The Bernstein homomorphism via Aguiar-Bergeron-Sottile universality

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Abstract

If H is a commutative connected graded Hopf algebra over a commutative ring $\mathbf{k}$, then a certain canonical $\mathbf{k}$-algebra homomorphism $H \to H \otimes \operatorname{QSym}_{\mathbf{k}}$ is defined, where $\operatorname{QSym}_{\mathbf{k}}$ denotes the Hopf algebra of quasisymmetric functions. This homomorphism generalizes the "internal comultiplication" on $\operatorname{QSym}_{\mathbf{k}}$, and extends what Hazewinkel (in §18.24 of his "Witt vectors") calls the Bernstein homomorphism.

We construct this homomorphism with the help of the universal property of $\operatorname{QSym}_{\mathbf{k}}$ as a combinatorial Hopf algebra (a well-known result by Aguiar, Bergeron and Sottile) and extension of scalars (the commutativity of H allows us to consider, for example, $H \otimes \operatorname{QSym}_{\mathbf{k}}$ as an H-Hopf algebra, and this change of viewpoint significantly extends the reach of the universal property).

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One of the most important aspects of QSym (the Hopf algebra of quasisymmetric functions) is a universal property discovered by Aguiar, Bergeron and Sottile in 2003 [ABS03]; among other applications, it gives a unifying framework for various quasisymmetric and symmetric functions constructed from combinatorial objects (e.g., the chromatic symmetric function of a graph).

On the other hand, let $\Lambda_{\mathbf{k}}$ be the Hopf algebra of symmetric functions over a commutative ring $\mathbf{k}$. If H is any commutative cocommutative connected graded $\mathbf{k}$-Hopf algebra, then a certain $\mathbf{k}$-algebra homomorphism $H \to H \otimes \Lambda_{\mathbf{k}}$ (not a Hopf algebra homomorphism!) was defined by Joseph N. Bernstein, and used by Zelevinsky in [Zelevi81, §5.2] to classify PSH-algebras. In [Haz08, §18.24], Hazewinkel observed that this homomorphism generalizes the second comultiplication of $\Lambda_{\mathbf{k}}$, and asked for "more study" and a better understanding of this homomorphism.

In this note, I shall define an extended version of this homomorphism: a $\mathbf{k}$-algebra homomorphism $H \to H \otimes \mathrm{QSym}_{\mathbf{k}}$ for any commutative (but not necessarily cocommutative) connected graded $\mathbf{k}$-Hopf algebra H. This homomorphism, which I will call the *Bernstein homomorphism*, will generalize the second comultiplication of $\mathrm{QSym}_{\mathbf{k}}$, or rather its variant with the two tensorands flipped. When H is cocommutative, this homomorphism has its image contained in $H \otimes \Lambda_{\mathbf{k}}$ and thus becomes Bernstein's original homomorphism.

The Bernstein homomorphism $H \to H \otimes \mathrm{QSym}_{\mathbf{k}}$ is not fully new (although I have not seen it appear explicitly in the literature). Its dual version is a coalgebra homomorphism $H' \otimes \mathrm{NSym}_{\mathbf{k}} \to H'$, where H' is a cocommutative connected graded Hopf algebra; i.e., it is an action of $\mathrm{NSym}_{\mathbf{k}}$ on any such H'. This action is implicit in the work of Patras and Reutenauer on descent algebras, and a variant of it for Hopf monoids instead of Hopf algebras appears in [Aguiar13, Propositions 84 and 88, and especially the Remark after Proposition 88]. What I believe to be new in this note is the way I will construct the Bernstein homomorphism: as a consequence of the Aguiar-Bergeron-Sottile universal property of QSym, but applied not to the $\mathbf{k}$-Hopf algebra $\mathrm{QSym}_{\mathbf{k}}$ but to the H-Hopf algebra QSym_H. The commutativity of H is being used here to deploy H as the base ring.

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1. Definitions and conventions

For the rest of this note, we fix a commutative ring[1] $\mathbf{k}$. All tensor signs ($\otimes$) without a subscript will mean $\otimes_{\mathbf{k}}$. We shall use the notions of $\mathbf{k}$-algebras, $\mathbf{k}$-coalgebras and $\mathbf{k}$-Hopf algebras as defined (e.g.) in [GriRei14, Chapter 1]. We shall also use the notions of graded $\mathbf{k}$-algebras, graded $\mathbf{k}$-coalgebras and graded $\mathbf{k}$-Hopf algebras as defined in [GriRei14, Chapter 1]; in particular, we shall not use the topologists' sign conventions[2]. The comultiplication and the counit of a $\mathbf{k}$-coalgebra C will be denoted by Δ_C and ε_C, respectively; when the C is unambiguously clear from the context, we will omit it from the notation (so we will just write Δ and ε).

If V and W are two $\mathbf{k}$-modules, then we let $\tau_{V,W}$ be the $\mathbf{k}$-linear map $V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$. This $\mathbf{k}$-linear map $\tau_{V,W}$ is called the *twist map*, and is a $\mathbf{k}$-module isomorphism.

The next two definitions are taken from [GriRei14, §1.4][3]:

Definition 1.1. Let A be a $\mathbf{k}$-algebra. Let m_A denote the $\mathbf{k}$-linear map $A \otimes A \to A$, $a \otimes b \mapsto ab$. Let u_A denote the $\mathbf{k}$-linear map $\mathbf{k} \to A$, $\lambda \mapsto \lambda \cdot 1_A$. (The maps m_A and u_A are often denoted by m and u when A is unambiguously clear from the context.) For any $k \in \mathbb{N}$, we define a $\mathbf{k}$-linear map $m^{(k-1)} : A^{\otimes k} \to A$ recursively as follows: We set $m^{(-1)} = u_A$, $m^{(0)} = \mathrm{id}_A$ and

$$m^{(k)} = m \circ \left(\mathrm{id}_A \otimes m^{(k-1)}\right) \qquad \text{for every } k \geq 1.$$

The maps $m^{(k-1)} : A^{\otimes k} \to A$ are called the *iterated multiplication maps* of A.

Notice that for every $\mathbf{k} \in \mathbb{N}$, the map $m^{(k-1)}$ is the $\mathbf{k}$-linear map $A^{\otimes k} \to A$ which sends every $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in A^{\otimes k}$ to $a_1 a_2 \cdots a_k$.

Definition 1.2. Let C be a $\mathbf{k}$-coalgebra. For any $k \in \mathbb{N}$, we define a $\mathbf{k}$-linear map $\Delta^{(k-1)} : C \to C^{\otimes k}$ recursively as follows: We set $\Delta^{(-1)} = \varepsilon_C$, $\Delta^{(0)} = \mathrm{id}_C$ and

$$\Delta^{(k)} = \left(\mathrm{id}_C \otimes \Delta^{(k-1)}\right) \circ \Delta \qquad \text{for every } k \geq 1.$$

[1] The word "ring" always means "associative ring with 1" in this note. Furthermore, a $\mathbf{k}$-algebra (when $\mathbf{k}$ is a commutative ring) means a $\mathbf{k}$-module A equipped with a ring structure such that the multiplication map $A \times A \to A$ is $\mathbf{k}$-bilinear.

[2] Thus, the twist map $V \otimes V \to V \otimes V$ for a graded $\mathbf{k}$-module V sends $v \otimes w \mapsto w \otimes v$, even if v and w are homogeneous of odd degree.

[3] The objects we are defining are classical and standard; however, the notation we are using for them is not. For example, what we call $\Delta^{(k-1)}$ in Definition 1.2 is denoted by Δ_{k-1} in [Sweed69], and is called $\Delta^{(k)}$ in [Fresse14, §7.1].

The maps $\Delta^{(k-1)} : C \to C^{\otimes k}$ are called the *iterated comultiplication maps* of C.

A *composition* shall mean a finite sequence of positive integers. The *size* of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is defined to be the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_k$, and is denoted by $|\alpha|$. Let Comp denote the set of all compositions.

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$.

Definition 1.3. Let H be a graded $\mathbf{k}$-module. For every $n \in \mathbb{N}$, we let $\pi_n : H \to H$ be the canonical projection of H onto the n-th graded component H_n of H. We shall always regard π_n as a map from H to H, not as a map from H to H_n, even though its image is H_n.

For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, we let $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ be the tensor product $\pi_{a_1} \otimes \pi_{a_2} \otimes \cdots \otimes \pi_{a_k}$ of the canonical projections $\pi_{a_i} : H \to H$. Thus, the image of π_α can be identified with $H_{a_1} \otimes H_{a_2} \otimes \cdots \otimes H_{a_k}$.

Let $\mathrm{QSym}_{\mathbf{k}}$ denote the $\mathbf{k}$-Hopf algebra of quasisymmetric functions defined over $\mathbf{k}$. (This is defined and denoted by *QSym* in [ABS03, §3]; it is also defined and denoted by QSym in [GriRei14, Chapter 5].) We shall follow the notations and conventions of [GriRei14, §5.1] as far as $\mathrm{QSym}_{\mathbf{k}}$ is concerned; in particular, we regard $\mathrm{QSym}_{\mathbf{k}}$ as a subring of the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates $x_1, x_2, x_3, \ldots$.

Let ε_P denote the $\mathbf{k}$-linear map $\mathrm{QSym}_{\mathbf{k}} \to \mathbf{k}$ sending every $f \in \mathrm{QSym}_{\mathbf{k}}$ to $f(1, 0, 0, 0, \ldots) \in \mathbf{k}$. (This map ε_P is denoted by ζ_Q in [ABS03, §4] and by ζ_Q in [GriRei14, Example 7.1.2].) Notice that ε_P is a $\mathbf{k}$-algebra homomorphism.

Definition 1.4. For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we define a power series $M_\alpha \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

(where the sum is over all strictly increasing ℓ-tuples $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers). It is well-known (and easy to check) that this M_α belongs to $\mathrm{QSym}_{\mathbf{k}}$. The power series M_α is called the *monomial quasisymmetric function* corresponding to α. The family $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module $\mathrm{QSym}_{\mathbf{k}}$; this is the so-called *monomial basis* of $\mathrm{QSym}_{\mathbf{k}}$. (See [ABS03, §3] and [GriRei14, §5.1] for more about this basis.)

It is well-known that every $(b_1, b_2, \ldots, b_\ell) \in \mathrm{Comp}$ satisfies

$$\Delta\left(M_{(b_1, b_2, \ldots, b_\ell)}\right) = \sum_{i=0}^{\ell} M_{(b_1, b_2, \ldots, b_i)} \otimes M_{(b_{i+1}, b_{i+2}, \ldots, b_\ell)} \tag{1}$$

and

$$\varepsilon\left(M_{(b_1, b_2, \ldots, b_\ell)}\right) = \begin{cases} 1, & \text{if } \ell = 0; \\ 0, & \text{if } \ell \neq 0 \end{cases}.$$

These two equalities can be used as a definition of the $\mathbf{k}$-coalgebra structure on $\operatorname{QSym}_{\mathbf{k}}$ (because $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module $\operatorname{QSym}_{\mathbf{k}}$, and thus the $\mathbf{k}$-linear maps Δ and ε are uniquely determined by their values on the M_α).

2. The Aguiar-Bergeron-Sottile theorem

The cornerstone of the Aguiar-Bergeron-Sottile paper [ABS03] is the following result:

Theorem 2.1. Let $\mathbf{k}$ be a commutative ring. Let H be a connected graded $\mathbf{k}$-Hopf algebra. Let $\zeta : H \to \mathbf{k}$ be a $\mathbf{k}$-algebra homomorphism.

(a) Then, there exists a unique graded $\mathbf{k}$-coalgebra homomorphism $\Psi : H \to \operatorname{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{tikzcd} H \arrow[rr, "\Psi"] \arrow[dr, "\zeta"'] & & \operatorname{QSym}_{\mathbf{k}} \arrow[dl, "\varepsilon_P"] \\ & \mathbf{k} & \end{tikzcd}$$

is commutative.

(b) This unique $\mathbf{k}$-coalgebra homomorphism $\Psi : H \to \operatorname{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-Hopf algebra homomorphism.

(c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\zeta_\alpha : H \to \mathbf{k}$ as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{\zeta^{\otimes k}} \mathbf{k}^{\otimes k} \xrightarrow{\cong} \mathbf{k} .$$

(Here, the map $\mathbf{k}^{\otimes k} \xrightarrow{\cong} \mathbf{k}$ is the canonical $\mathbf{k}$-algebra isomorphism from $\mathbf{k}^{\otimes k}$ to $\mathbf{k}$. Recall also that $\Delta^{(k-1)} : H \to H^{\otimes k}$ is the "iterated comultiplication map"; see [GriRei14, §1.4] for its definition. The map $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ is the one defined in Definition 1.3.)

Then, the unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 2.1 **(a)** is given by the formula

$$\Psi(h) = \sum_{\substack{\alpha \in \operatorname{Comp}; \\ |\alpha| = n}} \zeta_\alpha(h) \cdot M_\alpha \qquad \text{whenever } n \in \mathbb{N} \text{ and } h \in H_n.$$

(Recall that H_n denotes the n-th graded component of H.)

(d) The unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 2.1 **(a)** is also given by

$$\Psi(h) = \sum_{\alpha \in \operatorname{Comp}} \zeta_\alpha(h) \cdot M_\alpha \qquad \text{for every } h \in H$$

(in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).

(e) Assume that the $\mathbf{k}$-coalgebra H is cocommutative. Then, the unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 2.1 **(a)** satisfies $\Psi(H) \subseteq \Lambda_{\mathbf{k}}$, where $\Lambda_{\mathbf{k}}$ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$. (See [GriRei14, §2] for the definition of $\Lambda_{\mathbf{k}}$. We regard $\Lambda_{\mathbf{k}}$ as a $\mathbf{k}$-subalgebra of $\operatorname{QSym}_{\mathbf{k}}$ in the usual way.)

Parts **(a)**, **(b)** and **(c)** of Theorem 2.1 are proven in [ABS03, proof of Theorem 4.1] and [GriRei14, proof of Theorem 7.1.3] (although we are using different notations here[4], and avoiding the standing assumptions of [ABS03] which needlessly require $\mathbf{k}$ to be a field and H to be of finite type). Theorem 2.1 **(d)** easily follows from Theorem 2.1 **(c)**[5]. Theorem 2.1 **(e)** appears in [GriRei14, Remark 7.1.4] (and something very close is proven in [ABS03, Theorem 4.3]). For the sake of completeness, let me give some details on the proof of Theorem 2.1 **(e)**:

Proof of Theorem 2.1 ***(e)****.* Let $\varepsilon_p : \Lambda_{\mathbf{k}} \to \mathbf{k}$ be the restriction of the $\mathbf{k}$-algebra homomorphism $\varepsilon_P : \operatorname{QSym}_{\mathbf{k}} \to \mathbf{k}$ to $\Lambda_{\mathbf{k}}$. From [ABS03, Theorem 4.3], we know that there exists a unique graded $\mathbf{k}$-coalgebra homomorphism $\Psi' : H \to \Lambda_{\mathbf{k}}$ for

[4]The paper [ABS03] defines a *combinatorial coalgebra* to be a pair (H, ζ) consisting of a connected graded $\mathbf{k}$-coalgebra H (where "connected" means that $\varepsilon\mid_{H_0}: H_0 \to \mathbf{k}$ is a $\mathbf{k}$-module isomorphism) and a $\mathbf{k}$-linear map $\zeta : H \to \mathbf{k}$ satisfying $\zeta\mid_{H_0} = \varepsilon\mid_{H_0}$. Furthermore, it defines a *morphism* from a combinatorial coalgebra (H', ζ') to a combinatorial coalgebra (H, ζ) to be a homomorphism $\alpha : H' \to H$ of graded $\mathbf{k}$-coalgebras for which the diagram

$$\begin{tikzcd} H' \arrow[rr, "\alpha"] \arrow[dr, "\zeta'"'] & & H \arrow[dl, "\zeta"] \\ & \mathbf{k} & \end{tikzcd}$$

is commutative. Theorem 2.1 **(a)** translates into this language as follows: There exists a unique morphism from the combinatorial coalgebra (H, ζ) to the combinatorial coalgebra $(\operatorname{QSym}_{\mathbf{k}}, \varepsilon_P)$. (Apart from this, [ABS03] is also using the notations $\Bbbk$, $\mathcal{H}$, $\mathcal{QSym}$ and $\zeta_{\mathcal{Q}}$ for what we call $\mathbf{k}$, H, $\operatorname{QSym}_{\mathbf{k}}$ and ε_P.)

