A double Sylvester determinant

(detailed version)

Darij Grinberg

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Abstract. Given two $(n+1) \times (n+1)$-matrices A and B over a commutative ring, and some $k \in \{0, 1, \ldots, n\}$, we consider the $\binom{n}{k} \times \binom{n}{k}$-matrix W whose entries are $(k+1) \times (k+1)$-minors of A multiplied by corresponding $(k+1) \times (k+1)$-minors of B. Here we require the minors to use the last row and the last column (which is why we obtain an $\binom{n}{k} \times \binom{n}{k}$-matrix, not a $\binom{n+1}{k+1} \times \binom{n+1}{k+1}$-matrix). We prove that the determinant $\det W$ is a multiple of $\det A$ if the $(n+1, n+1)$-th entry of B is 0. Furthermore, if the $(n+1, n+1)$-th entries of both A and B are 0, then $\det W$ is a multiple of $(\det A)(\det B)$. This extends a previous result of Olver and the author.

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Contents

1. Introduction

Let n and k be nonnegative integers, and let $A = (a_{i,j})_{1 \leq i \leq n+1,\ 1 \leq j \leq n+1}$ be an $(n+1) \times (n+1)$-matrix over some commutative ring. Let P_k be the set of all k-

element subsets of $\{1, 2, \ldots, n\}$. For any such subset $K \in P_k$, let $K+$ denote the subset $K \cup \{n+1\}$ of $\{1, 2, \ldots, n+1\}$. If U and V are two subsets of $\{1, 2, \ldots, n+1\}$, then $\operatorname{sub}_U^V A$ shall denote the $|U| \times |V|$-submatrix of A containing only the entries $a_{u,v}$ with $u \in U$ and $v \in V$. Let W_A be the $P_k \times P_k$-matrix[1] whose (I, J)-th entry (for all $I \in P_k$ and $J \in P_k$) is

$$\det\left(\operatorname{sub}_{I+}^{J+} A\right).$$

(Thus, the entries of W_A are all $(k+1) \times (k+1)$-minors of A that use the last row and the last column.) A particular case of a celebrated result going back to Sylvester [Sylves51] (see [Prasol94, §2.7] or [Prasol15, Teorema 2.9.1] or [Mohr53] for modern proofs) then says that

$$\det(W_A) = a_{n+1,n+1}^p \cdot (\det A)^q, \qquad \text{where } p = \binom{n-1}{k} \text{ and } q = \binom{n-1}{k-1}.$$

Now, consider a second $(n+1) \times (n+1)$-matrix $B = (b_{i,j})_{1 \le i \le n+1,\ 1 \le j \le n+1}$ over the same ring. Let $W_{A,B}$ (later to be just called W) be the $P_k \times P_k$-matrix whose (I, J)-th entry (for all $I \in P_k$ and $J \in P_k$) is

$$\det\left(\operatorname{sub}_{I+}^{J+} A\right) \det\left(\operatorname{sub}_{I+}^{J+} B\right).$$

What can be said about $\det(W_{A,B})$? In general, very little[2]. However, under some assumptions, it splits off factors. Namely, we shall show (Theorem 2.1) that $\det(W_{A,B})$ is a multiple of $\det A$ if $b_{n+1,n+1} = 0$. We shall then conclude (Theorem 2.2) that if both $a_{n+1,n+1}$ and $b_{n+1,n+1}$ are 0, then $\det(W_{A,B})$ is a multiple of $(\det A)(\det B)$. In either case, the quotient (usually a much more complicated polynomial[3]) remains mysterious; our proofs are indirect and reveal little about it. Our second result generalizes a curious property of $\binom{n}{2} \times \binom{n}{2}$-determinants [GriOlv18, Theorem 10] that arose from the study of the n-body problem (see Example 2.4 for details).

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[1] This means a matrix whose rows and columns are indexed by the k-element subsets of $\{1, 2, \ldots, n\}$. If you pick a total order on the set P_k, then you can view such a matrix as an $\binom{n}{k} \times \binom{n}{k}$-matrix.

[2] For example, if $n = 3$ and $k = 2$, then $\det(W_{A,B})$ is an irreducible polynomial in the (altogether $2(n+1)^2 = 32$) variables $a_{i,j}$ and $b_{i,j}$ with 110268 monomials.

[3] again irreducible in the case when $n = 3$ and $k = 2$

2. The theorems

Let us first introduce the standing notations.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\mathbb{K}$ be a commutative ring. If a and b are two elements of $\mathbb{K}$, then we write $a \mid b$ when b is a multiple of a (that is, $b \in \mathbb{K}a$).

If $m \in \mathbb{N}$, then $[m]$ shall mean the set $\{1, 2, \ldots, m\}$.

Fix an $n \in \mathbb{N}$. If K is any subset of $[n]$, then $K+$ shall mean the subset $K \cup \{n+1\}$ of $[n+1]$.

Fix $k \in \{0, 1, \ldots, n\}$. Let P_k be the set of all k-element subsets of $[n]$. This is a finite set; thus, any $P_k \times P_k$-matrix (i.e., any matrix whose rows and columns are indexed by k-element subsets of $[n]$) has a well-defined determinant[4]. Such matrices appear frequently in classical determinant theory (see, e.g., the "k-th compound determinants" in [MuiMet60] and in [Prasol94, §2.6], as well as the related "Generalized Sylvester's identity" in [Prasol94, §2.7] and [Prasol15, Teorema 2.9.1] and [Mohr53]).

If $A \in \mathbb{K}^{u \times v}$ is a $u \times v$-matrix, and if $I \subseteq [u]$ and $J \subseteq [v]$, then $\operatorname{sub}_I^J A$ shall mean the submatrix of A obtained by removing all rows whose indices are not in I and removing all columns whose indices are not in J. (Rigorously speaking, if $A = (a_{i,j})_{1 \leq i \leq u,\ 1 \leq j \leq v}$ and $I = \{i_1 < i_2 < \cdots < i_p\}$ and $J = \{j_1 < j_2 < \cdots < j_q\}$, then $\operatorname{sub}_I^J A = \left(a_{i_x, j_y}\right)_{1 \leq x \leq p,\ 1 \leq y \leq q}$.) When $|I| = |J|$, then the submatrix $\operatorname{sub}_I^J A$ is square; its determinant $\det\left(\operatorname{sub}_I^J A\right)$ is called a *minor* of A.

Our main two results are the following:

Theorem 2.1. Let $A = (a_{i,j})_{1 \leq i \leq n+1,\ 1 \leq j \leq n+1} \in \mathbb{K}^{(n+1) \times (n+1)}$ and $B = (b_{i,j})_{1 \leq i \leq n+1,\ 1 \leq j \leq n+1} \in \mathbb{K}^{(n+1) \times (n+1)}$ be such that $b_{n+1,n+1} = 0$. Let W be the $P_k \times P_k$-matrix whose (I, J)-th entry (for all $I \in P_k$ and $J \in P_k$) is

$$\det\left(\operatorname{sub}_{I+}^{J+} A\right) \det\left(\operatorname{sub}_{I+}^{J+} B\right).$$

Then, $\det A \mid \det W$.

