A basis for a quotient of symmetric polynomials (draft)

Darij Grinberg

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Abstract. Consider the ring $\mathcal{S}$ of symmetric polynomials in k variables over an arbitrary base ring $\mathbf{k}$. Fix k scalars $a_1, a_2, \ldots, a_k \in \mathbf{k}$.

Let I be the ideal of $\mathcal{S}$ generated by $h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k$, where h_i is the i-th complete homogeneous symmetric polynomial.

The quotient ring $\mathcal{S}/I$ generalizes both the usual and the quantum cohomology of the Grassmannian.

We show that $\mathcal{S}/I$ has a $\mathbf{k}$-module basis consisting of (residue classes of) Schur polynomials fitting into an $(n-k) \times k$-rectangle; and that its multiplicative structure constants satisfy the same S_3-symmetry as those of the Grassmannian cohomology. We prove a Pieri rule and a "rim hook algorithm", and conjecture a positivity property generalizing that of Gromov-Witten invariants. We construct two further bases of $\mathcal{S}/I$ as well.

We also study the quotient of the whole polynomial ring (not just the symmetric polynomials) by the ideal generated by the same k polynomials as I.

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1. Introduction

This is still a draft – proofs are at various levels of detail, and the order of the results reflects the order in which I found them more than the order in which they are most reasonable to read. This draft will probably be split into several smaller papers for publication. **I recommend [Grinbe19] as a quick survey of the main results proved here**.

This work is devoted to a certain construction that generalizes both the regular and the quantum cohomology ring of the Grassmannian [Postni05]. This construction is purely algebraic – we do not know any geometric meaning for it at this point – but shares some basic properties with quantum cohomology, such as an S_3-symmetry of its structure constants (generalizing the S_3-symmetry for Littlewood-Richardson coefficients and Gromov-Witten invariants) and conjecturally a positivity as well. All our arguments are algebraic and combinatorial.

1.1. Acknowledgments

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2. The basis theorems

2.1. Definitions and notations

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$.

Let $\mathbf{k}$ be a commutative ring. Let $k \in \mathbb{N}$.

Let $\mathcal{P}$ denote the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_k]$. This is a graded ring, where the grading is by total degree (so $\deg x_i = 1$ for each $i \in \{1, 2, \ldots, k\}$).

For each $\alpha \in \mathbb{Z}^k$ and each $i \in \{1, 2, \ldots, k\}$, we denote the i-th entry of α by α_i (so that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$). For each $\alpha \in \mathbb{N}^k$, we define a monomial x^α by $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$.

Let $\mathcal{S}$ denote the ring of symmetric polynomials in $\mathcal{P}$; in other words, $\mathcal{S}$ is the ring of invariants of the symmetric group S_k acting on $\mathcal{P}$. (The action here is the one you would expect: A permutation $\sigma \in S_k$ sends a monomial $x_{i_1} x_{i_2} \cdots x_{i_m}$ to $x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots x_{\sigma(i_m)}$.)

The following fact is well-known (going back to Emil Artin):

Proposition 2.1. The $\mathcal{S}$-module $\mathcal{P}$ is free with basis $(x^\alpha)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$.

Proofs of Proposition 2.1 can be found in [LLPT95, (DIFF.1.3)], in [Bourba03, Chapter IV, §6, no. 1, Theorem 1 c)], in [Gailla21, Theorem, part (c)] and in [Macdon91, (5.1)][1]. The particular case when $\mathbf{k}$ is a field is also proved in [Artin71, result shown at the end of §II.G][2]. The particular case of Proposition 2.1 when $\mathbf{k} = \mathbb{Q}$ also appears in [Garsia02, Remark 3.2]. A related result is proven in [FoGePo97, Proposition 3.4] (for $\mathbf{k} = \mathbb{Z}$, but the proof applies equally over any $\mathbf{k}$).

Now, fix an integer $n \geq k$. For each $i \in \{1, 2, \ldots, k\}$, let a_i be an element of $\mathcal{P}$ with degree $< n - k + i$. (This is clearly satisfied when $a_1, a_2, \ldots, a_k$ are constants in $\mathbf{k}$, but also in some other cases. Note that the a_i do not have to be homogeneous.)

For each $\alpha \in \mathbb{Z}^k$, we let $|\alpha|$ denote the sum of the entries of the k-tuple α (that is, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$).

For each $m \in \mathbb{Z}$, we let h_m denote the m-th complete homogeneous symmetric polynomial; this is the element of $\mathcal{S}$ defined by

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha. \tag{1}$$

(Thus, $h_0 = 1$, and $h_m = 0$ when $m < 0$.)

Let J be the ideal of $\mathcal{P}$ generated by the k differences

$$h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k. \tag{2}$$

If M is a $\mathbf{k}$-module and N is a submodule of M, then the projection of any $m \in M$ onto the quotient M/N (that is, the congruence class of m modulo N) will be denoted by $\overline{m}$.

2.2. The basis theorem for $\mathcal{P}/J$

The following is our first result:

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[1] Strictly speaking, [Macdon91, (5.1)] is only the particular case of Proposition 2.1 for $\mathbf{k} = \mathbb{Z}$. However, with some minor modifications, the proof given in [Macdon91] works for any $\mathbf{k}$.

[2] To be more precise, Artin proves in [Artin71, §II.G, Example 2] that (when $\mathbf{k}$ is a field)

- the monomials x^α with $\alpha \in \mathbb{N}^k$ satisfying $\alpha_i < i$ for each i are linearly independent over the field $\mathcal{S}_{\text{rat}}$ of symmetric rational functions in $x_1, x_2, \ldots, x_k$ over $\mathbf{k}$ (and therefore also linearly independent over the ring $\mathcal{S}$ of symmetric polynomials), and
- each polynomial $g \in \mathcal{P}$ can be represented as a polynomial in $x_1, x_2, \ldots, x_k$ with coefficients in $\mathcal{S}$ and having degree $< i$ in each x_i (that is, as an $\mathcal{S}$-linear combination of the monomials x^α with $\alpha \in \mathbb{N}^k$ satisfying $\alpha_i < i$).

Combining these two facts yields Proposition 2.1 (when $\mathbf{k}$ is a field).

Theorem 2.2. The $\mathbf{k}$-module $\mathcal{P}/J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^k;\ \alpha_i<n-k+i \text{ for each } i}$.

Example 2.3. Let $n = 5$ and $k = 2$. Then, $\mathcal{P} = \mathbf{k}[x_1, x_2]$, and J is the ideal of $\mathcal{P}$ generated by the 2 differences

$$h_4 - a_1 = \left(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4\right) - a_1 \qquad \text{and}$$
$$h_5 - a_2 = \left(x_1^5 + x_1^4 x_2 + x_1^3 x_2^2 + x_1^2 x_2^3 + x_1 x_2^4 + x_2^5\right) - a_2.$$

Theorem 2.2 yields that the $\mathbf{k}$-module $\mathcal{P}/J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^2;\ \alpha_i<3+i \text{ for each } i}$; this basis can also be rewritten as $\left(\overline{x_1^{\alpha_1} x_2^{\alpha_2}}\right)_{\alpha_1\in\{0,1,2,3\};\ \alpha_2\in\{0,1,2,3,4\}}$. As a consequence, any $\overline{x_1^{\beta_1} x_2^{\beta_2}} \in \mathcal{P}/J$ can be written as a linear combination of elements of this basis. For example,

$$\overline{x_1^4} = a_1 - \overline{x_1^3 x_2} - \overline{x_1^2 x_2^2} - \overline{x_1 x_2^3} - \overline{x_2^4} \qquad \text{and}$$
$$\overline{x_2^5} = a_2 - a_1 \overline{x_1}.$$

These expressions will become more complicated for higher values of n and k.

Theorem 2.2 is related to the second part of [CoKrWa09, Proposition 2.9] (and our proof below can be viewed as an elaboration of the argument sketched in the last paragraph of [CoKrWa09, proof of Proposition 2.9]).

2.3. The basis theorem for $\mathcal{S}/I$

To state our next result, we need some more notations.

Definition 2.4. (a) We define the concept of *partitions* (of an integer) as in [GriRei20, Chapter 2]. Thus, a partition is a weakly decreasing infinite sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers such that all but finitely many i satisfy $\lambda_i = 0$. We identify each partition $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ with the finite list $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ whenever $p \in \mathbb{N}$ has the property that ($\lambda_i = 0$ for all $i > p$). For example, the partition $\left(3, 1, 1, \underbrace{0, 0, \ldots}_{\text{zeroes}}\right)$ is identified with $(3, 1, 1, 0)$ and with $(3, 1, 1)$.

(b) A *part* of a partition λ means a nonzero entry of λ.

(c) Let $P_{k,n}$ denote the set of all partitions that have at most k parts and have the property that each of their parts is $\leq n - k$. (Visually speaking, $P_{k,n}$ is the set of all partitions whose Young diagram fits into a $k \times (n-k)$-rectangle.)

(d) We let $\varnothing$ denote the empty partition $()$.

Example 2.5. If $n = 4$ and $k = 2$, then

$$P_{k,n} = P_{2,4} = \{\varnothing, (1), (2), (1,1), (2,1), (2,2)\}.$$

If $n = 5$ and $k = 2$, then

$$P_{k,n} = P_{2,5} = \{\varnothing, (1), (2), (3), (1,1), (2,1), (3,1), (2,2), (3,2), (3,3)\}.$$

It is well-known (and easy to see) that $P_{k,n}$ is a finite set of size $\dbinom{n}{k}$. (Indeed, the map

$$\begin{aligned} P_{k,n} &\to \left\{ (a_1, a_2, \ldots, a_k) \in \{1, 2, \ldots, n\}^k \mid a_1 > a_2 > \cdots > a_k \right\}, \\ \lambda &\mapsto (\lambda_1 + k, \lambda_2 + k - 1, \ldots, \lambda_k + 1) \end{aligned}$$

is easily seen to be well-defined and to be a bijection; but the set $\left\{ (a_1, a_2, \ldots, a_k) \in \{1, 2, \ldots, n\}^k \mid a_1 > a_2 > \cdots > a_k \right\}$ has size $\dbinom{n}{k}$.)

Definition 2.6. For any partition λ, we let s_λ denote the Schur polynomial in $x_1, x_2, \ldots, x_k$ corresponding to the partition λ. This Schur polynomial is what is called $s_\lambda(x_1, x_2, \ldots, x_k)$ in [GriRei20, Chapter 2]. Note that

$$s_\lambda = 0 \qquad \text{if } \lambda \text{ has more than } k \text{ parts.} \tag{3}$$

If λ is any partition, then the Schur polynomial $s_\lambda = s_\lambda(x_1, x_2, \ldots, x_k)$ is symmetric and thus belongs to $\mathcal{S}$.

We now state our next fundamental fact:

Theorem 2.7. Assume that $a_1, a_2, \ldots, a_k$ belong to $\mathcal{S}$. Let I be the ideal of $\mathcal{S}$ generated by the k differences (2). Then, the $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$.

We will prove Theorem 2.7 below; a different proof has been given by Weinfeld in [Weinfe19, Corollary 6.2].

The $\mathbf{k}$-algebra $\mathcal{S}/I$ generalizes several constructions in the literature:

- If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S}/I$ becomes the cohomology ring of the Grassmannian of k-dimensional subspaces in an n-dimensional space (see, e.g., [Fulton99, §9.4] or [Manive01, Exercise 3.2.12]); the elements of the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ correspond to the Schubert classes.

- If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then $\mathcal{S}/I$ becomes isomorphic to the quantum cohomology ring of the same Grassmannian (see [Postni05]). Indeed, our ideal I becomes the J^q_{kn} of [Postni05, (6)] in this case, and Theorem 2.7 generalizes the fact that the quotient $(\Lambda_k \otimes \mathbb{Z}[q]) / J^q_{kn}$ in [Postni05, (6)] has basis $(s_\lambda)_{\lambda \in P_{kn}}$.

One goal of this paper is to provide a purely algebraic foundation for the study of the standard and quantum cohomology rings of the Grassmannian, without having to resort to geometry for proofs of the basic properties of these rings. In particular, Theorem 2.7 shows that the "abstract Schubert classes" $\overline{s_\lambda}$ (with $\lambda \in P_{k,n}$) form a basis of the **k**-module $\mathcal{S}/I$, whereas Corollary 6.24 further below shows that the structure constants of the **k**-algebra $\mathcal{S}/I$ with respect to this basis (we may call them "generalized Gromov-Witten invariants") satisfy an S_3-symmetry. These two properties are two of the facts for whose proofs [Postni05] relies on algebro-geometric literature; thus, our paper helps provide an alternative footing for [Postni05] using only combinatorics and algebra[3].

Remark 2.8. The **k**-algebra $\mathcal{P}/J$ somewhat resembles the "splitting algebra" $\operatorname{Split}^d_A(p)$ from [LakTho12, §1.3]; further analogies between these concepts can be made as we study the former. For example, the basis we give in Theorem 2.2 is like the basis in [LakTho12, (1.5)]. It is not currently clear to us whether there is more than analogies.

3. A fundamental identity

Let us use the notations h_m and e_m for complete homogeneous symmetric polynomials and elementary symmetric polynomials in general. Thus, for any $m \in \mathbb{Z}$ and any p elements $y_1, y_2, \ldots, y_p$ of a commutative ring, we set

$$h_m(y_1, y_2, \ldots, y_p) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le p} y_{i_1} y_{i_2} \cdots y_{i_m} \qquad \text{and} \tag{4}$$

$$e_m(y_1, y_2, \ldots, y_p) = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le p} y_{i_1} y_{i_2} \cdots y_{i_m}. \tag{5}$$

(Thus, $h_0(y_1, y_2, \ldots, y_p) = 1$ and $e_0(y_1, y_2, \ldots, y_p) = 1$. Also, $e_m(y_1, y_2, \ldots, y_p) = 0$ for all $m > p$. Also, for any $m < 0$, we have $h_m(y_1, y_2, \ldots, y_p) = 0$ and $e_m(y_1, y_2, \ldots, y_p) = 0$. Finally, what we have previously called h_m without any arguments can now be rewritten as $h_m(x_1, x_2, \ldots, x_k)$. Similarly, we shall occasionally abbreviate $e_m(x_1, x_2, \ldots, x_k)$ as e_m.)

Lemma 3.1. Let $i \in \{1, 2, \ldots, k+1\}$ and $p \in \mathbb{N}$. Then,

$$h_p(x_i, x_{i+1}, \ldots, x_k) = \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) h_{p-t}(x_1, x_2, \ldots, x_k).$$

---

[3] This, of course, presumes that one is willing to forget the cohomological definition of the ring $\mathrm{QH}^*(\mathrm{Gr}_{kn})$, and instead to define it algebraically as the quotient ring $(\Lambda_k \otimes \mathbb{Z}[q]) / J^q_{kn}$, using the notations of [Postni05].

Notice that if $i = k + 1$, then the term $h_p(x_i, x_{i+1}, \ldots, x_k)$ on the left hand side of Lemma 3.1 is understood to be h_p of an empty list of vectors; this is 1 when $p = 0$ and 0 otherwise.

Lemma 3.1 is actually a particular case of [Grinbe16, detailed version, Theorem 3.15] (applied to $a = x_i \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b = h_p(x_1, x_2, x_3, \ldots) \in \operatorname{QSym}$) [4]. However, we shall give a more elementary proof of it here. This proof relies on the following two basic identities:

Lemma 3.2. Let A be a commutative ring. Let $y_1, y_2, \ldots, y_p$ be some elements of A. Consider the ring $A[[u]]$ of formal power series in one indeterminate u over A. Then, in this ring, we have

$$\sum_{q \in \mathbb{N}} h_q(y_1, y_2, \ldots, y_p) u^q = \prod_{j=1}^{p} \frac{1}{1 - y_j u} \tag{6}$$

and

$$\sum_{q \in \mathbb{N}} (-1)^q e_q(y_1, y_2, \ldots, y_p) u^q = \prod_{j=1}^{p} (1 - y_j u). \tag{7}$$

Proof of Lemma 3.2. The identity (6) can be obtained from the identities [GriRei20, (2.2.18)] by substituting $y_1, y_2, \ldots, y_p, 0, 0, 0, \ldots$ for the indeterminates $x_1, x_2, x_3, \ldots$ and substituting u for t. The identity (7) can be obtained from the identities [GriRei20, (2.2.19)] by substituting $y_1, y_2, \ldots, y_p, 0, 0, 0, \ldots$ for the indeterminates $x_1, x_2, x_3, \ldots$ and substituting $-u$ for t. Thus, Lemma 3.2 is proven. □

Proof of Lemma 3.1. Consider the ring $\mathcal{P}[[u]]$ of formal power series in one indeterminate u over $\mathcal{P}$. Applying (6) to $\mathcal{P}$ and $(x_i, x_{i+1}, \ldots, x_k)$ instead of A and $(y_1, y_2, \ldots, y_p)$, we obtain

$$\sum_{q \in \mathbb{N}} h_q(x_i, x_{i+1}, \ldots, x_k) u^q = \prod_{j=1}^{k-i+1} \frac{1}{1 - x_{i+j-1} u} = \prod_{j=i}^{k} \frac{1}{1 - x_j u}$$

(here, we have substituted j for $i + j - 1$ in the product). Applying (7) to $\mathcal{P}$ and $(x_1, x_2, \ldots, x_{i-1})$ instead of A and $(y_1, y_2, \ldots, y_p)$, we obtain

$$\sum_{q \in \mathbb{N}} (-1)^q e_q(x_1, x_2, \ldots, x_{i-1}) u^q = \prod_{j=1}^{i-1} (1 - x_j u). \tag{8}$$

Applying (6) to $\mathcal{P}$ and $(x_1, x_2, \ldots, x_k)$ instead of A and $(y_1, y_2, \ldots, y_p)$, we obtain

$$\sum_{q \in \mathbb{N}} h_q(x_1, x_2, \ldots, x_k) u^q = \prod_{j=1}^{k} \frac{1}{1 - x_j u}. \tag{9}$$

---

[4] Here, we are using the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in **infinitely** many variables $x_1, x_2, x_3, \ldots$, and its subring QSym of quasisymmetric functions. See [Grinbe16] for a brief introduction to both of these. Note that the symmetric function $h_p(x_1, x_2, x_3, \ldots)$ is called h_p in [Grinbe16].

Thus,

$$\begin{aligned}
&\sum_{q\in\mathbb{N}} h_q\left(x_i, x_{i+1}, \ldots, x_k\right) u^q \\
&= \prod_{j=i}^{k} \frac{1}{1-x_j u} = \left(\prod_{j=1}^{k} \frac{1}{1-x_j u}\right) / \left(\prod_{j=1}^{i-1} \frac{1}{1-x_j u}\right) \\
&= \underbrace{\left(\prod_{j=1}^{i-1} \left(1-x_j u\right)\right)}_{\substack{=\sum_{q\in\mathbb{N}} (-1)^q e_q(x_1,x_2,\ldots,x_{i-1}) u^q \\ \text{(by (8))}}} \underbrace{\left(\prod_{j=1}^{k} \frac{1}{1-x_j u}\right)}_{\substack{=\sum_{q\in\mathbb{N}} h_q(x_1,x_2,\ldots,x_k) u^q \\ \text{(by (9))}}} \\
&= \left(\sum_{q\in\mathbb{N}} (-1)^q e_q\left(x_1, x_2, \ldots, x_{i-1}\right) u^q\right) \left(\sum_{q\in\mathbb{N}} h_q\left(x_1, x_2, \ldots, x_k\right) u^q\right).
\end{aligned}$$

Comparing the coefficient before u^p in this equality of power series, we obtain

$$\begin{aligned}
h_p\left(x_i, x_{i+1}, \ldots, x_k\right) &= \sum_{t=0}^{p} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t}\left(x_1, x_2, \ldots, x_k\right) \\
&= \sum_{t=0}^{\infty} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t}\left(x_1, x_2, \ldots, x_k\right) \\
&\qquad \left(\text{since } h_{p-t}\left(x_1, x_2, \ldots, x_k\right) = 0 \text{ for all } t > p\right) \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t}\left(x_1, x_2, \ldots, x_k\right) \\
&\qquad \left(\text{since } e_t\left(x_1, x_2, \ldots, x_{i-1}\right) = 0 \text{ for all } t > i-1\right).
\end{aligned}$$

This proves Lemma 3.1. ☐

Corollary 3.3. Let p be a positive integer. Then,

$$h_p = -\sum_{t=1}^{k} (-1)^t e_t h_{p-t}.$$

Proof of Corollary 3.3. Lemma 3.1 (applied to $i = k+1$) yields

$$\begin{aligned}
h_p\left(x_{k+1}, x_{k+2}, \ldots, x_k\right) &= \sum_{t=0}^{k} (-1)^t \underbrace{e_t\left(x_1, x_2, \ldots, x_k\right)}_{=e_t} \underbrace{h_{p-t}\left(x_1, x_2, \ldots, x_k\right)}_{=h_{p-t}} \\
&= \sum_{t=0}^{k} (-1)^t e_t h_{p-t}.
\end{aligned}$$

Comparing this with

$$h_p(x_{k+1}, x_{k+2}, \ldots, x_k) = h_p(\text{an empty list of variables}) = 0 \qquad (\text{since } p > 0),$$

we obtain

$$0 = \sum_{t=0}^{k} (-1)^t e_t h_{p-t} = \underbrace{(-1)^0}_{=1} \underbrace{e_0}_{=1} \underbrace{h_{p-0}}_{=h_p} + \sum_{t=1}^{k} (-1)^t e_t h_{p-t} = h_p + \sum_{t=1}^{k} (-1)^t e_t h_{p-t}.$$

Hence,

$$h_p = -\sum_{t=1}^{k} (-1)^t e_t h_{p-t}.$$

This proves Corollary 3.3. ☐

4. Proof of Theorem 2.2

We shall next prove Theorem 2.2 using Gröbner bases. For the concept of Gröbner bases over a commutative ring, see [Grinbe17, detailed version, §3].

We define a degree-lexicographic term order on the monomials in $\mathcal{P}$, where the variables are ordered by $x_1 > x_2 > \cdots > x_k$. Explicitly, this term order is the total order on the set of monomials in $x_1, x_2, \ldots, x_k$ defined as follows: Two monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ satisfy $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} > x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ if and only if

- **either** $\alpha_1 + \alpha_2 + \cdots + \alpha_k > \beta_1 + \beta_2 + \cdots + \beta_k$,
- **or** $\alpha_1 + \alpha_2 + \cdots + \alpha_k = \beta_1 + \beta_2 + \cdots + \beta_k$ and there exists some $i \in \{1, 2, \ldots, k\}$ such that $\alpha_i > \beta_i$ and $\left(\alpha_j = \beta_j \text{ for all } j < i\right)$.

This total order is a term order (in the sense of [Grinbe17, detailed version, Definition 3.5]). Fix this term order; thus it makes sense to speak of Gröbner bases of ideals.

Proposition 4.1. The family

$$\left(h_{n-k+i}(x_i, x_{i+1}, \ldots, x_k) - \sum_{t=0}^{i-1} (-1)^t e_t(x_1, x_2, \ldots, x_{i-1}) a_{i-t} \right)_{i \in \{1, 2, \ldots, k\}}$$

is a Gröbner basis of the ideal J. (Recall that we are using the notations from (4) and (5).)

Proposition 4.1 is somewhat similar to [Sturmf08, Theorem 1.2.7] (or, equivalently, [CoLiOs15, §7.1, Proposition 5]), but not the same.[5] It is also similar to [LomQui21, comment at the end of §III.4]. Our proof of it relies on the following elementary fact:

Lemma 4.2. Let A be a commutative ring. Let $b_1, b_2, \ldots, b_k \in A$ and $c_1, c_2, \ldots, c_k \in A$. Assume that

$$b_i \in c_i + \sum_{t=1}^{i-1} c_{i-t} A \tag{10}$$

for each $i \in \{1, 2, \ldots, k\}$. Then, $b_1 A + b_2 A + \cdots + b_k A = c_1 A + c_2 A + \cdots + c_k A$ (as ideals of A).

Proof of Lemma 4.2. We claim that

$$\sum_{p=1}^{j} b_p A = \sum_{p=1}^{j} c_p A \qquad \text{for each } j \in \{0, 1, \ldots, k\}. \tag{11}$$

[*Proof of (11):* We shall prove (11) by induction on j:

Induction base: For $j = 0$, both sides of the equality (11) are the zero ideal of A (since they are empty sums of ideals of A). Thus, (11) holds for $j = 0$. This completes the induction base.

Induction step: Let $i \in \{1, 2, \ldots, k\}$. Assume that (11) holds for $j = i - 1$. We must prove that (11) holds for $j = i$.

We have assumed that (11) holds for $j = i - 1$. In other words, we have $\sum_{p=1}^{i-1} b_p A = \sum_{p=1}^{i-1} c_p A$. But (10) yields $b_i \in c_i + \sum_{t=1}^{i-1} c_{i-t} A = c_i + \sum_{p=1}^{i-1} c_p A$ (here, we have substituted p for $i - t$ in the sum). Thus,

$$c_i \in b_i - \sum_{p=1}^{i-1} c_p A = b_i + \sum_{p=1}^{i-1} c_p A,$$

so that

$$c_i A \subseteq \left(b_i + \sum_{p=1}^{i-1} c_p A \right) A \subseteq b_i A + \sum_{p=1}^{i-1} c_p A.$$

But from $b_i \in c_i + \sum_{p=1}^{i-1} c_p A$, we obtain

$$b_i A \subseteq \left(c_i + \sum_{p=1}^{i-1} c_p A \right) A \subseteq c_i A + \sum_{p=1}^{i-1} c_p A = \sum_{p=1}^{i} c_p A.$$

---

[5] For example, our $a_1, a_2, \ldots, a_k$ are elements of $\mathbf{k}$ rather than indeterminates (although they **can** be indeterminates if $\mathbf{k}$ itself is a polynomial ring), and our term order is degree-lexicographic rather than lexicographic. Thus, it should not be surprising that the families are different.

Now,

$$\sum_{p=1}^{i} b_pA = \underbrace{\sum_{p=1}^{i-1} b_pA}_{\substack{=\sum_{p=1}^{i-1} c_pA \subseteq \sum_{p=1}^{i} c_pA \\ \text{(since } i-1 \leq i\text{)}}} + \underbrace{b_iA}_{\subseteq \sum_{p=1}^{i} c_pA} \subseteq \sum_{p=1}^{i} c_pA + \sum_{p=1}^{i} c_pA = \sum_{p=1}^{i} c_pA.$$

Combining this inclusion with

$$\begin{aligned} \sum_{p=1}^{i} c_pA &= \sum_{p=1}^{i-1} c_pA + \underbrace{c_iA}_{\subseteq b_iA + \sum_{p=1}^{i-1} c_pA} \subseteq \sum_{p=1}^{i-1} c_pA + b_iA + \sum_{p=1}^{i-1} c_pA \\ &= \underbrace{\sum_{p=1}^{i-1} c_pA + \sum_{p=1}^{i-1} c_pA}_{=\sum_{p=1}^{i-1} c_pA = \sum_{p=1}^{i-1} b_pA} + b_iA = \sum_{p=1}^{i-1} b_pA + b_iA = \sum_{p=1}^{i} b_pA, \end{aligned}$$

we obtain $\sum_{p=1}^{i} b_pA = \sum_{p=1}^{i} c_pA$. In other words, (11) holds for $j = i$. This completes the induction step. Thus, (11) is proven by induction.]

Now, (11) (applied to $j = k$) yields

$$\sum_{p=1}^{k} b_pA = \sum_{p=1}^{k} c_pA.$$

Thus,

$$b_1A + b_2A + \cdots + b_kA = \sum_{p=1}^{k} b_pA = \sum_{p=1}^{k} c_pA = c_1A + c_2A + \cdots + c_kA.$$

This proves Lemma 4.2. □

Proof of Proposition 4.1 (sketched). For each $i \in \{1, 2, \ldots, k\}$, we define a polynomial $b_i \in \mathcal{P}$ by

$$b_i = h_{n-k+i}\left(x_i, x_{i+1}, \ldots, x_k\right) - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t}.$$

Then, we must prove that the family $(b_i)_{i \in \{1,2,\ldots,k\}}$ is a Gröbner basis of the ideal J. We shall first prove that this family generates J.

For each $i \in \{1, 2, \ldots, k\}$, we define $c_i \in \mathcal{P}$ by $c_i = h_{n-k+i} - a_i$. Then, J is the ideal of $\mathcal{P}$ generated by the k elements $c_1, c_2, \ldots, c_k$ (by the definition of J). In other words,

$$J = c_1\mathcal{P} + c_2\mathcal{P} + \cdots + c_k\mathcal{P}. \tag{12}$$

For each $i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned}
b_i &= \underbrace{h_{n-k+i}\left(x_i, x_{i+1}, \ldots, x_k\right)}_{\substack{=\sum_{t=0}^{i-1}(-1)^t e_t(x_1,x_2,\ldots,x_{i-1}) h_{n-k+i-t}(x_1,x_2,\ldots,x_k) \\ \text{(by Lemma 3.1 (applied to } p=n-k+i))}} - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t} \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) h_{n-k+i-t}\left(x_1, x_2, \ldots, x_k\right) \\
&\qquad - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t} \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) \left(\underbrace{h_{n-k+i-t}\left(x_1, x_2, \ldots, x_k\right)}_{=h_{n-k+i-t}} - a_{i-t} \right) \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) \underbrace{\left(h_{n-k+i-t} - a_{i-t}\right)}_{\substack{=c_{i-t} \\ \text{(by the definition of } c_{i-t})}} \\
&= \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) c_{i-t} \\
&= \underbrace{(-1)^0}_{=1} \underbrace{e_0\left(x_1, x_2, \ldots, x_{0-1}\right)}_{=1} \underbrace{c_{i-0}}_{=c_i} + \sum_{t=1}^{i-1} \underbrace{(-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right)}_{\in\mathcal{P}} c_{i-t} \\
&\in c_i + \sum_{t=1}^{i-1} \underbrace{\mathcal{P}c_{i-t}}_{=c_{i-t}\mathcal{P}} = c_i + \sum_{t=1}^{i-1} c_{i-t}\mathcal{P}.
\end{aligned}$$

Hence, Lemma 4.2 (applied to $A = \mathcal{P}$) yields that $b_1\mathcal{P} + b_2\mathcal{P} + \cdots + b_k\mathcal{P} = c_1\mathcal{P} + c_2\mathcal{P} + \cdots + c_k\mathcal{P}$ (as ideals of $\mathcal{P}$). Comparing this with (12), we obtain $J = b_1\mathcal{P} + b_2\mathcal{P} + \cdots + b_k\mathcal{P}$. Thus, the family $(b_i)_{i\in\{1,2,\ldots,k\}}$ generates the ideal J. Furthermore, for each $i \in \{1, 2, \ldots, k\}$, the i-th element

$$b_i = h_{n-k+i}\left(x_i, x_{i+1}, \ldots, x_k\right) - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t}$$

of this family has leading term x_i^{n-k+i} (because the polynomial

$\sum\limits_{t=0}^{i-1} (-1)^t e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}$ has degree $< n-k+i$ [6], whereas the polynomial $h_{n-k+i} (x_i, x_{i+1}, \ldots, x_k)$ is homogeneous of degree $n-k+i$ with leading term x_i^{n-k+i} [7]). Thus, the leading terms of the k elements of this family are disjoint (in the sense that no two of these leading terms have any indeterminates in common). Thus, clearly, Buchberger's first criterion (see, e.g., [Grinbe17, detailed version, Proposition 3.9]) shows that this family is a Gröbner basis. □

Proof of Theorem 2.2 (sketched). This follows using the Macaulay-Buchberger basis theorem (e.g., [Grinbe17, detailed version, Proposition 3.10]) from Proposition 4.1. (Indeed, if we let G be the Gröbner basis of J constructed in Proposition 4.1, then the monomials $\overline{x^\alpha}$ for all $\alpha \in \mathbb{N}^k$ satisfying ($\alpha_i < n-k+i$ for each i) are precisely the G-reduced monomials[8].) □

5. Proof of Theorem 2.7

Next, we shall prove Theorem 2.7.

> **Convention 5.1.** For the rest of Section 5, **we assume that $a_1, a_2, \ldots, a_k$ belong to $\mathcal{S}$.**

Thus, $a_1, a_2, \ldots, a_k$ are symmetric polynomials. Moreover, recall that for each $i \in \{1, 2, \ldots, k\}$, the polynomial a_i has degree $< n-k+i$. In other words, for each $i \in \{1, 2, \ldots, k\}$, we have

$$\deg (a_i) < n-k+i. \tag{13}$$

Substituting $i-n+k$ for i in this statement, we obtain the following: For each $i \in \{n-k+1, n-k+2, \ldots, n\}$, we have

$$\deg (a_{n-k+i}) < n-k+(i-n+k) = i. \tag{14}$$

---

[6] *Proof.* It clearly suffices to show that for each $t \in \{0, 1, \ldots, i-1\}$, the polynomial $e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}$ has degree $< n-k+i$.

So let us do this. Let $t \in \{0, 1, \ldots, i-1\}$. Then, the polynomial a_{i-t} has degree $< n-k+(i-t)$ (by the definition of $a_1, a_2, \ldots, a_k$). In other words, $\deg (a_{i-t}) < n-k+(i-t)$. Hence, the polynomial $e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}$ has degree

$$\deg (e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}) = \underbrace{\deg (e_t (x_1, x_2, \ldots, x_{i-1}))}_{\leq t} + \underbrace{\deg (a_{i-t})}_{< n-k+(i-t)}$$
$$< t + (n-k+(i-t)) = n-k+i.$$

In other words, the polynomial $e_t (x_1, x_2, \ldots, x_{i-1}) a_{i-t}$ has degree $< n-k+i$. Qed.

[7] Indeed, every term of the polynomial $h_{n-k+i} (x_i, x_{i+1}, \ldots, x_k)$ has the form $x_i^{u_i} x_{i+1}^{u_{i+1}} \cdots x_k^{u_k}$ for some nonnegative integers $u_i, u_{i+1}, \ldots, u_k \in \mathbb{N}$ satisfying $u_i + u_{i+1} + \cdots + u_k = n-k+i$. Among these terms, clearly the largest one is x_i^{n-k+i}.

[8] because the i-th entry of the Gröbner basis G has head term x_i^{n-k+i}

Let I be the ideal of $\mathcal{S}$ generated by the k differences (2). Hence, these differences belong to I. Thus,

$$h_{n-k+j} \equiv a_j \operatorname{mod} I \qquad \text{for each } j \in \{1,2,\ldots,k\}. \tag{15}$$

Renaming the index j as $i-n+k$ in this statement, we obtain

$$h_i \equiv a_{i-n+k} \operatorname{mod} I \qquad \text{for each } i \in \{n-k+1, n-k+2, \ldots, n\}. \tag{16}$$

Lemma 5.2. Let A be a commutative $\mathbf{k}$-algebra. Let B be a commutative A-algebra. Assume that the A-module B is spanned by the family $(b_u)_{u\in U} \in B^U$. Let $\mathcal{I}$ be an ideal of A. Let $(a_v)_{v\in V} \in A^V$ be a family of elements of A such that the $\mathbf{k}$-module $A/\mathcal{I}$ is spanned by the family $(\overline{a_v})_{v\in V} \in (A/\mathcal{I})^V$. Then, the $\mathbf{k}$-module $B/(\mathcal{I}B)$ is spanned by the family $\left(\overline{a_v b_u}\right)_{(u,v)\in U\times V} \in (B/(\mathcal{I}B))^{U\times V}$.

Proof of Lemma 5.2. Easy. Here is the proof under the assumption that the set U is finite[9]:

Let $x \in B/(\mathcal{I}B)$. Thus, $x = \overline{b}$ for some $b \in B$. Consider this b. Recall that the A-module B is spanned by the family $(b_u)_{u\in U}$. Hence, $b = \sum\limits_{u\in U} p_u b_u$ for some family $(p_u)_{u\in U} \in A^U$ of elements of A. Consider this family $(p_u)_{u\in U}$.

Recall that the $\mathbf{k}$-module $A/\mathcal{I}$ is spanned by the family $(\overline{a_v})_{v\in V} \in (A/\mathcal{I})^V$. Thus, for each $u \in U$, there exists a family $(q_{u,v})_{v\in V} \in \mathbf{k}^V$ of elements of $\mathbf{k}$ such that $\overline{p_u} = \sum\limits_{v\in V} q_{u,v}\overline{a_v}$ (and such that all but finitely many $v \in V$ satisfy $q_{u,v} = 0$). Consider this family $(q_{u,v})_{v\in V}$.

Now, recall that $B/(\mathcal{I}B)$ is an $A/\mathcal{I}$-module (since B is an A-module, but each $i \in \mathcal{I}$ clearly acts as 0 on $B/(\mathcal{I}B)$). Now,

$$\begin{aligned} x = \overline{b} &= \overline{\sum_{u\in U} p_u b_u} \qquad \left(\text{since } b = \sum_{u\in U} p_u b_u\right) \\ &= \sum_{u\in U} \underbrace{\overline{p_u}}_{=\sum\limits_{v\in V} q_{u,v}\overline{a_v}} \overline{b_u} = \underbrace{\sum_{u\in U}\sum_{v\in V}}_{=\sum\limits_{(u,v)\in U\times V}} q_{u,v} \underbrace{\overline{a_v}\,\overline{b_u}}_{=\overline{a_v b_u}} = \sum_{(u,v)\in U\times V} q_{u,v}\overline{a_v b_u}. \end{aligned}$$

Thus, x belongs to the $\mathbf{k}$-submodule of $B/(\mathcal{I}B)$ spanned by the family $\left(\overline{a_v b_u}\right)_{(u,v)\in U\times V}$. Since we have proven this for all $x \in B/(\mathcal{I}B)$, we thus conclude that the $\mathbf{k}$-module $B/(\mathcal{I}B)$ is spanned by the family $\left(\overline{a_v b_u}\right)_{(u,v)\in U\times V} \in (B/(\mathcal{I}B))^{U\times V}$. This proves Lemma 5.2. □

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[9] The case when U is infinite needs only minor modifications. But we shall only use the case when U is finite.

Lemma 5.3. Let M be a free $\mathbf{k}$-module with a finite basis $(b_s)_{s \in S}$. Let $(a_u)_{u \in U} \in M^U$ be a family that spans M. Assume that $|U| = |S|$. Then, $(a_u)_{u \in U}$ is a basis of the $\mathbf{k}$-module M. (In other words: A spanning family of M whose size equals the size of a basis must itself be a basis, as long as the sizes are finite.)

Proof of Lemma 5.3. Well-known (see, e.g., [GriRei20, Exercise 2.5.18 **(b)**]). ☐

Lemma 5.4. Let i be an integer such that $i > n - k$. Then,

$$h_i \equiv (\text{some symmetric polynomial of degree } < i) \bmod I.$$

Proof of Lemma 5.4 (sketched). We shall prove Lemma 5.4 by strong induction on i. Thus, we assume (as the induction hypothesis) that

$$h_j \equiv (\text{some symmetric polynomial of degree } < j) \bmod I \tag{17}$$

for every $j \in \{n-k+1, n-k+2, \ldots, i-1\}$.

If $i \leq n$, then (16) yields $h_i \equiv a_{i-n+k} \bmod I$ (since $i \in \{n-k+1, n-k+2, \ldots, n\}$), which clearly proves Lemma 5.4 (since a_{i-n+k} is a symmetric polynomial of degree $< i$ [10]). Thus, for the rest of this proof, we WLOG assume that $i > n$. Hence, each $t \in \{1, 2, \ldots, k\}$ satisfies
$i - t \in \{n-k+1, n-k+2, \ldots, i-1\}$ (since $\underbrace{i}_{>n} - \underbrace{t}_{\leq k} > n - k$ and $i - \underbrace{t}_{\geq 1} \leq i - 1$) and therefore

$$h_{i-t} \equiv (\text{some symmetric polynomial of degree } < i-t) \bmod I \tag{18}$$

(by (17), applied to $j = i - t$).

But i is a positive integer (since $i > n \geq 0$). Hence, Corollary 3.3 (applied to $p = i$) yields

$$\begin{aligned} h_i &= -\sum_{t=1}^{k} (-1)^t e_t \underbrace{h_{i-t}}_{\substack{\equiv (\text{some symmetric polynomial of degree } < i-t) \bmod I \\ (\text{by (18)})}} \\ &\equiv -\sum_{t=1}^{k} (-1)^t e_t \cdot (\text{some symmetric polynomial of degree } < i-t) \\ &= (\text{some symmetric polynomial of degree } < i) \bmod I. \end{aligned}$$

This completes the induction step. Thus, Lemma 5.4 is proven. ☐

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[10] by (14)

Definition 5.5. The *size* of a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is defined as $\lambda_1 + \lambda_2 + \lambda_3 + \cdots$, and is denoted by $|\lambda|$.

Definition 5.6. Let P_k denote the set of all partitions with at most k parts. Thus, the elements of P_k are weakly decreasing k-tuples of nonnegative integers.

Proposition 5.7. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition in P_k. Then:

(a) We have

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v} \right)_{1 \le u \le k,\ 1 \le v \le k} \right).$$

(b) Let $p \in \{0, 1, \ldots, k\}$ be such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$. Then,

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v} \right)_{1 \le u \le p,\ 1 \le v \le p} \right).$$

Proof of Proposition 5.7. **(b)** Proposition 5.7 **(b)** is the well-known Jacobi-Trudi identity, and is proven in various places. (For instance, [GriRei20, (2.4.16)] states a similar formula for skew Schur functions; if we set $\mu = \varnothing$ in it and apply both sides to the variables $x_1, x_2, \ldots, x_k$, then we recover the claim of Proposition 5.7 **(b)**.)

(a) We have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Hence, Proposition 5.7 **(a)** is the particular case of Proposition 5.7 **(b)** for $p = k$. □

Lemma 5.8. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be any partition. Let $i \in \{1, 2, \ldots, \ell\}$ and $j \in \{1, 2, \ldots, \ell\}$. Then,

$$\sum_{\substack{u \in \{1,2,\ldots,\ell\};\\ u \neq i}} (\lambda_u - u) + \sum_{\substack{u \in \{1,2,\ldots,\ell\};\\ u \neq j}} u = |\lambda| - (\lambda_i - i + j).$$

Proof of Lemma 5.8. We have

$$\begin{aligned}
&\underbrace{\sum_{\substack{u\in\{1,2,\ldots,\ell\};\\ u\neq i}} (\lambda_u - u)}_{=\sum\limits_{u\in\{1,2,\ldots,\ell\}} (\lambda_u-u)-(\lambda_i-i)} + \underbrace{\sum_{\substack{u\in\{1,2,\ldots,\ell\};\\ u\neq j}} u}_{=\sum\limits_{u\in\{1,2,\ldots,\ell\}} u-j} \\
&= \underbrace{\sum_{u\in\{1,2,\ldots,\ell\}} (\lambda_u - u)}_{=\sum\limits_{u\in\{1,2,\ldots,\ell\}} \lambda_u - \sum\limits_{u\in\{1,2,\ldots,\ell\}} u} - (\lambda_i - i) + \sum_{u\in\{1,2,\ldots,\ell\}} u - j \\
&= \sum_{u\in\{1,2,\ldots,\ell\}} \lambda_u - \sum_{u\in\{1,2,\ldots,\ell\}} u - (\lambda_i - i) + \sum_{u\in\{1,2,\ldots,\ell\}} u - j \\
&= \underbrace{\sum_{u\in\{1,2,\ldots,\ell\}} \lambda_u}_{=|\lambda|} - (\lambda_i - i) - j = |\lambda| - (\lambda_i - i) - j = |\lambda| - (\lambda_i - i + j) .
\end{aligned}$$

This proves Lemma 5.8. □

Next, let us recall the definition of a cofactor of a matrix:

Definition 5.9. Let $\ell \in \mathbb{N}$. Let R be a commutative ring. Let $A \in R^{\ell \times \ell}$ be any $\ell \times \ell$-matrix. Let $i \in \{1,2,\ldots,\ell\}$ and $j \in \{1,2,\ldots,\ell\}$. Then:

(a) The (i,j)*-th minor* of the matrix A is defined to be the determinant of the $(\ell-1) \times (\ell-1)$-matrix obtained from A by removing the i-th row and the j-th column.

(b) The (i,j)*-th cofactor* of the matrix A is defined to $(-1)^{i+j}$ times the (i,j)-th minor of A.

It is known that any $\ell \times \ell$-matrix $A = \left(a_{i,j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}$ over a commutative ring R satisfies

$$\det A = \sum_{j=1}^{\ell} a_{i,j} \cdot (\text{the } (i,j)\text{-th cofactor of } A) \tag{19}$$

for each $i \in \{1,2,\ldots,\ell\}$. (This is the Laplace expansion of the determinant of A along its i-th row.)

Lemma 5.10. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be any partition. Let $i \in \{1,2,\ldots,\ell\}$ and $j \in \{1,2,\ldots,\ell\}$. Then, the (i,j)-th cofactor of the matrix $\left(h_{\lambda_u - u + v}\right)_{1\leq u\leq \ell,\ 1\leq v\leq \ell}$ is a homogeneous symmetric polynomial of degree $|\lambda| - (\lambda_i - i + j)$.

Proof of Lemma 5.10 (sketched). This is a simple argument that inflates in length by a multiple when put on paper. You will probably have arrived at the proof long before you have finished reading the following.

For each $u \in \{1,2,\ldots,\ell\}$ and $v \in \{1,2,\ldots,\ell\}$, we define an integer $w(u,v)$ by

$$w(u,v) = \lambda_u - u + v. \tag{20}$$

Let A be the matrix $\left(h_{w(u,v)}\right)_{1\leq u\leq \ell,\ 1\leq v\leq \ell}$. Let μ be the (i,j)-th minor of the matrix A. Thus, μ is the determinant of the $(\ell-1)\times(\ell-1)$-matrix obtained from A by removing the i-th row and the j-th column (by Definition 5.9 **(a)**). The combinatorial definition of a determinant (i.e., the definition of a determinant as a sum over all permutations) thus shows that μ is a sum of $(\ell-1)!$ many products of the form

$$\pm h_{w(i_1,j_1)} h_{w(i_2,j_2)} \cdots h_{w(i_{\ell-1},j_{\ell-1})},$$

where $i_1,i_2,\ldots,i_{\ell-1}$ are $\ell-1$ distinct elements of the set $\{1,2,\ldots,\ell\}\setminus\{i\}$ and where $j_1,j_2,\ldots,j_{\ell-1}$ are $\ell-1$ distinct elements of the set $\{1,2,\ldots,\ell\}\setminus\{j\}$. Let us refer to such products as *diagonal products*. Hence, μ is a sum of diagonal products.

We shall now claim the following:

> *Claim 1:* Each diagonal product is a homogeneous symmetric polynomial of degree $|\lambda| - (\lambda_i - i + j)$.

[*Proof of Claim 1:* Let d be a diagonal product. We must show that d is a homogeneous symmetric polynomial of degree $|\lambda| - (\lambda_i - i + j)$.

We have assumed that d is a diagonal product. In other words, d is a product of the form

$$\pm h_{w(i_1,j_1)} h_{w(i_2,j_2)} \cdots h_{w(i_{\ell-1},j_{\ell-1})},$$

where $i_1,i_2,\ldots,i_{\ell-1}$ are $\ell-1$ distinct elements of the set $\{1,2,\ldots,\ell\}\setminus\{i\}$ and where $j_1,j_2,\ldots,j_{\ell-1}$ are $\ell-1$ distinct elements of the set $\{1,2,\ldots,\ell\}\setminus\{j\}$. Consider these $i_1,i_2,\ldots,i_{\ell-1}$ and these $j_1,j_2,\ldots,j_{\ell-1}$.

The numbers $i_1,i_2,\ldots,i_{\ell-1}$ are $\ell-1$ distinct elements of the set $\{1,2,\ldots,\ell\}\setminus\{i\}$; but the latter set has only $\ell-1$ elements altogether. Thus, these numbers $i_1,i_2,\ldots,i_{\ell-1}$ must be precisely the $\ell-1$ elements of the set $\{1,2,\ldots,\ell\}\setminus\{i\}$ in some order. Similarly, the numbers $j_1,j_2,\ldots,j_{\ell-1}$ must be precisely the $\ell-1$ elements of the set $\{1,2,\ldots,\ell\}\setminus\{j\}$ in some order.