[5]*Proof.* Let Ψ be the unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 2.1 **(a)**. It is easy to see that every $n \in \mathbb{N}$, every composition α with $|\alpha| \neq n$ and every $h \in H_n$ satisfy $\zeta_\alpha(h) = 0$ (because $\pi_\alpha\left(\Delta^{(k-1)}\left(\underbrace{h}_{\in H_n}\right)\right) \in \pi_\alpha\left(\Delta^{(k-1)}(H_n)\right) = 0$ (for reasons of gradedness)). Hence, for every $n \in \mathbb{N}$ and every $h \in H_n$, we have

$$\begin{aligned} \sum_{\alpha \in \operatorname{Comp}} \zeta_\alpha(h) \cdot M_\alpha &= \sum_{\substack{\alpha \in \operatorname{Comp}; \\ |\alpha| = n}} \zeta_\alpha(h) \cdot M_\alpha + \sum_{\substack{\alpha \in \operatorname{Comp}; \\ |\alpha| \neq n}} \underbrace{\zeta_\alpha(h)}_{=0} \cdot M_\alpha \\ &= \sum_{\substack{\alpha \in \operatorname{Comp}; \\ |\alpha| = n}} \zeta_\alpha(h) \cdot M_\alpha = \Psi(h) \qquad \text{(by Theorem 2.1 (c))}. \end{aligned}$$

Both sides of this equality are $\mathbf{k}$-linear in h; thus, it also holds for every $h \in H$ (even if h is not homogeneous). This proves Theorem 2.1 **(d)**.

which the diagram

$$\begin{tikzcd} H \arrow[rr, "\Psi'"] \arrow[dr, "\zeta"'] & & \Lambda_{\mathbf{k}} \arrow[dl, "\varepsilon_p"] \\ & \mathbf{k} & \end{tikzcd} \tag{2}$$

is commutative. Consider this Ψ'. Let $\iota : \Lambda_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}}$ be the canonical inclusion map; this is a $\mathbf{k}$-Hopf algebra homomorphism. Also, $\varepsilon_p = \varepsilon_P \circ \iota$ (by the definition of ε_p). The commutative diagram (2) yields $\zeta = \underbrace{\varepsilon_p}_{=\varepsilon_P \circ \iota} \circ \Psi' = \varepsilon_P \circ \iota \circ \Psi'$.

Now, consider the unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 2.1 **(a)**. Due to its uniqueness, it has the following property: If $\widetilde{\Psi}$ is any $\mathbf{k}$-coalgebra homomorphism $H \to \operatorname{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{tikzcd} H \arrow[rr, "\widetilde{\Psi}"] \arrow[dr, "\zeta"'] & & \operatorname{QSym}_{\mathbf{k}} \arrow[dl, "\varepsilon_P"] \\ & \mathbf{k} & \end{tikzcd} \tag{3}$$

is commutative, then $\widetilde{\Psi} = \Psi$. Applying this to $\widetilde{\Psi} = \iota \circ \Psi'$, we obtain $\iota \circ \Psi' = \Psi$ (since the diagram (3) is commutative for $\widetilde{\Psi} = \iota \circ \Psi'$ (because $\zeta = \varepsilon_P \circ \iota \circ \Psi'$)). Hence, $\underbrace{\Psi}_{=\iota \circ \Psi'}(H) = (\iota \circ \Psi')(H) = \iota\left(\underbrace{\Psi'(H)}_{\subseteq \Lambda_{\mathbf{k}}}\right) \subseteq \iota(\Lambda_{\mathbf{k}}) = \Lambda_{\mathbf{k}}$. This proves Theorem 2.1 **(e)**. □

Remark 2.2. Let $\mathbf{k}$, H and ζ be as in Theorem 2.1. Then, the $\mathbf{k}$-module $\operatorname{Hom}(H, \mathbf{k})$ of all $\mathbf{k}$-linear maps from H to $\mathbf{k}$ has a canonical structure of a $\mathbf{k}$-algebra; its unity is the map $\varepsilon \in \operatorname{Hom}(H, \mathbf{k})$, and its multiplication is the binary operation $\star$ defined by

$$f \star g = m_{\mathbf{k}} \circ (f \otimes g) \circ \Delta_H : H \to \mathbf{k} \qquad \text{for every } f, g \in \operatorname{Hom}(H, \mathbf{k})$$

(where $m_{\mathbf{k}}$ is the canonical isomorphism $\mathbf{k} \otimes \mathbf{k} \to \mathbf{k}$). This $\mathbf{k}$-algebra is called the *convolution algebra* of H and $\mathbf{k}$; it is a particular case of the construction in [GriRei14, Definition 1.4.1]. Using this convolution algebra, we can express the map ζ_α in Theorem 2.1 **(c)** as follows: For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, the map $\zeta_\alpha : H \to \mathbf{k}$ is given by

$$\zeta_\alpha = (\zeta \circ \pi_{a_1}) \star (\zeta \circ \pi_{a_2}) \star \cdots \star (\zeta \circ \pi_{a_k}).$$

(This follows from [GriRei14, Exercise 1.4.23].)

3. Extension of scalars and $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms

Various applications of Theorem 2.1 can be found in [ABS03] and [GriRei14, Chapter 7]. We are going to present another application, which we will obtain by "leveraging" Theorem 2.1 through an extension-of-scalars argument[6]. Let us first introduce some more notations.

Definition 3.1. Let H be a $\mathbf{k}$-algebra (possibly with additional structure, such as a grading or a Hopf algebra structure). Then, $\underline{H}$ will mean the $\mathbf{k}$-algebra H without any additional structure (for instance, the $\mathbf{k}$-coalgebra structure on H is forgotten if H was a $\mathbf{k}$-bialgebra, and the grading is forgotten if H was graded). Sometimes we will use the notation $\underline{H}$ even when H has no additional structure beyond being a H-algebra; in this case, it means the same as H, just stressing the fact that it is a plain $\mathbf{k}$-algebra with nothing up its sleeves.

In other words, $\underline{H}$ will denote the image of H under the forgetful functor from whatever category H belongs to to the category of $\mathbf{k}$-algebras. We shall often use $\underline{H}$ and H interchangeably, whenever H is merely a $\mathbf{k}$-algebra or the other structures on H cannot cause confusion.

Definition 3.2. Let A be a commutative $\mathbf{k}$-algebra.

(a) If H is a $\mathbf{k}$-module, then $\underline{A} \otimes H$ will be understood to mean the A-module $A \otimes H$, on which A acts by the rule

$$a\,(b \otimes h) = ab \otimes h \qquad \text{for all } a \in A,\ b \in A \text{ and } h \in H.$$

This A-module $\underline{A} \otimes H$ is called the $\mathbf{k}$-*module H with scalars extended to $\underline{A}$*.

We can define a functor $\operatorname{Mod}_{\mathbf{k}} \to \operatorname{Mod}_A$ (where Mod_B denotes the category of B-modules) which sends every object $H \in \operatorname{Mod}_{\mathbf{k}}$ to $\underline{A} \otimes H$ and every morphism $f \in \operatorname{Mod}_{\mathbf{k}}(H_1, H_2)$ to $\operatorname{id} \otimes f \in \operatorname{Mod}_A(\underline{A} \otimes H_1, \underline{A} \otimes H_2)$; this functor is called *extension of scalars* (from $\mathbf{k}$ to A).

(b) If H is a graded $\mathbf{k}$-module, then the A-module $\underline{A} \otimes H$ canonically becomes a graded $\underline{A}$-module (namely, its n-th graded component is $\underline{A} \otimes H_n$, where H_n is the n-th graded component of H). Notice that even if A is graded, we disregard its grading when defining the grading on $\underline{A} \otimes H$; this is why we are calling it $\underline{A} \otimes H$ and not $A \otimes H$.

As before, we can define a functor from the category of graded $\mathbf{k}$-modules to the category of graded A-modules (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars*.

(c) If H is a $\mathbf{k}$-algebra, then the A-module $\underline{A} \otimes H$ becomes an A-algebra according to the rule

$$(a \otimes h)\,(b \otimes g) = ab \otimes hg \qquad \text{for all } a \in A,\ b \in A,\ h \in H \text{ and } g \in H.$$

[6] I have learned this extension-of-scalars trick from Petracci's [Petra02, proof of Lemma 2.1.1]; similar ideas appear in various other algebraic arguments.

(This is, of course, the same rule as used in the standard definition of the tensor product $A \otimes H$; but notice that we are regarding $\underline{A} \otimes H$ as an A-algebra, not just as a $\mathbf{k}$-algebra.) This A-algebra $\underline{A} \otimes H$ is called the *$\mathbf{k}$-algebra H with scalars extended to $\underline{A}$*.

As before, we can define a functor from the category of $\mathbf{k}$-algebras to the category of A-algebras (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars*.

(d) If H is a $\mathbf{k}$-coalgebra, then the A-module $\underline{A} \otimes H$ becomes an A-coalgebra. Namely, its comultiplication is defined to be

$$\mathrm{id}_A \otimes \Delta_H : A \otimes H \to A \otimes (H \otimes H) \cong (A \otimes H) \otimes_A (A \otimes H),$$

and its counit is defined to be

$$\mathrm{id}_A \otimes \varepsilon_H : A \otimes H \to A \otimes \mathbf{k} \cong A$$

(recalling that Δ_H and ε_H are the comultiplication and the counit of H, respectively). Note that both the comultiplication and the counit of $\underline{A} \otimes H$ are A-linear, so that this A-coalgebra $\underline{A} \otimes H$ is well-defined. This A-coalgebra $\underline{A} \otimes H$ is called the *$\mathbf{k}$-coalgebra H with scalars extended to $\underline{A}$*.

As before, we can define a functor from the category of $\mathbf{k}$-coalgebras to the category of A-coalgebras (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars*.

Notice that $\underline{A} \otimes H$ is an A-coalgebra, not a $\mathbf{k}$-coalgebra. If A has a preexisting $\mathbf{k}$-coalgebra structure, then the A-coalgebra structure on $\underline{A} \otimes H$ usually has nothing to do with the $\mathbf{k}$-coalgebra structure on $A \otimes H$ obtained by tensoring the $\mathbf{k}$-coalgebras A and H.

(e) If H is a $\mathbf{k}$-bialgebra, then the A-module $\underline{A} \otimes H$ becomes an A-bialgebra. (Namely, the A-algebra structure and the A-coalgebra structure previously defined on $\underline{A} \otimes H$, combined, form an A-bialgebra structure.) This A-bialgebra $\underline{A} \otimes H$ is called the *$\mathbf{k}$-bialgebra H with scalars extended to $\underline{A}$*.

As before, we can define a functor from the category of $\mathbf{k}$-bialgebras to the category of A-bialgebras (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars*.

(f) Similarly, extension of scalars is defined for $\mathbf{k}$-Hopf algebras, graded $\mathbf{k}$-bialgebras, etc.. Again, all structures on A that go beyond the $\mathbf{k}$-algebra structure are irrelevant and can be forgotten.

Definition 3.3. Let A be a commutative $\mathbf{k}$-algebra.

(a) Let H be a $\mathbf{k}$-module, and let G be an A-module. For any $\mathbf{k}$-linear map $f : H \to G$, we let $f^\sharp$ denote the A-linear map

$$\underline{A} \otimes H \to G, \qquad a \otimes h \mapsto af(h).$$

(It is easy to see that this latter map is indeed well-defined and A-linear.) For any A-linear map $g : \underline{A} \otimes H \to G$, we let $g^\flat$ denote the $\mathbf{k}$-linear map

$$H \to G, \qquad h \mapsto g(1 \otimes h).$$

Sometimes we will use the notations $f^{\sharp(A,\mathbf{k})}$ and $g^{\flat(A,\mathbf{k})}$ instead of $f^\sharp$ and $g^\flat$ when the A and the $\mathbf{k}$ are not clear from the context.

It is easy to see that $\left(f^\sharp\right)^\flat = f$ for any $\mathbf{k}$-linear map $f : H \to G$, and that $\left(g^\flat\right)^\sharp = g$ for any A-linear map $g : \underline{A} \otimes H \to G$. Thus, the maps

$$\{\mathbf{k}\text{-linear maps } H \to G\} \to \{A\text{-linear maps } \underline{A} \otimes H \to G\}, \\ f \mapsto f^\sharp \tag{4}$$

and

$$\{A\text{-linear maps } \underline{A} \otimes H \to G\} \to \{\mathbf{k}\text{-linear maps } H \to G\}, \\ g \mapsto g^\flat \tag{5}$$

are mutually inverse.

This is a particular case of an adjunction between functors (namely, the Hom-tensor adjunction, with a slight simplification, also known as the induction-restriction adjunction); this is also the reason why we are using the $\sharp$ and $\flat$ notations. The maps (4) and (5) are natural in H and G.

(b) Let H be a $\mathbf{k}$-coalgebra, and let G be an A-coalgebra. A $\mathbf{k}$-linear map $f : H \to G$ is said to be a $(\mathbf{k}, \underline{A})$-*coalgebra homomorphism* if the A-linear map $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism.

Proposition 3.4. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-algebra. Let G be an A-algebra. Let $f : H \to G$ be a $\mathbf{k}$-linear map. Then, f is a $\mathbf{k}$-algebra homomorphism if and only if $f^\sharp$ is an A-algebra homomorphism.

Proof of Proposition 3.4. Straightforward and left to the reader. (The main step is to observe that $f^\sharp$ is an A-algebra homomorphism if and only if it satisfies the following two conditions:

1. We have $f^\sharp(1 \otimes 1) = 1$.
2. Every $a, b \in A$ and $h, g \in H$ satisfy $f^\sharp((a \otimes h)(b \otimes g)) = f^\sharp(a \otimes h) f^\sharp(b \otimes g)$.

This is because the tensor product $\underline{A} \otimes H$ is spanned by pure tensors.) □

Proposition 3.5. Let A be a commutative $\mathbf{k}$-algebra. Let H be a graded $\mathbf{k}$-module. Let G be an A-module. Let $f : H \to G$ be a $\mathbf{k}$-linear map. Then, the $\mathbf{k}$-linear map f is graded if and only if the $\mathbf{k}$-linear map $f^\sharp$ is graded.

Proof of Proposition 3.5. Again, straightforward and therefore omitted. □

Let us first prove some easily-checked properties of $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms.

Proposition 3.6. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-coalgebra. Let G and I be two A-coalgebras. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $g : G \to I$ be an A-coalgebra homomorphism. Then, $g \circ f$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism.

Proof of Proposition 3.6. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Now, straightforward elementwise computation (using the fact that the map f is $\mathbf{k}$-linear, and the map g is A-linear) shows that

$$(g \circ f)^\sharp = g \circ f^\sharp. \tag{6}$$

Thus, $(g \circ f)^\sharp$ is an A-coalgebra homomorphism (since g and $f^\sharp$ are A-coalgebra homomorphisms). In other words, $g \circ f$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. This proves Proposition 3.6. □

Proposition 3.7. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let F and H be two $\mathbf{k}$-coalgebras. Let G be an A-coalgebra. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $e : F \to H$ be a $\mathbf{k}$-coalgebra homomorphism. Then, $f \circ e$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism.

Proof of Proposition 3.7. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. The map $\mathrm{id}_A \otimes e : \underline{A} \otimes F \to \underline{A} \otimes H$ is an A-coalgebra homomorphism (since $e : F \to H$ is a $\mathbf{k}$-coalgebra homomorphism). Now, straightforward computation shows that $(f \circ e)^\sharp = f^\sharp \circ (\mathrm{id}_A \otimes e)$. Hence, $(f \circ e)^\sharp$ is an A-coalgebra homomorphism (since $f^\sharp$ and $\mathrm{id}_A \otimes e$ are A-coalgebra homomorphisms). In other words, $f \circ e$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. This proves Proposition 3.7. □

Proposition 3.8. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-coalgebra. Let G be an A-coalgebra. Let B be a commutative A-algebra. Let $p : A \to B$ be an A-algebra homomorphism. (Actually, p is uniquely determined by the A-algebra structure on B.) Let $p_G : G \to B \otimes_A G$ be the canonical A-module homomorphism defined as the composition

$$G \xrightarrow{\cong} A \otimes_A G \xrightarrow{p \otimes_A \mathrm{id}} B \otimes_A G.$$

Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Then, $p_G \circ f : H \to \underline{B} \otimes_A G$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

Proof of Proposition 3.8. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp = f^{\sharp(A,\mathbf{k})} : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Thus, the map $\mathrm{id}_B \otimes_A f^\sharp : \underline{B} \otimes_A (\underline{A} \otimes H) \to \underline{B} \otimes_A G$ is a B-coalgebra homomorphism.

Let $\kappa : \underline{B} \otimes H \to \underline{B} \otimes_A (\underline{A} \otimes H)$ be the canonical B-module isomorphism (sending each $b \otimes h \in \underline{B} \otimes H$ to $b \otimes_A (1 \otimes h)$). It is well-known that κ is a B-coalgebra isomorphism[7]. Thus, $\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ is a B-coalgebra homomorphism (since both $\mathrm{id}_B \otimes_A f^\sharp$ and κ are B-coalgebra homomorphisms).

The definition of p_G yields that

$$p_G(u) = 1 \otimes_A u \tag{7}$$

for every $u \in G$.

