# A note on bilinear forms

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## 1. Introduction

In this note, we shall prove some elementary linear-algebraic properties of bilinear forms. These properties generalize some of the standard facts about nondegenerate bilinear forms on finite-dimensional vector spaces (e.g., the fact that $\left(V^{\perp}\right)^{\perp} = V$ for a vector subspace $V$ of a vector space $A$ equipped with a nondegenerate symmetric bilinear form) to bilinear forms which may be degenerate. They are unlikely to be new, but I have not found them explored anywhere, whence this note.

## 2. Bilinear maps and forms

We fix a field $\mathbf{k}$. This field will be fixed for the rest of this note. The notion of a "vector space" will always be understood to mean "$\mathbf{k}$-vector space". The

word "subspace" will always mean "$\mathbf{k}$-vector subspace". The word "linear" will always mean "$\mathbf{k}$-linear". If $V$ and $W$ are two $\mathbf{k}$-vector spaces, then $\operatorname{Hom}(V, W)$ denotes the vector space of all linear maps from $V$ to $W$. If $S$ is a subset of a vector space $V$, then $\operatorname{span} S$ will mean the span of $S$ (that is, the subspace of $V$ spanned by $S$).

We recall the definition of a bilinear map:

**Definition 2.1.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be any map.

**(a)** We say that the map $f$ is *linear in its first argument* if, for every $w \in W$, the map $V \to U,\ v \mapsto f(v, w)$ is linear.

**(b)** We say that the map $f$ is *linear in its second argument* if, for every $v \in V$, the map $W \to U,\ w \mapsto f(v, w)$ is linear.

**(c)** We say that the map $f$ is *bilinear* if it is both linear in its first argument and linear in its second argument.

More explicitly, this definition can be rewritten as follows: A map $f : V \times W \to U$ is linear in its first argument if and only if every $\lambda_1, \lambda_2 \in \mathbf{k}$ and $v_1, v_2 \in V$ and $w \in W$ satisfy

$$f(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 f(v_1, w) + \lambda_2 f(v_2, w). \tag{1}$$

A map $f : V \times W \to U$ is linear in its second argument if and only if every $\lambda_1, \lambda_2 \in \mathbf{k}$ and $v \in V$ and $w_1, w_2 \in W$ satisfy

$$f(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 f(v, w_1) + \lambda_2 f(v, w_2). \tag{2}$$

A map $f : V \times W \to U$ is bilinear if and only if it satisfies both (1) and (2).

Bilinear maps are also known as "$\mathbf{k}$-bilinear maps" (actually, this latter notion is used when $\mathbf{k}$ is not clear from the context).

Here are some examples:

**Example 2.2.** Let $V$, $W$ and $U$ be three vector spaces. Let $\mathbf{0} : V \times W \to U$ be the map which sends every $p \in V \times W$ to $0$. Then, the map $\mathbf{0}$ is bilinear.

**Example 2.3.** Consider $\mathbf{k}^n$ as the vector space of all row vectors of size $n$ (with entries in $\mathbf{k}$).

- The map $\mathbf{k}^n \times \mathbf{k}^n \to \mathbf{k}$ sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in \mathbf{k}^n \times \mathbf{k}^n$ to $x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in \mathbf{k}$ is a bilinear map. (This map is called the *dot product map*, and the image of a pair $(v, w) \in \mathbf{k}^n \times \mathbf{k}^n$ under this map is called the *dot product* of $v$ and $w$.)

- The map $\mathbf{k}^n \times \mathbf{k}^n \to \mathbf{k}^n$ sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in \mathbf{k}^n \times \mathbf{k}^n$ to $(x_1 y_1, x_2 y_2, \ldots, x_n y_n) \in \mathbf{k}^n$ is a bilinear map.

- The map $\mathbf{k}^n \times \mathbf{k}^n \to \mathbf{k}^{n\times n}$ (where $\mathbf{k}^{n\times n}$ denotes the ring of $n \times n$-matrices over $\mathbf{k}$) sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in \mathbf{k}^n \times \mathbf{k}^n$ to the matrix
$$\begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{pmatrix} \in \mathbf{k}^{n\times n}$$
is a bilinear map.

- The map $\mathbf{k}^n \times \mathbf{k}^n \to \mathbf{k}$ sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in \mathbf{k}^n \times \mathbf{k}^n$ to $x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n \in \mathbf{k}$ is not bilinear (unless $n = 0$). It is neither linear in its first argument nor linear in its second argument.

- The map $\mathbf{k}^n \times \mathbf{k}^n \to \mathbf{k}$ sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in \mathbf{k}^n \times \mathbf{k}^n$ to $(x_1 + x_2 + \cdots + x_n) y_1y_2 \cdots y_n \in \mathbf{k}$ is linear in its first argument, but not bilinear (unless $n \leq 1$).

**Example 2.4.** Let $V$ and $W$ be two vector spaces. The map $\operatorname{Hom}(V, W) \times V \to W$, $(f, v) \mapsto f(v)$ is bilinear.

Let us see two ways to construct new bilinear maps out of old:

**Definition 2.5.** Let $V$, $W$ and $U$ be three sets. Let $f : V \times W \to U$ be any map. Then, $f^{\mathrm{op}}$ will be denote the map
$$W \times V \to U, \qquad (w, v) \mapsto f(v, w).$$
Thus, $f^{\mathrm{op}}(w, v) = f(v, w)$ for every $(w, v) \in W \times V$.

The following proposition is obvious:

**Proposition 2.6.** Let $V$, $W$ and $U$ be three sets. Let $f : V \times W \to U$ be any map.
**(a)** We have $(f^{\mathrm{op}})^{\mathrm{op}} = f$.
**(b)** Assume that the sets $V$, $W$ and $U$ are vector spaces. Assume that the map $f$ is bilinear. Then, the map $f^{\mathrm{op}} : W \times V \to U$ is bilinear.

The following proposition is equally trivial:

**Proposition 2.7.** Let $V$, $W$ and $U$ be three vector spaces. Let $A$ be a subspace of $V$, and let $B$ be a subspace of $W$. Let $f : V \times W \to U$ be a bilinear map. Then, the restriction $f|_{A\times B}$ is a bilinear map $A \times B \to U$.

Let us introduce some more terminology, for convenience:

**Definition 2.8.** Let $V$, $W$ and $U$ be three sets. Let $f : V \times W \to U$ be a bilinear map.

The map $f : V \times W \to U$ is bilinear. In other words, the map $f$ is both linear in its first argument and linear in its second argument (by the definition of "bilinear").

**(a)** The map $f$ is linear in its first argument. In other words, for every $w \in W$, the map $V \to U$, $v \mapsto f(v, w)$ is linear (by the definition of "linear in its first argument"). This latter map will be denoted by $f(-, w)$. This map $f(-, w)$ is thus a linear map $V \to U$; in other words, $f(-, w) \in \operatorname{Hom}(V, U)$.

**(b)** The map $f$ is linear in its second argument. In other words, for every $v \in V$, the map $W \to U$, $w \mapsto f(v, w)$ is linear (by the definition of "linear in its second argument"). This latter map will be denoted by $f(v, -)$. This map $f(v, -)$ is thus a linear map $W \to U$; in other words, $f(v, -) \in \operatorname{Hom}(W, U)$.

The notations $f(v, -)$ and $f(-, w)$ introduced in Definition 2.8 should not be confused with the notation $f(v, w)$ for the image of a pair $(v, w) \in V \times W$ under $f$. (Fortunately, they can easily be distinguished from the notation $f(v, w)$, because a dash cannot be mistaken for an element of $V$ or an element of $W$.) Of course, the dash in the notations $f(v, -)$ and $f(-, w)$ indicates "insert element here".

"Bilinear form" is just an alternative terminology for a bilinear map whose target is $\mathbf{k}$:

**Definition 2.9.** Let $V$ and $W$ be two vector spaces. A *bilinear form* on $V \times W$ means a bilinear map $V \times W \to \mathbf{k}$.

Using the notion of a bilinear form, we can restate Proposition 2.6 **(b)** and Proposition 2.7 in the particular case of $U = \mathbf{k}$ as follows:

**Proposition 2.10.** Let $V$ and $W$ be two vector spaces. Let $f : V \times W \to \mathbf{k}$ be a bilinear form.

**(a)** The map $f^{\mathrm{op}} : W \times V \to \mathbf{k}$ is a bilinear form.

**(b)** Let $A$ be a subspace of $V$, and let $B$ be a subspace of $W$. Then, the restriction $f|_{A \times B}$ is a bilinear form $A \times B \to \mathbf{k}$.

*Proof of Proposition 2.10.* Proposition 2.10 **(a)** follows immediately from Proposition 2.6 **(b)** (applied to $U = \mathbf{k}$). Proposition 2.10 **(b)** follows immediately from Proposition 2.7 (applied to $U = \mathbf{k}$). □

# 3. Left and right orthogonal spaces

We shall now introduce the notions of left and right orthogonal spaces with respect to a bilinear map:

**Definition 3.1.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

**(a)** If $B$ is a subset of $W$, then $\mathcal{L}_f(B)$ denotes the subset $\{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}$ of $V$. We call $\mathcal{L}_f(B)$ the *left orthogonal space of* $B$.

**(b)** If $A$ is a subset of $V$, then $\mathcal{R}_f(A)$ denotes the subset $\{w \in W \mid f(a,w) = 0 \text{ for all } a \in A\}$ of $W$. We call $\mathcal{R}_f(A)$ the *right orthogonal space of* $B$.

We will mostly use the notations $\mathcal{L}_f(B)$ and $\mathcal{R}_f(A)$ in the case when $f$ is a **bilinear** map. However, we have chosen to introduce them in full generality, since others occasionally use them in more general settings (e.g., for sesquilinear maps).

We shall first prove some simple facts:

**Proposition 3.2.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

**(a)** If $B$ is a subset of $W$, then $\mathcal{L}_f(B) = \mathcal{R}_{f^{\mathrm{op}}}(B)$.

**(b)** If $A$ is a subset of $V$, then $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A)$.

Notice that $A$ and $B$ are not required to be subspaces, only subsets, in Proposition 3.2; furthermore, the map $f$ is not required to be bilinear.

*Proof of Proposition 3.2.* **(b)** Let $A$ be a subset of $V$. Then, the definition of $\mathcal{L}_{f^{\mathrm{op}}}(A)$ yields

$$\begin{aligned}
\mathcal{L}_{f^{\mathrm{op}}}(A) &= \left\{ v \in W \mid \underbrace{f^{\mathrm{op}}(v,b)}_{\substack{=f(b,v) \\ \text{(by the definition of } f^{\mathrm{op}})}} = 0 \text{ for all } b \in A \right\} \\
&= \{v \in W \mid f(b,v) = 0 \text{ for all } b \in A\} \\
&= \{v \in W \mid f(a,v) = 0 \text{ for all } a \in A\} \\
&\qquad \text{(here, we have renamed the index } b \text{ as } a\text{)} \\
&= \{w \in W \mid f(a,w) = 0 \text{ for all } a \in A\} \\
&\qquad \text{(here, we have renamed the index } v \text{ as } w\text{)} \\
&= \mathcal{R}_f(A)
\end{aligned}$$

(since $\mathcal{R}_f(A) = \{w \in W \mid f(a,w) = 0 \text{ for all } a \in A\}$ (by the definition of $\mathcal{R}_f(A)$)). This proves Proposition 3.2 **(b)**.