For each $p \in \{1,2,\ldots,\ell-1\}$, the element $h_{w\left(i_p,j_p\right)}$ of $\mathcal{S}$ is homogeneous of degree $w\left(i_p,j_p\right)$ (because for each $m \in \mathbb{Z}$, the element h_m of $\mathcal{S}$ is homogeneous of degree m). Hence, the product $h_{w(i_1,j_1)} h_{w(i_2,j_2)} \cdots h_{w(i_{\ell-1},j_{\ell-1})}$ is homogeneous of degree

$$\begin{aligned}
& w\left(i_1,j_1\right) + w\left(i_2,j_2\right) + \cdots + w\left(i_{\ell-1},j_{\ell-1}\right) \\
&= \sum_{p\in\{1,2,\ldots,\ell-1\}} \underbrace{w\left(i_p,j_p\right)}_{\substack{=\lambda_{i_p}-i_p+j_p \\ \text{(by the definition of } w(i_p,j_p))}} = \sum_{p\in\{1,2,\ldots,\ell-1\}} \left(\lambda_{i_p} - i_p + j_p\right) \\
&= \underbrace{\sum_{p\in\{1,2,\ldots,\ell-1\}} \left(\lambda_{i_p}-i_p\right)}_{\substack{=(\lambda_{i_1}-i_1)+(\lambda_{i_2}-i_2)+\cdots+(\lambda_{i_{\ell-1}}-i_{\ell-1}) \\ = \sum_{u\in\{1,2,\ldots,\ell\}\setminus\{i\}} (\lambda_u-u) \\ \text{(since } i_1,i_2,\ldots,i_{\ell-1} \text{ are precisely} \\ \text{the } \ell-1 \text{ elements of the set } \{1,2,\ldots,\ell\}\setminus\{i\} \\ \text{in some order)}}} + \underbrace{\sum_{p\in\{1,2,\ldots,\ell-1\}} j_p}_{\substack{=j_1+j_2+\cdots+j_{\ell-1} \\ = \sum_{u\in\{1,2,\ldots,\ell\}\setminus\{j\}} u \\ \text{(since } j_1,j_2,\ldots,j_{\ell-1} \text{ are precisely} \\ \text{the } \ell-1 \text{ elements of the set } \{1,2,\ldots,\ell\}\setminus\{j\} \\ \text{in some order)}}} \\
&= \underbrace{\sum_{u\in\{1,2,\ldots,\ell\}\setminus\{i\}}}_{\substack{= \sum_{u\in\{1,2,\ldots,\ell\};\\ u\neq i}}} \left(\lambda_u - u\right) + \underbrace{\sum_{u\in\{1,2,\ldots,\ell\}\setminus\{j\}}}_{\substack{= \sum_{u\in\{1,2,\ldots,\ell\};\\ u\neq j}}} u \\
&= \sum_{\substack{u\in\{1,2,\ldots,\ell\};\\ u\neq i}} \left(\lambda_u - u\right) + \sum_{\substack{u\in\{1,2,\ldots,\ell\};\\ u\neq j}} u = \left|\lambda\right| - \left(\lambda_i - i + j\right)
\end{aligned}$$

(by Lemma 5.8). Thus, d is homogeneous of degree $\left|\lambda\right| - \left(\lambda_i - i + j\right)$ as well (since $d = \pm h_{w(i_1,j_1)} h_{w(i_2,j_2)} \cdots h_{w(i_{\ell-1},j_{\ell-1})}$). Hence, d is a homogeneous symmetric polynomial of degree $\left|\lambda\right| - \left(\lambda_i - i + j\right)$ (since d is clearly a symmetric polynomial). This proves Claim 1.]

Now, μ is a sum of diagonal products; but each such diagonal product is a homogeneous symmetric polynomial of degree $\left|\lambda\right| - \left(\lambda_i - i + j\right)$ (by Claim 1). Hence, their sum μ is also a homogeneous symmetric polynomial of degree $\left|\lambda\right| - \left(\lambda_i - i + j\right)$.

Recall that μ is the (i,j)-th minor of the matrix A. Hence, the (i,j)-th cofactor of the matrix A is $(-1)^{i+j}\mu$ (by Definition 5.9 **(b)**). Thus, this cofactor is a homogeneous symmetric polynomial of degree $\left|\lambda\right| - \left(\lambda_i - i + j\right)$ (since μ is a homogeneous symmetric polynomial of degree $\left|\lambda\right| - \left(\lambda_i - i + j\right)$).

But

$$A = \left(\underbrace{h_{w(u,v)}}_{\substack{=h_{\lambda_u-u+v} \\ \text{(by (20))}}} \right)_{1\leq u\leq \ell,\ 1\leq v\leq \ell} = \left(h_{\lambda_u-u+v}\right)_{1\leq u\leq \ell,\ 1\leq v\leq \ell}. \tag{21}$$

We have shown that the (i,j)-th cofactor of the matrix A is a homogeneous symmetric polynomial of degree $\left|\lambda\right| - \left(\lambda_i - i + j\right)$. In view of (21), this rewrites as follows: The (i, j)-th cofactor of the matrix $\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq \ell,\ 1 \leq v \leq \ell}$ is a homogeneous symmetric polynomial of degree $|\lambda| - (\lambda_i - i + j)$. This proves Lemma 5.10. $\square$

Lemma 5.11. Let $\lambda \in P_k$ be a partition such that $\lambda \notin P_{k,n}$. Then,

$$s_\lambda \equiv (\text{some symmetric polynomial of degree } < |\lambda|) \operatorname{mod} I.$$

Proof of Lemma 5.11 (sketched). Write the partition λ as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. (This can be done, since $\lambda \in P_k$.) Note that $k > 0$ (since otherwise, $\lambda \in P_k$ would lead to $\lambda = \varnothing \in P_{k,n}$, which would contradict $\lambda \notin P_{k,n}$).

From $\lambda \in P_k$ and $\lambda \notin P_{k,n}$, we conclude that not all parts of the partition λ are $\leq n - k$. Thus, the first entry λ_1 of λ is $> n - k$ (since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$). But $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Thus, Proposition 5.7 **(a)** yields

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = \sum_{j=1}^{k} h_{\lambda_1 - 1 + j} \cdot C_j, \tag{22}$$

where C_j denotes the $(1, j)$-th cofactor of the $k \times k$-matrix $\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$. (Here, the last equality sign follows from (19), applied to $\ell = k$ and $R = \mathcal{S}$ and $A = \left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$ and $a_{u,v} = h_{\lambda_u - u + v}$ and $i = 1$.)

For each $j \in \{1, 2, \ldots, k\}$, we have $\lambda_1 - 1 + j \geq \lambda_1 - 1 + 1 = \lambda_1 > n - k$ and therefore

$$\begin{aligned} & h_{\lambda_1 - 1 + j} \\ & \equiv (\text{some symmetric polynomial of degree } < \lambda_1 - 1 + j) \operatorname{mod} I \end{aligned} \tag{23}$$

(by Lemma 5.4, applied to $i = \lambda_1 - 1 + j$).

For each $j \in \{1, 2, \ldots, k\}$, the polynomial C_j is the $(1, j)$-th cofactor of the matrix $\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$ (by its definition), and thus is a homogeneous symmetric polynomial of degree $|\lambda| - (\lambda_1 - 1 + j)$ (by Lemma 5.10, applied to $\ell = k$ and $i = 1$). Hence,

$$C_j = (\text{some symmetric polynomial of degree } \leq |\lambda| - (\lambda_1 - 1 + j)) \tag{24}$$

for each $j \in \{1, 2, \ldots, k\}$.

Therefore, (22) becomes

$$\begin{aligned}
s_\lambda &= \sum_{j=1}^{k} \underbrace{h_{\lambda_1-1+j}}_{\substack{\equiv(\text{some symmetric polynomial of degree } <\lambda_1-1+j) \bmod I \\ (\text{by } (23))}} \cdot \underbrace{C_j}_{\substack{=(\text{some symmetric polynomial of degree } |\lambda|-(\lambda_1-1+j)) \\ (\text{by } (24))}} \\
&\equiv \sum_{j=1}^{k} (\text{some symmetric polynomial of degree } < \lambda_1 - 1 + j) \\
&\qquad \cdot (\text{some symmetric polynomial of degree } |\lambda| - (\lambda_1 - 1 + j)) \\
&= (\text{some symmetric polynomial of degree } < |\lambda|) \bmod I.
\end{aligned}$$

This proves Lemma 5.11. $\square$

Recall Definition 5.6.

Lemma 5.12. Let $N \in \mathbb{N}$. Let $f \in \mathcal{S}$ be a symmetric polynomial of degree $< N$. Then, there exists a family $(c_\kappa)_{\kappa \in P_k;\ |\kappa|<N}$ of elements of $\mathbf{k}$ such that $f = \sum\limits_{\substack{\kappa \in P_k; \\ |\kappa|<N}} c_\kappa s_\kappa$.

Proof of Lemma 5.12. For each $d \in \mathbb{N}$, we let $\mathcal{S}_{\deg=d}$ be the d-th graded part of the graded $\mathbf{k}$-module $\mathcal{S}$. This is the $\mathbf{k}$-submodule of $\mathcal{S}$ consisting of all homogeneous elements of $\mathcal{S}$ of degree d (including the zero vector 0, which is homogeneous of every degree).

Recall that the family $(s_\lambda)_{\lambda \in P_k}$ is a graded basis of the graded $\mathbf{k}$-module $\mathcal{S}$. In other words, for each $d \in \mathbb{N}$, the family $(s_\lambda)_{\lambda \in P_k;\ |\lambda|=d}$ is a basis of the $\mathbf{k}$-submodule $\mathcal{S}_{\deg=d}$ of $\mathcal{S}$. Hence, for each $d \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{S}_{\deg=d} &= \left(\text{the } \mathbf{k}\text{-linear span of the family } (s_\lambda)_{\lambda \in P_k;\ |\lambda|=d}\right) \\
&= \sum_{\substack{\lambda \in P_k; \\ |\lambda|=d}} \mathbf{k} s_\lambda. \qquad (25)
\end{aligned}$$

The polynomial f has degree $< N$. Hence, we can write f in the form $f = \sum\limits_{d=0}^{N-1} f_d$ for some $f_0, f_1, \ldots, f_{N-1} \in \mathcal{P}$, where each f_d is a homogeneous polynomial of degree d. Consider these $f_0, f_1, \ldots, f_{N-1}$. These N polynomials $f_0, f_1, \ldots, f_{N-1}$ are the first N homogeneous components of f, and thus are symmetric (since f is symmetric); in other words, $f_0, f_1, \ldots, f_{N-1}$ are elements of $\mathcal{S}$. Thus, for each $d \in \{0, 1, \ldots, N-1\}$, the polynomial f_d is an element of $\mathcal{S}$

and is homogeneous of degree d (as we already know). In other words, for each $d \in \{0, 1, \ldots, N-1\}$, we have

$$f_d \in \mathcal{S}_{\deg=d}. \tag{26}$$

Now,

$$f = \sum_{d=0}^{N-1} \underbrace{f_d}_{\substack{\in \mathcal{S}_{\deg=d} \\ \text{(by (26))}}} \in \sum_{d=0}^{N-1} \underbrace{\mathcal{S}_{\deg=d}}_{\substack{= \sum_{\lambda \in P_k;\ |\lambda|=d} \mathbf{k} s_\lambda \\ \text{(by (25))}}} = \underbrace{\sum_{d=0}^{N-1} \sum_{\substack{\lambda \in P_k; \\ |\lambda|=d}}}_{= \sum_{\lambda \in P_k;\ |\lambda|<N}} \mathbf{k} s_\lambda = \sum_{\substack{\lambda \in P_k; \\ |\lambda|<N}} \mathbf{k} s_\lambda = \sum_{\substack{\kappa \in P_k; \\ |\kappa|<N}} \mathbf{k} s_\kappa$$

(here, we have renamed the summation index λ as κ in the sum). In other words, there exists a family $(c_\kappa)_{\kappa \in P_k;\ |\kappa|<N}$ of elements of $\mathbf{k}$ such that $f = \sum_{\substack{\kappa \in P_k; \\ |\kappa|<N}} c_\kappa s_\kappa$. This proves Lemma 5.12. $\square$

Lemma 5.13. For each $\mu \in P_k$, the element $\overline{s_\mu} \in \mathcal{S}/I$ belongs to the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$.

Proof of Lemma 5.13. Let M be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$. We thus must prove that $\overline{s_\mu} \in M$ for each $\mu \in P_k$.

We shall prove this by strong induction on $|\mu|$. Thus, we fix some $N \in \mathbb{N}$, and we assume (as induction hypothesis) that

$$\overline{s_\kappa} \in M \qquad \text{for each } \kappa \in P_k \text{ satisfying } |\kappa| < N. \tag{27}$$

Now, let $\mu \in P_k$ be such that $|\mu| = N$. We then must show that $\overline{s_\mu} \in M$.

If $\mu \in P_{k,n}$, then this is obvious (since $\overline{s_\mu}$ then belongs to the family that spans M). Thus, for the rest of this proof, we WLOG assume that $\mu \notin P_{k,n}$. Hence, Lemma 5.11 (applied to $\lambda = \mu$) yields

$$s_\mu \equiv (\text{some symmetric polynomial of degree } < |\mu|) \bmod I.$$

In other words, there exists some symmetric polynomial $f \in \mathcal{S}$ of degree $< |\mu|$ such that $s_\mu \equiv f \bmod I$. Consider this f.

The polynomial f is a symmetric polynomial of degree $< |\mu|$. In other words, f is a symmetric polynomial of degree $< N$ (since $|\mu| = N$). Hence, Lemma 5.12 shows that there exists a family $(c_\kappa)_{\kappa \in P_k;\ |\kappa|<N}$ of elements of $\mathbf{k}$ such that $f = \sum_{\substack{\kappa \in P_k; \\ |\kappa|<N}} c_\kappa s_\kappa$. Consider this family. From $f = \sum_{\substack{\kappa \in P_k; \\ |\kappa|<N}} c_\kappa s_\kappa$, we obtain

$$\overline{f} = \overline{\sum_{\substack{\kappa \in P_k; \\ |\kappa|<N}} c_\kappa s_\kappa} = \sum_{\substack{\kappa \in P_k; \\ |\kappa|<N}} c_\kappa \underbrace{\overline{s_\kappa}}_{\substack{\in M \\ \text{(by (27))}}} \in \sum_{\substack{\kappa \in P_k; \\ |\kappa|<N}} c_\kappa M \subseteq M \qquad (\text{since } M \text{ is a } \mathbf{k}\text{-module}).$$

But from $s_\mu \equiv f \bmod I$, we obtain $\overline{s_\mu} = \overline{f} \in M$. This completes our induction step. Thus, we have proven by strong induction that $\overline{s_\mu} \in M$ for each $\mu \in P_k$. This proves Lemma 5.13. $\square$

Proof of Theorem 2.7 (sketched). Proposition 2.1 yields that $(x^\alpha)_{\alpha\in\mathbb{N}^k;\ \alpha_i<i \text{ for each } i}$ is a spanning set of the $\mathcal{S}$-module $\mathcal{P}$.

Recall Definition 5.6. It is well-known that $(s_\lambda)_{\lambda\in P_k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$. Hence, $(\overline{s_\lambda})_{\lambda\in P_k}$ is a spanning set of the $\mathbf{k}$-module $\mathcal{S}/I$. Thus, $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ is also a spanning set of the $\mathbf{k}$-module $\mathcal{S}/I$ (because Lemma 5.13 shows that every element of the first spanning set belongs to the span of the second). It remains to prove that this spanning set is also a basis.

In order to do so, we consider the family $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda\in P_{k,n};\ \alpha\in\mathbb{N}^k;\ \alpha_i<i \text{ for each } i}$ in the $\mathbf{k}$-module $\mathcal{P}/J$. This family spans $\mathcal{P}/J$ (by Lemma 5.2), because the family $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ spans $\mathcal{S}/I$ whereas the family $(x^\alpha)_{\alpha\in\mathbb{N}^k;\ \alpha_i<i \text{ for each } i}$ spans $\mathcal{P}$ over $\mathcal{S}$ (and because $I\mathcal{P} = J$). Moreover, this family $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda\in P_{k,n};\ \alpha\in\mathbb{N}^k;\ \alpha_i<i \text{ for each } i}$ has size

$$\underbrace{\left|P_{k,n}\right|}_{=\binom{n}{k}} \cdot \underbrace{\left|\left\{\alpha \in \mathbb{N}^k \ \mid\ \alpha_i < i \text{ for each } i\right\}\right|}_{=k!} = \binom{n}{k} \cdot k!$$

$$= n\left(n-1\right)\cdots\left(n-k+1\right),$$

which is exactly the size of the basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^k;\ \alpha_i<n-k+i \text{ for each } i}$ of the $\mathbf{k}$-module $\mathcal{P}/J$ (this is a basis by Theorem 2.2). Thus, this family $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda\in P_{k,n};\ \alpha\in\mathbb{N}^k;\ \alpha_i<i \text{ for each } i}$ must be a basis of the $\mathbf{k}$-module $\mathcal{P}/J$ (by Lemma 5.3), and hence is $\mathbf{k}$-linearly independent. Thus, its subfamily $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ is also $\mathbf{k}$-linearly independent.

The canonical $\mathbf{k}$-linear map $\mathcal{S}/I \to \mathcal{P}/J$ (obtained as a quotient of the inclusion $\mathcal{S} \to \mathcal{P}$) is injective (because it sends the spanning set $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ of $\mathcal{S}/I$ to the $\mathbf{k}$-linearly independent family $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ in $\mathcal{P}/J$). Hence, the $\mathbf{k}$-linear independency of the family $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ in $\mathcal{P}/J$ yields the $\mathbf{k}$-linear independency of the family $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ in $\mathcal{S}/I$. Thus, the family $(\overline{s_\lambda})_{\lambda\in P_{k,n}}$ in $\mathcal{S}/I$ is a basis of $\mathcal{S}/I$ (since it is $\mathbf{k}$-linearly independent and spans $\mathcal{S}/I$). This proves Theorem 2.7. $\square$

6. Symmetry of the multiplicative structure constants

> **Convention 6.1.** For the rest of Section 6, **we assume that** $a_1, a_2, \ldots, a_k$ **belong to k.**
>
> If $m \in \mathcal{S}$, then the notation $\overline{m}$ shall always mean the projection of $m \in \mathcal{S}$ onto the quotient $\mathcal{S}/I$ (and not the projection of $m \in \mathcal{P}$ onto the quotient

$\mathcal{P}/J$).

Definition 6.2. **(a)** Let ω be the partition $(n-k, n-k, \ldots, n-k)$ with k entries equal to $n-k$. (This is the largest partition in $P_{k,n}$.)

(b) Let I be the ideal of $\mathcal{S}$ generated by the k differences (2). For each $\mu \in P_{k,n}$, let $\operatorname{coeff}_\mu : \mathcal{S}/I \to \mathbf{k}$ be the $\mathbf{k}$-linear map that sends $\overline{s_\mu}$ to 1 while sending all other $\overline{s_\lambda}$ (with $\lambda \in P_{k,n}$) to 0. (This is well-defined by Theorem 2.7. Actually, $\left(\operatorname{coeff}_\mu\right)_{\mu \in P_{k,n}}$ is the dual basis to the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ of $\mathcal{S}/I$.)

(c) If λ is any partition and if p is a positive integer, then λ_p shall always denote the p-th entry of λ. Thus, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ for every partition λ.

(d) For every partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we let $\nu^\vee$ denote the partition $(n-k-\nu_k, n-k-\nu_{k-1}, \ldots, n-k-\nu_1) \in P_{k,n}$. This partition $\nu^\vee$ is called the *complement* of ν.

We can now make a more substantial claim:

Theorem 6.3. Each $\nu \in P_{k,n}$ and $f \in \mathcal{S}/I$ satisfy $\operatorname{coeff}_\omega\left(\overline{s_\nu} f\right) = \operatorname{coeff}_{\nu^\vee}(f)$.

The proof of this theorem requires some preliminary work.

We first recall some basic notations from [GriRei20, Chapter 2]. If λ and μ are two partitions, then we say that $\mu \subseteq \lambda$ if and only if each positive integer p satisfies $\mu_p \leq \lambda_p$. A *skew partition* means a pair (λ, μ) of two partitions satisfying $\mu \subseteq \lambda$; such a pair is denoted by λ/μ. We refer to [GriRei20, §2.7] for the definition of a *vertical i-strip* (where $i \in \mathbb{N}$).

Let Λ be the ring of symmetric functions in infinitely many indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. If $\mathbf{f} \in \Lambda$ is a symmetric function, then $\mathbf{f}(x_1, x_2, \ldots, x_k)$ is a symmetric polynomial in $\mathcal{S}$; the map

$$\Lambda \to \mathcal{S}, \qquad \mathbf{f} \mapsto \mathbf{f}(x_1, x_2, \ldots, x_k)$$

is a surjective $\mathbf{k}$-algebra homomorphism. We shall use boldfaced notations for symmetric functions in Λ in order to distinguish them from symmetric polynomials in $\mathcal{S}$. In particular:

- For any $i \in \mathbb{Z}$, we let $\mathbf{h}_i$ be the i-th complete homogeneous symmetric function in Λ. (This is called h_i in [GriRei20, Definition 2.2.1].)
- For any $i \in \mathbb{Z}$, we let $\mathbf{e}_i$ be the i-th elementary symmetric function in Λ. (This is called e_i in [GriRei20, Definition 2.2.1].)
- For any partition λ, we let $\mathbf{e}_\lambda$ be the corresponding elementary symmetric function in Λ. (This is called e_λ in [GriRei20, Definition 2.2.1].)
- For any partition λ, we let $\mathbf{s}_\lambda$ be the corresponding Schur function in Λ. (This is called s_λ in [GriRei20, Definition 2.2.1].)

- For any partitions λ and μ, we let $\mathbf{s}_{\lambda/\mu}$ be the corresponding skew Schur function in Λ. (This is called $s_{\lambda/\mu}$ in [GriRei20, §2.3]. Note that $\mathbf{s}_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$.)

Also, we shall use the *skewing operators* as defined (e.g.) in [GriRei20, §2.8]. We recall their main properties:

- For each $\mathbf{f} \in \Lambda$, the skewing operator $\mathbf{f}^{\perp}$ is a $\mathbf{k}$-linear map $\Lambda \to \Lambda$. It depends $\mathbf{k}$-linearly on $\mathbf{f}$ (that is, we have $(\alpha \mathbf{f} + \beta \mathbf{g})^{\perp} = \alpha \mathbf{f}^{\perp} + \beta \mathbf{g}^{\perp}$ for any $\alpha, \beta \in \mathbf{k}$ and $\mathbf{f}, \mathbf{g} \in \Lambda$).
- For any partitions λ and μ, we have
$$\left(\mathbf{s}_{\mu}\right)^{\perp}\left(\mathbf{s}_{\lambda}\right) = \mathbf{s}_{\lambda/\mu}. \tag{28}$$
(This is [GriRei20, (2.8.2)].)
- For any $\mathbf{f}, \mathbf{g} \in \Lambda$, we have
$$(\mathbf{f}\mathbf{g})^{\perp} = \mathbf{g}^{\perp} \circ \mathbf{f}^{\perp}. \tag{29}$$
(This is [GriRei20, Proposition 2.8.2(ii)], applied to $A = \Lambda$.)
- We have $1^{\perp} = \mathrm{id}$.

For each partition λ, let λ^{t} denote the *conjugate partition* of λ; see [GriRei20, Definition 2.2.8] for its definition.

Recall the second Jacobi-Trudi identity ([GriRei20, (2.4.17)]):

Proposition 6.4. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be two partitions. Then,
$$\mathbf{s}_{\lambda^t/\mu^t} = \det\left(\left(\mathbf{e}_{\lambda_i - \mu_j - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right).$$

Corollary 6.5. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. Then,
$$\mathbf{s}_{\lambda^t} = \det\left(\left(\mathbf{e}_{\lambda_i - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right).$$

Proof of Corollary 6.5. This follows from Proposition 6.4, applied to $\mu = \varnothing$ (since $\varnothing^t = \varnothing$ and thus $\mathbf{s}_{\lambda^t/\varnothing^t} = \mathbf{s}_{\lambda^t/\varnothing} = \mathbf{s}_{\lambda^t}$). ☐

We also recall one of the Pieri rules ([GriRei20, (2.7.2)]):

Proposition 6.6. Let λ be a partition, and let $i \in \mathbb{N}$. Then,

$$\mathbf{s}_\lambda \mathbf{e}_i = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .$$

From this, we can easily derive the following:

Corollary 6.7. Let λ be a partition, and let $i \in \mathbb{N}$. Then,

$$(\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .$$

Corollary 6.7 is also proven in [GriRei20, (2.8.4)].

The next proposition is the claim of [GriRei20, Exercise 2.9.1(b)]:

Proposition 6.8. Let λ be a partition. Let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Then,

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{m+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \mathbf{s}_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)} .$$

We shall use this to derive the following corollary:

Corollary 6.9. Let λ be a partition with at most k parts. Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then,

$$\mathbf{s}_\lambda = \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .$$

Proof of Corollary 6.9. The partition $\overline{\lambda}$ is obtained from λ by removing the first part. Hence, this partition $\overline{\lambda}$ has at most $k-1$ parts (since λ has at most k parts). Thus, if $i \in \mathbb{N}$ satisfies $i \geq k$, then

$$\text{there exists no partition } \mu \text{ such that } \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip.} \tag{30}$$

We have $\overline{\lambda} = (\lambda_2, \lambda_3, \lambda_4, \ldots)$, so that $(\lambda_2, \lambda_3, \lambda_4, \ldots) = \overline{\lambda} = \left(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots\right)$. Hence,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots) = \left(\lambda_1, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots\right).$$

Also, clearly, $\lambda_1 \geq \overline{\lambda}_1$ (since $\lambda_1 \geq \lambda_2 = \overline{\lambda}_1$). Hence, Proposition 6.8 (applied to $\overline{\lambda}$ and λ_1 instead of λ and m) yields

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{\lambda_1+i} (\mathbf{e}_i)^\perp \mathbf{s}_{\overline{\lambda}} = \mathbf{s}_{\left(\lambda_1,\overline{\lambda}_1,\overline{\lambda}_2,\overline{\lambda}_3,\ldots\right)} = \mathbf{s}_\lambda$$

(since $\left(\lambda_1, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \ldots\right) = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots\right) = \lambda$). Therefore,

$$\begin{aligned}
\mathbf{s}_\lambda &= \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{\lambda_1+i} \underbrace{(\mathbf{e}_i)^\perp \mathbf{s}_{\overline{\lambda}}}_{\substack{= \sum\limits_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\ \text{(by Corollary 6.7)}}} = \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu + \sum_{i \geq k} (-1)^i \mathbf{h}_{\lambda_1+i} \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu}_{\substack{=0 \\ \text{(by (30))}}} \\
&= \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .
\end{aligned}$$

This proves Corollary 6.9. ☐

Convention 6.10. We WLOG assume that $k > 0$ for the rest of Section 6 (since otherwise, Theorem 6.3 is trivial).

Next, we define a filtration on the $\mathbf{k}$-module $\mathcal{S}/I$:

Definition 6.11. For each $p \in \mathbb{Z}$, we let Q_p denote the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$.

Thus, $0 = Q_{-1} \subseteq Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots$. Theorem 2.7 shows that the $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$; hence, $\mathcal{S}/I = Q_{n-k}$ (since each $\lambda \in P_{k,n}$ satisfies $\lambda_k \leq n-k$).

Note that $(Q_0, Q_1, Q_2, \ldots)$ is a filtration of the $\mathbf{k}$-module $\mathcal{S}/I$, but not (in general) of the $\mathbf{k}$-algebra $\mathcal{S}/I$.

Lemma 6.12. We have $\operatorname{coeff}_\omega \left(Q_{n-k-1}\right) = 0$.

Proof of Lemma 6.12. The map $\operatorname{coeff}_\omega$ is $\mathbf{k}$-linear; thus, it suffices to prove that $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$ for each $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq n-k-1$ (because the $\mathbf{k}$-module Q_{n-k-1} is spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq n-k-1$). So let us fix some $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq n-k-1$. We must then prove that $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$.

We have $\lambda_k \leq n-k-1 < n-k = \omega_k$. Thus, $\lambda_k \neq \omega_k$, so that $\lambda \neq \omega$.

The definition of the map $\operatorname{coeff}_\omega$ yields $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = \begin{cases} 1, & \text{if } \lambda = \omega; \\ 0, & \text{if } \lambda \neq \omega \end{cases} = 0$ (since $\lambda \neq \omega$). This completes our proof of Lemma 6.12. ☐

Lemma 6.13. Let λ be a partition with at most k parts. Assume that $\lambda_1 = n-k+1$. Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then,

$$\overline{s_\lambda} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Proof of Lemma 6.13. Corollary 6.9 yields

$$\mathbf{s}_\lambda = \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$$

This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$\begin{aligned} s_\lambda &= \sum_{i=0}^{k-1} (-1)^i \underbrace{h_{\lambda_1+i}}_{\substack{=h_{n-k+1+i} \\ (\text{since } \lambda_1 = n-k+1)}} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \\ &= \sum_{i=0}^{k-1} (-1)^i \underbrace{h_{n-k+1+i}}_{\substack{\equiv a_{1+i} \operatorname{mod} I \\ (\text{by (15)})}} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \\ &\equiv \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \operatorname{mod} I. \end{aligned}$$

Projecting both sides of this equality from $\mathcal{S}$ to $\mathcal{S}/I$, we obtain

$$\overline{s_\lambda} = \overline{\sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} s_\mu} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

This proves Lemma 6.13. ☐

Lemma 6.14. Let λ be a partition with at most k parts. Assume that $\lambda_1 = n-k+1$. Then, $\overline{s_\lambda} \in \mathcal{Q}_0$.

Proof of Lemma 6.14. We shall prove Lemma 6.14 by strong induction on $|\lambda|$. Thus, we fix some $N \in \mathbb{N}$, and we assume (as induction hypothesis) that Lemma 6.14 is already proven whenever $|\lambda| < N$. We now must prove Lemma 6.14 in the case when $|\lambda| = N$.

So let λ be as in Lemma 6.14, and assume that $|\lambda| = N$. Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then, Lemma 6.13 yields

$$\overline{s_{\overline{\lambda}}} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}. \tag{31}$$

But if μ is a partition such that $\overline{\lambda}/\mu$ is a vertical i-strip, then

$$\overline{s_\mu} \in Q_0. \tag{32}$$

[*Proof of (32):* The partition λ has at most k parts; thus, the partition $\overline{\lambda}$ has at most $k-1$ parts.

Now, let μ be a partition such that $\overline{\lambda}/\mu$ is a vertical i-strip. Then, $\mu \subseteq \overline{\lambda}$, so that μ has at most $k-1$ parts (since $\overline{\lambda}$ has at most $k-1$ parts). Thus, $\mu_k = 0 \leq 0$. Also, μ has at most k parts (since μ has at most $k-1$ parts). If $\mu_1 \leq n-k$, then this yields that $\mu \in P_{k,n}$ and therefore $\overline{s_\mu} \in Q_0$ (since $\mu \in P_{k,n}$ and $\mu_k \leq 0$). Thus, (32) is proven if $\mu_1 \leq n-k$. Hence, for the rest of this proof, we WLOG assume that we don't have $\mu_1 \leq n-k$. Hence, $\mu_1 > n-k$.

But $\mu \subseteq \overline{\lambda}$, so that $\mu_1 \leq \overline{\lambda}_1 = \lambda_2 \leq \lambda_1 = n-k+1$. Combining this with $\mu_1 > n-k$, we obtain $\mu_1 = n-k+1$. Also, $\mu \subseteq \overline{\lambda}$, so that

$$|\mu| \leq \left|\overline{\lambda}\right| = |\lambda| - \underbrace{\lambda_1}_{=n-k+1 \geq 1 > 0} < |\lambda| = N.$$

Hence, we can apply Lemma 6.14 to μ instead of λ (by the induction hypothesis). We thus obtain $\overline{s_\mu} \in Q_0$. This completes the proof of (32).]

Now, (31) becomes

$$\overline{s_{\overline{\lambda}}} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \underbrace{\overline{s_\mu}}_{\substack{\in Q_0 \\ \text{(by (32))}}} \in Q_0.$$

Thus, we have proven Lemma 6.14 for our λ. This completes the induction step; thus, Lemma 6.14 is proven. ☐

Lemma 6.15. Let $i \in \mathbb{N}$ and $\lambda \in P_{k,n}$. Then,

$$\overline{e_i s_\lambda} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \operatorname{mod} Q_0.$$

Proof of Lemma 6.15. If μ is a partition such that μ/λ is a vertical i-strip and $\mu \notin P_{k,n}$, then

$$\overline{s_\mu} \equiv 0 \operatorname{mod} Q_0. \tag{33}$$

[*Proof of (33):* Let μ be a partition such that μ/λ is a vertical i-strip and $\mu \notin P_{k,n}$. We must prove (33).

If the partition μ has more than k parts, then (33) easily follows[11]. Hence, for the rest of this proof, we WLOG assume that the partition μ has at most k parts.

Since μ/λ is a vertical strip, we have $\mu_1 \leq \lambda_1 + 1$. But $\lambda_1 \leq n-k$ (since $\lambda \in P_{k,n}$). If $\mu_1 = n-k+1$, then (33) easily follows[12]. Hence, for the rest of this proof, we WLOG assume that $\mu_1 \neq n-k+1$. Combining this with $\mu_1 \leq \underbrace{\lambda_1}_{\leq n-k} +1 \leq n-k+1$, we obtain $\mu_1 < n-k+1$, so that $\mu_1 \leq n-k$. Hence, $\mu \in P_{k,n}$ (since μ has at most k parts). This contradicts $\mu \notin P_{k,n}$. Thus, $\overline{s_\mu} \equiv 0 \mod Q_0$ (because *ex falso quodlibet*). Hence, (33) is proven.]

Proposition 6.6 yields

$$\mathbf{s}_\lambda \mathbf{e}_i = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu .$$

This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$s_\lambda e_i = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} s_\mu .$$

Projecting both sides of this equality from $\mathcal{S}$ to $\mathcal{S}/I$, we obtain

$$\begin{aligned}
\overline{s_\lambda e_i} &= \overline{\sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} s_\mu} = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \\
&= \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip;} \\ \mu \in P_{k,n}}} \overline{s_\mu} + \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip;} \\ \mu \notin P_{k,n}}} \underbrace{\overline{s_\mu}}_{\substack{\equiv 0 \mod Q_0 \\ \text{(by (33))}}} \\
&\equiv \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip;} \\ \mu \in P_{k,n}}} \overline{s_\mu} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \mod Q_0 .
\end{aligned}$$

Thus, $\overline{e_i s_\lambda} = \overline{s_\lambda e_i} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \mod Q_0$. This proves Lemma 6.15. $\square$

Lemma 6.16. Let $i \in \mathbb{Z}$ and $p \in \mathbb{Z}$. Then, $\overline{e_i} Q_p \subseteq Q_{p+1}$.

Proof of Lemma 6.16. Due to the definition of Q_p, it suffices to prove that every $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$ satisfies $\overline{e_i s_\lambda} \in Q_{p+1}$. So let us fix $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$. We must prove that $\overline{e_i s_\lambda} \in Q_{p+1}$.

---

[11] *Proof.* Assume that the partition μ has more than k parts. Thus, (3) (applied to μ instead of λ) yields $s_\mu = 0$. Thus, $\overline{s_\mu} = 0 \equiv 0 \mod Q_0$. Thus, (33) holds.

[12] *Proof.* Assume that $\mu_1 = n-k+1$. Then, Lemma 6.14 (applied to μ instead of λ) yields $\overline{s_\mu} \in Q_0$. Hence, $\overline{s_\mu} \equiv 0 \mod Q_0$. Thus, (33) holds.

From $\lambda_k \geq 0$, we obtain $0 \leq \lambda_k \leq p \leq p+1$.

We WLOG assume that $i \in \mathbb{N}$ (since otherwise, we have $e_i = 0$ and thus $\overline{e_i s_\lambda} = \overline{0 s_\lambda} = 0 \in Q_{p+1}$).

If $\mu \in P_{k,n}$ is such that μ / λ is a vertical i-strip, then

$$\overline{s_\mu} \equiv 0 \operatorname{mod} Q_{p+1}. \tag{34}$$

[*Proof of (34):* Let $\mu \in P_{k,n}$ be such that μ / λ is a vertical i-strip. We must prove (34).

Since μ / λ is a vertical strip, we have $\mu_k \leq \underbrace{\lambda_k}_{\leq p} + 1 \leq p+1$. From $\mu \in P_{k,n}$ and $\mu_k \leq p+1$, we obtain $\overline{s_\mu} \in Q_{p+1}$. In other words, $\overline{s_\mu} \equiv 0 \operatorname{mod} Q_{p+1}$. Thus, (34) is proven.]

Lemma 6.15 yields

$$\overline{e_i s_\lambda} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \operatorname{mod} Q_0.$$

Hence,

$$\overline{e_i s_\lambda} - \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \in Q_0 \subseteq Q_{p+1} \qquad (\text{since } 0 \leq p+1).$$

Thus,

$$\overline{e_i s_\lambda} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \underbrace{\overline{s_\mu}}_{\substack{\equiv 0 \operatorname{mod} Q_{p+1} \\ \text{(by (34))}}} \equiv 0 \operatorname{mod} Q_{p+1}.$$

In other words, $\overline{e_i s_\lambda} \in Q_{p+1}$. This completes our proof of Lemma 6.16. ☐

The next fact that we use from the theory of symmetric functions are some basic properties of the Littlewood-Richardson coefficients. For any partitions λ, μ, ν, we let $c^\lambda_{\mu,\nu}$ be the Littlewood-Richardson coefficient as defined in [GriRei20, Definition 2.5.8]. Then, we have the following fact (part of [GriRei20, Remark 2.5.9]):

> **Proposition 6.17.** Let λ and μ be two partitions.
>
> **(a)** We have
>
> $$\mathbf{s}_{\lambda/\mu} = \sum_{\nu \text{ is a partition}} c^\lambda_{\mu,\nu} \mathbf{s}_\nu.$$
>
> **(b)** If ν is a partition, then $c^\lambda_{\mu,\nu} = 0$ unless $\nu \subseteq \lambda$.
>
> **(c)** If ν is a partition, then $c^\lambda_{\mu,\nu} = 0$ unless $|\mu| + |\nu| = |\lambda|$.

Next, let $\mathcal{Z}$ be the $\mathbf{k}$-submodule of Λ spanned by the $\mathbf{s}_\lambda$ with $\lambda \in P_{k,n}$. Then, $(\mathbf{s}_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{Z}$ (since $(\mathbf{s}_\lambda)_{\lambda \text{ is a partition}}$ is a basis of the $\mathbf{k}$-module Λ). We thus can define a $\mathbf{k}$-linear map $\delta : \mathcal{Z} \to \mathcal{S}/I$ by setting

$$\delta(\mathbf{s}_\lambda) = \overline{s_{\lambda^\vee}} \qquad \text{for every } \lambda \in P_{k,n}.$$

Notice that a partition λ satisfies $\lambda \in P_{k,n}$ if and only if $\lambda \subseteq \omega$.

Lemma 6.18. We have $\mathbf{f}^\perp(\mathcal{Z}) \subseteq \mathcal{Z}$ for each $\mathbf{f} \in \Lambda$.

Proof of Lemma 6.18. Since $\mathbf{f}^\perp$ depends $\mathbf{k}$-linearly on $\mathbf{f}$, it suffices to check that $(\mathbf{s}_\mu)^\perp(\mathcal{Z}) \subseteq \mathcal{Z}$ for each partition μ. So let us fix a partition μ; we then must prove that $(\mathbf{s}_\mu)^\perp(\mathcal{Z}) \subseteq \mathcal{Z}$.

Recall that $\mathcal{Z}$ is the $\mathbf{k}$-module spanned by the $\mathbf{s}_\lambda$ with $\lambda \in P_{k,n}$. Hence, in order to prove that $(\mathbf{s}_\mu)^\perp(\mathcal{Z}) \subseteq \mathcal{Z}$, it suffices to check that $(\mathbf{s}_\mu)^\perp(\mathbf{s}_\lambda) \in \mathcal{Z}$ for each $\lambda \in P_{k,n}$. So let us fix $\lambda \in P_{k,n}$; we must then prove that $(\mathbf{s}_\mu)^\perp(\mathbf{s}_\lambda) \in \mathcal{Z}$.

From (28), we obtain

$$\begin{aligned}(\mathbf{s}_\mu)^\perp(\mathbf{s}_\lambda) = \mathbf{s}_{\lambda/\mu} &= \sum_{\nu \text{ is a partition}} c^\lambda_{\mu,\nu}\mathbf{s}_\nu \qquad (\text{by Proposition 6.17 } \textbf{(a)})\\ &= \sum_{\substack{\nu \text{ is a partition;}\\ \nu\subseteq\lambda}} c^\lambda_{\mu,\nu}\mathbf{s}_\nu + \sum_{\substack{\nu \text{ is a partition;}\\ \text{not } \nu\subseteq\lambda}} \underbrace{c^\lambda_{\mu,\nu}}_{\substack{=0\\ (\text{by Proposition 6.17 } \textbf{(b)})}} \mathbf{s}_\nu\\ &= \sum_{\substack{\nu \text{ is a partition;}\\ \nu\subseteq\lambda}} c^\lambda_{\mu,\nu} \underbrace{\mathbf{s}_\nu}_{\substack{\in\mathcal{Z}\\ (\text{because } \nu\subseteq\lambda\\ \text{and } \lambda\in P_{k,n} \text{ lead to}\\ \nu\in P_{k,n})}} \in \mathcal{Z}.\end{aligned}$$

This completes our proof of Lemma 6.18. □

Lemma 6.19. Let $i \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. Then,

$$\delta\left((\mathbf{e}_i)^\perp \mathbf{f}\right) \equiv \overline{e_i}\delta(\mathbf{f}) \bmod Q_0.$$

(Note that $\delta\left((\mathbf{e}_i)^\perp \mathbf{f}\right)$ is well-defined, since Lemma 6.18 yields $(\mathbf{e}_i)^\perp \mathbf{f} \in \mathcal{Z}$.)

Proof of Lemma 6.19. Both sides of the claim are $\mathbf{k}$-linear in $\mathbf{f}$. Hence, we can WLOG assume that $\mathbf{f} = \mathbf{s}_\lambda$ for some $\lambda \in P_{k,n}$ (since $(\mathbf{s}_\lambda)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{Z}$). Assume this, and consider this λ.

We must prove that $\delta\left((\mathbf{e}_i)^\perp \mathbf{f}\right) \equiv \overline{e_i}\delta(\mathbf{f}) \bmod Q_0$. If $i < 0$, then this is obvious (because if $i < 0$, then both $\mathbf{e}_i$ and e_i equal 0, and therefore both sides of the congruence $\delta\left((\mathbf{e}_i)^\perp \mathbf{f}\right) \equiv \overline{e_i}\delta(\mathbf{f}) \bmod Q_0$ are equal to 0). Hence, for the rest of this proof, we WLOG assume that we don't have $i < 0$. Thus, $i \geq 0$, so that $i \in \mathbb{N}$.

It is easy to see that if $\mu \in P_{k,n}$, then we have the following equivalence of statements:

$$\left(\lambda / \mu \text{ is a vertical } i\text{-strip}\right) \iff \left(\mu^\vee / \lambda^\vee \text{ is a vertical } i\text{-strip}\right). \tag{35}$$

(Indeed, the skew Young diagram of $\mu^\vee / \lambda^\vee$ is obtained from the skew Young diagram of λ / μ by a rotation by 180°.)

From $\mathbf{f} = \mathbf{s}_\lambda$, we obtain

$$\begin{aligned}\left(\mathbf{e}_i\right)^\perp \mathbf{f} = \left(\mathbf{e}_i\right)^\perp \mathbf{s}_\lambda &= \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \qquad \text{(by Corollary 6.7)} \\ &= \sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu\end{aligned}$$

(because if μ is a partition such that λ / μ is a vertical i-strip, then $\mu \in P_{k,n}$ (since $\mu \subseteq \lambda$ and $\lambda \in P_{k,n}$)). Applying the map δ to both sides of this equality, we find

$$\begin{aligned}\delta\left(\left(\mathbf{e}_i\right)^\perp \mathbf{f}\right) = \delta\left(\sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu\right) &= \underbrace{\sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}}}_{\substack{= \sum\limits_{\substack{\mu \in P_{k,n}; \\ \mu^\vee/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \\ \text{(by (35))}}} \underbrace{\delta\left(\mathbf{s}_\mu\right)}_{\substack{= \overline{s_{\mu^\vee}} \\ \text{(by the definition of } \delta\text{)}}} \\ &= \sum_{\substack{\mu \in P_{k,n}; \\ \mu^\vee/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_{\mu^\vee}} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}\end{aligned} \tag{36}$$

(here, we have substituted μ for $\mu^\vee$ in the sum, since the map $P_{k,n} \to P_{k,n}$, $\mu \mapsto \mu^\vee$ is a bijection).

On the other hand, from $\mathbf{f} = \mathbf{s}_\lambda$, we obtain $\delta\left(\mathbf{f}\right) = \delta\left(\mathbf{s}_\lambda\right) = \overline{s_{\lambda^\vee}}$ (by the definition of δ) and thus

$$\overline{e_i} \delta\left(\mathbf{f}\right) = \overline{e_i s_{\lambda^\vee}} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda^\vee \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \bmod Q_0 \qquad \left(\begin{array}{c}\text{by Lemma 6.15, applied} \\ \text{to } \lambda^\vee \text{ instead of } \lambda\end{array}\right).$$

Comparing this with (36), we obtain $\delta\left(\left(\mathbf{e}_i\right)^\perp \mathbf{f}\right) \equiv \overline{e_i} \delta\left(\mathbf{f}\right) \bmod Q_0$. This proves Lemma 6.19. $\square$

Lemma 6.20. Let $p \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_p \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. Then,

$$\delta \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_p} \right)^{\perp} \mathbf{f} \right) \equiv \overline{e_{i_1} e_{i_2} \cdots e_{i_p}} \delta \left(\mathbf{f} \right) \operatorname{mod} Q_{p-1}.$$

Proof of Lemma 6.20. We proceed by induction on p.

The *induction base* (the case $p = 0$) is obvious (since $1^{\perp} = \operatorname{id}$ and thus $1^{\perp} \mathbf{f} = \mathbf{f}$).

Induction step: Let $q \in \mathbb{N}$. Assume (as the induction hypothesis) that Lemma 6.20 holds for $p = q$. We must now prove that Lemma 6.20 holds for $p = q + 1$. In other words, we must prove that every $i_1, i_2, \ldots, i_{q+1} \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$ satisfy

$$\delta \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_{q+1}} \right)^{\perp} \mathbf{f} \right) \equiv \overline{e_{i_1} e_{i_2} \cdots e_{i_{q+1}}} \delta \left(\mathbf{f} \right) \operatorname{mod} Q_q. \tag{37}$$

So let $i_1, i_2, \ldots, i_{q+1} \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. We must prove (37).

Lemma 6.16 (applied to i_{q+1} and $q - 1$ instead of i and p) yields $\overline{e_{i_{q+1}}} Q_{q-1} \subseteq Q_q$.

The induction hypothesis yields

$$\delta \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} \mathbf{f} \right) \equiv \overline{e_{i_1} e_{i_2} \cdots e_{i_q}} \delta \left(\mathbf{f} \right) \operatorname{mod} Q_{q-1}.$$

Multiplying both sides of this congruence by $\overline{e_{i_{q+1}}}$, we obtain

$$\overline{e_{i_{q+1}}} \delta \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} \mathbf{f} \right) \equiv \overline{e_{i_{q+1}} e_{i_1} e_{i_2} \cdots e_{i_q}} \delta \left(\mathbf{f} \right) \operatorname{mod} Q_q \tag{38}$$

(since $\overline{e_{i_{q+1}}} Q_{q-1} \subseteq Q_q$).

Applying Lemma 6.18 to $\mathbf{f} = \mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q}$, we obtain $\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} (\mathcal{Z}) \subseteq \mathcal{Z}$. Hence, $\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} \mathbf{f} \in \mathcal{Z}$ (since $\mathbf{f} \in \mathcal{Z}$).

But (29) (applied to $\mathbf{f} = \mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q}$ and $\mathbf{g} = \mathbf{e}_{i_{q+1}}$) yields

$$\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_{q+1}} \right)^{\perp} = \left(\mathbf{e}_{i_{q+1}} \right)^{\perp} \circ \left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp}.$$

Hence,

$$\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_{q+1}} \right)^{\perp} \mathbf{f} = \left(\left(\mathbf{e}_{i_{q+1}} \right)^{\perp} \circ \left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} \right) \mathbf{f} = \left(\mathbf{e}_{i_{q+1}} \right)^{\perp} \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} \mathbf{f} \right).$$

Applying the map δ to both sides of this equality, we find

$$\begin{aligned} \delta \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_{q+1}} \right)^{\perp} \mathbf{f} \right) &= \delta \left(\left(\mathbf{e}_{i_{q+1}} \right)^{\perp} \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} \mathbf{f} \right) \right) \\ &\equiv \overline{e_{i_{q+1}}} \delta \left(\left(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_q} \right)^{\perp} \mathbf{f} \right) \operatorname{mod} Q_0 \end{aligned}$$

(by Lemma 6.19, applied to i_{q+1} and $\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}$ instead of i and $\mathbf{f}$). Since $Q_0 \subseteq Q_q$, this yields

$$\begin{aligned}\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{q+1}}\right)^{\perp}\mathbf{f}\right) &\equiv \overline{e_{i_{q+1}}}\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_q}\right)^{\perp}\mathbf{f}\right)\\ &\equiv \overline{e_{i_{q+1}}e_{i_1}e_{i_2}\cdots e_{i_q}}\delta\left(\mathbf{f}\right) \qquad \text{(by (38))}\\ &= \overline{e_{i_1}e_{i_2}\cdots e_{i_{q+1}}}\delta\left(\mathbf{f}\right) \bmod Q_q.\end{aligned}$$

Thus, (37) is proven. This completes the induction step. Thus, Lemma 6.20 is proven. □

Lemma 6.21. Let $\lambda \in P_{k,n}$ and $\mathbf{f} \in \mathcal{Z}$. Then,

$$\delta\left(\left(\mathbf{s}_\lambda\right)^{\perp}\mathbf{f}\right) \equiv \overline{s_\lambda}\delta\left(\mathbf{f}\right) \bmod Q_{n-k-1}.$$

Proof of Lemma 6.21. Let $\ell = n-k$. From $\lambda \in P_{k,n}$, we have $\lambda_1 \leq n-k = \ell$.

