and equals 2 if $t = u$. Thus,

$$N(t,u,v) = \begin{cases} 2, & \text{if } t = u; \\ 1, & \text{if } t \neq u \end{cases} = 1 + \underbrace{\begin{cases} 1, & \text{if } t = u; \\ 0, & \text{if } t \neq u \end{cases}}_{=[t=u]} = 1 + [t=u],$$

qed.

Now, (A3) becomes

$$Z_t \star Z_u = \underbrace{\sum_{v \text{ tree } \neq \bullet} N(t,u,v) Z_v}_{\substack{= Z_{\sum_{v \text{ tree } \neq \bullet} N(t,u,v) v} = Z_{t \triangleright u} \\ \text{(by (A4))}}} + (1 + [t = u]) Z_{tu}$$
$$= Z_{t \triangleright u} + (1 + [t = u]) Z_{tu} = (1 + [t = u]) Z_{tu} + Z_{t \triangleright u}.$$

This proves Lemma B.1. □

Next let us notice an easy consequence of (A1):

Lemma B.2. Any two trees $t \neq \bullet$ and $u \neq \bullet$ satisfy

$$Z_t \star Z_u - Z_u \star Z_t = Z_{t \triangleright u - u \triangleright t}. \tag{A9}$$

Proof. Let $t \neq \bullet$ and $u \neq \bullet$ be two trees. Applying (A1) to u and t instead of t and u gives us $Z_u \star Z_t = \left(1 + \underbrace{[u = t]}_{= [t = u]}\right) \underbrace{Z_{ut}}_{= Z_{tu}} + Z_{u \triangleright t} = (1 + [t = u]) Z_{tu} + Z_{u \triangleright t}$. But applying (A1) directly gives us $Z_t \star Z_u = (1 + [t = u]) Z_{tu} + Z_{t \triangleright u}$. Subtracting the first of these two equations from the second, we obtain

$$Z_t \star Z_u - Z_u \star Z_t = ((1 + [t = u]) Z_{tu} + Z_{t \triangleright u}) - ((1 + [t = u]) Z_{tu} + Z_{u \triangleright t})$$
$$= Z_{t \triangleright u} - Z_{u \triangleright t} = Z_{t \triangleright u - u \triangleright t}.$$

This proves Lemma B.2. □

Now recall that $\mathcal{H}$ is the symmetric algebra of T'. Hence, we can regard $k\bullet$ and T' as k-vector subspaces of $\mathcal{H}$, respectively, where $k\bullet$ is contained in the 0-th graded component of $\mathcal{H}$ while T' is contained in the direct sum of the other graded components. Hence, $T = T' \oplus k\bullet$ becomes a k-vector subspace of $\mathcal{H}$ as well. We notice that $\Delta(k\bullet) \subseteq \mathcal{H} \otimes T$ (since $\Delta(\bullet) = \underbrace{\bullet}_{\in \mathcal{H}} \otimes \underbrace{\bullet}_{\in T} \in \mathcal{H} \otimes T$) and $\Delta(T') \subseteq \mathcal{H} \otimes T$ (since every tree $t \in T'$ satisfies

$$\Delta(t) = \sum_{s \subseteq t} \underbrace{s}_{\in \mathcal{H}} \otimes \underbrace{(t / s)}_{\substack{\in T \\ \text{(indeed, } t/s \text{ is a quotient of a tree,} \\ \text{thus a tree, hence an element of } T)}}$$
$$\in \sum_{s \subseteq t} \mathcal{H} \otimes T \subseteq \mathcal{H} \otimes T \qquad \text{(since } \mathcal{H} \otimes T \text{ is a vector space)}$$

). Hence,

$$\Delta\left(\underbrace{T}_{= T' \oplus k\bullet = T' + k\bullet}\right) = \Delta(T' + k\bullet) = \underbrace{\Delta(T')}_{\subseteq \mathcal{H} \otimes T} + \underbrace{\Delta(k\bullet)}_{\subseteq \mathcal{H} \otimes T} \subseteq \mathcal{H} \otimes T + \mathcal{H} \otimes T \subseteq \mathcal{H} \otimes T$$

(since $\mathcal{H} \otimes T$ is a vector space). In other words, T is a left coideal of $\mathcal{H}$.

Now, for every subspace U of $\mathcal{H}$, let $U^\perp$ denote the subspace $\{f \in \mathcal{H}^\circ \mid f(U) = 0\}$ of $\mathcal{H}^\circ$. By well-known properties of graded duals of coalgebras, we know that whenever U is a left coideal of $\mathcal{H}$, the subspace $U^\perp$ is a left ideal of $\mathcal{H}^\circ$. Applied to $U = T$, this yields that $T^\perp$ is a left ideal of $\mathcal{H}^\circ$.