The map $p_G \circ f : H \to \underline{B} \otimes_A G$ gives rise to a map $(p_G \circ f)^{\sharp(B,\mathbf{k})} : \underline{B} \otimes H \to \underline{B} \otimes_A G$. But easy computations show that $(p_G \circ f)^{\sharp(B,\mathbf{k})} = \left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ (because the map $(p_G \circ f)^{\sharp(B,\mathbf{k})}$ sends a pure tensor $b \otimes h \in \underline{B} \otimes H$ to $b \underbrace{(p_G \circ f)(h)}_{\substack{=p_G(f(h)) \\ =1 \otimes_A f(h) \\ \text{(by (7))}}} =$ $b(1 \otimes_A f(h)) = b \otimes_A f(h)$, whereas the map $\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ sends a pure tensor $b \otimes h \in \underline{B} \otimes H$ to

$$\left(\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa\right)(b \otimes h) = \left(\mathrm{id}_B \otimes_A f^\sharp\right)\left(\underbrace{\kappa(b \otimes h)}_{=b \otimes_A (1 \otimes h)}\right) = \left(\mathrm{id}_B \otimes_A f^\sharp\right)(b \otimes_A (1 \otimes h))$$
$$= b \otimes_A \underbrace{f^\sharp(1 \otimes h)}_{=1 f(h) = f(h)} = b \otimes_A f(h)$$

as well). Thus, $(p_G \circ f)^{\sharp(B,\mathbf{k})}$ is a B-coalgebra homomorphism (since $\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ is a B-coalgebra homomorphism). In other words, $p_G \circ f$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism. This proves Proposition 3.8. $\square$

Proposition 3.9. Let $\mathbf{k}$ be a commutative ring. Let A and B be two commutative $\mathbf{k}$-algebras. Let H and G be two $\mathbf{k}$-coalgebras. Let $f : H \to \underline{A} \otimes G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $p : A \to B$ be a $\mathbf{k}$-algebra homomorphism. Then, $(p \otimes \mathrm{id}) \circ f : H \to \underline{B} \otimes G$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

Proof of Proposition 3.9. Consider B as an A-algebra by means of the $\mathbf{k}$-algebra homomorphism $p : A \to B$. Thus, p becomes an A-algebra homomorphism $A \to B$. Now, $\underline{A} \otimes G$ is an A-coalgebra. Let $p_{\underline{A} \otimes G} : \underline{A} \otimes G \to B \otimes_A (\underline{A} \otimes G)$ be the canonical A-module homomorphism defined as the composition

$$\underline{A} \otimes G \xrightarrow{\cong} A \otimes_A (\underline{A} \otimes G) \xrightarrow{p \otimes_A \mathrm{id}} B \otimes_A (\underline{A} \otimes G).$$

[7] In fact, it is part of the natural isomorphism $\mathrm{Ind}_A^B \circ \mathrm{Ind}_{\mathbf{k}}^A \cong \mathrm{Ind}_{\mathbf{k}}^B$, where Ind_P^Q means extension of scalars from P to Q (as a functor from the category of P-coalgebras to the category of Q-coalgebras).

Proposition 3.8 (applied to $\underline{A} \otimes G$ and $p_{\underline{A} \otimes G}$ instead of G and p_G) shows that $p_{\underline{A} \otimes G} \circ f : H \to \underline{B} \otimes_A (\underline{A} \otimes G)$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

Now, let ϕ be the canonical B-module isomorphism

$$\underline{B} \otimes_A (\underline{A} \otimes G) \to \underbrace{(\underline{B} \otimes_A \underline{A})}_{\cong \underline{B}} \otimes G \to \underline{B} \otimes G.$$

Then, ϕ is a B-coalgebra homomorphism, and has the property that $p \otimes \mathrm{id} = \phi \circ p_{\underline{A} \otimes G}$ as maps $\underline{A} \otimes G \to \underline{B} \otimes G$ (this can be checked by direct computation). Now,

$$\underbrace{(p \otimes \mathrm{id})}_{= \phi \circ p_{\underline{A} \otimes G}} \circ f = \phi \circ p_{\underline{A} \otimes G} \circ f = \phi \circ \left(p_{\underline{A} \otimes G} \circ f \right)$$

must be a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism (by Proposition 3.6, since $p_{\underline{A} \otimes G} \circ f$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism and since ϕ is a B-coalgebra homomorphism). This proves Proposition 3.9. $\square$

Proposition 3.10. Let $\mathbf{k}$ be a commutative ring. Let A and B be two commutative $\mathbf{k}$-algebras. Let H be a $\mathbf{k}$-coalgebra. Let G be an A-coalgebra. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Then, $\mathrm{id} \otimes f : \underline{B} \otimes H \to \underline{B} \otimes G$ is a $(\underline{B}, \underline{B} \otimes \underline{A})$-coalgebra homomorphism.

Proof of Proposition 3.10. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp = f^{\sharp(A, \mathbf{k})} : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Thus, the map $\mathrm{id}_B \otimes f^\sharp : \underline{B} \otimes (\underline{A} \otimes H) \to \underline{B} \otimes G$ is a B-coalgebra homomorphism.

But the $\underline{B}$-linear map $\mathrm{id} \otimes f : \underline{B} \otimes H \to \underline{B} \otimes G$ gives rise to a $\underline{B} \otimes \underline{A}$-linear map $(\mathrm{id} \otimes f)^{\sharp(\underline{B} \otimes \underline{A}, \underline{B})} : (\underline{B} \otimes \underline{A}) \otimes_B (\underline{B} \otimes H) \to \underline{B} \otimes G$.

Now, let γ be the canonical B-module isomorphism $(\underline{B} \otimes \underline{A}) \otimes_B (\underline{B} \otimes H) \to \underline{B} \otimes (\underline{A} \otimes H)$ (sending each $(b \otimes a) \otimes_B (b' \otimes h) \in (\underline{B} \otimes \underline{A}) \otimes_B (\underline{B} \otimes H)$ to $bb' \otimes (a \otimes h)$). Then, γ is a B-coalgebra isomorphism (this is easy to check). Hence, $\left(\mathrm{id}_B \otimes f^\sharp\right) \circ \gamma$ is a B-coalgebra isomorphism (since $\mathrm{id}_B \otimes f^\sharp$ and γ are B-coalgebra isomorphisms).

Now, it is straightforward to see that $(\mathrm{id} \otimes f)^{\sharp(\underline{B} \otimes \underline{A}, \underline{B})} = \left(\mathrm{id}_B \otimes f^\sharp\right) \circ \gamma$ [8]. Hence, the map $(\mathrm{id} \otimes f)^{\sharp(\underline{B} \otimes \underline{A}, \underline{B})}$ is a B-coalgebra homomorphism (since $\left(\mathrm{id}_B \otimes f^\sharp\right) \circ \gamma$ is a B-coalgebra homomorphism). In other words, $\mathrm{id} \otimes f : \underline{B} \otimes H \to \underline{B} \otimes G$ is a $(\underline{B}, \underline{B} \otimes \underline{A})$-coalgebra homomorphism. This proves Proposition 3.10. $\square$

[8] Indeed, it suffices to check it on pure tensors, i.e., to prove that

$$(\mathrm{id} \otimes f)^{\sharp(\underline{B} \otimes \underline{A}, \underline{B})} \left((b \otimes a) \otimes_B (b' \otimes h) \right) = \left(\left(\mathrm{id}_B \otimes f^\sharp \right) \circ \gamma \right) \left((b \otimes a) \otimes_B (b' \otimes h) \right)$$

for each $b \in B$, $a \in A$, $b' \in B$ and $h \in H$. But this is easy (both sides turn out to be $bb' \otimes_B af(h)$).

Proposition 3.11. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let B be a commutative A-algebra. Let H be a $\mathbf{k}$-coalgebra. Let G be an A-coalgebra. Let I be a B-coalgebra. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $g : G \to I$ be an $(\underline{A}, \underline{B})$-coalgebra homomorphism. Then, $g \circ f : H \to I$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

Proof of Proposition 3.11. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^{\sharp(A,\mathbf{k})} : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Thus, the map $\mathrm{id}_B \otimes_A f^{\sharp(A,\mathbf{k})} : \underline{B} \otimes_A (\underline{A} \otimes H) \to \underline{B} \otimes_A G$ is a B-coalgebra homomorphism.

Since $g : G \to I$ is an $(\underline{A}, \underline{B})$-coalgebra homomorphism, the map $g^{\sharp(\underline{B},\underline{A})} : \underline{B} \otimes_A G \to I$ is a B-coalgebra homomorphism.

Let $\delta : \underline{B} \otimes H \to \underline{B} \otimes_A (\underline{A} \otimes H)$ be the canonical B-module isomorphism (sending each $b \otimes h$ to $b \otimes_A (1 \otimes h)$). Then, δ is a B-coalgebra isomorphism. Straightforward elementwise computation shows that $(g \circ f)^{\sharp(\underline{B},\mathbf{k})} = g^{\sharp(\underline{B},\underline{A})} \circ \left(\mathrm{id}_B \otimes_A f^{\sharp(A,\mathbf{k})}\right) \circ \delta$. Hence, $(g \circ f)^{\sharp(\underline{B},\mathbf{k})}$ is a B-coalgebra homomorphism (since $g^{\sharp(\underline{B},\underline{A})}$, $\mathrm{id}_B \otimes_A f^{\sharp(A,\mathbf{k})}$ and δ are B-coalgebra homomorphisms). In other words, $g \circ f : H \to I$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism. This proves Proposition 3.11. $\square$

With these basics in place, we can now "escalate" Theorem 2.1 to the following setting:

Corollary 3.12. Let $\mathbf{k}$ be a commutative ring. Let H be a connected graded $\mathbf{k}$-Hopf algebra. Let A be a commutative $\mathbf{k}$-algebra. Let $\xi : H \to \underline{A}$ be a $\mathbf{k}$-algebra homomorphism.

(a) Then, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{tikzcd} H \arrow[rr, "\Xi"] \arrow[dr, "\xi"'] & & \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[dl, "\mathrm{id}_A \otimes \varepsilon_P"] \\ & \underline{A} & \end{tikzcd} \tag{8}$$

is commutative (where we regard $\mathrm{id}_A \otimes \varepsilon_P : \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{A} \otimes \mathbf{k}$ as a map from $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ to $\underline{A}$, by canonically identifying $\underline{A} \otimes \mathbf{k}$ with $\underline{A}$).

(b) This unique $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-algebra homomorphism.

(c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to A$ (not to $\mathbf{k}$!) as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{\xi^{\otimes k}} A^{\otimes k} \xrightarrow{m^{(k-1)}} A .$$

(Recall that $\Delta^{(k-1)} : H \to H^{\otimes k}$ and $m^{(k-1)} : A^{\otimes k} \to A$ are the "iterated comultiplication and multiplication maps"; see [GriRei14, §1.4] for their definitions. The map $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ is the one defined in Definition 1.3.)

Then, the unique $(\mathbf{k}, \underline{A})$-coalgebra homomorphism Ξ of Corollary 3.12 **(a)** is given by

$$\Xi(h) = \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H$$

(in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).

(d) If the $\mathbf{k}$-coalgebra H is cocommutative, then $\Xi(H)$ is a subset of the subring $\underline{A} \otimes \Lambda_{\mathbf{k}}$ of $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$, where $\Lambda_{\mathbf{k}}$ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$.

Proof of Corollary 3.12. We have $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \cong \mathrm{QSym}_{\underline{A}}$ as A-bialgebras canonically (since $\mathrm{QSym}_{\mathbf{k}}$ is defined functorially in $\mathbf{k}$, with a basis that is independent of $\mathbf{k}$).

Recall that we have defined a $\mathbf{k}$-algebra homomorphism $\varepsilon_P : \mathrm{QSym}_{\mathbf{k}} \to \mathbf{k}$. We shall now denote this ε_P by $\varepsilon_{P,\mathbf{k}}$ in order to stress that it depends on $\mathbf{k}$. Similarly, an $\mathbf{m}$-algebra homomorphism $\varepsilon_{P,\mathbf{m}} : \mathrm{QSym}_{\mathbf{m}} \to \mathbf{m}$ is defined for any commutative ring $\mathbf{m}$. In particular, an $\underline{A}$-algebra homomorphism $\varepsilon_{P,\underline{A}} : \mathrm{QSym}_{\underline{A}} \to \underline{A}$ is defined. The definitions of $\varepsilon_{P,\mathbf{m}}$ for all $\mathbf{m}$ are essentially identical; thus, the map $\varepsilon_{P,\underline{A}} : \mathrm{QSym}_{\underline{A}} \to \underline{A}$ can be identified with the map $\mathrm{id}_A \otimes \varepsilon_{P,\mathbf{k}} : \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{A} \otimes \mathbf{k}$ (if we identify $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ with $\mathrm{QSym}_{\underline{A}}$ and identify $\underline{A} \otimes \mathbf{k}$ with $\underline{A}$). We shall use this identification below.

The $\mathbf{k}$-linear map $\zeta : H \to \underline{A}$ induces an A-linear map $\zeta^\sharp : \underline{A} \otimes H \to \underline{A}$ (defined by $\zeta^\sharp(a \otimes h) = a\zeta(h)$ for all $a \in A$ and $h \in H$). Proposition 3.4 (applied to $G = \underline{A}$ and $f = \zeta$) shows that $\zeta^\sharp$ is an A-algebra homomorphism (since ζ is a $\mathbf{k}$-algebra homomorphism).

Theorem 2.1 **(a)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\zeta^\sharp$ instead of $\mathbf{k}$, H and ζ) shows that there exists a unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \mathrm{QSym}_{\underline{A}}$ for which the diagram

$$\begin{tikzcd} \underline{A} \otimes H \arrow[rr, "\Psi"] \arrow[dr, "\zeta^\sharp"'] & & \mathrm{QSym}_{\underline{A}} \arrow[dl, "\varepsilon_{P,\underline{A}}"] \\ & \underline{A} & \end{tikzcd} \tag{9}$$

is commutative. Since we are identifying the map $\varepsilon_{P,\underline{A}} : \mathrm{QSym}_{\underline{A}} \to \underline{A}$ with the map $\mathrm{id}_A \otimes \varepsilon_{P,\mathbf{k}} : \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{A} \otimes \mathbf{k} = \underline{A}$, we can rewrite this as follows: There exists a unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{tikzcd} \underline{A} \otimes H \arrow[rr, "\Psi"] \arrow[dr, "\zeta^\sharp"'] & & \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[dl, "\mathrm{id}_A \otimes \varepsilon_{P,\mathbf{k}}"] \\ & \underline{A} & \end{tikzcd}$$

is commutative. In other words, there exists a unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $(\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Psi = \xi^{\sharp}$. Let us refer to this observation as the *intermediate universal property*.

The $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms $H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ are in a 1-to-1 correspondence with the A-coalgebra homomorphisms $\underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$, which is the same as the A-coalgebra homomorphisms $\underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$ (since $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}} \cong \operatorname{QSym}_{\underline{A}}$). The correspondence is given by sending a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ to the A-coalgebra homomorphism $\Xi^{\sharp} : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$. Moreover, this correspondence has the property that Ξ is graded if and only if $\Xi^{\sharp}$ is (according to Proposition 3.5). Thus, this correspondence restricts to a correspondence between the graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms $H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ and the graded A-coalgebra homomorphisms $\underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$. Using this correspondence, we can rewrite the intermediate universal property as follows: There exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $(\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi^{\sharp} = \xi^{\sharp}$. In other words, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $((\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi)^{\sharp} = \xi^{\sharp}$ (since (6) shows that $((\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi)^{\sharp} = (\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi^{\sharp}$). In other words, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $(\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi = \xi$ (since the map (4) is a bijection). In other words, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ for which the diagram (8) is commutative. This proves Corollary 3.12 **(a)**.

By tracing back the above argument, we see that it yields an explicit construction of the unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ for which the diagram (8) is commutative: Namely, it is defined by $\Xi^{\sharp} = \Psi$, where Ψ is the unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$ for which the diagram (9) is commutative. Consider these Ξ and Ψ.

Theorem 2.1 **(b)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^{\sharp}$ instead of $\mathbf{k}$, H and ζ) shows that $\Psi : \underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$ is an $\underline{A}$-Hopf algebra homomorphism, thus an $\underline{A}$-algebra homomorphism. In other words, $\Xi^{\sharp} : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ is an $\underline{A}$-algebra homomorphism (since $\Xi^{\sharp} : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ is the same as $\Psi : \underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$, up to our identifications). Hence, $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-algebra homomorphism as well (by Proposition 3.4, applied to $\underline{A}$, $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ and Ξ instead of A, G and f). This proves Corollary 3.12 **(b)**.