Consider the conjugate partition λ^t of λ. Then, λ^t has exactly λ_1 parts. Thus, λ^t has $\leq \ell$ parts (since $\lambda_1 \leq \ell$). Therefore, $\lambda^t = \left(\left(\lambda^t\right)_1, \left(\lambda^t\right)_2, \ldots, \left(\lambda^t\right)_\ell\right)$. Hence, Corollary 6.5 (applied to λ^t instead of λ) yields

$$\mathbf{s}_{\left(\lambda^t\right)^t} = \det\left(\left(\mathbf{e}_{\left(\lambda^t\right)_i - i + j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right).$$

In view of $\left(\lambda^t\right)^t = \lambda$, this rewrites as

$$\mathbf{s}_\lambda = \det\left(\left(\mathbf{e}_{\left(\lambda^t\right)_i - i + j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right) = \sum_{\sigma\in S_\ell}(-1)^\sigma \prod_{i=1}^{\ell}\mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)} \tag{39}$$

(where S_ℓ denotes the symmetric group of the set $\{1,2,\ldots,\ell\}$, and where $(-1)^\sigma$ denotes the sign of a permutation $\sigma \in S_\ell$). Hence,

$$\left(\mathbf{s}_\lambda\right)^{\perp}\mathbf{f} = \left(\sum_{\sigma\in S_\ell}(-1)^\sigma \prod_{i=1}^{\ell}\mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)}\right)^{\perp}\mathbf{f} = \sum_{\sigma\in S_\ell}(-1)^\sigma \left(\prod_{i=1}^{\ell}\mathbf{e}_{\left(\lambda^t\right)_i - i + \sigma(i)}\right)^{\perp}\mathbf{f}.$$

Applying the map δ to this equality, we obtain

$$\begin{aligned}
\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{f}\right) &= \delta\left(\sum_{\sigma\in S_\ell}(-1)^\sigma \left(\prod_{i=1}^{\ell}\mathbf{e}_{(\lambda^t)_i - i+\sigma(i)}\right)^\perp \mathbf{f}\right) \\
&= \sum_{\sigma\in S_\ell}(-1)^\sigma \underbrace{\delta\left(\left(\prod_{i=1}^{\ell}\mathbf{e}_{(\lambda^t)_i - i+\sigma(i)}\right)^\perp \mathbf{f}\right)}_{\substack{\equiv \overline{\prod_{i=1}^{\ell} e_{(\lambda^t)_i - i+\sigma(i)}}\delta(\mathbf{f}) \bmod Q_{\ell-1} \\ \text{(by Lemma 6.20, applied} \\ \text{to } p=\ell \text{ and } i_j=\left(\lambda^t\right)_j - j+\sigma(j))}} \qquad \text{(since } \delta \text{ is } \mathbf{k}\text{-linear)} \\
&\equiv \sum_{\sigma\in S_\ell}(-1)^\sigma \overline{\prod_{i=1}^{\ell} e_{(\lambda^t)_i - i+\sigma(i)}}\delta\left(\mathbf{f}\right) \bmod Q_{\ell-1}.
\end{aligned} \tag{40}$$

On the other hand, (39) is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$s_\lambda = \sum_{\sigma\in S_\ell}(-1)^\sigma \prod_{i=1}^{\ell} e_{(\lambda^t)_i - i+\sigma(i)}.$$

Hence,

$$\overline{s_\lambda}\delta\left(\mathbf{f}\right) = \overline{\sum_{\sigma\in S_\ell}(-1)^\sigma \prod_{i=1}^{\ell} e_{(\lambda^t)_i - i+\sigma(i)}}\delta\left(\mathbf{f}\right) = \sum_{\sigma\in S_\ell}(-1)^\sigma \overline{\prod_{i=1}^{\ell} e_{(\lambda^t)_i - i+\sigma(i)}}\delta\left(\mathbf{f}\right).$$

Thus, (40) rewrites as $\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{f}\right) \equiv \overline{s_\lambda}\delta\left(\mathbf{f}\right) \bmod Q_{\ell-1}$. In other words, $\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{f}\right) \equiv \overline{s_\lambda}\delta\left(\mathbf{f}\right) \bmod Q_{n-k-1}$ (since $\ell = n-k$). This proves Lemma 6.21. ☐

Lemma 6.22. Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$. Then,

$$\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}.$$

Proof of Lemma 6.22. From $\mu \in P_{k,n}$, we obtain $\mu^\vee \in P_{k,n}$. Hence, $\mathbf{s}_{\mu^\vee} \in \mathcal{Z}$ and

$$\begin{aligned}
\delta\left(\mathbf{s}_{\mu^\vee}\right) &= \overline{s_{(\mu^\vee)^\vee}} && \text{(by the definition of } \delta\text{)} \\
&= \overline{s_\mu} && \left(\text{since } \left(\mu^\vee\right)^\vee = \mu\right).
\end{aligned}$$

Also, Lemma 6.21 (applied to $\mathbf{f} = \mathbf{s}_{\mu^\vee}$) yields

$$\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{s}_{\mu^\vee}\right) \equiv \overline{s_\lambda}\delta\left(\mathbf{s}_{\mu^\vee}\right) \bmod Q_{n-k-1}$$

(since $\mathbf{s}_{\mu^\vee} \in \mathcal{Z}$). In other words, $\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{s}_{\mu^\vee}\right) - \overline{s_\lambda}\delta\left(\mathbf{s}_{\mu^\vee}\right) \in Q_{n-k-1}$. Hence,

$$\operatorname{coeff}_\omega\left(\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{s}_{\mu^\vee}\right) - \overline{s_\lambda}\delta\left(\mathbf{s}_{\mu^\vee}\right)\right) \in \operatorname{coeff}_\omega\left(Q_{n-k-1}\right) = 0$$

(by Lemma 6.12). Thus,

$$\begin{aligned}\operatorname{coeff}_\omega\left(\delta\left(\left(\mathbf{s}_\lambda\right)^\perp \mathbf{s}_{\mu^\vee}\right)\right) &= \operatorname{coeff}_\omega\left(\overline{s_\lambda}\underbrace{\delta\left(\mathbf{s}_{\mu^\vee}\right)}_{=\overline{s_\mu}}\right) = \operatorname{coeff}_\omega\left(\overline{s_\lambda}\,\overline{s_\mu}\right)\\ &= \operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right). \qquad (41)\end{aligned}$$

Applying (28) to λ and $\mu^\vee$ instead of μ and λ, we obtain $\left(\mathbf{s}_\lambda\right)^\perp \mathbf{s}_{\mu^\vee} = \mathbf{s}_{\mu^\vee/\lambda}$. Thus, (41) rewrites as

$$\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right)\right) = \operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right). \qquad (42)$$

We are in one of the following three cases:

Case 1: We have $\lambda = \mu^\vee$.

Case 2: We have $\lambda \subseteq \mu^\vee$ but not $\lambda = \mu^\vee$.

Case 3: We don't have $\lambda \subseteq \mu^\vee$.

Let us first consider Case 1. In this case, we have $\lambda = \mu^\vee$. Thus, $\mathbf{s}_{\mu^\vee/\lambda} = \mathbf{s}_{\mu^\vee/\mu^\vee} = 1 = \mathbf{s}_\varnothing$ and thus

$$\begin{aligned}\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right) &= \delta\left(\mathbf{s}_\varnothing\right) = \overline{s_{\varnothing^\vee}} \qquad \text{(by the definition of } \delta\text{)}\\ &= \overline{s_\omega} \qquad \left(\text{since } \varnothing^\vee = \omega\right).\end{aligned}$$

Therefore, $\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right)\right) = \operatorname{coeff}_\omega\left(\overline{s_\omega}\right) = 1$ (by the definition of $\operatorname{coeff}_\omega$). Comparing this with

$$\begin{cases}1, & \text{if } \lambda = \mu^\vee;\\ 0, & \text{if } \lambda \neq \mu^\vee\end{cases} = 1 \qquad \left(\text{since } \lambda = \mu^\vee\right),$$

we obtain $\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases}1, & \text{if } \lambda = \mu^\vee;\\ 0, & \text{if } \lambda \neq \mu^\vee\end{cases}$. Hence, Lemma 6.22 is proven in Case 1.

Let us next consider Case 2. In this case, we have $\lambda \subseteq \mu^\vee$ but not $\lambda = \mu^\vee$. Hence, $|\lambda| < |\mu^\vee|$ and $\lambda \neq \mu^\vee$.

Now, every partition ν satisfying $|\lambda| + |\nu| = |\mu^\vee|$ and $\nu \subseteq \mu^\vee$ must satisfy

$$\nu \in P_{k,n} \text{ and } \operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_\nu\right)\right) = 0. \qquad (43)$$

[*Proof of (43):* Let ν be a partition satisfying $|\lambda| + |\nu| = |\mu^\vee|$ and $\nu \subseteq \mu^\vee$. We must prove (43).

First of all, from $\nu \subseteq \mu^\vee$ and $\mu^\vee \in P_{k,n}$, we obtain $\nu \in P_{k,n}$. It thus remains to show that $\operatorname{coeff}_\omega(\delta(\mathbf{s}_\nu)) = 0$.

The definition of δ yields $\delta(\mathbf{s}_\nu) = \overline{s_{\nu^\vee}}$ (since $\nu \in P_{k,n}$). But $|\lambda| + |\nu| = |\mu^\vee|$ yields $|\nu| = |\mu^\vee| - |\lambda| > 0$ (since $|\lambda| < |\mu^\vee|$).

But every partition $\kappa \in P_{k,n}$ satisfies $|\kappa^\vee| = \underbrace{k(n-k)}_{=|\omega|} - |\kappa| = |\omega| - |\kappa|$. Applying this to $\kappa = \nu$, we obtain

$$\left|\nu^\vee\right| = |\omega| - \underbrace{|\nu|}_{>0} < |\omega|.$$

Hence, $|\nu^\vee| \neq |\omega|$, so that $\nu^\vee \neq \omega$.

But the definition of $\operatorname{coeff}_\omega$ yields $\operatorname{coeff}_\omega(\overline{s_{\nu^\vee}}) = \begin{cases} 1, & \text{if } \nu^\vee = \omega; \\ 0, & \text{if } \nu^\vee \neq \omega \end{cases} = 0$ (since $\nu^\vee \neq \omega$). In view of $\delta(\mathbf{s}_\nu) = \overline{s_{\nu^\vee}}$, this rewrites as $\operatorname{coeff}_\omega(\delta(\mathbf{s}_\nu)) = 0$. This completes the proof of (43).]

Proposition 6.17 **(a)** (applied to $\mu^\vee$ and λ instead of λ and μ) yields

$$\begin{aligned}
\mathbf{s}_{\mu^\vee/\lambda} &= \sum_{\nu \text{ is a partition}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu + \sum_{\substack{\nu \text{ is a partition;} \\ \text{not } \nu \subseteq \mu^\vee}} \underbrace{c^{\mu^\vee}_{\lambda,\nu}}_{\substack{=0 \\ \text{(by Proposition 6.17 (b),} \\ \text{applied to } \mu^\vee \text{ and } \lambda \text{ instead of } \lambda \text{ and } \mu)}} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu + \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ \text{not } |\lambda|+|\nu|=|\mu^\vee|}} \underbrace{c^{\mu^\vee}_{\lambda,\nu}}_{\substack{=0 \\ \text{(by Proposition 6.17 (c),} \\ \text{applied to } \mu^\vee \text{ and } \lambda \text{ instead of } \lambda \text{ and } \mu)}} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu .
\end{aligned}$$

Applying the map δ to this equality, we find

$$\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right) = \delta\left(\sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \mathbf{s}_\nu \right) = \sum_{\substack{\nu \text{ is a partition;} \\ \nu \subseteq \mu^\vee; \\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu} \delta(\mathbf{s}_\nu)$$
$$\left(\begin{array}{c} \text{since every partition } \nu \text{ satisfying } \nu \subseteq \mu^\vee \text{ and } |\lambda| + |\nu| = |\mu^\vee| \\ \text{must satisfy } \nu \in P_{k,n} \text{ (by (43)) and thus } \mathbf{s}_\nu \in \mathcal{Z} \end{array} \right).$$

Applying the map $\operatorname{coeff}_\omega$ to this equality, we find

$$\begin{aligned}\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right)\right) &= \operatorname{coeff}_\omega\left(\sum_{\substack{\nu \text{ is a partition;}\\ \nu\subseteq\mu^\vee;\\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu}\delta\left(\mathbf{s}_\nu\right)\right)\\ &= \sum_{\substack{\nu \text{ is a partition;}\\ \nu\subseteq\mu^\vee;\\ |\lambda|+|\nu|=|\mu^\vee|}} c^{\mu^\vee}_{\lambda,\nu}\underbrace{\operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_\nu\right)\right)}_{\substack{=0\\ \text{(by (43))}}} = 0.\end{aligned}$$

Comparing this with

$$\begin{cases}1, & \text{if } \lambda=\mu^\vee;\\ 0, & \text{if } \lambda\neq\mu^\vee\end{cases} = 0 \qquad \left(\text{since } \lambda\neq\mu^\vee\right),$$

we obtain $\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases}1, & \text{if } \lambda=\mu^\vee;\\ 0, & \text{if } \lambda\neq\mu^\vee\end{cases}$. Hence, Lemma 6.22 is proven in Case 2.

Let us finally consider Case 3. In this case, we don't have $\lambda\subseteq\mu^\vee$. Hence, we don't have $\lambda=\mu^\vee$ either. Thus, $\lambda\neq\mu^\vee$.

Also, $\mathbf{s}_{\mu^\vee/\lambda}=0$ (since we don't have $\lambda\subseteq\mu^\vee$). Thus,

$$\operatorname{coeff}_\omega\left(\delta\left(\underbrace{\mathbf{s}_{\mu^\vee/\lambda}}_{=0}\right)\right) = \operatorname{coeff}_\omega\left(\delta\left(0\right)\right) = 0.$$

Comparing this with

$$\begin{cases}1, & \text{if } \lambda=\mu^\vee;\\ 0, & \text{if } \lambda\neq\mu^\vee\end{cases} = 0 \qquad \left(\text{since } \lambda\neq\mu^\vee\right),$$

we obtain $\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \begin{cases}1, & \text{if } \lambda=\mu^\vee;\\ 0, & \text{if } \lambda\neq\mu^\vee\end{cases}$. Hence, Lemma 6.22 is proven in Case 3.

We have now proven Lemma 6.22 in all three Cases 1, 2 and 3. Thus, Lemma 6.22 always holds. □

Proof of Theorem 6.3. Write $f\in\mathcal{S}/I$ in the form $f=\sum_{\lambda\in P_{k,n}}\alpha_\lambda\overline{s_\lambda}$ with $\alpha_\lambda\in\mathbf{k}$. (This is possible, since $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.) Then, the definition of $\operatorname{coeff}_{\nu^\vee}$ yields $\operatorname{coeff}_{\nu^\vee}(f)=\alpha_{\nu^\vee}$.

On the other hand,

$$\begin{aligned}
\operatorname{coeff}_{\omega}\left(\overline{s_\nu}\underbrace{f}_{=\sum\limits_{\lambda\in P_{k,n}}\alpha_\lambda\overline{s_\lambda}}\right) &= \operatorname{coeff}_{\omega}\left(\overline{s_\nu}\sum_{\lambda\in P_{k,n}}\alpha_\lambda\overline{s_\lambda}\right) = \sum_{\lambda\in P_{k,n}}\alpha_\lambda \operatorname{coeff}_{\omega}\left(\underbrace{\overline{s_\nu s_\lambda}}_{=\overline{s_\lambda s_\nu}}\right)\\
&= \sum_{\lambda\in P_{k,n}}\alpha_\lambda \underbrace{\operatorname{coeff}_{\omega}\left(\overline{s_\lambda s_\nu}\right)}_{\substack{=\begin{cases}1, & \text{if } \lambda=\nu^\vee;\\ 0, & \text{if } \lambda\neq\nu^\vee\end{cases}\\ \text{(by Lemma 6.22, applied to } \mu=\nu)}}\\
&= \sum_{\lambda\in P_{k,n}}\alpha_\lambda \begin{cases}1, & \text{if } \lambda=\nu^\vee;\\ 0, & \text{if } \lambda\neq\nu^\vee\end{cases} = \alpha_{\nu^\vee}
\end{aligned}$$

(since $\nu^\vee \in P_{k,n}$). Comparing this with $\operatorname{coeff}_{\nu^\vee}(f) = \alpha_{\nu^\vee}$, we obtain $\operatorname{coeff}_{\omega}\left(\overline{s_\nu}f\right) = \operatorname{coeff}_{\nu^\vee}(f)$. This proves Theorem 6.3. □

Definition 6.23. For any three partitions $\alpha, \beta, \gamma \in P_{k,n}$, let $g_{\alpha,\beta,\gamma} = \operatorname{coeff}_{\gamma^\vee}\left(\overline{s_\alpha s_\beta}\right) \in \mathbf{k}$.

These scalars $g_{\alpha,\beta,\gamma}$ are thus the structure constants of the $\mathbf{k}$-algebra $\mathcal{S}/I$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ (although slightly reindexed). As a consequence of Theorem 6.3, we obtain the following S_3-property of these structure constants:

Corollary 6.24. We have

$$g_{\alpha,\beta,\gamma} = g_{\alpha,\gamma,\beta} = g_{\beta,\alpha,\gamma} = g_{\beta,\gamma,\alpha} = g_{\gamma,\alpha,\beta} = g_{\gamma,\beta,\alpha} = \operatorname{coeff}_{\omega}\left(\overline{s_\alpha s_\beta s_\gamma}\right)$$

for any $\alpha, \beta, \gamma \in P_{k,n}$.

Proof of Corollary 6.24. Let $\alpha, \beta, \gamma \in P_{k,n}$. It clearly suffices to prove $g_{\alpha,\beta,\gamma} = \operatorname{coeff}_{\omega}\left(\overline{s_\alpha s_\beta s_\gamma}\right)$, since the rest of the claim then follows by analogy.

Theorem 6.3 (applied to $\nu = \gamma$ and $f = \overline{s_\alpha s_\beta}$) yields

$$\operatorname{coeff}_{\omega}\left(\overline{s_\gamma s_\alpha s_\beta}\right) = \operatorname{coeff}_{\gamma^\vee}\left(\overline{s_\alpha s_\beta}\right) = g_{\alpha,\beta,\gamma}$$

(by the definition of $g_{\alpha,\beta,\gamma}$). Thus, $g_{\alpha,\beta,\gamma} = \operatorname{coeff}_{\omega}\left(\underbrace{\overline{s_\gamma s_\alpha s_\beta}}_{=\overline{s_\alpha s_\beta s_\gamma}}\right) = \operatorname{coeff}_{\omega}\left(\overline{s_\alpha s_\beta s_\gamma}\right)$.

This completes our proof of Corollary 6.24. □

7. Complete homogeneous symmetric polynomials

In this section, we shall further explore the projections $\overline{h_i}$ of complete homogeneous symmetric polynomials h_i onto $\mathcal{S}/I$. This exploration will culminate in a second proof of Theorem 6.3.

Convention 7.1. Convention 6.1 remains in place for the whole Section 7.

We shall also use all the notations introduced in Section 6.

If $j \in \mathbb{N}$, then the expression "1^j" in a tuple stands for j consecutive entries equal to 1 (that is, $\underbrace{1,1,\ldots,1}_{j \text{ times}}$). Thus, $\left(m, 1^j\right) = \left(m, \underbrace{1,1,\ldots,1}_{j \text{ times}}\right)$ for any $m \in \mathbb{N}$ and $j \in \mathbb{N}$.

7.1. A reduction formula for h_{n+m}

The following result helps us reduce complete homogeneous symmetric polynomials h_{n+m} modulo the ideal I:

Proposition 7.2. Let m be a positive integer. Then,

$$h_{n+m} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{\left(m,1^j\right)} \mod I.$$

We shall derive Proposition 7.2 from the following identity between symmetric functions in Λ:

Proposition 7.3. Let m be a positive integer. Then,

$$\mathbf{h}_{n+m} = \sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j} \mathbf{s}_{\left(m,1^j\right)}.$$

Proof of Proposition 7.3. Let $j \in \mathbb{N}$.

In [GriRei20, Exercise 2.9.14(b)], it is shown that

$$\sum_{i=0}^{b} (-1)^i \mathbf{h}_{a+i+1} \mathbf{e}_{b-i} = \mathbf{s}_{\left(a+1,1^b\right)} \tag{44}$$

for all $a, b \in \mathbb{N}$. Applying this equality to $a = m-1$ and $b = j$, we obtain

$$\sum_{i=0}^{j} (-1)^i \mathbf{h}_{m+i} \mathbf{e}_{j-i} = \mathbf{s}_{\left(m,1^j\right)}. \tag{45}$$

Now, forget that we fixed j. We thus have proven (45) for each $j \in \mathbb{N}$.

Also, for any $N \in \mathbb{N}$, we have

$$\sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=N}} (-1)^i \mathbf{e}_i \mathbf{h}_j = \delta_{0,N}$$

(where $\delta_{0,N}$ is a Kronecker delta). (This is [GriRei20, (2.4.4)], with n renamed as N.) Thus, for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\delta_{0,N} &= \sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=N}} (-1)^i \mathbf{e}_i \mathbf{h}_j = \sum_{\substack{(i,j)\in\mathbb{N}^2;\\ i+j=N}} (-1)^i \mathbf{h}_j \mathbf{e}_i \\
&= \sum_{i=0}^{N} (-1)^i \mathbf{h}_{N-i} \mathbf{e}_i \qquad \left(\begin{array}{c}\text{here, we have substituted } (i, N-i) \\ \text{for } (i,j) \text{ in the sum}\end{array}\right) \\
&= \sum_{j=0}^{N} (-1)^j \mathbf{h}_{N-j} \mathbf{e}_j \qquad \left(\begin{array}{c}\text{here, we have renamed the} \\ \text{summation index } i \text{ as } j\end{array}\right).
\end{aligned}$$

Thus, for any $N \in \mathbb{N}$, we have

$$\sum_{j=0}^{N} (-1)^j \mathbf{h}_{N-j} \mathbf{e}_j = \delta_{0,N}. \tag{46}$$

For each $i \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned}
&\sum_{j=i}^{n} (-1)^{j-i} \mathbf{h}_{n-j} \mathbf{e}_{j-i} \\
&= \sum_{j=0}^{n-i} (-1)^j \mathbf{h}_{n-i-j} \mathbf{e}_j \qquad (\text{here, we have substituted } j \text{ for } j-i \text{ in the sum}) \\
&= \delta_{0,n-i} \qquad (\text{by (46), applied to } n-i \text{ instead of } N) \\
&= \delta_{i,n}.
\end{aligned} \tag{47}$$

Now,

$$\begin{aligned}
&\sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j} \underbrace{\mathbf{s}_{\left(m,1^j\right)}}_{\substack{=\sum\limits_{i=0}^{j} (-1)^i \mathbf{h}_{m+i}\mathbf{e}_{j-i} \\ \text{(by (45))}}} \\
&= \sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j} \sum_{i=0}^{j} (-1)^i \mathbf{h}_{m+i}\mathbf{e}_{j-i} = \underbrace{\sum_{j=0}^{n} \sum_{i=0}^{j}}_{=\sum\limits_{i=0}^{n} \sum\limits_{j=i}^{n}} (-1)^j \mathbf{h}_{n-j} (-1)^i \mathbf{h}_{m+i}\mathbf{e}_{j-i} \\
&= \sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^j \mathbf{h}_{n-j} (-1)^i \mathbf{h}_{m+i}\mathbf{e}_{j-i} = \sum_{i=0}^{n} \mathbf{h}_{m+i} \sum_{j=i}^{n} (-1)^{j-i} \mathbf{h}_{n-j}\mathbf{e}_{j-i} \\
&= \sum_{i=0}^{n} \mathbf{h}_{m+i} \underbrace{\sum_{j=i}^{n} (-1)^{j-i} \mathbf{h}_{n-j}\mathbf{e}_{j-i}}_{\substack{=\delta_{i,n} \\ \text{(by (47))}}} = \sum_{i=0}^{n} \mathbf{h}_{m+i}\delta_{i,n} = \mathbf{h}_{m+n} = \mathbf{h}_{n+m}.
\end{aligned}$$

This proves Proposition 7.3. □

Proof of Proposition 7.2. For each integer $j \geq k$, we have

$$s_{\left(m,1^j\right)} = 0. \tag{48}$$

[*Proof of (48):* Let $j \geq k$ be an integer. Then, the partition $\left(m, 1^j\right)$ has $j+1$ parts; thus, this partition has more than k parts (since $j+1 > j \geq k$). Thus, (3) (applied to $\lambda = \left(m, 1^j\right)$) yields $s_{\left(m,1^j\right)} = 0$. This proves (48).]

Proposition 7.3 yields

$$\mathbf{h}_{n+m} = \sum_{j=0}^{n} (-1)^j \mathbf{h}_{n-j}\mathbf{s}_{\left(m,1^j\right)}.$$

This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$\begin{aligned}
h_{n+m} &= \sum_{j=0}^{n} (-1)^j h_{n-j} s_{(m,1^j)} \\
&= \sum_{j=0}^{k-1} (-1)^j h_{n-j} s_{(m,1^j)} + \sum_{j=k}^{n} (-1)^j h_{n-j} \underbrace{s_{(m,1^j)}}_{\substack{=0 \\ \text{(by (48))}}} \\
&= \sum_{j=0}^{k-1} (-1)^j \underbrace{h_{n-j}}_{\substack{=h_{n-k+(k-j)} \equiv a_{k-j} \bmod I \\ \text{(by (15), applied to } k-j \\ \text{instead of } j)}} s_{(m,1^j)} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{(m,1^j)} \bmod I.
\end{aligned}$$

This proves Proposition 7.2. □

7.2. Lemmas on free modules

Next, we state a basic lemma from commutative algebra:

Lemma 7.4. Let $r \in \mathbb{N}$. Let X and Y be two free $\mathbf{k}$-modules of rank r. Then, every surjective $\mathbf{k}$-linear map from X to Y is a $\mathbf{k}$-module isomorphism.

Proof of Lemma 7.4. Let $f : X \to Y$ be a surjective $\mathbf{k}$-linear map from X to Y. We must prove that f is a $\mathbf{k}$-module isomorphism.

There is clearly a $\mathbf{k}$-module isomorphism $j : Y \to X$ (since X and Y are free $\mathbf{k}$-modules of the same rank). Consider this j. Then, the composition $j \circ f$ is surjective (since j and f are surjective), and thus is a surjective endomorphism of the finitely generated $\mathbf{k}$-module X. But [GriRei20, Exercise 2.5.18(a)] shows that any surjective endomorphism of a finitely generated $\mathbf{k}$-module is a $\mathbf{k}$-module isomorphism. Hence, we conclude that $j \circ f$ is a $\mathbf{k}$-module isomorphism. Thus, f is a $\mathbf{k}$-module isomorphism (since j is a $\mathbf{k}$-module isomorphism). This proves Lemma 7.4. □

Lemma 7.5. Let Z be a $\mathbf{k}$-module. Let U, X and Y be $\mathbf{k}$-submodules of Z such that $Z = X \oplus Y$ and $X \subseteq U$. Let $r \in \mathbb{N}$. Assume that the $\mathbf{k}$-module X has a basis with r elements, whereas the $\mathbf{k}$-module U can be spanned by r elements. Then, $X = U$.

Proof of Lemma 7.5. Let $\pi : Z \to X$ be the canonical projection from the direct sum $Z = X \oplus Y$ onto its addend X. Let $\iota : X \to U$ be the canonical injection. Then, the composition

$$X \xrightarrow{\iota} U \xrightarrow{\pi|_U} X$$

is just id_X (since $\pi |_X = \operatorname{id}_X$). Hence, the map $\pi |_U$ is surjective.

We assumed that the $\mathbf{k}$-module U can be spanned by r elements. Thus, there is a surjective $\mathbf{k}$-module homomorphism $u : \mathbf{k}^r \to U$. Consider this u.

Both $\mathbf{k}$-modules $\mathbf{k}^r$ and X are free of rank r (since X has a basis with r elements). The composition

$$\mathbf{k}^r \xrightarrow{u} U \xrightarrow{\pi|_U} X$$

is surjective (since both u and $\pi\mid_U$ are surjective), and thus is a $\mathbf{k}$-module isomorphism (by Lemma 7.4, applied to $\mathbf{k}^r$ and X instead of X and Y). Hence, it is injective. Thus, u is injective. Since u is also surjective, we thus conclude that u is bijective, and therefore a $\mathbf{k}$-module isomorphism. Since both u and the composition $\mathbf{k}^r \xrightarrow{u} U \xrightarrow{\pi|_U} X$ are $\mathbf{k}$-module isomorphisms, we now conclude that the map $\pi\mid_U$ is a $\mathbf{k}$-module isomorphism. Hence, it has an inverse. But this inverse must be ι (since the composition $X \xrightarrow{\iota} U \xrightarrow{\pi|_U} X$ is id_X). Thus, ι is a $\mathbf{k}$-module isomorphism, too. Thus, in particular, ι is surjective. Therefore, $U = \iota(X) = X$. This proves Lemma 7.5. $\square$

7.3. The symmetric polynomials h_ν

Definition 7.6. Let $\ell \in \mathbb{N}$, and let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ be any ℓ-tuple of integers. Then, we define the symmetric polynomial $h_\nu \in \mathcal{S}$ as follows:

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell}.$$

Note that the polynomial h_ν does not change if we permute the entries of the ℓ-tuple ν. If an ℓ-tuple ν of integers contains any negative entries, then $h_\nu = 0$ (since $h_i = 0$ for any $i < 0$). Also, if an ℓ-tuple ν of integers contains any entry $= 0$, then we can remove this entry without changing h_ν (since $h_0 = 1$).

7.4. The submodules L_p and H_p of $\mathcal{S}/I$

It is time to define two further filtrations of the $\mathbf{k}$-module $\mathcal{S}/I$ (in addition to the filtration $(Q_p)_{p\in\mathbb{Z}}$ from Definition 6.11):

Definition 7.7. **(a)** If λ is a partition, then $\ell(\lambda)$ shall denote the *length* of λ; this is defined as the number of positive entries of λ. Note that $\ell(\lambda) \leq k$ for each $\lambda \in P_{k,n}$.

(b) For each $p \in \mathbb{Z}$, we let L_p denote the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$.

(c) For each $p \in \mathbb{Z}$, we let H_p denote the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$.

The only partition λ satisfying $\ell(\lambda) \leq 0$ is the empty partition $\varnothing = ()$; it belongs to $P_{k,n}$ and satisfies $\overline{s_\lambda} = 1$. Hence, L_0 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by 1. Similarly, H_0 is the same $\mathbf{k}$-submodule.

Also, L_k is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by all $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ (because each $\lambda \in P_{k,n}$ satisfies $\ell(\lambda) \leq k$). But the latter $\mathbf{k}$-submodule is $\mathcal{S}/I$ itself (by Theorem 2.7). Thus, we conclude that L_k is $\mathcal{S}/I$ itself. In other words,

$$L_k = \mathcal{S}/I.$$

Clearly, $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$ and $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$. We shall soon see that the families $(L_p)_{p\in\mathbb{Z}}$ and $(H_p)_{p\in\mathbb{Z}}$ are identical (Proposition 7.11) and are filtrations of the $\mathbf{k}$-algebra $\mathcal{S}/I$ (Proposition 7.15). First let us show a basic fact:

Lemma 7.8. Let $p \in \mathbb{N}$ be such that $p \leq k$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_p) \in \mathbb{Z}^p$. Assume that $\nu_i \leq n$ for each $i \in \{1, 2, \ldots, p\}$. Then, $\overline{h_\nu} \in H_p$.

(The condition "$p \leq k$" can be removed from this lemma, but we aren't yet at the point where this is easy to see. We will show this in Proposition 7.14 below.)

Proof of Lemma 7.8. We WLOG assume that $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p$ (since otherwise, we can just permute the entries of ν to achieve this). Let j be the number of $i \in \{1, 2, \ldots, p\}$ satisfying $\nu_i > n - k$. Then,

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_j > n - k \geq \nu_{j+1} \geq \nu_{j+2} \geq \cdots \geq \nu_p.$$

We WLOG assume that all of the $\nu_1, \nu_2, \ldots, \nu_p$ are nonnegative (since otherwise, we have $h_\nu = 0$ and thus $\overline{h_\nu} = 0 \in H_p$).

Now,

$$\overline{h_{\nu_i}} \in \mathbf{k} \qquad \text{for each } i \in \{1, 2, \ldots, j\}. \tag{49}$$

[*Proof of (49):* Let $i \in \{1, 2, \ldots, j\}$. Then, $\nu_i > n - k$ (since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_j > n - k$), but also $\nu_i \leq n$ (by the assumptions of Lemma 7.8). Thus, $n - k < \nu_i \leq n$, so that $\nu_i \in \{n-k+1, n-k+2, \ldots, n\}$ and thus $\nu_i - (n-k) \in \{1, 2, \ldots, k\}$. Hence, (15) (applied to $\nu_i - (n-k)$ instead of j) yields $h_{\nu_i} \equiv a_{\nu_i - (n-k)} \bmod I$. Hence, $\overline{h_{\nu_i}} = \overline{a_{\nu_i - (n-k)}} \in \mathbf{k}$. This proves (49).]

Furthermore, $(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p)$ is a partition (since $\nu_{j+1} \geq \nu_{j+2} \geq \cdots \geq \nu_p$ and since all of the $\nu_1, \nu_2, \ldots, \nu_p$ are nonnegative) with at most k entries (indeed, its number of entries is $\leq p - j \leq p \leq k$), and all of its entries are $\leq n - k$ (since $n - k \geq \nu_{j+1} \geq \nu_{j+2} \geq \cdots \geq \nu_p$). Hence, $(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p)$ belongs to $P_{k,n}$.

From $(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p) \in P_{k,n}$ and $\ell(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p) \leq p - j \leq p$, we obtain $\overline{h_{(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p)}} \in H_p$ (by the definition of H_p).

Now, the definition of h_ν yields $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}$, so that

$$\overline{h_\nu} = \overline{h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}} = \overline{h_{\nu_1}}\, \overline{h_{\nu_2}} \cdots \overline{h_{\nu_p}} = \underbrace{\left(\overline{h_{\nu_1}}\, \overline{h_{\nu_2}} \cdots \overline{h_{\nu_j}}\right)}_{\substack{\in \mathbf{k} \\ \text{(by (49))}}} \underbrace{\left(\overline{h_{\nu_{j+1}}}\, \overline{h_{\nu_{j+2}}} \cdots \overline{h_{\nu_p}}\right)}_{\substack{= \overline{h_{\nu_{j+1}} h_{\nu_{j+2}} \cdots h_{\nu_p}} \\ = \overline{h_{\left(\nu_{j+1}, \nu_{j+2}, \ldots, \nu_p\right)}} \in H_p}}$$

$$\in \mathbf{k} H_p \subseteq H_p.$$

This proves Lemma 7.8. ☐

Lemma 7.9. Let $p \in \mathbb{Z}$. Then, the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p.

Proof of Lemma 7.9. Theorem 2.7 yields that $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. Hence, this family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ is $\mathbf{k}$-linearly independent. Thus, its subfamily $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is $\mathbf{k}$-linearly independent as well. Moreover, this subfamily $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ spans the $\mathbf{k}$-module L_p (by the definition of L_p). Hence, this subfamily $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p. This proves Lemma 7.9. ☐

Lemma 7.10. Let $p \in \{0, 1, \ldots, k\}$. Then, $L_p = H_p$.

(This lemma holds more generally for all $p \in \mathbb{Z}$, as we shall see in Lemma 7.11 below.)

Proof of Lemma 7.10. Let $\lambda \in P_{k,n}$ be such that $\ell(\lambda) \leq p$. We shall show that $\overline{s_\lambda} \in H_p$.

Indeed, let S_p denote the group of permutations of $\{1, 2, \ldots, p\}$. For each $\sigma \in S_p$, let $(-1)^\sigma$ denote the sign of σ.

For each $\sigma \in S_p$, we have

$$\overline{\prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)}} \in H_p. \tag{50}$$

[*Proof of (50):* Let $\sigma \in S_p$. Then, each $i \in \{1, 2, \ldots, p\}$ satisfies

$$\underbrace{\lambda_i}_{\substack{\leq n-k \\ \text{(since } \lambda \in P_{k,n})}} - \underbrace{i}_{\geq 0} + \underbrace{\sigma(i)}_{\leq p \leq k} \leq n - k + 0 + k = n.$$

Thus, Lemma 7.8 (applied to $\left(\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \ldots, \lambda_p - p + \sigma(p)\right)$ and $\lambda_i - i + \sigma(i)$ instead of ν and ν_i) yields

$$\overline{h_{\left(\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \ldots, \lambda_p - p + \sigma(p)\right)}} \in H_p$$

(since $p \leq k$). In view of

$$h_{\left(\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \ldots, \lambda_p - p + \sigma(p)\right)} = \prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)},$$

this rewrites as $\overline{\prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)}} \in H_p$. Thus, (50) is proven.]

We have $\ell(\lambda) \leq p$ and thus $\lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_p\right)$. Hence, Proposition 5.7 **(b)** yields

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq p,\ 1 \leq v \leq p}\right) = \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)}$$

(by the definition of a determinant). Projecting both sides of this equality onto $\mathcal{S}/I$, we obtain

$$\overline{s_\lambda} = \overline{\sum_{\sigma \in S_p} (-1)^\sigma \prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)}} = \sum_{\sigma \in S_p} (-1)^\sigma \underbrace{\overline{\prod_{i=1}^{p} h_{\lambda_i - i + \sigma(i)}}}_{\substack{\in H_p \\ \text{(by (50))}}} \in H_p.$$

Now, forget that we fixed λ. We thus have proven that

$$\overline{s_\lambda} \in H_p \qquad \text{for each } \lambda \in P_{k,n} \text{ satisfying } \ell(\lambda) \leq p.$$

Therefore, $L_p \subseteq H_p$ (since L_p is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$).

Lemma 7.9 yields that the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p.

Now, let L'_p be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) > p$. Recall (from Theorem 2.7) that $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. Hence, $\mathcal{S}/I = L_p \oplus L'_p$ (since each $\lambda \in P_{k,n}$ satisfies either $\ell(\lambda) \leq p$ or $\ell(\lambda) > p$ but not both). Let r be the number of all $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$. Then, the $\mathbf{k}$-module H_p can be spanned by r elements (namely, by the $\overline{h_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$), whereas the $\mathbf{k}$-module L_p has a basis with r elements (namely, the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$). Thus, Lemma 7.5 (applied to $Z = \mathcal{S}/I$, $X = L_p$, $Y = L'_p$ and $U = H_p$) yields $L_p = H_p$. This proves Lemma 7.10. □

Proposition 7.11. Let $p \in \mathbb{Z}$. Then, $L_p = H_p$.

Proof of Proposition 7.11. If p is negative, then both L_p and H_p equal 0 (since there exists no $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq p$ in this case). Thus, if p is negative, then

$L_p = H_p$ is obviously true. Hence, for the rest of this proof, we WLOG assume that p is not negative. Thus, $p \in \mathbb{N}$.

If $p \in \{0, 1, \ldots, k\}$, then $L_p = H_p$ follows from Lemma 7.10. Hence, for the rest of this proof, we WLOG assume that $p \notin \{0, 1, \ldots, k\}$. Thus, $p > k$ (since $p \in \mathbb{N}$). Hence, $k < p$, so that $H_k \subseteq H_p$ (since $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$). But Lemma 7.10 (applied to k instead of p) yields $L_k = H_k$.

But recall that $L_k = \mathcal{S}/I$. Thus, $\mathcal{S}/I = L_k = H_k \subseteq H_p$. Thus, $H_p \supseteq \mathcal{S}/I \supseteq L_p$.

On the other hand, $k < p$ and thus $L_k \subseteq L_p$ (since $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$). Hence, $L_p \supseteq L_k = \mathcal{S}/I \supseteq H_p$. Combining this with $H_p \supseteq L_p$, we obtain $L_p = H_p$. This proves Proposition 7.11. □

Corollary 7.12. Let $p \in \mathbb{Z}$. Then, the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p.

Proof of Corollary 7.12. Lemma 7.9 yields that the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p. On the other hand, the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ spans the $\mathbf{k}$-module H_p (by the definition of H_p). In other words, the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ spans the $\mathbf{k}$-module L_p (since Proposition 7.11 yields $L_p = H_p$). Since $\left|\{\lambda \in P_{k,n} \mid \ell(\lambda) \leq p\}\right| = \left|\{\lambda \in P_{k,n} \mid \ell(\lambda) \leq p\}\right|$, we can therefore apply Lemma 5.3 to L_p, $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ and $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ instead of M, $(b_s)_{s \in S}$ and $(a_u)_{u \in U}$. We thus conclude that $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq p}$ is a basis of the $\mathbf{k}$-module L_p. This proves Corollary 7.12. □

Theorem 7.13. The family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

Proof of Theorem 7.13. Corollary 7.12 (applied to $p = k$) shows that the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq k}$ is a basis of the $\mathbf{k}$-module L_k. In view of $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n};\ \ell(\lambda) \leq k} = \left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ (since each $\lambda \in P_{k,n}$ satisfies $\ell(\lambda) \leq k$) and $L_k = \mathcal{S}/I$, this rewrites as follows: The family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. This proves Theorem 7.13. □

Proposition 7.14. Let $p \in \mathbb{N}$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_p) \in \mathbb{Z}^p$. Assume that $\nu_i \leq n$ for each $i \in \{1, 2, \ldots, p\}$. Then, $\overline{h_\nu} \in H_p$.

Proof of Proposition 7.14. If $p \leq k$, then this follows from Lemma 7.8. Thus, for the rest of this proof, we WLOG assume that $p > k$. Hence, $k < p$, so that $H_k \subseteq H_p$ (since $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$). But Proposition 7.11 (applied to k instead of p) yields $H_k = L_k = \mathcal{S}/I$. Now, $\overline{h_\nu} \in \mathcal{S}/I = H_k \subseteq H_p$. This proves Proposition 7.14. □

We recall that the $\mathbf{k}$-submodules of a given $\mathbf{k}$-algebra A form a monoid under multiplication: The product XY of two $\mathbf{k}$-submodules X and Y of A is defined as the $\mathbf{k}$-linear span of all products xy with $x \in X$ and $y \in Y$. The neutral element of this monoid is $\mathbf{k} \cdot 1_A$. We shall specifically use this monoid in the case when $A = \mathcal{S}/I$.

Proposition 7.15. The family $(L_p)_{p \in \mathbb{N}}$ is a filtration of the $\mathbf{k}$-algebra $\mathcal{S}/I$; that is, we have

$$\begin{aligned} &L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots, \qquad \bigcup_{p \in \mathbb{N}} L_p = \mathcal{S}/I, \\ &1 \in L_0, \qquad \text{and} \\ &L_a L_b \subseteq L_{a+b} \qquad \text{for every } a, b \in \mathbb{N}. \end{aligned} \tag{51}$$

Proof of Proposition 7.15. We already know that $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$. Also, $1 \in L_0$ (since L_0 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by 1). Also, $L_k = \mathcal{S}/I$, so that $\mathcal{S}/I = L_k \subseteq \bigcup_{p \in \mathbb{N}} L_p$. Combining this with $\bigcup_{p \in \mathbb{N}} L_p \subseteq \mathcal{S}/I$, we obtain $\bigcup_{p \in \mathbb{N}} L_p = \mathcal{S}/I$.

Hence, it remains to prove that $L_a L_b \subseteq L_{a+b}$ for every $a, b \in \mathbb{N}$. So let us fix $a, b \in \mathbb{N}$. We must prove that $L_a L_b \subseteq L_{a+b}$.

If $a + b \geq k$, then this is obvious (because if $a + b \geq k$, then $k \leq a + b$, hence $L_k \subseteq L_{a+b}$ (since $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$), hence $L_a L_b \subseteq \mathcal{S}/I = L_k \subseteq L_{a+b}$). Hence, we WLOG assume that $a + b < k$.

We must prove that $L_a L_b \subseteq L_{a+b}$. It clearly suffices to show that $fg \in L_{a+b}$ for each $f \in L_a$ and $g \in L_b$. So let us fix $f \in L_a$ and $g \in L_b$; we must prove that $fg \in L_{a+b}$.

Proposition 7.11 yields that $L_a = H_a$. Thus, $f \in L_a = H_a$, so that f is a $\mathbf{k}$-linear combination of the $\overline{h_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq a$ (because H_a is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by these $\overline{h_\lambda}$). Since the claim we are proving (that is, $fg \in L_{a+b}$) depends $\mathbf{k}$-linearly on f, we can thus WLOG assume that f is one of those $\overline{h_\lambda}$. In other words, we can WLOG assume that $f = \overline{h_\alpha}$ for some $\alpha \in P_{k,n}$ satisfying $\ell(\alpha) \leq a$. Assume this, and consider this α. For similar reasons, we WLOG assume that $g = \overline{h_\beta}$ for some $\beta \in P_{k,n}$ satisfying $\ell(\beta) \leq b$. Consider this β.

Note that each entry of α is $\leq n - k$ (since $\alpha \in P_{k,n}$), and therefore $\leq n$. Thus, we can consider α as an a-tuple of elements of $\{0, 1, \ldots, n\}$ (since $\ell(\alpha) \leq a$). Likewise, consider β as a b-tuple of elements of $\{0, 1, \ldots, n\}$.

Let γ be the concatenation of the a-tuple α with the b-tuple β. Thus, γ is an $(a + b)$-tuple of elements of $\{0, 1, \ldots, n\}$ (since α is an a-tuple of elements of $\{0, 1, \ldots, n\}$ and since β is a b-tuple of elements of $\{0, 1, \ldots, n\}$), and satisfies $h_\gamma = h_\alpha h_\beta$. (But γ is not necessarily a partition.) Moreover, $a + b \leq k$ (since $a + b < k$). Finally, write γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{a+b})$; then, we have $\gamma_i \leq n$ for each $i \in \{1, 2, \ldots, a + b\}$ (because γ is an $(a + b)$-tuple of elements of

$\{0,1,\ldots,n\}$). Hence, Lemma 7.8 (applied to $p = a+b$, $\nu = \gamma$ and $\nu_i = \gamma_i$) yields $\overline{h_\gamma} \in H_{a+b}$. But Proposition 7.11 yields that $L_{a+b} = H_{a+b}$.

From $f = \overline{h_\alpha}$ and $g = \overline{h_\beta}$, we obtain $fg = \overline{h_\alpha}\,\overline{h_\beta} = \overline{h_\alpha h_\beta} = \overline{h_\gamma}$ (since $h_\alpha h_\beta = h_\gamma$). Thus, $fg = \overline{h_\gamma} \in H_{a+b} = L_{a+b}$ (since $L_{a+b} = H_{a+b}$). This completes our proof of Proposition 7.15. □

Corollary 7.16. We have $(L_1)^m \subseteq L_m$ for each $m \in \mathbb{N}$.

Proof of Corollary 7.16. This follows by induction on m, using the facts (which we proved in Proposition 7.15) that $1 \in L_0$ and that $L_a L_b \subseteq L_{a+b}$ for every $a, b \in \mathbb{N}$. □

7.5. A formula for hook-shaped Schur functions

Lemma 7.17. Let m be a positive integer. Let $j \in \mathbb{N}$. Then,

$$\mathbf{s}_{(m,1^j)} = \sum_{i=1}^{m} (-1)^{i-1} \mathbf{h}_{m-i} \mathbf{e}_{j+i}.$$

Proof of Lemma 7.17. For each $N \in \mathbb{N}$, we have

$$\sum_{p=0}^{N} (-1)^p \mathbf{h}_{N-p} \mathbf{e}_p = \delta_{0,N}. \tag{52}$$

(This is just the equality (46), with j renamed as p.)

From $m > 0$ and $j \geq 0$, we obtain $m + j > 0$, so that $\delta_{0,m+j} = 0$. The equality (52) (applied to $N = m + j$) becomes

$$\sum_{p=0}^{m+j} (-1)^p \mathbf{h}_{m+j-p} \mathbf{e}_p = \delta_{0,m+j} = 0.$$

Thus,

$$\begin{aligned}
0 &= \sum_{p=0}^{m+j} (-1)^p \mathbf{h}_{m+j-p}\mathbf{e}_p \\
&= \sum_{i=-m}^{j} (-1)^{j-i} \mathbf{h}_{m+i}\mathbf{e}_{j-i} \qquad \left(\begin{array}{c}\text{here, we have substituted } j-i \\ \text{for } p \text{ in the sum}\end{array}\right) \\
&= \underbrace{\sum_{i=-m}^{-1} (-1)^{j-i} \mathbf{h}_{m+i}\mathbf{e}_{j-i}}_{\substack{=\sum_{i=1}^{m}(-1)^{j+i}\mathbf{h}_{m-i}\mathbf{e}_{j+i} \\ \text{(here, we have substituted } -i \text{ for } i \\ \text{in the sum)}}} + \sum_{i=0}^{j} \underbrace{(-1)^{j-i}}_{=(-1)^j(-1)^i} \mathbf{h}_{m+i}\mathbf{e}_{j-i} \\
&= \sum_{i=1}^{m} (-1)^{j+i} \mathbf{h}_{m-i}\mathbf{e}_{j+i} + (-1)^j \underbrace{\sum_{i=0}^{j} (-1)^i \mathbf{h}_{m+i}\mathbf{e}_{j-i}}_{\substack{=\mathbf{s}_{\left(m,1^j\right)} \\ \text{(by (45))}}} \\
&= \sum_{i=1}^{m} (-1)^{j+i} \mathbf{h}_{m-i}\mathbf{e}_{j+i} + (-1)^j \mathbf{s}_{\left(m,1^j\right)}.
\end{aligned}$$

Solving this equality for $\mathbf{s}_{\left(m,1^j\right)}$, we obtain

$$\mathbf{s}_{\left(m,1^j\right)} = -\frac{1}{(-1)^j}\sum_{i=1}^{m} (-1)^{j+i} \mathbf{h}_{m-i}\mathbf{e}_{j+i} = \sum_{i=1}^{m} (-1)^{i-1} \mathbf{h}_{m-i}\mathbf{e}_{j+i}.$$

This proves Lemma 7.17. ☐

7.6. The submodules C and R_p of $\mathcal{S}/I$

Next, we introduce some more $\mathbf{k}$-submodules of $\mathcal{S}/I$:

Definition 7.18. **(a)** Let C be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{e_i}$ with $i \in \mathbb{N}$.

(b) For each $p \in \mathbb{Z}$, we let R_p be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq p$.

We recall that $e_i = 0$ for every $i > k$. Thus, $\overline{e_i} = 0$ for every $i > k$. Hence, the $\mathbf{k}$-module C is spanned by $\overline{e_0}, \overline{e_1}, \ldots, \overline{e_k}$ (because all the other among its designated generators $\overline{e_i}$ are 0). Also, the definition of C yields $\overline{e_0} \in C$, so that $1 = \overline{e_0} \in C$. Thus, each $i \in \mathbb{N}$ satisfies $C^i = \underbrace{1}_{\in C} C^i \subseteq CC^i = C^{i+1}$. In other words, $C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots$.