**(a)** Let $B$ be a subset of $W$. Proposition 2.6 **(a)** yields $(f^{\mathrm{op}})^{\mathrm{op}} = f$. But Proposition 3.2 **(b)** (applied to $W$, $V$, $f^{\mathrm{op}}$ and $B$ instead of $V$, $W$, $f$ and $A$) shows that $\mathcal{R}_{f^{\mathrm{op}}}(B) = \mathcal{L}_{(f^{\mathrm{op}})^{\mathrm{op}}}(B)$. Since $(f^{\mathrm{op}})^{\mathrm{op}} = f$, this rewrites as $\mathcal{R}_{f^{\mathrm{op}}}(B) = \mathcal{L}_f(B)$. This proves Proposition 3.2 **(a)**. $\square$

**Proposition 3.3.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a bilinear map.

**(a)** If $B$ is a subset of $W$, then $\mathcal{L}_f(B)$ is a subspace of $V$.

**(b)** If $A$ is a subset of $V$, then $\mathcal{R}_f(A)$ is a subspace of $W$.

Notice that $A$ and $B$ are not required to be subspaces, only subsets, in Proposition 3.3.

*Proof of Proposition 3.3.* **(a)** Let $B$ be a subset of $W$.

Recall that a linear map $f(-,w) \in \operatorname{Hom}(V,U)$ is defined for every $w \in W$ (according to Definition 2.8 **(a)**). Thus, $\operatorname{Ker}(f(-,w))$ is a subspace of $V$ for every $w \in W$ (because the kernel of a linear map is always a subspace of its domain). Consequently, $\bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w))$ is an intersection of subspaces of $V$, and therefore itself a subspace of $V$.

We claim that

$$\mathcal{L}_f(B) = \bigcap_{w\in B} \operatorname{Ker}(f(-,w)) \tag{3}$$

[1].

*Proof of (3):* Let $x \in \mathcal{L}_f(B)$. Thus,

$$x \in \mathcal{L}_f(B) = \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}$$

(by the definition of $\mathcal{L}_f(B)$). In other words, $x$ is an element of $V$ and satisfies

$$f(x,b) = 0 \qquad \text{for all } b \in B. \tag{4}$$

Now, we have $x \in \operatorname{Ker}(f(-,w))$ for every $w \in B$ [2]. In other words, $x \in \bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w))$.

Let us now forget that we fixed $x$. We thus have shown that $x \in \bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w))$ for every $x \in \mathcal{L}_f(B)$. In other words,

$$\operatorname{Ker}(f(-,w)) \subseteq \bigcap_{w\in B} \operatorname{Ker}(f(-,w)). \tag{5}$$

On the other hand, let $y \in \bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w))$. Thus, $y \in \bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w)) \subseteq V$. Moreover, we have $f(y,b) = 0$ for all $b \in B$ [3]. Thus, $y$ is an el-

[1] Here, the intersection $\bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w))$ is taken in the ambient set $V$. Thus, if $B = \varnothing$, then $\bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w))$ is understood to be $V$.

[2] *Proof.* Let $w \in B$. Applying (4) to $b = w$, we obtain $f(x,w) = 0$.

But the definition of $f(-,w)$ yields $(f(-,w))(x) = f(x,w) = 0$. In other words, $x \in \operatorname{Ker}(f(-,w))$, qed.

[3] *Proof.* Let $b \in B$. We have $y \in \bigcap\limits_{w\in B} \operatorname{Ker}(f(-,w)) \subseteq \operatorname{Ker}(f(-,b))$ (since $b \in B$). Thus, $(f(-,b))(y) = 0$. But the definition of $f(-,b)$ yields $(f(-,b))(y) = f(y,b)$. Hence, $f(y,b) = (f(-,b))(y) = 0$, qed.

ement of $V$ and satisfies $f(y,b) = 0$ for all $b \in B$. In other words, $y \in \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}$.

Since $\mathcal{L}_f(B) = \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}$, this rewrites as $y \in \mathcal{L}_f(B)$.

Let us now forget that we fixed $y$. We thus have shown that $y \in \mathcal{L}_f(B)$ for every $y \in \bigcap\limits_{w \in B} \operatorname{Ker}(f(-,w))$. In other words,

$$\mathcal{L}_f(B) \subseteq \bigcap_{w \in B} \operatorname{Ker}(f(-,w)).$$

Combining this with (5), we obtain $\mathcal{L}_f(B) = \bigcap\limits_{w \in B} \operatorname{Ker}(f(-,w))$. This proves (3).

Now, recall that the subset $\bigcap\limits_{w \in B} \operatorname{Ker}(f(-,w))$ of $V$ is a subspace of $V$. In view of (3), this rewrites as follows: The subset $\mathcal{L}_f(B)$ of $V$ is a subspace of $V$. This proves Proposition 3.3 **(a)**.

**(b)** Let $A$ be a subset of $V$. Proposition 3.2 **(b)** yields $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A)$. But the map $f^{\mathrm{op}}$ is bilinear (by Proposition 2.6 **(b)**). Hence, Proposition 3.3 **(a)** (applied to $W$, $V$, $f^{\mathrm{op}}$ and $A$ instead of $V$, $W$, $f$ and $B$) shows that $\mathcal{L}_{f^{\mathrm{op}}}(A)$ is a subspace of $W$. Since $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A)$, this yields that $\mathcal{R}_f(A)$ is a subspace of $W$. This proves Proposition 3.3 **(b)**. □

Of course, Proposition 3.3 **(b)** is an analogue of Proposition 3.3 **(a)**, and could be proven in the same way as we proved Proposition 3.3 **(a)**. However, we have chosen to derive it from Proposition 3.3 **(a)** instead (using Proposition 3.2 **(b)**), since this way is shorter.

**Proposition 3.4.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

**(a)** If $B$ and $B'$ are two subsets of $W$ satisfying $B \subseteq B'$, then $\mathcal{L}_f(B) \supseteq \mathcal{L}_f(B')$.

**(b)** If $A$ and $A'$ are two subsets of $V$ satisfying $A \subseteq A'$, then $\mathcal{R}_f(A) \supseteq \mathcal{R}_f(A')$.

*Proof of Proposition 3.4.* **(a)** Let $B$ and $B'$ be two subsets of $W$ satisfying $B \subseteq B'$. The definition of $\mathcal{L}_f(B)$ yields

$$\mathcal{L}_f(B) = \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}. \tag{6}$$

The definition of $\mathcal{L}_f(B')$ yields

$$\mathcal{L}_f(B') = \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B'\}.$$

Now, let $x \in \mathcal{L}_f(B')$. Thus, $x \in \mathcal{L}_f(B') = \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B'\}$. In other words, $x$ is an element $v$ of $V$ satisfying $f(v,b) = 0$ for all $b \in B'$. In other words, $x$ is an element of $V$ and satisfies

$$f(x,b) = 0 \qquad \text{for all } b \in B'. \tag{7}$$

Therefore, $f(x,b) = 0$ for all $b \in B$ [4]. Hence, $x$ is an element of $V$ and satisfies $f(x,b) = 0$ for all $b \in B$. In other words, $x$ is an element $v$ of $V$ satisfying $f(v,b) = 0$ for all $b \in B$. In other words, $x \in \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}$. In light of (6), this rewrites as $x \in \mathcal{L}_f(B)$.

Now, let us forget that we fixed $x$. We thus have proven that $x \in \mathcal{L}_f(B)$ for every $x \in \mathcal{L}_f(B')$. In other words, $\mathcal{L}_f(B) \supseteq \mathcal{L}_f(B')$. This proves Proposition 3.4 **(a)**.

**(b)** Let $A$ and $A'$ be two subsets of $V$ satisfying $A \subseteq A'$. Proposition 3.2 **(b)** yields $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A)$. Proposition 3.2 **(b)** (applied to $A'$ instead of $A$) yields $\mathcal{R}_f(A') = \mathcal{L}_{f^{\mathrm{op}}}(A')$. But Proposition 3.4 **(a)** (applied to $W$, $V$, $f^{\mathrm{op}}$, $A$ and $A'$ instead of $V$, $W$, $f$, $B$ and $B'$) shows that $\mathcal{L}_{f^{\mathrm{op}}}(A) \supseteq \mathcal{L}_{f^{\mathrm{op}}}(A')$. Thus, $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A) \supseteq \mathcal{L}_{f^{\mathrm{op}}}(A') = \mathcal{R}_f(A')$. This proves Proposition 3.4 **(b)**. □

**Proposition 3.5.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map. Let $A$ be a subset of $V$. Let $B$ be a subset of $W$. Then, $A \subseteq \mathcal{L}_f(B)$ holds if and only if $B \subseteq \mathcal{R}_f(A)$.

*Proof of Proposition 3.5.* We have the following chain of logical equivalences:

$$\begin{aligned}
&\left(A \subseteq \mathcal{L}_f(B)\right) \\
\iff &\left(\text{every } x \in A \text{ satisfies } x \in \underbrace{\mathcal{L}_f(B)}_{\substack{=\{v \in V \mid f(v,b)=0 \text{ for all } b \in B\} \\ \text{(by the definition of } \mathcal{L}_f(B))}}\right) \\
\iff &\left(\text{every } x \in A \text{ satisfies } x \in \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}\right) \\
\iff &\left(\text{every } x \in A \text{ satisfies } f(x,b) = 0 \text{ for all } b \in B\right) \\
&\quad \left(\text{since every } x \in A \text{ satisfies } x \in V \text{ already (because } A \subseteq V)\right) \\
\iff &\left(\text{every } x \in A \text{ and } b \in B \text{ satisfy } f(x,b) = 0\right) \\
\iff &\left(\text{every } a \in A \text{ and } b \in B \text{ satisfy } f(a,b) = 0\right) \qquad (8) \\
&\quad \left(\text{here, we renamed the index } x \text{ as } a\right).
\end{aligned}$$

[4] *Proof.* Let $b \in B$. Then, $b \in B \subseteq B'$. Hence, $f(x,b) = 0$ (by (7)). Qed.

Also, we have the following chain of logical equivalences:

$$\begin{aligned}
&\left(B \subseteq \mathcal{R}_f(A)\right) \\
&\iff \left(\text{every } y \in B \text{ satisfies } y \in \underbrace{\mathcal{R}_f(A)}_{\substack{=\{w \in W \mid f(a,w)=0 \text{ for all } a \in A\} \\ \text{(by the definition of } \mathcal{R}_f(A))}}\right) \\
&\iff (\text{every } y \in B \text{ satisfies } y \in \{w \in W \mid f(a,w) = 0 \text{ for all } a \in A\}) \\
&\iff (\text{every } y \in B \text{ satisfies } f(a,y) = 0 \text{ for all } a \in A) \\
&\qquad (\text{since every } y \in B \text{ satisfies } y \in W \text{ already (because } B \subseteq W)) \\
&\iff (\text{every } a \in A \text{ and } y \in B \text{ satisfy } f(a,y) = 0) \\
&\iff (\text{every } a \in A \text{ and } b \in B \text{ satisfy } f(a,b) = 0) \\
&\qquad (\text{here, we renamed the index } y \text{ as } b).
\end{aligned}$$

Comparing this with (8), we obtain $\left(A \subseteq \mathcal{L}_f(B)\right) \iff \left(B \subseteq \mathcal{R}_f(A)\right)$. This proves Proposition 3.5. □

**Corollary 3.6.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

**(a)** If $B$ is a subset of $W$, then $\mathcal{L}_f(B) = \left\{v \in V \mid B \subseteq \mathcal{R}_f(\{v\})\right\}$.

**(b)** If $A$ is a subset of $V$, then $\mathcal{R}_f(A) = \left\{w \in W \mid A \subseteq \mathcal{L}_f(\{w\})\right\}$.