(c) Theorem 2.1 **(d)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^{\sharp}$ instead of $\mathbf{k}$, H and ζ) shows that Ψ is given by

$$\Psi(h) = \sum_{\alpha \in \operatorname{Comp}} \left(\xi^{\sharp}\right)_{\alpha}(h) \cdot M_{\alpha} \qquad \text{for every } h \in \underline{A} \otimes H, \tag{10}$$

where the map $\left(\xi^{\sharp}\right)_{\alpha} : \underline{A} \otimes H \to \underline{A}$ is defined in the same way as the map $\zeta_{\alpha} : H \to \mathbf{k}$ was defined in Theorem 2.1 **(d)** (but with $\mathbf{k}$, H and ζ replaced by $\underline{A}$, $\underline{A} \otimes H$ and $\xi^{\sharp}$). Notice that (10) is an equality inside $\operatorname{QSym}_{\underline{A}}$. Recalling that we are identifying $\operatorname{QSym}_{\underline{A}}$ with $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$, we can rewrite it as an equality in $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$; it then takes the form

$$\Psi(h) = \sum_{\alpha \in \operatorname{Comp}} \left(\xi^{\sharp}\right)_{\alpha}(h) \otimes M_{\alpha} \qquad \text{for every } h \in \underline{A} \otimes H. \tag{11}$$

Let ι_H be the $\mathbf{k}$-module homomorphism

$$H \to \underline{A} \otimes H, \qquad h \mapsto 1 \otimes h.$$

Also, for every $k \in \mathbb{N}$, we let ι_k be the $\mathbf{k}$-module homomorphism

$$H^{\otimes k} \to (\underline{A} \otimes H)^{\otimes_{\underline{A}} k}, \qquad g \mapsto 1 \otimes g \in \underline{A} \otimes H^{\otimes k} \cong (\underline{A} \otimes H)^{\otimes_{\underline{A}} k}$$

(where $U^{\otimes_{\underline{A}} k}$ denotes the k-th tensor power of an $\underline{A}$-module U); this homomorphism sends every $h_1 \otimes h_2 \otimes \cdots \otimes h_k \in H^{\otimes k}$ to $(1 \otimes h_1) \otimes_{\underline{A}} (1 \otimes h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} (1 \otimes h_k)$.

On the other hand, fix some $\alpha \in \operatorname{Comp}$. Write the composition α in the form $\alpha = (a_1, a_2, \ldots, a_k)$. The diagram

$$\begin{tikzcd}
H \arrow[r, "\Delta^{(k-1)}"'] \arrow[d, "\iota_H"] \arrow[rrrr, bend left, "\xi_\alpha"] & H^{\otimes k} \arrow[r, "\pi_\alpha"] \arrow[d, "\iota_k"] & H^{\otimes k} \arrow[r, "\xi^{\otimes k}"'] \arrow[d, "\iota_k"] & A^{\otimes k} \arrow[r, "m^{(k-1)}"'] & A \arrow[d, "\mathrm{id}"] \\
\underline{A} \otimes H \arrow[r, "\Delta^{(k-1)}"] \arrow[rrrr, bend right, "\left(\xi^{\sharp}\right)_\alpha"'] & (\underline{A} \otimes H)^{\otimes_{\underline{A}} k} \arrow[r, "\pi_\alpha"] & (\underline{A} \otimes H)^{\otimes_{\underline{A}} k} \arrow[r, "\left(\xi^{\sharp}\right)^{\otimes_{\underline{A}} k}"] & \underline{A}^{\otimes_{\underline{A}} k} \arrow[r, "\cong"] & \underline{A}
\end{tikzcd}$$

is commutative[9]. Therefore, $\left(\xi^{\sharp}\right)_{\alpha} \circ \iota_H = \mathrm{id} \circ \xi_{\alpha} = \xi_{\alpha}$.

Now, forget that we fixed α. We thus have shown that

$$\left(\xi^{\sharp}\right)_{\alpha} \circ \iota_H = \xi_{\alpha} \qquad \text{for every } \alpha \in \operatorname{Comp}. \tag{12}$$

[9] *Proof.* In fact:

- Its upper pentagon is commutative (by the definition of ξ_α).
- Its lower pentagon is commutative (by the definition of $\left(\xi^{\sharp}\right)_{\alpha}$).
- Its left square is commutative (since the operation $\Delta^{(k-1)}$ on a $\mathbf{k}$-coalgebra is functorial with respect to the base ring, i.e., commutes with extension of scalars).
- Its middle square is commutative (since the operation π_α on a graded $\mathbf{k}$-module is functorial with respect to the base ring, i.e., commutes with extension of scalars).

Now, every $h \in H$ satisfies

$$\begin{aligned}
\Xi(h) &= \underbrace{\Xi^\sharp}_{=\Psi}(1 \otimes h) = \Psi(1 \otimes h) \\
&= \sum_{\alpha \in \mathrm{Comp}} \left(\xi^\sharp\right)_\alpha \underbrace{(1 \otimes h)}_{=\iota_H(h)} \otimes M_\alpha \qquad \text{(by (11), applied to } 1 \otimes h \text{ instead of } h\text{)} \\
&= \sum_{\alpha \in \mathrm{Comp}} \underbrace{\left(\xi^\sharp\right)_\alpha (\iota_H(h))}_{=\left(\left(\xi^\sharp\right)_\alpha \circ \iota_H\right)(h)} \otimes M_\alpha = \sum_{\alpha \in \mathrm{Comp}} \underbrace{\left(\left(\xi^\sharp\right)_\alpha \circ \iota_H\right)}_{\substack{=\xi_\alpha \\ \text{(by (12))}}}(h) \otimes M_\alpha \\
&= \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha .
\end{aligned}$$

This proves Corollary 3.12 **(c)**.

(d) Assume that the $\mathbf{k}$-coalgebra H is cocommutative. Then, the A-coalgebra $\underline{A} \otimes H$ is cocommutative as well.

Let us first see why $\underline{A} \otimes \Lambda_{\mathbf{k}}$ is a subring of $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$. Indeed, recall that we are using the standard A-Hopf algebra isomorphism $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\underline{A}}$

- Its right rectangle is commutative. (Indeed, every $h_1, h_2, \ldots, h_k \in H$ satisfy

$$\begin{aligned}
&\left(\mathrm{id} \circ m^{(k-1)} \circ \xi^{\otimes k}\right)(h_1 \otimes h_2 \otimes \cdots \otimes h_k) \\
&= m^{(k-1)} \left(\underbrace{\xi^{\otimes k}(h_1 \otimes h_2 \otimes \cdots \otimes h_k)}_{=\xi(h_1) \otimes \xi(h_2) \otimes \cdots \otimes \xi(h_k)} \right) = m^{(k-1)} (\xi(h_1) \otimes \xi(h_2) \otimes \cdots \otimes \xi(h_k)) \\
&= \xi(h_1)\, \xi(h_2) \cdots \xi(h_k)
\end{aligned}$$

and thus

$$\begin{aligned}
&\left(\left(\xi^\sharp\right)^{\otimes_{\underline{A}} k} \circ \iota_k \right)(h_1 \otimes h_2 \otimes \cdots \otimes h_k) \\
&= \left(\xi^\sharp\right)^{\otimes_{\underline{A}} k} \underbrace{(\iota_k(h_1 \otimes h_2 \otimes \cdots \otimes h_k))}_{=(1 \otimes h_1) \otimes_{\underline{A}} (1 \otimes h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} (1 \otimes h_k)} \\
&= \left(\xi^\sharp\right)^{\otimes_{\underline{A}} k} ((1 \otimes h_1) \otimes_{\underline{A}} (1 \otimes h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} (1 \otimes h_k)) \\
&= \xi^\sharp(1 \otimes h_1) \otimes_{\underline{A}} \xi^\sharp(1 \otimes h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \xi^\sharp(1 \otimes h_k) \\
&= \xi(h_1) \otimes_{\underline{A}} \xi(h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \xi(h_k) \qquad \left(\text{since } \xi^\sharp(1 \otimes y) = \xi(y) \text{ for every } y \in H\right) \\
&= \xi(h_1)\, \xi(h_2) \cdots \xi(h_k) \qquad \left(\text{since } \underline{A}^{\otimes_{\underline{A}} k} \cong \underline{A}\right) \\
&= \left(\mathrm{id} \circ m^{(k-1)} \circ \xi^{\otimes k}\right)(h_1 \otimes h_2 \otimes \cdots \otimes h_k).
\end{aligned}$$

Hence, $\left(\xi^\sharp\right)^{\otimes_{\underline{A}} k} \circ \iota_k = \mathrm{id} \circ m^{(k-1)} \circ \xi^{\otimes k}$. In other words, the right rectangle is commutative.)

to identify $\operatorname{QSym}_{\underline{A}}$ with $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$. Similarly, let us use the standard A-Hopf algebra isomorphism $\underline{A} \otimes \Lambda_{\mathbf{k}} \to \Lambda_{\underline{A}}$ to identify $\Lambda_{\underline{A}}$ with $\underline{A} \otimes \Lambda_{\mathbf{k}}$. Now, $\underline{A} \otimes \Lambda_{\mathbf{k}} = \Lambda_{\underline{A}} \subseteq \operatorname{QSym}_{\underline{A}} = \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$.

Theorem 2.1 **(e)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^{\sharp}$ instead of $\mathbf{k}$, H and ζ) shows that $\Psi(\underline{A} \otimes H) \subseteq \Lambda_{\underline{A}} = \underline{A} \otimes \Lambda_{\mathbf{k}}$. Since $\Psi = \Xi^{\sharp}$, this rewrites as $\Xi^{\sharp}(\underline{A} \otimes H) \subseteq \underline{A} \otimes \Lambda_{\mathbf{k}}$. But $\Xi(H) \subseteq \Xi^{\sharp}(\underline{A} \otimes H)$ (since every $h \in H$ satisfies $\Xi(h) = \Xi^{\sharp}(1 \otimes h) \in \Xi^{\sharp}(\underline{A} \otimes H)$). Hence, $\Xi(H) \subseteq \Xi^{\sharp}(\underline{A} \otimes H) \subseteq \underline{A} \otimes \Lambda_{\mathbf{k}}$. This proves Corollary 3.12 **(d)**. $\square$

Remark 3.13. Let $\mathbf{k}$, H, A and ξ be as in Corollary 3.12. Then, the $\mathbf{k}$-module $\operatorname{Hom}(H, A)$ of all $\mathbf{k}$-linear maps from H to A has a canonical structure of a $\mathbf{k}$-algebra; its unity is the map $u_A \circ \varepsilon_H \in \operatorname{Hom}(H, A)$ (where $u_A : \mathbf{k} \to A$ is the $\mathbf{k}$-linear map sending 1 to 1), and its multiplication is the binary operation $\star$ defined by

$$f \star g = m_A \circ (f \otimes g) \circ \Delta_H : H \to A \qquad \text{for every } f, g \in \operatorname{Hom}(H, A)$$

(where m_A is the $\mathbf{k}$-linear map $A \otimes A \to A$, $a \otimes b \mapsto ab$). This $\mathbf{k}$-algebra is called the *convolution algebra* of H and A; it is precisely the $\mathbf{k}$-algebra defined in [GriRei14, Definition 1.4.1]. Using this $\mathbf{k}$-algebra, we can express the map ξ_α in Theorem 2.1 **(c)** as follows: For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, the map $\xi_\alpha : H \to A$ is given by

$$\xi_\alpha = (\xi \circ \pi_{a_1}) \star (\xi \circ \pi_{a_2}) \star \cdots \star (\xi \circ \pi_{a_k}).$$

(This follows easily from [GriRei14, Exercise 1.4.23].)

4. The second comultiplication on $\operatorname{QSym}_{\mathbf{k}}$

Convention 4.1. In the following, we do **not** identify compositions with infinite sequences, as several authors do. As a consequence, the composition $(1, 3)$ does not equal the vector $(1, 3, 0)$ or the infinite sequence $(1, 3, 0, 0, 0, \ldots)$.

We now recall the definition of the *second comultiplication* (a.k.a. *internal comultiplication*) of $\operatorname{QSym}_{\mathbf{k}}$. Several definitions of this operation appear in the literature; we shall use the one in [Haz08, §11.39]:[10]

Definition 4.2. **(a)** Given a $u \times v$-matrix $A = (a_{i,j})_{1 \leq i \leq u,\ 1 \leq j \leq v} \in \mathbb{N}^{u \times v}$ (where $u, v \in \mathbb{N}$) with nonnegative entries, we define three tuples of nonnegative integers:

[10]The second comultiplication seems to be as old as $\operatorname{QSym}_{\mathbf{k}}$; it first appeared in Gessel's [Gessel84, §4] (the same article where $\operatorname{QSym}_{\mathbf{k}}$ was first defined).

- The v-tuple column $A \in \mathbb{N}^v$ is the v-tuple whose j-th entry is $\sum_{i=1}^{u} a_{i,j}$ (that is, the sum of all entries in the j-th column of A) for each j. (In other words, column A is the sum of all rows of A, regarded as vectors.)

- The u-tuple row $A \in \mathbb{N}^u$ is the u-tuple whose i-th entry is $\sum_{j=1}^{v} a_{i,j}$ (that is, the sum of all entries in the i-th row of A) for each i. (In other words, row A is the sum of all columns of A, regarded as vectors.)

- The uv-tuple read $A \in \mathbb{N}^{uv}$ is the uv-tuple whose $(v(i-1)+j)$-th entry is $a_{i,j}$ for all $i \in \{1,2,\ldots,u\}$ and $j \in \{1,2,\ldots,v\}$. In other words,

$$\begin{aligned}&\operatorname{read} A\\ &= (a_{1,1}, a_{1,2}, \ldots, a_{1,v}, a_{2,1}, a_{2,2}, \ldots, a_{2,v}, \ldots, a_{u,1}, a_{u,2}, \ldots, a_{u,v}).\end{aligned}$$

We say that the matrix A is *column-reduced* if column A is a composition (i.e., contains no zero entries). Equivalently, A is column-reduced if and only if no column of A is the 0 vector.

We say that the matrix A is *row-reduced* if row A is a composition (i.e., contains no zero entries). Equivalently, A is row-reduced if and only if no row of A is the 0 vector.

We say that the matrix A is *reduced* if A is both column-reduced and row-reduced.

(b) If $w \in \mathbb{N}^k$ is a k-tuple of nonnegative integers (for some $k \in \mathbb{N}$), then w^{red} shall mean the composition obtained from w by removing each entry that equals 0. For instance, $(3,1,0,1,0,0,2)^{\text{red}} = (3,1,1,2)$.

(c) Let $\mathbb{N}^{\bullet,\bullet}_{\text{red}}$ denote the set of all reduced matrices in $\mathbb{N}^{u\times v}$, where u and v both range over $\mathbb{N}$. In other words, we set

$$\mathbb{N}^{\bullet,\bullet}_{\text{red}} = \bigcup_{(u,v)\in\mathbb{N}^2} \left\{A \in \mathbb{N}^{u\times v} \mid A \text{ is reduced}\right\}.$$

(d) Let $\Delta_P : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}}$ be the $\mathbf{k}$-linear map defined by setting

$$\Delta_P(M_\alpha) = \sum_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\text{red}};\\ (\operatorname{read} A)^{\text{red}}=\alpha}} M_{\operatorname{row} A} \otimes M_{\operatorname{column} A} \qquad \text{for each } \alpha \in \operatorname{Comp}.$$

This map Δ_P is called the *second comultiplication* (or *internal comultiplication*) of $\operatorname{QSym}_{\mathbf{k}}$.

(e) Let τ denote the twist map $\tau_{\operatorname{QSym}_{\mathbf{k}},\operatorname{QSym}_{\mathbf{k}}} : \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}}$. Let $\Delta'_P = \tau \circ \Delta_P : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}}$.

Example 4.3. The matrix $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 5 \\ 0 & 0 & 3 & 1 \end{pmatrix} \in \mathbb{N}^{3\times 4}$ is row-reduced but not column-reduced (and thus not reduced). If we denote it by A, then $\operatorname{row} A = (3,7,4)$ and $\operatorname{column} A = (3,0,5,6)$ and $\operatorname{read} A = (1,0,2,0,2,0,0,5,0,0,3,1)$.

Proposition 4.4. The $\mathbf{k}$-algebra $\operatorname{QSym}_{\mathbf{k}}$, equipped with comultiplication Δ_P and counit ε_P, is a $\mathbf{k}$-bialgebra (albeit not a connected graded one, and not a Hopf algebra).

Proposition 4.4 is a well-known fact (appearing, for example, in [MalReu95, first paragraph of §3]), but we shall actually derive it further below using our results.

5. The (generalized) Bernstein homomorphism

Let us now define the Bernstein homomorphism of a commutative connected graded $\mathbf{k}$-Hopf algebra, generalizing [Haz08, §18.24]:

Definition 5.1. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to H$ (not to $\mathbf{k}$!) as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{m^{(k-1)}} H .$$

(Recall that $\Delta^{(k-1)} : H \to H^{\otimes k}$ and $m^{(k-1)} : H^{\otimes k} \to H$ are the "iterated comultiplication and multiplication maps"; see [GriRei14, §1.4] for their definitions. The map $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ is the one defined in Definition 1.3.) Define a map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ by

$$\beta_H(h) = \sum_{\alpha \in \operatorname{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H. \tag{13}$$

It is easy to see that this map β_H is well-defined (i.e., the sum on the right hand side of (13) has only finitely many nonzero addends[11]) and $\mathbf{k}$-linear.