Theorem 2.2. Let $A = (a_{i,j})_{1 \leq i \leq n+1,\ 1 \leq j \leq n+1} \in \mathbb{K}^{(n+1) \times (n+1)}$ and $B = (b_{i,j})_{1 \leq i \leq n+1,\ 1 \leq j \leq n+1} \in \mathbb{K}^{(n+1) \times (n+1)}$ be such that $a_{n+1,n+1} = 0$ and $b_{n+1,n+1} = 0$. Define the $P_k \times P_k$-matrix W as in Theorem 2.1. Then, $(\det A)(\det B) \mid \det W$.

Example 2.3. For this example, set $k = 1$. Then, $P_k = P_1 = \{\{1\}, \{2\}, \ldots, \{n\}\}$. Thus, the map

$$[n] \to P_k, \qquad i \mapsto \{i\}$$

[4] Here, we are using the concepts of $P \times P$-matrices (where P is a finite set) and their determinants. Both of these concepts are folklore; a brief introduction can be found in [Grinbe18, §1].

is a bijection. Use this bijection to identify the elements $1, 2, \ldots, n$ of $[n]$ with the elements $\{1\}, \{2\}, \ldots, \{n\}$ of P_k. Thus, the $P_k \times P_k$-matrix W in Theorem 2.1 becomes the $n \times n$-matrix

$$\left(\underbrace{\det\left(\operatorname{sub}_{\{i\}+}^{\{j\}+} A\right)}_{=a_{i,j}a_{n+1,n+1}-a_{i,n+1}a_{n+1,j}} \quad \underbrace{\det\left(\operatorname{sub}_{\{i\}+}^{\{j\}+} B\right)}_{=b_{i,j}b_{n+1,n+1}-b_{i,n+1}b_{n+1,j}} \right)_{1\le i\le n,\ 1\le j\le n}$$
$$= \left(\left(a_{i,j}a_{n+1,n+1} - a_{i,n+1}a_{n+1,j}\right) \left(b_{i,j} \underbrace{b_{n+1,n+1}}_{=0} - b_{i,n+1}b_{n+1,j} \right) \right)_{1\le i\le n,\ 1\le j\le n}$$
$$= \left(\left(a_{i,j}a_{n+1,n+1} - a_{i,n+1}a_{n+1,j}\right) \left(-b_{i,n+1}b_{n+1,j} \right) \right)_{1\le i\le n,\ 1\le j\le n}.$$

This is the matrix obtained from $\left(\left(a_{i,j}a_{n+1,n+1} - a_{i,n+1}a_{n+1,j}\right)\right)_{1\le i\le n,\ 1\le j\le n}$ by multiplying the i-th row with $-b_{i,n+1}$ for all $i \in [n]$ and multiplying the j-th column with $b_{n+1,j}$ for all $j \in [n]$. Thus, the claim of Theorem 2.1 follows from the classical fact that

$$\det\left(\left(a_{i,j}a_{n+1,n+1} - a_{i,n+1}a_{n+1,j}\right)_{1\le i\le n,\ 1\le j\le n} \right) = a_{n+1,n+1}^{n-1} \cdot \det A.$$

This fact is known as Chio pivotal condensation (see, e.g., [KarZha16, Theorem 0.1]), and is a particular case of Sylvester's identity ([Prasol94, §2.7]).

Example 2.4. For this example, set $k = 2$, and consider the situation of Theorem 2.1 again. Then, $P_k = P_2 = \{\{i, j\} \mid 1 \le i < j \le n\}$. If $\{i, j\} \in P_2$ and $\{k, l\} \in P_2$ satisfy $i < j$ and $k < l$, then the $(\{i, j\}, \{k, l\})$-th entry of W is

$$\det \underbrace{\left(\operatorname{sub}_{\{i,j\}+}^{\{k,l\}+} A\right)}_{=\begin{pmatrix} a_{i,k} & a_{i,l} & a_{i,n+1} \\ a_{j,k} & a_{j,l} & a_{j,n+1} \\ a_{n+1,k} & a_{n+1,l} & a_{n+1,n+1} \end{pmatrix}} \det \underbrace{\left(\operatorname{sub}_{\{i,j\}+}^{\{k,l\}+} B\right)}_{\substack{=\begin{pmatrix} b_{i,k} & b_{i,l} & b_{i,n+1} \\ b_{j,k} & b_{j,l} & b_{j,n+1} \\ b_{n+1,k} & b_{n+1,l} & 0 \end{pmatrix} \\ \text{(since } b_{n+1,n+1}=0\text{)}}}$$
$$= \det \begin{pmatrix} a_{i,k} & a_{i,l} & a_{i,n+1} \\ a_{j,k} & a_{j,l} & a_{j,n+1} \\ a_{n+1,k} & a_{n+1,l} & a_{n+1,n+1} \end{pmatrix} \det \begin{pmatrix} b_{i,k} & b_{i,l} & b_{i,n+1} \\ b_{j,k} & b_{j,l} & b_{j,n+1} \\ b_{n+1,k} & b_{n+1,l} & 0 \end{pmatrix}.$$

If we furthermore assume that

$$\begin{aligned} a_{n+1,n+1} &= 0, && \text{and} \\ a_{n+1,i} = a_{i,n+1} &= 1 && \text{for all } i \in [n], \qquad \text{and} \\ b_{n+1,i} = b_{i,n+1} &= 1 && \text{for all } i \in [n], \end{aligned}$$

then this entry rewrites as

$$\underbrace{\det\begin{pmatrix} a_{i,k} & a_{i,l} & 1 \\ a_{j,k} & a_{j,l} & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{=a_{j,k}+a_{i,l}-a_{i,k}-a_{j,l}} \underbrace{\det\begin{pmatrix} b_{i,k} & b_{i,l} & 1 \\ b_{j,k} & b_{j,l} & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{=b_{j,k}+b_{i,l}-b_{i,k}-b_{j,l}}$$
$$= \left(a_{j,k} + a_{i,l} - a_{i,k} - a_{j,l}\right) \left(b_{j,k} + b_{i,l} - b_{i,k} - b_{j,l}\right).$$

Hence, [GriOlv18, Theorem 10] can be obtained from Theorem 2.2 by setting $k = 2$ and $A = C_S$ and $B = C_T$ (and observing that the matrix W then equals to $W_{S,T}$).

3. The proofs

Our proofs of Theorem 2.1 and Theorem 2.2 will rely on some basic commutative algebra: the notion of a unique factorization domain ("UFD"); the concepts of coprime, prime and irreducible elements; the localization of a commutative ring at a multiplicative subset. This all appears in most textbooks on abstract algebra; for example, [Knapp16, Sections VIII.4 and VIII.10] is a good reference[5].