*Proof of Corollary 3.6.* **(a)** Let $B$ be a subset of $W$. For every $v \in V$, we have the following logical equivalence:

$$\left(B \subseteq \mathcal{R}_f(\{v\})\right) \iff \left(v \in \mathcal{L}_f(B)\right) \tag{9}$$

[5]. Now,

$$\begin{aligned}
&\left\{v \in V \mid \underbrace{B \subseteq \mathcal{R}_f(\{v\})}_{\substack{\text{this is equivalent to } \left(v \in \mathcal{L}_f(B)\right) \\ \text{(by (9))}}}\right\} \\
&= \left\{v \in V \mid v \in \mathcal{L}_f(B)\right\} = \mathcal{L}_f(B) \qquad \left(\text{since } \mathcal{L}_f(B) \subseteq V\right).
\end{aligned}$$

[5] *Proof of (9):* Let $v \in V$. Proposition 3.5 (applied to $A = \{v\}$) yields that $\{v\} \subseteq \mathcal{L}_f(B)$ holds if and only if $B \subseteq \mathcal{R}_f(\{v\})$. In other words, we have the following logical equivalence $\left(\{v\} \subseteq \mathcal{L}_f(B)\right) \iff \left(B \subseteq \mathcal{R}_f(\{v\})\right)$. But clearly, we have the logical equivalence $\left(\{v\} \subseteq \mathcal{L}_f(B)\right) \iff \left(v \in \mathcal{L}_f(B)\right)$. Thus, we have the chain of logical equivalences

$$\left(v \in \mathcal{L}_f(B)\right) \iff \left(\{v\} \subseteq \mathcal{L}_f(B)\right) \iff \left(B \subseteq \mathcal{R}_f(\{v\})\right).$$

This proves (9).

This proves Corollary 3.6 **(a)**.

**(b)** Let $A$ be a subset of $V$. Then, Corollary 3.6 **(a)** (applied to $W$, $V$, $f^{\mathrm{op}}$ and $A$ instead of $V$, $W$, $f$ and $B$) shows that

$$\mathcal{L}_{f^{\mathrm{op}}}(A) = \left\{v \in W \mid A \subseteq \mathcal{R}_{f^{\mathrm{op}}}(\{v\})\right\} = \left\{w \in W \mid A \subseteq \mathcal{R}_{f^{\mathrm{op}}}(\{w\})\right\} \quad (10)$$

(here, we renamed the index $v$ as $w$).

But Proposition 3.2 **(b)** shows that $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A)$. Furthermore, every $w \in W$ satisfies $\mathcal{L}_f(\{w\}) = \mathcal{R}_{f^{\mathrm{op}}}(\{w\})$ (by Proposition 3.2 **(a)**, applied to $B = \{w\}$). Hence,

$$\begin{aligned}\left\{w \in W \mid A \subseteq \underbrace{\mathcal{L}_f(\{w\})}_{=\mathcal{R}_{f^{\mathrm{op}}}(\{w\})}\right\} &= \left\{w \in W \mid A \subseteq \mathcal{R}_{f^{\mathrm{op}}}(\{w\})\right\} \\ &= \mathcal{L}_{f^{\mathrm{op}}}(A) \qquad \text{(by (10))} \\ &= \mathcal{R}_f(A).\end{aligned}$$

This proves Corollary 3.6 **(b)**. □

**Corollary 3.7.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

**(a)** If $B$ is a subset of $W$, then $B \subseteq \mathcal{R}_f\left(\mathcal{L}_f(B)\right)$.

**(b)** If $A$ is a subset of $V$, then $A \subseteq \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$.

*Proof of Corollary 3.7.* **(a)** Let $B$ be a subset of $W$. Proposition 3.5 (applied to $A = \mathcal{L}_f(B)$) shows that $\mathcal{L}_f(B) \subseteq \mathcal{L}_f(B)$ holds if and only if $B \subseteq \mathcal{R}_f\left(\mathcal{L}_f(B)\right)$. Hence, $B \subseteq \mathcal{R}_f\left(\mathcal{L}_f(B)\right)$ (since $\mathcal{L}_f(B) \subseteq \mathcal{L}_f(B)$). This proves Corollary 3.7 **(a)**.

**(b)** Let $A$ be a subset of $V$. Proposition 3.5 (applied to $B = \mathcal{R}_f(A)$) shows that $A \subseteq \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$ holds if and only if $\mathcal{R}_f(A) \subseteq \mathcal{R}_f(A)$. Hence, $A \subseteq \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$ (since $\mathcal{R}_f(A) \subseteq \mathcal{R}_f(A)$). This proves Corollary 3.7 **(b)**. □

We have so far not focussed on computing the subspaces $\mathcal{L}_f(B)$ and $\mathcal{R}_f(A)$ in Definition 3.1. This is straightforward to do when $B$ (resp., $A$) is a finite set and $V$, $W$ and $U$ are finite-dimensional vector spaces (because in this case, the statement that "$f(v, b) = 0$ for all $b \in B$" is a conjunction of finitely many linear equations, and thus easy to solve using linear algebra). However, when $B$ (resp., $A$) is infinite, this becomes harder (as one would need to solve an infinite system of linear equations). However, when $B$ (resp., $A$) is a finite-dimensional subspace of $W$ (resp. $V$), then it is still easy, due to the following fact:

**Proposition 3.8.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a bilinear map.

**(a)** If $B$ is a subset of $W$, then $\mathcal{L}_f(B) = \mathcal{L}_f(\operatorname{span} B)$.

**(b)** If $A$ is a subset of $V$, then $\mathcal{R}_f(A) = \mathcal{R}_f(\operatorname{span} A)$.

Using Proposition 3.8 **(a)**, it is easy to compute $\mathcal{L}_f(B)$ whenever $V$, $W$ and $U$ are three finite-dimensional vector spaces and $B$ is a subspace of $W$ [6]. Similarly, using Proposition 3.8 **(b)**, it is easy to compute $\mathcal{R}_f(A)$ whenever $V$, $W$ and $U$ are three finite-dimensional vector spaces and $A$ is a subspace of $V$.

*Proof of Proposition 3.8.* **(a)** Let $B$ be a subset of $W$. Recall that $\operatorname{span} B$ is the smallest subspace of $W$ containing $B$ as a subset. Thus, if $G$ is any subspace of $W$ containing $B$ as a subset, then

$$\operatorname{span} B \subseteq G. \tag{11}$$

Clearly, $B \subseteq \operatorname{span} B$. Thus, Proposition 3.4 **(a)** (applied to $B' = \operatorname{span} B$) shows that $\mathcal{L}_f(B) \supseteq \mathcal{L}_f(\operatorname{span} B)$.

Corollary 3.6 **(a)** yields $\mathcal{L}_f(B) = \{v \in V \mid B \subseteq \mathcal{R}_f(\{v\})\}$. Also, Corollary 3.6 **(a)** (applied to $\operatorname{span} B$ instead of $B$) yields

$$\mathcal{L}_f(\operatorname{span} B) = \{v \in V \mid \operatorname{span} B \subseteq \mathcal{R}_f(\{v\})\}. \tag{12}$$

But for any $v \in V$, we have the following logical equivalence:

$$\left(B \subseteq \mathcal{R}_f(\{v\})\right) \iff \left(\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})\right) \tag{13}$$

[6] *Proof.* Namely, let us assume that $V$, $W$ and $U$ are three finite-dimensional vector spaces, and that $B$ is a subspace of $W$. The vector space $B$ is a subspace of the finite-dimensional vector space $W$, and thus itself finite-dimensional. Hence, $B$ has a finite basis $B'$. Consider this $B'$.

Proposition 3.8 **(a)** (applied to $B'$ instead of $B$) shows that $\mathcal{L}_f(B') = \mathcal{L}_f\left(\underbrace{\operatorname{span}(B')}_{\substack{=B \\ \text{(since } B' \text{ is a basis of } B)}}\right) = \mathcal{L}_f(B)$. But since $B'$ is finite, it is easy to compute $\mathcal{L}_f(B')$ using linear algebra. In other words, it is easy to compute $\mathcal{L}_f(B)$ using linear algebra (since $\mathcal{L}_f(B') = \mathcal{L}_f(B)$). Qed.

[7]. Now,

$$\mathcal{L}_f(B) = \left\{ v \in V \mid \underbrace{B \subseteq \mathcal{R}_f(\{v\})}_{\substack{\text{this is equivalent to } \left(\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})\right) \\ \text{(by (13))}}} \right\}$$
$$= \left\{ v \in V \mid \operatorname{span} B \subseteq \mathcal{R}_f(\{v\}) \right\} = \mathcal{L}_f(\operatorname{span} B) \qquad \text{(by (12))}.$$

This proves Proposition 3.8 **(a)**.

**(b)** Let $A$ be a subset of $W$. Then, Proposition 3.2 **(b)** yields $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A)$. Also, Proposition 3.2 **(b)** (applied to $\operatorname{span} A$ instead of $A$) yields $\mathcal{R}_f(\operatorname{span} A) = \mathcal{L}_{f^{\mathrm{op}}}(\operatorname{span} A)$.

But the map $f^{\mathrm{op}}$ is bilinear (by Proposition 2.6 **(b)**). Hence, Proposition 3.8 **(a)** (applied to $W$, $V$, $f^{\mathrm{op}}$ and $A$ instead of $V$, $W$, $f$ and $B$) shows that $\mathcal{L}_{f^{\mathrm{op}}}(A) = \mathcal{L}_{f^{\mathrm{op}}}(\operatorname{span} A)$. Thus, $\mathcal{R}_f(A) = \mathcal{L}_{f^{\mathrm{op}}}(A) = \mathcal{L}_{f^{\mathrm{op}}}(\operatorname{span} A) = \mathcal{R}_f(\operatorname{span} A)$. This proves Proposition 3.8 **(b)**. □

Next, we explore the orthogonal spaces of unions:

**Proposition 3.9.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

**(a)** If $B_1$ and $B_2$ are two subsets of $W$, then $\mathcal{L}_f(B_1 \cup B_2) = \mathcal{L}_f(B_1) \cap \mathcal{L}_f(B_2)$.

**(b)** If $A_1$ and $A_2$ are two subsets of $V$, then $\mathcal{R}_f(A_1 \cup A_2) = \mathcal{R}_f(A_1) \cap \mathcal{R}_f(A_2)$.

Of course, Proposition 3.9 does not hold (in general) when the $\cup$ and $\cap$ signs are switched.

*Proof of Proposition 3.9.* **(a)** Let $B_1$ and $B_2$ be two subsets of $W$. The definition of

[7] *Proof of (13):* Let $v \in V$.

Assume that $B \subseteq \mathcal{R}_f(\{v\})$. We shall show that $\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})$.

We know that $\mathcal{R}_f(\{v\})$ is a subspace of $W$ (by Proposition 3.3 **(b)**, applied to $A = \{v\}$). Hence, $\mathcal{R}_f(\{v\})$ is a subspace of $W$ containing $B$ as a subset (since $B \subseteq \mathcal{R}_f(\{v\})$). Thus, $\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})$ (by (11), applied to $G = \mathcal{R}_f(\{v\})$).

Now, let us forget that we assumed that $B \subseteq \mathcal{R}_f(\{v\})$. We thus have proven $\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})$ under the assumption that $B \subseteq \mathcal{R}_f(\{v\})$. In other words, we have proven the implication $\left(B \subseteq \mathcal{R}_f(\{v\})\right) \Longrightarrow \left(\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})\right)$.

On the other hand, if $\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})$, then $B \subseteq \operatorname{span} B \subseteq \mathcal{R}_f(\{v\})$. Hence, we have proven the implication $\left(\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})\right) \Longrightarrow \left(B \subseteq \mathcal{R}_f(\{v\})\right)$. Combining this implication with the (already proven) implication $\left(B \subseteq \mathcal{R}_f(\{v\})\right) \Longrightarrow \left(\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})\right)$, we obtain the equivalence $\left(B \subseteq \mathcal{R}_f(\{v\})\right) \Longleftrightarrow \left(\operatorname{span} B \subseteq \mathcal{R}_f(\{v\})\right)$. This proves (13).