Note that $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$. Also:

Proposition 7.19. We have $R_{n-k} = L_1$.

Proof of Proposition 7.19. We WLOG assume that $k \neq 0$, because the case when $k = 0$ is trivial for its own reasons[13]. Thus, $k > 0$, and therefore the partition (i) belongs to $P_{k,n}$ for each $i \in \{0, 1, \ldots, n-k\}$.

Recall that L_1 was defined as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq 1$. But the $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq 1$ are exactly the partitions of the form (i) for $i \in \{0, 1, \ldots, n-k\}$. Hence, L_1 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_{(i)}}$ with $i \in \{0, 1, \ldots, n-k\}$. Since we have $s_{(i)} = h_i$ for each $i \in \{0, 1, \ldots, n-k\}$, we can rewrite this as follows: L_1 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \{0, 1, \ldots, n-k\}$. In other words, L_1 is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq n-k$. But this is precisely the definition of the $\mathbf{k}$-submodule R_{n-k}. Hence, $L_1 = R_{n-k}$. This proves Proposition 7.19. □

It is easy to see that $R_{n-k} = R_{n-k+1} = \cdots = R_n$, but the sequence $(R_0, R_1, R_2, \ldots)$ may and may not grow after its n-th term depending on the choice of $a_1, a_2, \ldots, a_k$. So the family $(R_p)_{p \in \mathbb{Z}}$ is a filtration of some $\mathbf{k}$-submodule of $\mathcal{S}/I$, but it isn't easy to say which specific $\mathbf{k}$-submodule it is.

Lemma 7.20. We have $R_p \subseteq C^p$ for each $p \in \mathbb{N}$.

Proof of Lemma 7.20. We have

$$\overline{e_i} \in C \qquad \text{for each } i \in \mathbb{N} \tag{53}$$

(by the definition of C).

Let $p \in \mathbb{N}$. Recall that R_p is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq p$. Hence, in order to prove that $R_p \subseteq C^p$, it suffices to show that $\overline{h_i} \in C^p$ for each $i \in \mathbb{N}$ satisfying $i \leq p$.

We first claim that

$$\overline{h_i} \in C^i \qquad \text{for each } i \in \mathbb{N}. \tag{54}$$

[*Proof of (54):* We shall prove (54) by strong induction on i. So we fix $j \in \mathbb{N}$, and we assume (as induction hypothesis) that (54) holds for all $i < j$. We must now prove that (54) holds for $i = j$. In other words, we must prove that $\overline{h_j} \in C^j$.

If $j = 0$, then this is obvious (because in this case, we have $\overline{h_j} = \overline{h_0} = \overline{1} = 1 \in C^0$). Thus, we WLOG assume that $j \neq 0$. Hence, j is a positive integer. Thus, Corollary 3.3 (applied to j instead of p) yields

$$h_j = -\sum_{t=1}^{k} (-1)^t e_t h_{j-t}.$$

---

[13] *Proof.* Assume that $k = 0$. Then, $\mathcal{S} = \mathbf{k}$ and $I = 0$, whence $\mathcal{S}/I = \mathbf{k} \cdot 1$. Both $\mathbf{k}$-submodules R_{n-k} and L_1 contain 1 (since $1 = \overline{h_0}$ and since $1 = \overline{s_\varnothing}$); hence, both of these $\mathbf{k}$-submodules must be the whole $\mathcal{S}/I$ (since $\mathcal{S}/I = \mathbf{k} \cdot 1$) and therefore must be equal. So we have proven $R_{n-k} = L_1$. In other words, we have proven Proposition 7.19 under the assumption that $k = 0$.

Hence,

$$\overline{h_j} = \overline{-\sum_{t=1}^{k}(-1)^t e_t h_{j-t}} = -\sum_{t=1}^{k}(-1)^t \underbrace{\overline{e_t}}_{\substack{\in C \\ \text{(by (53))}}} \underbrace{\overline{h_{j-t}}}_{\substack{\in C^{j-t} \\ \text{(by the induction} \\ \text{hypothesis, since } j-t<j)}}$$

$$\in -\sum_{t=1}^{k}(-1)^t \underbrace{CC^{j-t}}_{\substack{=C^{j-t+1}\subseteq C^j \\ \text{(since } j-t+1\leq j \text{ and} \\ C^0\subseteq C^1\subseteq C^2\subseteq\cdots)}} \subseteq -\sum_{t=1}^{k}(-1)^t C^j \subseteq C^j.$$

In other words, (54) holds for $i = j$. This completes the induction step. Thus, (54) is proven.]

Now, let us fix $i \in \mathbb{N}$ satisfying $i \leq p$. Then, $C^i \subseteq C^p$ (since $i \leq p$ and $C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots$). But (54) yields $\overline{h_i} \in C^i \subseteq C^p$.

Now, forget that we fixed i. We thus have shown that $\overline{h_i} \in C^p$ for each $i \in \mathbb{N}$ satisfying $i \leq p$. As we have said, this proves Lemma 7.20. ☐

Lemma 7.21. Let m be a positive integer. Let $j \in \mathbb{N}$. Then, $\overline{s_{(m,1^j)}} \in R_{m-1}C$.

Proof of Lemma 7.21. Lemma 7.17 yields

$$\mathbf{s}_{(m,1^j)} = \sum_{i=1}^{m}(-1)^{i-1}\mathbf{h}_{m-i}\mathbf{e}_{j+i}.$$

This is an equality in Λ. If we evaluate both of its sides at $x_1, x_2, \ldots, x_k$, then we obtain

$$s_{(m,1^j)} = \sum_{i=1}^{m}(-1)^{i-1}h_{m-i}e_{j+i}.$$

Thus,

$$\overline{s_{(m,1^j)}} = \overline{\sum_{i=1}^{m}(-1)^{i-1}h_{m-i}e_{j+i}} = \sum_{i=1}^{m}(-1)^{i-1} \underbrace{\overline{h_{m-i}}}_{\substack{\in R_{m-1} \\ \text{(by the definition of } R_{m-1}\text{,} \\ \text{since } m-i\leq m-1)}} \underbrace{\overline{e_{j+i}}}_{\substack{\in C \\ \text{(by the definition of } C)}}$$

$$\in \sum_{i=1}^{m}(-1)^{i-1}R_{m-1}C \subseteq R_{m-1}C.$$

This proves Lemma 7.21. ☐

Corollary 7.22. Let m be a positive integer. Then, $\overline{h_{n+m}} \in R_{m-1}C$.

Proof of Corollary 7.22. Proposition 7.2 yields

$$h_{n+m} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{\left(m,1^j\right)} \operatorname{mod} I.$$

Thus,

$$\overline{h_{n+m}} = \overline{\sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{\left(m,1^j\right)}} = \sum_{j=0}^{k-1} (-1)^j a_{k-j} \underbrace{\overline{s_{\left(m,1^j\right)}}}_{\substack{\in R_{m-1}C \\ \text{(by Lemma 7.21)}}}$$
$$\in \sum_{j=0}^{k-1} (-1)^j a_{k-j} R_{m-1} C \subseteq R_{m-1} C.$$

This proves Corollary 7.22. □

Lemma 7.23. Let $j \in \mathbb{N}$ be such that $j \leq n$.

(a) We have $\overline{h_j} \in L_1$.

(b) Assume that $n > k$ and $j \neq n-k$. Then, $\overline{h_j} \in R_{n-k-1}$.

Proof of Lemma 7.23. **(a)** We are in one of the following two cases:

Case 1: We have $j \leq n-k$.

Case 2: We have $j > n-k$.

Let us first consider Case 1. In this case, we have $j \leq n-k$. Recall that R_{n-k} was defined as the **k**-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq n-k$. Hence, $\overline{h_j} \in R_{n-k}$ (since $j \in \mathbb{N}$ and $j \leq n-k$). Thus, $\overline{h_j} \in R_{n-k} = L_1$ (by Proposition 7.19). Thus, Lemma 7.23 **(a)** is proven in Case 1.

Let us now consider Case 2. In this case, we have $j > n-k$. Hence, $n-k < j \leq n$, so that $j \in \{n-k+1, n-k+2, \ldots, n\}$ and therefore $j-(n-k) \in \{1,2,\ldots,k\}$. Hence, (15) (applied to $j-(n-k)$ instead of j) yields $h_j \equiv a_{j-(n-k)} \operatorname{mod} I$. Hence, $\overline{h_j} = \overline{a_{j-(n-k)}} \in \mathbf{k}$.

But $0 \leq n-k$ and thus $\overline{h_0} \in R_{n-k}$ (by the definition of R_{n-k}). Hence, $1 = \overline{h_0} \in R_{n-k}$, so that $\mathbf{k} \subseteq R_{n-k}$ and thus $\overline{h_j} \in \mathbf{k} \subseteq R_{n-k} = L_1$ (by Proposition 7.19). Thus, Lemma 7.23 **(a)** is proven in Case 2.

We have now proven Lemma 7.23 **(a)** in each of the two Cases 1 and 2. Thus, Lemma 7.23 **(a)** is proven.

(b) We are in one of the following two cases:

Case 1: We have $j \leq n-k$.

Case 2: We have $j > n-k$.

Let us first consider Case 1. In this case, we have $j \leq n-k$. Thus, $j < n-k$ (since $j \neq n-k$), so that $j \leq n-k-1$. Thus, $n-k-1 \geq j \geq 0$, so that $n-k-1 \in \mathbb{N}$. Recall that R_{n-k-1} is defined as the **k**-submodule of $\mathcal{S}/I$ spanned by the $\overline{h_i}$ with $i \in \mathbb{N}$ satisfying $i \leq n-k-1$. Hence, $\overline{h_j} \in R_{n-k-1}$ (since $j \in \mathbb{N}$ and $j \leq n-k-1$). Thus, Lemma 7.23 **(b)** is proven in Case 1.

Let us now consider Case 2. In this case, we have $j > n - k$. Hence, $n - k < j \leq n$, so that $j \in \{n-k+1, n-k+2, \ldots, n\}$ and therefore $j - (n-k) \in \{1, 2, \ldots, k\}$. Hence, (15) (applied to $j - (n-k)$ instead of j) yields $h_j \equiv a_{j-(n-k)} \operatorname{mod} I$. Hence, $\overline{h_j} = \overline{a_{j-(n-k)}} \in \mathbf{k}$.

But $n - k > 0$ (since $n > k$), and thus $1 \leq n - k$, so that $0 \leq n - k - 1$. Hence, $\overline{h_0} \in R_{n-k-1}$ (by the definition of R_{n-k-1}). Hence, $1 = \overline{h_0} \in R_{n-k-1}$, so that $\mathbf{k} \subseteq R_{n-k-1}$ and thus $\overline{h_j} \in \mathbf{k} \subseteq R_{n-k-1}$. Thus, Lemma 7.23 **(b)** is proven in Case 2.

We have now proven Lemma 7.23 **(b)** in each of the two Cases 1 and 2. Thus, Lemma 7.23 **(b)** is proven. □

7.7. Connection to the Q_p

Convention 7.24. We WLOG assume that $k > 0$ from now on.

Now, let us recall Definition 6.11.

Proposition 7.25. We have $L_{k-1} = Q_0$.

Proof of Proposition 7.25. Recall the following:

- We have defined L_{k-1} as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\ell(\lambda) \leq k-1$.
- We have defined Q_0 as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq 0$.

Comparing these two definitions, we conclude that $L_{k-1} = Q_0$ (because for any $\lambda \in P_{k,n}$, the statement $(\ell(\lambda) \leq k-1)$ is equivalent to the statement $(\lambda_k \leq 0)$). This proves Proposition 7.25. □

Lemma 7.26. We have $(L_1)^{k-1} \subseteq Q_0$.

Proof of Lemma 7.26. Corollary 7.16 yields $(L_1)^{k-1} \subseteq L_{k-1} = Q_0$ (by Proposition 7.25). This proves Lemma 7.26. □

Lemma 7.27. Let $p \in \mathbb{Z}$. Then, $CQ_p \subseteq Q_{p+1}$.

Proof of Lemma 7.27. Lemma 6.16 shows that $\overline{e_i} Q_p \subseteq Q_{p+1}$ for each $i \in \mathbb{N}$. Thus, $CQ_p \subseteq Q_{p+1}$ (since the $\mathbf{k}$-module C is spanned by the $\overline{e_i}$ with $i \in \mathbb{N}$). This proves Lemma 7.27. □

Corollary 7.28. Let $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then, $C^q Q_p \subseteq Q_{p+q}$.

Proof of Corollary 7.28. This follows by induction on q, where the induction step uses Lemma 7.27. □

7.8. Criteria for $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$

We shall now show two sufficient criteria for when a p-tuple $\nu \in \mathbb{Z}^p$ satisfies $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$.

Theorem 7.29. Let $p \in \mathbb{N}$ be such that $p \leq k$. Let $\nu = \left(\nu_1, \nu_2, \ldots, \nu_p\right) \in \mathbb{Z}^p$ be an p-tuple of integers. Let $q \in \{1, 2, \ldots, p\}$ be such that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_q > n \geq \nu_{q+1} \geq \nu_{q+2} \geq \cdots \geq \nu_p$$

and $\nu_q \leq 2n - k - q$.

Assume also that

$$\nu_i \leq 2n - k + 1 \qquad \text{for each } i \in \{1, 2, \ldots, p\}. \tag{55}$$

Then, $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$.

Proof of Theorem 7.29. From $\nu_q \leq 2n - k - q$, we obtain $2n - k - q \geq \nu_q > n$, so that $n - k - q > 0$. Thus, $n - k - q - 1 \in \mathbb{N}$.

If any of the entries $\nu_1, \nu_2, \ldots, \nu_p$ of ν is negative, then Theorem 7.29 holds for easy reasons[14]. Hence, we WLOG assume that none of the entries $\nu_1, \nu_2, \ldots, \nu_p$ of ν is negative. Thus, all of the entries $\nu_1, \nu_2, \ldots, \nu_p$ are nonnegative integers.

From $p \leq k$, we obtain $p - 1 \leq k - 1$ and thus $L_{p-1} \subseteq L_{k-1}$ (since $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$). Thus,

$$L_{p-1} \subseteq L_{k-1} = Q_0 \tag{56}$$

(by Proposition 7.25).

From $n \geq \nu_{q+1} \geq \nu_{q+2} \geq \cdots \geq \nu_p$, we conclude that $\nu_j \leq n$ for each $j \in \{q+1, q+2, \ldots, p\}$. In other words, $\nu_{q+i} \leq n$ for each $i \in \{1, 2, \ldots, p-q\}$. Hence, Proposition 7.14 (applied to $p - q$, $\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)$ and ν_{q+i} instead of p, ν and ν_i) yields $\overline{h_{\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)}} \in H_{p-q}$. But Proposition 7.11 (applied to $p - q$ instead of p) yields $L_{p-q} = H_{p-q}$. Thus,

$$\overline{h_{\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)}} \in H_{p-q} = L_{p-q}. \tag{57}$$

Next, we claim that

$$\overline{h_{\nu_i}} \in L_1 C \qquad \text{for each } i \in \{1, 2, \ldots, q-1\}. \tag{58}$$

---

[14] Indeed, in this case we have $\nu_i < 0$ for some $i \in \{1, 2, \ldots, p\}$, and therefore $h_{\nu_i} = 0$ for this i, and therefore

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p} = \left(h_{\nu_1} h_{\nu_2} \cdots h_{\nu_{i-1}}\right) \underbrace{h_{\nu_i}}_{=0} \left(h_{\nu_{i+1}} h_{\nu_{i+2}} \cdots h_{\nu_p}\right) = 0,$$

and therefore $\operatorname{coeff}_\omega\left(\overline{h_\nu}\right) = 0$, qed.

[*Proof of (58):* Let $i \in \{1, 2, \ldots, q-1\}$. Then, $\nu_i > n$ (since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_q > n$), so that $\nu_i - n$ is a positive integer. Thus, Corollary 7.22 (applied to $m = \nu_i - n$) yields $\overline{h_{\nu_i}} \in R_{\nu_i - n - 1} C$.

But $\nu_i \leq 2n - k + 1$ (by (55)), so that $\nu_i - n - 1 \leq n - k$. Thus, $R_{\nu_i - n - 1} \subseteq R_{n-k}$ (since $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$). Thus, $R_{\nu_i - n - 1} \subseteq R_{n-k} = L_1$ (by Proposition 7.19). Hence, $\overline{h_{\nu_i}} \in \underbrace{R_{\nu_i - n - 1}}_{\subseteq L_1} C \subseteq L_1 C$. This proves (58).]

From (58), we obtain

$$\overline{h_{\nu_1}}\,\overline{h_{\nu_2}} \cdots \overline{h_{\nu_{q-1}}} \in (L_1 C)^{q-1} = \underbrace{(L_1)^{q-1}}_{\substack{\subseteq L_{q-1} \\ \text{(by Corollary 7.16)}}} C^{q-1} \subseteq L_{q-1} C^{q-1}.$$

Also, $\nu_q > n$, so that $\nu_q - n$ is a positive integer. Thus, Corollary 7.22 (applied to $m = \nu_q - n$) yields $\overline{h_{\nu_q}} \in R_{\nu_q - n - 1} C$. But $\nu_q \leq 2n - k - q$ and thus $\nu_q - n - 1 \leq n - k - q - 1$. Hence, $R_{\nu_q - n - 1} \subseteq R_{n-k-q-1}$ (since $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$). Thus, $\overline{h_{\nu_q}} \in \underbrace{R_{\nu_q - n - 1}}_{\subseteq R_{n-k-q-1}} C \subseteq R_{n-k-q-1} C$.

Recall that $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}$. Thus,

$$\begin{aligned}
\overline{h_\nu} &= \overline{h_{\nu_1} h_{\nu_2} \cdots h_{\nu_p}} = \overline{h_{\nu_1}}\,\overline{h_{\nu_2}} \cdots \overline{h_{\nu_p}} \\
&= \underbrace{\left(\overline{h_{\nu_1}}\,\overline{h_{\nu_2}} \cdots \overline{h_{\nu_{q-1}}}\right)}_{\in L_{q-1} C^{q-1}} \underbrace{\overline{h_{\nu_q}}}_{\in R_{n-k-q-1} C} \underbrace{\left(\overline{h_{\nu_{q+1}}}\,\overline{h_{\nu_{q+2}}} \cdots \overline{h_{\nu_p}}\right)}_{\substack{= \overline{h_{\nu_{q+1}} h_{\nu_{q+2}} \cdots h_{\nu_p}} \\ = \overline{h_{\left(\nu_{q+1}, \nu_{q+2}, \ldots, \nu_p\right)}} \in L_{p-q} \\ \text{(by (57))}}} \\
&\in L_{q-1} C^{q-1} R_{n-k-q-1} C L_{p-q} = \underbrace{C^{q-1} C}_{= C^q} \underbrace{R_{n-k-q-1}}_{\substack{\subseteq C^{n-k-q-1} \\ \text{(by Lemma 7.20,} \\ \text{applied to } n-k-q-1 \text{ instead of } p\text{)}}} \underbrace{L_{q-1} L_{p-q}}_{\substack{\subseteq L_{(q-1)+(p-q)} \\ \text{(by (51))}}} \\
&\subseteq \underbrace{C^q C^{n-k-q-1}}_{= C^{q+(n-k-q-1)} = C^{n-k-1}} \underbrace{L_{(q-1)+(p-q)}}_{\substack{= L_{p-1} \subseteq Q_0 \\ \text{(by (56))}}} \subseteq C^{n-k-1} Q_0 \subseteq Q_{0+(n-k-1)}
\end{aligned}$$

(by Corollary 7.28, applied to $n - k - 1$ and 0 instead of q and p). In other words, $\overline{h_\nu} \in Q_{n-k-1}$. Hence, $\operatorname{coeff}_\omega \left(\overline{h_\nu}\right) \in \operatorname{coeff}_\omega \left(Q_{n-k-1}\right) = 0$ (by Lemma 6.12), and thus $\operatorname{coeff}_\omega \left(\overline{h_\nu}\right) = 0$. This proves Theorem 7.29. □

Theorem 7.30. Assume that $n > k$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k$ be a k-tuple of integers such that $\gamma \neq \omega$.

Assume that

$$\gamma_i \leq 2n - k - i \qquad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{59}$$

Then, $\operatorname{coeff}_{\omega}\left(\overline{h_{\gamma}}\right) = 0$.

Proof of Theorem 7.30. We have $k \neq 0$ [15]. Thus, $k > 0$; hence, γ_1 is well-defined.

If any of the entries $\gamma_1, \gamma_2, \ldots, \gamma_k$ of γ is negative, then Theorem 7.30 holds for easy reasons[16]. Hence, we WLOG assume that none of the entries $\gamma_1, \gamma_2, \ldots, \gamma_k$ of γ is negative. Thus, all of the entries $\gamma_1, \gamma_2, \ldots, \gamma_k$ are nonnegative integers. In other words, $(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$.

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{Z}^k$ be the weakly decreasing permutation of the k-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. Thus, $h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k}$. Hence, $h_{\nu} = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = h_{\gamma}$.

Recall that $(\nu_1, \nu_2, \ldots, \nu_k)$ is a permutation of $(\gamma_1, \gamma_2, \ldots, \gamma_k)$. In other words, there exists a permutation $\sigma \in S_k$ such that

$$\left(\nu_i = \gamma_{\sigma(i)} \text{ for each } i \in \{1, 2, \ldots, k\}\right). \tag{60}$$

Consider this σ.

Recall that $(\nu_1, \nu_2, \ldots, \nu_k)$ is weakly decreasing. Thus, $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$. Also, $(\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{N}^k$ (since $(\nu_1, \nu_2, \ldots, \nu_k)$ is a permutation of $(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$).

For each $i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned} \nu_i &= \gamma_{\sigma(i)} \qquad \text{(by (60))} \\ &\leq 2n - k - \sigma(i) \end{aligned} \tag{61}$$

(by (59), applied to $\sigma(i)$ instead of i).

We are in one of the following two cases:

Case 1: We have $\nu_1 \leq n$.

Case 2: We have $\nu_1 > n$.

Let us first consider Case 1. In this case, we have $\nu_1 \leq n$. But recall that $\gamma \neq \omega$. Hence, there exists at least one $q \in \{1, 2, \ldots, k\}$ satisfying $\nu_q \neq n - k$ [17].

---

[15] *Proof.* Assume the contrary. Thus, $k = 0$. Now, $\gamma \in \mathbb{Z}^k = \mathbb{Z}^0$ (since $k = 0$), whence $\gamma = ()$. But $k = 0$ also leads to $\omega = ()$, and thus $\gamma = () = \omega$. But this contradicts $\gamma \neq \omega$. This contradiction shows that our assumption was false. Qed.

[16] Indeed, in this case we have $\gamma_i < 0$ for some $i \in \{1, 2, \ldots, k\}$, and therefore $h_{\gamma_i} = 0$ for this i, and therefore

$$h_{\gamma} = h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_k} = \left(h_{\gamma_1} h_{\gamma_2} \cdots h_{\gamma_{i-1}}\right) \underbrace{h_{\gamma_i}}_{=0} \left(h_{\gamma_{i+1}} h_{\gamma_{i+2}} \cdots h_{\gamma_k}\right) = 0,$$

and therefore $\operatorname{coeff}_{\omega}\left(\overline{h_{\gamma}}\right) = 0$, qed.

[17] *Proof.* Assume the contrary. Thus, $\nu_i = n - k$ for each $i \in \{1, 2, \ldots, k\}$. Now, let $j \in \{1, 2, \ldots, k\}$ be arbitrary. Then, $\nu_{\sigma^{-1}(j)} = n - k$ (since $\nu_i = n - k$ for each $i \in \{1, 2, \ldots, k\}$). But (60) (applied to $i = \sigma^{-1}(j)$) yields $\nu_{\sigma^{-1}(j)} = \gamma_{\sigma\left(\sigma^{-1}(j)\right)} = \gamma_j$. Hence, $\gamma_j = \nu_{\sigma^{-1}(j)} = n - k$. Now, forget that we fixed j. We thus have proven that $\gamma_j = n - k$ for each $j \in \{1, 2, \ldots, k\}$. Hence, $\gamma = (n - k, n - k, \ldots, n - k) = \omega$. This contradicts $\gamma \neq \omega$. This contradiction shows that our assumption was false, qed.

Consider such a q.

Next, we claim that

$$\overline{h_{\nu_i}} \in L_1 \qquad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{62}$$

[*Proof of (62):* Let $i \in \{1, 2, \ldots, k\}$. We have $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$, thus $\nu_i \leq \nu_1 \leq n$. Now, $\nu_i \leq n$ and $\nu_i \in \mathbb{N}$ (since $(\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{N}^k$). Hence, Lemma 7.23 **(a)** (applied to $j = \nu_i$) yields $\overline{h_{\nu_i}} \in L_1$. This proves (62).]

Also, $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$, thus $\nu_q \leq \nu_1 \leq n$. Also, $n > k$ and $\nu_q \in \mathbb{N}$ (since $(\nu_1, \nu_2, \ldots, \nu_k) \in \mathbb{N}^k$) and $\nu_q \neq n - k$. Hence, Lemma 7.23 **(b)** (applied to $j = \nu_q$) yields $\overline{h_{\nu_q}} \in R_{n-k-1}$. From $n > k$, we obtain $n - k > 0$, so that $n - k \geq 1$, and thus $n - k - 1 \in \mathbb{N}$.

Now, $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} = \prod\limits_{i=1}^{k} h_{\nu_i}$, so that

$$\begin{aligned}
\overline{h_\nu} &= \overline{\prod_{i=1}^{k} h_{\nu_i}} = \prod_{i=1}^{k} \overline{h_{\nu_i}} = \left(\prod_{\substack{i \in \{1,2,\ldots,k\};\\ i \neq q}} \underbrace{\overline{h_{\nu_i}}}_{\substack{\in L_1 \\ \text{(by (62))}}} \right) \underbrace{\overline{h_{\nu_q}}}_{\in R_{n-k-1}} \\
&\in \underbrace{\left(\prod_{\substack{i \in \{1,2,\ldots,k\};\\ i \neq q}} L_1 \right)}_{\substack{= (L_1)^{k-1} \subseteq L_{k-1} \\ \text{(by Corollary 7.16)}}} \underbrace{R_{n-k-1}}_{\substack{\subseteq C^{n-k-1} \\ \text{(by Lemma 7.20,} \\ \text{applied to } n-k-1 \text{ instead of } p)}} \subseteq \underbrace{L_{k-1}}_{\substack{= Q_0 \\ \text{(by Proposition 7.25)}}} C^{n-k-1} \\
&= Q_0 C^{n-k-1} = C^{n-k-1} Q_0 \subseteq Q_{0+(n-k-1)}
\end{aligned}$$

(by Corollary 7.28, applied to $n - k - 1$ and 0 instead of q and p). In other words, $\overline{h_\nu} \in Q_{n-k-1}$. In view of $h_\nu = h_\gamma$, this rewrites as $\overline{h_\gamma} \in Q_{n-k-1}$. Hence, $\operatorname{coeff}_\omega \left(\overline{h_\gamma} \right) \in \operatorname{coeff}_\omega \left(Q_{n-k-1} \right) = 0$ (by Lemma 6.12), and thus $\operatorname{coeff}_\omega \left(\overline{h_\gamma} \right) = 0$. Thus, Theorem 7.30 is proven in Case 1.

Let us now consider Case 2. In this case, we have $\nu_1 > n$. Hence, there exists at least one $r \in \{1, 2, \ldots, k\}$ such that $\nu_r > n$ (namely, $r = 1$). Let q be the **largest** such r. Thus, $\nu_q > n$, but each $r > q$ satisfies $\nu_r \leq n$. Hence,

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_q > n \geq \nu_{q+1} \geq \nu_{q+2} \geq \cdots \geq \nu_k$$

(since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$). Also, $\nu_q \leq 2n - k - q$ [18]. Furthermore, (61) shows that

$$\nu_i \leq 2n - k - \underbrace{\sigma(i)}_{\geq -1} \leq 2n - k + 1$$

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[18] *Proof.* The map σ is a permutation, and thus injective. Hence, $|\sigma(\{1, 2, \ldots, q\})| = |\{1, 2, \ldots, q\}| = q$. Thus, $\sigma(\{1, 2, \ldots, q\})$ cannot be a subset of $\{1, 2, \ldots, q-1\}$ (because this would lead to $|\sigma(\{1, 2, \ldots, q\})| \leq |\{1, 2, \ldots, q-1\}| = q - 1 < q$, which would contradict $|\sigma(\{1, 2, \ldots, q\})| = q$). In other words, not every $i \in \{1, 2, \ldots, q\}$ satisfies

for each $i \in \{1, 2, \ldots, k\}$. Hence, Theorem 7.29 (applied to $p = k$) yields $\operatorname{coeff}_\omega \left(\overline{h_\nu}\right) = 0$. In view of $h_\nu = h_\gamma$, this rewrites as $\operatorname{coeff}_\omega \left(\overline{h_\gamma}\right) = 0$. Thus, Theorem 7.30 is proven in Case 2.

We have now proven Theorem 7.30 in both Cases 1 and 2. Hence, Theorem 7.30 always holds. $\square$

7.9. A criterion for $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$

Theorem 7.31. Let λ be a partition with at most k parts. Assume that $\lambda_1 \leq 2(n-k)$ and $\lambda \neq \omega$. Then, $\operatorname{coeff}_\omega \left(\overline{s_\lambda}\right) = 0$.

Proof of Theorem 7.31. We have $n > k$ [19].

We have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ (since the partition λ has at most k parts). Proposition 5.7 **(a)** yields

$$s_\lambda = \det\left(\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}$$

(by the definition of a determinant). Hence,

$$\overline{s_\lambda} = \overline{\sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}} = \sum_{\sigma \in S_k} (-1)^\sigma \overline{\prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}}. \tag{63}$$

Now, we claim that each $\sigma \in S_k$ satisfies

$$\operatorname{coeff}_\omega \left(\overline{\prod_{i=1}^{k} h_{\lambda_i - i + \sigma(i)}}\right) = 0. \tag{64}$$

[*Proof of (64):* Let $\sigma \in S_k$. Define a k-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k$ of integers by

$$\left(\gamma_i = \lambda_i - i + \sigma(i) \qquad \text{for each } i \in \{1, 2, \ldots, k\}\right). \tag{65}$$

$\sigma(i) \in \{1, 2, \ldots, q-1\}$. In other words, there exists some $i \in \{1, 2, \ldots, q\}$ that satisfies $\sigma(i) \notin \{1, 2, \ldots, q-1\}$. Consider such an i.

From $i \in \{1, 2, \ldots, q\}$, we obtain $i \leq q$ and thus $\nu_i \geq \nu_q$ (since $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$). Hence, $\nu_q \leq \nu_i$. From $\sigma(i) \notin \{1, 2, \ldots, q-1\}$, we obtain $\sigma(i) > q-1$, so that $\sigma(i) \geq q$. Now, (61) yields $\nu_i \leq 2n - k - \underbrace{\sigma(i)}_{\geq q} \leq 2n - k - q$. Now, $\nu_q \leq \nu_i \leq 2n - k - q$, qed.

[19] *Proof.* Assume the contrary. Thus, $n \leq k$ and therefore $n = k$ (since $n \geq k$). Hence, $n - k = 0$. Thus, $\lambda_1 \leq 2\underbrace{(n-k)}_{=0} = 0$, so that $\lambda_1 = 0$ and thus $\lambda = \varnothing$ (since λ is a partition). But from $n - k = 0$, we also obtain $\omega = \varnothing$ (since $\omega = (n-k, n-k, \ldots, n-k)$). Thus, $\lambda = \varnothing = \omega$. But this contradicts $\lambda \neq \omega$. This contradiction shows that our assumption was wrong, qed.

Then, $\gamma \neq \omega$ [20]. Moreover,

$$\gamma_i \leq 2n-k-i \qquad \text{for each } i \in \{1,2,\ldots,k\}$$

[21]. Hence, Theorem 7.30 yields $\operatorname{coeff}_\omega\left(\overline{h_\gamma}\right) = 0$. In view of

$$h_\gamma = h_{\gamma_1}h_{\gamma_2}\cdots h_{\gamma_k} = \prod_{i=1}^{k} \underbrace{h_{\gamma_i}}_{\substack{=h_{\lambda_i-i+\sigma(i)} \\ \text{(by (65))}}} = \prod_{i=1}^{k} h_{\lambda_i-i+\sigma(i)},$$

this rewrites as $\operatorname{coeff}_\omega\left(\overline{\prod\limits_{i=1}^{k} h_{\lambda_i-i+\sigma(i)}}\right) = 0$. Thus, (64) is proven.]

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[20] *Proof.* Assume the contrary. Thus, $\gamma = \omega$.

Let $i \in \{1,2,\ldots,k-1\}$. From $\gamma = \omega$, we obtain $\gamma_i = \omega_i = n-k$. Comparing this with (65), we find $\lambda_i - i + \sigma(i) = n-k$. The same argument (applied to $i+1$ instead of i) yields $\lambda_{i+1} - (i+1) + \sigma(i+1) = n-k$. But $\lambda_i \geq \lambda_{i+1}$ (since λ is a partition). Hence,

$$\underbrace{\lambda_i}_{\geq \lambda_{i+1}} - \underbrace{i}_{<i+1} + \sigma(i+1) > \lambda_{i+1} - (i+1) + \sigma(i+1) = n-k = \lambda_i - i + \sigma(i)$$

(since $\lambda_i - i + \sigma(i) = n-k$). If we subtract $\lambda_i - i$ from this inequality, we obtain $\sigma(i+1) > \sigma(i)$. In other words, $\sigma(i) < \sigma(i+1)$.

Now, forget that we fixed i. We thus have shown that each $i \in \{1,2,\ldots,k-1\}$ satisfies $\sigma(i) < \sigma(i+1)$. In other words, we have $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$. Hence, σ is a strictly increasing map from $\{1,2,\ldots,k\}$ to $\{1,2,\ldots,k\}$. But the only such map is id. Thus, $\sigma = \operatorname{id}$. Hence, for each $i \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned} \gamma_i &= \lambda_i - i + \underbrace{\sigma}_{=\operatorname{id}}(i) \qquad \text{(by (65))} \\ &= \lambda_i - i + \operatorname{id}(i) = \lambda_i - i + i = \lambda_i. \end{aligned}$$

Thus, $\gamma = \lambda$. Comparing this with $\gamma = \omega$, we obtain $\lambda = \omega$. This contradicts $\lambda \neq \omega$. This contradiction shows that our assumption was wrong, qed.

[21] *Proof.* Let $i \in \{1,2,\ldots,k\}$. Then, $\lambda_1 \geq \lambda_i$ (since λ is a partition), so that $\lambda_i \leq \lambda_1 \leq 2(n-k)$. Now, (65) yields

$$\gamma_i = \underbrace{\lambda_i}_{\leq 2(n-k)} - i + \underbrace{\sigma(i)}_{\leq k} \leq 2(n-k) - i + k = 2n-k-i,$$

qed.

From (63), we obtain

$$\begin{aligned}\operatorname{coeff}_{\omega}\left(\overline{s_{\lambda}}\right) &= \operatorname{coeff}_{\omega}\left(\sum_{\sigma\in S_k}(-1)^{\sigma}\overline{\prod_{i=1}^{k}h_{\lambda_i-i+\sigma(i)}}\right)\\ &= \sum_{\sigma\in S_k}(-1)^{\sigma}\underbrace{\operatorname{coeff}_{\omega}\left(\overline{\prod_{i=1}^{k}h_{\lambda_i-i+\sigma(i)}}\right)}_{\substack{=0\\ \text{(by (64))}}}=0.\end{aligned}$$

This proves Theorem 7.31. □

8. Another proof of Theorem 6.3

We can use Theorem 7.31 to obtain a second proof of Theorem 6.3. To that end, we shall use a few more basic facts about Littlewood-Richardson coefficients. First we introduce a few notations (only for this section):

Convention 8.1. Convention 6.1 remains in place for the whole Section 8. We shall also use all the notations introduced in Section 6.

8.1. Some basics on Littlewood-Richardson coefficients

Definition 8.2. Let $a \in \mathbb{N}$.

(a) We let Par_a denote the set of all partitions with size a. (That is, $\operatorname{Par}_a = \{\lambda \text{ is a partition} \mid |\lambda| = a\}$.)

(b) If λ and μ are two partitions with size a, then we write $\lambda \triangleright \mu$ if and only if we have

$$\lambda_1+\lambda_2+\cdots+\lambda_i \geq \mu_1+\mu_2+\cdots+\mu_i \qquad \text{for each } i \in \{1,2,\ldots,a\}.$$

This defines a binary relation $\triangleright$ on Par_a. This relation is the smaller-or-equal relation of a partial order on Par_a, which is called the *dominance order*.

Here is another way to describe the dominance order:

Remark 8.3. Let $a \in \mathbb{N}$. Let λ and μ be two partitions with size a. Then, we have $\lambda \triangleright \mu$ if and only if we have

$$\lambda_1+\lambda_2+\cdots+\lambda_i \geq \mu_1+\mu_2+\cdots+\mu_i \qquad \text{for each } i \geq 1. \tag{66}$$

Proof of Remark 8.3. $\Longleftarrow$: Assume that we have (66). We must prove that $\lambda \triangleright \mu$.

For each $i \in \{1, 2, \ldots, a\}$, we have $i \geq 1$ and therefore $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ (by (66)). In other words, we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \text{for each } i \in \{1, 2, \ldots, a\}.$$

In other words, we have $\lambda \triangleright \mu$ (by the definition of the relation $\triangleright$). This proves the "$\Longleftarrow$" direction of Remark 8.3.

$\Longrightarrow$: Assume that $\lambda \triangleright \mu$. We must prove that we have (66).

We have assumed that $\lambda \triangleright \mu$. In other words, we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \text{for each } i \in \{1, 2, \ldots, a\} \tag{67}$$

(by the definition of the relation $\triangleright$).

Now, let $i \geq 1$. Our goal is to show that $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$. If $i \in \{1, 2, \ldots, a\}$, then this follows from (67). Hence, for the rest of this proof, we WLOG assume that we don't have $i \in \{1, 2, \ldots, a\}$. Hence, $i \geq a + 1$ (because $i \geq 1$), so that $a + 1 \leq i < i + 1$. But λ is a partition; thus, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Now, recall that λ is a partition of size a; hence, $|\lambda| = a$. Thus,

$$\begin{aligned} a = |\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \cdots = \sum_{p=1}^{\infty} \lambda_p = \sum_{p=1}^{i+1} \underbrace{\lambda_p}_{\substack{\geq \lambda_{i+1} \\ \text{(since } p \leq i+1 \\ \text{and } \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)}} + \underbrace{\sum_{p=i+2}^{\infty} \lambda_p}_{\substack{\geq 0 \\ \text{(since all } \lambda_p \text{ are } \geq 0)}} \\ \geq \sum_{p=1}^{i+1} \lambda_{i+1} = (i+1)\,\lambda_{i+1}. \end{aligned}$$

Hence, $\lambda_{i+1} \leq \dfrac{a}{i+1} < 1$ (since $a < a + 1 < i + 1$). Thus, $\lambda_{i+1} = 0$ (since $\lambda_{i+1} \in \mathbb{N}$). Furthermore, from $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$, we conclude that each $p \in \{i+1, i+2, i+3, \ldots\}$ satisfies $\lambda_{i+1} \geq \lambda_p$ and thus $\lambda_p \leq \lambda_{i+1} = 0$ and therefore $\lambda_p = 0$ (since $\lambda_p \in \mathbb{N}$). Hence, $\sum\limits_{p=i+1}^{\infty} \underbrace{\lambda_p}_{=0} = \sum\limits_{p=i+1}^{\infty} 0 = 0$. Thus,

$$a = \sum_{p=1}^{\infty} \lambda_p = \sum_{p=1}^{i} \lambda_p + \underbrace{\sum_{p=i+1}^{\infty} \lambda_p}_{=0} = \sum_{p=1}^{i} \lambda_p = \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

The same argument (applied to the partition μ instead of λ) yields

$$a = \mu_1 + \mu_2 + \cdots + \mu_i.$$

Comparing these two equalities, we find $\lambda_1 + \lambda_2 + \cdots + \lambda_i = \mu_1 + \mu_2 + \cdots + \mu_i$. Hence, $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$.

Now, forget that we fixed i. We thus have shown that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \text{for each } i \geq 1.$$

In other words, (66). This proves the "$\Longrightarrow$" direction of Remark 8.3. □

Definition 8.4. Let μ and ν be two partitions. Then, we define two new partitions $\mu + \nu$ and $\mu \sqcup \nu$ as follows:

- The partition $\mu + \nu$ is defined as $(\mu_1 + \nu_1, \mu_2 + \nu_2, \mu_3 + \nu_3, \ldots)$.
- The partition $\mu \sqcup \nu$ is defined as the result of sorting the list $\left(\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}, \nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}\right)$ in decreasing order.

We shall use the following fact:

Proposition 8.5. Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be such that $a \leq b$. Let $\mu \in \operatorname{Par}_a$, $\nu \in \operatorname{Par}_{b-a}$ and $\lambda \in \operatorname{Par}_b$ be such that $c^{\lambda}_{\mu,\nu} \neq 0$. Then, $\mu + \nu \rhd \lambda \rhd \mu \sqcup \nu$.

Proposition 8.5 is precisely [GriRei20, Exercise 2.9.17(c)] (with k and n renamed as a and b).

Corollary 8.6. Let λ, μ and ν be three partitions such that $\lambda_1 > \mu_1 + \nu_1$. Then, $c^{\lambda}_{\mu,\nu} = 0$.

Proof of Corollary 8.6. Assume the contrary. Thus, $c^{\lambda}_{\mu,\nu} \neq 0$.

Let $a = |\mu|$; thus, $\mu \in \operatorname{Par}_a$. Let $b = |\lambda|$; thus, $\lambda \in \operatorname{Par}_b$.

Proposition 6.17 **(c)** shows that $c^{\lambda}_{\mu,\nu} = 0$ unless $|\mu| + |\nu| = |\lambda|$. Hence, $|\mu| + |\nu| = |\lambda|$ (since $c^{\lambda}_{\mu,\nu} \neq 0$). Thus, $|\nu| = \underbrace{|\lambda|}_{=b} - \underbrace{|\mu|}_{=a} = b - a$. Hence, $b - a = |\nu| \geq 0$, so that $a \leq b$. Also, from $|\nu| = b - a$, we obtain $\nu \in \operatorname{Par}_{b-a}$. Thus, Proposition 8.5 yields $\mu + \nu \rhd \lambda \rhd \mu \sqcup \nu$.

But $b = |\lambda| \geq \lambda_1 > \mu_1 + \nu_1 \geq 0$, so that $1 \in \{1, 2, \ldots, b\}$.

Now, from $\mu + \nu \rhd \lambda$, we conclude that

$$(\mu + \nu)_1 + (\mu + \nu)_2 + \cdots + (\mu + \nu)_i \geq \lambda_1 + \lambda_2 + \cdots + \lambda_i \qquad \text{for each } i \in \{1, 2, \ldots, b\}$$

(by the definition of the relation $\rhd$, since $\mu + \nu$ and λ are two partitions of size b). Applying this to $i = 1$, we obtain $(\mu + \nu)_1 \geq \lambda_1$ (since $1 \in \{1, 2, \ldots, b\}$). But the definition of $\mu + \nu$ yields $(\mu + \nu)_1 = \mu_1 + \nu_1 < \lambda_1$ (since $\lambda_1 > \mu_1 + \nu_1$). This contradicts $(\mu + \nu)_1 \geq \lambda_1$. This contradiction shows that our assumption was false. Hence, Corollary 8.6 is proven. $\square$

Next, we recall the Littlewood-Richardson rule itself:

Proposition 8.7. Let λ and μ be two partitions. Then,

$$\mathbf{s}_\lambda \mathbf{s}_\mu = \sum_{\rho \text{ is a partition}} c^{\rho}_{\lambda,\mu} \mathbf{s}_\rho .$$

Proposition 8.7 is precisely [GriRei20, (2.5.6)] (with λ, μ and ν renamed as ρ, λ and μ).

Corollary 8.8. Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$. Then,

$$s_\lambda s_\mu = \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} s_\rho .$$

Proof of Corollary 8.8. If ρ is a partition satisfying $\rho_1 > 2(n-k)$, then

$$c^\rho_{\lambda,\mu} = 0. \tag{68}$$

[*Proof of (68):* Let ρ be a partition satisfying $\rho_1 > 2(n-k)$.

We have $\lambda \in P_{k,n}$; thus, each part of λ is $\leq n-k$. Thus, $\lambda_1 \leq n-k$. Similarly, $\mu_1 \leq n-k$. Hence, $\underbrace{\lambda_1}_{\leq n-k} + \underbrace{\mu_1}_{\leq n-k} \leq 2(n-k) < \rho_1$. In other words, $\rho_1 > \lambda_1 + \mu_1$. Hence, Corollary 8.6 (applied to ρ, λ and μ instead of λ, μ and ν) yields $c^\rho_{\lambda,\mu} = 0$. This proves (68).]

Proposition 8.7 yields

$$\mathbf{s}_\lambda \mathbf{s}_\mu = \sum_{\rho \text{ is a partition}} c^\rho_{\lambda,\mu} \mathbf{s}_\rho .$$

This is an equality in Λ. Evaluating both of its sides at the k indeterminates $x_1, x_2, \ldots, x_k$, we find

$$\begin{aligned}
s_\lambda s_\mu &= \sum_{\rho \text{ is a partition}} c^\rho_{\lambda,\mu} s_\rho \\
&= \sum_{\substack{\rho \text{ is a partition;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} s_\rho + \sum_{\substack{\rho \text{ is a partition;} \\ \rho_1 > 2(n-k)}} \underbrace{c^\rho_{\lambda,\mu}}_{\substack{=0 \\ \text{(by (68))}}} s_\rho \\
&\qquad \left(\begin{array}{c} \text{since each partition } \rho \text{ satisfies either } \rho_1 \leq 2(n-k) \\ \text{or } \rho_1 > 2(n-k) \text{ (but not both)} \end{array} \right) \\
&= \sum_{\substack{\rho \text{ is a partition;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} s_\rho \\
&= \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} s_\rho + \sum_{\substack{\rho \text{ is a partition with more than } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} \underbrace{s_\rho}_{\substack{=0 \\ \text{(by (3))}}} \\
&= \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} s_\rho .
\end{aligned}$$

This proves Corollary 8.8. ☐

Next, let us recall another known fact on skew Schur functions:

Proposition 8.9. Let λ be any partition. Then, $\mathbf{s}_{\omega/\lambda^\vee} = \mathbf{s}_\lambda$.

Proof of Proposition 8.9. From [GriRei20, Exercise 2.9.15(a)] (applied to $n-k$ and $\varnothing$ instead of m and μ), we obtain $\mathbf{s}_{\lambda/\varnothing} = \mathbf{s}_{\varnothing^\vee/\lambda^\vee}$. In view of $\varnothing^\vee = \omega$, this rewrites as $\mathbf{s}_{\lambda/\varnothing} = \mathbf{s}_{\omega/\lambda^\vee}$. Thus, $\mathbf{s}_{\omega/\lambda^\vee} = \mathbf{s}_{\lambda/\varnothing} = \mathbf{s}_\lambda$. This proves Proposition 8.9. □

Corollary 8.10. Let λ and μ be two partitions. Then,

$$c^{\omega}_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \mu = \lambda^\vee; \\ 0, & \text{else} \end{cases}.$$

Proof of Corollary 8.10. Proposition 6.17 **(a)** (applied to ω and λ instead of λ and μ) shows that

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} c^{\omega}_{\lambda,\nu} \mathbf{s}_\nu. \tag{69}$$

On the other hand, it is easy to see that

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \mathbf{s}_\nu. \tag{70}$$

[*Proof of (70):* We are in one of the following two cases:

Case 1: We have $\lambda \in P_{k,n}$.

Case 2: We have $\lambda \notin P_{k,n}$.

Let us first consider Case 1. In this case, we have $\lambda \in P_{k,n}$. Thus, $\lambda^\vee$ is well-defined, and we have $(\lambda^\vee)^\vee = \lambda$. Hence, Proposition 8.9 (applied to $\lambda^\vee$ instead of λ) yields

$$\mathbf{s}_{\omega/(\lambda^\vee)^\vee} = \mathbf{s}_{\lambda^\vee} = \sum_{\nu \text{ is a partition}} \underbrace{\begin{cases} 1, & \text{if } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases}}_{\substack{= \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \\ \text{(since } \lambda \in P_{k,n} \text{ holds)}}} \mathbf{s}_\nu$$

$$= \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \mathbf{s}_\nu.$$

In view of $(\lambda^\vee)^\vee = \lambda$, this rewrites as

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \mathbf{s}_\nu.$$

Thus, (70) is proven in Case 1.

Now, let us consider Case 2. In this case, we have $\lambda \notin P_{k,n}$. Hence, $\lambda \not\subseteq \omega$ (since $\lambda \subseteq \omega$ holds if and only if $\lambda \in P_{k,n}$). Thus, $\mathbf{s}_{\omega/\lambda} = 0$. Comparing this with

$$\sum_{\nu \text{ is a partition}} \underbrace{\begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases}}_{\substack{=0 \\ (\text{since } \lambda \notin P_{k,n})}} \mathbf{s}_\nu = 0,$$

we obtain

$$\mathbf{s}_{\omega/\lambda} = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \mathbf{s}_\nu.$$

Thus, (70) is proven in Case 2.

We have now proven (70) in each of the two Cases 1 and 2. Thus, (70) always holds.]