The *content* of a polynomial p over a UFD is defined to be the greatest common divisor of the coefficients of p. For example, the polynomial $4x^2 + 6y^2 \in \mathbb{Z}[x, y]$ has content $\gcd(4, 6) = 2$. (Of course, in a general UFD, the greatest common divisor is defined only up to multiplication by a unit.) The following known facts are crucial to us:

Proposition 3.1. A polynomial ring over $\mathbb{Z}$ in finitely many indeterminates is always a UFD.

Proof of Proposition 3.1. Proposition 3.1 appears, e.g., in [Knapp16, Remark after Corollary 8.21]. For a constructive proof of Proposition 3.1, we refer to [MiRiRu87, Chapter IV, Theorems 4.8 and 4.9] or to [Edward05, Essay 1.4, Corollary of Theorem 1 and Corollary 1 of Theorem 2]. □

Proposition 3.2. Let p be an irreducible element of a UFD $\mathbb{K}$. Then, the quotient ring $\mathbb{K}/(p)$ is an integral domain.

Proof of Proposition 3.2. First of all, we recall that any irreducible element of a UFD is prime (indeed, this follows from [Knapp16, Proposition 8.13]). Thus, the element p of $\mathbb{K}$ is prime. Hence, [Knapp16, Proposition 8.14] shows that the ideal (p) of $\mathbb{K}$ is prime. Therefore, the quotient ring $\mathbb{K}/(p)$ is an integral domain. This proves Proposition 3.2. □

[5] We call "multiplicative subset" what Knapp (in [Knapp16, Section VIII.10]) calls a "multiplicative system".

We shall furthermore use the following properties of contents (whose proofs are easy):

Proposition 3.3. Let $\mathbb{U}$ be a UFD. Let $\mathbb{F}$ be the field of fractions of $\mathbb{U}$. Let $p \in \mathbb{U}[x_1, x_2, \ldots, x_m]$ be a polynomial over $\mathbb{U}$. Assume that the content of p is 1. Also assume that p is irreducible when considered as a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_m]$. Then, p is also irreducible when considered as a polynomial in $\mathbb{U}[x_1, x_2, \ldots, x_m]$.

Proposition 3.4. Let $\mathbb{U}$ be a UFD. Let $p, q \in \mathbb{U}[x_1, x_2, \ldots, x_m]$ be two polynomials over $\mathbb{U}$. Assume that both p and q have content 1, and assume furthermore that p and q don't have any indeterminates in common (i.e., there is no $i \in [m]$ such that $\deg_{x_i} p > 0$ and $\deg_{x_i} q > 0$). Then, p and q are coprime.

The next simple fact states that for any positive integer n, the determinant of the "generic $n \times n$-matrix" (i.e., of the $n \times n$-matrix whose n^2 entries are distinct indeterminates in a polynomial ring over $\mathbb{Z}$) is irreducible as a polynomial over $\mathbb{Z}$:

Corollary 3.5. Let n be a positive integer. Let $\mathbb{G}$ be the polynomial ring $\mathbb{Z}\left[a_{i,j} \mid (i,j) \in [n]^2\right]$. Let $\overline{A} \in \mathbb{G}^{n \times n}$ be the $n \times n$-matrix $\left(a_{i,j}\right)_{1 \le i \le n,\ 1 \le j \le n}$. Then, the element $\det \overline{A}$ of $\mathbb{G}$ is irreducible.

Proof of Corollary 3.5. A well-known fact (e.g., [DeKuRo78, Lemma 5.12]) shows that $\det \overline{A}$ is irreducible as an element of $\mathbb{Q}\left[a_{i,j} \mid (i,j) \in [n]^2\right]$. This yields (using Proposition 3.3) that $\det \overline{A}$ is irreducible as an element of $\mathbb{Z}\left[a_{i,j} \mid (i,j) \in [n]^2\right]$ as well, since the polynomial $\det \overline{A}$ has content 1. This proves Corollary 3.5. $\square$

An element a of a commutative ring $\mathbb{A}$ is said to be *regular* if every $b \in \mathbb{A}$ satisfying $ab = 0$ must satisfy $b = 0$. (Regular elements are also known as *non-zero-divisors*.) In a polynomial ring, each indeterminate is regular; hence, each monomial (without coefficient) is regular (since any product of two regular elements is regular). The following fact is easy to see:[6]

[6]We recall a few standard concepts from commutative algebra:

Let $\mathbb{K}$ be a commutative ring. A *multiplicative subset* of $\mathbb{K}$ means a subset S of $\mathbb{K}$ that contains the unity $1_{\mathbb{K}}$ of $\mathbb{K}$ and has the property that every $a, b \in S$ satisfy $ab \in S$.

If S is a multiplicative subset of $\mathbb{K}$, then the *localization* of $\mathbb{K}$ at S is defined as follows: Let $\sim$ be the binary relation on the set $\mathbb{K} \times S$ defined by

$$\left((r,s) \sim (r',s')\right) \iff \left(t\left(rs' - sr'\right) = 0 \text{ for some } t \in S\right).$$

Then, it is easy to see that $\sim$ is an equivalence relation. The set $\mathbb{L}$ of its equivalence classes $[(r,s)]$ can be equipped with a ring structure via the rules $[(r,s)] + [(r',s')] = [(rs' + sr', ss')]$ and $[(r,s)] \cdot [(r',s')] = [(rr', ss')]$ (with zero element $[(0,1)]$ and unity $[(1,1)]$). The resulting ring $\mathbb{L}$ is commutative, and is known as the localization of $\mathbb{K}$ at S. (This generalizes the construction of $\mathbb{Q}$ from $\mathbb{Z}$ known from high school.)

Proposition 3.6. Let $\mathbb{K}$ be a commutative ring. Let S be a multiplicative subset of $\mathbb{K}$ such that all elements of S are regular. Let $\mathbb{L}$ be the localization of the ring $\mathbb{K}$ at S. Then:

(a) The canonical ring homomorphism from $\mathbb{K}$ to $\mathbb{L}$ is injective. We shall thus consider it as an embedding.

(b) If $\mathbb{K}$ is an integral domain, then $\mathbb{L}$ is an integral domain.

(c) Let a and b be two elements of $\mathbb{K}$. Then, we have the following logical equivalence:

$$(a \mid b \text{ in } \mathbb{L}) \iff (a \mid sb \text{ in } \mathbb{K} \text{ for some } s \in S).$$

Matrices over arbitrary commutative rings can behave a lot less predictably than matrices over fields. However, matrices over integral domains still show a lot of the latter good behavior, such as the following:

Proposition 3.7. Let P be a finite set. Let $\mathbb{M}$ be an integral domain. Let $W \in \mathbb{M}^{P \times P}$ be a $P \times P$-matrix over $\mathbb{M}$. Let $\mathbf{u} \in \mathbb{M}^P$ be a vector such that $\mathbf{u} \neq 0$ and $W\mathbf{u} = 0$. Here, $\mathbf{u}$ is considered as a "column vector", so that $W\mathbf{u}$ is defined by

$$W\mathbf{u} = \left(\sum_{q \in P} w_{p,q} u_q\right)_{p \in P}, \qquad \text{where } W = \left(w_{p,q}\right)_{(p,q) \in P \times P} \text{ and } \mathbf{u} = \left(u_p\right)_{p \in P}.$$

Then, $\det W = 0$.