Another easy consequence of (A1): Any two trees $t \neq \bullet$ and $u \neq \bullet$ satisfy

$$Z_t \star Z_u \equiv Z_{t \triangleright u} \operatorname{mod} T^\perp. \tag{A11}$$

[4] By linearity, this yields the following: Any two elements $x \in T'$ and $y \in T'$ satisfy

$$Z_x \star Z_y \equiv Z_{x \triangleright y} \operatorname{mod} T^\perp. \tag{A12}$$

(In fact, (A12) follows from (A11) because the term $Z_x \star Z_y - Z_{x \triangleright y}$ is bilinear with respect to x and y, because the vector space T' is generated by trees $\neq \bullet$, and because $T^\perp$ is a vector space.)

Finally, let us notice that

$$\text{if } x \in T' \text{ and } y \in T' \text{ are two elements satisfying } Z_x \equiv Z_y \operatorname{mod} T^\perp \text{, then } x = y. \tag{A15}$$

[5]

[4] *Proof of (A11).* Let $t \neq \bullet$ and $u \neq \bullet$ be two trees. Every tree $v \in T$ satisfies $v \neq tu$ (since v is a tree, whereas tu is not a tree). Hence, every tree $v \in T$ satisfies $Z_{tu}(v) = [v = tu] = 0$ (since $v = tu$ is false (because $v \neq tu$)). Thus, $Z_{tu}(T) = 0$, so that $Z_{tu} \in \{f \in \mathcal{H}^\circ \mid f(T) = 0\} = T^\perp$ (by the definition of $T^\perp$). But (A1) yields

$$Z_t \star Z_u - Z_{t \triangleright u} = (1 + [t = u]) \underbrace{Z_{tu}}_{\in T^\perp} \in T^\perp \qquad \left(\text{since } T^\perp \text{ is a vector space}\right).$$

In other words, $Z_t \star Z_u \equiv Z_{t \triangleright u} \operatorname{mod} T^\perp$. This proves (A11).

[5] *Proof of (A15).* Let $x \in T'$ and $y \in T'$ be two elements satisfying $Z_x \equiv Z_y \operatorname{mod} T^\perp$. Since $x \in T'$, we can write x in the form $x = \sum_{t \text{ tree } \neq \bullet} \lambda_t t$ for some $\lambda_t \in k$ (because T' is the k-vector space spanned by rooted trees $\neq \bullet$). Consider these $\lambda_t \in k$. Since $y \in T'$, we can write y in the form $y = \sum_{t \text{ tree } \neq \bullet} \mu_t t$ for some $\mu_t \in k$ (because T' is the k-vector space spanned by rooted trees $\neq \bullet$). Consider these $\mu_t \in k$. Every tree $s \neq \bullet$ satisfies $Z_x(s) = \lambda_s$ (since $x = \sum_{t \text{ tree } \neq \bullet} \lambda_t t$ and thus

$$\begin{aligned} Z_x(s) &= Z_{\sum_{t \text{ tree } \neq \bullet} \lambda_t t}(s) = \sum_{t \text{ tree } \neq \bullet} \lambda_t \underbrace{Z_t(s)}_{=[s=t]} = \sum_{t \text{ tree } \neq \bullet} \lambda_t [s = t] \\ &= \sum_{\substack{t \text{ tree } \neq \bullet; \\ s=t}} \lambda_t \underbrace{[s = t]}_{=1 \text{ (since } s=t)} + \sum_{\substack{t \text{ tree } \neq \bullet; \\ s \neq t}} \lambda_t \underbrace{[s = t]}_{=0 \text{ (since } s \neq t)} = \underbrace{\sum_{\substack{t \text{ tree } \neq \bullet; \\ s=t}} \lambda_t 1}_{=\lambda_s 1 = \lambda_s} + \underbrace{\sum_{\substack{t \text{ tree } \neq \bullet; \\ s \neq t}} \lambda_t 0}_{=0} = \lambda_s \end{aligned}$$

) and $Z_y(s) = \mu_s$ (similarly). But $Z_x \equiv Z_y \operatorname{mod} T^\perp$, so that $Z_x - Z_y \in T^\perp = \{f \in \mathcal{H}^\circ \mid f(T) = 0\}$ (by the definition of $T^\perp$), so that $(Z_x - Z_y)(T) = 0$. Hence, every

Now, let us finally prove that the product $\triangleright$ defined in §4.3 satisfies the left pre-Lie relation (5). In fact, let a, b and c be three trees in T'. Then, (A9) (applied to $t = a$ and $u = b$) yields

$$Z_a \star Z_b - Z_b \star Z_a = Z_{a \triangleright b - b \triangleright a}.$$

Thus,

$$\begin{aligned} (Z_a \star Z_b - Z_b \star Z_a) \star Z_c &= Z_{a \triangleright b - b \triangleright a} \star Z_c \\ &\equiv Z_{(a \triangleright b - b \triangleright a) \triangleright c} \operatorname{mod} T^{\perp} \end{aligned} \tag{A16}$$