Now, comparing (70) with (69), we obtain

$$\sum_{\nu \text{ is a partition}} c^{\omega}_{\lambda,\nu} \mathbf{s}_\nu = \sum_{\nu \text{ is a partition}} \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \nu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \mathbf{s}_\nu.$$

Since the family $(\mathbf{s}_\nu)_{\nu \text{ is a partition}}$ is a basis of the $\mathbf{k}$-module Λ, we can compare the coefficients of $\mathbf{s}_\mu$ on both sides of this equality. We thus obtain

$$c^{\omega}_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \mu = \lambda^\vee; \\ 0, & \text{else} \end{cases}.$$

This proves Corollary 8.10. □

8.2. Another proof of Theorem 6.3

We are now ready to prove Theorem 6.3 again. More precisely, we shall prove Lemma 6.22 (as we know that Theorem 6.3 quickly follows from Lemma 6.22).

Second proof of Lemma 6.22. If $k = 0$, then Lemma 6.22 holds[22]. Hence, for the rest of this proof, we WLOG assume that $k \neq 0$. Thus, $k > 0$. Hence, $\omega_1 =$

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[22] *Proof.* Assume that $k = 0$. Then, $P_{k,n} = \{\varnothing\}$, so that $\lambda \in P_{k,n} = \{\varnothing\}$ and thus $\lambda = \varnothing$. Similarly, $\mu = \varnothing$. Therefore, $\lambda = \mu^\vee$ holds. Also, $\omega = \varnothing$. Moreover, from $\lambda = \varnothing$, we obtain $s_\lambda = s_\varnothing = 1$; similarly, $s_\mu = 1$. Thus, $\underbrace{s_\lambda}_{=1} \underbrace{s_\mu}_{=1} = 1 = s_\varnothing = s_\omega$ (since $\varnothing = \omega$). Hence, $\operatorname{coeff}_\omega \left(\overline{s_\lambda s_\mu}\right) = \operatorname{coeff}_\omega \left(\overline{s_\omega}\right) = 1$. Comparing this with $\begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases} = 1$ (since $\lambda = \mu^\vee$ holds), we obtain $\operatorname{coeff}_\omega \left(\overline{s_\lambda s_\mu}\right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$. Thus, Lemma 6.22 holds. Qed.

$n-k \leq 2(n-k)$. Thus, ω is a partition ρ with at most k parts that satisfies $\rho_1 \leq 2(n-k)$ (since $\omega_1 \leq 2(n-k)$).

Corollary 8.8 yields

$$s_\lambda s_\mu = \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} s_\rho .$$

Hence,

$$\overline{s_\lambda s_\mu} = \overline{\sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} s_\rho} = \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} \overline{s_\rho} .$$

Thus,

$$\begin{aligned}
\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) &= \operatorname{coeff}_\omega\left(\sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} \overline{s_\rho}\right) \\
&= \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k)}} c^\rho_{\lambda,\mu} \operatorname{coeff}_\omega\left(\overline{s_\rho}\right) \\
&= c^\omega_{\lambda,\mu} \operatorname{coeff}_\omega\left(\overline{s_\omega}\right) + \sum_{\substack{\rho \text{ is a partition with at most } k \text{ parts;} \\ \rho_1 \leq 2(n-k); \\ \rho \neq \omega}} c^\rho_{\lambda,\mu} \underbrace{\operatorname{coeff}_\omega\left(\overline{s_\rho}\right)}_{\substack{=0 \\ \text{(by Theorem 7.31,} \\ \text{applied to } \rho \text{ instead of } \lambda)}} \\
&\qquad \left(\begin{array}{c} \text{here, we have split off the addend for } \rho = \omega \\ \text{from the sum, since } \omega \text{ is a partition } \rho \text{ with} \\ \text{at most } k \text{ parts that satisfies } \rho_1 \leq 2(n-k) \end{array}\right) \\
&= c^\omega_{\lambda,\mu} \underbrace{\operatorname{coeff}_\omega\left(\overline{s_\omega}\right)}_{\substack{=1 \\ \text{(by the definition of } \operatorname{coeff}_\omega)}} \\
&= c^\omega_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda \in P_{k,n} \text{ and } \mu = \lambda^\vee; \\ 0, & \text{else} \end{cases} \qquad \text{(by Corollary 8.10)} \\
&= \begin{cases} 1, & \text{if } \mu = \lambda^\vee; \\ 0, & \text{if } \mu \neq \lambda^\vee \end{cases} \qquad (\text{since } \lambda \in P_{k,n} \text{ holds}) \\
&= \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}
\end{aligned}$$

(since $\mu = \lambda^\vee$ holds if and only if $\lambda = \mu^\vee$). Thus, Lemma 6.22 is proven again. □

9. The h-basis and the m-basis

Convention 9.1. For the rest of Section 9, **we assume that** $a_1, a_2, \ldots, a_k$ **belong to** $\mathcal{S}$.

9.1. A lemma on the s-basis

For future use, we shall show a technical lemma, which improves on Lemma 5.13:

Lemma 9.2. Let $N \in \mathbb{N}$. Let $f \in \mathcal{S}$ be a symmetric polynomial of degree $< N$. Then, in $\mathcal{S}/I$, we have

$$\overline{f} \in \sum_{\substack{\kappa \in P_{k,n};\\ |\kappa| < N}} \mathbf{k}\overline{s_\kappa}.$$

Proof of Lemma 9.2. We shall prove Lemma 9.2 by strong induction on N. Thus, we fix some $M \in \mathbb{N}$, and we assume (as the induction hypothesis) that Lemma 9.2 holds whenever $N < M$. We now must prove that Lemma 9.2 holds for $N = M$.

Let $f \in \mathcal{S}$ be a symmetric polynomial of degree $< M$. Then, in $\mathcal{S}/I$, we shall show that $\overline{f} \in \sum\limits_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}$.

Indeed, let U be the $\mathbf{k}$-submodule $\sum\limits_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}$ of $\mathcal{S}/I$. Hence, U is the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the family $\left(\overline{s_\kappa}\right)_{\kappa \in P_{k,n};\ |\kappa| < M}$. Hence,

$$\overline{s_\kappa} \in U \qquad \text{for each } \kappa \in P_{k,n} \text{ satisfying } |\kappa| < M. \tag{71}$$

We are going to show that $\overline{f} \in U$.

Lemma 5.12 (applied to $N = M$) shows that there exists a family $(c_\kappa)_{\kappa \in P_k;\ |\kappa| < M}$ of elements of $\mathbf{k}$ such that $f = \sum\limits_{\substack{\kappa \in P_k;\\ |\kappa| < M}} c_\kappa s_\kappa$. Consider this family. Thus,

$$f = \sum_{\substack{\kappa \in P_k;\\ |\kappa| < M}} c_\kappa s_\kappa = \sum_{\substack{\mu \in P_k;\\ |\mu| < M}} c_\mu s_\mu \tag{72}$$

(here, we have renamed the summation index κ as μ).

Now, let $\mu \in P_k$ satisfy $|\mu| < M$. We shall show that $\overline{s_\mu} \in U$.

[*Proof:* If $\mu \in P_{k,n}$, then this follows directly from (71) (applied to $\kappa = \mu$). Hence, for the rest of this proof, we WLOG assume that $\mu \notin P_{k,n}$. Thus, Lemma 5.11 (applied to μ instead of λ) shows that

$$s_\mu \equiv (\text{some symmetric polynomial of degree } < |\mu|) \bmod I.$$

In other words, there exists a symmetric polynomial $g \in \mathcal{S}$ of degree $< |\mu|$ such that $s_\mu \equiv g \bmod I$. Consider this g. We have $|\mu| < M$. Hence, Lemma 9.2 holds for $N = |\mu|$ (by our induction hypothesis). Thus, we can apply Lemma 9.2 to g and $|\mu|$ instead of f and N. We thus conclude that

$$\overline{g} \in \sum_{\substack{\kappa \in P_{k,n};\\ |\kappa| < |\mu|}} \mathbf{k}\overline{s_\kappa}.$$

But from $s_\mu \equiv g \bmod I$, we obtain

$$\overline{s_\mu} = \overline{g} \in \sum_{\substack{\kappa \in P_{k,n};\\ |\kappa| < |\mu|}} \mathbf{k}\overline{s_\kappa} \subseteq \sum_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}$$

(since each $\kappa \in P_{k,n}$ satisfying $|\kappa| < |\mu|$ must also satisfy $|\kappa| < M$ (because $|\kappa| < |\mu| < M$), and therefore the sum $\sum\limits_{\substack{\kappa \in P_{k,n};\\ |\kappa| < |\mu|}} \mathbf{k}\overline{s_\kappa}$ is a subsum of the sum $\sum\limits_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}$). Hence,

$$\overline{s_\mu} \in \sum_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa} = U \qquad \left(\text{since } U \text{ is defined as } \sum_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}\right),$$

qed.]

Forget that we fixed μ. We thus have shown that

$$\overline{s_\mu} \in U \qquad \text{for each } \mu \in P_k \text{ satisfying } |\mu| < M. \tag{73}$$

Now, (72) yields

$$\overline{f} = \overline{\sum_{\substack{\mu \in P_k;\\ |\mu| < M}} c_\mu s_\mu} = \sum_{\substack{\mu \in P_k;\\ |\mu| < M}} c_\mu \underbrace{\overline{s_\mu}}_{\substack{\in U\\ \text{(by (73))}}} \in \sum_{\substack{\mu \in P_k;\\ |\mu| < M}} c_\mu U \subseteq U$$

(since U is a $\mathbf{k}$-module). In other words, $\overline{f} \in \sum\limits_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}$ (since $U = \sum\limits_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}$).

Forget that we fixed f. We thus have shown that if $f \in \mathcal{S}$ is a symmetric polynomial of degree $< M$, then $\overline{f} \in \sum\limits_{\substack{\kappa \in P_{k,n};\\ |\kappa| < M}} \mathbf{k}\overline{s_\kappa}$. In other words, Lemma 9.2 holds for $N = M$. This completes the induction step. Hence, Lemma 9.2 is proven by induction. □

9.2. The h-basis

In Theorem 7.13, we have shown that the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$ under the condition that $a_1, a_2, \ldots, a_k \in \mathbf{k}$. We shall soon prove this again, this time under the weaker condition that $a_1, a_2, \ldots, a_k \in \mathcal{S}$. The vehicle of the proof will be a triangularity property for the change-of-basis matrix between the bases $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ and $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ of $\mathcal{S}/I$. We refer to [GriRei20, Definition 11.1.16(c)] for the concepts that we shall be using. The triangularity is defined with respect to a certain partial order on the set $P_{k,n}$:

> **Definition 9.3.** We define a binary relation $\geq^*$ on the set $P_{k,n}$ as follows: For two partitions $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$, we set $\lambda \geq^* \mu$ if and only if
>
> - **either** $|\lambda| > |\mu|$
> - **or** $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$.
>
> It is clear that this relation $\geq^*$ is the greater-or-equal relation of a partial order on $P_{k,n}$. This order will be called the *size-then-antidominance order*.

Note that the condition "$|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$" in Definition 9.3 can also be restated as "$\mu \rhd \lambda$", where $\rhd$ means the dominance relation (defined in Definition 8.2 **(b)**). Indeed, this follows easily from Remark 8.3 (applied to μ and λ instead of λ and μ).

For future reference, we need two simple criteria for the $\geq^*$ relation:

> **Remark 9.4.** Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$.
>
> **(a)** If $|\lambda| > |\mu|$, then $\lambda \geq^* \mu$.
>
> **(b)** Let $a \in \mathbb{N}$. If both λ and μ are partitions of size a and satisfy $\mu \rhd \lambda$, then $\lambda \geq^* \mu$. (See Definition 8.2 **(b)** for the meaning of "$\rhd$".)

Proof of Remark 9.4. **(a)** This follows immediately from the definition of the relation $\geq^*$.

(b) Assume that both λ and μ are partitions of size a and satisfy $\mu \rhd \lambda$. Now, both partitions λ and μ have size a; in other words, $|\lambda| = a$ and $|\mu| = a$. Hence, $|\lambda| = a = |\mu|$.

We have $\mu \rhd \lambda$. In other words, we have

$$\mu_1 + \mu_2 + \cdots + \mu_i \geq \lambda_1 + \lambda_2 + \cdots + \lambda_i \qquad \text{for each } i \geq 1$$

(by Remark 8.3, applied to μ and λ instead of λ and μ). In other words, $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$. Hence, we have $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$. Therefore, $\lambda \geq^* \mu$ (by the definition of the relation $\geq^*$). This proves Remark 9.4 **(b)**. □

Now, we can put the size-then-antidominance order to use. Recall that Theorem 2.7 yields that the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

Theorem 9.5. The family $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ expands unitriangularly in the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$. Here, the word "expands unitriangularly" is understood according to [GriRei20, Definition 11.1.16(c)], with the poset structure on $P_{k,n}$ being given by the size-then-antidominance order.

Example 9.6. For this example, let $n = 5$ and $k = 3$. Assume that $a_1, a_2 \in \mathbf{k}$. Then, the expansion of the $\overline{h_\lambda}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ looks as follows:

$$\begin{aligned}
\overline{h_\varnothing} &= \overline{s_\varnothing};\\
\overline{h_{(1)}} &= \overline{s_{(1)}};\\
\overline{h_{(2)}} &= \overline{s_{(2)}};\\
\overline{h_{(1,1)}} &= \overline{s_{(2)}} + \overline{s_{(1,1)}};\\
\overline{h_{(2,1)}} &= a_1\overline{s_\varnothing} + \overline{s_{(2,1)}};\\
\overline{h_{(1,1,1)}} &= a_1\overline{s_\varnothing} + \overline{s_{(1,1,1)}} + 2\overline{s_{(2,1)}};\\
\overline{h_{(2,2)}} &= a_1\overline{s_{(1)}} + \overline{s_{(2,2)}};\\
\overline{h_{(2,1,1)}} &= -a_2\overline{s_\varnothing} + 2a_1\overline{s_{(1)}} + \overline{s_{(2,1,1)}} + \overline{s_{(2,2)}};\\
\overline{h_{(2,2,1)}} &= -a_2\overline{s_{(1)}} + a_1\overline{s_{(1,1)}} + 2a_1\overline{s_{(2)}} + \overline{s_{(2,2,1)}};\\
\overline{h_{(2,2,2)}} &= a_1^2\overline{s_\varnothing} - a_2\overline{s_{(1,1)}} + 2a_1\overline{s_{(2,1)}} + \overline{s_{(2,2,2)}}.
\end{aligned}$$

These equalities hold for arbitrary $a_1, a_2 \in \mathcal{S}$, not only for $a_1, a_2 \in \mathbf{k}$; but in the general case they are not expansions in the basis $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$, since a_1, a_2 themselves can be expanded further.

Our proof of Theorem 9.5 will use the concept of Kostka numbers. Let us recall their definition:

Definition 9.7. **(a)** See [GriRei20, §2.2] for the definition of a *column-strict tableau of shape* λ (where λ is a partition), and also for a definition of $\operatorname{cont}(T)$ where T is such a tableau.

(b) Let λ and μ be two partitions. Then, the *Kostka number* $K_{\lambda,\mu}$ is defined to be the number of all column-strict tableaux T of shape λ having $\operatorname{cont}(T) = \mu$.

This definition of $K_{\lambda,\mu}$ is a particular case of the definition of $K_{\lambda,\mu}$ in [GriRei20, Exercise 2.2.13].

We shall use the following properties of Kostka numbers:

Lemma 9.8. **(a)** If $a \in \mathbb{N}$, then we have $K_{\lambda,\mu} = 0$ for any partitions $\lambda \in \operatorname{Par}_a$ and $\mu \in \operatorname{Par}_a$ that don't satisfy $\lambda \triangleright \mu$.

(b) If $a \in \mathbb{N}$, then we have $K_{\lambda,\lambda} = 1$ for any $\lambda \in \operatorname{Par}_a$.

(c) If λ and μ are two partitions such that $|\lambda| \neq |\mu|$, then $K_{\lambda,\mu} = 0$.

(d) For any partition μ, we have

$$\mathbf{h}_\mu = \sum_{\lambda \in \operatorname{Par}} K_{\lambda,\mu} \mathbf{s}_\lambda,$$

where Par denotes the set of all partitions.

(e) For any $a \in \mathbb{N}$ and any $\lambda \in \operatorname{Par}_a$, we have

$$\mathbf{h}_\lambda = \sum_{\mu \in \operatorname{Par}_a} K_{\mu,\lambda} \mathbf{s}_\mu.$$

(f) For any $a \in \mathbb{N}$ and any $\lambda \in \operatorname{Par}_a$, we have

$$h_\lambda = \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a}} K_{\mu,\lambda} s_\mu.$$

Proof of Lemma 9.8. **(a)** This is [GriRei20, Exercise 2.2.13(d)], applied to a instead of n.

(b) This is [GriRei20, Exercise 2.2.13(e)], applied to a instead of n.

(c) Let λ and μ be two partitions such that $|\lambda| \neq |\mu|$. Let T be a column-strict tableau of shape λ having $\operatorname{cont}(T) = \mu$. We shall derive a contradiction.

Indeed, the tableau T has shape λ, and thus has $|\lambda|$ many cells. Hence,

$$\begin{aligned} |\lambda| &= (\text{the number of cells of } T) = (\text{the number of entries of } T) \\ &= |\operatorname{cont}(T)| = |\mu| \qquad (\text{since } \operatorname{cont}(T) = \mu). \end{aligned}$$

This contradicts $|\lambda| \neq |\mu|$.

Now, forget that we fixed T. We thus have found a contradiction whenever T is a column-strict tableau of shape λ having $\operatorname{cont}(T) = \mu$. Hence, there exists no such tableau. In other words, the number of such tableaux is 0. In other words, $K_{\lambda,\mu} = 0$ (since $K_{\lambda,\mu}$ is defined to be the number of such tableaux). This proves Lemma 9.8 **(c)**.

(d) This is [GriRei20, Exercise 2.7.10(a)].

(e) Let Par denote the set of all partitions. Then, Lemma 9.8 **(d)** yields that

$$\mathbf{h}_\mu = \sum_{\lambda \in \operatorname{Par}} K_{\lambda,\mu} \mathbf{s}_\lambda \qquad \text{for any partition } \mu.$$

Hence, for any partition μ, we have

$$\begin{aligned}\mathbf{h}_\mu &= \sum_{\lambda\in\operatorname{Par}} K_{\lambda,\mu}\mathbf{s}_\lambda = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=|\mu|}} K_{\lambda,\mu}\mathbf{s}_\lambda + \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|\neq|\mu|}} \underbrace{K_{\lambda,\mu}}_{\substack{=0\\ \text{(by Lemma 9.8 (c))}}} \mathbf{s}_\lambda \\ &= \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=|\mu|}} K_{\lambda,\mu}\mathbf{s}_\lambda + \underbrace{\sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|\neq|\mu|}} 0\mathbf{s}_\lambda}_{=0} = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=|\mu|}} K_{\lambda,\mu}\mathbf{s}_\lambda .\end{aligned}$$

Renaming μ and λ as λ and μ in this equality, we obtain the following: For any partition λ, we have

$$\mathbf{h}_\lambda = \sum_{\substack{\mu\in\operatorname{Par};\\ |\mu|=|\lambda|}} K_{\mu,\lambda}\mathbf{s}_\mu . \tag{74}$$

Now, let $a \in \mathbb{N}$ and $\lambda \in \operatorname{Par}_a$. Then, $|\lambda| = a$. Now, (74) becomes

$$\begin{aligned}\mathbf{h}_\lambda &= \sum_{\substack{\mu\in\operatorname{Par};\\ |\mu|=|\lambda|}} K_{\mu,\lambda}\mathbf{s}_\mu = \underbrace{\sum_{\substack{\mu\in\operatorname{Par};\\ |\mu|=a}}}_{=\sum_{\mu\in\operatorname{Par}_a}} K_{\mu,\lambda}\mathbf{s}_\mu \qquad (\text{since } |\lambda| = a) \\ &= \sum_{\mu\in\operatorname{Par}_a} K_{\mu,\lambda}\mathbf{s}_\mu .\end{aligned}$$

This proves Lemma 9.8 **(e)**.

(f) Let $a \in \mathbb{N}$ and $\lambda \in \operatorname{Par}_a$. Lemma 9.8 **(e)** yields $\mathbf{h}_\lambda = \sum_{\mu\in\operatorname{Par}_a} K_{\mu,\lambda}\mathbf{s}_\mu$. This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$\begin{aligned}h_\lambda &= \sum_{\mu\in\operatorname{Par}_a} K_{\mu,\lambda}s_\mu = \underbrace{\sum_{\substack{\mu\in\operatorname{Par}_a;\\ \mu \text{ has at most } k \text{ parts}}}}_{=\sum_{\substack{\mu\in\operatorname{Par}_a;\\ \mu\in P_k}} = \sum_{\substack{\mu\in P_k;\\ \mu\in\operatorname{Par}_a}}} K_{\mu,\lambda}s_\mu + \sum_{\substack{\mu\in\operatorname{Par}_a;\\ \mu \text{ has more than } k \text{ parts}}} K_{\mu,\lambda} \underbrace{s_\mu}_{\substack{=0\\ \text{(by (3), applied to } \mu\\ \text{instead of } \lambda)}} \\ &= \sum_{\substack{\mu\in P_k;\\ \mu\in\operatorname{Par}_a}} K_{\mu,\lambda}s_\mu + \underbrace{\sum_{\substack{\mu\in\operatorname{Par}_a;\\ \mu \text{ has more than } k \text{ parts}}} K_{\mu,\lambda}0}_{=0} = \sum_{\substack{\mu\in P_k;\\ \mu\in\operatorname{Par}_a}} K_{\mu,\lambda}s_\mu .\end{aligned}$$

This proves Lemma 9.8 **(f)**. ☐

Proof of Theorem 9.5. Let $<^*$ denote the smaller relation of the size-then-antidominance order on $P_{k,n}$. Thus, two partitions λ and μ satisfy $\mu <^* \lambda$ if and only if $\mu \neq \lambda$ and $\lambda \geq^* \mu$.

Our goal is to show that the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$. In other words, our goal is to show that each $\lambda \in P_{k,n}$ satisfies

$$\overline{h_\lambda} = \overline{s_\lambda} + \left(\text{a } \mathbf{k}\text{-linear combination of the elements } \overline{s_\mu} \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <^* \lambda\right) \tag{75}$$

(because [GriRei20, Remark 11.1.17(c)] shows that the family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ if and only if every $\lambda \in P_{k,n}$ satisfies (75)). So let us prove (75).

Fix $\lambda \in P_{k,n}$. Define $a \in \mathbb{N}$ by $a = |\lambda|$. Thus, $\lambda \in \operatorname{Par}_a$. Hence, Lemma 9.8 **(f)** yields

$$h_\lambda = \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a}} K_{\mu,\lambda} s_\mu = K_{\lambda,\lambda} s_\lambda + \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a; \\ \mu \neq \lambda}} K_{\mu,\lambda} s_\mu \tag{76}$$

(here, we have split off the addend for $\mu = \lambda$, since $\lambda \in P_{k,n} \subseteq P_k$ and $\lambda \in \operatorname{Par}_a$).

Now, let M be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the elements $\overline{s_\mu}$ for $\mu \in P_{k,n}$ satisfying $\mu <^* \lambda$. Thus, we have

$$\overline{s_\mu} \in M \qquad \text{for each } \mu \in P_{k,n} \text{ satisfying } \mu <^* \lambda. \tag{77}$$

Also, $0 \in M$ (since M is a $\mathbf{k}$-submodule of $\mathcal{S}/I$).

We shall next show that

$$K_{\mu,\lambda} \overline{s_\mu} \in M \qquad \text{for each } \mu \in P_k \text{ satisfying } \mu \in \operatorname{Par}_a \text{ and } \mu \neq \lambda. \tag{78}$$

[*Proof of (78):* Let $\mu \in P_k$ be such that $\mu \in \operatorname{Par}_a$ and $\mu \neq \lambda$. We must prove that $K_{\mu,\lambda} \overline{s_\mu} \in M$.

If $K_{\mu,\lambda} = 0$, then this follows immediately from $\underbrace{K_{\mu,\lambda}}_{=0} \overline{s_\mu} = 0 \overline{s_\mu} = 0 \in M$. Hence, for the rest of this proof, we WLOG assume that $K_{\mu,\lambda} \neq 0$.

If μ and λ would not satisfy $\mu \rhd \lambda$, then we would have $K_{\mu,\lambda} = 0$ (by Lemma 9.8 **(a)**, applied to μ and λ instead of λ and μ), which would contradict $K_{\mu,\lambda} \neq 0$. Hence, μ and λ must satisfy $\mu \rhd \lambda$. Both λ and μ are partitions of size a (since $\lambda \in \operatorname{Par}_a$ and $\mu \in \operatorname{Par}_a$). Thus, $|\lambda| = a$ and $|\mu| = a$.

Now, we are in one of the following two cases:

Case 1: We have $\mu \in P_{k,n}$.

Case 2: We have $\mu \notin P_{k,n}$.

Let us first consider Case 1. In this case, we have $\mu \in P_{k,n}$. Thus, $\lambda \geq^* \mu$ (by Remark 9.4 **(b)**) and thus $\mu <^* \lambda$ (since $\mu \neq \lambda$). Hence, (77) shows that $\overline{s_\mu} \in M$. Thus, $K_{\mu,\lambda} \underbrace{\overline{s_\mu}}_{\in M} \in K_{\mu,\lambda} M \subseteq M$ (since M is a $\mathbf{k}$-submodule of $\mathcal{S}/I$). Thus, (78) is proven in Case 1.

Let us next consider Case 2. In this case, we have $\mu \notin P_{k,n}$. Hence, Lemma 5.11 (applied to μ instead of λ) shows that

$$s_\mu \equiv (\text{some symmetric polynomial of degree } < |\mu|) \bmod I.$$

In other words, there exists some symmetric polynomial $f \in \mathcal{S}$ of degree $< |\mu|$ such that $s_\mu \equiv f \bmod I$. Consider this f. Lemma 9.2 (applied to $N = |\mu|$) yields that in $\mathcal{S}/I$, we have

$$\overline{f} \in \sum_{\substack{\kappa \in P_{k,n}; \\ |\kappa| < |\mu|}} \mathbf{k}\overline{s_\kappa}. \tag{79}$$

Now, let $\kappa \in P_{k,n}$ be such that $|\kappa| < |\mu|$. Then, $|\kappa| < |\mu| = a = |\lambda|$, so that $|\lambda| > |\kappa|$. Thus, Remark 9.4 **(a)** (applied to κ instead of μ) yields $\lambda \geq^* \kappa$. Also, $|\kappa| \neq |\lambda|$ (since $|\kappa| < |\lambda|$) and thus $\kappa \neq \lambda$. Combining this with $\lambda \geq^* \kappa$, we obtain $\kappa <^* \lambda$. Hence, (77) (applied to κ instead of μ) yields $\overline{s_\kappa} \in M$. Hence, $\mathbf{k} \underbrace{\overline{s_\kappa}}_{\in M} \subseteq \mathbf{k}M \subseteq M$ (since M is a $\mathbf{k}$-module).

Forget that we fixed κ. We thus have shown that $\mathbf{k}\overline{s_\kappa} \subseteq M$ for each $\kappa \in P_{k,n}$ satisfying $|\kappa| < |\mu|$. Hence, $\sum_{\substack{\kappa \in P_{k,n}; \\ |\kappa| < |\mu|}} \underbrace{\mathbf{k}\overline{s_\kappa}}_{\subseteq M} \subseteq \sum_{\substack{\kappa \in P_{k,n}; \\ |\kappa| < |\mu|}} M \subseteq M$ (since M is a $\mathbf{k}$-module). Thus, (79) becomes $\overline{f} \in \sum_{\substack{\kappa \in P_{k,n}; \\ |\kappa| < |\mu|}} \mathbf{k}\overline{s_\kappa} = M$. But $s_\mu \equiv f \bmod I$ and thus $\overline{s_\mu} = \overline{f} \in M$. Thus, $K_{\mu,\lambda} \underbrace{\overline{s_\mu}}_{\in M} \in K_{\mu,\lambda} M \subseteq M$ (since M is a $\mathbf{k}$-submodule of $\mathcal{S}/I$). Hence, (78) is proven in Case 2.

We have now proven (78) in both Cases 1 and 2. Hence, (78) always holds.]

Now, from (76), we obtain

$$\begin{aligned} \overline{h_\lambda} &= \overline{K_{\lambda,\lambda} s_\lambda + \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a; \\ \mu \neq \lambda}} K_{\mu,\lambda} s_\mu} = \underbrace{K_{\lambda,\lambda}}_{\substack{=1 \\ \text{(by Lemma 9.8 (b))}}} \overline{s_\lambda} + \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a; \\ \mu \neq \lambda}} \underbrace{K_{\mu,\lambda} \overline{s_\mu}}_{\substack{\in M \\ \text{(by (78))}}} \\ &\in \overline{s_\lambda} + \underbrace{\sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a; \\ \mu \neq \lambda}} M}_{\substack{\subseteq M \\ \text{(since } M \text{ is a } \mathbf{k}\text{-module)}}} \subseteq \overline{s_\lambda} + M. \end{aligned}$$

In other words, $\overline{h_\lambda} - \overline{s_\lambda} \in M$. In other words, $\overline{h_\lambda} - \overline{s_\lambda}$ is a $\mathbf{k}$-linear combination of the elements $\overline{s_\mu}$ for $\mu \in P_{k,n}$ satisfying $\mu <^* \lambda$ (since M was defined as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by these elements). In other words,

$$\overline{h_\lambda} = \overline{s_\lambda} + \left(\text{a } \mathbf{k}\text{-linear combination of the elements } \overline{s_\mu} \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <^* \lambda\right).$$

Thus, (75) is proven. As we already have explained, this completes the proof of Theorem 9.5. ☐

We can now prove Theorem 7.13 again. Better yet, we can prove the following more general fact:

Theorem 9.9. The family $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

Theorem 9.9 makes the exact same claim as Theorem 7.13, but is nevertheless more general because we have stated it in a more general context (namely, $a_1, a_2, \ldots, a_k \in \mathcal{S}$ rather than $a_1, a_2, \ldots, a_k \in \mathbf{k}$).

Proof of Theorem 9.9. Consider the finite set $P_{k,n}$ as a poset (using the size-then-antidominance order).

Theorem 9.5 says that the family $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ expands unitriangularly in the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$. Hence, the family $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ expands invertibly triangularly[23] in the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$. Thus, [GriRei20, Corollary 11.1.19(e)] (applied to $\mathcal{S}/I$, $P_{k,n}$, $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ and $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ instead of M, S, $(e_s)_{s\in S}$ and $(f_s)_{s\in S}$) shows that the family $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$ if and only if the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. Hence, the family $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$ (since the family $\left(\overline{s_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$). Thus, Theorem 9.9 is proven. (And therefore, Theorem 7.13 is proven again.) ☐

9.3. The m-basis

Next, we recall another well-known family of symmetric polynomials:

Definition 9.10. For any partition λ, we let m_λ denote the monomial symmetric polynomial in $x_1, x_2, \ldots, x_k$ corresponding to the partition λ. This monomial symmetric polynomial is what is called $m_\lambda(x_1, x_2, \ldots, x_k)$ in [GriRei20, Chapter 2]. Note that

$$m_\lambda = 0 \qquad \text{if } \lambda \text{ has more than } k \text{ parts.} \tag{80}$$

If λ is any partition, then the monomial symmetric polynomial $m_\lambda = m_\lambda(x_1, x_2, \ldots, x_k)$ is symmetric and thus belongs to $\mathcal{S}$.

We now claim the following:

Theorem 9.11. The family $\left(\overline{m_\lambda}\right)_{\lambda\in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

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[23] See [GriRei20, Definition 11.1.16(b)] for the meaning of this word.

We shall prove this further below; a different proof has been given in by Weinfeld in [Weinfe19, Corollary 6.2].

Our proof of Theorem 9.11 will again rely on the concept of unitriangularity and on a partial order on the set $P_{k,n}$. The partial order, this time, is not the size-then-antidominance order, but a simpler one (the "graded dominance order"):

Definition 9.12. We define a binary relation $\geq_*$ on the set $P_{k,n}$ as follows: For two partitions $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$, we set $\lambda \geq_* \mu$ if and only if

- $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$.

It is clear that this relation $\geq_*$ is the greater-or-equal relation of a partial order on $P_{k,n}$. This order will be called the *graded dominance order*.

Note that the condition "$|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$" in Definition 9.12 can also be restated as "$\lambda \rhd \mu$", where $\rhd$ means the dominance relation (defined in Definition 8.2 **(b)**). Indeed, this follows easily from Remark 8.3.

For future reference, we need a simple criterion for the $\geq_*$ relation:

Remark 9.13. Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$.

Let $a \in \mathbb{N}$. If both λ and μ are partitions of size a and satisfy $\lambda \rhd \mu$, then $\lambda \geq_* \mu$. (See Definition 8.2 **(b)** for the meaning of "$\rhd$".)

Proof of Remark 9.13. Assume that both λ and μ are partitions of size a and satisfy $\lambda \rhd \mu$. Now, both partitions λ and μ have size a; in other words, $|\lambda| = a$ and $|\mu| = a$. Hence, $|\lambda| = a = |\mu|$.

We have $\lambda \rhd \mu$. In other words, we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \text{for each } i \geq 1$$

(by Remark 8.3). Hence, we have $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \geq 1$. Therefore, $\lambda \geq_* \mu$ (by the definition of the relation $\geq_*$). This proves Remark 9.13. □

Now, we can put the graded dominance order to use. Recall that Theorem 2.7 yields that the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

Theorem 9.14. The family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$. Here, the word "expands unitriangularly" is understood according to [GriRei20, Definition 11.1.16(c)], with the poset structure on $P_{k,n}$ being given by the graded dominance order.

Example 9.15. For this example, let $n = 5$ and $k = 3$. Then, the expansion of the $\overline{s_\lambda}$ in the basis $\left(\overline{m_\lambda}\right)_{\lambda \in P_{k,n}}$ looks as follows:

$$\begin{aligned}
\overline{s_\varnothing} &= \overline{m_\varnothing}; \\
\overline{s_{(1)}} &= \overline{m_{(1)}}; \\
\overline{s_{(2)}} &= \overline{m_{(1,1)}} + \overline{m_{(2)}}; \\
\overline{s_{(1,1)}} &= \overline{m_{(1,1)}}; \\
\overline{s_{(2,1)}} &= 2\overline{m_{(1,1,1)}} + \overline{m_{(2,1)}}; \\
\overline{s_{(1,1,1)}} &= \overline{m_{(1,1,1)}}; \\
\overline{s_{(2,2)}} &= \overline{m_{(2,1,1)}} + \overline{m_{(2,2)}}; \\
\overline{s_{(2,1,1)}} &= \overline{m_{(2,1,1)}}; \\
\overline{s_{(2,2,1)}} &= \overline{m_{(2,2,1)}}; \\
\overline{s_{(2,2,2)}} &= \overline{m_{(2,2,2)}}.
\end{aligned}$$

The coefficients in these expansions are Kostka numbers; the $a_1, a_2, \ldots, a_k$ do not appear in them. (This will become clear in the proof of Theorem 9.14.)

To prove Theorem 9.14, we shall use the monomial symmetric functions $\mathbf{m}_\lambda$:

- For any partition λ, we let $\mathbf{m}_\lambda$ be the corresponding monomial symmetric function in Λ. (This is called m_λ in [GriRei20, (2.1.1)].)

We shall furthermore use the following property of the dominance order:

Lemma 9.16. Let $a \in \mathbb{N}$. Let $\lambda \in \operatorname{Par}_a$ and $\mu \in \operatorname{Par}_a$ be such that $\lambda \rhd \mu$. Assume that $\lambda \in P_{k,n}$ and $\mu \in P_k$. Then, $\mu \in P_{k,n}$.

Proof of Lemma 9.16. We have $\lambda \rhd \mu$. In other words, we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \text{for each } i \geq 1$$

(by Remark 8.3). Applying this to $i = 1$, we obtain $\lambda_1 \geq \mu_1$. Hence, $\mu_1 \leq \lambda_1 \leq n - k$ (since $\lambda \in P_{k,n}$). Thus, all parts of the partition μ are $\leq n - k$ (since $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$). Hence, $\mu \in P_{k,n}$ (since $\mu \in P_k$). This proves Lemma 9.16. □

Also, we shall again use Kostka numbers, specifically their following properties:

Lemma 9.17. **(a)** For any $a \in \mathbb{N}$ and any $\lambda \in \operatorname{Par}_a$, we have

$$\mathbf{s}_\lambda = \sum_{\mu \in \operatorname{Par}_a} K_{\lambda,\mu} \mathbf{m}_\mu.$$

(b) For any $a \in \mathbb{N}$ and $\lambda \in \operatorname{Par}_a$, we have

$$s_\lambda = \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu .$$

(c) For any $a \in \mathbb{N}$ and $\lambda \in \operatorname{Par}_a$ satisfying $\lambda \in P_{k,n}$, we have

$$s_\lambda = \sum_{\substack{\mu \in P_{k,n}; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu .$$

Proof of Lemma 9.17. **(a)** This is [GriRei20, Exercise 2.2.13(c)].

(b) Let $a \in \mathbb{N}$ and $\lambda \in \operatorname{Par}_a$. Lemma 9.17 **(a)** yields $\mathbf{s}_\lambda = \sum_{\mu \in \operatorname{Par}_a} K_{\lambda,\mu} \mathbf{m}_\mu$. This is an identity in Λ. Evaluating both of its sides at the k variables $x_1, x_2, \ldots, x_k$, we obtain

$$\begin{aligned} s_\lambda &= \sum_{\mu \in \operatorname{Par}_a} K_{\lambda,\mu} m_\mu = \underbrace{\sum_{\substack{\mu \in \operatorname{Par}_a; \\ \mu \text{ has at most } k \text{ parts}}}}_{= \sum_{\substack{\mu \in \operatorname{Par}_a; \\ \mu \in P_k}} = \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a}}} K_{\lambda,\mu} m_\mu + \sum_{\substack{\mu \in \operatorname{Par}_a; \\ \mu \text{ has more than } k \text{ parts}}} K_{\lambda,\mu} \underbrace{m_\mu}_{\substack{=0 \\ \text{(by (80), applied to } \mu \\ \text{instead of } \lambda)}} \\ &= \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu + \underbrace{\sum_{\substack{\mu \in \operatorname{Par}_a; \\ \mu \text{ has more than } k \text{ parts}}} K_{\lambda,\mu} 0}_{=0} = \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu . \end{aligned}$$

This proves Lemma 9.17 **(b)**.

(c) Let $a \in \mathbb{N}$ and $\lambda \in \operatorname{Par}_a$ satisfy $\lambda \in P_{k,n}$.

Fix some $\mu \in P_k$ such that $\mu \in \operatorname{Par}_a$ and $\mu \notin P_{k,n}$. Then, we don't have $\lambda \rhd \mu$ (since otherwise, Lemma 9.16 would yield that $\mu \in P_{k,n}$; but this would contradict $\mu \notin P_{k,n}$). Hence, Lemma 9.8 **(a)** yields $K_{\lambda,\mu} = 0$.

Forget that we fixed μ. We thus have shown that

$$K_{\lambda,\mu} = 0 \qquad \text{for every } \mu \in P_k \text{ satisfying } \mu \in \operatorname{Par}_a \text{ and } \mu \notin P_{k,n}. \tag{81}$$

Now, Lemma 9.17 **(b)** yields

$$s_\lambda = \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu = \underbrace{\sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a; \\ \mu \in P_{k,n}}}}_{\substack{= \sum_{\substack{\mu \in P_{k,n}; \\ \mu \in \operatorname{Par}_a}} \\ (\text{since } P_{k,n} \subseteq P_k)}} K_{\lambda,\mu} m_\mu + \sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a; \\ \mu \notin P_{k,n}}} \underbrace{K_{\lambda,\mu}}_{\substack{=0 \\ (\text{by (81)})}} m_\mu$$

$$= \sum_{\substack{\mu \in P_{k,n}; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu + \underbrace{\sum_{\substack{\mu \in P_k; \\ \mu \in \operatorname{Par}_a; \\ \mu \notin P_{k,n}}} 0 m_\mu}_{=0} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu.$$

This proves Lemma 9.17 **(c)**. □

Proof of Theorem 9.14. Let $<_*$ denote the smaller relation of the graded dominance order on $P_{k,n}$. Thus, two partitions λ and μ satisfy $\mu <_* \lambda$ if and only if $\mu \neq \lambda$ and $\lambda \geq_* \mu$.

Our goal is to show that the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$. In other words, our goal is to show that each $\lambda \in P_{k,n}$ satisfies

$$\overline{s_\lambda} = \overline{m_\lambda} + \left(\text{a } \mathbf{k}\text{-linear combination of the elements } \overline{m_\mu} \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <_* \lambda\right) \tag{82}$$

(because [GriRei20, Remark 11.1.17(c)] shows that the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$ if and only if every $\lambda \in P_{k,n}$ satisfies (82)). So let us prove (82).

Fix $\lambda \in P_{k,n}$. Define $a \in \mathbb{N}$ by $a = |\lambda|$. Thus, $\lambda \in \operatorname{Par}_a$. Hence, Lemma 9.17 **(c)** yields

$$s_\lambda = \sum_{\substack{\mu \in P_{k,n}; \\ \mu \in \operatorname{Par}_a}} K_{\lambda,\mu} m_\mu = K_{\lambda,\lambda} m_\lambda + \sum_{\substack{\mu \in P_{k,n}; \\ \mu \in \operatorname{Par}_a; \\ \mu \neq \lambda}} K_{\lambda,\mu} m_\mu \tag{83}$$

(here, we have split off the addend for $\mu = \lambda$, since $\lambda \in P_{k,n}$ and $\lambda \in \operatorname{Par}_a$).

Now, let M be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the elements $\overline{m_\mu}$ for $\mu \in P_{k,n}$ satisfying $\mu <_* \lambda$. Thus, we have

$$\overline{m_\mu} \in M \qquad \text{for each } \mu \in P_{k,n} \text{ satisfying } \mu <_* \lambda. \tag{84}$$

Also, $0 \in M$ (since M is a $\mathbf{k}$-submodule of $\mathcal{S}/I$).

We shall next show that

$$K_{\lambda,\mu} \overline{m_\mu} \in M \qquad \text{for each } \mu \in P_{k,n} \text{ satisfying } \mu \in \operatorname{Par}_a \text{ and } \mu \neq \lambda. \tag{85}$$

[*Proof of (85):* Let $\mu \in P_{k,n}$ be such that $\mu \in \operatorname{Par}_a$ and $\mu \neq \lambda$. We must prove that $K_{\lambda,\mu} \overline{m_\mu} \in M$.

If $K_{\lambda,\mu} = 0$, then this follows immediately from $\underbrace{K_{\lambda,\mu}}_{=0} \overline{m_\mu} = 0\overline{m_\mu} = 0 \in M$. Hence, for the rest of this proof, we WLOG assume that $K_{\lambda,\mu} \neq 0$.

If λ and μ would not satisfy $\lambda \trianglerighteq \mu$, then we would have $K_{\lambda,\mu} = 0$ (by Lemma 9.8 **(a)**), which would contradict $K_{\lambda,\mu} \neq 0$. Hence, λ and μ must satisfy $\lambda \trianglerighteq \mu$. Both λ and μ are partitions of size a (since $\lambda \in \operatorname{Par}_a$ and $\mu \in \operatorname{Par}_a$). Thus, $|\lambda| = a$ and $|\mu| = a$. Thus, $\lambda \geq_* \mu$ (by Remark 9.13) and thus $\mu <_* \lambda$ (since $\mu \neq \lambda$). Hence, (84) shows that $\overline{m_\mu} \in M$. Thus, $K_{\lambda,\mu} \underbrace{\overline{m_\mu}}_{\in M} \in K_{\lambda,\mu} M \subseteq M$ (since M is a $\mathbf{k}$-submodule of $\mathcal{S}/I$). Thus, (85) is proven.]

Now, from (83), we obtain

$$\begin{aligned}\overline{s_\lambda} &= \overline{K_{\lambda,\lambda} m_\lambda + \sum_{\substack{\mu \in P_{k,n};\\ \mu \in \operatorname{Par}_a;\\ \mu \neq \lambda}} K_{\lambda,\mu} m_\mu} = \underbrace{K_{\lambda,\lambda}}_{\substack{=1\\ \text{(by Lemma 9.8 (b))}}} \overline{m_\lambda} + \sum_{\substack{\mu \in P_{k,n};\\ \mu \in \operatorname{Par}_a;\\ \mu \neq \lambda}} \underbrace{K_{\lambda,\mu} \overline{m_\mu}}_{\substack{\in M\\ \text{(by (85))}}} \\ &\in \overline{m_\lambda} + \underbrace{\sum_{\substack{\mu \in P_{k,n};\\ \mu \in \operatorname{Par}_a;\\ \mu \neq \lambda}} M}_{\substack{\subseteq M\\ \text{(since } M \text{ is a } \mathbf{k}\text{-module)}}} \subseteq \overline{m_\lambda} + M.\end{aligned}$$

In other words, $\overline{s_\lambda} - \overline{m_\lambda} \in M$. In other words, $\overline{s_\lambda} - \overline{m_\lambda}$ is a $\mathbf{k}$-linear combination of the elements $\overline{m_\mu}$ for $\mu \in P_{k,n}$ satisfying $\mu <_* \lambda$ (since M was defined as the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by these elements). In other words,

$$\overline{s_\lambda} = \overline{m_\lambda} + \left(\text{a } \mathbf{k}\text{-linear combination of the elements } \overline{m_\mu} \text{ for } \mu \in P_{k,n} \text{ satisfying } \mu <_* \lambda\right).$$

Thus, (82) is proven. As we already have explained, this completes the proof of Theorem 9.14. □

Proof of Theorem 9.11. Consider the finite set $P_{k,n}$ as a poset (using the graded dominance order).

Theorem 9.14 says that the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ expands unitriangularly in the family $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$. Hence, the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ expands invertibly triangularly[24] in the family $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$. Thus, [GriRei20, Corollary 11.1.19(e)] (applied to $\mathcal{S}/I$, $P_{k,n}$, $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ and $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$ instead of M, S, $(e_s)_{s \in S}$ and $(f_s)_{s \in S}$) shows that the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$ if and only if the family $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$. Hence, the family $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$ (since the family $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$). Thus, Theorem 9.11 is proven. □

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[24] See [GriRei20, Definition 11.1.16(b)] for the meaning of this word.

9.4. The e-basis

We recall one more classical basis of $\mathcal{S}$:

Definition 9.18. For any partition λ, we let e_λ denote the elementary symmetric polynomial in $x_1, x_2, \ldots, x_k$ corresponding to the partition λ. This elementary symmetric polynomial is what is called $e_\lambda(x_1, x_2, \ldots, x_k)$ in [GriRei20, Chapter 2]. It is explicitly given by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} e_{\lambda_3} \cdots .$$

Note that

$$e_\lambda = 0 \qquad \text{if } \lambda_1 > k.$$

If λ is any partition, then the elementary symmetric polynomial $e_\lambda = e_\lambda(x_1, x_2, \ldots, x_k)$ is symmetric and thus belongs to $\mathcal{S}$.

It is well-known (and goes back to Gauss) that the $\mathbf{k}$-algebra $\mathcal{S}$ is generated by the algebraically independent elements $e_1, e_2, \ldots, e_k$. Equivalently, the family $(e_{\lambda^t})_{\lambda \in P_k}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}$ (see Definition 5.6 for the meaning of P_k). Again, we can obtain a basis of $\mathcal{S}/I$ by restricting this family:

Theorem 9.19. The family $(\overline{e_{\lambda^t}})_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.

This is a result of Weinfeld, proved in [Weinfe19, Theorem 6.2].

9.5. Non-bases

What other known families of symmetric functions give rise to bases of $\mathcal{S}/I$? Here is an example of a family that does not lead to such a basis (at least not in an obvious way):

Remark 9.20. Let $n = 4$ and $k = 2$. Let $a_1, a_2 \in \mathbf{k}$. For each partition λ, let p_λ be the corresponding power sum symmetric polynomial, i.e., the $p_\lambda(x_1, x_2, \ldots, x_k)$ from [GriRei20, Definition 2.2.1]. Then, the family $(\overline{p_\lambda})_{\lambda \in P_{k,n}}$ is not a basis of the $\mathbf{k}$-module $\mathcal{S}/I$ (unless $\mathbf{k} = 0$).

Proof of Remark 9.20. Straightforward computations yield the following expansions of the $\overline{p_\lambda}$ in the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of $\mathcal{S}/I$:

$$\begin{aligned}
\overline{p_\varnothing} &= \overline{s_\varnothing}; \\
\overline{p_{(1)}} &= \overline{s_{(1)}}; \\
\overline{p_{(2)}} &= -\overline{s_{(1,1)}} + \overline{s_{(2)}}; \\
\overline{p_{(1,1)}} &= \overline{s_{(1,1)}} + \overline{s_{(2)}}; \\
\overline{p_{(2,1)}} &= a_1 \overline{s_\varnothing}; \\
\overline{p_{(2,2)}} &= 2a_2 \overline{s_\varnothing} - a_1 \overline{s_{(1)}} + 2\overline{s_{(2,2)}}.
\end{aligned}$$

Thus, $\overline{p_{(2,1)}} - a_1 \overline{p_\varnothing} = 0$. Hence, the family $\left(\overline{p_\lambda}\right)_{\lambda \in P_{k,n}}$ fails to be **k**-linearly independent, and thus cannot be a basis of $\mathcal{S}/I$. This proves Remark 9.20. $\square$

It is natural to wonder for which pairs (n,k) the family $\left(\overline{p_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of $\mathcal{S}/I$. The following table (made using SageMath) collects some answers:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=2$	yes						
$n=3$	yes	yes					
$n=4$	yes	no	yes				
$n=5$	yes	st	st	yes			
$n=6$	yes	st	no	no	yes		
$n=7$	yes	st	st	st	st	yes	
$n=8$	yes	st	st	no	st	no	yes

Here, "yes" means that the family is a basis; "no" means that the family is not a basis; "st" means that the answer depends on the characteristic of **k**. Interestingly, the answer never depends on $a_1, a_2, \ldots, a_k$ in the cases tabulated above. (We have omitted the trivial cases $k = 0$ and $k = n$ from the table, since $\mathcal{S}/I \cong \mathbf{k}$ in these cases. We note that the "yes"es for $k = n - 1$ are fairly obvious, since $\left(\overline{p_\lambda}\right)_{\lambda \in P_{k,n}} = \left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ in this case. The "yes"es for $k = 1$ hold for the same reason.)