Proof of Proposition 3.7. Let $m = |P|$. Then, we can view the $P \times P$-matrix W as an $m \times m$-matrix (by "numerical reindexing", as explained in [Grinbe18, §1]), and we can view the vector $\mathbf{u}$ as a column vector of size m. Let us do this from here on.

Let $\mathbb{F}$ be the quotient field of the integral domain $\mathbb{M}$. Thus, there is a canonical embedding of $\mathbb{M}$ into $\mathbb{F}$. Hence, we can view the matrix $W \in \mathbb{M}^{m \times m}$ as a matrix over $\mathbb{F}$, and we can view the vector $\mathbf{u} \in \mathbb{M}^m$ as a vector over $\mathbb{F}$. Let us do so from here on. We are now in the realm of classical linear algebra over fields: The vector $\mathbf{u} \in \mathbb{F}^m$ is nonzero (since $\mathbf{u} \neq 0$) and belongs to the kernel of the $m \times m$-matrix $W \in \mathbb{F}^{m \times m}$ (since $W\mathbf{u} = 0$). Hence, the kernel of the matrix W is nontrivial. In other words, this matrix W is singular. Thus, $\det W = 0$ by a classical fact of linear algebra. This proves Proposition 3.7. $\square$

Let us next recall an identity for determinants (a version of the Cauchy–Binet formula):

The element $[(r,s)]$ of $\mathbb{L}$ is denoted by $\frac{r}{s}$. There is a canonical ring homomorphism from $\mathbb{K}$ to $\mathbb{L}$ that sends each $r \in \mathbb{K}$ to $[(r,1)] = \frac{r}{1} \in \mathbb{L}$.

When all elements of the multiplicative subset S are regular, the statement "$t(rs' - sr') = 0$ for some $t \in S$" in the definition of the relation $\sim$ can be rewritten in the equivalent (but much simpler) form "$rs' = sr'$" (which is even more reminiscent of the construction of $\mathbb{Q}$).

Lemma 3.8. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $p \in \mathbb{N}$. Let $A \in \mathbb{K}^{n \times p}$ be an $n \times p$-matrix. Let $B \in \mathbb{K}^{p \times m}$ be a $p \times m$-matrix. Let $k \in \mathbb{N}$. Let P be a subset of $[n]$ such that $|P| = k$. Let Q be a subset of $[m]$ such that $|Q| = k$. Then,

$$\det\left(\operatorname{sub}_P^Q (AB)\right) = \sum_{\substack{R \subseteq [p];\\ |R| = k}} \det\left(\operatorname{sub}_P^R A\right) \cdot \det\left(\operatorname{sub}_R^Q B\right).$$

Lemma 3.8 is [Grinbe17, Corollary 7.251] (except that we are using the notation $\operatorname{sub}_I^J C$ for what is called $\operatorname{sub}_{w(I)}^{w(J)} C$ in [Grinbe17]). It also appears in [Gantma00, Chapter I, (19)] (where it is stated using p-tuples instead of subsets). $\square$

The next lemma is just a particular case of Theorem 2.1, but it is a helpful stepping stone on the way to proving the latter theorem:

Lemma 3.9. Let $A = \left(a_{i,j}\right)_{1 \leq i \leq n+1,\ 1 \leq j \leq n+1} \in \mathbb{K}^{(n+1) \times (n+1)}$ and $B = \left(b_{i,j}\right)_{1 \leq i \leq n+1,\ 1 \leq j \leq n+1} \in \mathbb{K}^{(n+1) \times (n+1)}$ be such that $b_{n+1,n+1} = 0$. Assume further that

$$a_{n+1,j} = 0 \qquad \text{for all } j \in [n]. \tag{1}$$

Define the $P_k \times P_k$-matrix W as in Theorem 2.1. Then, $\det A \mid \det W$.

The following proof is inspired by [GriOlv18, proof of Theorem 10].

Proof of Lemma 3.9. We WLOG assume that $\mathbb{K}$ is the polynomial ring over $\mathbb{Z}$ in $n^2 + (n+1) + \left((n+1)^2 - 1\right)$ indeterminates

$$\begin{aligned} &a_{i,j} && \text{for all } i \in [n] \text{ and } j \in [n]; \\ &a_{i,n+1} && \text{for all } i \in [n+1]; \\ &b_{i,j} && \text{for all } i \in [n+1] \text{ and } j \in [n+1] \text{ except for } b_{n+1,n+1}. \end{aligned}$$

And, of course, we assume that the entries of A and B that are not zero by assumption are these indeterminates.[7]

The ring $\mathbb{K}$ is a UFD (by Proposition 3.1).

We WLOG assume that $n > 0$ (otherwise, the result follows from $\det W = \det\begin{pmatrix} 0 \end{pmatrix} = 0$).

The set P_k is nonempty (since $k \in \{0, 1, \ldots, n\}$); thus, $|P_k| \geq 1$.

Let $\overline{A}$ be the $n \times n$-matrix $\left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n} \in \mathbb{K}^{n \times n}$. Then, because of (1), we have

$$\det A = a_{n+1,n+1} \cdot \det \overline{A} \tag{2}$$

[7] These assumptions are legitimate, because if we can prove Lemma 3.9 under these assumptions, then the universal property of polynomial rings shows that Lemma 3.9 holds in the general case.

(by [Grinbe17, Theorem 6.43], applied to $n+1$ instead of n).

The matrix $\overline{A}$ is a completely generic $n \times n$-matrix (i.e., its entries are distinct indeterminates); thus, its determinant $\det \overline{A}$ is an irreducible polynomial in the polynomial ring $\mathbb{Z}\left[a_{i,j} \mid (i,j) \in [n]^2\right]$ (by Corollary 3.5). Hence, $\det \overline{A}$ also is an irreducible polynomial in the ring $\mathbb{K}$ (since $\mathbb{K}$ differs from $\mathbb{Z}\left[a_{i,j} \mid (i,j) \in [n]^2\right]$ only in having more variables, which clearly cannot contribute any factors to $\det \overline{A}$). Thus, Proposition 3.2 (applied to $p = \det \overline{A}$) shows that the quotient ring $\mathbb{K}/\left(\det \overline{A}\right)$ is an integral domain.

Let $\mathbb{M}$ be the quotient ring $\mathbb{K}/\left(\det \overline{A}\right)$. Then, $\mathbb{M}$ is an integral domain (since $\mathbb{K}/\left(\det \overline{A}\right)$ is an integral domain). All monomials in the variables $b_{i,j}$ (with $(i,j) \neq (n+1,n+1)$) are nonzero in $\mathbb{M}$. Likewise, $a_{n+1,n+1} \neq 0$ in $\mathbb{M}$.