$\mathcal{L}_f(B_1 \cup B_2)$ yields

$$\begin{aligned}
&\mathcal{L}_f(B_1 \cup B_2) \\
&= \left\{ v \in V \mid \underbrace{f(v,b) = 0 \text{ for all } b \in B_1 \cup B_2}_{\substack{\text{this is equivalent to} \\ (f(v,b)=0 \text{ for all } b \in B_1) \wedge (f(v,b)=0 \text{ for all } b \in B_2)}} \right\} \\
&= \{ v \in V \mid (f(v,b) = 0 \text{ for all } b \in B_1) \wedge (f(v,b) = 0 \text{ for all } b \in B_2) \} \\
&= \underbrace{\{ v \in V \mid f(v,b) = 0 \text{ for all } b \in B_1 \}}_{\substack{=\mathcal{L}_f(B_1) \\ \text{(by the definition of } \mathcal{L}_f(B_1))}} \cap \underbrace{\{ v \in V \mid f(v,b) = 0 \text{ for all } b \in B_2 \}}_{\substack{=\mathcal{L}_f(B_2) \\ \text{(by the definition of } \mathcal{L}_f(B_2))}} \\
&= \mathcal{L}_f(B_1) \cap \mathcal{L}_f(B_2).
\end{aligned}$$

This proves Proposition 3.9 **(a)**.

**(b)** Proposition 3.9 **(b)** can be derived from Proposition 3.9 **(a)** using the same tactic that we used (for example) to derive Proposition 3.4 **(b)** from Proposition 3.4 **(a)**. (Alternatively, Proposition 3.9 **(b)** can be proven analogously to Proposition 3.9 **(a)**.) □

A similar result holds for orthogonal spaces of sums of subspaces when $f$ is bilinear:

**Proposition 3.10.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a bilinear map.

**(a)** If $B_1$ and $B_2$ are two subspaces of $W$, then $\mathcal{L}_f(B_1 + B_2) = \mathcal{L}_f(B_1) \cap \mathcal{L}_f(B_2)$.

**(b)** If $A_1$ and $A_2$ are two subspaces of $V$, then $\mathcal{R}_f(A_1 + A_2) = \mathcal{R}_f(A_1) \cap \mathcal{R}_f(A_2)$.

*Proof of Proposition 3.10.* **(a)** Let $B_1$ and $B_2$ be two subspaces of $W$. Then, we have $\operatorname{span}(B_1 \cup B_2) = B_1 + B_2$ [8]. Now, Proposition 3.8 **(a)** (applied to $B = B_1 \cup B_2$)

[8] *Proof.* Let $B = B_1 \cup B_2$. A well-known fact from linear algebra tells us that $\operatorname{span} B$ is the smallest subspace of $W$ containing $B$ as a subset. Thus, if $G$ is any subspace of $W$ containing $B$ as a subset, then

$$\operatorname{span} B \subseteq G. \tag{14}$$

Now, $B_1$ and $B_2$ are two subspaces of $W$. Thus, $B_1 + B_2$ is again a subspace of $W$. Also, $B = B_1 \cup B_2 \subseteq B_1 + B_2$ (since $B_1 \subseteq B_1 + B_2$ and $B_2 \subseteq B_1 + B_2$). Thus, $B_1 + B_2$ is a subspace of $W$ containing $B_1 \cup B_2$ as a subset. Therefore, (14) (applied to $G = B_1 + B_2$) shows that $\operatorname{span} B \subseteq B_1 + B_2$.

On the other hand, combining $B_1 \subseteq B_1 \cup B_2 = B \subseteq \operatorname{span} B$ and $B_2 \subseteq B_1 \cup B_2 = B \subseteq \operatorname{span} B$, we obtain $B_1 + B_2 \subseteq \operatorname{span} B$ (since $\operatorname{span} B$ is a vector subspace of $W$). Combined with $\operatorname{span} B \subseteq B_1 + B_2$, this yields $B_1 + B_2 = \operatorname{span} B$. Since $B = B_1 \cup B_2$, this rewrites as $B_1 + B_2 = \operatorname{span}(B_1 \cup B_2)$. Qed.

shows that $\mathcal{L}_f(B_1 \cup B_2) = \mathcal{L}_f\left(\underbrace{\operatorname{span}(B_1 \cup B_2)}_{=B_1+B_2}\right) = \mathcal{L}_f(B_1 + B_2)$. Hence,

$$\mathcal{L}_f(B_1 + B_2) = \mathcal{L}_f(B_1 \cup B_2) = \mathcal{L}_f(B_1) \cap \mathcal{L}_f(B_2)$$

(by Proposition 3.9 **(a)**). This proves Proposition 3.10 **(a)**.

**(b)** The map $f^{\mathrm{op}}$ is bilinear (by Proposition 2.6 **(b)**). Hence, Proposition 3.10 **(b)** can be derived from Proposition 3.10 **(a)** using the same tactic that we used (for example) to derive Proposition 3.4 **(b)** from Proposition 3.4 **(a)**. (Alternatively, Proposition 3.10 **(b)** can be proven analogously to Proposition 3.10 **(a)**.) □

# 4. Interlude: Symmetric and antisymmetric bilinear forms

We make a digression to define the notions of *symmetric* and *antisymmetric* bilinear forms and maps:

**Definition 4.1.** Let $V$ and $U$ be two vector spaces. Let $f : V \times V \to U$ be a map.

**(a)** The map $f$ is said to be *symmetric* if and only if it satisfies

$$(f(v,w) = f(w,v) \qquad \text{for all } (v,w) \in V \times W).$$

**(b)** The map $f$ is said to be *antisymmetric* if and only if it satisfies

$$(f(v,w) = -f(w,v) \qquad \text{for all } (v,w) \in V \times W).$$

Antisymmetric maps are also called *skew-symmetric* maps.

**Proposition 4.2.** Let $V$ and $U$ be two vector spaces. Let $f : V \times V \to U$ be a map. Let $A$ be a subset of $V$.

**(a)** If $f$ is symmetric, then $\mathcal{L}_f(A) = \mathcal{R}_f(A)$.

**(b)** If $f$ is antisymmetric, then $\mathcal{L}_f(A) = \mathcal{R}_f(A)$.

Proposition 4.2 allows for the following definition (which is actually standard):

**Definition 4.3.** Let $V$ and $U$ be two vector spaces. Let $f : V \times V \to U$ be a map. Let $A$ be a subset of $V$. Assume that $f$ is symmetric or antisymmetric. Then, Proposition 4.2 shows that $\mathcal{L}_f(A) = \mathcal{R}_f(A)$. The subset $\mathcal{L}_f(A) = \mathcal{R}_f(A)$ of $V$ is denoted by $A^{\perp}$, at least when $f$ is clear from the context.

*Proof of Proposition 4.2.* Straightforward and left to the reader. □

# 5. The morphism of quotients I

We shall now construct a certain morphism between quotient spaces induced by any bilinear map. First, we recall the universal property of quotient spaces:

**Proposition 5.1.** Let $V$ be a vector space. Let $A$ be a subspace of $V$. Let $\pi_{V,A}$ be the canonical projection $V \to V/A$.

Let $W$ be a further vector space. Let $g : V \to W$ be a linear map such that $g(A) = 0$. Then, there exists a unique linear map $g' : V/A \to W$ such that $g = g' \circ \pi_{V,A}$.

Let us fix a notation for projections onto quotient spaces:

**Definition 5.2.** Let $V$ be a vector space. Let $A$ be a subspace of $V$. Let $v \in V$. Then, the residue class of $v$ modulo $A$ (that is, the image of $v$ under the canonical projection $V \to V/A$) will be denoted by $[v]_A$. (Other widespread notations for this residue class are $v \bmod A$, $v + A$ and $\overline{v}_A$.)

We now state the main theorem of this section:

**Theorem 5.3.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a bilinear map.

**(a)** Then, there exists a unique linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying

$$\left(\left(\alpha\left([v]_{\mathcal{L}_f(W)}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = f(v,w) \qquad \text{for all } (v,w) \in V \times W\right). \tag{15}$$

**(b)** This map $\alpha$ is injective.

*Proof of Theorem 5.3.* First, we notice that $\mathcal{L}_f(W)$ is a subspace of $V$ (by Proposition 3.3 **(a)**, applied to $B = W$). Hence, the quotient space $V/\mathcal{L}_f(W)$ is well-defined. Furthermore, $\mathcal{R}_f(V)$ is a subspace of $W$ (by Proposition 3.3 **(b)**, applied to $A = V$). Hence, the quotient space $W/\mathcal{R}_f(V)$ is well-defined.

**(a)** It is easy to see that there exists **at most one** linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15)[9].

[9] *Proof.* Let $\alpha_1$ and $\alpha_2$ be two linear maps $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15). We shall show that $\alpha_1 = \alpha_2$.

We know that $\alpha_1$ is a linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15). In other words, $\alpha_1$ is a linear map $V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ and satisfies

$$\left(\left(\alpha_1\left([v]_{\mathcal{L}_f(W)}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = f(v,w) \qquad \text{for all } (v,w) \in V \times W\right). \tag{16}$$

Now, let $x \in V/\mathcal{L}_f(W)$. Let $y \in W/\mathcal{R}_f(V)$.

We have $x \in V/\mathcal{L}_f(W)$. Hence, we can write $x$ in the form $[v]_{\mathcal{L}_f(W)}$ for some $v \in V$.

We shall now construct such a map.

Let $\pi_{V,\mathcal{L}_f(W)}$ be the canonical projection $V \to V/\mathcal{L}_f(W)$. Thus,

$$\pi_{V,\mathcal{L}_f(W)}(v) = [v]_{\mathcal{L}_f(W)} \qquad \text{for every } v \in V. \tag{17}$$

Let $\pi_{W,\mathcal{R}_f(V)}$ be the canonical projection $W \to W/\mathcal{R}_f(V)$. Thus,

$$\pi_{W,\mathcal{R}_f(V)}(w) = [w]_{\mathcal{R}_f(V)} \qquad \text{for every } w \in W. \tag{18}$$

Let $v \in V$. The map $f$ is bilinear.

Recall that a linear map $f(v,-) \in \operatorname{Hom}(W,U)$ is defined (according to Definition 2.8 **(b)**). We have $(f(v,-))(\mathcal{R}_f(V)) = 0$ [10]. Hence, Proposition 5.1 (applied to $W$, $\mathcal{R}_f(V)$, $\pi_{W,\mathcal{R}_f(V)}$, $U$ and $f(v,-)$ instead of $V$, $A$, $\pi_{V,A}$, $W$ and $g$) shows that there exists a unique linear map $g' : W/\mathcal{R}_f(V) \to U$ such that $f(v,-) = g' \circ \pi_{W,\mathcal{R}_f(V)}$. Let us denote this $g'$ by $g_v$. Thus, $g_v$ is a linear map $W/\mathcal{R}_f(V) \to U$ and satisfies $f(v,-) = g_v \circ \pi_{W,\mathcal{R}_f(V)}$.

We have

$$g_v\left([w]_{\mathcal{R}_f(V)}\right) = f(v,w) \qquad \text{for all } w \in W \tag{19}$$

---

Consider this $v$. Thus, $x = [v]_{\mathcal{L}_f(W)}$.

We have $y \in W/\mathcal{R}_f(V)$. Hence, we can write $y$ in the form $[w]_{\mathcal{R}_f(V)}$ for some $w \in W$. Consider this $w$. Thus, $y = [w]_{\mathcal{R}_f(V)}$.