Question 9.21. Which other patterns in the above table can be explained? Is there a reason why the "no"s appear for even n's?

Another non-basis is the family $\left(\overline{h_{\lambda^t}}\right)_{\lambda \in P_{k,n}}$:

Remark 9.22. Let $n = 3$ and $k = 2$. Let $a_1, a_2 \in \mathbf{k}$. Then, the family $\left(\overline{h_{\lambda^t}}\right)_{\lambda \in P_{k,n}}$ is not a basis of the **k**-module $\mathcal{S}/I$ (unless $\mathbf{k} = 0$).

Proof of Remark 9.22. It is easy to see that $\overline{h_{(1,1)^t}} - a_1 \overline{h_{\varnothing^t}} = 0$ (indeed, this follows from $h_{(1,1)^t} - a_1 h_{\varnothing^t} = h_2 - a_1 \in I$). Hence, the family $\left(\overline{h_{\lambda^t}}\right)_{\lambda \in P_{k,n}}$ fails to be **k**-linearly independent, and thus cannot be a basis of $\mathcal{S}/I$. This proves Remark 9.22. $\square$

The followng table shows for which pairs (n,k) the family $\left(\overline{h_{\lambda^t}}\right)_{\lambda \in P_{k,n}}$ is a basis of $\mathcal{S}/I$. The following table collects some answers:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=2$	yes						
$n=3$	yes	no					
$n=4$	yes	yes	no				
$n=5$	yes	st	no	no			
$n=6$	yes	no	yes	no	no		
$n=7$	yes	st	st	no	no	no	
$n=8$	yes	st	no	yes	no	no	no

.

The “yes”s in the $k = 1$ column are easily explained (they are saying that $(1, x, x^2, \ldots, x^{n-k})$ is a basis of $\mathbf{k}[x] \,/\, (x^{n-k+1} - a_1)$), and so are the “no”s in the $k > n/2$ region (indeed, in these cases, $\overline{h_{n-k+1}} - a_1\overline{h_\varnothing} = 0$ provides a $\mathbf{k}$-linear dependence relation between the $\overline{h_{\lambda^t}}$, as in our above proof of Remark 9.22). The “yes”s in the $k = n/2$ cases follow from Theorem 7.13 (since the family $\left(\overline{h_{\lambda^t}}\right)_{\lambda\in P_{k,n}}$ is a relabeling of the family $\left(\overline{h_\lambda}\right)_{\lambda\in P_{k,n}}$ in these cases).

Question 9.23. Which other patterns exist in the above table?

10. Pieri rules for multiplying by $\overline{h_j}$

Convention 10.1. Convention 6.1 remains in place for the whole Section 10. We shall also use all the notations introduced in Section 6.

In this section, we shall explore formulas for expanding products of the form $\overline{s_\lambda h_j}$ in the basis $\left(\overline{s_\mu}\right)_{\mu\in P_{k,n}}$. We begin with the simplest case – that of $j = 1$:

10.1. Multiplying by $\overline{h_1}$

Proposition 10.2. Let $\lambda \in P_{k,n}$. Assume that $k > 0$.

(a) If $\lambda_1 < n - k$, then

$$\overline{s_\lambda h_1} = \sum_{\substack{\mu\in P_{k,n};\\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu}.$$

(b) Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. If $\lambda_1 = n - k$, then

$$\overline{s_\lambda h_1} = \sum_{\substack{\mu\in P_{k,n};\\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} + \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu\in P_{k,n};\\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Proof of Proposition 10.2. We have $\mathbf{h}_1 = \mathbf{e}_1$, thus

$$\mathbf{s}_\lambda \mathbf{h}_1 = \mathbf{s}_\lambda \mathbf{e}_1 = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical 1-strip}}} \mathbf{s}_\mu$$

(by Proposition 6.6, applied to $i = 1$). Evaluating both sides of this identity at the k variables $x_1, x_2, \ldots, x_k$, we find

$$s_\lambda h_1 = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical 1-strip}}} s_\mu = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box}}} s_\mu$$

(because a skew diagram μ/λ is a vertical 1-strip if and only if it is a single box). This becomes

$$\begin{aligned} s_\lambda h_1 &= \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box}}} s_\mu \\ &= \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} s_\mu + \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has more than } k \text{ parts}}} \underbrace{s_\mu}_{\substack{=0 \\ \text{(by (3)} \\ \text{(applied to } \mu \text{ instead of } \lambda))}} \\ &= \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} s_\mu . \end{aligned} \tag{86}$$

(a) Assume that $\lambda_1 < n - k$. Then, each partition μ satisfying

$$(\mu/\lambda \text{ is a single box}) \wedge (\mu \text{ has at most } k \text{ parts}) \tag{87}$$

must satisfy

$$\mu \in P_{k,n}. \tag{88}$$

[*Proof of (88):* Let μ be a partition satisfying (87). We must prove that $\mu \in P_{k,n}$.

We have $\mu_1 \leq \lambda_1 + 1$ (since μ/λ is a single box) and thus $\mu_1 \leq \lambda_1 + 1 \leq n - k$ (since $\lambda_1 < n - k$). Hence, each part of μ is $\leq n - k$ (since μ is a partition). Thus, $\mu \in P_{k,n}$ (since μ has at most k parts). This proves (88).]

Now, (86) becomes

$$s_\lambda h_1 = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} s_\mu = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu$$

(because (88) yields the equality $\sum\limits_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} = \sum\limits_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}}$ of summation signs). Projecting both sides of this equality onto $\mathcal{S}/I$, we obtain

$$\overline{s_\lambda h_1} = \overline{\sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu}.$$

This proves Proposition 10.2 **(a)**.

(b) Assume that $\lambda_1 = n - k$. Let ν be the partition $(\lambda_1 + 1, \lambda_2, \lambda_3, \ldots)$. Then, ν/λ is a single box, which lies in the first row. The definition of ν yields $\nu_1 = \lambda_1 + 1 = n - k + 1$ (since $\lambda_1 = n - k$) and thus $\nu_1 > n - k$; hence, not all parts of ν are $\leq n - k$. Thus, $\nu \notin P_{k,n}$.

Clearly, $\overline{\lambda} \in P_{k,n}$. Hence, if $i \in \mathbb{N}$, and if μ is any partition such that $\overline{\lambda}/\mu$ is a vertical i-strip, then $\mu \in P_{k,n}$ (since $\mu \subseteq \overline{\lambda}$). Thus, for each $i \in \mathbb{N}$, we have the following equality of summation signs:

$$\sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} = \sum_{\substack{\mu \in P_{k,n}; \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} . \tag{89}$$

The partition ν has at most k parts (since λ has at most k parts, and since $k > 0$). The definition of ν yields $\nu_1 = \lambda_1 + 1 = n - k + 1$ (since $\lambda_1 = n - k$) and $(\nu_2, \nu_3, \nu_4, \ldots) = (\lambda_2, \lambda_3, \lambda_4, \ldots) = \overline{\lambda}$. Hence, Lemma 6.13 (applied to ν and ν_i instead of λ and λ_i) yields

$$\overline{s_\nu} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} = \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \in P_{k,n}; \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \tag{90}$$

(by (89)).

Each partition μ satisfying

$$(\mu/\lambda \text{ is a single box}) \wedge (\mu \text{ has at most } k \text{ parts}) \wedge (\mu \neq \nu) \tag{91}$$

must satisfy

$$\mu \in P_{k,n}. \tag{92}$$

[*Proof of (92):* Let μ be a partition satisfying (91). We must prove that $\mu \in P_{k,n}$.

We know that μ/λ is a single box. If we had $\mu_1 > \lambda_1$, then this box would lie in the first row, which would yield that $\mu = \nu$ (because ν is the partition obtained from λ by adding a box in the first row); but this would contradict $\mu \neq \nu$. Hence, we cannot have $\mu_1 > \lambda_1$. Thus, we have $\mu_1 \leq \lambda_1 = n - k$. Hence, each part of μ is $\leq n - k$ (since μ is a partition). Thus, $\mu \in P_{k,n}$ (since μ has at most k parts). This proves (92).]

Conversely, each $\mu \in P_{k,n}$ satisfies $\mu \neq \nu$ (because $\nu \notin P_{k,n}$) and has at most k parts. Combining this with (92), we obtain the following equality of summation signs:

$$\sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts;} \\ \mu \neq \nu}} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} . \tag{93}$$

Now, (86) becomes

$$s_\lambda h_1 = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts}}} s_\mu = s_\nu + \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a single box;} \\ \mu \text{ has at most } k \text{ parts;} \\ \mu \neq \nu}} s_\mu$$

$$\left(\begin{array}{c}\text{here, we have split off the addend for } \mu = \nu \text{ from the sum} \\ \text{(since } \nu/\lambda \text{ is a single box, and since } \nu \text{ has at most } k \text{ parts)}\end{array}\right)$$

$$= s_\nu + \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu \qquad \text{(by (93))}.$$

Projecting both sides of this equality onto $\mathcal{S}/I$, we obtain

$$\overline{s_\lambda h_1} = \overline{s_\nu + \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} s_\mu} = \overline{s_\nu} + \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} + \overline{s_\nu}$$

$$= \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} + \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \in P_{k,n}; \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}$$

(by (90)). This proves Proposition 10.2 **(b)**. ☐

10.2. Multiplying by $\overline{h_{n-k}}$

On the other end of the spectrum is the case of $j = n - k$; this case also turns out to have a simple answer:

Proposition 10.3. Let $\lambda \in P_{k,n}$. Assume that $k > 0$.

(a) We have

$$\overline{s_\lambda h_{n-k}} = \overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

(b) If $\lambda_k > 0$, then

$$\overline{s_\lambda h_{n-k}} = -\sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Proof of Proposition 10.3. We have $\lambda \in P_{k,n}$, thus $\lambda_1 \leq n - k$. Hence, $n - k \geq \lambda_1$. Thus, $(n - k, \lambda_1, \lambda_2, \lambda_3, \ldots)$ is a partition.

(a) We have

$$(\mathbf{e}_i)^\perp \mathbf{s}_\lambda = 0 \qquad \text{for every integer } i > k. \tag{94}$$

[*Proof of (94):* Let $i > k$ be an integer. The partition λ has at most k parts (since $\lambda \in P_{k,n}$). In other words, the Young diagram of λ contains at most k rows. Hence, this diagram contains no vertical i-strip (since a vertical i-strip would involve more than k rows (because $i > k$)). Thus, there exists no partition μ such that λ / μ is a vertical i-strip. Hence, $\sum\limits_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu = (\text{empty sum}) = 0$. But Corollary 6.7 yields $(\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \sum\limits_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu = 0$. This proves (94).]

Recall that $\mathbf{e}_0 = 1$ and thus $(\mathbf{e}_0)^\perp = 1^\perp = \text{id}$. Hence, $(\mathbf{e}_0)^\perp \mathbf{s}_\lambda = \text{id}\, \mathbf{s}_\lambda = \mathbf{s}_\lambda$.

But $n - k \geq \lambda_1$. Hence, Proposition 6.8 (applied to $m = n - k$) yields

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \mathbf{s}_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}.$$

Hence,

$$\begin{aligned}
\mathbf{s}_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} &= \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda \\
&= \sum_{i=0}^{k} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda + \sum_{i=k+1}^{\infty} (-1)^i \mathbf{h}_{n-k+i} \underbrace{(\mathbf{e}_i)^\perp \mathbf{s}_\lambda}_{\substack{=0 \\ \text{(by (94))}}} \\
&= \sum_{i=0}^{k} (-1)^i \mathbf{h}_{n-k+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda \\
&= \underbrace{(-1)^0}_{=1} \underbrace{\mathbf{h}_{n-k+0}}_{=\mathbf{h}_{n-k}} \underbrace{(\mathbf{e}_0)^\perp \mathbf{s}_\lambda}_{=\mathbf{s}_\lambda} + \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \underbrace{(\mathbf{e}_i)^\perp \mathbf{s}_\lambda}_{\substack{= \sum\limits_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\ \text{(by Corollary 6.7)}}} \\
&= \underbrace{\mathbf{h}_{n-k} \mathbf{s}_\lambda}_{=\mathbf{s}_\lambda \mathbf{h}_{n-k}} + \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu \\
&= \mathbf{s}_\lambda \mathbf{h}_{n-k} + \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu,
\end{aligned}$$

so that

$$\mathbf{s}_\lambda \mathbf{h}_{n-k} = \mathbf{s}_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$$

This is an equality in Λ. If we evaluate both of its sides at $x_1, x_2, \ldots, x_k$, then we obtain

$$s_\lambda h_{n-k} = s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i \underbrace{h_{n-k+i}}_{\substack{\equiv a_i \bmod I \\ \text{(by (15))}}} \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}}}_{\substack{= \sum\limits_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \\ \text{(because if } \mu \text{ is a partition such} \\ \text{that } \lambda/\mu \text{ is a vertical } i\text{-strip, then } \mu\in P_{k,n} \\ \text{(since } \mu\subseteq\lambda \text{ and } \lambda\in P_{k,n}\text{))}}} s_\mu$$

$$\equiv s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} s_\mu \bmod I.$$

In other words,

$$\overline{s_\lambda h_{n-k}} = \overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} s_\mu}$$
$$= \overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

This proves Proposition 10.3 **(a)**.

(b) Assume that $\lambda_k > 0$. Hence, the partition $(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)$ has more than k parts (since its $(k+1)$-st entry is $\lambda_k > 0$). Thus, (3) (applied to $(n-k, \lambda_1, \lambda_2, \lambda_3, \ldots)$ instead of λ) yields $s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)} = 0$. Hence, $\overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}} = \overline{0} = 0$. Now, Proposition 10.3 **(a)** yields

$$\overline{s_\lambda h_{n-k}} = \underbrace{\overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}}}_{=0} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}$$
$$= -\sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu\in P_{k,n}; \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

This proves Proposition 10.3 **(b)**. □

10.3. Multiplying by $\overline{h_j}$

At last, let us give an explicit expansion for $\overline{s_\lambda h_j}$ in the basis $\left(\overline{s_\mu}\right)_{\mu\in P_{k,n}}$ that holds for all $j \in \{0, 1, \ldots, n-k\}$. Before we state it, we need a notation:

Definition 10.4. Let $\mathbf{f} \in \Lambda$ be any symmetric function. Then, $\overline{\mathbf{f}} \in \mathcal{S}/I$ is defined to be $\overline{f}$, where $f \in \mathcal{S}$ is the result of evaluating the symmetric function $\mathbf{f} \in \Lambda$ at the k variables $x_1, x_2, \ldots, x_k$. Thus, for every partition λ, we have $\overline{\mathbf{s}_\lambda} = \overline{s_\lambda}$. Likewise, for any $m \in \mathbb{N}$, we have $\overline{\mathbf{h}_m} = \overline{h_m}$ and $\overline{\mathbf{e}_m} = \overline{e_m}$.

Theorem 10.5. Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{\left(n-k-j+1, 1^{i-1}\right)}\right)^\perp \mathbf{s}_\lambda}.$$

Example 10.6. If $n = 7$ and $k = 3$, then

$$\begin{aligned} &\overline{s_{(4,3,2)} h_2} \\ &= \overline{s_{(4,4,3)}} + a_1 \left(\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}\right) - a_2 \left(\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}\right) \\ &\qquad + a_3 \left(\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}\right). \end{aligned}$$

It is not hard to reveal Propositions 10.2 and 10.3 as particular cases of Theorem 10.5 (by setting $j = 1$ or $j = n-k$, respectively). Likewise, one can see that Theorem 10.5 generalizes [BeCiFu99, (22)]. Indeed, [BeCiFu99, (22)] says that if $a_1 = a_2 = \cdots = a_{k-1} = 0$, then every $\lambda \in P_{k,n}$ and $j \in \{0, 1, \ldots, n-k\}$ satisfy

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - (-1)^k a_k \sum_{\nu} \overline{s_\nu},$$

where the second sum runs over all $\nu \in P_{k,n}$ satisfying

$$\begin{aligned} &(\lambda_i - 1 \geq \nu_i \text{ for all } i \in \{1, 2, \ldots, k\}) \qquad \text{and} \\ &(\nu_i \geq \lambda_{i+1} - 1 \text{ for all } i \in \{1, 2, \ldots, k-1\}) \qquad \text{and} \\ &|\nu| = |\lambda| + j - n. \end{aligned}$$

Note, however, that the sums in Theorem 10.5 contain multiplicities (see the "$2\overline{s_{(3,2)}}$" in Example 10.6), unlike those in [BeCiFu99, (22)].

We shall prove Theorem 10.5 by deriving it from an identity between genuine symmetric functions (in Λ, not in $\mathcal{S}$ or $\mathcal{S}/I$):

Theorem 10.7. Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition}; \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu - \sum_{i=1}^{k} (-1)^i \mathbf{h}_{n-k+i} \left(\mathbf{s}_{\left(n-k-j+1, 1^{i-1}\right)}\right)^\perp \mathbf{s}_\lambda.$$

Before we prove this theorem, we need several auxiliary results. First, we recall one of the Pieri rules ([GriRei20, (2.7.1)]):

Proposition 10.8. Let λ be a partition, and let $i \in \mathbb{N}$. Then,

$$\mathbf{s}_\lambda \mathbf{h}_i = \sum_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a horizontal } i\text{-strip}}} \mathbf{s}_\mu .$$

From this, we can easily derive the following:

Corollary 10.9. Let λ be a partition, and let $i \in \mathbb{N}$. Then,

$$\left(\mathbf{h}_i\right)^\perp \mathbf{s}_\lambda = \sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a horizontal } i\text{-strip}}} \mathbf{s}_\mu .$$

Corollary 10.9 is also proven in [GriRei20, (2.8.3)].

Next, let us show some further lemmas:

Lemma 10.10. Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Let g be a positive integer. Then,

$$\sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 = n-k+g; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = \sum_{w \geq 1} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda .$$

Proof of Lemma 10.10 (sketched). First, we observe that $\lambda_1 \leq n-k$ (since $\lambda \in P_{k,n}$). Now, every partition μ satisfying $\mu_1 = n-k+g$ must automatically satisfy $\mu_1 \geq \lambda_1$ (because $\mu_1 = n-k+\underbrace{g}_{\geq 0} \geq n-k \geq \lambda_1$).

Let A be the set of all partitions μ such that $\mu_1 = n-k+g$ and such that μ/λ is a horizontal j-strip. Let B be the set of all partitions ν such that λ/ν is a horizontal $(n-k+g-j)$-strip. Then,[25]

$$\begin{aligned} A &= \{\mu \text{ is a partition} \mid \mu_1 = n-k+g \text{ and } |\mu| - |\lambda| = j \\ &\qquad \text{and } \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \lambda_3 \geq \cdots\} \\ &= \{\mu \text{ is a partition} \mid \mu_1 = n-k+g \text{ and } |\mu| - |\lambda| = j \\ &\qquad \text{and } \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \lambda_3 \geq \mu_4 \geq \cdots\} \end{aligned}$$

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[25] We are using Definition 6.2 **(c)** here.

(since every partition μ satisfying $\mu_1 = n-k+g$ must automatically satisfy $\mu_1 \geq \lambda_1$) and

$$B = \{\nu \text{ is a partition } \mid |\lambda| - |\nu| = n-k+g-j \text{ and } \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \lambda_3 \geq \nu_3 \geq \cdots\}.$$

Hence, it is easy to check that the map

$$B \to A,$$
$$\nu \mapsto (n-k+g, \nu_1, \nu_2, \nu_3, \ldots)$$

is well-defined (because every $\nu \in B$ satisfies $\lambda_1 \geq \nu_1$ and thus $n-k+\underbrace{g}_{\geq 0} \geq n-k \geq \lambda_1 \geq \nu_1$) and is a bijection (its inverse map just sends each $\mu \in A$ to $(\mu_2, \mu_3, \mu_4, \ldots) \in B$). Thus, we can substitute $(n-k+g, \nu_1, \nu_2, \nu_3, \ldots)$ for μ in the sum $\sum\limits_{\mu \in A} \mathbf{s}_\mu$. We thus obtain

$$\sum_{\mu \in A} \mathbf{s}_\mu = \sum_{\nu \in B} \mathbf{s}_{(n-k+g,\nu_1,\nu_2,\nu_3,\ldots)}. \tag{95}$$

But each $\nu \in B$ satisfies $n-k+\underbrace{g}_{\geq 0} \geq n-k \geq \lambda_1 \geq \nu_1$ and thus

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \mathbf{s}_\nu = \mathbf{s}_{(n-k+g,\nu_1,\nu_2,\nu_3,\ldots)} \tag{96}$$

(by Proposition 6.8, applied to ν and $n-k+g$ instead of λ and m). Hence, (95) becomes

$$\sum_{\mu \in A} \mathbf{s}_\mu = \sum_{\nu \in B} \underbrace{\mathbf{s}_{(n-k+g,\nu_1,\nu_2,\nu_3,\ldots)}}_{\substack{= \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \mathbf{s}_\nu \\ \text{(by (96))}}} = \sum_{\nu \in B} \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \mathbf{s}_\nu$$
$$= \sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{n-k+g+i} (\mathbf{e}_i)^\perp \left(\sum_{\nu \in B} \mathbf{s}_\nu \right). \tag{97}$$

But Corollary 10.9 (applied to $i = n-k+g-j$) yields[26]

$$\left(\mathbf{h}_{n-k+g-j}\right)^\perp \mathbf{s}_\lambda = \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a horizontal } (n-k+g-j)\text{-strip}}}}_{\substack{= \sum_{\mu \in B} \\ \text{(by the definition of } B)}} \mathbf{s}_\mu = \sum_{\mu \in B} \mathbf{s}_\mu$$
$$= \sum_{\nu \in B} \mathbf{s}_\nu. \tag{98}$$

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[26] More precisely: This follows from Corollary 10.9 (applied to $i = n-k+g-j$) when $n-k+g-j \in \mathbb{N}$. But otherwise, it is obvious for trivial reasons ($0 = 0$).

Hence, (97) becomes

$$\begin{aligned}\sum_{\mu\in A}\mathbf{s}_\mu &= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\,(\mathbf{e}_i)^\perp\left(\underbrace{\sum_{\nu\in B}\mathbf{s}_\nu}_{\substack{=\left(\mathbf{h}_{n-k+g-j}\right)^\perp\mathbf{s}_\lambda\\ \text{(by (98))}}}\right)\\
&= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\,\underbrace{(\mathbf{e}_i)^\perp\left(\left(\mathbf{h}_{n-k+g-j}\right)^\perp\mathbf{s}_\lambda\right)}_{=\left((\mathbf{e}_i)^\perp\circ\left(\mathbf{h}_{n-k+g-j}\right)^\perp\right)\mathbf{s}_\lambda}\\
&= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\qquad\underbrace{\left((\mathbf{e}_i)^\perp\circ\left(\mathbf{h}_{n-k+g-j}\right)^\perp\right)}_{\substack{=\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_i\right)^\perp\\ \text{(since (29)}\\ \text{yields }\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_i\right)^\perp=(\mathbf{e}_i)^\perp\circ\left(\mathbf{h}_{n-k+g-j}\right)^\perp)}}\qquad\mathbf{s}_\lambda\\
&= \sum_{i\in\mathbb{N}}(-1)^i\,\mathbf{h}_{n-k+g+i}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_i\right)^\perp\mathbf{s}_\lambda\\
&= \sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&\qquad\qquad\left(\text{here, we have substituted } w-g \text{ for } i \text{ in the sum}\right).\end{aligned}$$

Comparing this with

$$\begin{aligned}&\sum_{w\geq 1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&= \sum_{w=1}^{g-1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\qquad\underbrace{\mathbf{e}_{w-g}}_{\substack{=0\\ \text{(since } w-g<0\\ \text{(since } w\leq g-1<g))}}\right)^\perp\mathbf{s}_\lambda\\
&\qquad+\sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&\qquad\left(\text{since } g \text{ is a positive integer}\right)\\
&= \underbrace{\sum_{w=1}^{g-1}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}0\right)^\perp\mathbf{s}_\lambda}_{=0}+\sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda\\
&= \sum_{w\geq g}(-1)^{w-g}\,\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda,\end{aligned}$$

we obtain

$$\sum_{\mu\in A}\mathbf{s}_\mu=\sum_{w\geq 1}(-1)^{w-g}\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda. \tag{99}$$

In view of

$$\sum_{\mu\in A}=\sum_{\substack{\mu\text{ is a partition;}\\ \mu_1=n-k+g;\\ \mu/\lambda\text{ is a horizontal }j\text{-strip}}}\qquad\text{(by the definition of }A\text{)},$$

this rewrites as

$$\sum_{\substack{\mu\text{ is a partition;}\\ \mu_1=n-k+g;\\ \mu/\lambda\text{ is a horizontal }j\text{-strip}}}\mathbf{s}_\mu=\sum_{w\geq 1}(-1)^{w-g}\mathbf{h}_{n-k+w}\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^\perp\mathbf{s}_\lambda.$$

This proves Lemma 10.10. ☐

Our next lemma will be a slight generalization of Lemma 7.17; but first we extend our definition of $\mathbf{s}_{\left(m,1^j\right)}$:

Convention 10.11. Let $m\in\mathbb{N}$, and let j be a negative integer. Then, we shall understand the (otherwise undefined) expression $\mathbf{s}_{\left(m,1^j\right)}$ to mean $0\in\Lambda$.

We can now generalize Lemma 7.17 as follows:

Lemma 10.12. Let m be a positive integer. Let $j\in\mathbb{Z}$ be such that $m+j>0$. Then,

$$\mathbf{s}_{\left(m,1^j\right)}=\sum_{i=1}^{m}(-1)^{i-1}\mathbf{h}_{m-i}\mathbf{e}_{j+i}.$$

Proof of Lemma 10.12. If $j\in\mathbb{N}$, then this follows directly from Lemma 7.17. Hence, for the rest of this proof, we WLOG assume that $j\notin\mathbb{N}$. Hence, $j<0$. Now, the proof of Lemma 10.12 is the same as our above proof of Lemma 7.17, with two changes:

- The inequality $m+j>0$ no longer follows from $m>0$ and $j\geq 0$, but rather comes straight from the assumptions.
- The equality $\sum_{i=0}^{j}(-1)^i\mathbf{h}_{m+i}\mathbf{e}_{j-i}=\mathbf{s}_{\left(m,1^j\right)}$ no longer follows from (45), but rather comes from comparing $\sum_{i=0}^{j}(-1)^i\mathbf{h}_{m+i}\mathbf{e}_{j-i}=(\text{empty sum})=0$ with $\mathbf{s}_{\left(m,1^j\right)}=0$.

Thus, Lemma 10.12 is proven. ☐

Lemma 10.13. Let $j \in \{0, 1, \ldots, n-k\}$, and let w be a positive integer. Then,

$$\sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g} = (-1)^{w-j} \mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^{w} \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}.$$

Proof of Lemma 10.13. From $j \in \{0, 1, \ldots, n-k\}$, we obtain $0 \leq j \leq n-k \leq n-k+1$.

We have $\underbrace{n}_{\geq k} -k+1 \geq k-k+1 = 1$; thus, $n-k+1$ is a positive integer. Also, $(n-k+1) + \left(w - \underbrace{j}_{\leq n-k} -1 \right) \geq (n-k+1) + (w-(n-k)-1) = w > 0$ (since w is a positive integer). Hence, Lemma 10.12 (applied to $n-k+1$ and $w-j-1$ instead of m and j) yields

$$\begin{aligned} \mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} &= \sum_{i=1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \\ &= \sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} + \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \end{aligned} \tag{100}$$

(since $0 \leq j \leq n-k+1$). Also, $n-k-\underbrace{j}_{\leq n-k}+1 \geq n-k-(n-k)+1 = 1$; thus, $n-k-j+1$ is a positive integer. Furthermore, $w-1 \in \mathbb{N}$ (since w is a positive integer). Hence, Lemma 7.17 (applied to $n-k-j+1$ and $w-1$ instead of m and j) yields

$$\begin{aligned} \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} &= \sum_{i=1}^{n-k-j+1} (-1)^{i-1} \mathbf{h}_{n-k-j+1-i} \mathbf{e}_{w-1+i} \\ &= \sum_{i=j+1}^{n-k+1} \underbrace{(-1)^{i-j-1}}_{=(-1)^{j}(-1)^{i-1}} \underbrace{\mathbf{h}_{n-k-j+1-(i-j)}}_{=\mathbf{h}_{n-k+1-i}} \underbrace{\mathbf{e}_{w-1+i-j}}_{=\mathbf{e}_{w-j-1+i}} \\ &\qquad \text{(here, we have substituted } i-j \text{ for } i \text{ in the sum)} \\ &= (-1)^{j} \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}. \end{aligned}$$

Multiplying this equality by $(-1)^{j}$, we find

$$(-1)^{j} \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} = \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}.$$

Subtracting this equality from (100), we obtain

$$\begin{aligned}
&\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^j \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \\
&= \left(\sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} + \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \right) \\
&\qquad - \sum_{i=j+1}^{n-k+1} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i} \\
&= \sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}. \qquad (101)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g} \\
&= \sum_{i=1}^{j} \underbrace{(-1)^{w-(j+1-i)}}_{=(-1)^{w-j}(-1)^{i-1}} \underbrace{\mathbf{h}_{n-k+(j+1-i)-j}}_{=\mathbf{h}_{n-k+1-i}} \underbrace{\mathbf{e}_{w-(j+1-i)}}_{=\mathbf{e}_{w-j-1+i}} \\
&\qquad \left(\text{here, we have substituted } j+1-i \text{ for } g \text{ in the sum}\right) \\
&= (-1)^{w-j} \underbrace{\sum_{i=1}^{j} (-1)^{i-1} \mathbf{h}_{n-k+1-i} \mathbf{e}_{w-j-1+i}}_{\substack{=\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^j \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \\ \text{(by (101))}}} \\
&= (-1)^{w-j} \left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^j \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \right) \\
&= (-1)^{w-j} \mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - \underbrace{(-1)^{w-j} (-1)^j}_{=(-1)^w} \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)} \\
&= (-1)^{w-j} \mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} - (-1)^w \mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}.
\end{aligned}$$

This proves Lemma 10.13. □

Proof of Theorem 10.7. We have $j \in \{0, 1, \ldots, n-k\}$, thus $0 \leq j \leq n-k$. Also, we have $\lambda \in P_{k,n}$; thus, the partition λ has at most k parts and satisfies $\lambda_1 \leq n-k$.

Let g be an integer such that $g \geq j+1$. If μ is a partition such that μ / λ is a horizontal j-strip, then $\mu_1 \leq \underbrace{\lambda_1}_{\leq n-k} + j \leq n-k+ \underbrace{j}_{<j+1\leq g} < n-k+g$ and thus $\mu_1 \neq n-k+g$. Thus, there exists no partition μ such that $\mu_1 = n-k+g$ and such that μ / λ is a horizontal j-strip. Hence,

$$\sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 = n-k+g; \\ \mu / \lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = (\text{empty sum}) = 0. \tag{102}$$

Now, forget that we fixed g. We thus have proven the equality (102) for every integer g satisfying $g \geq j+1$.

On the other hand, let $g \in \{1, 2, \ldots, j\}$. Thus, $g \leq j \leq n-k$. If w is an integer satisfying $w \geq n+1$, then $\underbrace{w}_{\geq n+1} - \underbrace{g}_{\leq n-k} \geq (n+1) - (n-k) = k+1 > k$, and thus the partition (1^{w-g}) does **not** satisfy $(1^{w-g}) \subseteq \lambda$ (because the partition λ has at most k parts, whereas the partition (1^{w-g}) has $w-g > k$ parts), and therefore we have

$$\begin{aligned} \Big(\underbrace{\mathbf{e}_{w-g}}_{= \mathbf{s}_{(1^{w-g})}}\Big)^\perp (\mathbf{s}_\lambda) &= \left(\mathbf{s}_{(1^{w-g})}\right)^\perp (\mathbf{s}_\lambda) = \mathbf{s}_{\lambda / (1^{w-g})} \qquad \text{(by (28))} \\ &= 0 \qquad \left(\text{since we don't have } \left(1^{w-g}\right) \subseteq \lambda\right). \end{aligned} \tag{103}$$

Hence, if w is an integer satisfying $w \geq n+1$, then

$$\begin{aligned} \Big(\underbrace{\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}}_{= \mathbf{e}_{w-g} \mathbf{h}_{n-k+g-j}}\Big)^\perp \mathbf{s}_\lambda &= \underbrace{\left(\mathbf{e}_{w-g} \mathbf{h}_{n-k+g-j}\right)^\perp}_{\substack{= \left(\mathbf{h}_{n-k+g-j}\right)^\perp \circ \left(\mathbf{e}_{w-g}\right)^\perp \\ \text{(by (29))}}} \mathbf{s}_\lambda = \left(\left(\mathbf{h}_{n-k+g-j}\right)^\perp \circ \left(\mathbf{e}_{w-g}\right)^\perp\right) (\mathbf{s}_\lambda) \\ &= \left(\mathbf{h}_{n-k+g-j}\right)^\perp \underbrace{\left(\left(\mathbf{e}_{w-g}\right)^\perp (\mathbf{s}_\lambda)\right)}_{\substack{=0 \\ \text{(by (103))}}} = 0. \end{aligned} \tag{104}$$

Now,

$$\begin{aligned}
&\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 = n-k+g;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{w\geq 1} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda \qquad \text{(by Lemma 10.10)} \\
&= \sum_{w=1}^{n} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda \\
&\qquad + \sum_{w\geq n+1} (-1)^{w-g} \mathbf{h}_{n-k+w} \underbrace{\left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda}_{\substack{=0\\ \text{(by (104))}}} \\
&= \sum_{w=1}^{n} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j}\mathbf{e}_{w-g}\right)^{\perp} \mathbf{s}_\lambda . \qquad (105)
\end{aligned}$$

Now, forget that we fixed g. We thus have proven the equality (105) for each $g \in \{1, 2, \ldots, j\}$.

Proposition 10.8 (applied to $i = j$) yields

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu + \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 > n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu$$

(since each partition μ satisfies either $\mu_1 \leq n-k$ or $\mu_1 > n-k$). Hence,

$$\begin{aligned}
&\mathbf{s}_\lambda \mathbf{h}_j - \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 > n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu = \sum_{g \geq 1} \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 = n-k+g; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&\qquad \left(\begin{array}{c} \text{because the partitions } \mu \text{ satisfying } \mu_1 > n-k \text{ are precisely} \\ \text{the partitions } \mu \text{ satisfying } \mu_1 = n-k+g \text{ for some } g \geq 1, \\ \text{and moreover the } g \text{ is uniquely determined by the partition} \end{array} \right) \\
&= \sum_{g=1}^{j} \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 = n-k+g; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu}_{\substack{= \sum_{w=1}^{n} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda \\ \text{(by (105))}}} + \sum_{g=j+1}^{\infty} \underbrace{\sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 = n-k+g; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu}_{\substack{=0 \\ \text{(by (102))}}} \\
&= \underbrace{\sum_{g=1}^{j} \sum_{w=1}^{n}}_{= \sum_{w=1}^{n} \sum_{g=1}^{j}} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+w} \left(\mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}\right)^\perp \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} \left(\underbrace{\sum_{g=1}^{j} (-1)^{w-g} \mathbf{h}_{n-k+g-j} \mathbf{e}_{w-g}}_{\substack{=(-1)^{w-j} \mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} - (-1)^{w} \mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)} \\ \text{(by Lemma 10.13)}}} \right)^\perp \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} \left((-1)^{w-j} \mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} - (-1)^{w} \mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)} \right)^\perp \mathbf{s}_\lambda \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} (-1)^{w-j} \left(\mathbf{s}_{\left(n-k+1, 1^{w-j-1}\right)} \right)^\perp \mathbf{s}_\lambda \\
&\qquad - \sum_{w=1}^{n} \mathbf{h}_{n-k+w} (-1)^{w} \left(\mathbf{s}_{\left(n-k-j+1, 1^{w-1}\right)} \right)^\perp \mathbf{s}_\lambda . \qquad (106)
\end{aligned}$$

Next, we claim that

$$\left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)}\right)^{\perp} \mathbf{s}_{\lambda} = 0 \qquad \text{for each } w \in \{1,2,\ldots,n\} . \tag{107}$$

[*Proof of (107):* Let $w \in \{1,2,\ldots,n\}$. If $w-j-1$ is a negative integer, then $\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)} = 0$ (by Convention 10.11), and thus (107) holds in this case. Hence, for the rest of this proof of (107), we WLOG assume that $w-j-1$ is not a negative integer. Thus, $w-j-1 \in \mathbb{N}$. Now, the partition $\left(n-k+1,1^{w-j-1}\right)$ has a bigger first entry than the partition λ (since its first entry is $n-k+1 > n-k \geq \lambda_1$). Thus, we do not have $\left(n-k+1,1^{w-j-1}\right) \subseteq \lambda$. Hence, $\mathbf{s}_{\lambda/\left(n-k+1,1^{w-j-1}\right)} = 0$. But (28) yields $\left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)}\right)^{\perp} \mathbf{s}_{\lambda} = \mathbf{s}_{\lambda/\left(n-k+1,1^{w-j-1}\right)} = 0$. This proves (107).]

Next, we claim that

$$\left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_{\lambda} = 0 \qquad \text{for each } w \in \{k+1,k+2,\ldots,n\} . \tag{108}$$

[*Proof of (108):* Let $w \in \{k+1,k+2,\ldots,n\}$. Then, $w \geq k+1$. Now, the number of parts of the partition $\left(n-k-j+1,1^{w-1}\right)$ is $1+(w-1) = w \geq k+1 > k$, which is bigger than the number of parts of λ (since λ has at most k parts). Hence, we don't have $\left(n-k-j+1,1^{w-1}\right) \subseteq \lambda$. Thus, $\mathbf{s}_{\lambda/\left(n-k-j+1,1^{w-1}\right)} = 0$. But (28) yields $\left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_{\lambda} = \mathbf{s}_{\lambda/\left(n-k-j+1,1^{w-1}\right)} = 0$. This proves (108).]

Now, (106) becomes

$$\begin{aligned}
&\mathbf{s}_\lambda \mathbf{h}_j - \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu \\
&= \sum_{w=1}^{n} \mathbf{h}_{n-k+w} \left(-1\right)^{w-j} \underbrace{\left(\mathbf{s}_{\left(n-k+1,1^{w-j-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}_{\substack{=0 \\ \text{(by (107))}}} \\
&\qquad - \sum_{w=1}^{n} \mathbf{h}_{n-k+w} \left(-1\right)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \sum_{w=1}^{n} \mathbf{h}_{n-k+w} \left(-1\right)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \left(\sum_{w=1}^{k} \mathbf{h}_{n-k+w} \left(-1\right)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \right. \\
&\qquad\qquad \left. + \sum_{w=k+1}^{n} \mathbf{h}_{n-k+w} \left(-1\right)^{w} \underbrace{\left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}_{\substack{=0 \\ \text{(by (108))}}} \right) \\
&\qquad \left(\text{since } 0 \leq k \leq n\right) \\
&= - \sum_{w=1}^{k} \mathbf{h}_{n-k+w} \left(-1\right)^{w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \sum_{w=1}^{k} \left(-1\right)^{w} \mathbf{h}_{n-k+w} \left(\mathbf{s}_{\left(n-k-j+1,1^{w-1}\right)}\right)^{\perp} \mathbf{s}_\lambda \\
&= - \sum_{i=1}^{k} \left(-1\right)^{i} \mathbf{h}_{n-k+i} \left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda
\end{aligned}$$

(here, we have renamed the summation index w as i). Hence,

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu - \sum_{i=1}^{k} \left(-1\right)^{i} \mathbf{h}_{n-k+i} \left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda .$$

This proves Theorem 10.7. □

Proof of Theorem 10.5. Theorem 10.7 yields

$$\mathbf{s}_\lambda \mathbf{h}_j = \sum_{\substack{\mu \text{ is a partition;} \\ \mu_1 \leq n-k; \\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \mathbf{s}_\mu - \sum_{i=1}^{k} \left(-1\right)^{i} \mathbf{h}_{n-k+i} \left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda .$$

Both sides of this equality are symmetric functions in Λ. If we evaluate them at $x_1, x_2, \ldots, x_k$ and project the resulting symmetric polynomials onto $\mathcal{S}/I$, then we obtain

$$\begin{aligned}
\overline{s_\lambda h_j} &= \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i \underbrace{\overline{h_{n-k+i}}}_{\substack{=a_i\\ \text{(since (15)}\\ \text{yields } h_{n-k+i} \equiv a_i \bmod I)}} \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda} \\
&= \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}. \qquad (109)
\end{aligned}$$

But every partition μ has either at most k parts or more than k parts. Hence,

$$\begin{aligned}
&\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} \\
&= \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu \text{ has at most } k \text{ parts;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} + \sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu \text{ has more than } k \text{ parts;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \underbrace{\overline{s_\mu}}_{\substack{=0\\ \text{(because (3) (applied to } \mu\\ \text{instead of } \lambda) \text{ yields } s_\mu = 0)}} \\
&= \underbrace{\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu \text{ has at most } k \text{ parts;}\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}}}_{\substack{= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}}\\ \text{(because the partitions } \mu \text{ such that } \mu_1 \leq n-k\\ \text{and such that } \mu \text{ has at most } k \text{ parts}\\ \text{are precisely the partitions } \mu \in P_{k,n})}} \overline{s_\mu} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu}.
\end{aligned}$$

Hence, (109) becomes

$$\begin{aligned}
\overline{s_\lambda h_j} &= \underbrace{\sum_{\substack{\mu \text{ is a partition;}\\ \mu_1 \leq n-k;\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu}}_{= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu}} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda} \\
&= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}.
\end{aligned}$$

This proves Theorem 10.5. □

Let us again use the notation $c^{\gamma}_{\alpha,\beta}$ for a Littlewood–Richardson coefficient (defined as in [GriRei20, Definition 2.5.8], for example). Then, we can restate Theorem 10.5 as follows:

Theorem 10.14. Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{\left(n-k-j+1,1^{i-1}\right),\nu} \overline{s_\nu},$$

where the last sum ranges over all partitions ν satisfying $\nu \subseteq \lambda$.

Proof of Theorem 10.14. Let μ be a partition. Then, (28) yields

$$\begin{aligned}
\left(\mathbf{s}_\mu\right)^{\perp} \mathbf{s}_\lambda = \mathbf{s}_{\lambda/\mu} &= \sum_{\nu \text{ is a partition}} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu \\
&= \sum_{\substack{\nu \text{ is a partition;}\\ \nu \subseteq \lambda}} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu + \sum_{\substack{\nu \text{ is a partition;}\\ \text{we don't have } \nu \subseteq \lambda}} \underbrace{c^{\lambda}_{\mu,\nu}}_{\substack{=0\\ \text{(by Proposition 6.17 } \textbf{(b)}\text{)}}} \mathbf{s}_\nu \\
&= \underbrace{\sum_{\substack{\nu \text{ is a partition;}\\ \nu \subseteq \lambda}}}_{= \sum_{\nu \subseteq \lambda}} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu = \sum_{\nu \subseteq \lambda} c^{\lambda}_{\mu,\nu} \mathbf{s}_\nu. \qquad (110)
\end{aligned}$$

Both sides of this equality are symmetric functions in Λ. If we evaluate them at $x_1, x_2, \ldots, x_k$ and project the resulting symmetric polynomials onto $\mathcal{S}/I$, then we obtain

$$\overline{\left(\mathbf{s}_\mu\right)^{\perp} \mathbf{s}_\lambda} = \sum_{\nu \subseteq \lambda} c^{\lambda}_{\mu,\nu} \overline{s_\nu}. \qquad (111)$$

Now, forget that we fixed μ. We thus have proven (111) for each partition μ. Theorem 10.5 yields

$$\begin{aligned}
\overline{s_\lambda h_j} &= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \underbrace{\overline{\left(\mathbf{s}_{\left(n-k-j+1,1^{i-1}\right)}\right)^{\perp} \mathbf{s}_\lambda}}_{\substack{= \sum_{\nu \subseteq \lambda} c^{\lambda}_{\left(n-k-j+1,1^{i-1}\right),\nu} \overline{s_\nu}\\ \text{(by (111), applied to } \mu = \left(n-k-j+1,1^{i-1}\right)\text{)}}} \\
&= \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{\left(n-k-j+1,1^{i-1}\right),\nu} \overline{s_\nu}.
\end{aligned}$$

This proves Theorem 10.14. □

Note that Theorem 10.14 can also be used to prove Theorem 9.5.

10.4. Positivity?

Let us recall some background about the quantum cohomology ring $\operatorname{QH}^*(\operatorname{Gr}_{kn})$ discussed in [Postni05]. The structure constants of the $\mathbb{Z}[q]$-algebra $\operatorname{QH}^*(\operatorname{Gr}_{kn})$ are polynomials in the indeterminate q, whose coefficients are the famous Gromov-Witten invariants $C^d_{\lambda\mu\nu}$. These Gromov-Witten invariants $C^d_{\lambda\mu\nu}$ are nonnegative integers (as follows from their geometric interpretation, but also from the "Quantum Littlewood-Richardson Rule" [BKPT16, Theorem 2]). This appears to generalize to the general case of $\mathcal{S}/I$:

> **Conjecture 10.15.** Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1, 2, \ldots, k\}$. Let λ, μ and ν be three partitions in $P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|} \operatorname{coeff}_\nu \left(\overline{s_\lambda s_\mu}\right)$ is a polynomial in $b_1, b_2, \ldots, b_k$ with nonnegative integer coefficients. (See Definition 6.2 **(b)** for the meaning of coeff_ν.)

We have verified this conjecture for all $n \leq 8$ using SageMath.

11. The "rim hook algorithm"

We shall next take aim at a recursive formula for "straightening" a Schur polynomial – i.e., representing an $\overline{s_\mu}$, where μ is a partition that does not belong to $P_{k,n}$, as a $\mathbf{k}$-linear combination of "smaller" $\overline{s_\lambda}$'s. However, before we can state this formula, we will have to introduce several new notations.

11.1. Schur polynomials for non-partitions

Recall Definition 5.6. Thus, the elements of P_k are weakly decreasing k-tuples in $\mathbb{N}^k$. For each $\lambda \in P_k$, a Schur polynomial $s_\lambda \in \mathcal{S}$ is defined. Let us extend this definition by defining s_λ for each $\lambda \in \mathbb{Z}^k$:

> **Definition 11.1.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$. Then, we define a symmetric polynomial $s_\lambda \in \mathcal{S}$ by
>
> $$s_\lambda = \det\left(\left(h_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right). \tag{112}$$
>
> This new definition does not clash with the previous use of the notation s_λ, because when $\lambda \in P_k$, both definitions yield the same result (because of Proposition 5.7 **(a)**).

This definition is similar to the definition of $\overline{s}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ in [GriRei20, Exercise 2.9.1 (c)], but we are working with symmetric polynomials rather than symmetric functions here.

Definition 11.1 does not really open the gates to a new world of symmetric polynomials; indeed, each s_α (with $\alpha \in \mathbb{Z}^k$) defined in Definition 11.1 is either

0 or can be rewritten in the form $\pm s_\lambda$ for some $\lambda \in P_k$. Here is a more precise statement of this:

Proposition 11.2. Let $\alpha \in \mathbb{Z}^k$. Define a k-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ by

$$(\beta_i = \alpha_i + k - i \qquad \text{for each } i \in \{1, 2, \ldots, k\}).$$

(a) If β has at least one negative entry, then $s_\alpha = 0$.

(b) If β has two equal entries, then $s_\alpha = 0$.

(c) Assume that β has no negative entries and no two equal entries. Let $\sigma \in S_k$ be the permutation such that $\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}$. (Such a permutation σ exists and is unique, since β has no two equal entries.) Define a k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$ by

$$\left(\lambda_i = \beta_{\sigma(i)} - k + i \qquad \text{for each } i \in \{1, 2, \ldots, k\}\right).$$

Then, $\lambda \in P_k$ and $s_\alpha = (-1)^\sigma s_\lambda$.

Proof of Proposition 11.2. For each $u \in \{1, 2, \ldots, k\}$, we have $\beta_u = \alpha_u + k - u$ (by the definition of β_u) and thus

$$\underbrace{\beta_u}_{=\alpha_u + k - u} - k = (\alpha_u + k - u) - k = \alpha_u - u. \tag{113}$$

The definition of s_α yields

$$\begin{aligned} s_\alpha &= \det\left(\left(\underbrace{h_{\alpha_u - u + v}}_{\substack{= h_{\beta_u - k + v} \\ \text{(since (113) yields } \alpha_u - u = \beta_u - k)}}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) \\ &= \det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right). \end{aligned} \tag{114}$$

(b) Assume that β has two equal entries. In other words, there are two distinct elements i and j of $\{1, 2, \ldots, k\}$ such that $\beta_i = \beta_j$. Consider these i and j. The i-th and j-th rows of the matrix $\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$ are equal (since $\beta_i = \beta_j$). Hence, this matrix has two equal rows. Thus, its determinant is 0. In other words, $\det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = 0$. Now, (114) becomes $s_\alpha = \det\left(\left(h_{\beta_u - k + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = 0$. This proves Proposition 11.2 **(b)**.

(a) Assume that β has at least one negative entry. In other words, there exists some $i \in \{1, 2, \ldots, k\}$ such that $\beta_i < 0$. Consider this i. For each $v \in \{1, 2, \ldots, k\}$, we have $\beta_i - k + \underbrace{v}_{\leq k} \leq \beta_i - k + k = \beta_i < 0$ and thus

$h_{\beta_i - k+v} = 0$. Hence, all entries of the i-th row of the matrix $\left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}$ are 0. Hence, this matrix has a zero row. Thus, its determinant is 0. In other words, $\det\left(\left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) = 0$. Now, the equality (114) becomes $s_\alpha = \det\left(\left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) = 0$. This proves Proposition 11.2 **(a)**.