Let w be the element $\prod\limits_{j\in[n]} b_{n+1,j} \in \mathbb{M}$. (Strictly speaking, we mean the canonical projection of $\prod\limits_{j\in[n]} b_{n+1,j} \in \mathbb{K}$ onto the quotient ring $\mathbb{M}$.) Then, w is a nonzero element of the integral domain $\mathbb{M}$ (since $b_{n+1,j} \neq 0$ in $\mathbb{M}$ for all $j \in [n]$).

For each $i \in [n]$, we define $z_i \in \mathbb{M}$ by $z_i = \prod\limits_{\substack{j\in[n];\\ j\neq i}} b_{n+1,j}$ (projected onto $\mathbb{M}$). This is a nonzero element of $\mathbb{M}$. In $\mathbb{M}$, we have

$$b_{n+1,i} z_i = b_{n+1} \prod_{\substack{j\in[n];\\ j\neq i}} b_{n+1,j} = \prod_{j\in[n]} b_{n+1,j} = w \tag{3}$$

for all $i \in [n]$.

We need another piece of notation: If M is a $p \times q$-matrix, and if $u \in [p]$ and $v \in [q]$, then $M_{\sim u,\sim v}$ denotes the $(p-1) \times (q-1)$-matrix obtained from M by removing the u-th row and the v-th column.

The matrix $A_{\sim 1,\sim(n+1)}$ has determinant 0 (because (1) shows that its last row consists of zeroes). In other words, $\det\left(A_{\sim 1,\sim(n+1)}\right) = 0$.

Also, due to (1), we see that each $i \in [n]$ satisfies

$$\det\left(A_{\sim 1,\sim i}\right) = a_{n+1,n+1} \cdot \det\left(\overline{A}_{\sim 1,\sim i}\right) \tag{4}$$

(by [Grinbe17, Theorem 6.43], applied to $A_{\sim 1,\sim i}$ instead of A), because the last row of the matrix $A_{\sim 1,\sim i}$ is $(0,0,\ldots,0,a_{n+1,n+1})$.

For each $i \in [n+1]$, we define an element $u_i \in \mathbb{M}$ by

$$u_i = \begin{cases} z_i (-1)^i \det\left(A_{\sim 1,\sim i}\right), & \text{if } i \in [n]; \\ 1, & \text{if } i = n+1 \end{cases}.$$

All these $n+1$ elements $u_1, u_2, \ldots, u_{n+1}$ of $\mathbb{M}$ are nonzero[8].

Let $\mathbf{u} = (u_J)_{J \in P_k} \in \mathbb{M}^{P_k}$ be the vector defined by

$$u_J = \prod_{j \in J} u_j.$$

Then, the entries of the vector $\mathbf{u}$ are nonzero (because they are products of the nonzero elements $u_1, u_2, \ldots, u_{n+1}$ of the integral domain $\mathbb{M}$). Since the vector $\mathbf{u}$ has at least one entry (because $|P_k| \geq 1$), we thus conclude that $\mathbf{u} \neq 0$.

Let Δ be the diagonal matrix $\Delta = \operatorname{diag}(u_1, u_2, \ldots, u_{n+1}) \in \mathbb{M}^{(n+1)\times(n+1)}$.

Let $\mathbf{x} \in \mathbb{M}^{n+1}$ be the column vector defined by

$$\mathbf{x} = \left((-1)^1 \det (A_{\sim 1, \sim 1}), (-1)^2 \det (A_{\sim 1, \sim 2}), \ldots, (-1)^{n+1} \det \left(A_{\sim 1, \sim (n+1)} \right) \right)^T.$$

Let $(e_1, e_2, \ldots, e_{n+1})$ be the standard basis of the free $\mathbb{M}$-module $\mathbb{M}^{n+1}$. Thus, for any $(n+1) \times (n+1)$-matrix $C \in \mathbb{M}^{(n+1)\times(n+1)}$ and any $j \in \{1, 2, \ldots, n+1\}$, we have

$$(\text{the } j\text{-th column of the matrix } C) = Ce_j. \tag{5}$$

Now, using Laplace expansion, it is easy to see that

$$A\mathbf{x} = -\det A \cdot e_1. \tag{6}$$

[*Proof of (6):* The quickest way to check this is to use the adjugate $\operatorname{adj} A$ of the matrix A. A standard fact ([Grinbe17, Theorem 6.100]) says that $A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = \det A \cdot I_{n+1}$. But the definition of $\operatorname{adj} A$ shows that the (i,j)-th entry of $\operatorname{adj} A$ is $(-1)^{i+j} \deg \left(A_{\sim j, \sim i} \right)$ for each $i \in \{1, 2, \ldots, n+1\}$ and each $j \in \{1, 2, \ldots, n+1\}$. Hence, the entries in the first column of the matrix $\operatorname{adj} A$ are $(-1)^{1+1} \det (A_{\sim 1, \sim 1}), (-1)^{2+1} \det (A_{\sim 1, \sim 2}), \ldots, (-1)^{(n+1)+1} \det \left(A_{\sim 1, \sim (n+1)} \right)$ (from

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[8] *Proof.* Each $i \in [n]$ satisfies

$$u_i = z_i (-1)^i \underbrace{\det (A_{\sim 1, \sim i})}_{\substack{= a_{n+1,n+1} \cdot \det\left(\overline{A}_{\sim 1, \sim i}\right) \\ \text{(by (4))}}} = \underbrace{z_i}_{\neq 0 \text{ in } \mathbb{M}} \underbrace{(-1)^i}_{\neq 0 \text{ in } \mathbb{M}} \underbrace{a_{n+1,n+1}}_{\neq 0 \text{ in } \mathbb{M}} \cdot \underbrace{\det \left(\overline{A}_{\sim 1, \sim i} \right)}_{\substack{\neq 0 \text{ in } \mathbb{M} \\ \text{(since } \det\left(\overline{A}_{\sim 1, \sim i}\right) \text{ is a polynomial} \\ \text{of smaller degree than } \det \overline{A}\text{, and thus} \\ \text{is not a multiple of } \det \overline{A}\text{)}}}$$

$$\neq 0 \text{ in } \mathbb{M}$$

(since $\mathbb{M}$ is an integral domain). Thus, $u_1, u_2, \ldots, u_n$ are nonzero. Moreover, u_{n+1} is nonzero (since $u_{n+1} = 1$). Thus, we are done.