(by (A12), applied to $x = a \triangleright b - b \triangleright a$ and $y = c$). On the other hand, (A12) (applied to $x = b$ and $y = c$) yields $Z_b \star Z_c \equiv Z_{b \triangleright c} \operatorname{mod} T^{\perp}$, so that $Z_b \star Z_c - Z_{b \triangleright c} \in T^{\perp}$. This yields $Z_a \star (Z_b \star Z_c - Z_{b \triangleright c}) \in T^{\perp}$ (since $T^{\perp}$ is a left ideal). This rewrites as $Z_a \star (Z_b \star Z_c) - Z_a \star Z_{b \triangleright c} \in T^{\perp}$. In other words, $Z_a \star (Z_b \star Z_c) \equiv Z_a \star Z_{b \triangleright c} \operatorname{mod} T^{\perp}$. Now,

$$\begin{aligned} (Z_a \star Z_b) \star Z_c &= Z_a \star (Z_b \star Z_c) && \text{(since } \star \text{ is associative)} \\ &\equiv Z_a \star Z_{b \triangleright c} \\ &\equiv Z_{a \triangleright (b \triangleright c)} \operatorname{mod} T^{\perp} && \text{(by (A12), applied to } x = a \text{ and } y = b \triangleright c) \end{aligned}$$

and similarly

$$(Z_b \star Z_a) \star Z_c \equiv Z_{b \triangleright (a \triangleright c)} \operatorname{mod} T^{\perp}.$$

Now, (A16) yields

$$\begin{aligned} Z_{(a \triangleright b - b \triangleright a) \triangleright c} &\equiv (Z_a \star Z_b - Z_b \star Z_a) \star Z_c = \underbrace{(Z_a \star Z_b) \star Z_c}_{\equiv Z_{a \triangleright (b \triangleright c)} \operatorname{mod} T^{\perp}} - \underbrace{(Z_b \star Z_a) \star Z_c}_{\equiv Z_{b \triangleright (a \triangleright c)} \operatorname{mod} T^{\perp}} \\ &\equiv Z_{a \triangleright (b \triangleright c)} - Z_{b \triangleright (a \triangleright c)} = Z_{a \triangleright (b \triangleright c) - b \triangleright (a \triangleright c)} \operatorname{mod} T^{\perp}. \end{aligned}$$

By (A15) (applied to $x = (a \triangleright b - b \triangleright a) \triangleright c$ and $y = a \triangleright (b \triangleright c) - b \triangleright (a \triangleright c)$), this yields that

$$(a \triangleright b - b \triangleright a) \triangleright c = a \triangleright (b \triangleright c) - b \triangleright (a \triangleright c).$$

Since $(a \triangleright b - b \triangleright a) \triangleright c = (a \triangleright b) \triangleright c - (b \triangleright a) \triangleright c$, this rewrites as

$$(a \triangleright b) \triangleright c - (b \triangleright a) \triangleright c = a \triangleright (b \triangleright c) - b \triangleright (a \triangleright c).$$

tree $s \neq \bullet$ satisfies $(Z_x - Z_y)(s) = 0$ (since $s \in T' \subseteq T$). Thus, every tree $s \neq \bullet$ satisfies

$$0 = (Z_x - Z_y)(s) = \underbrace{Z_x(s)}_{= \lambda_s} - \underbrace{Z_y(s)}_{= \mu_s} = \lambda_s - \mu_s.$$

In other words, every tree $s \neq \bullet$ satisfies $\lambda_s = \mu_s$. Renaming s as t, we thus conclude: Every tree $t \neq \bullet$ satisfies $\lambda_t = \mu_t$. Hence, $\sum_{t \text{ tree } \neq \bullet} \lambda_t t = \sum_{t \text{ tree } \neq \bullet} \mu_t t$. So we have $x = \sum_{t \text{ tree } \neq \bullet} \lambda_t t = \sum_{t \text{ tree } \neq \bullet} \mu_t t = y$. This proves (A15).

In other words,

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c) . \qquad \text{(A20)}$$

We have thus proven that any three trees a, b and c in T' satisfy (A20). But since the equation (A20) is multilinear in a, b and c, and since T' is generated (as a k-vector space) by the trees in T', this yields that any three elements a, b and c in T' (not necessarily trees) satisfy (A20). In other words, we have shown that the product $\triangleright$ defined in §4.3 satisfies the left pre-Lie relation (5). Qed.