(c) It is well-known that if we permute the rows of a $k\times k$-matrix using a permutation τ, then the determinant of the matrix gets multiplied by $(-1)^\tau$. In other words, every $k\times k$-matrix $(b_{u,v})_{1\leq u\leq k,\ 1\leq v\leq k}$ and every $\tau\in S_k$ satisfy $\det\left(\left(b_{\tau(u),v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) = (-1)^\tau \det\left((b_{u,v})_{1\leq u\leq k,\ 1\leq v\leq k}\right)$. Applying this to $(b_{u,v})_{1\leq u\leq k,\ 1\leq v\leq k} = \left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}$ and $\tau = \sigma$, we obtain

$$\det\left(\left(h_{\beta_{\sigma(u)} - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) = (-1)^\sigma \det\left(\left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right).$$

Multiplying both sides of this equality by $(-1)^\sigma$, we find

$$\begin{aligned}(-1)^\sigma \det\left(\left(h_{\beta_{\sigma(u)} - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) &= \underbrace{(-1)^\sigma (-1)^\sigma}_{=\left((-1)^\sigma\right)^2=1} \det\left(\left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)\\ &= \det\left(\left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right). \qquad (115)\end{aligned}$$

For each $u\in\{1,2,\ldots,k\}$, we have $\lambda_u = \beta_{\sigma(u)} - k + u$ (by the definition of λ_u) and thus

$$\lambda_u - u = \beta_{\sigma(u)} - k. \qquad (116)$$

Now, (114) becomes

$$\begin{aligned}s_\alpha &= \det\left(\left(h_{\beta_u - k+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)\\ &= (-1)^\sigma \det\left(\left(\underbrace{h_{\beta_{\sigma(u)} - k+v}}_{\substack{=h_{\lambda_u - u+v}\\ \text{(since (116) yields } \beta_{\sigma(u)}-k=\lambda_u - u)}}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right) \qquad \text{(by (115))}\\ &= (-1)^\sigma \underbrace{\det\left(\left(h_{\lambda_u - u+v}\right)_{1\leq u\leq k,\ 1\leq v\leq k}\right)}_{\substack{=s_\lambda\\ \text{(by (112))}}} = (-1)^\sigma s_\lambda.\end{aligned}$$

It remains to prove that $\lambda\in P_k$.

Let $i\in\{1,2,\ldots,k-1\}$. Then, $\beta_{\sigma(i)} > \beta_{\sigma(i+1)}$ (since $\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}$) and thus $\beta_{\sigma(i)} \geq \beta_{\sigma(i+1)} + 1$ (since $\beta_{\sigma(i)}$ and $\beta_{\sigma(i+1)}$ are integers). The definition of λ_{i+1} yields $\lambda_{i+1} = \beta_{\sigma(i+1)} - k + (i+1)$. The definition of λ_i yields

$$\lambda_i = \underbrace{\beta_{\sigma(i)}}_{\geq \beta_{\sigma(i+1)}+1} - k + i \geq \beta_{\sigma(i+1)} + 1 - k + i = \beta_{\sigma(i+1)} - k + (i+1) = \lambda_{i+1}.$$

Now, forget that we fixed i. We thus have proven that $\lambda_i \geq \lambda_{i+1}$ for each $i \in \{1,2,\ldots,k-1\}$. In other words, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

Let $i \in \{1,2,\ldots,k\}$. Then, $1 \leq i \leq k$ and thus $k \geq 1$, so that λ_k is well-defined. Furthermore, from $i \leq k$, we obtain $\lambda_i \geq \lambda_k$ (since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$). But the definition of λ_k yields $\lambda_k = \beta_{\sigma(k)} - k + k = \beta_{\sigma(k)} \geq 0$ (since all entries of β are nonnegative (since β has no negative entries)). Thus, $\lambda_i \geq \lambda_k \geq 0$.

Now, forget that we fixed i. We thus have proven that $\lambda_i \geq 0$ for each $i \in \{1,2,\ldots,k\}$. In other words, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are nonnegative integers (since they are clearly integers). Hence, $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$. Combining this with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, we obtain $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_k$. Hence, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_k$. This completes the proof of Proposition 11.2 **(c)**. □

Let us next recall the bialternant formula for Schur polynomials. We need a few definitions first:

Definition 11.3. **(a)** Let ρ denote the k-tuple $(k-1, k-2, \ldots, 0) \in \mathbb{N}^k$.

(b) We regard $\mathbb{Z}^k$ as a $\mathbb{Z}$-module in the obvious way: Addition is define entrywise (i.e., we set $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_k + \beta_k)$ for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{Z}^k$ and any $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$). This also defines subtraction on $\mathbb{Z}^k$ (which, too, works entrywise). We let $\mathbf{0}$ denote the k-tuple $\left(\underbrace{0,0,\ldots,0}_{k \text{ entries}}\right) \in \mathbb{N}^k \subseteq \mathbb{Z}^k$; this is the zero vector of $\mathbb{Z}^k$.

Definition 11.4. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. Then, we define the *alternant* $a_\alpha \in \mathcal{P}$ by

$$a_\alpha = \det\left(\left(x_i^{\alpha_j}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right).$$

The two definitions we have just made match the notations in [GriRei20, §2.6], except that we are using k instead of n for the number of indeterminates.

Note that the element a_ρ of $\mathcal{P}$ is the Vandermonde determinant $\det\left(\left(x_i^{k-j}\right)_{1\leq i\leq k,\ 1\leq j\leq k}\right) = \prod_{1\leq i<j\leq k} (x_i - x_j)$; it is a regular element of $\mathcal{P}$ (that is, a non-zero-divisor).

We recall the *bialternant formula* for Schur polynomials ([GriRei20, Corollary 2.6.7]):

Proposition 11.5. For any $\lambda \in P_k$, we have $s_\lambda = a_{\lambda+\rho}/a_\rho$ in $\mathcal{P}$.

Let us extend this fact to arbitrary $\lambda \in \mathbb{Z}^k$ satisfying $\lambda + \rho \in \mathbb{N}^k$ (and rename λ as α):

Proposition 11.6. Let $\alpha \in \mathbb{Z}^k$ be such that $\alpha + \rho \in \mathbb{N}^k$. Then, $s_\alpha = a_{\alpha+\rho}/a_\rho$ in $\mathcal{P}$.

Proof of Proposition 11.6. We have $\rho = (k-1, k-2, \ldots, 0)$. Thus,

$$\rho_i = k - i \qquad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{117}$$

Define a k-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ as in Proposition 11.2. Thus, for each $i \in \{1, 2, \ldots, k\}$, we have

$$\beta_i = \alpha_i + \underbrace{k - i}_{\substack{=\rho_i \\ \text{(by (117))}}} = \alpha_i + \rho_i = (\alpha + \rho)_i .$$

In other words, $\beta = \alpha + \rho$. Hence, $\beta = \alpha + \rho \in \mathbb{N}^k$. Thus, the k-tuple β has no negative entries.

Moreover, from $\alpha + \rho = \beta$, we obtain

$$\begin{aligned} a_{\alpha+\rho} = a_\beta &= \det\left(\left(x_i^{\beta_j} \right)_{1 \le i \le k,\ 1 \le j \le k} \right) \qquad \left(\text{by the definition of } a_\beta\right) \\ &= \det\left(\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k} \right) \end{aligned} \tag{118}$$

(here, we have renamed the indices i and j as u and v). Now, we are in one of the following two cases:

Case 1: The k-tuple β has two equal entries.

Case 2: The k-tuple β has no two equal entries.

Let us first consider Case 1. In this case, the k-tuple β has two equal entries. In other words, there are two distinct elements i and j of $\{1, 2, \ldots, k\}$ such that $\beta_i = \beta_j$. Consider these i and j. The i-th and j-th columns of the matrix $\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k}$ are equal (since $\beta_i = \beta_j$). Hence, this matrix has two equal columns. Thus, its determinant is 0. In other words, $\det\left(\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k} \right) = 0$. Now, (118) becomes $a_{\alpha+\rho} = \det\left(\left(x_u^{\beta_v} \right)_{1 \le u \le k,\ 1 \le v \le k} \right) = 0$. Hence, $a_{\alpha+\rho}/a_\rho = 0/a_\rho = 0$. Comparing this with $s_\alpha = 0$ (which follows from Proposition 11.2 **(b)**), we obtain $s_\alpha = a_{\alpha+\rho}/a_\rho$. Thus, Proposition 11.6 is proven in Case 1.

Let us next consider Case 2. In this case, the k-tuple β has no two equal entries. Thus, there is a unique permutation $\sigma \in S_k$ that sorts this k-tuple into strictly decreasing order. In other words, there is a unique permutation $\sigma \in S_k$

such that $\beta_{\sigma(1)} > \beta_{\sigma(2)} > \cdots > \beta_{\sigma(k)}$. Consider this σ. Define a k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$ by

$$\left(\lambda_i = \beta_{\sigma(i)} - k + i \qquad \text{for each } i \in \{1, 2, \ldots, k\}\right).$$

Then, Proposition 11.2 **(c)** yields $\lambda \in P_k$ and $s_\alpha = (-1)^\sigma s_\lambda$.

It is well-known that if we permute the columns of a $k \times k$-matrix using a permutation τ, then the determinant of the matrix gets multiplied by $(-1)^\tau$. In other words, every $k \times k$-matrix $(b_{u,v})_{1 \leq u \leq k,\ 1 \leq v \leq k}$ and every $\tau \in S_k$ satisfy $\det\left(\left(b_{u,\tau(v)}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = (-1)^\tau \det\left((b_{u,v})_{1 \leq u \leq k,\ 1 \leq v \leq k}\right)$. Applying this to $(b_{u,v})_{1 \leq u \leq k,\ 1 \leq v \leq k} = \left(x_u^{\beta_v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$ and $\tau = \sigma$, we obtain

$$\det\left(\left(x_u^{\beta_{\sigma(v)}}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) = (-1)^\sigma \det\left(\left(x_u^{\beta_v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right). \tag{119}$$

But each $v \in \{1, 2, \ldots, k\}$ satisfies

$$\begin{aligned}(\lambda + \rho)_v &= \underbrace{\lambda_v}_{\substack{=\beta_{\sigma(v)} - k + v \\ \text{(by the definition of } \lambda_v)}} + \underbrace{\rho_v}_{\substack{=k - v \\ \text{(by (117))}}} = \left(\beta_{\sigma(v)} - k + v\right) + (k - v) \\ &= \beta_{\sigma(v)}.\end{aligned} \tag{120}$$

Now, the definition of $a_{\lambda+\rho}$ yields

$$\begin{aligned}a_{\lambda+\rho} &= \det\left(\left(x_i^{(\lambda+\rho)_j}\right)_{1 \leq i \leq k,\ 1 \leq j \leq k}\right) = \det\left(\left(\underbrace{x_u^{(\lambda+\rho)_v}}_{\substack{=x_u^{\beta_{\sigma(v)}} \\ \text{(by (120))}}}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) \\ &\qquad \text{(here, we have renamed the indices } i \text{ and } j \text{ as } u \text{ and } v) \\ &= \det\left(\left(x_u^{\beta_{\sigma(v)}}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right) \\ &= (-1)^\sigma \underbrace{\det\left(\left(x_u^{\beta_v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}\right)}_{\substack{=a_{\alpha+\rho} \\ \text{(by (118))}}} \qquad \text{(by (119))} \\ &= (-1)^\sigma a_{\alpha+\rho}.\end{aligned}$$

But $\lambda \in P_k$. Hence, Proposition 11.5 yields

$$s_\lambda = \underbrace{a_{\lambda+\rho}}_{=(-1)^\sigma a_{\alpha+\rho}} / a_\rho = (-1)^\sigma a_{\alpha+\rho} / a_\rho.$$

Hence,

$$s_\alpha = (-1)^\sigma \underbrace{s_\lambda}_{=(-1)^\sigma a_{\alpha+\rho}/a_\rho} = \underbrace{(-1)^\sigma (-1)^\sigma}_{=\left((-1)^\sigma\right)^2=1} a_{\alpha+\rho}/a_\rho = a_{\alpha+\rho}/a_\rho .$$

Thus, Proposition 11.6 is proven in Case 2.

We have now proven Proposition 11.6 in both Cases 1 and 2. Thus, Proposition 11.6 is proven. □

11.2. The uncancelled Pieri rule

Having defined s_λ for all $\lambda \in \mathbb{Z}^k$ (rather than merely for partitions), we can state a nonstandard version of the Pieri rule for products of the form $s_\lambda h_i$, which will turn out rather useful:

Theorem 11.7. Let $\lambda \in \mathbb{Z}^k$ be such that $\lambda + \rho \in \mathbb{N}^k$. Let $m \in \mathbb{N}$. Then,

$$s_\lambda h_m = \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = m}} s_{\lambda+\nu} .$$

Example 11.8. For this example, let $k = 3$ and $\lambda = (-2, 2, 1)$. Then, $\lambda + \rho = (-2, 2, 1) + (2, 1, 0) = (0, 3, 1)$. It is easy to see (using Proposition 11.2 **(c)**) that $s_\lambda = s_{(1)}$.

Furthermore, set $m = 2$. Then, the $\nu \in \mathbb{N}^k$ satisfying $|\nu| = m$ are the six 3-tuples

$$(2,0,0), \qquad (0,2,0), \qquad (0,0,2), \qquad (1,1,0), \qquad (1,0,1), \qquad (0,1,1).$$

Hence, Theorem 11.7 yields

$$\begin{aligned} s_{(-2,2,1)} h_2 &= \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = m}} s_{(-2,2,1)+\nu} \\ &= \underbrace{s_{(-2,2,1)+(2,0,0)}}_{\substack{=s_{(0,2,1)}=-s_{(1,1,1)} \\ \text{(by Proposition 11.2 \textbf{(c)})}}} + \underbrace{s_{(-2,2,1)+(0,2,0)}}_{\substack{=s_{(-2,4,1)}=s_{(3)} \\ \text{(by Proposition 11.2 \textbf{(c)})}}} + \underbrace{s_{(-2,2,1)+(0,0,2)}}_{\substack{=s_{(-2,2,3)}=0 \\ \text{(by Proposition 11.2 \textbf{(b)})}}} \\ &\qquad + \underbrace{s_{(-2,2,1)+(1,1,0)}}_{\substack{=s_{(-1,3,1)}=0 \\ \text{(by Proposition 11.2 \textbf{(b)})}}} + \underbrace{s_{(-2,2,1)+(1,0,1)}}_{\substack{=s_{(-1,2,2)}=s_{(1,1,1)} \\ \text{(by Proposition 11.2 \textbf{(c)})}}} + \underbrace{s_{(-2,2,1)+(0,1,1)}}_{\substack{=s_{(-2,3,2)}=s_{(2,1)} \\ \text{(by Proposition 11.2 \textbf{(c)})}}} \\ &= -s_{(1,1,1)} + s_{(3)} + 0 + 0 + s_{(1,1,1)} + s_{(2,1)} = s_{(2,1)} + s_{(3)} . \end{aligned}$$

In view of $s_{(-2,2,1)} = s_{(1)}$, this rewrites as $s_{(1)} h_2 = s_{(2,1)} + s_{(3)}$, which is exactly what the usual Pieri rule would yield. Note that the expression we obtained from Theorem 11.7 involves both vanishing addends (here, $s_{(-2,2,1)+(0,0,2)}$ and $s_{(-2,2,1)+(1,1,0)}$) and mutually cancelling addends (here, $s_{(-2,2,1)+(2,0,0)}$ and $s_{(-2,2,1)+(1,0,1)}$); this is why I call it the "uncancelled Pieri rule".

We note that the idea of such an "uncancelled Pieri rule" as our Theorem 11.7 is not new (similar things appeared in [LakTho07, §2] and [Tamvak13]), but we have not seen it stated in this exact form anywhere in the literature. Thus, let us give a proof:

Proof of Theorem 11.7. Define $\beta \in \mathbb{N}^k$ by $\beta = \lambda + \rho$. (This is well-defined, since $\lambda + \rho \in \mathbb{N}^k$.)

From (1), we obtain

$$h_m = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \underbrace{x^\alpha}_{\substack{= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \\ = \prod_{i=1}^k x_i^{\alpha_i}}} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \prod_{i=1}^k x_i^{\alpha_i}. \tag{121}$$

For each permutation $\sigma \in S_k$, we have

$$\begin{aligned} h_m &= h_m\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}\right) \qquad \text{(since the polynomial } h_m \text{ is symmetric)} \\ &= \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} \prod_{i=1}^k x_{\sigma(i)}^{\alpha_i} \end{aligned} \tag{122}$$

(here, we have substituted $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}$ for $x_1, x_2, \ldots, x_k$ in the equality (121)).

But Proposition 11.6 (applied to $\alpha = \lambda$) yields $s_\lambda = a_{\lambda+\rho}/a_\rho$ in $\mathcal{P}$. Thus,

$$\begin{aligned} a_\rho s_\lambda &= a_{\lambda+\rho} = a_\beta \qquad (\text{since } \lambda + \rho = \beta) \\ &= \det\left(\left(x_i^{\beta_j}\right)_{1 \le i \le k,\ 1 \le j \le k}\right) \qquad \left(\text{by the definition of } a_\beta\right) \\ &= \det\left(\left(x_j^{\beta_i}\right)_{1 \le i \le k,\ 1 \le j \le k}\right) \\ &\qquad \left(\begin{array}{c} \text{since the determinant of a matrix equals} \\ \text{the determinant of its transpose} \end{array}\right) \\ &= \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{\beta_i} \qquad \left(\text{by the definition of a determinant}\right). \end{aligned}$$

Multiplying both sides of this equality with h_m, we find

$$\begin{aligned}
a_\rho s_\lambda h_m &= \left(\sum_{\sigma\in S_k} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{\beta_i}\right) h_m = \sum_{\sigma\in S_k} (-1)^\sigma \left(\prod_{i=1}^{k} x_{\sigma(i)}^{\beta_i}\right) \underbrace{h_m}_{\substack{=\sum\limits_{\substack{\alpha\in\mathbb{N}^k;\\ |\alpha|=m}} \prod\limits_{i=1}^{k} x_{\sigma(i)}^{\alpha_i} \\ \text{(by (122))}}} \\
&= \sum_{\sigma\in S_k} (-1)^\sigma \left(\prod_{i=1}^{k} x_{\sigma(i)}^{\beta_i}\right) \sum_{\substack{\alpha\in\mathbb{N}^k;\\ |\alpha|=m}} \prod_{i=1}^{k} x_{\sigma(i)}^{\alpha_i} \\
&= \sum_{\substack{\alpha\in\mathbb{N}^k;\\ |\alpha|=m}} \sum_{\sigma\in S_k} (-1)^\sigma \underbrace{\left(\prod_{i=1}^{k} x_{\sigma(i)}^{\beta_i}\right) \prod_{i=1}^{k} x_{\sigma(i)}^{\alpha_i}}_{=\prod\limits_{i=1}^{k}\left(x_{\sigma(i)}^{\beta_i} x_{\sigma(i)}^{\alpha_i}\right)} \\
&= \sum_{\substack{\alpha\in\mathbb{N}^k;\\ |\alpha|=m}} \sum_{\sigma\in S_k} (-1)^\sigma \prod_{i=1}^{k} \underbrace{\left(x_{\sigma(i)}^{\beta_i} x_{\sigma(i)}^{\alpha_i}\right)}_{\substack{=x_{\sigma(i)}^{\beta_i+\alpha_i}=x_{\sigma(i)}^{(\beta+\alpha)_i} \\ \text{(since } \beta_i+\alpha_i=(\beta+\alpha)_i\text{)}}} \\
&= \sum_{\substack{\alpha\in\mathbb{N}^k;\\ |\alpha|=m}} \sum_{\sigma\in S_k} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{(\beta+\alpha)_i} \\
&= \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} \sum_{\sigma\in S_k} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(i)}^{(\beta+\nu)_i} \qquad (123)
\end{aligned}$$

(here, we have renamed the summation index α as ν).

On the other hand, let $\nu \in \mathbb{N}^k$. Then, $(\lambda+\nu)+\rho = \underbrace{(\lambda+\rho)}_{\in\mathbb{N}^k} + \underbrace{\nu}_{\in\mathbb{N}^k} \in \mathbb{N}^k$. Thus, Proposition 11.6 (applied to $\alpha = \lambda+\nu$) yields $s_{\lambda+\nu} = a_{(\lambda+\nu)+\rho}/a_\rho$ in $\mathcal{P}$.

Thus,

$$\begin{aligned}
a_\rho s_{\lambda+\nu} &= a_{(\lambda+\nu)+\rho} = a_{\beta+\nu} && \left(\text{since } (\lambda+\nu)+\rho = \underbrace{\lambda+\rho}_{=\beta}+\nu = \beta+\nu\right) \\
&= \det\left(\left(x_i^{(\beta+\nu)_j}\right)_{1\le i\le k,\ 1\le j\le k}\right) && \left(\text{by the definition of } a_{\beta+\nu}\right) \\
&= \det\left(\left(x_j^{(\beta+\nu)_i}\right)_{1\le i\le k,\ 1\le j\le k}\right) \\
&\qquad \left(\begin{array}{c}\text{since the determinant of a matrix equals}\\ \text{the determinant of its transpose}\end{array}\right) \\
&= \sum_{\sigma\in S_k}(-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{(\beta+\nu)_i} && (124)
\end{aligned}$$

(by the definition of a determinant).

Now, forget that we fixed ν. We thus have proven (124) for each $\nu \in \mathbb{N}^k$.

Now, (123) becomes

$$a_\rho s_\lambda h_m = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} \underbrace{\sum_{\sigma\in S_k}(-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{(\beta+\nu)_i}}_{\substack{=a_\rho s_{\lambda+\nu}\\ \text{(by (124))}}} = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} a_\rho s_{\lambda+\nu} = a_\rho \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} s_{\lambda+\nu}.$$

We can cancel a_ρ from this equality (since a_ρ is a regular element of $\mathcal{P}$), and thus obtain

$$s_\lambda h_m = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=m}} s_{\lambda+\nu}.$$

This proves Theorem 11.7. □

11.3. The "rim hook algorithm"

For the rest of this section, we assume that $k > 0$.

We need one more weird definition:

> **Definition 11.9.** Let V be the set of all k-tuples $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying
>
> $$(\tau_i \in \{0,1\} \qquad \text{for each } i \in \{2,3,\ldots,k\}). \tag{125}$$

Example 11.10. If $n = 6$ and $k = 3$, then

$$V = \{(-6,0,0), (-6,0,1), (-6,1,0), (-6,1,1)\}. \tag{126}$$

Proposition 11.11. Let $\tau \in V$. Then, $-|\tau| \in \{n-k+1, n-k+2, \ldots, n\}$.

Proof of Proposition 11.11. We have $\tau \in V$. Thus, τ has the form

$$\tau = (-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$$

for some $\tau_2, \tau_3, \ldots, \tau_k$ satisfying (125) (by the definition of V). Consider these $\tau_2, \tau_3, \ldots, \tau_k$. We have $\tau = (-n, \tau_2, \tau_3, \ldots, \tau_k)$ and thus

$$|\tau| = (-n) + \underbrace{\tau_2 + \tau_3 + \cdots + \tau_k}_{= \sum_{i=2}^{k} \tau_i} = (-n) + \sum_{i=2}^{k} \tau_i.$$

Therefore,

$$\begin{aligned} -|\tau| &= -\left((-n) + \sum_{i=2}^{k} \tau_i\right) = n - \sum_{i=2}^{k} \underbrace{\tau_i}_{\substack{\leq 1 \\ \text{(since (125)} \\ \text{yields } \tau_i \in \{0,1\})}} \geq n - \underbrace{\sum_{i=2}^{k} 1}_{=k-1} \\ &= n - (k-1) = n - k + 1. \end{aligned}$$

Combining this with

$$-|\tau| = n - \sum_{i=2}^{k} \underbrace{\tau_i}_{\substack{\geq 0 \\ \text{(since (125)} \\ \text{yields } \tau_i \in \{0,1\})}} \leq n - \underbrace{\sum_{i=2}^{k} 0}_{=0} = n,$$

we obtain $n - k + 1 \leq -|\tau| \leq n$. Thus, $-|\tau| \in \{n-k+1, n-k+2, \ldots, n\}$ (since $-|\tau|$ is an integer). This proves Proposition 11.11. $\square$

We are now ready to state the main theorem of this section: a generalization of the "rim hook algorithm" from [BeCiFu99, §2, Main Lemma]:

Theorem 11.12. Assume that $a_1, a_2, \ldots, a_k$ belong to $\mathbf{k}$.

Let $\mu \in P_k$ be such that $\mu_1 > n - k$. Then,

$$\overline{s_\mu} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} \overline{s_{\mu+\tau}}.$$

Example 11.13. For this example, set $n = 6$ and $k = 3$ and $\mu = (5, 4, 1)$. Then, Theorem 11.12 yields

$$\begin{aligned}
&\overline{s_{(5,4,1)}} \\
&= \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} \overline{s_{(5,4,1)+\tau}} \\
&= (-1)^{3-1} a_1 \overline{s_{(5,4,1)+(-6,1,1)}} + (-1)^{3-2} a_2 \left(\overline{s_{(5,4,1)+(-6,0,1)}} + \overline{s_{(5,4,1)+(-6,1,0)}} \right) \\
&\qquad + (-1)^{3-3} a_3 \overline{s_{(5,4,1)+(-6,0,0)}} \qquad \text{(by (126))} \\
&= a_1 \underbrace{\overline{s_{(5,4,1)+(-6,1,1)}}}_{\substack{= \overline{s_{(-1,5,2)}} \\ = \overline{s_{(4,1,1)}} \\ \text{(by Proposition 11.2 } \textbf{(c)}\text{)}}} - a_2 \left(\underbrace{\overline{s_{(5,4,1)+(-6,0,1)}}}_{\substack{= \overline{s_{(-1,4,2)}} \\ = \overline{s_{(3,1,1)}} \\ \text{(by Proposition 11.2 } \textbf{(c)}\text{)}}} + \underbrace{\overline{s_{(5,4,1)+(-6,1,0)}}}_{\substack{= \overline{s_{(-1,5,1)}} \\ = 0 \\ \text{(by Proposition 11.2 } \textbf{(b)}\text{)}}} \right) \\
&\qquad + a_3 \underbrace{\overline{s_{(5,4,1)+(-6,0,0)}}}_{\substack{= \overline{s_{(-1,4,1)}} \\ = 0 \\ \text{(by Proposition 11.2 } \textbf{(b)}\text{)}}} \\
&= a_1 \overline{s_{(4,1,1)}} - a_2 \overline{s_{(3,1,1)}}.
\end{aligned}$$

Note that this is **not** yet an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$. Indeed, we still have a term $\overline{s_{(4,1,1)}}$ on the right hand side which has $(4, 1, 1) \notin P_{k,n}$. But this term can, in turn, be rewritten using Theorem 11.12, and so on until we end up with an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$, namely

$$\overline{s_{(5,4,1)}} = -a_2 \overline{s_{(3,1,1)}} + a_1^2 \overline{s_{(1,1)}} - a_1 a_2 \overline{s_{(1)}} + a_1 a_3 \overline{s_{()}}.$$

As we saw in this example, when we apply Theorem 11.12, some of the $\overline{s_{\mu+\tau}}$ addends on the right hand side may be 0 (by Proposition 11.2 **(b)**). Once these addends are removed, the remaining addends can be rewritten in the form $\pm\overline{s_\lambda}$ for some $\lambda \in P_k$ satisfying $|\lambda| < |\mu|$ (using Proposition 11.2 **(c)**). The resulting sum is multiplicity-free – in the sense that no $\overline{s_\lambda}$ occurs more than once in it. (This is not difficult to check, but would take us too far afield.) However, this sum is (in general) not an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ yet, because it often contains terms $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$. If we keep applying Theorem 11.12 multiple times until we reach an expansion of $\overline{s_\mu}$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$, then this latter expansion may contain multiplicities: For example, for $n = 6$ and $k = 3$, we have

$$\overline{s_{(4,4,3)}} = -a_2\overline{s_{(3,3)}} + a_3\overline{s_{(3,2)}} + a_1^2\overline{s_{(3)}} - 2a_1a_2\overline{s_{(2)}} + a_2^2\overline{s_{(1)}}.$$

We owe the reader an explanation of why we call Theorem 11.12 a "rim hook algorithm". It owes this name to the fact that it generalizes the "rim hook algorithm" for quantum cohomology [BeCiFu99, §2, Main Lemma] (which can be obtained from it with some work by setting $a_i = 0$ for all $i < k$). Nevertheless, it does not visibly involve any rim hooks itself. I am, in fact, unaware of a way to restate it in the language of Young diagrams; the operation $\mu \mapsto \mu + \tau$ for $\tau \in V$ resembles both the removal of an n-rim hook (since it lowers the first entry by n) and the addition of a vertical strip (since it increases each of the remaining entries by 0 or 1), but it cannot be directly stated as one of these operations followed by the other.

We shall prove Theorem 11.12 by deriving it from an identity in $\mathcal{S}$:

Theorem 11.14. Let $\mu \in P_k$ be such that $\mu_1 > n - k$. Then,

$$s_\mu = \sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} s_{\mu+\tau}.$$

Our proof of this identity, in turn, will rely on the following combinatorial lemmas:

Lemma 11.15. Let $j \in \{2, 3, \ldots, k\}$. Let Δ be the vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k$, where 1 is the j-th entry.

(a) If $\tau \in V$ satisfies $\tau_j = 0$, then $\tau + \Delta \in V$ and $(\tau + \Delta)_j = 1$.

(b) If $\tau \in V$ satisfies $\tau_j = 1$, then $\tau - \Delta \in V$ and $(\tau - \Delta)_j = 0$.

(c) If $\nu \in \mathbb{N}^k$ satisfies $\nu_j \neq 0$, then $\nu - \Delta \in \mathbb{N}^k$.

(d) If $\nu \in \mathbb{N}^k$, then $\nu + \Delta \in \mathbb{N}^k$.

Proof of Lemma 11.15. We have $j \in \{2, 3, \ldots, k\}$, thus $j \neq 1$.

We have $\Delta = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{N}^k$. Thus, $\Delta_j = 1$ and

$$(\Delta_i = 0 \qquad \text{for each } i \in \{1, 2, \ldots, k\} \text{ satisfying } i \neq j). \tag{127}$$

Applying (127) to $i = 1$, we obtain $\Delta_1 = 0$ (since $1 \neq j$).

(a) Let $\tau \in V$ be such that $\tau_j = 0$.

We have $\tau \in V$. According to the definition of V, this means that τ is a k-tuple $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (125). In other words, $\tau \in \mathbb{Z}^k$ and $\tau_1 = -n$ and

$$(\tau_i \in \{0, 1\} \qquad \text{for each } i \in \{2, 3, \ldots, k\}). \tag{128}$$

Define a k-tuple $\sigma \in \mathbb{Z}^k$ by $\sigma = \tau + \Delta$. Thus, $\sigma_1 = (\tau + \Delta)_1 = \tau_1 + \underbrace{\Delta_1}_{=0} = \tau_1 = -n$.

Furthermore, from $\sigma = \tau + \Delta$, we obtain $\sigma_j = (\tau + \Delta)_j = \underbrace{\tau_j}_{=0} + \underbrace{\Delta_j}_{=1} = 1 \in \{0, 1\}$.

Next, we have $\sigma_i \in \{0, 1\}$ for each $i \in \{2, 3, \ldots, k\}$ [27]. Altogether, we thus have shown that $\sigma \in \mathbb{Z}^k$ and $\sigma_1 = -n$ and

$$(\sigma_i \in \{0, 1\} \qquad \text{for each } i \in \{2, 3, \ldots, k\}). \tag{129}$$

In other words, σ is a k-tuple $(-n, \sigma_2, \sigma_3, \ldots, \sigma_k) \in \mathbb{Z}^k$ satisfying (129). In other words, $\sigma \in V$ (by the definition of V). Thus, $\tau + \Delta = \sigma \in V$. So we have proven that $\tau + \Delta \in V$ and $(\tau + \Delta)_j = 1$. Thus, Lemma 11.15 **(a)** is proven.

(b) The proof of Lemma 11.15 **(b)** is analogous to the above proof of Lemma 11.15 **(a)**, and is left to the reader.

(c) Let $\nu \in \mathbb{N}^k$ be such that $\nu_j \neq 0$. We must prove that $\nu - \Delta \in \mathbb{N}^k$.

We have $\nu_j \in \mathbb{N}$ (since $\nu \in \mathbb{N}^k$). Hence, from $\nu_j \neq 0$, we conclude that $\nu_j \geq 1$. Thus, $\nu_j - 1 \in \mathbb{N}$. Also, the entries $\nu_1, \nu_2, \ldots, \nu_{j-1}, \nu_{j+1}, \nu_{j+2}, \ldots, \nu_k$ of ν belong to $\mathbb{N}$ (since $\nu \in \mathbb{N}^k$).

Recall that Δ is the vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k$, where 1 is the j-th entry. Hence,

$$\begin{aligned} \nu - \Delta &= \nu - (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \\ &= \left(\nu_1, \nu_2, \ldots, \nu_{j-1}, \nu_j - 1, \nu_{j+1}, \nu_{j+2}, \ldots, \nu_k\right) \in \mathbb{N}^k \end{aligned}$$

(since $\nu_j - 1 \in \mathbb{N}$ and since the entries $\nu_1, \nu_2, \ldots, \nu_{j-1}, \nu_{j+1}, \nu_{j+2}, \ldots, \nu_k$ of ν belong to $\mathbb{N}$). This proves Lemma 11.15 **(c)**.

(d) Let $\nu \in \mathbb{N}^k$. Also, $\Delta \in \mathbb{N}^k$. Thus, $\underbrace{\nu}_{\in \mathbb{N}^k} + \underbrace{\Delta}_{\in \mathbb{N}^k} \in \mathbb{N}^k$. This proves Lemma 11.15 **(d)**. ☐

Lemma 11.16. Let $\gamma \in \mathbb{Z}^k$. Then,

$$\sum_{\tau \in V} \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu + \tau = \gamma}} (-1)^{n + |\tau|} = \begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases}.$$

(Recall that $\mathbf{0}$ denotes the vector $\left(\underbrace{0, 0, \ldots, 0}_{k \text{ zeroes}}\right) \in \mathbb{Z}^k$.)

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[27] *Proof.* Let $i \in \{2, 3, \ldots, k\}$. We must prove $\sigma_i \in \{0, 1\}$.

If $i = j$, then this follows from $\sigma_j \in \{0, 1\}$. Hence, for the rest of this proof, we WLOG assume that $i \neq j$. Thus, (127) yields $\Delta_i = 0$. Now, from $\sigma = \tau + \Delta$, we obtain $\sigma_i = (\tau + \Delta)_i = \tau_i + \underbrace{\Delta_i}_{=0} = \tau_i \in \{0, 1\}$ (by (128)). Qed.

Proof of Lemma 11.16. Let Q be the set of all pairs $(\tau, \nu) \in V \times \mathbb{N}^k$ satisfying $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$. We have the following equality of summation signs:

$$\sum_{\tau \in V} \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu + \tau = \gamma}} = \sum_{\substack{(\tau,\nu) \in V \times \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu + \tau = \gamma}} = \sum_{(\tau,\nu) \in Q} \tag{130}$$

(since Q is the set of all pairs $(\tau, \nu) \in V \times \mathbb{N}^k$ satisfying $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$).

We are in one of the following three cases:

Case 1: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) \neq (0, 0, \ldots, 0)$.

Case 2: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$ and $\gamma_1 \neq 0$.

Case 3: We have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$ and $\gamma_1 = 0$.

Let us first consider Case 1. In this case, we have $(\gamma_2, \gamma_3, \ldots, \gamma_k) \neq (0, 0, \ldots, 0)$. In other words, there exists a $j \in \{2, 3, \ldots, k\}$ such that $\gamma_j \neq 0$. Consider such a j. Clearly, $\gamma \neq \mathbf{0}$ (since $\gamma_j \neq 0$). Hence, $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 0$.

Let Δ be the vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^k$, where 1 is the j-th entry. Clearly, $\Delta \in \mathbb{N}^k$ and $|\Delta| = 1$.

Let Q_0 be the set of all $(\tau, \nu) \in Q$ satisfying $\tau_j = 0$. (Recall that τ_j denotes the j-th entry of the k-tuple $\tau \in V \subseteq \mathbb{Z}^k$.) Let Q_1 be the set of all $(\tau, \nu) \in Q$ satisfying $\tau_j = 1$. Each $(\tau, \nu) \in Q$ satisfies $(\tau, \nu) \in V \times \mathbb{N}^k$ (by the definition of Q) and thus $\tau \in V$ and thus $\tau_j \in \{0, 1\}$ (by (125), applied to $i = j$). In other words, each $(\tau, \nu) \in Q$ satisfies either $\tau_j = 0$ or $\tau_j = 1$ (but not both at the same time). In other words, each $(\tau, \nu) \in Q$ belongs to either Q_0 or Q_1 (but not both at the same time).

For each $(\tau, \nu) \in Q_0$, we have $(\tau + \Delta, \nu - \Delta) \in Q_1$ [28]. Thus, the map

$$Q_0 \to Q_1, \qquad (\tau, \nu) \mapsto (\tau + \Delta, \nu - \Delta) \tag{131}$$

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[28] *Proof.* Let $(\tau, \nu) \in Q_0$. According to the definition of Q_0, this means that $(\tau, \nu) \in Q$ and $\tau_j = 0$.

We have $(\tau, \nu) \in Q$. According to the definition of Q, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

From $\nu + \tau = \gamma$, we obtain $(\nu + \tau)_j = \gamma_j$. Hence, $\gamma_j = (\nu + \tau)_j = \nu_j + \underbrace{\tau_j}_{=0} = \nu_j$. Thus, $\nu_j = \gamma_j \neq 0$. Thus, Lemma 11.15 **(c)** yields $\nu - \Delta \in \mathbb{N}^k$. Also, Lemma 11.15 **(a)** yields that $\tau + \Delta \in V$ and $(\tau + \Delta)_j = 1$. Also, any two k-tuples $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^k$ satisfy $|\alpha + \beta| = |\alpha| + |\beta|$ and $|\alpha - \beta| = |\alpha| - |\beta|$. Thus, $|\tau + \Delta| = |\tau| + |\Delta|$ and $|\nu - \Delta| = \underbrace{|\nu|}_{=-|\tau|} - |\Delta| = -|\tau| - |\Delta| = -\underbrace{(|\tau| + |\Delta|)}_{=|\tau+\Delta|} = -|\tau + \Delta|$. Also, $(\nu - \Delta) + (\tau + \Delta) = \nu + \tau = \gamma$.

From $\tau + \Delta \in V$ and $\nu - \Delta \in \mathbb{N}^k$ and $|\nu - \Delta| = -|\tau + \Delta|$ and $(\nu - \Delta) + (\tau + \Delta) = \gamma$, we obtain $(\tau + \Delta, \nu - \Delta) \in Q$ (by the definition of Q). Combining this with $(\tau + \Delta)_j = 1$, we obtain $(\tau + \Delta, \nu - \Delta) \in Q_1$ (by the definition of Q_1), qed.

is well-defined.

For each $(\tau,\nu) \in Q_1$, we have $(\tau - \Delta, \nu + \Delta) \in Q_0$ [29]. Thus, the map

$$Q_1 \to Q_0, \qquad (\tau,\nu) \mapsto (\tau - \Delta, \nu + \Delta) \tag{132}$$

is well-defined.

The two maps (131) and (132) are mutually inverse (this is clear from their definitions), and thus are bijections. Hence, in particular, the map (131) is a bijection.

Also, each $\tau \in \mathbb{Z}^k$ satisfies

$$\begin{aligned} |\tau + \Delta| &= |\tau| + \underbrace{|\Delta|}_{=1} \qquad \left(\text{since } |\alpha+\beta| = |\alpha| + |\beta| \text{ for all } \alpha \in \mathbb{Z}^k \text{ and } \beta \in \mathbb{Z}^k\right) \\ &= |\tau| + 1 \end{aligned}$$

and thus

$$(-1)^{n+|\tau+\Delta|} = (-1)^{n+|\tau|+1} = -(-1)^{n+|\tau|}. \tag{133}$$

Now, recall that Q_0 and Q_1 are two subsets of Q such that each $(\tau,\nu) \in Q$ belongs to either Q_0 or Q_1 (but not both at the same time). In other words, Q_0 and Q_1 are two disjoint subsets of Q whose union is the whole set Q. Hence, we

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[29] *Proof.* Let $(\tau,\nu) \in Q_1$. According to the definition of Q_1, this means that $(\tau,\nu) \in Q$ and $\tau_j = 1$.

We have $(\tau,\nu) \in Q$. According to the definition of Q, this means that $(\tau,\nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau,\nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

Lemma 11.15 **(d)** yields $\nu + \Delta \in \mathbb{N}^k$. Also, Lemma 11.15 **(b)** yields that $\tau - \Delta \in V$ and $(\tau - \Delta)_j = 0$. Also, any two k-tuples $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^k$ satisfy $|\alpha+\beta| = |\alpha| + |\beta|$ and $|\alpha - \beta| = |\alpha| - |\beta|$. Thus, $|\tau - \Delta| = |\tau| - |\Delta|$ and $|\nu + \Delta| = \underbrace{|\nu|}_{=-|\tau|} + |\Delta| = -|\tau| + |\Delta| = -\underbrace{(|\tau| - |\Delta|)}_{=|\tau-\Delta|} = -|\tau - \Delta|$. Also, $(\nu+\Delta) + (\tau - \Delta) = \nu + \tau = \gamma$.

From $\tau - \Delta \in V$ and $\nu + \Delta \in \mathbb{N}^k$ and $|\nu + \Delta| = -|\tau - \Delta|$ and $(\nu+\Delta)+(\tau-\Delta) = \gamma$, we obtain $(\tau - \Delta, \nu + \Delta) \in Q$ (by the definition of Q). Combining this with $(\tau - \Delta)_j = 0$, we obtain $(\tau - \Delta, \nu + \Delta) \in Q_0$ (by the definition of Q_0), qed.

can split the sum $\sum_{(\tau,\nu)\in Q}(-1)^{n+|\tau|}$ as follows:

$$\begin{aligned}
\sum_{(\tau,\nu)\in Q}(-1)^{n+|\tau|} &= \sum_{(\tau,\nu)\in Q_0}(-1)^{n+|\tau|} + \underbrace{\sum_{(\tau,\nu)\in Q_1}(-1)^{n+|\tau|}}_{\substack{=\sum_{(\tau,\nu)\in Q_0}(-1)^{n+|\tau+\Delta|} \\ \text{(here, we have substituted } (\tau+\Delta,\nu-\Delta) \text{ for } (\tau,\nu) \\ \text{in the sum, since the map (131) is a bijection)}}} \\
&= \sum_{(\tau,\nu)\in Q_0}(-1)^{n+|\tau|} + \sum_{(\tau,\nu)\in Q_0}\underbrace{(-1)^{n+|\tau+\Delta|}}_{\substack{=-(-1)^{n+|\tau|} \\ \text{(by (133))}}} \\
&= \sum_{(\tau,\nu)\in Q_0}(-1)^{n+|\tau|} + \sum_{(\tau,\nu)\in Q_0}\left(-(-1)^{n+|\tau|}\right) \\
&= \sum_{(\tau,\nu)\in Q_0}(-1)^{n+|\tau|} - \sum_{(\tau,\nu)\in Q_0}(-1)^{n+|\tau|} = 0.
\end{aligned}$$

Now, (130) yields

$$\sum_{\tau\in V}\sum_{\substack{\nu\in\mathbb{N}^k; \\ |\nu|=-|\tau|; \\ \nu+\tau=\gamma}}(-1)^{n+|\tau|} = \sum_{(\tau,\nu)\in Q}(-1)^{n+|\tau|} = 0 = \begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases}$$

(since $\begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases} = 0$). Thus, Lemma 11.16 is proven in Case 1.

Let us now consider Case 2. In this case, we have $(\gamma_2,\gamma_3,\ldots,\gamma_k)=(0,0,\ldots,0)$ and $\gamma_1\neq 0$. From $\gamma_1\neq 0$, we obtain $\gamma\neq\mathbf{0}$ and thus $\begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases} = 0$.

Now, $Q = \varnothing$ [30]. But (130) yields

$$\sum_{\tau \in V} \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|; \\ \nu + \tau = \gamma}} (-1)^{n+|\tau|} = \sum_{(\tau,\nu) \in Q} (-1)^{n+|\tau|} = (\text{empty sum}) \qquad (\text{since } Q = \varnothing)$$

$$= 0 = \begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases}$$

(since $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 0$). Thus, Lemma 11.16 is proven in Case 2.

Let us finally consider Case 3. In this case, we have $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$

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[30] *Proof.* Let $(\tau, \nu) \in Q$. We shall derive a contradiction.

Indeed, we have $(\tau, \nu) \in Q$. According to the definition of Q, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

We have $\tau \in V$. According to the definition of V, this means that τ is a k-tuple $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (125). In other words, $\tau \in \mathbb{Z}^k$ and $\tau_1 = -n$ and the condition (125) holds.

Now, fix $j \in \{2, 3, \ldots, k\}$. Then, $\tau_j \in \{0, 1\}$ (by (125), applied to $i = j$). Hence, $\tau_j \geq 0$. Also, $\nu_j \in \mathbb{N}$ (since $\nu \in \mathbb{N}^k$), so that $\nu_j \geq 0$. But $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$, and thus $\gamma_j = 0$ (since $j \in \{2, 3, \ldots, k\}$). But $\gamma = \nu + \tau$, and thus $\gamma_j = (\nu + \tau)_j = \nu_j + \tau_j$. Hence, $\nu_j + \tau_j = \gamma_j = 0$, so that $\nu_j = -\underbrace{\tau_j}_{\geq 0} \leq 0$. Combining this with $\nu_j \geq 0$, we obtain $\nu_j = 0$. Hence, $\nu_j = -\tau_j$ rewrites as $0 = -\tau_j$, so that $\tau_j = 0$.

Now, forget that we fixed j. Thus, we have shown that each $j \in \{2, 3, \ldots, k\}$ satisfies

$$\nu_j = 0 \tag{134}$$

and

$$\tau_j = 0. \tag{135}$$

Now,

$$|\tau| = \tau_1 + \tau_2 + \cdots + \tau_k = \sum_{j=1}^{k} \tau_j = \tau_1 + \sum_{j=2}^{k} \underbrace{\tau_j}_{\substack{=0 \\ (\text{by (135)})}} = \tau_1 = -n,$$

so that $-|\tau| = n$. Furthermore,

$$|\nu| = \nu_1 + \nu_2 + \cdots + \nu_k = \sum_{j=1}^{k} \nu_j = \nu_1 + \sum_{j=2}^{k} \underbrace{\nu_j}_{\substack{=0 \\ (\text{by (134)})}} = \nu_1,$$

so that $\nu_1 = |\nu| = -|\tau| = n$.

Now, from $\gamma = \nu + \tau$, we obtain $\gamma_1 = (\nu + \tau)_1 = \underbrace{\nu_1}_{=n} + \underbrace{\tau_1}_{=-n} = n + (-n) = 0$. This contradicts $\gamma_1 \neq 0$.

Now, forget that we fixed (τ, ν). We thus have found a contradiction for each $(\tau, \nu) \in Q$. Thus, there exists no $(\tau, \nu) \in Q$. In other words, $Q = \varnothing$.

and $\gamma_1 = 0$. Combining these two equalities, we obtain $\gamma_i = 0$ for all $i \in \{1, 2, \ldots, k\}$. In other words, $\gamma = \mathbf{0}$. Hence, $\begin{cases} 1, & \text{if } \gamma = \mathbf{0}; \\ 0, & \text{if } \gamma \neq \mathbf{0} \end{cases} = 1$.

Now, define two k-tuples $\tau_0 \in \mathbb{Z}^k$ and $\nu_0 \in \mathbb{Z}^k$ by

$$\tau_0 = (-n, 0, 0, \ldots, 0) \qquad \text{and} \qquad \nu_0 = (n, 0, 0, \ldots, 0).$$

Clearly, $\tau_0 \in V$ (by the definition of V) and $\nu_0 \in \mathbb{N}^k$ and $|\tau_0| = -n$ and $|\nu_0| = n$ and $\nu_0 + \tau_0 = \mathbf{0}$.

From $\tau_0 \in V$ and $\nu_0 \in \mathbb{N}^k$, we obtain $(\tau_0, \nu_0) \in V \times \mathbb{N}^k$. Also, $|\nu_0| = -|\tau_0|$ (since $\underbrace{|\nu_0|}_{=n} + \underbrace{|\tau_0|}_{=-n} = n + (-n) = 0$) and $\nu_0 + \tau_0 = \mathbf{0} = \gamma$. Thus, we have shown that $(\tau_0, \nu_0) \in V \times \mathbb{N}^k$ and $|\nu_0| = -|\tau_0|$ and $\nu_0 + \tau_0 = \gamma$. In other words, $(\tau_0, \nu_0) \in Q$ (by the definition of Q). In other words, $\{(\tau_0, \nu_0)\} \subseteq Q$.

On the other hand, $Q \subseteq \{(\tau_0, \nu_0)\}$ [31]. Combining this with $\{(\tau_0, \nu_0)\} \subseteq Q$, we obtain $Q = \{(\tau_0, \nu_0)\}$.

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[31] *Proof.* Let $(\tau, \nu) \in Q$. We shall prove that $(\tau, \nu) = (\tau_0, \nu_0)$.

Most of the following argument is copypasted from the previous footnote.

We have $(\tau, \nu) \in Q$. According to the definition of Q, this means that $(\tau, \nu) \in V \times \mathbb{N}^k$ and $|\nu| = -|\tau|$ and $\nu + \tau = \gamma$.

From $(\tau, \nu) \in V \times \mathbb{N}^k$, we obtain $\tau \in V$ and $\nu \in \mathbb{N}^k$.

We have $\tau \in V$. According to the definition of V, this means that τ is a k-tuple $(-n, \tau_2, \tau_3, \ldots, \tau_k) \in \mathbb{Z}^k$ satisfying (125). In other words, $\tau \in \mathbb{Z}^k$ and $\tau_1 = -n$ and the condition (125) holds.