top to bottom). Thus,

$$\begin{aligned}
&(\text{the first column of the matrix } \operatorname{adj} A) \\
&= \left((-1)^{1+1}\det(A_{\sim 1,\sim 1}), (-1)^{2+1}\det(A_{\sim 1,\sim 2}), \ldots, (-1)^{(n+1)+1}\det\left(A_{\sim 1,\sim(n+1)}\right)\right)^T \\
&= \left(-(-1)^{1}\det(A_{\sim 1,\sim 1}), -(-1)^{2}\det(A_{\sim 1,\sim 2}), \ldots, -(-1)^{n+1}\det\left(A_{\sim 1,\sim(n+1)}\right)\right)^T \\
&\qquad \left(\text{since } (-1)^{i+1} = -(-1)^i \text{ for each } i \in \{1,2,\ldots,n+1\}\right) \\
&= -\underbrace{\left((-1)^{1}\det(A_{\sim 1,\sim 1}), (-1)^{2}\det(A_{\sim 1,\sim 2}), \ldots, (-1)^{n+1}\det\left(A_{\sim 1,\sim(n+1)}\right)\right)}_{\substack{=\mathbf{x} \\ \text{(by the definition of } \mathbf{x})}} \\
&= -\mathbf{x}.
\end{aligned}$$

Hence,

$$-\mathbf{x} = (\text{the first column of the matrix } \operatorname{adj} A) = (\operatorname{adj} A)\, e_1$$

(by (5), applied to $C = \operatorname{adj} A$ and $j = 1$). Now,

$$A \underbrace{\mathbf{x}}_{=-(-\mathbf{x})} = A\left(-(-\mathbf{x})\right) = -A \underbrace{(-\mathbf{x})}_{=(\operatorname{adj} A)e_1} = -\underbrace{A \cdot \operatorname{adj} A}_{=\det A \cdot I_{n+1}} \cdot e_1 = -\det A \cdot e_1.$$

This proves (6).]

Also, (5) (applied to $C = B^T$ and $j = n+1$) yields

$$B^T e_{n+1} = \left(\text{the } (n+1)\text{-st column of the matrix } B^T\right) = \left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,n+1}\right)^T.$$

Hence,

$$\begin{aligned}
\Delta B^T e_{n+1} &= \Delta\left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,n+1}\right)^T \\
&= \left(u_1 b_{n+1,1}, u_2 b_{n+1,2}, \ldots, u_{n+1} b_{n+1,n+1}\right)^T \qquad (7)
\end{aligned}$$

(since $\Delta = \operatorname{diag}(u_1, u_2, \ldots, u_{n+1})$).

Now, we claim that

$$u_i b_{n+1,i} = w \cdot (-1)^i \det(A_{\sim 1,\sim i}) \qquad \text{for each } i \in [n+1]. \qquad (8)$$

[*Proof of (8):* Let $i \in [n+1]$. We must prove (8). If $i = n+1$, then this is easy (indeed, in this case, both sides are zero, because $b_{n+1,n+1} = 0$ and $\det\left(A_{\sim 1,\sim(n+1)}\right) = 0$). Hence, we WLOG assume that $i \neq n+1$. Hence, $i \in [n]$. Thus, the definition of u_i yields $u_i = z_i (-1)^i \det(A_{\sim 1,\sim i})$. Hence,

$$\begin{aligned}
u_i b_{n+1,i} &= z_i (-1)^i \det(A_{\sim 1,\sim i})\, b_{n+1,i} = \underbrace{b_{n+1,i} z_i}_{\substack{=w \\ \text{(by (3))}}} (-1)^i \det(A_{\sim 1,\sim i}) \\
&= w \cdot (-1)^i \det(A_{\sim 1,\sim i}).
\end{aligned}$$

This proves (8).]

Now, (7) becomes

$$\begin{aligned}
&\Delta B^T e_{n+1} \\
&= \left(u_1 b_{n+1,1}, u_2 b_{n+1,2}, \ldots, u_{n+1} b_{n+1,n+1}\right)^T \\
&= \left(w \cdot (-1)^1 \det\left(A_{\sim 1,\sim 1}\right), w \cdot (-1)^2 \det\left(A_{\sim 1,\sim 2}\right), \ldots, w \cdot (-1)^{n+1} \det\left(A_{\sim 1,\sim (n+1)}\right)\right)^T \\
&\qquad \text{(by (8))} \\
&= w \cdot \underbrace{\left((-1)^1 \det\left(A_{\sim 1,\sim 1}\right), (-1)^2 \det\left(A_{\sim 1,\sim 2}\right), \ldots, (-1)^{n+1} \det\left(A_{\sim 1,\sim (n+1)}\right)\right)^T}_{\substack{=\mathbf{x} \\ \text{(by the definition of } \mathbf{x}\text{)}}} \\
&= w\mathbf{x}.
\end{aligned}$$

Hence,

$$\begin{aligned}
A\Delta B^T e_{n+1} = Aw\mathbf{x} = w \cdot \underbrace{A\mathbf{x}}_{\substack{=-\det A \cdot e_1 \\ \text{(by (6))}}} &= -w \cdot \underbrace{\det A}_{\substack{=a_{n+1,n+1} \cdot \det \overline{A} \\ \text{(by (2))}}} \cdot e_1 \\
&= -w \cdot a_{n+1,n+1} \cdot \underbrace{\det \overline{A}}_{\substack{=0 \\ \text{(since we are in } \mathbb{M}\text{)}}} \cdot e_1 = 0.
\end{aligned}$$

In other words, the $(n+1)$-st column of the matrix $A\Delta B^T$ is 0 (since the $(n+1)$-st column of the matrix $A\Delta B^T$ is $A\Delta B^T e_{n+1}$ (by (5), applied to $C = A\Delta B^T$ and $j = n+1$)).

Now, fix $I \in P_k$. Then, the last column of the matrix $\operatorname{sub}_{I+}^{I+}\left(A\Delta B^T\right)$ is 0 (because this column is a piece of the $(n+1)$-st column of the matrix $A\Delta B^T$, but as we have just shown the latter column is 0). Thus, $\det\left(\operatorname{sub}_{I+}^{I+}\left(A\Delta B^T\right)\right) = 0$.

But $A\Delta B^T = A\left(\Delta B^T\right)$. Hence, Lemma 3.8 (applied to $\mathbb{M}$, $n+1$, $n+1$, $n+1$, ΔB^T, $k+1$, $I+$ and $I+$ instead of $\mathbb{K}$, n, m, p, B, k, P and Q) yields

$$\det\left(\operatorname{sub}_{I+}^{I+}\left(A\Delta B^T\right)\right) = \sum_{\substack{R \subseteq [n+1]; \\ |R| = k+1}} \det\left(\operatorname{sub}_{I+}^{R} A\right) \det\left(\operatorname{sub}_{R}^{I+}\left(\Delta B^T\right)\right).$$