[11] *Proof.* Let $h \in H$. Then, there exists some $N \in \mathbb{N}$ such that $h \in H_0 + H_1 + \cdots + H_{N-1}$ (since $h \in H = \bigoplus_{i \in \mathbb{N}} H_i$). Consider this N. Now, it is easy to see that every composition $\alpha = (a_1, a_2, \ldots, a_k)$ of size $\geq N$ satisfies $\left(\pi_\alpha \circ \Delta^{(k-1)}\right)(h) = 0$ (because $\Delta^{(k-1)}(h)$ is concentrated in the first N homogeneous components of the graded $\mathbf{k}$-module $H^{\otimes k}$, and all of these components are annihilated by π_α) and therefore $\xi_\alpha(h) = 0$. Thus, the sum on the right hand side of (13) has only finitely many nonzero addends (namely, all its addends with $|\alpha| \geq N$ are 0).

Remark 5.2. Let $\mathbf{k}$ and H be as in Definition 5.1. Then, the $\mathbf{k}$-module $\operatorname{Hom}(H, H)$ of all $\mathbf{k}$-linear maps from H to H has a canonical structure of a $\mathbf{k}$-algebra, defined as in Remark 3.13 (for $A = H$). Using this $\mathbf{k}$-algebra, we can express the map ξ_α in Theorem 2.1 **(c)** as follows: For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, the map $\xi_\alpha : H \to A$ is given by

$$\xi_\alpha = \pi_{a_1} \star \pi_{a_2} \star \cdots \star \pi_{a_k}.$$

(This follows easily from [GriRei14, Exercise 1.4.23].)

The graded $\mathbf{k}$-Hopf algebra $\operatorname{QSym}_{\mathbf{k}}$ is commutative and connected; thus, Definition 5.1 (applied to $H = \operatorname{QSym}_{\mathbf{k}}$) constructs a $\mathbf{k}$-linear map $\beta_{\operatorname{QSym}_{\mathbf{k}}} : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}}$. We shall now prove that this map is identical with the Δ'_P from Definition 4.2 **(e)**:

Proposition 5.3. We have $\beta_{\operatorname{QSym}_{\mathbf{k}}} = \Delta'_P$.

Before we prove this, let us recall a basic formula for multiplication of monomial quasisymmetric functions:

Proposition 5.4. Let $k \in \mathbb{N}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be k compositions. Let $\mathbb{N}^{k,\bullet}_{\text{Cred}}$ denote the set of all column-reduced matrices in $\mathbb{N}^{k \times v}$ with v ranging over $\mathbb{N}$. In other words, let

$$\mathbb{N}^{k,\bullet}_{\text{Cred}} = \bigcup_{v \in \mathbb{N}} \left\{ A \in \mathbb{N}^{k \times v} \mid A \text{ is column-reduced} \right\}.$$

Then,

$$M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} = \sum_{\substack{A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} M_{\text{column } A}.$$

Here, $A_{i,\bullet}$ denotes the i-th row of A (regarded as a list of nonnegative integers).

Notice that the $k = 2$ case of Proposition 5.4 is a restatement of the standard formula for the multiplication of monomial quasisymmetric functions (e.g., [GriRei14, Proposition 5.1.3][12] or [Haz08, §11.26]). The general case is still classical, but since an explicit proof is hard to locate in the literature, let me sketch it here.

Proof of Proposition 5.4. We begin by introducing notations:

[12] Actually, [GriRei14, Proposition 5.1.3] is slightly more general (the $k = 2$ case of Proposition 5.4 is obtained from [GriRei14, Proposition 5.1.3] by setting $I = \{1, 2, 3, \ldots\}$). That said, our proof can easily be extended to work in this greater generality.

- Let $\mathbb{N}^{k,\infty}$ denote the set of all matrices with k rows (labelled $1,2,\ldots,k$) and countably many columns (labelled $1,2,3,\ldots$) whose entries all belong to $\mathbb{N}$.

- Let $\mathbb{N}^{k,\infty}_{\text{fin}}$ denote the set of all matrices in $\mathbb{N}^{k,\infty}$ which have only finitely many nonzero entries.

- Let $\mathbb{N}^{\infty}$ denote the set of all infinite sequences $(a_1, a_2, a_3, \ldots)$ of elements of $\mathbb{N}$.

- Let $\mathbb{N}^{\infty}_{\text{fin}}$ denote the set of all sequences in $\mathbb{N}^{\infty}$ which have only finitely many nonzero entries.

- For every $B \in \mathbb{N}^{k,\infty}_{\text{fin}}$ and $i \in \{1,2,\ldots,k\}$, we let $B_{i,\bullet} \in \mathbb{N}^{\infty}_{\text{fin}}$ be the i-th row of B.

- For every $B = (b_{i,j})_{1\leq i\leq k,\ 1\leq j} \in \mathbb{N}^{k,\infty}_{\text{fin}}$, we let $\operatorname{column} B \in \mathbb{N}^{\infty}_{\text{fin}}$ be the sequence whose j-th entry is $\sum\limits_{i=1}^{k} a_{i,j}$ (that is, the sum of all entries in the j-th column of B) for each j. (In other words, $\operatorname{column} B$ is the sum of all rows of B, regarded as vectors.)

- We extend Definition 4.2 **(b)** to the case when $w \in \mathbb{N}^{\infty}_{\text{fin}}$: If $w \in \mathbb{N}^{\infty}_{\text{fin}}$, then w^{red} shall mean the composition obtained from w by removing each entry that equals 0 [13].

- For every $\beta = (b_1, b_2, b_3, \ldots) \in \mathbb{N}^{\infty}_{\text{fin}}$, we define a monomial $\mathbf{x}^{\beta}$ in the indeterminates $x_1, x_2, x_3, \ldots$ by
$$\mathbf{x}^{\beta} = x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots .$$

Then, it is easy to see that
$$M_{\alpha} = \sum_{\substack{\beta \in \mathbb{N}^{\infty}_{\text{fin}};\\ \beta^{\text{red}} = \alpha}} \mathbf{x}^{\beta} \qquad \text{for every composition } \alpha. \tag{14}$$

[13] Here is a more rigorous definition of w^{red}: Let $w = (w_1, w_2, w_3, \ldots)$. Let $\mathcal{J}$ be the set of all positive integers j such that $w_j \neq 0$. Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements of $\mathcal{J}$, in increasing order. Then, w^{red} is defined to be the composition $(w_{j_1}, w_{j_2}, \ldots, w_{j_h})$.

This rigorous definition of w^{red} has the additional advantage of making sense in greater generality than "remove each entry that equals 0"; namely, it still works when $w \in \mathbb{N}^{I}_{\text{fin}}$ for some totally ordered set I.

Now,

$$
\begin{aligned}
& M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} \\
&= \prod_{g=1}^{k} \underbrace{M_{\alpha_g}}_{\substack{= \sum\limits_{\substack{\beta \in \mathbb{N}^{\infty}_{\mathrm{fin}}; \\ \beta^{\mathrm{red}} = \alpha_g}} \mathbf{x}^{\beta} \\ \text{(by (14))}}} = \prod_{g=1}^{k} \sum_{\substack{\beta \in \mathbb{N}^{\infty}_{\mathrm{fin}}; \\ \beta^{\mathrm{red}} = \alpha_g}} \mathbf{x}^{\beta} \\
&= \sum_{\substack{(\beta_1, \beta_2, \ldots, \beta_k) \in \left(\mathbb{N}^{\infty}_{\mathrm{fin}}\right)^k; \\ \left(\beta_g\right)^{\mathrm{red}} = \alpha_g \text{ for each } g}} \mathbf{x}^{\beta_1} \mathbf{x}^{\beta_2} \cdots \mathbf{x}^{\beta_k} \qquad \text{(by the product rule)} \\
&= \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\mathrm{fin}}; \\ \left(B_{g,\bullet}\right)^{\mathrm{red}} = \alpha_g \text{ for each } g}} \underbrace{\mathbf{x}^{B_{1,\bullet}} \mathbf{x}^{B_{2,\bullet}} \cdots \mathbf{x}^{B_{k,\bullet}}}_{\substack{= \mathbf{x}^{\operatorname{column} B} \\ \text{(since } \operatorname{column} B \text{ is the sum of the} \\ \text{rows of } B \text{ (as vectors))}}} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } \left(B_{1,\bullet}, B_{2,\bullet}, \ldots, B_{k,\bullet}\right) \text{ for} \\ \left(\beta_1, \beta_2, \ldots, \beta_k\right) \text{ in the sum, since the map} \\ \mathbb{N}^{k,\infty}_{\mathrm{fin}} \to \left(\mathbb{N}^{\infty}_{\mathrm{fin}}\right)^k,\ B \mapsto \left(B_{1,\bullet}, B_{2,\bullet}, \ldots, B_{k,\bullet}\right) \text{ is a bijection} \end{array} \right) \\
&= \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\mathrm{fin}}; \\ \left(B_{g,\bullet}\right)^{\mathrm{red}} = \alpha_g \text{ for each } g}} \mathbf{x}^{\operatorname{column} B}. \qquad (15)
\end{aligned}
$$

Now, let us introduce one more notation: For every matrix $B \in \mathbb{N}^{k,\infty}_{\mathrm{fin}}$, let B^{Cred} be the matrix obtained from B by removing all zero columns (i.e., all columns containing only zeroes)[14]. It is easy to see that $B^{\mathrm{Cred}} \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$ for every $B \in \mathbb{N}^{k,\infty}_{\mathrm{fin}}$. Moreover, every $B \in \mathbb{N}^{k,\infty}_{\mathrm{fin}}$ satisfies the following fact: If $A = B^{\mathrm{Cred}}$, then

$$\left(B_{g,\bullet}\right)^{\mathrm{red}} = \left(A_{g,\bullet}\right)^{\mathrm{red}} \text{ for each } g \qquad (16)$$

(indeed, $A_{g,\bullet}$ is obtained from $B_{g,\bullet}$ by removing some zero entries).

[14] Again, we can define B^{Cred} more rigorously as follows: Let $\mathcal{J}$ be the set of all positive integers j such that the j-th column of B is nonzero. Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements of $\mathcal{J}$, in increasing order. Then, B^{Cred} is defined to be the $k \times h$-matrix whose columns (from left to right) are the j_1-th column of B, the j_2-nd column of B, ..., the j_h-th column of B.

Now, (15) becomes

$$
\begin{aligned}
M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} &= \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \mathbf{x}^{\text{column } B} \\
&= \sum_{A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}} \underbrace{\sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g; \\ B^{\text{Cred}} = A}}}_{\substack{= \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ B^{\text{Cred}} = A; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} = \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ B^{\text{Cred}} = A; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \\ \text{(because if } B^{\text{Cred}} = A \text{, then } (B_{g,\bullet})^{\text{red}} = (A_{g,\bullet})^{\text{red}} \\ \text{for each } g \text{ (because of (16)))}}} \mathbf{x}^{\text{column } B} \\
&\qquad \left(\text{since } B^{\text{Cred}} \in \mathbb{N}^{k,\bullet}_{\text{Cred}} \text{ for each } B \in \mathbb{N}^{k,\infty}_{\text{fin}}\right) \\
&= \underbrace{\sum_{A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}} \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ B^{\text{Cred}} = A; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}}}_{= \sum_{\substack{A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ B^{\text{Cred}} = A}}} \mathbf{x}^{\text{column } B} \\
&= \sum_{\substack{A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ B^{\text{Cred}} = A}} \mathbf{x}^{\text{column } B}.
\end{aligned}
\tag{17}
$$

However, for every matrix $A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}$, we have

$$
\sum_{\substack{B \in \mathbb{N}^{k,\infty}_{\text{fin}}; \\ B^{\text{Cred}} = A}} \mathbf{x}^{\text{column } B} = M_{\text{column } A}. \tag{18}
$$

Proof of (18): Let $A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}$. We need to prove (18).

For every $B \in \mathbb{N}^{k,\infty}_{\text{fin}}$, we have $(\text{column } B)^{\text{red}} = \text{column}\left(B^{\text{Cred}}\right)$ (because first taking the sum of each column of B and then removing the zeroes among these sums results in the same list as first removing the zero columns of B and then taking the sum of each remaining column). Thus, for every $B \in \mathbb{N}^{k,\infty}_{\text{fin}}$ satisfying $B^{\text{Cred}} = A$, we have $\text{column } B \in \mathbb{N}^{\infty}_{\text{fin}}$ and $(\text{column } B)^{\text{red}} = \text{column} \underbrace{\left(B^{\text{Cred}}\right)}_{=A} =$

column A. Hence, the map

$$\left\{B \in \mathbb{N}_{\text{fin}}^{k,\infty} \mid B^{\text{Cred}} = A\right\} \to \left\{\beta \in \mathbb{N}_{\text{fin}}^{\infty} \mid \beta^{\text{red}} = \text{column}\, A\right\}, \\ B \mapsto \text{column}\, B \tag{19}$$

is well-defined.

On the other hand, if $\beta \in \mathbb{N}_{\text{fin}}^{\infty}$ satisfies $\beta^{\text{red}} = \text{column}\, A$, then there exists a unique $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$ satisfying $B^{\text{Cred}} = A$ and $\text{column}\, B = \beta$ [15]. In other words,

[15]Namely, this B can be computed as follows: Write the sequence β in the form $\beta = (\beta_1, \beta_2, \beta_3, \ldots)$. Let $(i_1 < i_2 < \cdots < i_h)$ be the list of all c satisfying $\beta_c \neq 0$, written in increasing order. Then, B shall be the matrix whose i_1-st, i_2-nd, ..., i_h-th columns are the columns of A (from left to right), whereas all its other columns are 0.

Let us briefly sketch a proof of the fact that this B is indeed an element of $\mathbb{N}_{\text{fin}}^{k,\infty}$ satisfying $B^{\text{Cred}} = A$ and $\text{column}\, B = \beta$:

Indeed, it is clear that $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$.

We shall now show that

$$(\text{the } j\text{-th entry of column}\, B) = \beta_j \tag{20}$$

for every $j \in \{1, 2, 3, \ldots\}$.

Proof of (20): Let $j \in \{1, 2, 3, \ldots\}$. We must prove (20). We are in one of the following two cases:

Case 1: We have $j \in \{i_1, i_2, \ldots, i_h\}$.

Case 2: We have $j \notin \{i_1, i_2, \ldots, i_h\}$.

Let us first consider Case 1. In this case, we have $j \in \{i_1, i_2, \ldots, i_h\}$. Hence, there exists a $g \in \{1, 2, \ldots, h\}$ such that $j = i_g$. Consider this g. Now,

$$\begin{aligned}
&(\text{the } j\text{-th entry of column}\, B) \\
&= \left(\text{the sum of the entries of the } \underbrace{j}_{=i_g}\text{-th column of } B\right) \\
&= \left(\text{the sum of the entries of } \underbrace{\text{the } i_g\text{-th column of } B}_{\substack{=(\text{the } g\text{-th column of } A) \\ (\text{by the definition of } B)}}\right) \\
&= (\text{the sum of the entries of the } g\text{-th column of } A) \\
&= \left(\text{the } g\text{-th entry of } \underbrace{\text{column}\, A}_{\substack{=\beta^{\text{red}}=\left(\beta_{i_1},\beta_{i_2},\ldots,\beta_{i_h}\right) \\ (\text{by the definition of } \beta^{\text{red}})}}\right) \\
&= \left(\text{the } g\text{-th entry of } \left(\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_h}\right)\right) = \beta_{i_g} = \beta_j \qquad (\text{since } i_g = j).
\end{aligned}$$

Thus, (20) is proven in Case 1.

Let us now consider Case 2. In this case, we have $j \notin \{i_1, i_2, \ldots, i_h\}$. Hence, j does not the map (19) is bijective. Thus, we can substitute β for $\operatorname{column} B$ in the sum $\sum\limits_{\substack{B\in\mathbb{N}_{\text{fin}}^{k,\infty};\\ B^{\text{Cred}}=A}} \mathbf{x}^{\operatorname{column} B}$, and obtain

$$\sum_{\substack{B\in\mathbb{N}_{\text{fin}}^{k,\infty};\\ B^{\text{Cred}}=A}} \mathbf{x}^{\operatorname{column} B} = \sum_{\substack{\beta\in\mathbb{N}_{\text{fin}}^{\infty};\\ \beta^{\text{red}}=\operatorname{column} A}} \mathbf{x}^{\beta} = M_{\operatorname{column} A}$$

(by (14), applied to $\alpha = \operatorname{column} A$). This proves (18).