Now,

$$\left(\alpha_1\left(\underbrace{x}_{=[v]_{\mathcal{L}_f(W)}}\right)\right)\left(\underbrace{y}_{=[w]_{\mathcal{R}_f(V)}}\right) = \left(\alpha_1\left([v]_{\mathcal{L}_f(W)}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = f(v,w)$$

(by (16)). The same reasoning (but applied to $\alpha_2$ instead of $\alpha_1$) shows that $(\alpha_2(x))(y) = f(v,w)$. Thus, $(\alpha_1(x))(y) = f(v,w) = (\alpha_2(x))(y)$.

Let us now forget that we fixed $y$. We thus have shown that $(\alpha_1(x))(y) = (\alpha_2(x))(y)$ for every $y \in W/\mathcal{R}_f(V)$. In other words, $\alpha_1(x) = \alpha_2(x)$.

Let us now forget that we fixed $x$. We thus have shown that $\alpha_1(x) = \alpha_2(x)$ for every $x \in V/\mathcal{L}_f(W)$. In other words, $\alpha_1 = \alpha_2$.

Let us now forget that we fixed $\alpha_1$ and $\alpha_2$. We thus have shown that $\alpha_1 = \alpha_2$ whenever $\alpha_1$ and $\alpha_2$ are two linear maps $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15). In other words, there exists **at most one** linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15). Qed.

[10] *Proof.* Let $y \in \mathcal{R}_f(V)$. We shall prove that $(f(v,-))(y) = 0$.

We have $y \in \mathcal{R}_f(V) = \{w \in W \mid f(a,w) = 0 \text{ for all } a \in V\}$ (by the definition of $\mathcal{R}_f(V)$). In other words, $y$ is an element $w$ of $W$ satisfying $f(a,w) = 0$ for all $a \in V$. In other words, $y$ is an element of $W$ and satisfies $f(a,y) = 0$ for all $a \in V$.

We know that $f(a,y) = 0$ for all $a \in V$. Applying this to $a = v$, we obtain $f(v,y) = 0$. But the definition of $f(v,-)$ yields $(f(v,-))(y) = f(v,y) = 0$.

Now, let us forget that we fixed $y$. We thus have shown that $(f(v,-))(y) = 0$ for every $y \in \mathcal{R}_f(V)$. In other words, $(f(v,-))\left(\mathcal{R}_f(V)\right) = 0$. Qed.

[11].

Now, let us forget that we fixed $v$. We thus have constructed a linear map $g_v : W/\mathcal{R}_f(V) \to U$ for every $v \in V$. We have furthermore shown that this map satisfies (19) for every $v \in V$. For every $v \in V$, the map $g_v$ is a linear map $W/\mathcal{R}_f(V) \to U$, and thus belongs to $\operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$.

Now, we define a map $g : V \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ by

$$\left(g(v) = g_v \qquad \text{for every } v \in V\right).$$

(This is well-defined, because for every $v \in V$, the map $g_v$ belongs to $\operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$.) Thus, for every $v \in V$ and $w \in W$, we have

$$\underbrace{(g(v))}_{=g_v}\left([w]_{\mathcal{R}_f(V)}\right) = g_v\left([w]_{\mathcal{R}_f(V)}\right) = f(v, w) \tag{20}$$

(by (19)).

[11] *Proof of (19):* Let $w \in W$. From (18), we obtain $[w]_{\mathcal{R}_f(V)} = \pi_{W,\mathcal{R}_f(V)}(w)$. Applying the map $g_v$ to both sides of this equality, we obtain

$$\begin{aligned} g_v\left([w]_{\mathcal{R}_f(V)}\right) &= g_v\left(\pi_{W,\mathcal{R}_f(V)}(w)\right) = \underbrace{\left(g_v \circ \pi_{W,\mathcal{R}_f(V)}\right)}_{=f(v,-)}(w) \\ &= (f(v,-))(w) = f(v,w) \qquad \text{(by the definition of } f(v,-)). \end{aligned}$$

This proves (19).

The map $g$ is linear[12]. Moreover, $g\left(\mathcal{L}_f(W)\right) = 0$ [13]. Hence, Proposition 5.1 (applied to $\mathcal{L}_f(W)$, $\pi_{V,\mathcal{L}_f(W)}$ and $\operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ instead of $A$, $\pi_{V,A}$ and $g$) shows that there exists a unique linear map $g' : V/\mathcal{L}_f(W) \to$

[12] *Proof.* Let $v_1$ and $v_2$ be two elements of $V$. Let $\lambda_1$ and $\lambda_2$ be two elements of $\mathbf{k}$. We shall show that $g(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g(v_1) + \lambda_2 g(v_2)$.

Let $y \in W/\mathcal{R}_f(V)$. Thus, $y$ can be written in the form $[w]_{\mathcal{R}_f(V)}$ for some $w \in W$. Consider this $w$. Thus, $y = [w]_{\mathcal{R}_f(V)}$. Hence, for every $v \in V$, we have

$$(g(v))\left(\underbrace{y}_{=[w]_{\mathcal{R}_f(V)}}\right) = (g(v))\left([w]_{\mathcal{R}_f(V)}\right) = f(v,w) \qquad \text{(by (20))}. \tag{21}$$

Recall that a linear map $f(-,w) \in \operatorname{Hom}(V,U)$ is defined (according to Definition 2.8 **(a)**). This map $f(-,w)$ is linear, and thus we have $(f(-,w))(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 (f(-,w))(v_1) + \lambda_2 (f(-,w))(v_2)$. Comparing this with $(f(-,w))(\lambda_1 v_1 + \lambda_2 v_2) = f(\lambda_1 v_1 + \lambda_2 v_2, w)$ (by the definition of $f(-,w)$), we obtain

$$\begin{aligned} f(\lambda_1 v_1 + \lambda_2 v_2, w) &= \lambda_1 \underbrace{(f(-,w))(v_1)}_{\substack{=f(v_1,w) \\ \text{(by the definition of } f(-,w))}} + \lambda_2 \underbrace{(f(-,w))(v_2)}_{\substack{=f(v_2,w) \\ \text{(by the definition of } f(-,w))}} \\ &= \lambda_1 f(v_1,w) + \lambda_2 f(v_2,w). \end{aligned}$$

Now, (21) (applied to $v = v_1$) yields $(g(v_1))(y) = f(v_1,w)$. Also, (21) (applied to $v = v_2$) yields $(g(v_2))(y) = f(v_2,w)$. But (21) (applied to $v = \lambda_1 v_1 + \lambda_2 v_2$) yields

$$\begin{aligned} (g(\lambda_1 v_1 + \lambda_2 v_2))(y) &= f(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 \underbrace{f(v_1,w)}_{=(g(v_1))(y)} + \lambda_2 \underbrace{f(v_2,w)}_{=(g(v_2))(y)} \\ &= \lambda_1 (g(v_1))(y) + \lambda_2 (g(v_2))(y) = (\lambda_1 g(v_1) + \lambda_2 g(v_2))(y). \end{aligned}$$

Let us now forget that we fixed $y$. We thus have shown that $(g(\lambda_1 v_1 + \lambda_2 v_2))(y) = (\lambda_1 g(v_1) + \lambda_2 g(v_2))(y)$ for every $y \in W/\mathcal{R}_f(V)$. In other words, $g(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g(v_1) + \lambda_2 g(v_2)$.

Let us now forget that we fixed $v_1$, $v_2$, $\lambda_1$ and $\lambda_2$. Thus, we have proven that $g(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g(v_1) + \lambda_2 g(v_2)$ for every $v_1 \in V$, $v_2 \in V$, $\lambda_1 \in \mathbf{k}$ and $\lambda_2 \in \mathbf{k}$. In other words, the map $g$ is linear, qed.

[13] *Proof.* Let $p \in \mathcal{L}_f(W)$. Thus, $p \in \mathcal{L}_f(W) = \{v \in V \mid f(v,b) = 0 \text{ for all } b \in W\}$ (by the definition of $\mathcal{L}_f(W)$). In other words, $p$ is an element $v$ of $V$ satisfying $f(v,b) = 0$ for all $b \in W$. In other words, $p$ is an element of $V$ and satisfies $f(p,b) = 0$ for all $b \in W$.

Let $y \in W/\mathcal{R}_f(V)$. Thus, $y$ can be written in the form $[w]_{\mathcal{R}_f(V)}$ for some $w \in W$. Consider this $w$. Thus, $y = [w]_{\mathcal{R}_f(V)}$.

But we have $f(p,b) = 0$ for all $b \in W$. Applying this to $b = w$, we obtain $f(p,w) = 0$. But (20) (applied to $v = p$) yields $(g(p))\left([w]_{\mathcal{R}_f(V)}\right) = f(p,w) = 0$. Hence,

$$(g(p))\left(\underbrace{y}_{=[w]_{\mathcal{R}_f(V)}}\right) = (g(p))\left([w]_{\mathcal{R}_f(V)}\right) = 0.$$

Now, let us forget that we fixed $y$. We thus have shown that $(g(p))(y) = 0$ for every

$\operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ such that $g = g' \circ \pi_{V,\mathcal{L}_f(W)}$. Let us denote this $g'$ by $\beta$. Thus, $\beta$ is a linear map $V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ and satisfies $g = \beta \circ \pi_{V,\mathcal{L}_f(W)}$.

Now, for all $(v, w) \in V \times W$, we have

$$\begin{aligned}
&\left(\beta\left(\underbrace{[v]_{\mathcal{L}_f(W)}}_{\substack{=\pi_{V,\mathcal{L}_f(W)}(v) \\ \text{(by (17))}}}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) \\
&= \left(\underbrace{\beta\left(\pi_{V,\mathcal{L}_f(W)}(v)\right)}_{=\left(\beta\circ\pi_{V,\mathcal{L}_f(W)}\right)(v)}\right)\left([w]_{\mathcal{R}_f(V)}\right) = \left(\underbrace{\left(\beta \circ \pi_{V,\mathcal{L}_f(W)}\right)}_{=g}(v)\right)\left([w]_{\mathcal{R}_f(V)}\right) \\
&= (g(v))\left([w]_{\mathcal{R}_f(V)}\right) = f(v, w) \qquad \text{(by (20))}.
\end{aligned}$$

In other words, the linear map $\beta$ satisfies

$$\left(\left(\beta\left([v]_{\mathcal{L}_f(W)}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = f(v, w) \qquad \text{for all } (v, w) \in V \times W\right).$$

Hence, the map $\beta$ is a linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15). Therefore, there exists at least one linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15) (namely, $\beta$). Consequently, there exists a **unique** linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15) (since we already know that there exists **at most one** linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15)). This proves Theorem 5.3 **(a)**.

**(b)** Consider the unique linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), U\right)$ satisfying (15).

We have

$$\begin{aligned}
\mathcal{L}_f(W) &= \{v \in V \mid f(v, b) = 0 \text{ for all } b \in W\} \qquad \left(\text{by the definition of } \mathcal{L}_f(W)\right) \\
&= \{p \in V \mid f(p, b) = 0 \text{ for all } b \in W\} \\
&\qquad \text{(here, we have renamed the index } v \text{ as } p) \\
&= \{p \in V \mid f(p, w) = 0 \text{ for all } w \in W\} \qquad (22) \\
&\qquad \text{(here, we have renamed the index } b \text{ as } w).
\end{aligned}$$

$y \in W/\mathcal{R}_f(V)$. In other words, $g(p) = 0$.

Now, let us forget that we fixed $p$. We thus have shown that $g(p) = 0$ for every $p \in \mathcal{L}_f(W)$. In other words, $g\left(\mathcal{L}_f(W)\right) = 0$, qed.

Let $x \in \operatorname{Ker} \alpha$. We have $x \in \operatorname{Ker} \alpha \subseteq V / \mathcal{L}_f(W)$. Hence, we can write $x$ in the form $[v]_{\mathcal{L}_f(W)}$ for some $v \in V$. Consider this $v$. Thus, $x = [v]_{\mathcal{L}_f(W)}$.