Comparing this with $\det\left(\operatorname{sub}_{I+}^{I+}\left(A\Delta B^T\right)\right) = 0$, we obtain

$$\begin{aligned}
0 &= \sum_{\substack{R\subseteq[n+1];\\ |R|=k+1}} \det\left(\operatorname{sub}_{I+}^{R} A\right) \det\left(\operatorname{sub}_{R}^{I+}\left(\Delta B^T\right)\right) \\
&= \sum_{\substack{R\subseteq[n+1];\\ |R|=k+1;\\ n+1\in R}} \det\left(\operatorname{sub}_{I+}^{R} A\right) \det\left(\operatorname{sub}_{R}^{I+}\left(\Delta B^T\right)\right) \\
&\qquad + \sum_{\substack{R\subseteq[n+1];\\ |R|=k+1;\\ n+1\notin R}} \underbrace{\det\left(\operatorname{sub}_{I+}^{R} A\right)}_{\substack{=0\\ \text{(because the last row of the}\\ \text{matrix }\operatorname{sub}_{I+}^{R} A\text{ consists of zeroes}\\ \text{(by (1), since }n+1\notin R\text{, but }n+1\in I+\text{))}}} \det\left(\operatorname{sub}_{R}^{I+}\left(\Delta B^T\right)\right) \\
&= \sum_{\substack{R\subseteq[n+1];\\ |R|=k+1;\\ n+1\in R}} \det\left(\operatorname{sub}_{I+}^{R} A\right) \det\left(\operatorname{sub}_{R}^{I+}\left(\Delta B^T\right)\right) \\
&= \sum_{J\in P_k} \det\left(\operatorname{sub}_{I+}^{J+} A\right) \underbrace{\det\left(\operatorname{sub}_{J+}^{I+}\left(\Delta B^T\right)\right)}_{\substack{=\left(\prod_{j\in J+} u_j\right)\det\left(\operatorname{sub}_{J+}^{I+}\left(B^T\right)\right)\\ \text{(since }\Delta\text{ is a diagonal matrix,}\\ \text{and thus }\Delta B^T\text{ is just }B^T\text{ with rescaled rows)}}} \\
&\qquad \left(\begin{array}{c}\text{since the subsets } R \text{ of } [n+1] \text{ satisfying } |R| = k+1\\ \text{and } n+1\in R \text{ can be parametrized as } J+\\ \text{with } J \text{ ranging over } P_k\end{array}\right) \\
&= \sum_{J\in P_k} \det\left(\operatorname{sub}_{I+}^{J+} A\right) \underbrace{\left(\prod_{j\in J+} u_j\right)}_{\substack{=\prod_{j\in J} u_j\\ \text{(since }u_{n+1}=1\text{)}}} \underbrace{\det\left(\operatorname{sub}_{J+}^{I+}\left(B^T\right)\right)}_{\substack{=\det\left(\operatorname{sub}_{I+}^{J+} B\right)\\ \text{(since }\det\left(C^T\right)=\det C\\ \text{for any square matrix }C\text{)}}} \\
&= \sum_{J\in P_k} \det\left(\operatorname{sub}_{I+}^{J+} A\right) \underbrace{\left(\prod_{j\in J} u_j\right)}_{\substack{=u_J\\ \text{(by the definition of }u_J\text{)}}} \det\left(\operatorname{sub}_{I+}^{J+} B\right) \\
&= \sum_{J\in P_k} \det\left(\operatorname{sub}_{I+}^{J+} A\right) \det\left(\operatorname{sub}_{I+}^{J+} B\right) u_J.
\end{aligned}$$

Now, forget that we fixed I. We thus have proven that

$$0 = \sum_{J\in P_k} \det\left(\operatorname{sub}_{I+}^{J+} A\right) \det\left(\operatorname{sub}_{I+}^{J+} B\right) u_J \qquad (9)$$

for each $I \in P_k$. This rewrites as $W\mathbf{u} = 0$ (indeed, the left hand side of (9) is the I-th entry of the zero vector 0, whereas the right hand side of (9) is the I-th entry of $W\mathbf{u}$).

Now, consider the matrix W as a matrix in $\mathbb{M}^{P_k \times P_k}$. Then, Proposition 3.7 (applied to $P = P_k$) yields $\det W = 0$ in $\mathbb{M}$ (since $\mathbf{u} \neq 0$ and $W\mathbf{u} = 0$). In view of the definition of $\mathbb{M}$, this rewrites as $\det \overline{A} \mid \det W$ in $\mathbb{K}$.

Let us consider the matrix W again as a matrix over $\mathbb{K}$. Each entry of W has the form $\det \left(\operatorname{sub}_{I+}^{J+} A \right) \det \left(\operatorname{sub}_{I+}^{J+} B \right)$ for some $I, J \in P_k$. Thus, all entries of W are multiples of $a_{n+1,n+1}$ (since $\det \left(\operatorname{sub}_{I+}^{J+} A \right)$ is a multiple of $a_{n+1,n+1}$ for all $I, J \in P_k$ [9]). Hence, the determinant of W is a multiple of $(a_{n+1,n+1})^{|P_k|}$, thus a multiple of $a_{n+1,n+1}$ (since $|P_k| \geq 1$). In other words, $a_{n+1,n+1} \mid \det W$ in $\mathbb{K}$.

Recall that $\mathbb{K}$ is a UFD. Also, the two polynomials $a_{n+1,n+1}$ and $\det \overline{A}$ in $\mathbb{K}$ both have content 1, and don't have any indeterminates in common; thus, these two polynomials are coprime (by Proposition 3.4). Hence, any polynomial in $\mathbb{K}$ that is divisible by both $a_{n+1,n+1}$ and $\det \overline{A}$ must be divisible by the product $a_{n+1,n+1} \cdot \det \overline{A}$ as well. Thus, from $a_{n+1,n+1} \mid \det W$ and $\det \overline{A} \mid \det W$, we obtain $a_{n+1,n+1} \cdot \det \overline{A} \mid \det W$. In view of (2), this rewrites as $\det A \mid \det W$. This proves Lemma 3.9. □

We shall now derive Theorem 2.2 from Lemma 3.9, following the same idea as in [Prasol94, §2.7] and [Prasol15, Teorema 2.9.1] and [Mohr53]:

Proof of Theorem 2.1. We WLOG assume that $n > 0$ (otherwise, the result follows from $\det W = \det \left(\begin{array}{c} 0 \end{array} \right) = 0$).

We WLOG assume that $\mathbb{K}$ is the polynomial ring over $\mathbb{Z}$ in $(n+1)^2 + \left((n+1)^2 - 1 \right)$ indeterminates

$$\begin{aligned} &a_{i,j} && \text{for all } i \in [n+1] \text{ and } j \in [n+1]\,; \\ &b_{i,j} && \text{for all } i \in [n+1] \text{ and } j \in [n+1] \text{ except for } b_{n+1,n+1}. \end{aligned}$$

And, of course, we assume that the entries of A and B that are not zero by assumption are these indeterminates. Proposition 3.1 shows that the ring $\mathbb{K}$ is a UFD (since it is a polynomial ring over $\mathbb{Z}$).

Let S be the multiplicative subset $\left\{ a_{n+1,n+1}^p \mid p \in \mathbb{N} \right\}$ of $\mathbb{K}$. Then, all elements of S are regular (since they are monomials in a polynomial ring).