Now, (17) becomes

$$\begin{aligned} M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} &= \sum_{\substack{A\in\mathbb{N}_{\text{Cred}}^{k,\bullet};\\ \left(A_{g,\bullet}\right)^{\text{red}}=\alpha_g \text{ for each } g}} \underbrace{\sum_{\substack{B\in\mathbb{N}_{\text{fin}}^{k,\infty};\\ B^{\text{Cred}}=A}} \mathbf{x}^{\operatorname{column} B}}_{\substack{=M_{\operatorname{column} A}\\ \text{(by (18))}}} \\ &= \sum_{\substack{A\in\mathbb{N}_{\text{Cred}}^{k,\bullet};\\ \left(A_{g,\bullet}\right)^{\text{red}}=\alpha_g \text{ for each } g}} M_{\operatorname{column} A}. \end{aligned}$$

---

belong to the list $(i_1 < i_2 < \cdots < i_h)$. In other words, j does not belong to the list of all $c \in \{1, 2, 3, \ldots\}$ satisfying $\beta_c \neq 0$ (since this list is $(i_1 < i_2 < \cdots < i_h)$). Hence, $\beta_j = 0$.

Recall that $j \notin \{i_1, i_2, \ldots, i_h\}$. Hence, the j-th column of B is the 0 vector (by the definition of B). Now,

$$\begin{aligned} &(\text{the } j\text{-th entry of } \operatorname{column} B) \\ &= \left(\text{the sum of the entries of } \underbrace{\text{the } j\text{-th column of } B}_{=(\text{the 0 vector})}\right) \\ &= (\text{the sum of the entries of the 0 vector}) \\ &= 0 = \beta_j. \end{aligned}$$

Thus, (20) is proven in Case 2.

Hence, (20) is proven in both Cases 1 and 2. Thus, the proof of (20) is complete.

Now, from (20), we immediately obtain $\operatorname{column} B = (\beta_1, \beta_2, \beta_3, \ldots) = \beta$.

It remains to prove that $B^{\text{Cred}} = A$. This can be done as follows: We have $A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}$; thus, the matrix A is column-reduced. Hence, no column of A is the zero vector. Therefore, none of the i_1-st, i_2-nd, …, i_h-th columns of B is the zero vector (since these columns are the columns of A). On the other hand, each of the remaining columns of B is the zero vector (due to the definition of B). Thus, the set of all positive integers j such that the j-th column of B is nonzero is precisely $\{i_1, i_2, \ldots, i_h\}$. The list of all elements of this set, in increasing order, is $(i_1 < i_2 < \cdots < i_h)$. Hence, the definition of B^{Cred} shows that B^{Cred} is the $k \times h$-matrix whose columns (from left to right) are the i_1-th column of B, the i_2-nd column of B, …, the i_h-th column of B. Since these columns are precisely the columns of A, this entails that B^{Cred} is the matrix A. In other words, $B^{\text{Cred}} = A$.

Thus, we have proven that B is an element of $\mathbb{N}_{\text{fin}}^{k,\infty}$ satisfying $B^{\text{Cred}} = A$ and $\operatorname{column} B = \beta$. It is fairly easy to see that it is the only such element (because the condition $\operatorname{column} B = \beta$ determines which columns of B are nonzero, whereas the condition $B^{\text{Cred}} = A$ determines the precise values of these columns).

This proves Proposition 5.4. □

We need one more piece of notation:

Definition 5.5. We define a (multiplicative) monoid structure on the set Comp as follows: If $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ are two compositions, then we set $\alpha\beta = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$. Thus, Comp becomes a monoid with neutral element $\varnothing = ()$ (the empty composition). (This monoid is actually the free monoid on the set $\{1, 2, 3, \ldots\}$.)

Proposition 5.6. Let $\gamma \in \operatorname{Comp}$ and $k \in \mathbb{N}$. Then,

$$\Delta^{(k-1)} M_\gamma = \sum_{\substack{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \operatorname{Comp}^k; \\ \gamma_1 \gamma_2 \cdots \gamma_k = \gamma}} M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k}.$$

Proof of Proposition 5.6 (sketched). We can rewrite (1) as follows:

$$\Delta M_\beta = \sum_{\substack{(\sigma, \tau) \in \operatorname{Comp} \times \operatorname{Comp}; \\ \sigma\tau = \beta}} M_\sigma \otimes M_\tau \qquad \text{for every } \beta \in \operatorname{Comp}. \tag{21}$$

Proposition 5.6 can easily be proven by induction using (21). □

Proof of Proposition 5.3. Fix $\alpha \in \operatorname{Comp}$ and $\gamma \in \operatorname{Comp}$. Write α in the form $\alpha = (a_1, a_2, \ldots, a_k)$; thus, a $\mathbf{k}$-linear map $\xi_\alpha : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}}$ is defined (as in Definition 5.1, applied to $H = \operatorname{QSym}_{\mathbf{k}}$).

We shall prove that

$$\xi_\alpha (M_\gamma) = \sum_{\substack{A \in \mathbb{N}^{\bullet, \bullet}_{\mathrm{red}}; \\ (\operatorname{read} A)^{\mathrm{red}} = \gamma; \\ \operatorname{row} A = \alpha}} M_{\operatorname{column} A}. \tag{22}$$

Proof of (22): The definition of ξ_α yields $\xi_\alpha = m^{(k-1)} \circ \pi_\alpha \circ \Delta^{(k-1)}$. Thus,

$$\begin{aligned}
\xi_\alpha \left(M_\gamma\right) &= \left(m^{(k-1)} \circ \pi_\alpha \circ \Delta^{(k-1)}\right) \left(M_\gamma\right) \\
&= \left(m^{(k-1)} \circ \pi_\alpha\right) \underbrace{\left(\Delta^{(k-1)} \left(M_\gamma\right)\right)}_{\substack{= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} M_{\gamma_1}\otimes M_{\gamma_2}\otimes\cdots\otimes M_{\gamma_k} \\ \text{(by Proposition 5.6)}}} \\
&= \left(m^{(k-1)} \circ \pi_\alpha\right) \left(\sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k} \right) \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} m^{(k-1)} \underbrace{\left(\pi_\alpha \left(M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k}\right)\right)}_{\substack{=\pi_{a_1}\left(M_{\gamma_1}\right)\otimes\pi_{a_2}\left(M_{\gamma_2}\right)\otimes\cdots\otimes\pi_{a_k}\left(M_{\gamma_k}\right) \\ \text{(since } \pi_\alpha=\pi_{a_1}\otimes\pi_{a_2}\otimes\cdots\otimes\pi_{a_k}\text{)}}} \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \underbrace{m^{(k-1)} \left(\pi_{a_1}\left(M_{\gamma_1}\right) \otimes \pi_{a_2}\left(M_{\gamma_2}\right) \otimes \cdots \otimes \pi_{a_k}\left(M_{\gamma_k}\right)\right)}_{=\pi_{a_1}\left(M_{\gamma_1}\right)\cdot\pi_{a_2}\left(M_{\gamma_2}\right)\cdot\cdots\cdot\pi_{a_k}\left(M_{\gamma_k}\right)=\prod_{g=1}^{k}\pi_{a_g}\left(M_{\gamma_g}\right)} \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \prod_{g=1}^{k} \underbrace{\pi_{a_g}\left(M_{\gamma_g}\right)}_{\substack{=\begin{cases} M_{\gamma_g}, & \text{if } \left|\gamma_g\right| = a_g; \\ 0, & \text{if } \left|\gamma_g\right| \neq a_g \end{cases} \\ \text{(since the power series } M_{\gamma_g} \text{ is} \\ \text{homogeneous of degree } \left|\gamma_g\right|\text{)}}}
\end{aligned}$$

$$= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \underbrace{\prod_{g=1}^{k} \begin{cases} M_{\gamma_g}, & \text{if } \left|\gamma_g\right| = a_g; \\ 0, & \text{if } \left|\gamma_g\right| \neq a_g \end{cases}}_{=\begin{cases} \prod_{g=1}^{k} M_{\gamma_g}, & \text{if } \left|\gamma_g\right| = a_g \text{ for all } g; \\ 0, & \text{otherwise} \end{cases}}$$

$$= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \begin{cases} \prod_{g=1}^{k} M_{\gamma_g}, & \text{if } \left|\gamma_g\right| = a_g \text{ for all } g; \\ 0, & \text{otherwise} \end{cases}$$

$$= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ \left|\gamma_g\right|=a_g \text{ for all } g}} \underbrace{\prod_{g=1}^{k} M_{\gamma_g}}_{\substack{=M_{\gamma_1}M_{\gamma_2}\cdots M_{\gamma_k} = \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ \left(A_{g,\bullet}\right)^{\mathrm{red}}=\gamma_g \text{ for each } g}} M_{\operatorname{column} A} \\ \text{(by Proposition 5.4, applied to } \alpha_i=\gamma_i\text{)}}}$$

(here, we have filtered out the zero addends)

$$
\begin{aligned}
&= \underbrace{\sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathrm{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ |\gamma_g|=a_g \text{ for all } g}} \ \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ (A_{g,\bullet})^{\mathrm{red}}=\gamma_g \text{ for each } g}}}_{=\sum_{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathrm{Comp}^k} \ \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ (A_{g,\bullet})^{\mathrm{red}}=\gamma_g \text{ for each } g;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ |\gamma_g|=a_g \text{ for all } g}}} M_{\mathrm{column}\,A} \\
&= \sum_{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathrm{Comp}^k} \underbrace{\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ (A_{g,\bullet})^{\mathrm{red}}=\gamma_g \text{ for each } g;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ |\gamma_g|=a_g \text{ for all } g}}}_{\substack{=\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ (A_{g,\bullet})^{\mathrm{red}}=\gamma_g \text{ for each } g;\\ (A_{1,\bullet})^{\mathrm{red}}(A_{2,\bullet})^{\mathrm{red}}\cdots(A_{k,\bullet})^{\mathrm{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\mathrm{red}}\right|=a_g \text{ for all } g}} \\ \text{(here, we have replaced every } \gamma_g \\ \text{in the conditions } (\gamma_1\gamma_2\cdots\gamma_k=\gamma) \\ \text{and } \left(|\gamma_g|=a_g \text{ for all } g\right) \text{ by the} \\ \text{corresponding } (A_{g,\bullet})^{\mathrm{red}}\text{, because} \\ \text{of the condition that } (A_{g,\bullet})^{\mathrm{red}}=\gamma_g)}} M_{\mathrm{column}\,A} \\
&= \underbrace{\sum_{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathrm{Comp}^k} \ \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ (A_{g,\bullet})^{\mathrm{red}}=\gamma_g \text{ for each } g;\\ (A_{1,\bullet})^{\mathrm{red}}(A_{2,\bullet})^{\mathrm{red}}\cdots(A_{k,\bullet})^{\mathrm{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\mathrm{red}}\right|=a_g \text{ for all } g}}}_{=\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ (A_{1,\bullet})^{\mathrm{red}}(A_{2,\bullet})^{\mathrm{red}}\cdots(A_{k,\bullet})^{\mathrm{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\mathrm{red}}\right|=a_g \text{ for all } g}}} M_{\mathrm{column}\,A} \\
&= \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\mathrm{Cred}};\\ (A_{1,\bullet})^{\mathrm{red}}(A_{2,\bullet})^{\mathrm{red}}\cdots(A_{k,\bullet})^{\mathrm{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\mathrm{red}}\right|=a_g \text{ for all } g}} M_{\mathrm{column}\,A}. \qquad (23)
\end{aligned}
$$

Now, we observe that every $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$ satisfies

$$
(A_{1,\bullet})^{\mathrm{red}} (A_{2,\bullet})^{\mathrm{red}} \cdots (A_{k,\bullet})^{\mathrm{red}} = (\mathrm{read}\, A)^{\mathrm{red}} \qquad (24)
$$

[16].

Also, for every every $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$, we have the logical equivalence

$$\left(\left| \left(A_{g,\bullet}\right)^{\mathrm{red}} \right| = a_g \text{ for all } g \right) \iff \left(\operatorname{row} A = \alpha\right) \tag{25}$$

[17].

Also, every $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$ satisfying $\operatorname{row} A = \alpha$ belongs to $\mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}$ [18]. Conversely, every $A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}$ satisfying $\operatorname{row} A = \alpha$ belongs to $\mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$ [19]. Combining these

---

[16] *Proof of (24):* Let $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$.

Let $\mathbb{N}^{\bullet}$ be the set of all finite lists of nonnegative integers. Then, $\mathrm{Comp} \subseteq \mathbb{N}^{\bullet}$. In Definition 5.5, we have defined a monoid structure on the set Comp. We can extend this monoid structure to the set $\mathbb{N}^{\bullet}$ (by the same rule: namely, if $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$, then $\alpha\beta = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$). (Of course, this monoid $\mathbb{N}^{\bullet}$ is just the free monoid on the set $\mathbb{N}$.) Using the latter structure, we can rewrite the definition of $\operatorname{read} A$ as follows:

$$\operatorname{read} A = A_{1,\bullet} A_{2,\bullet} \cdots A_{k,\bullet}.$$

Clearly, the map $\mathbb{N}^{\bullet} \to \mathrm{Comp}, \ \beta \mapsto \beta^{\mathrm{red}}$ is a monoid homomorphism. Thus,

$$\left(A_{1,\bullet}\right)^{\mathrm{red}} \left(A_{2,\bullet}\right)^{\mathrm{red}} \cdots \left(A_{k,\bullet}\right)^{\mathrm{red}} = \left(\underbrace{A_{1,\bullet} A_{2,\bullet} \cdots A_{k,\bullet}}_{= \operatorname{read} A} \right)^{\mathrm{red}} = \left(\operatorname{read} A\right)^{\mathrm{red}}.$$

This proves (24).

[17] *Proof of (25):* Let $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$. Then, every $g \in \{1, 2, \ldots, k\}$ satisfies

$$\begin{aligned}
\left| \left(A_{g,\bullet}\right)^{\mathrm{red}} \right| &= \left(\text{sum of all entries of } \left(A_{g,\bullet}\right)^{\mathrm{red}}\right) = \left(\text{sum of all nonzero entries in } A_{g,\bullet}\right) \\
&\qquad \left(\text{by the definition of } \left(A_{g,\bullet}\right)^{\mathrm{red}}\right) \\
&= \left(\text{sum of all nonzero entries in the } g\text{-th row of } A\right) \\
&= \left(\text{sum of all entries in the } g\text{-th row of } A\right) = \left(\text{the } g\text{-th entry of } \operatorname{row} A\right).
\end{aligned}$$

Hence, we have the following chain of equivalences:

$$\begin{aligned}
&\left(\underbrace{\left| \left(A_{g,\bullet}\right)^{\mathrm{red}} \right|}_{= (\text{the } g\text{-th entry of } \operatorname{row} A)} = a_g \text{ for all } g \right) \\
\iff &\left(\left(\text{the } g\text{-th entry of } \operatorname{row} A\right) = a_g \text{ for all } g \right) \\
\iff &\left(\operatorname{row} A = \underbrace{(a_1, a_2, \ldots, a_k)}_{= \alpha} \right) = \left(\operatorname{row} A = \alpha\right).
\end{aligned}$$

This proves (25).

[18] *Proof.* Let $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$ be such that $\operatorname{row} A = \alpha$. We must show that $A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}$. The sequence $\operatorname{row} A = \alpha$ is a composition; hence, A is row-reduced. Since A is also column-reduced (because $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$), this shows that A is reduced. Hence, $A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}$, qed.

[19] *Proof.* Let $A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}$ be such that $\operatorname{row} A = \alpha$. We must show that $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$. The number of

two observations, we see that

$$\left(\begin{array}{c}\text{the matrices } A\in\mathbb{N}^{k,\bullet}_{\text{Cred}} \text{ satisfying } \operatorname{row} A=\alpha\\ \text{are precisely the matrices } A\in\mathbb{N}^{\bullet,\bullet}_{\text{red}} \text{ satisfying } \operatorname{row} A=\alpha\end{array}\right). \tag{26}$$

Now, (23) becomes

$$\begin{aligned}\xi_\alpha\left(M_\gamma\right) &= \underbrace{\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{1,\bullet})^{\text{red}}(A_{2,\bullet})^{\text{red}}\cdots\left(A_{k,\bullet}\right)^{\text{red}}=\gamma;\\ \left|\left(A_{g,\bullet}\right)^{\text{red}}\right|=a_g \text{ for all } g}}}_{\substack{=\sum\limits_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \text{row } A=\alpha}}\\ \text{(by (24) and (25))}}} M_{\text{column } A}\\ &= \underbrace{\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \text{row } A=\alpha}}}_{\substack{=\sum\limits_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\text{red}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \text{row } A=\alpha}}\\ \text{(by (26))}}} M_{\text{column } A} = \sum_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\text{red}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \text{row } A=\alpha}} M_{\text{column } A}.\end{aligned}$$

This proves (22).

rows of A is clearly the length of the vector row A (where the "length" of a vector just means its number of entries). But this length is k (since row $A=\alpha=(a_1,a_2,\ldots,a_k)$). Therefore, the number of rows of A is k. Also, A is reduced (since $A\in\mathbb{N}^{\bullet,\bullet}_{\text{red}}$) and therefore column-reduced. Hence, $A\in\mathbb{N}^{k,\bullet}_{\text{Cred}}$ (since A is column-reduced and the number of rows of A is k), qed.