We have $x \in \operatorname{Ker} \alpha$, so that $\alpha(x) = 0$. Now, let $w \in W$. Then, $(v, w) \in V \times W$ (since $v \in V$ and $w \in W$). Thus, (15) shows that $\left(\alpha\left([v]_{\mathcal{L}_f(W)}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = f(v, w)$ (since we know that $\alpha$ satisfies (15)). Hence,

$$f(v, w) = \left(\alpha\left(\underbrace{[v]_{\mathcal{L}_f(W)}}_{=x}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = \underbrace{(\alpha(x))}_{=0}\left([w]_{\mathcal{R}_f(V)}\right) = 0\left([w]_{\mathcal{R}_f(V)}\right) = 0.$$

Let us now forget that we fixed $w$. We thus have shown that $f(v, w) = 0$ for all $w \in W$. Thus, $v$ is an element of $V$ and satisfies $f(v, w) = 0$ for all $w \in W$. In other words, $v$ is an element $p$ of $V$ satisfying $f(p, w) = 0$ for all $w \in W$. In other words,

$$v \in \{p \in V \mid f(p, w) = 0 \text{ for all } w \in W\}.$$

In view of (22), this rewrites as $v \in \mathcal{L}_f(W)$. Hence, $[v]_{\mathcal{L}_f(W)} = 0$. Thus, $x = [v]_{\mathcal{L}_f(W)} = 0$.

Let us now forget that we fixed $x$. We thus have shown that $x = 0$ for every $x \in \operatorname{Ker} \alpha$. In other words, $\operatorname{Ker} \alpha = 0$. Thus, the linear map $\alpha$ is injective. This proves Theorem 5.3 **(b)**. □

# 6. The morphism of quotients II

So far, we have not used the assumption that $\mathbf{k}$ is a field; we could have just as well let $\mathbf{k}$ be any commutative ring. (Of course, we would have to talk about $\mathbf{k}$-modules instead of vector spaces, and similarly; but apart from this, the results and proofs would have been the same.) We shall now prove some slightly deeper result where we will actually need this assumption (that $\mathbf{k}$ is a field). We will also make some assumptions on finite-dimensionality.

**Definition 6.1.** Let $V$ be a vector space. Then, $V^*$ will denote the vector space $\operatorname{Hom}(V, \mathbf{k})$. This space $V^*$ is called the *dual space* of $V$. It is well-known that $\dim(V^*) = \dim V$ when $V$ is finite-dimensional. Moreover, there is a canonical injective linear map $V \to V^{**}$ (which sends every $v \in V$ to the linear map $\widetilde{v} \in V^{**}$ which sends every $f \in V^*$ to $f(v) \in \mathbf{k}$). When $V$ is finite-dimensional, this linear map $V \to V^{**}$ is a vector space isomorphism.

We shall first state some well-known properties of finite-dimensional vector spaces. The first of these properties is the fact that an injective linear map between two finite-dimensional vector spaces of equal dimension must be an isomorphism:

**Proposition 6.2.** Let $A$ and $B$ be two finite-dimensional vector spaces. Let $i : A \to B$ be an injective linear map. Assume that $\dim A = \dim B$. Then, $i$ is a vector space isomorphism.

The next property is the particular case of Proposition 6.2 when the map $i$ is an inclusion:

**Proposition 6.3.** Let $B$ be a finite-dimensional vector space. Let $A$ be a subspace of $B$ such that $\dim A = \dim B$. Then, $A = B$.

Furthermore, the domain of an injective linear map always has at most the same dimension as its target:

**Proposition 6.4.** Let $A$ and $B$ be two finite-dimensional vector spaces. Let $i : A \to B$ be an injective linear map. Then, $\dim A \leq \dim B$.

Now, we state the next crucial theorem:

**Theorem 6.5.** Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to \mathbf{k}$ be a bilinear form.

**(a)** Then, there exists a unique linear map $\alpha : V/\mathcal{L}_f(W) \to \left(W/\mathcal{R}_f(V)\right)^*$ satisfying (15).

**(b)** This map $\alpha$ is a vector space isomorphism.

**(c)** We have $\dim\left(V/\mathcal{L}_f(W)\right) = \dim\left(W/\mathcal{R}_f(V)\right)$.

*Proof of Theorem 6.5.* The map $f : V \times W \to \mathbf{k}$ is a bilinear form. In other words, the map $f : V \times W \to \mathbf{k}$ is a bilinear map (according to the definition of a "bilinear form").

**(a)** Theorem 5.3 **(a)** (applied to $U = \mathbf{k}$) yields that there exists a unique linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), \mathbf{k}\right)$ satisfying (15). In other words, there exists a unique linear map $\alpha : V/\mathcal{L}_f(W) \to \left(W/\mathcal{R}_f(V)\right)^*$ satisfying (15) (since $\left(W/\mathcal{R}_f(V)\right)^* = \operatorname{Hom}\left(W/\mathcal{R}_f(V), \mathbf{k}\right)$). This proves Theorem 6.5 **(a)**.

Theorem 5.3 **(b)** (applied to $U = \mathbf{k}$) yields that the unique linear map $\alpha : V/\mathcal{L}_f(W) \to \operatorname{Hom}\left(W/\mathcal{R}_f(V), \mathbf{k}\right)$ satisfying (15) is injective. In other words, the unique linear map $\alpha : V/\mathcal{L}_f(W) \to \left(W/\mathcal{R}_f(V)\right)^*$ satisfying (15) is injective (since $\left(W/\mathcal{R}_f(V)\right)^* = \operatorname{Hom}\left(W/\mathcal{R}_f(V), \mathbf{k}\right)$). Consider this map $\alpha$.

The vector spaces $V/\mathcal{L}_f(W)$ and $W/\mathcal{R}_f(V)$ are finite-dimensional (since $V$ and $W$ are finite-dimensional). The vector space $\left(W/\mathcal{R}_f(V)\right)^*$ is finite-dimensional (since $W/\mathcal{R}_f(V)$ is finite-dimensional). We know that the map $\alpha$ is injective. Thus, Proposition 6.4 (applied to $A = V/\mathcal{L}_f(W)$, $B = \left(W/\mathcal{R}_f(V)\right)^*$ and $i = \alpha$) shows that $\dim\left(V/\mathcal{L}_f(W)\right) \leq \dim\left(\left(W/\mathcal{R}_f(V)\right)^*\right)$.

But every finite-dimensional vector space $G$ satisfies $\dim\left(G^*\right) = \dim G$. Applying this to $G = W/\mathcal{R}_f(V)$, we obtain $\dim\left(\left(W/\mathcal{R}_f(V)\right)^*\right) = \dim\left(W/\mathcal{R}_f(V)\right)$.

Hence,

$$\dim\left(V/\mathcal{L}_f\left(W\right)\right) \leq \dim\left(\left(W/\mathcal{R}_f\left(V\right)\right)^*\right) = \dim\left(W/\mathcal{R}_f\left(V\right)\right). \tag{23}$$

We thus have proven the inequality (23) for every two finite-dimensional vector spaces $V$ and $W$ and every bilinear form $f : V \times W \to \mathbf{k}$.

But $f^{\mathrm{op}} : W \times V \to \mathbf{k}$ is a bilinear form (by Proposition 2.10 **(a)**). Hence, we can apply (23) to $W$, $V$ and $f^{\mathrm{op}}$ instead of $V$, $W$ and $f$. As a result, we obtain

$$\dim\left(W/\mathcal{L}_{f^{\mathrm{op}}}\left(V\right)\right) \leq \dim\left(V/\mathcal{R}_{f^{\mathrm{op}}}\left(W\right)\right).$$

But Proposition 3.2 **(a)** (applied to $U = \mathbf{k}$ and $B = W$) yields $\mathcal{L}_f\left(W\right) = \mathcal{R}_{f^{\mathrm{op}}}\left(W\right)$. Also, Proposition 3.2 **(b)** (applied to $U = \mathbf{k}$ and $A = V$) yields $\mathcal{R}_f\left(V\right) = \mathcal{L}_{f^{\mathrm{op}}}\left(V\right)$. Now,

$$\begin{aligned}\dim\left(W/\underbrace{\mathcal{R}_f\left(V\right)}_{=\mathcal{L}_{f^{\mathrm{op}}}\left(V\right)}\right) &= \dim\left(W/\mathcal{L}_{f^{\mathrm{op}}}\left(V\right)\right) \\ &\leq \dim\left(V/\underbrace{\mathcal{R}_{f^{\mathrm{op}}}\left(W\right)}_{=\mathcal{L}_f\left(W\right)}\right) = \dim\left(V/\mathcal{L}_f\left(W\right)\right).\end{aligned}$$

Combining this inequality with (23), we obtain $\dim\left(V/\mathcal{L}_f\left(W\right)\right) = \dim\left(W/\mathcal{R}_f\left(V\right)\right)$. This proves Theorem 6.5 **(c)**.

It remains to prove Theorem 6.5 **(b)**. But this is now easy: We have

$$\dim\left(V/\mathcal{L}_f\left(W\right)\right) = \dim\left(W/\mathcal{R}_f\left(V\right)\right) = \dim\left(\left(W/\mathcal{R}_f\left(V\right)\right)^*\right).$$

Hence, Proposition 6.2 (applied to $A = V/\mathcal{L}_f\left(W\right)$, $B = \left(W/\mathcal{R}_f\left(V\right)\right)^*$ and $i = \alpha$) shows that $\alpha$ is a vector space isomorphism (since $\alpha$ is injective). This proves Theorem 6.5 **(b)**. □

Theorem 6.5 **(c)** leads to a particularly useful corollary:

**Corollary 6.6.** Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to \mathbf{k}$ be a bilinear form. Let $A$ be a subspace of $V$. Let $B$ be a subspace of $W$. Then,

$$\dim\left(A/\left(A \cap \mathcal{L}_f\left(B\right)\right)\right) = \dim\left(B/\left(B \cap \mathcal{R}_f\left(A\right)\right)\right).$$

We shall derive this corollary from Theorem 6.5 **(c)** using the following simple fact:

**Proposition 6.7.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map. Let $A$ be a subspace of $V$. Let $B$ be a subspace of $W$.

**(a)** We have $A \cap \mathcal{L}_f(B) = \mathcal{L}_{f|_{A\times B}}(B)$.

**(b)** We have $B \cap \mathcal{R}_f(A) = \mathcal{R}_{f|_{A\times B}}(A)$.

*Proof of Proposition 6.7.* **(a)** The definition of $\mathcal{L}_f(B)$ yields $\mathcal{L}_f(B) = \{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}$.

The map $f|_{A\times B}$ is a map $A \times B \to U$. Hence, the definition of $\mathcal{L}_{f|_{A\times B}}(B)$ yields

$$\begin{aligned}
\mathcal{L}_{f|_{A\times B}}(B) &= \left\{ v \in A \mid \underbrace{(f|_{A\times B})(v,b)}_{=f(v,b)} = 0 \text{ for all } b \in B \right\} \\
&= \{v \in A \mid f(v,b) = 0 \text{ for all } b \in B\} \\
&= \{v \in V \mid v \in A \text{ and } f(v,b) = 0 \text{ for all } b \in B\} \\
&\qquad (\text{since } A \subseteq V) \\
&= \underbrace{\{v \in V \mid v \in A\}}_{\substack{=A \\ (\text{since } A\subseteq V)}} \cap \underbrace{\{v \in V \mid f(v,b) = 0 \text{ for all } b \in B\}}_{=\mathcal{L}_f(B)} \\
&= A \cap \mathcal{L}_f(B).
\end{aligned}$$

This proves Proposition 6.7 **(a)**.