Let $\mathbb{L}$ be the localization of the commutative ring $\mathbb{K}$ at the multiplicative subset S. Then, Proposition 3.6 **(a)** shows that the canonical ring homomorphism from $\mathbb{K}$ to $\mathbb{L}$ is injective; we shall thus consider it as an embedding. Also, Proposition 3.6 **(b)** shows that $\mathbb{L}$ is an integral domain.

[9] *Proof.* Let $I, J \in P_k$. Then, the equality (1) shows that the last row of the matrix $\operatorname{sub}_{I+}^{J+} A$ is $(0, 0, \ldots, 0, a_{n+1,n+1})$. Hence, an application of [Grinbe17, Theorem 6.43] shows that $\det \left(\operatorname{sub}_{I+}^{J+} A \right) = a_{n+1,n+1} \det \left(\operatorname{sub}_{I}^{J} A \right)$. Thus, $\det \left(\operatorname{sub}_{I+}^{J+} A \right)$ is a multiple of $a_{n+1,n+1}$, qed.

We claim that

$$\det A \mid \det W \text{ in } \mathbb{L}. \tag{10}$$

[*Proof of (10):* Consider A, B and W as matrices over $\mathbb{L}$. The entry $a_{n+1,n+1}$ of A is invertible in $\mathbb{L}$ (by the construction of $\mathbb{L}$). Hence, we can subtract appropriate scalar multiples[10] of the $(n+1)$-st column of A from each other column of A to ensure that all entries of the last row of A become 0, except for $a_{n+1,n+1}$. (Specifically, for each $j \in [n]$, we have to subtract $a_{j,n+1}/a_{n+1,n+1}$ times the $(n+1)$-st column of A from the j-th column of A.) All these column transformations preserve the determinant $\det A$, and also preserve the minors $\det\left(\operatorname{sub}_{I+}^{J+} A\right)$ for all $I, J \in P_k$ (because when the $(n+1)$-st column of A is subtracted from another column of A, the matrix $\operatorname{sub}_{I+}^{J+} A$ either stays the same or undergoes an analogous column transformation[11], which preserves its determinant); thus, they preserve the matrix W. Hence, we can replace A by the result of these transformations. This new matrix A satisfies (1). Hence, Lemma 3.9 (applied to $\mathbb{L}$ instead of $\mathbb{K}$) yields that $\det A \mid \det W$ in $\mathbb{L}$. This proves (10).]

But we must prove that $\det A \mid \det W$ in $\mathbb{K}$. Fortunately, this is easy: Since $\mathbb{K}$ embeds into $\mathbb{L}$, we can translate our result "$\det A \mid \det W$ in $\mathbb{L}$" as "$\det A \mid a_{n+1,n+1}^p \det W$ in $\mathbb{K}$ for an appropriate $p \in \mathbb{N}$" (by Proposition 3.6 **(c)**, applied to $a = \det A$ and $b = \det W$). Consider this p. The polynomial $a_{n+1,n+1} \in \mathbb{K}$ is coprime to $\det A$ (this is easily checked[12]); thus, its power $a_{n+1,n+1}^p$ is coprime to $\det A$ as well. Hence, we can cancel the $a_{n+1,n+1}^p$ from the divisibility $\det A \mid a_{n+1,n+1}^p \det W$, and conclude that $\det A \mid \det W$ in $\mathbb{K}$. This proves Theorem 2.1. □

Proof of Theorem 2.2. We WLOG assume that $\mathbb{K}$ is the polynomial ring over $\mathbb{Z}$ in the $\left((n+1)^2 - 1\right) + \left((n+1)^2 - 1\right)$ indeterminates

$$\begin{array}{ll} a_{i,j} & \text{for all } i \in [n+1] \text{ and } j \in [n+1] \text{ except for } a_{n+1,n+1}; \\ b_{i,j} & \text{for all } i \in [n+1] \text{ and } j \in [n+1] \text{ except for } b_{n+1,n+1}. \end{array}$$

And, of course, we assume that the entries of A and B that are not zero by assumption are these indeterminates. The ring $\mathbb{K}$ is a UFD (by Proposition 3.1).

WLOG assume that $n > 0$ (otherwise, the result follows from $\det W = \det\begin{pmatrix} 0 \end{pmatrix} = 0$). Thus, the monomial $a_{1,n+1}a_{2,n}\cdots a_{n+1,1} = \prod_{i\in[n+1]} a_{i,n+2-i}$ occurs in the polynomial $\det A$ with coefficient ± 1. Hence, the polynomial $\det A$ has content 1. Similarly, the polynomial $\det B$ has content 1.

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[10]The scalars, of course, come from $\mathbb{L}$ here.

[11]Here we are using the fact that $n+1 \in J+$ (so that the matrix $\operatorname{sub}_{I+}^{J+} A$ contains part of the $(n+1)$-st column of A).

[12]*Proof.* The polynomial $\det A$ contains the monomial $a_{1,n+1}a_{2,n}\cdots a_{n+1,1} = \prod_{i\in[n+1]} a_{i,n+2-i}$, and thus is not a multiple of $a_{n+1,n+1}$. Hence, it is coprime to $a_{n+1,n+1}$ (since the only non-unit divisor of $a_{n+1,n+1}$ is $a_{n+1,n+1}$ itself, up to scaling by units).

Theorem 2.1 yields $\det A \mid \det W$. The same argument yields $\det B \mid \det W$ (since the matrices A and B play symmetric roles in the construction of W). But Proposition 3.4 shows that the polynomials $\det A$ and $\det B$ in $\mathbb{K}$ are coprime (because they have content 1, and don't have any indeterminates in common). Thus, any polynomial in $\mathbb{K}$ that is divisible by both $\det A$ and $\det B$ must be divisible by the product $(\det A)(\det B)$ as well. Thus, from $\det A \mid \det W$ and $\det B \mid \det W$, we obtain $(\det A)(\det B) \mid \det W$. This proves Theorem 2.2. $\square$

4. Further questions

While Theorems 2.1 and 2.2 are now proven, the field appears far from fully harvested. Three questions readily emerge:

Question 4.1. What can be said about $\dfrac{\det W}{\det A}$ (in Theorem 2.1) and $\dfrac{\det W}{(\det A)(\det B)}$ (in Theorem 2.2)? Are there formulas?

Question 4.2. Are there more direct proofs of Theorems 2.1 and 2.2, avoiding the use of polynomial rings and their properties and instead "staying inside $\mathbb{K}$"? Such proofs might help answer the previous question.

Question 4.3. The entries of our matrix W were products of minors of two $(n+1) \times (n+1)$-matrices that each use the last row and the last column. What can be said about products of minors of two $(n+m) \times (n+m)$-matrices that each use the last m rows and the last m columns, where m is an arbitrary positive integer? The "Generalized Sylvester's identity" in [Prasol94, §2.7] answers this for the case of one matrix. It is not quite obvious what the right analogues of the conditions $a_{n+1,n+1} = 0$ and $b_{n+1,n+1} = 0$ are; furthermore, nontrivial examples become even more computationally challenging.

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