Now, forget that we fixed α and γ. For every $\gamma \in \operatorname{Comp}$, we have

$$\begin{aligned}
\beta_{\operatorname{QSym}_{\mathbf{k}}}\left(M_\gamma\right) &= \sum_{\alpha \in \operatorname{Comp}} \underbrace{\xi_\alpha\left(M_\gamma\right)}_{\substack{= \sum\limits_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma;\\ \operatorname{row} A = \alpha}} M_{\operatorname{column} A} \\ \text{(by (22))}}} \otimes M_\alpha \qquad \left(\text{by the definition of } \beta_{\operatorname{QSym}_{\mathbf{k}}}\right) \\
&= \sum_{\alpha \in \operatorname{Comp}} \sum_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma;\\ \operatorname{row} A = \alpha}} M_{\operatorname{column} A} \otimes \underbrace{M_\alpha}_{\substack{= M_{\operatorname{row} A} \\ \text{(since } \operatorname{row} A = \alpha\text{)}}} \\
&= \underbrace{\sum_{\alpha \in \operatorname{Comp}} \sum_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma;\\ \operatorname{row} A = \alpha}}}_{= \sum\limits_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma;\\ \operatorname{row} A \in \operatorname{Comp}}}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A} \\
&= \sum_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma;\\ \operatorname{row} A \in \operatorname{Comp}}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A}. \qquad (27)
\end{aligned}$$

But every $A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet}$ satisfies $\operatorname{row} A \in \operatorname{Comp}$ [20]. Hence, the summation sign $\sum\limits_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma;\\ \operatorname{row} A \in \operatorname{Comp}}}$ on the right hand side of (27) can be replaced by $\sum\limits_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma}}$. Thus, (27) becomes

$$\begin{aligned}
\beta_{\operatorname{QSym}_{\mathbf{k}}}\left(M_\gamma\right) &= \underbrace{\sum_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma;\\ \operatorname{row} A \in \operatorname{Comp}}}}_{= \sum\limits_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma}}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A} \\
&= \sum_{\substack{A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}} = \gamma}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A}. \qquad (28)
\end{aligned}$$

[20] *Proof.* Let $A \in \mathbb{N}_{\mathrm{red}}^{\bullet,\bullet}$. Then, the matrix A is reduced, and therefore row-reduced. In other words, $\operatorname{row} A$ is a composition. In other words, $\operatorname{row} A \in \operatorname{Comp}$, qed.

On the other hand, every $\gamma \in \mathrm{Comp}$ satisfies

$$\underbrace{\Delta'_P}_{=\tau\circ\Delta_P}(M_\gamma) = (\tau \circ \Delta_P)(M_\gamma) = \tau \left(\underbrace{\Delta_P(M_\gamma)}_{\substack{= \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}; \\ (\mathrm{read}\, A)^{\mathrm{red}} = \gamma}} M_{\mathrm{row}\, A} \otimes M_{\mathrm{column}\, A} \\ \text{(by the definition of } \Delta_P)}} \right)$$
$$= \tau \left(\sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}; \\ (\mathrm{read}\, A)^{\mathrm{red}} = \gamma}} M_{\mathrm{row}\, A} \otimes M_{\mathrm{column}\, A} \right)$$
$$= \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}; \\ (\mathrm{read}\, A)^{\mathrm{red}} = \gamma}} M_{\mathrm{column}\, A} \otimes M_{\mathrm{row}\, A} \qquad \text{(by the definition of } \tau)$$
$$= \beta_{\mathrm{QSym}_{\mathbf{k}}}(M_\gamma) \qquad \text{(by (28))}.$$

Since both maps Δ'_P and $\beta_{\mathrm{QSym}_{\mathbf{k}}}$ are $\mathbf{k}$-linear, this yields $\Delta'_P = \beta_{\mathrm{QSym}_{\mathbf{k}}}$ (since $(M_\gamma)_{\gamma \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module $\mathrm{QSym}_{\mathbf{k}}$). This proves Proposition 5.3. $\square$

The next theorem is an analogue for QSym of the Bernstein homomorphism ([Haz08, §18.24]) for the symmetric functions:

Theorem 5.7. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. For every composition α, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to H$ as in Definition 5.1. Define a map $\beta_H : H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ as in Definition 5.1.

(a) The map β_H is a $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ and a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism.

(b) We have $(\mathrm{id} \otimes \varepsilon_P) \circ \beta_H = \mathrm{id}$, where we regard $\mathrm{id} \otimes \varepsilon_P : \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{H} \otimes \mathbf{k}$ as a map from $\underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ to $\underline{H}$ (by identifying $\underline{H} \otimes \mathbf{k}$ with $\underline{H}$).

(c) Define a map $\Delta'_P : \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$ as in Definition 4.2 **(e)**. The diagram

$$\begin{tikzcd} H \arrow[r, "\beta_H"] \arrow[d, "\beta_H"'] & \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[d, "\beta_H \otimes \mathrm{id}"] \\ \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[r, "\mathrm{id} \otimes \Delta'_P"'] & \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}} \end{tikzcd}$$

is commutative.

(d) If the $\mathbf{k}$-coalgebra H is cocommutative, then $\beta_H(H)$ is a subset of the subring $\underline{H} \otimes \Lambda_{\mathbf{k}}$ of $\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$, where $\Lambda_{\mathbf{k}}$ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$.

Parts **(b)** and **(c)** of Theorem 5.7 can be combined into "β_H makes H into a QSym_2-comodule, where QSym_2 is the coalgebra $(\operatorname{QSym}, \Delta'_P, \varepsilon_P)$" (the fact that this QSym_2 is actually a coalgebra follows from Proposition 4.4).

What Hazewinkel actually calls the Bernstein homomorphism in [Haz08, §18.24] is the $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \Lambda_{\mathbf{k}}$ obtained from our map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ by restricting the codomain when H is both commutative and cocommutative[21]. His observation that the second comultiplication of $\Lambda_{\mathbf{k}}$ is a particular case of the Bernstein homomorphism is what gave the original motivation for the present note; its analogue for $\operatorname{QSym}_{\mathbf{k}}$ is our Proposition 5.3.

Proof of Theorem 5.7. Set $A = \underline{H}$ and $\xi = \operatorname{id}$. Then, the map ξ_α defined in Corollary 3.12 **(c)** is precisely the map ξ_α defined in Definition 5.1 (because $\xi^{\otimes k} = \operatorname{id}^{\otimes k} = \operatorname{id}$). Thus, we can afford calling both maps ξ_α without getting confused.

(a) Corollary 3.12 **(a)** shows that there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ for which the diagram (8) is commutative. Since $A = \underline{H}$ and $\xi = \operatorname{id}$, we can rewrite this as follows: There exists a unique graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism $\Xi : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{tikzcd} H \arrow[rr, "\Xi"] \arrow[dr, "\operatorname{id}"'] & & \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}} \arrow[dl, "\operatorname{id}_H \otimes \varepsilon_P"] \\ & \underline{H} & \end{tikzcd} \qquad (29)$$

is commutative. Consider this Ξ. Corollary 3.12 **(c)** shows that this homomorphism Ξ is given by

$$\Xi(h) = \sum_{\alpha \in \operatorname{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H.$$

Comparing this equality with (13), we obtain $\Xi(h) = \beta_H(h)$ for every $h \in H$. In other words, $\Xi = \beta_H$. Thus, β_H is a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism (since Ξ is a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism).

Corollary 3.12 **(b)** shows that Ξ is a $\mathbf{k}$-algebra homomorphism. In other words, β_H is a $\mathbf{k}$-algebra homomorphism (since $\Xi = \beta_H$). This completes the proof of Theorem 5.7 **(a)**.

(b) Consider the map Ξ defined in our above proof of Theorem 5.7 **(a)**. We have shown that $\Xi = \beta_H$.

[21] Hazewinkel neglects to require the cocommutativity of H in [Haz08, §18.24], but he uses it nevertheless.

The commutative diagram (29) shows that $(\mathrm{id} \otimes \varepsilon_P) \circ \Xi = \mathrm{id}$. In other words, $(\mathrm{id} \otimes \varepsilon_P) \circ \beta_H = \mathrm{id}$ (since $\Xi = \beta_H$). This proves Theorem 5.7 **(b)**.

(c) Theorem 5.7 **(a)** shows that the map β_H is a $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ and a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism. Theorem 5.7 **(a)** (applied to $\mathrm{QSym}_{\mathbf{k}}$ instead of H) shows that the map $\beta_{\mathrm{QSym}_{\mathbf{k}}}$ is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ and a graded $\left(\mathbf{k}, \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. Since $\Delta'_P = \beta_{\mathrm{QSym}_{\mathbf{k}}}$ (by Proposition 5.3), this rewrites as follows: The map Δ'_P is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ and a graded $\left(\mathbf{k}, \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism.

Applying Corollary 3.12 **(a)** to $\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}$ and β_H instead of A and ξ, we see that there exists a unique graded $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism $\Xi : H \to \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{tikzcd}
H \arrow[rr, "\Xi"] \arrow[dr, "\beta_H"'] & & \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[dl, "\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P"] \\
& \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} &
\end{tikzcd} \tag{30}$$

is commutative. Thus, if we have two graded $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphisms $\Xi : H \to \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram (30) is commutative, then these two homomorphisms must be identical. We will now show that the two homomorphisms $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ and $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ both fit the bill; this will then yield that $(\beta_H \otimes \mathrm{id}) \circ \beta_H = (\mathrm{id} \otimes \Delta'_P) \circ \beta_H$, and thus Theorem 5.7 **(c)** will follow.

Recall that β_H and Δ'_P are graded maps. Thus, so are $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ and $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$. Moreover, β_H is a $(\mathbf{k}, \underline{H})$-coalgebra homomorphism, and Δ'_P is a $\left(\mathbf{k}, \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. From this, it is easy to see that $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ and $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ are $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphisms[22].

[22]*Proof.* Proposition 3.9 (applied to H, $\underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$, H, $\mathrm{QSym}_{\mathbf{k}}$, β_H and β_H instead of A, B, H, G, f and p) shows that $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ is a $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. It remains to show that $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ is a $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism.

Recall that Δ'_P is a $\left(\mathbf{k}, \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$. Hence, Proposition 3.10 (applied to $\underline{\mathrm{QSym}_{\mathbf{k}}}$, $\underline{H}$, $\mathrm{QSym}_{\mathbf{k}}$, $\underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ and Δ'_P instead of A, B, H, G and f) shows that $\mathrm{id} \otimes \Delta'_P : \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ is an $\left(\underline{H}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. Therefore, Proposition 3.11 (applied to $\underline{H}$, $\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}$, H, $\underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$, $\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$, β_H and $\mathrm{id} \otimes \Delta'_P$ instead of A, B, H, G, I, Now, we shall show that the diagrams

$$\begin{tikzcd} H \arrow[rr, "(\beta_H \otimes \mathrm{id}) \circ \beta_H"] \arrow[dr, "\beta_H"'] & & \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[dl, "\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P"] \\ & \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} & \end{tikzcd} \tag{31}$$

and

$$\begin{tikzcd} H \arrow[rr, "\left(\mathrm{id} \otimes \Delta'_P\right) \circ \beta_H"] \arrow[dr, "\beta_H"'] & & \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[dl, "\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P"] \\ & \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} & \end{tikzcd} \tag{32}$$

are commutative. This follows from the computations

$$\underbrace{\left(\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P\right) \circ (\beta_H \otimes \mathrm{id})}_{= \beta_H \otimes \varepsilon_P = \beta_H \circ (\mathrm{id} \otimes \varepsilon_P)} \circ \beta_H = \beta_H \circ \underbrace{(\mathrm{id} \otimes \varepsilon_P) \circ \beta_H}_{\substack{= \mathrm{id} \\ \text{(by Theorem 5.7 } \textbf{(b)}\text{)}}} = \beta_H$$

and

$$\begin{aligned} &\left(\underbrace{\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}}}_{= \mathrm{id}_{\underline{H}} \otimes \mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}}} \otimes \varepsilon_P \right) \circ \left(\underbrace{\mathrm{id}}_{= \mathrm{id}_{\underline{H}}} \otimes \underbrace{\Delta'_P}_{= \beta_{\mathrm{QSym}_{\mathbf{k}}}} \right) \circ \beta_H \\ &= \underbrace{\left(\mathrm{id}_{\underline{H}} \otimes \mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P \right) \circ \left(\mathrm{id}_{\underline{H}} \otimes \beta_{\mathrm{QSym}_{\mathbf{k}}} \right)}_{= \mathrm{id}_{\underline{H}} \otimes \left(\left(\mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P \right) \circ \beta_{\mathrm{QSym}_{\mathbf{k}}} \right)} \circ \beta_H \\ &= \left(\mathrm{id}_{\underline{H}} \otimes \underbrace{\left(\left(\mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P \right) \circ \beta_{\mathrm{QSym}_{\mathbf{k}}} \right)}_{\substack{= \mathrm{id} \\ \text{(by Theorem 5.7 } \textbf{(b)}\text{,} \\ \text{applied to } \mathrm{QSym}_{\mathbf{k}} \text{ instead of } H\text{)}}} \right) \circ \beta_H \\ &= \underbrace{(\mathrm{id}_{\underline{H}} \otimes \mathrm{id})}_{= \mathrm{id}} \circ \beta_H = \beta_H. \end{aligned}$$

Thus, we know that $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ and $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ are two graded $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphisms $\Xi : H \to \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram (30) is commutative (since the diagrams (31) and (32) are commutative).

---

f and g) shows that $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ is a $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. This completes the proof.

But we have shown before that any two such homomorphisms must be identical. Thus, we conclude that $(\beta_H \otimes \mathrm{id}) \circ \beta_H = (\mathrm{id} \otimes \Delta'_P) \circ \beta_H$. This completes the proof of Theorem 5.7 **(c)**.

(d) Consider the map Ξ defined in our above proof of Theorem 5.7 **(a)**. We have shown that $\Xi = \beta_H$.

Assume that H is cocommutative. Corollary 3.12 **(d)** then shows that $\Xi(H)$ is a subset of the subring $\underline{A} \otimes \Lambda_{\mathbf{k}}$ of $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$. In other words, $\beta_H(H)$ is a subset of the subring $\underline{H} \otimes \Lambda_{\mathbf{k}}$ of $\underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ (since $\Xi = \beta_H$ and $A = H$). This proves Theorem 5.7 **(d)**.

(Alternatively, we could prove **(d)** by checking that for any element h of a commutative cocommutative Hopf algebra H, the element $\xi_\alpha(h)$ of H depends only on the result of sorting α, rather than on the composition α itself.) $\square$

Proof of Proposition 4.4. Let τ be the twist map $\tau_{\mathrm{QSym}_{\mathbf{k}},\mathrm{QSym}_{\mathbf{k}}} : \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$. This twist map clearly satisfies $\tau \circ \tau = \mathrm{id}$. Hence, $\tau \circ \underbrace{\Delta'_P}_{=\tau\circ\Delta_P} = \underbrace{\tau \circ \tau}_{=\mathrm{id}} \circ \Delta_P = \Delta_P$.

Theorem 5.7 **(c)** (applied to $H = \mathrm{QSym}_{\mathbf{k}}$) shows that the diagram

$$\begin{tikzcd}
\mathrm{QSym}_{\mathbf{k}} \arrow[r, "\beta_{\mathrm{QSym}_{\mathbf{k}}}"] \arrow[d, "\beta_{\mathrm{QSym}_{\mathbf{k}}}"'] & \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[d, "\beta_{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{id}"'] \\
\underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}} \arrow[r, "\mathrm{id} \otimes \Delta'_P"'] & \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}
\end{tikzcd}$$

is commutative. In other words, $(\mathrm{id} \otimes \Delta'_P) \circ \beta_{\mathrm{QSym}_{\mathbf{k}}} = \left(\beta_{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{id}\right) \circ \beta_{\mathrm{QSym}_{\mathbf{k}}}$. Since $\beta_{\mathrm{QSym}_{\mathbf{k}}} = \Delta'_P$ (by Proposition 5.3), this rewrites as $(\mathrm{id} \otimes \Delta'_P) \circ \Delta'_P = (\Delta'_P \otimes \mathrm{id}) \circ \Delta'_P$. Thus, the operation Δ'_P is coassociative. Therefore, the operation $\Delta_P = \tau \circ \Delta'_P$ is also coassociative (because the coassociativity of a map $H \to H \otimes H$ does not change if we compose this map with the twist map $\tau_{H,H} : H \otimes H \to H \otimes H$). It is furthermore easy to see that the operation ε_P is counital with respect to the operation Δ_P (see, for example, [Haz08, §11.45]). Hence, the $\mathbf{k}$-module $\mathrm{QSym}_{\mathbf{k}}$, equipped with the comultiplication Δ_P and the counit ε_P, is a $\mathbf{k}$-coalgebra. Our goal is to prove that it is a $\mathbf{k}$-bialgebra. Hence, it remains to show that Δ_P and ε_P are $\mathbf{k}$-algebra homomorphisms. For ε_P, this is again obvious (indeed, ε_P sends any $f \in \mathrm{QSym}_{\mathbf{k}}$ to $f(1,0,0,0,\ldots)$). It remains to prove that Δ_P is a $\mathbf{k}$-algebra homomorphism.