**(b)** This proof is similar to that of Proposition 6.7 **(a)**.

□

*Proof of Corollary 6.6.* The vector spaces $V$ and $W$ are finite-dimensional. Hence, their subspaces $A$ and $B$ also are finite-dimensional. Proposition 2.10 **(b)** shows that the restriction $f|_{A\times B}$ is a bilinear form $A \times B \to \mathbf{k}$. Hence, Theorem 6.5 **(c)** (applied to $A$, $B$ and $f|_{A\times B}$ instead of $V$, $W$ and $f$) shows that $\dim\left(A/\mathcal{L}_{f|_{A\times B}}(B)\right) = \dim\left(B/\mathcal{R}_{f|_{A\times B}}(A)\right)$.

But Proposition 6.7 **(a)** (applied to $U = \mathbf{k}$) shows that $A \cap \mathcal{L}_f(B) = \mathcal{L}_{f|_{A\times B}}(B)$. Also, Proposition 6.7 **(b)** (applied to $U = \mathbf{k}$) shows that $B \cap \mathcal{R}_f(A) = \mathcal{R}_{f|_{A\times B}}(A)$. Hence,

$$\begin{aligned}
\dim\left(A/\underbrace{(A \cap \mathcal{L}_f(B))}_{=\mathcal{L}_{f|_{A\times B}}(B)}\right) &= \dim\left(A/\mathcal{L}_{f|_{A\times B}}(B)\right) = \dim\left(B/\underbrace{\mathcal{R}_{f|_{A\times B}}(A)}_{=B\cap\mathcal{R}_f(A)}\right) \\
&= \dim\left(B/\left(B \cap \mathcal{R}_f(A)\right)\right).
\end{aligned}$$

This proves Corollary 6.6. □

# 7. More on orthogonal spaces

We are now in the position to prove less obvious results about orthogonal spaces.

First of all, using Corollary 6.6, we can derive a formula for the dimension of of an orthogonal space:

**Corollary 7.1.** Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to \mathbf{k}$ be a bilinear form.

**(a)** If $A$ is a subspace of $V$, then

$$\dim\left(\mathcal{R}_f(A)\right) = \dim W - \dim A + \dim\left(A \cap \mathcal{L}_f(W)\right).$$

**(b)** If $B$ is a subspace of $W$, then

$$\dim\left(\mathcal{L}_f(B)\right) = \dim V - \dim B + \dim\left(B \cap \mathcal{R}_f(V)\right).$$

*Proof of Corollary 7.1.* **(a)** Let $A$ be a subspace of $V$. Then, Corollary 6.6 (applied to $B = W$) yields

$$\begin{aligned}\dim\left(A/\left(A \cap \mathcal{L}_f(W)\right)\right) &= \dim\left(W/\underbrace{\left(W \cap \mathcal{R}_f(A)\right)}_{\substack{=\mathcal{R}_f(A) \\ (\text{since } \mathcal{R}_f(A) \subseteq W)}}\right) \\ &= \dim\left(W/\mathcal{R}_f(A)\right) = \dim W - \dim\left(\mathcal{R}_f(A)\right),\end{aligned}$$

so that

$$\dim W - \dim\left(\mathcal{R}_f(A)\right) = \dim\left(A/\left(A \cap \mathcal{L}_f(W)\right)\right) = \dim A - \dim\left(A \cap \mathcal{L}_f(W)\right).$$

Solving this equation for $\dim\left(\mathcal{R}_f(A)\right)$, we obtain

$$\begin{aligned}\dim\left(\mathcal{R}_f(A)\right) &= \dim W - \left(\dim A - \dim\left(A \cap \mathcal{L}_f(W)\right)\right) \\ &= \dim W - \dim A + \dim\left(A \cap \mathcal{L}_f(W)\right).\end{aligned}$$

This proves Corollary 7.1 **(a)**.

**(b)** The proof of Corollary 7.1 **(b)** is analogous to that of Corollary 7.1 **(a)** (but now we need to apply Corollary 6.6 to $A = V$).

□

Let us now state another fact from linear algebra:

**Proposition 7.2.** Let $V$ be a vector space. Let $A$ and $B$ be two subspaces of $V$.

**(a)** We have $(A+B)/B \cong A/(A \cap B)$ as vector spaces.

**(b)** Assume that $V$ is finite-dimensional. Then, $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$.

*Proof of Proposition 7.2.* **(a)** Let $\iota$ be the canonical inclusion $A \to A+B$. (This is well-defined, since $A \subseteq A+B$.) Let $\pi_{A,A\cap B}$ be the canonical projection $A \to A/(A \cap B)$. Let $\pi_{A+B,B}$ be the canonical projection $A+B \to (A+B)/B$.

The map $\pi_{A+B,B} \circ \iota : A \to (A+B)/B$ is linear (since it is the composition of the two linear maps $\pi_{A+B,B}$ and $\iota$). It furthermore satisfies $(\pi_{A+B,B} \circ \iota)(A \cap B) = 0$ [14]. Proposition 5.1 (applied to $A$, $A \cap B$, $\pi_{A,A\cap B}$, $(A+B)/B$ and $\pi_{A+B,B} \circ \iota$ instead of $V$, $A$, $\pi_{V,A}$, $W$ and $g$) thus shows that there exists a unique linear map $g' : A/(A \cap B) \to (A+B)/B$ such that $\pi_{A+B,B} \circ \iota = g' \circ \pi_{A,A\cap B}$. Consider this $g'$.

We have

$$g'([a]_{A\cap B}) = [a]_B \qquad \text{for every } a \in A \tag{24}$$

[14] *Proof.* Let $a \in A \cap B$. Thus, $a \in A \cap B \subseteq A$; therefore, $\iota(a)$ is well-defined. Now, $\iota$ is an inclusion map; therefore, $\iota(a) = a$. But $\pi_{A+B,B}$ is the canonical projection $A+B \to (A+B)/B$. Hence, $\pi_{A+B,B}(a) = [a]_B = 0$ (since $a \in A \cap B \subseteq B$). Therefore, $(\pi_{A+B,B} \circ \iota)(a) = \pi_{A+B,B}\left(\underbrace{\iota(a)}_{=a}\right) = \pi_{A+B,B}(a) = 0$.

Now, let us forget that we fixed $a$. We thus have shown that $(\pi_{A+B,B} \circ \iota)(a) = 0$ for every $a \in A \cap B$. In other words, $(\pi_{A+B,B} \circ \iota)(A \cap B) = 0$, qed.

[15]. Now, the map $g'$ is surjective[16] and injective[17]. Hence, the map $g'$ is bijective, and thus a vector space isomorphism (since $g'$ is linear). Thus, there exists a vector space isomorphism $A/(A\cap B) \to (A+B)/B$ (namely, $g'$). In other words, $(A+B)/B \cong A/(A\cap B)$ as vector spaces. Proposition 7.2 **(a)** is now proven.

**(b)** Proposition 7.2 **(a)** yields $(A+B)/B \cong A/(A\cap B)$. Thus we conclude $\dim((A+B)/B) = \dim(A/(A\cap B))$ (since isomorphic vector spaces have equal dimensions). Thus, $\dim((A+B)/B) = \dim(A/(A\cap B)) = \dim A - \dim(A\cap B)$. Comparing this with $\dim((A+B)/B) = \dim(A+B) - \dim B$, we obtain

$$\dim(A+B) - \dim B = \dim A - \dim(A\cap B).$$

[15] *Proof of (24):* Let $a \in A$. Recall that $\pi_{A,A\cap B}$ is the canonical projection $A \to A/(A\cap B)$. Hence, $\pi_{A,A\cap B}(a) = [a]_{A\cap B}$. Thus, $[a]_{A\cap B} = \pi_{A,A\cap B}(a)$. Applying the map $g'$ to both sides of this equality, we obtain

$$g'([a]_{A\cap B}) = g'(\pi_{A,A\cap B}(a)) = \underbrace{(g'\circ\pi_{A,A\cap B})}_{=\pi_{A+B,B}\circ\iota}(a) = (\pi_{A+B,B}\circ\iota)(a)$$

$$= \pi_{A+B,B}\left(\underbrace{\iota(a)}_{\substack{=a \\ \text{(since } \iota \text{ is an inclusion map)}}}\right) = \pi_{A+B,B}(a) = [a]_B$$

(since $\pi_{A+B,B}$ is the canonical projection $A+B \to (A+B)/B$). This proves (24).

[16] *Proof.* Let $x \in (A+B)/B$. We shall show that $x \in g'(A/(A\cap B))$.

We have $x \in (A+B)/B$. Thus, $x$ can be written in the form $[u]_B$ for some $u \in A+B$. Consider this $u$. Thus, $x = [u]_B$.

We have $u \in A+B$. Thus, $u$ can be written in the form $a+b$ for some $a \in A$ and $b \in B$. Consider these $a$ and $b$. Thus, $u = a+b$. Now,

$$x = \left[\underbrace{u}_{=a+b}\right]_B = [a+b]_B = [a]_B + \underbrace{[b]_B}_{\substack{=0 \\ \text{(since } b\in B)}} = [a]_B = g'\left(\underbrace{[a]_{A\cap B}}_{\in A/(A\cap B)}\right) \qquad \text{(by (24))}$$

$$\in g'(A/(A\cap B)).$$

Now, let us forget that we fixed $x$. We thus have shown that every $x \in (A+B)/B$ satisfies $x \in g'(A/(A\cap B))$. In other words, $(A+B)/B \subseteq g'(A/(A\cap B))$. In other words, the map $g'$ is surjective, qed.

[17] *Proof.* Let $x \in \operatorname{Ker}(g')$. We shall show that $x = 0$.

We have $x \in \operatorname{Ker}(g') \subseteq A/(A\cap B)$. Thus, $x$ can be written in the form $[a]_{A\cap B}$ for some $a \in A$. Consider this $a$. Thus, $x = [a]_{A\cap B}$.

We have $x \in \operatorname{Ker}(g')$ and thus $g'(x) = 0$. Hence, $0 = g'\left(\underbrace{x}_{=[a]_{A\cap B}}\right) = g'([a]_{A\cap B}) = [a]_B$ (by (24)). Hence, $[a]_B = 0$. In other words, $a \in B$. Combining this with $a \in A$, we obtain $a \in A\cap B$. Hence, $[a]_{A\cap B} = 0$. Thus, $x = [a]_{A\cap B} = 0$.

Let us now forget that we fixed $x$. We thus have proven that every $x \in \operatorname{Ker}(g')$ satisfies $x = 0$. In other words, $\operatorname{Ker}(g') = 0$. This shows that $g'$ is injective (since $g'$ is a linear map), qed.

Thus,

$$\begin{aligned}\dim(A+B) &= \dim B + \dim A - \dim(A \cap B) \\ &= \dim A + \dim B - \dim(A \cap B).\end{aligned}$$

This proves Proposition 7.2 **(b)**. □

We notice that Proposition 7.2 **(a)** holds even if $\mathbf{k}$ is not a field but just a commutative ring (and vector spaces are replaced by $\mathbf{k}$-modules).

We shall now prove the following fact:

**Proposition 7.3.** Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to \mathbf{k}$ be a bilinear form.

**(a)** If $A$ is a subspace of $V$, then $\mathcal{L}_f(\mathcal{R}_f(A)) = A + \mathcal{L}_f(W)$.

**(b)** If $B$ is a subspace of $W$, then $\mathcal{R}_f(\mathcal{L}_f(B)) = B + \mathcal{R}_f(V)$.

*Proof of Proposition 7.3.* The map $f : V \times W \to \mathbf{k}$ is a bilinear form. In other words, the map $f : V \times W \to \mathbf{k}$ is a bilinear map (according to the definition of a "bilinear form").

**(a)** Let $A$ be a subspace of $V$.

Straightforward applications of Proposition 3.3 **(a)** show that both $\mathcal{L}_f(W)$ and $\mathcal{L}_f(\mathcal{R}_f(A))$ are subspaces of $V$, and thus finite-dimensional.

Corollary 3.7 **(b)** yields $A \subseteq \mathcal{L}_f(\mathcal{R}_f(A))$. Also, $\mathcal{R}_f(A) \subseteq W$, and thus $\mathcal{L}_f(\mathcal{R}_f(A)) \supseteq \mathcal{L}_f(W)$ (by Proposition 3.4 **(a)**, applied to $U = \mathbf{k}$, $B = \mathcal{R}_f(A)$ and $B' = W$). In other words, $\mathcal{L}_f(W) \subseteq \mathcal{L}_f(\mathcal{R}_f(A))$. Combined with $A \subseteq \mathcal{L}_f(\mathcal{R}_f(A))$, this yields $A + \mathcal{L}_f(W) \subseteq \mathcal{L}_f(\mathcal{R}_f(A))$ [18].

We have $A \subseteq V$. Thus, Proposition 3.4 **(b)** (applied to $A' = V$ and $U = \mathbf{k}$) yields $\mathcal{R}_f(A) \supseteq \mathcal{R}_f(V)$. Therefore, $\mathcal{R}_f(A) \cap \mathcal{R}_f(V) = \mathcal{R}_f(V)$.

Theorem 6.5 **(c)** shows that $\dim(V/\mathcal{L}_f(W)) = \dim(W/\mathcal{R}_f(V)) = \dim W - \dim(\mathcal{R}_f(V))$. Hence,

$$\dim W - \dim(\mathcal{R}_f(V)) = \dim(V/\mathcal{L}_f(W)) = \dim V - \dim(\mathcal{L}_f(W)). \tag{25}$$

Proposition 7.2 (applied to $B = \mathcal{L}_f(W)$) shows that

$$\dim(A + \mathcal{L}_f(W)) = \dim A + \dim(\mathcal{L}_f(W)) - \dim(A \cap \mathcal{L}_f(W)). \tag{26}$$

[18] *Proof.* Recall the following simple fact from linear algebra: If $X$, $Y$ and $Z$ are three subspaces of $V$ satisfying $X \subseteq Z$ and $Y \subseteq Z$, then $X + Y \subseteq Z$. This fact (applied to $X = A$, $Y = \mathcal{L}_f(W)$ and $Z = \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$) shows that $A + \mathcal{L}_f(W) \subseteq \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$ (since $A \subseteq \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$ and $\mathcal{L}_f(W) \subseteq \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$). Qed.

But $\mathcal{R}_f(A)$ is a subspace of $W$ (by Proposition 3.3 **(a)**, applied to $U = \mathbf{k}$). Hence, Corollary 7.1 **(b)** (applied to $B = \mathcal{R}_f(A)$) yields

$$\begin{aligned}
&\dim\left(\mathcal{L}_f\left(\mathcal{R}_f(A)\right)\right) \\
&= \dim V - \underbrace{\dim\left(\mathcal{R}_f(A)\right)}_{\substack{=\dim W - \dim A + \dim\left(A \cap \mathcal{L}_f(W)\right) \\ \text{(by Corollary 7.1 (a))}}} + \dim\left(\underbrace{\mathcal{R}_f(A) \cap \mathcal{R}_f(V)}_{=\mathcal{R}_f(V)}\right) \\
&= \dim V - \left(\dim W - \dim A + \dim\left(A \cap \mathcal{L}_f(W)\right)\right) + \dim\left(\mathcal{R}_f(V)\right) \\
&= \dim V - \dim W + \dim A - \dim\left(A \cap \mathcal{L}_f(W)\right) + \dim\left(\mathcal{R}_f(V)\right) \\
&= \dim V - \underbrace{\left(\dim W - \dim\left(\mathcal{R}_f(V)\right)\right)}_{\substack{=\dim V - \dim\left(\mathcal{L}_f(W)\right) \\ \text{(by (25))}}} + \dim A - \dim\left(A \cap \mathcal{L}_f(W)\right) \\
&= \dim V - \left(\dim V - \dim\left(\mathcal{L}_f(W)\right)\right) + \dim A - \dim\left(A \cap \mathcal{L}_f(W)\right) \\
&= \dim A + \dim\left(\mathcal{L}_f(W)\right) - \dim\left(A \cap \mathcal{L}_f(W)\right) \\
&= \dim\left(A + \mathcal{L}_f(W)\right) \qquad (27)
\end{aligned}$$

(by (26)).

Thus, we know that $\mathcal{L}_f\left(\mathcal{R}_f(A)\right)$ is a finite-dimensional vector space; we know that $A + \mathcal{L}_f(W)$ is a subspace of $\mathcal{L}_f\left(\mathcal{R}_f(A)\right)$ (since $A + \mathcal{L}_f(W) \subseteq \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$); we furthermore know that $\dim\left(A + \mathcal{L}_f(W)\right) = \dim\left(\mathcal{L}_f\left(\mathcal{R}_f(A)\right)\right)$ (by (27)). Thus, Proposition 6.3 (applied to $\mathcal{L}_f\left(\mathcal{R}_f(A)\right)$ and $A + \mathcal{L}_f(W)$ instead of $B$ and $A$) shows that $A + \mathcal{L}_f(W) = \mathcal{L}_f\left(\mathcal{R}_f(A)\right)$. This proves Proposition 7.3 **(a)**.

**(b)** Let $B$ be a subspace of $V$. Proposition 2.10 **(a)** shows that the map $f^{\mathrm{op}} : W \times V \to \mathbf{k}$ is a bilinear form. Hence, we can apply Proposition 7.3 **(a)** to $W$, $V$, $f^{\mathrm{op}}$ and $B$ instead of $V$, $W$, $f$ and $A$. As a result, we obtain $\mathcal{L}_{f^{\mathrm{op}}}\left(\mathcal{R}_{f^{\mathrm{op}}}(B)\right) = B + \mathcal{L}_{f^{\mathrm{op}}}(V)$.

But using straightforward (by now) applications of Proposition 3.2, we can rewrite this as $\mathcal{R}_f\left(\mathcal{L}_f(B)\right) = B + \mathcal{R}_f(V)$. This proves Proposition 7.3 **(b)**.

□

**Corollary 7.4.** Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to \mathbf{k}$ be a bilinear form.

**(a)** If $A$ is a subspace of $V$ satisfying $\mathcal{L}_f(W) \subseteq A$, then $\mathcal{L}_f\left(\mathcal{R}_f(A)\right) = A$.

**(b)** If $B$ is a subspace of $W$ satisfying $\mathcal{R}_f(V) \subseteq B$, then $\mathcal{R}_f\left(\mathcal{L}_f(B)\right) = B$.

*Proof of Corollary 7.4.* **(a)** Let $A$ be a subspace of $V$ satisfying $\mathcal{L}_f(W) \subseteq A$. Proposition 7.3 **(a)** shows that $\mathcal{L}_f\left(\mathcal{R}_f(A)\right) = A + \mathcal{L}_f(W) = A$ (since $\mathcal{L}_f(W) \subseteq A$). Corollary 7.4 **(a)** is proven.

**(b)** Let $B$ be a subspace of $W$ satisfying $\mathcal{R}_f(V) \subseteq B$, then $\mathcal{R}_f\left(\mathcal{L}_f(B)\right) = B$. Proposition 7.3 **(b)** shows that $\mathcal{R}_f\left(\mathcal{L}_f(B)\right) = B + \mathcal{R}_f(V) = B$ (since $\mathcal{R}_f(V) \subseteq B$). Corollary 7.4 **(b)** is proven. □

Our next claim is an analogue of Proposition 3.10 with the roles of $\cap$ and $+$ interchanged (but also with stricter assumptions, since in general it would not hold):

**Proposition 7.5.** Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to \mathbf{k}$ be a bilinear form.

**(a)** If $B_1$ and $B_2$ are two subspaces of $W$ satisfying $\mathcal{R}_f(V) \subseteq B_1$ and $\mathcal{R}_f(V) \subseteq B_2$, then $\mathcal{L}_f(B_1 \cap B_2) = \mathcal{L}_f(B_1) + \mathcal{L}_f(B_2)$.

**(b)** If $A_1$ and $A_2$ are two subspaces of $V$ satisfying $\mathcal{L}_f(W) \subseteq A_1$ and $\mathcal{L}_f(W) \subseteq A_2$, then $\mathcal{R}_f(A_1 \cap A_2) = \mathcal{R}_f(A_1) + \mathcal{R}_f(A_2)$.

*Proof of Proposition 7.5.* **(a)** Let $B_1$ and $B_2$ be two subspaces of $W$ satisfying $\mathcal{R}_f(V) \subseteq B_1$ and $\mathcal{R}_f(V) \subseteq B_2$.

Straightforward applications of Proposition 3.3 **(a)** show that $\mathcal{L}_f(B_1)$ and $\mathcal{L}_f(B_2)$ are subspaces of $V$.

From $B_1 \subseteq W$, we obtain $\mathcal{L}_f(B_1) \supseteq \mathcal{L}_f(W)$ (by Proposition 3.4 **(a)**, applied to $U = \mathbf{k}$, $B = B_1$ and $B' = W$). In other words, $\mathcal{L}_f(W) \subseteq \mathcal{L}_f(B_1)$.

Corollary 7.4 **(b)** (applied to $B = B_1$) yields $\mathcal{R}_f\left(\mathcal{L}_f(B_1)\right) = B_1$ (since $\mathcal{R}_f(V) \subseteq B_1$). Corollary 7.4 **(b)** (applied to $B = B_2$) yields $\mathcal{R}_f\left(\mathcal{L}_f(B_2)\right) = B_2$ (since $\mathcal{R}_f(V) \subseteq B_2$).

Now, $\mathcal{L}_f(W) \subseteq \mathcal{L}_f(B_1) \subseteq \mathcal{L}_f(B_1) + \mathcal{L}_f(B_2)$. Hence, Corollary 7.4 **(a)** (applied to $A = \mathcal{L}_f(B_1) + \mathcal{L}_f(B_2)$) yields

$$\mathcal{L}_f\left(\mathcal{R}_f\left(\mathcal{L}_f(B_1) + \mathcal{L}_f(B_2)\right)\right) = \mathcal{L}_f(B_1) + \mathcal{L}_f(B_2).$$

Hence,

$$\begin{aligned}
\mathcal{L}_f(B_1) + \mathcal{L}_f(B_2) &= \mathcal{L}_f\left(\underbrace{\mathcal{R}_f\left(\mathcal{L}_f(B_1) + \mathcal{L}_f(B_2)\right)}_{\substack{=\mathcal{R}_f\left(\mathcal{L}_f(B_1)\right)\cap\mathcal{R}_f\left(\mathcal{L}_f(B_2)\right) \\ \text{(by Proposition 3.10 (b), applied to} \\ A_1=\mathcal{L}_f(B_1) \text{ and } A_2=\mathcal{L}_f(B_2))}}\right) \\
&= \mathcal{L}_f\left(\underbrace{\mathcal{R}_f\left(\mathcal{L}_f(B_1)\right)}_{=B_1} \cap \underbrace{\mathcal{R}_f\left(\mathcal{L}_f(B_2)\right)}_{=B_2}\right) = \mathcal{L}_f(B_1 \cap B_2).
\end{aligned}$$

This proves Proposition 7.5 **(a)**.

**(b)** The map $f^{\mathrm{op}}$ is a bilinear form (by Proposition 2.10 **(a)**). Hence, Proposition 7.5 **(b)** can be derived from Proposition 7.5 **(a)** using the same tactic that we used (for example) to derive Proposition 3.4 **(b)** from Proposition 3.4 **(a)**.

□