The map $\beta_{\mathrm{QSym}_{\mathbf{k}}}$ is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ (by Theorem 5.7 **(a)**, applied to $H = \mathrm{QSym}_{\mathbf{k}}$). In other words, the map Δ'_P is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$ (since $\beta_{\mathrm{QSym}_{\mathbf{k}}} = \Delta'_P$, and since $\underline{\mathrm{QSym}_{\mathbf{k}}} = \mathrm{QSym}_{\mathbf{k}}$ as $\mathbf{k}$-algebras). Thus, $\Delta_P = \tau \circ \Delta'_P$ is also a $\mathbf{k}$-algebra homomorphism (since both τ and Δ'_P are $\mathbf{k}$-algebra homomorphisms). This completes the proof of Proposition 4.4. $\square$

6. Remark on antipodes

We have hitherto not really used the antipode of a Hopf algebra; thus, we could just as well have replaced the words "Hopf algebra" by "bialgebra" throughout the entire preceding text[23]. Let us now connect the preceding results with antipodes.

The antipode of any Hopf algebra H will be denoted by S_H.

Proposition 6.1. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-Hopf algebra. Let G be an A-Hopf algebra. Then, every $\mathbf{k}$-algebra homomorphism $f : H \to G$ which is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism must also satisfy $f \circ S_H = S_G \circ f$.

Proof of Proposition 6.1. We know that H is a $\mathbf{k}$-Hopf algebra. Thus, $\underline{A} \otimes H$ is an A-Hopf algebra. Its definition by extending scalars yields that its antipode is given by $S_{\underline{A} \otimes H} = \mathrm{id}_A \otimes S_H$.

Let $f : H \to G$ be a $\mathbf{k}$-algebra homomorphism which is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Then, $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism (since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism) and an A-algebra homomorphism (by Proposition 3.4). Hence, $f^\sharp$ is an A-bialgebra homomorphism, thus an A-Hopf algebra homomorphism (since every A-bialgebra homomorphism between two A-Hopf algebras is an A-Hopf algebra homomorphism). Thus, $f^\sharp$ commutes with the antipodes, i.e., satisfies $f^\sharp \circ S_{\underline{A} \otimes H} = S_G \circ f^\sharp$.

Now, let ι be the canonical $\mathbf{k}$-module homomorphism $H \to \underline{A} \otimes H$, $h \mapsto 1 \otimes h$. Then, $(\mathrm{id}_A \otimes S_H) \circ \iota = \iota \circ S_H$. On the other hand, $f^\sharp \circ \iota = f$ (this is easy to check). Thus,

$$\begin{aligned} \underbrace{f}_{= f^\sharp \circ \iota} \circ S_H &= f^\sharp \circ \underbrace{\iota \circ S_H}_{= (\mathrm{id}_A \otimes S_H) \circ \iota} = f^\sharp \circ \underbrace{(\mathrm{id}_A \otimes S_H)}_{= S_{\underline{A} \otimes H}} \circ \iota = \underbrace{f^\sharp \circ S_{\underline{A} \otimes H}}_{= S_G \circ f^\sharp} \circ \iota \\ &= S_G \circ \underbrace{f^\sharp \circ \iota}_{= f} = S_G \circ f. \end{aligned}$$

This proves Proposition 6.1. □

Corollary 6.2. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. Define a map $\beta_H : H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ as in Definition 5.1. Then,

$$\beta_H \circ S_H = \left(\mathrm{id}_H \otimes S_{\mathrm{QSym}_{\mathbf{k}}} \right) \circ \beta_H.$$

[23] That said, we would not have gained anything this way, because any connected graded $\mathbf{k}$-bialgebra is a $\mathbf{k}$-Hopf algebra (see [GriRei14, Proposition 1.4.16]).

Proof of Corollary 6.2. Theorem 5.7 **(a)** shows that the map β_H is a $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ and a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism. Thus, Proposition 6.1 (applied to $A = H$, $G = \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ and $f = \beta_H$) shows that $\beta_H \circ S_H = S_{\underline{H}\otimes \operatorname{QSym}_{\mathbf{k}}} \circ \beta_H$.

But the H-Hopf algebra $\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ is defined by extension of scalars; thus, its antipode is given by $S_{\underline{H}\otimes \operatorname{QSym}_{\mathbf{k}}} = \operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}$. Hence,

$$\beta_H \circ S_H = \underbrace{S_{\underline{H}\otimes \operatorname{QSym}_{\mathbf{k}}}}_{=\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}} \circ \beta_H = \left(\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}\right) \circ \beta_H.$$

This proves Corollary 6.2. $\square$

Corollary 6.3. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. Define a map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ as in Definition 5.1. Then,

$$S_H = \left(\operatorname{id}_H \otimes \left(\varepsilon_P \circ S_{\operatorname{QSym}_{\mathbf{k}}}\right)\right) \circ \beta_H.$$

Proof of Corollary 6.3. We have

$$\begin{aligned} \underbrace{\left(\operatorname{id}_H \otimes \left(\varepsilon_P \circ S_{\operatorname{QSym}_{\mathbf{k}}}\right)\right)}_{=(\operatorname{id}_H \otimes \varepsilon_P)\circ\left(\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}\right)} \circ \beta_H &= (\operatorname{id}_H \otimes \varepsilon_P) \circ \underbrace{\left(\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}\right) \circ \beta_H}_{\substack{=\beta_H \circ S_H \\ \text{(by Corollary 6.2)}}} \\ &= \underbrace{(\operatorname{id}_H \otimes \varepsilon_P) \circ \beta_H}_{\substack{=\operatorname{id} \\ \text{(by Theorem 5.7 (b))}}} \circ S_H = S_H, \end{aligned}$$

and thus Corollary 6.3 is proven. $\square$

Remark 6.4. What I find remarkable about Corollary 6.3 is that it provides a formula for the antipode S_H of H in terms of β_H and $\operatorname{QSym}_{\mathbf{k}}$. Thus, in order to understand the antipode of H, it suffices to study the map β_H and the antipode of $\operatorname{QSym}_{\mathbf{k}}$ well enough.

Similar claims can be made about other endomorphisms of H, such as the Dynkin idempotent or the Eulerian idempotent (when $\mathbf{k}$ is a $\mathbb{Q}$-algebra). Better yet, we can regard the map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ as an "embedding" of the $\mathbf{k}$-Hopf algebra H into the H-Hopf algebra $\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}} \cong \operatorname{QSym}_H$. Here, I am using the word "embedding" in scare quotes, since this map is not a Hopf algebra homomorphism (its domain and its target are Hopf algebras over different base rings); nevertheless, the map β_H is injective (by Theorem 5.7 **(b)**), and the corresponding map $(\beta_H)^\sharp : \underline{H} \otimes H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ (sending every $a \otimes h$ to $a\beta_H(h)$) is a graded H-Hopf algebra homomorphism (because it is graded, an H-algebra homomorphism and an H-coalgebra homomorphism);

this shows that β_H commutes with various maps defined canonically in terms of a commutative connected graded Hopf algebra. It appears possible to use this for proving identities in commutative connected graded Hopf algebra.

Note that Corollary 6.3 is not completely new. Indeed, it can also be obtained from Takeuchi's formula ([GriRei14, Proposition 1.4.24]) by breaking up the map $f = \mathrm{id} - u\epsilon$ into $\pi_1 + \pi_2 + \pi_3 + \cdots$. However, in its above form, it is more suited to algebraic applications as discussed in Remark 6.4.

7. Questions and final remarks

I shall finish with some remarks and open questions which may or may not be worth further study.

7.1. Dualizing QSym_2

It is well-known (see, e.g., [GriRei14, §5.4]) that the graded Hopf-algebraic dual of the graded Hopf algebra QSym is a graded Hopf algebra NSym. The second comultiplication Δ_P and the second counit ε_P on QSym dualize to a second multiplication m_P and a second unit u_P on NSym, albeit u_P is not a map from $\mathbf{k}$ to NSym but rather a map from $\mathbf{k}$ to the completion $\widehat{\mathrm{NSym}}$ (specifically, to the completion of NSym with respect to its grading). We denote the "almost-$\mathbf{k}$-bialgebra" $(\mathrm{NSym}, m_P, u_P, \Delta, \varepsilon)$ ("almost" because u_P does not go into NSym) by $\mathrm{NSym}^{(2)}$. Explicitly, its operations are given as follows:

- Its multiplication m_P is given by

$$m_P\left(H_\beta \otimes H_\gamma\right) = \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}};\\ \text{row } A = \beta;\\ \text{column } A = \gamma}} H_{(\text{read } A)^{\mathrm{red}}} \qquad \text{for all } \beta, \gamma \in \mathrm{Comp},$$

  where $(H_\alpha)_{\alpha \in \mathrm{Comp}}$ is the basis of NSym dual to the basis $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ of QSym. Thus, it is precisely the *internal product* $*$ introduced in [GKLLRT94, Section 5.1] (by [GKLLRT94, Proposition 5.1]). The canonical projection $\mathrm{NSym} \to \Lambda$ (which sends each H_α to the complete homogeneous symmetric function h_α) intertwines this internal product m_P with the Kronecker multiplication on Λ.

- Its counit u_P sends $1 \in \mathbf{k}$ to the element

$$u_P = H_{()} + H_{(1)} + H_{(2)} + H_{(3)} + \cdots$$

  of the completion $\widehat{\mathrm{NSym}}$ of NSym.

Forgetting Δ and ε for a moment, we can identify the "almost-algebra" $\operatorname{NSym}^{(2)} = (\operatorname{NSym}, m_P, u_P)$ with the direct sum of the *descent algebras* of the symmetric groups $S_0, S_1, S_2, \ldots$ (see, e.g., [GKLLRT94, Section 5.1]).

We can (more or less) dualize Theorem 5.7. As a result, instead of a QSym_2-comodule structure on every commutative graded connected Hopf algebra H, we obtain an $\operatorname{NSym}^{(2)}$-module structure on every cocommutative graded connected Hopf algebra H. This structure is rather well-known: It has $H_\alpha \in \operatorname{NSym}^{(2)}$ act as the convolution product

$$\pi_{a_1} \star \pi_{a_2} \star \cdots \star \pi_{a_k} \in \operatorname{End} H$$

for every composition $\alpha = (a_1, a_2, \ldots, a_k)$ (where $\star$ denotes the convolution product in $\operatorname{End} H$). This $\operatorname{NSym}^{(2)}$-module structure is well-known; it appears implicitly in [Patras94, Théorème II.7] (and is the map $\mathfrak{W}$ in [GriRei14, Hint to Exercise 5.4.6], although [GriRei14] does not prove that it is an action). It provides a way to transfer information from **the** descent algebra NSym_2 to **a** descent algebra $(\operatorname{End}_{\text{graded}} H, \circ)$ of a cocommutative graded connected Hopf algebra H.

Question 7.1. Is it possible to prove that this works using universal properties like I have done above for Theorem 5.7? (Just saying "dualize Theorem 5.7" is not enough, because dualization over arbitrary commutative rings is a heuristic, not a proof strategy; there does not seem to be a general theorem stating that "the dual of a correct result is correct", at least when the result has assumptions about gradedness and similar things.)

If the answer is positive, can we use this to give a slick proof of Solomon's Mackey formula? (I am not saying that there is need for slick proofs of this formula – not after those by Gessel and Bidigare –, but it would be interesting to have a new one. I am thinking of letting both $\operatorname{NSym}^{(2)}$ and the symmetric groups act on the tensor algebra $T(V)$ of an infinite-dimensional free $\mathbf{k}$-module V; one then only needs to check that the actions match.)

Note that if u and v are two elements of $\operatorname{NSym}^{(2)}$, then the action of the NSym-product uv (not the $\operatorname{NSym}^{(2)}$-product!) on H is the convolution of the actions of u and v. So the action map $\operatorname{NSym}^{(2)} \to \operatorname{End} H$ takes the multiplication of $\operatorname{NSym}^{(2)}$ to composition, and the multiplication of NSym to convolution.

7.2. Natural transformations

Question 7.2. In Question 7.1, we found a $\mathbf{k}$-algebra homomorphism $\operatorname{NSym}^{(2)} \to (\operatorname{End} H, \circ)$ for every cocommutative connected graded Hopf algebra H. This is functorial in H, and so is really a map (i.e., natural transformation) from the constant functor $\operatorname{NSym}^{(2)}$ to the functor

$$\{\text{cocommutative connected graded Hopf algebras}\} \to \{\mathbf{k}\text{-modules}\},$$
$$H \mapsto \operatorname{End} H.$$

Does the image of this action span (up to topology) the whole functor? I guess I am badly abusing categorical language here, so let me restate the question in simpler terms: If a natural endomorphism of the $\mathbf{k}$-module H is given for every cocommutative connected graded Hopf algebra H, and this endomorphism is known to annihilate all homogeneous components H_m for sufficiently high m (this is what I mean by "up to topology"), then must there be an element v of $\mathrm{NSym}^{(2)}$ such that this endomorphism is the action of v ?

If the answer is "No", then does it change if we require the endomorphism of H to be graded? If we require $\mathbf{k}$ to be a field of characteristic 0 ?

What if we restrict ourselves to commutative cocommutative connected graded Hopf algebras? At least then, if $\mathbf{k}$ is a finite field $\mathbb{F}_q$, there are more natural endomorphisms of H, such as the Frobenius morphism $x \mapsto x^q$ and its powers. One can then ask for the graded endomorphisms of H, but actually it is also interesting to see how the full $\mathbf{k}$-algebra of natural endomorphisms looks like (how do the endomorphisms coming from $\mathrm{NSym}^{(2)}$ interact with the Frobenii?). And what about characteristic 0 here?

7.3. Dropping commutativity

Question 7.3. What are the natural endomorphisms of connected graded Hopf algebras, without any cocommutativity or commutativity assumption? I suspect that they will form a connected graded Hopf algebra, with two multiplications (one for composition and the other for convolution), but now with a basis indexed by "mopiscotions" (i.e., pairs (α, σ) of a composition α and a permutation $\sigma \in \mathfrak{S}_{\ell(\alpha)}$). Is this a known combinatorial Hopf algebra?

7.4. Other combinatorial Hopf algebras?

Question 7.4. Can we extend the map $\beta_H : H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ to a map $H \to \underline{H} \otimes U$ for some combinatorial Hopf algebra U bigger than $\mathrm{QSym}_{\mathbf{k}}$? What if we require some additional (say, dendriform?) structure on H ? Can we achieve $U = \mathrm{NCQSym}_{\mathbf{k}}$ or $U = \mathrm{DoublePosets}_{\mathbf{k}}$ (the combinatorial Hopf algebra of double posets, which is defined for $\mathbf{k} = \mathbb{Z}$ and denoted by $\mathbb{Z}\mathbf{D}$ in [MalReu11], and can be similarly defined over any $\mathbf{k}$) ? (I am singling out these two Hopf algebras because they have fairly nice internal comultiplications. Actually, the internal comultiplication of $\mathrm{NCQSym}_{\mathbf{k}}$ is the key to Bidigare's proof of Solomon's Mackey formula [Schock04, §2], and I feel it will tell us more if we listen to it.)

Aguiar suggests that the map $H \to \underline{H} \otimes \mathrm{NCQSym}_{\mathbf{k}}$ I am looking for is the dual of his action of the Tits algebra on Hopf monoids [Aguiar13, Proposition 88].

7.5. Further consequences?

Question 7.5. Do we gain anything from applying Corollary 6.2 to $H = \mathrm{QSym}_{\mathbf{k}}$ (thus getting a statement about Δ'_P) ? Probably not much for Δ'_P that the Marne-la-Vallée school has not already discovered using virtual alphabets (the dual version is the statement that $S(a * b) = a * S(b)$ for all $a, b \in \mathrm{NSym}_{\mathbf{k}}$, where $*$ is the internal product).

Question 7.6. From Theorem 5.7 **(a)** and Proposition 5.3, we can conclude that Δ'_P is a $\left(\mathbf{k}, \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. If I am not mistaken, this can be rewritten as the equality

$$(AB) * G = \sum_{(G)} \left(A * G_{(1)}\right) \left(B * G_{(2)}\right)$$

(using Sweedler's notation) for any three elements A, B and G of NSym. This is the famous splitting formula.

Now, it is known from [DHNT08, §7] that the same splitting formula holds when A and B are elements of FQSym (into which NSym is known to inject), as long as G is still an element of NSym (actually, it can be an element of the bigger Patras-Reutenauer algebra, but let us settle for NSym so far). Can this be proven in a similar vein? How much of the Marne-la-Vallée theory follows from Theorem 2.1?

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