Now, fix $j \in \{2, 3, \ldots, k\}$. Then, $\tau_j \in \{0, 1\}$ (by (125), applied to $i = j$). Hence, $\tau_j \geq 0$. Also, $\nu_j \in \mathbb{N}$ (since $\nu \in \mathbb{N}^k$), so that $\nu_j \geq 0$. But $(\gamma_2, \gamma_3, \ldots, \gamma_k) = (0, 0, \ldots, 0)$, and thus $\gamma_j = 0$ (since $j \in \{2, 3, \ldots, k\}$). But $\gamma = \nu + \tau$, and thus $\gamma_j = (\nu + \tau)_j = \nu_j + \tau_j$. Hence, $\nu_j + \tau_j = \gamma_j = 0$, so that $\nu_j = -\underbrace{\tau_j}_{\geq 0} \leq 0$. Combining this with $\nu_j \geq 0$, we obtain $\nu_j = 0$. Hence, $\nu_j = -\tau_j$ rewrites as $0 = -\tau_j$, so that $\tau_j = 0$.

Now, forget that we fixed j. Thus, we have shown that each $j \in \{2, 3, \ldots, k\}$ satisfies $\tau_j = 0$. In other words, $(\tau_2, \tau_3, \ldots, \tau_k) = (0, 0, \ldots, 0)$. Combining this with $\tau_1 = -n$, we obtain $\tau = (-n, 0, 0, \ldots, 0) = \tau_0$.

From $\nu + \tau = \gamma$, we obtain $\nu = \gamma - \underbrace{\tau}_{=\tau_0} = \gamma - \tau_0 = \nu_0$ (since $\nu_0 + \tau_0 = \gamma$). Combining this with $\tau = \tau_0$, we obtain $(\tau, \nu) = (\tau_0, \nu_0) \in \{(\tau_0, \nu_0)\}$.

Now, forget that we fixed (τ, ν). We thus have proven that $(\tau, \nu) \in \{(\tau_0, \nu_0)\}$ for each $(\tau, \nu) \in Q$. In other words, $Q \subseteq \{(\tau_0, \nu_0)\}$.

But (130) yields

$$\begin{aligned}\sum_{\tau\in V}\ \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} (-1)^{n+|\tau|} &= \sum_{(\tau,\nu)\in Q}(-1)^{n+|\tau|} = (-1)^{n+|\tau_0|} \qquad (\text{since } Q=\{(\tau_0,\nu_0)\})\\ &= (-1)^0 \qquad \left(\text{since } n+\underbrace{|\tau_0|}_{=-n} = n+(-n)=0\right)\\ &= 1 = \begin{cases}1, & \text{if } \gamma=\mathbf{0};\\ 0, & \text{if } \gamma\neq\mathbf{0}\end{cases}\end{aligned}$$

(since $\begin{cases}1, & \text{if } \gamma=\mathbf{0};\\ 0, & \text{if } \gamma\neq\mathbf{0}\end{cases} = 1$). Thus, Lemma 11.16 is proven in Case 3.

We have now proven Lemma 11.16 in each of the three Cases 1, 2 and 3. Hence, Lemma 11.16 always holds. □

Proof of Theorem 11.14. Each $\tau\in V$ satisfies $-|\tau|\in\{n-k+1,n-k+2,\ldots,n\}$ (by Proposition 11.11). Thus, we have the following equality of summation signs:

$$\sum_{\tau\in V} = \sum_{i=n-k+1}^{n}\ \sum_{\substack{\tau\in V;\\ -|\tau|=i}} = \sum_{j=1}^{k}\ \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} \tag{136}$$

(here, we have substituted $n-k+j$ for i in the outer sum). Now,

$$\begin{aligned}&\sum_{j=1}^{k}(-1)^{k-j}h_{n-k+j}\sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau}\\ &= \underbrace{\sum_{j=1}^{k}\ \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}}}_{\substack{=\sum\limits_{\tau\in V}\\ \text{(by (136))}}} \underbrace{(-1)^{k-j}}_{\substack{=(-1)^{n+|\tau|}\\ \text{(since } k-j=n+|\tau|\\ \text{(because } -|\tau|=n-k+j))}} \underbrace{h_{n-k+j}}_{\substack{=h_{-|\tau|}\\ \text{(since } n-k+j=-|\tau|\\ \text{(because } -|\tau|=n-k+j))}} s_{\mu+\tau}\\ &= \sum_{\tau\in V}(-1)^{n+|\tau|}h_{-|\tau|}s_{\mu+\tau}.\end{aligned} \tag{137}$$

But each $\tau\in V$ satisfies

$$h_{-|\tau|}s_{\mu+\tau} = \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|}} s_{\mu+(\nu+\tau)}. \tag{138}$$

[*Proof of (138):* Let $\tau\in V$. According to the definition of V, this means that τ is a k-tuple $(-n,\tau_2,\tau_3,\ldots,\tau_k)\in\mathbb{Z}^k$ satisfying (125). In other words, $\tau\in\mathbb{Z}^k$ and $\tau_1=-n$ and the relation (125) holds.

Proposition 11.11 yields $-|\tau| \in \{n-k+1, n-k+2, \ldots, n\} \subseteq \mathbb{N}$. Also, $\mu \in P_k \subseteq \mathbb{N}^k$; hence,

$$\mu_i \geq 0 \qquad \text{for each } i \in \{1, 2, \ldots, k\}. \tag{139}$$

Also, $\rho_1 = k-1$ (by the definition of ρ) and $\rho \in \mathbb{N}^k$ (likewise). Now,

$$(\mu + \tau + \rho)_1 = \underbrace{\mu_1}_{> n-k} + \underbrace{\tau_1}_{=-n} + \underbrace{\rho_1}_{=k-1} > (n-k) + (-n) + (k-1) = -1.$$

Thus, $(\mu + \tau + \rho)_1 \geq 0$ (since $(\mu + \tau + \rho)_1$ is an integer). In other words, $(\mu + \tau + \rho)_1 \in \mathbb{N}$. Furthermore, for each $i \in \{2, 3, \ldots, k\}$, we have

$$(\mu + \tau + \rho)_i = \underbrace{\mu_i}_{\substack{\in \mathbb{N} \\ (\text{since } \mu \in \mathbb{N}^k)}} + \underbrace{\tau_i}_{\substack{\in \mathbb{N} \\ (\text{since (125)} \\ \text{yields } \tau_i \in \{0,1\} \subseteq \mathbb{N})}} + \underbrace{\rho_i}_{\substack{\in \mathbb{N} \\ (\text{since } \rho \in \mathbb{N}^k)}} \in \mathbb{N}.$$

This also holds for $i = 1$ (since $(\mu + \tau + \rho)_1 \in \mathbb{N}$). Thus, we have $(\mu + \tau + \rho)_i \in \mathbb{N}$ for each $i \in \{1, 2, \ldots, k\}$. In other words, $\mu + \tau + \rho \in \mathbb{N}^k$. Hence, Theorem 11.7 (applied to $\lambda = \mu + \tau$ and $m = -|\tau|$) yields

$$s_{\mu+\tau} h_{-|\tau|} = \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|}} \underbrace{s_{\mu+\tau+\nu}}_{= s_{\mu+(\nu+\tau)}} = \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|}} s_{\mu+(\nu+\tau)}.$$

Thus,

$$h_{-|\tau|} s_{\mu+\tau} = s_{\mu+\tau} h_{-|\tau|} = \sum_{\substack{\nu \in \mathbb{N}^k; \\ |\nu| = -|\tau|}} s_{\mu+(\nu+\tau)}.$$

This proves (138).]

Now, (137) becomes

$$\begin{aligned}
&\sum_{j=1}^{k} (-1)^{k-j} h_{n-k+j} \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau} \\
&= \sum_{\tau\in V} (-1)^{n+|\tau|} \underbrace{h_{-|\tau|} s_{\mu+\tau}}_{\substack{= \sum\limits_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|}} s_{\mu+(\nu+\tau)} \\ \text{(by (138))}}} \\
&= \sum_{\tau\in V} (-1)^{n+|\tau|} \underbrace{\sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|}}}_{\substack{= \sum\limits_{\gamma\in\mathbb{Z}^k} \sum\limits_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}}}} s_{\mu+(\nu+\tau)} = \sum_{\tau\in V} (-1)^{n+|\tau|} \sum_{\gamma\in\mathbb{Z}^k} \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} \underbrace{s_{\mu+(\nu+\tau)}}_{\substack{=s_{\mu+\gamma} \\ \text{(since } \nu+\tau=\gamma)}} \\
&= \sum_{\tau\in V} (-1)^{n+|\tau|} \sum_{\gamma\in\mathbb{Z}^k} \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} s_{\mu+\gamma} = \sum_{\gamma\in\mathbb{Z}^k} \underbrace{\left(\sum_{\tau\in V} \sum_{\substack{\nu\in\mathbb{N}^k;\\ |\nu|=-|\tau|;\\ \nu+\tau=\gamma}} (-1)^{n+|\tau|} \right)}_{\substack{= \begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases} \\ \text{(by Lemma 11.16)}}} s_{\mu+\gamma} \\
&= \sum_{\gamma\in\mathbb{Z}^k} \begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases} s_{\mu+\gamma} = \underbrace{\begin{cases} 1, & \text{if } \mathbf{0}=\mathbf{0}; \\ 0, & \text{if } \mathbf{0}\neq\mathbf{0} \end{cases}}_{\substack{=1 \\ \text{(since } \mathbf{0}=\mathbf{0})}} s_{\mu+\mathbf{0}} + \sum_{\substack{\gamma\in\mathbb{Z}^k;\\ \gamma\neq\mathbf{0}}} \underbrace{\begin{cases} 1, & \text{if } \gamma=\mathbf{0}; \\ 0, & \text{if } \gamma\neq\mathbf{0} \end{cases}}_{\substack{=0 \\ \text{(since } \gamma\neq\mathbf{0})}} s_{\mu+\gamma} \\
&\qquad \left(\text{here, we have split off the addend for } \gamma=\mathbf{0} \text{ from the sum}\right) \\
&= s_{\mu+\mathbf{0}} = s_\mu .
\end{aligned}$$

This proves Theorem 11.14. ☐

Proof of Theorem 11.12. Theorem 11.14 yields

$$s_\mu = \sum_{j=1}^{k} (-1)^{k-j} \underbrace{h_{n-k+j}}_{\substack{\equiv a_j \operatorname{mod} I \\ \text{(by (15))}}} \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau} \equiv \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau\in V;\\ -|\tau|=n-k+j}} s_{\mu+\tau} \operatorname{mod} I.$$

Thus, in $\mathcal{S}/I$, we have

$$\overline{s_\mu} = \overline{\sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} s_{\mu+\tau}} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\tau \in V; \\ -|\tau| = n-k+j}} \overline{s_{\mu+\tau}}.$$

This proves Theorem 11.12. $\square$

12. Deforming symmetric functions

12.1. The basis theorem

> **Convention 12.1.** Let R be any commutative ring. Let $(a_1, a_2, \ldots, a_p)$ be any list of elements of R. Then, $\langle a_1, a_2, \ldots, a_p \rangle_R$ shall denote the ideal of R generated by these elements $a_1, a_2, \ldots, a_p$. When it is clear from the context what R is, we will simply write $\langle a_1, a_2, \ldots, a_p \rangle$ for this ideal (thus omitting the mention of R); for example, when we write "$R/\langle a_1, a_2, \ldots, a_p \rangle$", we will always mean $R/\langle a_1, a_2, \ldots, a_p \rangle_R$.

We have so far studied a quotient $\mathcal{S}/I$ of the ring $\mathcal{S}$ of symmetric polynomials in k variables $x_1, x_2, \ldots, x_k$. But $\mathcal{S}$ itself is a quotient of a larger ring – the ring Λ of symmetric functions in infinitely many variables. More precisely,

$$\mathcal{S} \cong \Lambda / \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle$$

(and the canonical $\mathbf{k}$-algebra isomorphism $\mathcal{S} \to \Lambda / \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle$ sends $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ to $e_1, e_2, \ldots, e_k, 0, 0, 0, \ldots$). Hence, at least when $a_1, a_2, \ldots, a_k \in \mathbf{k}$, we have

$$\mathcal{S}/I \cong \Lambda / \left(\langle \mathbf{h}_{n-k+1} - a_1, \mathbf{h}_{n-k+2} - a_2, \ldots, \mathbf{h}_n - a_k \rangle + \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \right).$$

If $a_1, a_2, \ldots, a_k$ are themselves elements of $\mathcal{S}$, then we need to lift them to elements $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ of Λ in order for such an isomorphism to hold.

This suggests a further generalization: What if we replace $\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots$ by $\mathbf{e}_{k+1} - \mathbf{b}_1, \mathbf{e}_{k+2} - \mathbf{b}_2, \mathbf{e}_{k+3} - \mathbf{b}_3, \ldots$ for some $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots \in \Lambda$? Let us take a look at this generalization:

> **Definition 12.2.** Throughout Section 12, we shall use the following notations:
>
> Let Λ be the ring of symmetric functions in infinitely many indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. (See [GriRei20, Chapter 2] for more about this ring Λ.) Let $\mathbf{e}_m$ and $\mathbf{h}_m$ be the elementary symmetric functions and the complete homogeneous symmetric functions in Λ. For each partition λ, let $\mathbf{s}_\lambda$ be the Schur function in Λ corresponding to λ.
>
> For each $i \in \{1, 2, \ldots, k\}$, let $\mathbf{a}_i$ be an element of Λ with degree $< n - k + i$.

For each $i \in \{1, 2, 3, \ldots\}$, let $\mathbf{b}_i$ be an element of Λ with degree $< k + i$. Let K be the ideal

$$\langle \mathbf{h}_{n-k+1} - \mathbf{a}_1, \mathbf{h}_{n-k+2} - \mathbf{a}_2, \ldots, \mathbf{h}_n - \mathbf{a}_k \rangle + \langle \mathbf{e}_{k+1} - \mathbf{b}_1, \mathbf{e}_{k+2} - \mathbf{b}_2, \mathbf{e}_{k+3} - \mathbf{b}_3, \ldots \rangle$$

of Λ. For each $f \in \Lambda$, we let $\overline{f}$ denote the projection of f onto the quotient Λ / K.

Theorem 12.3. The $\mathbf{k}$-module Λ / K is a free $\mathbf{k}$-module with basis $\left(\overline{\mathbf{s}_\lambda} \right)_{\lambda \in P_{k,n}}$.

12.2. Spanning

Proving Theorem 12.3 will take us a while. We start with some easy observations:

- For each $i \in \{1, 2, \ldots, k\}$, we have
$$\mathbf{a}_i = (\text{some symmetric function of degree } < n - k + i) . \tag{140}$$
(This follows from the definition of $\mathbf{a}_i$.)

- For each $i \in \{1, 2, 3, \ldots\}$, we have
$$\mathbf{b}_i = (\text{some symmetric function of degree } < k + i) . \tag{141}$$
(This follows from the definition of $\mathbf{b}_i$.)

- For each $i \in \{1, 2, 3, \ldots\}$, we have
$$\mathbf{e}_{k+i} - \mathbf{b}_i \in K \tag{142}$$
(because of how K was defined). In other words, for each $i \in \{1, 2, 3, \ldots\}$, we have
$$\mathbf{e}_{k+i} \equiv \mathbf{b}_i \mod K. \tag{143}$$
Substituting $j - k$ for i in this statement, we obtain the following: For each $j \in \{k + 1, k + 2, k + 3, \ldots\}$, we have
$$\mathbf{e}_j \equiv \mathbf{b}_{j-k} \mod K. \tag{144}$$

- For each $i \in \{1, 2, \ldots, k\}$, we have
$$\mathbf{h}_{n-k+i} - \mathbf{a}_i \in K \tag{145}$$
(because of how K was defined). In other words, for each $i \in \{1, 2, \ldots, k\}$, we have
$$\mathbf{h}_{n-k+i} \equiv \mathbf{a}_i \mod K. \tag{146}$$
Substituting $j - (n - k)$ for i in this statement, we obtain the following: For each $j \in \{n - k + 1, n - k + 2, \ldots, n\}$, we have
$$\mathbf{h}_j \equiv \mathbf{a}_{j-(n-k)} \mod K. \tag{147}$$

Let Par denote the set of all partitions.

For each $m \in \mathbb{Z}$, we let $\Lambda_{\deg \leq m}$ denote the $\mathbf{k}$-submodule of Λ that consists of all symmetric functions $f \in \Lambda$ of degree $\leq m$. Thus, $\left(\Lambda_{\deg \leq m}\right)_{m \in \mathbb{N}}$ is a filtration of the $\mathbf{k}$-algebra Λ. In particular, $1 \in \Lambda_{\deg \leq 0}$ and

$$\Lambda_{\deg \leq i} \Lambda_{\deg \leq j} \subseteq \Lambda_{\deg \leq i+j} \qquad \text{for all } i, j \in \mathbb{N}. \tag{148}$$

Also, $\Lambda_{\deg \leq m}$ is the $\mathbf{k}$-submodule 0 of Λ whenever $m \in \mathbb{Z}$ is negative; thus, in particular, $\Lambda_{\deg \leq -1} = 0$.

We state an analogue of Lemma 5.10:

Lemma 12.4. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be any partition. Let $i \in \{1, 2, \ldots, \ell\}$ and $j \in \{1, 2, \ldots, \ell\}$. Then:

(a) The (i, j)-th cofactor of the matrix $\left(\mathbf{h}_{\lambda_u - u + v}\right)_{1 \leq u \leq \ell,\ 1 \leq v \leq \ell}$ is a homogeneous element of Λ of degree $|\lambda| - (\lambda_i - i + j)$.

(b) The (i, j)-th cofactor of the matrix $\left(\mathbf{e}_{\lambda_u - u + v}\right)_{1 \leq u \leq \ell,\ 1 \leq v \leq \ell}$ is a homogeneous element of Λ of degree $|\lambda| - (\lambda_i - i + j)$.

Proof of Lemma 12.4. Each of the two parts of Lemma 12.4 is proven in the same way as Lemma 5.10, with the obvious modifications to the argument (viz., replacing $\mathcal{S}$ by Λ, and replacing h_m by $\mathbf{h}_m$ or by $\mathbf{e}_m$). □

Next, we claim a lemma that will yield one half of Theorem 12.3 (namely, that the family $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda \in P_{k,n}}$ spans the $\mathbf{k}$-module Λ / K):

Lemma 12.5. Let λ be a partition such that $\lambda \notin P_{k,n}$. Then,

$$\mathbf{s}_\lambda \equiv (\text{some symmetric function of degree } < |\lambda|) \operatorname{mod} K.$$

We will not prove Lemma 12.5 immediately; instead, let us show a weakening of it first:

Lemma 12.6. Let λ be a partition such that $\lambda \notin P_k$. Then,

$$\mathbf{s}_\lambda \equiv (\text{some symmetric function of degree } < |\lambda|) \operatorname{mod} K.$$

Proof of Lemma 12.6 (sketched). We have $\lambda \notin P_k$. Hence, the partition λ has more than k parts.

Define a partition μ by $\mu = \lambda^t$. Hence, μ_1 is the number of parts of λ. Thus, we have $\mu_1 > k$ (since λ has more than k parts), so that $\mu_1 \geq k + 1$. Moreover, $|\mu| = |\lambda|$ (since $\mu = \lambda^t$).

Write the partition μ in the form $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. For each $j \in \{1, 2, \ldots, \ell\}$, we have $\mu_1 - 1 + \underbrace{j}_{\geq 1} \geq \mu_1 - 1 + 1 = \mu_1 \geq k + 1$ and thus $\mu_1 - 1 + j \in$

$\{k+1, k+2, k+3, \ldots\}$ and therefore

$$\begin{aligned}
&\mathbf{e}_{\mu_1-1+j} \\
&\equiv \mathbf{b}_{\mu_1-1+j-k} \qquad \text{(by (144), applied to } \mu_1 - 1 + j \text{ instead of } j) \\
&= \left(\text{some symmetric function of degree } < \underbrace{k + (\mu_1 - 1 + j - k)}_{=\mu_1-1+j} \right) \\
&\qquad \text{(by (141), applied to } i = \mu_1 - 1 + j - k) \\
&= (\text{some symmetric function of degree } < \mu_1 - 1 + j) \bmod K. \qquad (149)
\end{aligned}$$

From $\mu = \lambda^t$, we obtain $\mu^t = \left(\lambda^t\right)^t = \lambda$. But Corollary 6.5 (applied to μ and μ_i instead of λ and λ_i) yields

$$\begin{aligned}
\mathbf{s}_{\mu^t} &= \det\left(\left(\mathbf{e}_{\mu_i-i+j}\right)_{1\le i\le \ell,\ 1\le j\le \ell}\right) = \det\left(\left(\mathbf{e}_{\mu_u-u+v}\right)_{1\le u\le \ell,\ 1\le v\le \ell}\right) \\
&= \sum_{j=1}^{\ell} \mathbf{e}_{\mu_1-1+j} \cdot C_j, \qquad (150)
\end{aligned}$$

where C_j denotes the $(1, j)$-th cofactor of the $\ell \times \ell$-matrix $\left(\mathbf{e}_{\mu_u-u+v}\right)_{1\le u\le \ell,\ 1\le v\le \ell}$. (Here, the last equality sign follows from (19), applied to $R = \Lambda$ and $A = \left(\mathbf{e}_{\mu_u-u+v}\right)_{1\le u\le \ell,\ 1\le v\le \ell}$ and $a_{u,v} = \mathbf{e}_{\mu_u-u+v}$ and $i = 1$.)

For each $j \in \{1, 2, \ldots, \ell\}$, the element C_j is the $(1, j)$-th cofactor of the matrix $\left(\mathbf{e}_{\mu_u-u+v}\right)_{1\le u\le \ell,\ 1\le v\le \ell}$ (by its definition), and thus is a homogeneous element of Λ of degree $|\mu| - (\mu_1 - 1 + j)$ (by Lemma 12.4 **(b)**, applied to 1 and μ instead of i and λ). Hence,

$$C_j = (\text{some symmetric function of degree } \le |\mu| - (\mu_1 - 1 + j)) \qquad (151)$$

for each $j \in \{1, 2, \ldots, \ell\}$. Therefore, (150) becomes

$$\begin{aligned}
\mathbf{s}_{\mu^t} &= \sum_{j=1}^{\ell} \underbrace{\mathbf{e}_{\mu_1-1+j}}_{\substack{\equiv(\text{some symmetric function of degree } <\mu_1-1+j) \bmod K \\ \text{(by (149))}}} \\
&\qquad \cdot \underbrace{C_j}_{\substack{=(\text{some symmetric function of degree } \le|\mu|-(\mu_1-1+j)) \\ \text{(by (151))}}} \\
&\equiv \sum_{j=1}^{k} (\text{some symmetric function of degree } < \mu_1 - 1 + j) \\
&\qquad \cdot (\text{some symmetric function of degree } \le |\mu| - (\mu_1 - 1 + j)) \\
&= (\text{some symmetric function of degree } < |\mu|) \bmod K.
\end{aligned}$$

In view of $\mu^t = \lambda$ and $|\mu| = |\lambda|$, this rewrites as

$$\mathbf{s}_\lambda \equiv (\text{some symmetric function of degree } < |\lambda|) \operatorname{mod} K.$$

This proves Lemma 12.6. ☐

Our next lemma is an analogue of Lemma 5.4:

Lemma 12.7. Let i be an integer such that $i > n - k$. Then,

$$\mathbf{h}_i \equiv (\text{some symmetric function of degree } < i) \operatorname{mod} K.$$

Proof of Lemma 12.7 (sketched). We shall prove Lemma 12.7 by strong induction on i. Thus, we assume (as the induction hypothesis) that

$$\mathbf{h}_j \equiv (\text{some symmetric function of degree } < j) \operatorname{mod} K \tag{152}$$

for every $j \in \{n-k+1, n-k+2, \ldots, i-1\}$.

If $i \leq n$, then (147) (applied to $j = i$) yields $\mathbf{h}_i \equiv \mathbf{a}_{i-(n-k)} \operatorname{mod} K$ (since $i \in \{n-k+1, n-k+2, \ldots, n\}$), which clearly proves Lemma 12.7 (since $\mathbf{a}_{i-(n-k)}$ is a symmetric function of degree $< i$ [32]). Thus, for the rest of this proof, we WLOG assume that $i > n$. Hence, each $t \in \{1, 2, \ldots, k\}$ satisfies $i - t \in \{n-k+1, n-k+2, \ldots, i-1\}$ (since $\underbrace{i}_{>n} - \underbrace{t}_{\leq k} > n - k$ and $i - \underbrace{t}_{\geq 1} \leq i - 1$) and therefore

$$\mathbf{h}_{i-t} \equiv (\text{some symmetric function of degree } < i - t) \operatorname{mod} K \tag{153}$$

(by (152), applied to $j = i - t$).

On the other hand, each $t \in \{k+1, k+2, k+3, \ldots\}$ satisfies

$$\mathbf{e}_t \equiv (\text{some symmetric function of degree } < t) \operatorname{mod} K. \tag{154}$$

[*Proof of (154):* Let $t \in \{k+1, k+2, k+3, \ldots\}$. Thus, $t > k$. Hence, the partition $\left(1^t\right)$ has more than k parts (since it has t parts), and therefore we have $\left(1^t\right) \notin P_k$. Hence, Lemma 12.6 (applied to $\lambda = \left(1^t\right)$) yields

$$\mathbf{s}_{(1^t)} \equiv \left(\text{some symmetric function of degree } < \left|1^t\right|\right) \operatorname{mod} K.$$

In view of $\mathbf{s}_{(1^t)} = \mathbf{e}_t$ and $\left|1^t\right| = t$, this rewrites as

$$\mathbf{e}_t \equiv (\text{some symmetric function of degree } < t) \operatorname{mod} K.$$

This proves (154).]

---

[32] by (140)

Now, we claim that each $t \in \{1, 2, \ldots, i\}$ satisfies

$$\mathbf{h}_{i-t}\mathbf{e}_t \equiv (\text{some symmetric function of degree } < i) \bmod K. \tag{155}$$

[*Proof of (155):* Let $t \in \{1, 2, \ldots, i\}$. We are in one of the following two cases:

Case 1: We have $t \leq k$.

Case 2: We have $t > k$.

Let us first consider Case 1. In this case, we have $t \leq k$. Hence, $t \in \{1, 2, \ldots, k\}$. Thus,

$$\begin{aligned}
&\underbrace{\mathbf{h}_{i-t}}_{\substack{\equiv (\text{some symmetric function of degree } < i-t) \bmod K \\ (\text{by (153)})}} \mathbf{e}_t \\
&\equiv (\text{some symmetric function of degree } < i-t) \cdot \mathbf{e}_t \\
&= (\text{some symmetric function of degree } < i) \bmod K
\end{aligned}$$

(since $\mathbf{e}_t$ is a symmetric function of degree t). Hence, (155) is proven in Case 1.

Let us next consider Case 2. In this case, we have $t > k$. Hence, $t \in \{k+1, k+2, k+3, \ldots\}$. Thus,

$$\begin{aligned}
&\mathbf{h}_{i-t} \underbrace{\mathbf{e}_t}_{\substack{\equiv (\text{some symmetric function of degree } < t) \bmod K \\ (\text{by (154)})}} \\
&\equiv \mathbf{h}_{i-t} \cdot (\text{some symmetric function of degree } < t) \\
&= (\text{some symmetric function of degree } < i) \bmod K
\end{aligned}$$

(since $\mathbf{h}_{i-t}$ is a symmetric function of degree $i - t$). Thus, (155) is proven in Case 2.

We have now proven (155) in both Cases 1 and 2. Thus, (155) always holds.]

On the other hand, $i > n - k \geq 0$ (since $n \geq k$), so that $i \neq 0$. Now, (46) (applied to $N = i$) yields

$$\sum_{j=0}^{i} (-1)^j \mathbf{h}_{i-j}\mathbf{e}_j = \delta_{0,i} = 0 \qquad (\text{since } i \neq 0).$$

Hence,

$$\begin{aligned}
0 &= \sum_{j=0}^{i} (-1)^j \mathbf{h}_{i-j}\mathbf{e}_j = \sum_{t=0}^{i} (-1)^t \mathbf{h}_{i-t}\mathbf{e}_t \\
&\qquad (\text{here, we have renamed the summation index } j \text{ as } t) \\
&= \underbrace{(-1)^0}_{=1} \underbrace{\mathbf{h}_{i-0}}_{=\mathbf{h}_i} \underbrace{\mathbf{e}_0}_{=1} + \sum_{t=1}^{i} (-1)^t \mathbf{h}_{i-t}\mathbf{e}_t \\
&\qquad (\text{here, we have split off the addend for } t = 0 \text{ from the sum}) \\
&= \mathbf{h}_i + \sum_{t=1}^{i} (-1)^t \mathbf{h}_{i-t}\mathbf{e}_t.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{h}_i &= -\sum_{t=1}^{i} (-1)^t \underbrace{\mathbf{h}_{i-t}\mathbf{e}_t}_{\substack{\equiv (\text{some symmetric function of degree } <i) \bmod K \\ (\text{by (155)})}} \\
&\equiv -\sum_{t=1}^{i} (-1)^t (\text{some symmetric function of degree } < i) \\
&= (\text{some symmetric function of degree } < i) \bmod K.
\end{aligned}$$

This completes the induction step. Thus, Lemma 12.7 is proven. □

Recall the first Jacobi-Trudi identity ([GriRei20, (2.4.16)]):

Proposition 12.8. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be two partitions. Then,

$$\mathbf{s}_{\lambda/\mu} = \det\left(\left(\mathbf{h}_{\lambda_i - \mu_j - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right).$$

Next, we are ready to prove Lemma 12.5:

Proof of Lemma 12.5 (sketched). We must prove that

$$\mathbf{s}_\lambda \equiv (\text{some symmetric function of degree } < |\lambda|) \bmod K.$$

If $\lambda \notin P_k$, then this follows from Lemma 12.6. Thus, for the rest of this proof, we WLOG assume that $\lambda \in P_k$.

From $\lambda \in P_k$ and $\lambda \notin P_{k,n}$, we conclude that not all parts of the partition λ are $\le n - k$. Thus, the first entry λ_1 of λ is $> n - k$ (since $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots$). But $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ (since $\lambda \in P_k$). Thus, Proposition 12.8 (applied to $\ell = k$, $\mu = \varnothing$ and $\mu_i = 0$) yields

$$\begin{aligned}
\mathbf{s}_{\lambda/\varnothing} &= \det\left(\left(\mathbf{h}_{\lambda_i - 0 - i + j}\right)_{1 \le i \le k,\ 1 \le j \le k}\right) = \det\left(\left(\mathbf{h}_{\lambda_i - i + j}\right)_{1 \le i \le k,\ 1 \le j \le k}\right) \\
&= \det\left(\left(\mathbf{h}_{\lambda_u - u + v}\right)_{1 \le u \le k,\ 1 \le v \le k}\right) \\
&\qquad \left(\begin{array}{c}\text{here, we have renamed the indices } i \text{ and } j \\ \text{as } u \text{ and } v \text{ in the matrix}\end{array}\right) \\
&= \sum_{j=1}^{k} \mathbf{h}_{\lambda_1 - 1 + j} \cdot C_j, \qquad (156)
\end{aligned}$$

where C_j denotes the $(1, j)$-th cofactor of the $k \times k$-matrix $\left(\mathbf{h}_{\lambda_u - u + v}\right)_{1 \le u \le k,\ 1 \le v \le k}$. (Here, the last equality sign follows from (19), applied to $\ell = k$ and $R = \Lambda$ and $A = \left(\mathbf{h}_{\lambda_u - u + v}\right)_{1 \le u \le k,\ 1 \le v \le k}$ and $a_{u,v} = \mathbf{h}_{\lambda_u - u + v}$ and $i = 1$.)

For each $j \in \{1, 2, \ldots, k\}$, we have $\lambda_1 - 1 + j \geq \lambda_1 - 1 + 1 = \lambda_1 > n - k$ and therefore

$$\begin{aligned}&\mathbf{h}_{\lambda_1 - 1 + j} \\ &\quad \equiv (\text{some symmetric function of degree } < \lambda_1 - 1 + j) \bmod K \qquad (157)\end{aligned}$$

(by Lemma 12.7, applied to $i = \lambda_1 - 1 + j$).

For each $j \in \{1, 2, \ldots, k\}$, the polynomial C_j is the $(1, j)$-th cofactor of the matrix $\left(\mathbf{h}_{\lambda_u - u + v}\right)_{1 \leq u \leq k,\ 1 \leq v \leq k}$ (by its definition), and thus is a homogeneous element of Λ of degree $|\lambda| - (\lambda_1 - 1 + j)$ (by Lemma 12.4 **(a)**, applied to $\ell = k$ and $i = 1$). Hence,

$$C_j = (\text{some symmetric function of degree } \leq |\lambda| - (\lambda_1 - 1 + j)) \qquad (158)$$

for each $j \in \{1, 2, \ldots, k\}$.

Therefore, (156) becomes

$$\begin{aligned}\mathbf{s}_{\lambda/\varnothing} &= \sum_{j=1}^{k} \underbrace{\mathbf{h}_{\lambda_1 - 1 + j}}_{\substack{\equiv (\text{some symmetric function of degree } < \lambda_1 - 1 + j) \bmod K \\ (\text{by (157)})}} \\ &\qquad \cdot \underbrace{C_j}_{\substack{= (\text{some symmetric function of degree } \leq |\lambda| - (\lambda_1 - 1 + j)) \\ (\text{by (158)})}} \\ &\equiv \sum_{j=1}^{k} (\text{some symmetric function of degree } < \lambda_1 - 1 + j) \\ &\qquad \cdot (\text{some symmetric function of degree } \leq |\lambda| - (\lambda_1 - 1 + j)) \\ &= (\text{some symmetric function of degree } < |\lambda|) \bmod K.\end{aligned}$$

In view of $\mathbf{s}_{\lambda/\varnothing} = \mathbf{s}_\lambda$, this rewrites as

$$\mathbf{s}_\lambda \equiv (\text{some symmetric function of degree } < |\lambda|) \bmod K.$$

This proves Lemma 12.5. ☐

Lemma 12.9. Let $N \in \mathbb{N}$. Let $f \in \Lambda$ be a symmetric function of degree $< N$. Then, there exists a family $(c_\kappa)_{\kappa \in \operatorname{Par};\ |\kappa| < N}$ of elements of $\mathbf{k}$ such that $f = \sum\limits_{\substack{\kappa \in \operatorname{Par}; \\ |\kappa| < N}} c_\kappa \mathbf{s}_\kappa$.

Proof of Lemma 12.9. For each $d \in \mathbb{N}$, we let $\Lambda_{\deg = d}$ be the d-th graded part of the graded $\mathbf{k}$-module Λ. This is the $\mathbf{k}$-submodule of Λ consisting of all homogeneous elements of Λ of degree d (including the zero vector 0, which is homogeneous of every degree).

Recall that the family $(\mathbf{s}_\lambda)_{\lambda\in\operatorname{Par}}$ is a graded basis of the graded $\mathbf{k}$-module Λ. In other words, for each $d \in \mathbb{N}$, the family $(\mathbf{s}_\lambda)_{\lambda\in\operatorname{Par};\ |\lambda|=d}$ is a basis of the $\mathbf{k}$-submodule $\Lambda_{\deg=d}$ of Λ. Hence, for each $d \in \mathbb{N}$, we have

$$\begin{aligned}\Lambda_{\deg=d} &= \left(\text{the } \mathbf{k}\text{-linear span of the family } (\mathbf{s}_\lambda)_{\lambda\in\operatorname{Par};\ |\lambda|=d}\right) \\ &= \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=d}} \mathbf{k}\mathbf{s}_\lambda. \end{aligned}\qquad (159)$$

The symmetric function f has degree $< N$. Hence, we can write f in the form $f = \sum\limits_{d=0}^{N-1} f_d$ for some $f_0, f_1, \ldots, f_{N-1} \in \Lambda$, where each f_d is a homogeneous symmetric function of degree d. Consider these $f_0, f_1, \ldots, f_{N-1}$. For each $d \in \{0, 1, \ldots, N-1\}$, the symmetric function f_d is an element of Λ and is homogeneous of degree d (as we already know). In other words, for each $d \in \{0, 1, \ldots, N-1\}$, we have

$$f_d \in \Lambda_{\deg=d}. \qquad (160)$$

Now,

$$f = \sum_{d=0}^{N-1} \underbrace{f_d}_{\substack{\in\Lambda_{\deg=d}\\ \text{(by (160))}}} \in \sum_{d=0}^{N-1} \underbrace{\Lambda_{\deg=d}}_{\substack{=\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=d}} \mathbf{k}\mathbf{s}_\lambda\\ \text{(by (159))}}} = \underbrace{\sum_{d=0}^{N-1} \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=d}}}_{=\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|<N}}} \mathbf{k}\mathbf{s}_\lambda = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|<N}} \mathbf{k}\mathbf{s}_\lambda = \sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<N}} \mathbf{k}\mathbf{s}_\kappa$$

(here, we have renamed the summation index λ as κ in the sum). In other words, there exists a family $(c_\kappa)_{\kappa\in\operatorname{Par};\ |\kappa|<N}$ of elements of $\mathbf{k}$ such that $f = \sum\limits_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<N}} c_\kappa \mathbf{s}_\kappa$. This proves Lemma 12.9. ☐

Lemma 12.10. For each $\mu \in \operatorname{Par}$, the element $\overline{\mathbf{s}_\mu} \in \Lambda / K$ belongs to the $\mathbf{k}$-submodule of Λ / K spanned by the family $(\overline{\mathbf{s}_\lambda})_{\lambda\in P_{k,n}}$.

Proof of Lemma 12.10. Let M be the $\mathbf{k}$-submodule of Λ / K spanned by the family $(\overline{\mathbf{s}_\lambda})_{\lambda\in P_{k,n}}$. We thus must prove that $\overline{\mathbf{s}_\mu} \in M$ for each $\mu \in \operatorname{Par}$.

We shall prove this by strong induction on $|\mu|$. Thus, we fix some $N \in \mathbb{N}$, and we assume (as induction hypothesis) that

$$\overline{\mathbf{s}_\kappa} \in M \qquad \text{for each } \kappa \in \operatorname{Par} \text{ satisfying } |\kappa| < N. \qquad (161)$$

Now, let $\mu \in \operatorname{Par}$ be such that $|\mu| = N$. We then must show that $\overline{\mathbf{s}_\mu} \in M$.

If $\mu \in P_{k,n}$, then this is obvious (since $\overline{\mathbf{s}_\mu}$ then belongs to the family that spans M). Thus, for the rest of this proof, we WLOG assume that $\mu \notin P_{k,n}$. Hence, Lemma 12.5 (applied to $\lambda = \mu$) yields

$$\mathbf{s}_\mu \equiv (\text{some symmetric function of degree } < |\mu|) \operatorname{mod} K.$$

In other words, there exists some symmetric function $f \in \Lambda$ of degree $< |\mu|$ such that $\mathbf{s}_\mu \equiv f \operatorname{mod} K$. Consider this f.

But f is a symmetric function of degree $< |\mu|$. In other words, f is a symmetric function of degree $< N$ (since $|\mu| = N$). Hence, Lemma 12.9 shows that there exists a family $(c_\kappa)_{\kappa \in \operatorname{Par};\ |\kappa| < N}$ of elements of $\mathbf{k}$ such that $f = \sum\limits_{\substack{\kappa \in \operatorname{Par}; \\ |\kappa| < N}} c_\kappa \mathbf{s}_\kappa$. Consider this family. From $f = \sum\limits_{\substack{\kappa \in \operatorname{Par}; \\ |\kappa| < N}} c_\kappa \mathbf{s}_\kappa$, we obtain

$$\overline{f} = \overline{\sum_{\substack{\kappa \in \operatorname{Par}; \\ |\kappa| < N}} c_\kappa \mathbf{s}_\kappa} = \sum_{\substack{\kappa \in \operatorname{Par}; \\ |\kappa| < N}} c_\kappa \underbrace{\overline{\mathbf{s}_\kappa}}_{\substack{\in M \\ (\text{by } (161))}} \in \sum_{\substack{\kappa \in \operatorname{Par}; \\ |\kappa| < N}} c_\kappa M \subseteq M \qquad (\text{since } M \text{ is a } \mathbf{k}\text{-module}).$$

But from $\mathbf{s}_\mu \equiv f \operatorname{mod} K$, we obtain $\overline{\mathbf{s}_\mu} = \overline{f} \in M$. This completes our induction step. Thus, we have proven by strong induction that $\overline{\mathbf{s}_\mu} \in M$ for each $\mu \in \operatorname{Par}$. This proves Lemma 12.10. □

Corollary 12.11. The family $(\overline{\mathbf{s}_\lambda})_{\lambda \in P_{k,n}}$ spans the $\mathbf{k}$-module Λ / K.

Proof of Corollary 12.11. It is well-known that $(\mathbf{s}_\lambda)_{\lambda \in \operatorname{Par}}$ is a basis of the $\mathbf{k}$-module Λ. Hence, $(\overline{\mathbf{s}_\lambda})_{\lambda \in \operatorname{Par}}$ is a spanning set of the $\mathbf{k}$-module Λ / K. Thus, $(\overline{\mathbf{s}_\lambda})_{\lambda \in P_{k,n}}$ is also a spanning set of the $\mathbf{k}$-module Λ / K (because Lemma 12.10 shows that every element of the first spanning set belongs to the span of the second). This proves Corollary 12.11. □

With Corollary 12.11, we have proven "one half" of Theorem 12.3.

12.3. A lemma on filtrations

Next, we recall the definition of a filtration of a k-module:

Definition 12.12. Let V be a $\mathbf{k}$-module. A $\mathbf{k}$*-module filtration* of V means a sequence $(V_m)_{m \in \mathbb{N}}$ of $\mathbf{k}$-submodules of V such that $\bigcup\limits_{m \in \mathbb{N}} V_m = V$ and $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$.

For example, $\left(\Lambda_{\deg \leq m}\right)_{m \in \mathbb{N}}$ is a $\mathbf{k}$-module filtration of Λ.

The filtered $\mathbf{k}$-modules are the objects of a category, whose morphisms are $\mathbf{k}$-linear maps respecting the filtration. Here is how they are defined:

Definition 12.13. Let V and W be two $\mathbf{k}$-modules. Let $f : V \to W$ be a $\mathbf{k}$-module homomorphism. Let $(V_m)_{m\in\mathbb{N}}$ be a $\mathbf{k}$-module filtration of V, and let $(W_m)_{m\in\mathbb{N}}$ be a $\mathbf{k}$-module filtration of W.

We say that *the map f respects the filtrations $(V_m)_{m\in\mathbb{N}}$ and $(W_m)_{m\in\mathbb{N}}$* if it satisfies $(f(V_m) \subseteq W_m$ for every $m \in \mathbb{N})$. Sometimes we abbreviate "the map f respects the filtrations $(V_m)_{m\geq 0}$ and $(W_m)_{m\geq 0}$" to "*the map f respects the filtration*", as long as the filtrations $(V_m)_{m\in\mathbb{N}}$ and $(W_m)_{m\in\mathbb{N}}$ are clear from the context.

The following elementary fact about filtrations of $\mathbf{k}$-modules will be crucial to us:

Proposition 12.14. Let V be a $\mathbf{k}$-module. Let $(V_m)_{m\in\mathbb{N}}$ be a $\mathbf{k}$-module filtration of V. Let $f : V \to V$ be a $\mathbf{k}$-module homomorphism which satisfies

$$(f(V_m) \subseteq V_{m-1} \qquad \text{for every } m \in \mathbb{N}),$$

where V_{-1} denotes the $\mathbf{k}$-submodule 0 of V. Then:

(a) The $\mathbf{k}$-module homomorphism $\operatorname{id} - f$ is an isomorphism.

(b) Each of the maps $\operatorname{id} - f$ and $(\operatorname{id} - f)^{-1}$ respects the filtration.

Proposition 12.14 is classical; a proof can be found in [Grinbe11, Proposition 1.99] (see the detailed version of [Grinbe11] for a detailed proof). Let us restate this proposition in a form adapted for our use:

Proposition 12.15. Let V be a $\mathbf{k}$-module. Let $(V_m)_{m\in\mathbb{N}}$ be a $\mathbf{k}$-module filtration of V. Let $g : V \to V$ be a $\mathbf{k}$-module homomorphism which satisfies

$$(g(v) \in v + V_{m-1} \qquad \text{for every } m \in \mathbb{N} \text{ and each } v \in V_m), \tag{162}$$

where V_{-1} denotes the $\mathbf{k}$-submodule 0 of V. Then:

(a) The $\mathbf{k}$-module homomorphism g is an isomorphism.

(b) Each of the maps g and g^{-1} respects the filtration.

Proof of Proposition 12.15. Let $f : V \to V$ be the $\mathbf{k}$-module homomorphism $\operatorname{id} - g$. Then, $f = \operatorname{id} - g$, so that $g = \operatorname{id} - f$. Now, for each $m \in \mathbb{N}$, we have $f(V_m) \subseteq V_{m-1}$ (since each $v \in V_m$ satisfies

$$\begin{aligned}
\underbrace{f}_{=\operatorname{id} - g}(v) &= (\operatorname{id} - g)(v) = \underbrace{\operatorname{id}(v)}_{=v} - g(v) = v - g(v) \\
&= - \underbrace{(g(v) - v)}_{\substack{\in V_{m-1} \\ (\text{since } g(v) \in v + V_{m-1} \\ (\text{by } (162)))}} \in -V_{m-1} \\
&\subseteq V_{m-1} \qquad (\text{since } V_{m-1} \text{ is a } \mathbf{k}\text{-module})
\end{aligned}$$

). Hence, Proposition 12.14 **(a)** yields that the $\mathbf{k}$-module homomorphism $\operatorname{id} - f$ is an isomorphism. In other words, the $\mathbf{k}$-module homomorphism g is an isomorphism (since $g = \operatorname{id} - f$). This proves Proposition 12.15 **(a)**.

(b) Proposition 12.14 **(b)** yields that each of the maps $\operatorname{id} - f$ and $(\operatorname{id} - f)^{-1}$ respects the filtration. In other words, each of the maps g and g^{-1} respects the filtration (since $g = \operatorname{id} - f$). This proves Proposition 12.15 **(b)**. □

We next move back to symmetric functions. Recall that $\left(\Lambda_{\deg \leq m}\right)_{m\in\mathbb{N}}$ is a $\mathbf{k}$-module filtration of Λ. Whenever we say that a map $\varphi : \Lambda \to \Lambda$ "respects the filtration", we shall be referring to this filtration.

Lemma 12.16. Let $N \in \mathbb{N}$. Let $f \in \Lambda$ be a symmetric function of degree $< N$. Then, there exists a family $(c_\kappa)_{\kappa\in\operatorname{Par};\ |\kappa|<N}$ of elements of $\mathbf{k}$ such that $f = \sum\limits_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<N}} c_\kappa \mathbf{e}_\kappa$.

Proof of Lemma 12.16. This can be proved using the same argument that we used to prove Lemma 12.9, as long as we replace every Schur function $\mathbf{s}_\mu$ by the corresponding $\mathbf{e}_\mu$. □

Recall one of our notations defined long time ago: For any partition λ, we let $\mathbf{e}_\lambda$ be the corresponding elementary symmetric function in Λ. (This is called e_λ in [GriRei20, Definition 2.2.1].)

Lemma 12.17. Let $\varphi : \Lambda \to \Lambda$ be a $\mathbf{k}$-algebra homomorphism. Assume that

$$\varphi\left(\mathbf{e}_i\right) \in \mathbf{e}_i + \Lambda_{\deg \leq i-1} \qquad \text{for each } i \in \{1,2,3,\ldots\}. \tag{163}$$

Then:

(a) We have $\varphi(v) \in v + \Lambda_{\deg \leq m-1}$ for each $m \in \mathbb{N}$ and $v \in \Lambda_{\deg \leq m}$. (Here, $\Lambda_{\deg \leq -1}$ denotes the $\mathbf{k}$-submodule 0 of Λ.)

(b) The map $\varphi : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra isomorphism.

(c) Each of the maps φ and φ^{-1} respects the filtration.

Proof of Lemma 12.17. We shall use the notation $\ell(\lambda)$ defined in Definition 7.7 **(a)**.

Let us first prove a few auxiliary claims:

Claim 1: Let $i, j \in \{-1, 0, 1, \ldots\}$. Then, $\Lambda_{\deg \leq i} \Lambda_{\deg \leq j} \subseteq \Lambda_{\deg \leq i+j}$.

[*Proof of Claim 1:* If one of i and j is -1, then Claim 1 holds for obvious reasons (since $\Lambda_{\deg \leq -1} = 0$ and thus $\Lambda_{\deg \leq i} \Lambda_{\deg \leq j} = 0$ in this case). Hence, for the rest of this proof, we WLOG assume that none of i and j is -1. Hence, i and j belong to $\mathbb{N}$ (since $i, j \in \{-1, 0, 1, \ldots\}$). Thus, (148) yields $\Lambda_{\deg \leq i} \Lambda_{\deg \leq j} \subseteq \Lambda_{\deg \leq i+j}$. This proves Claim 1.]

Claim 2: Let $\alpha, \beta \in \mathbb{N}$. Let $a \in \Lambda_{\deg \leq \alpha}$ and $b \in \Lambda_{\deg \leq \beta}$. Let $u \in a + \Lambda_{\deg \leq \alpha - 1}$ and $v \in b + \Lambda_{\deg \leq \beta - 1}$. Then, $uv \in ab + \Lambda_{\deg \leq \alpha + \beta - 1}$.

[*Proof of Claim 2:* For every $m \in \mathbb{N}$, we have $\Lambda_{\deg \leq m-1} \subseteq \Lambda_{\deg \leq m}$ (indeed, this is clear from the definitions of $\Lambda_{\deg \leq m-1}$ and $\Lambda_{\deg \leq m}$). Thus, $\Lambda_{\deg \leq \alpha - 1} \subseteq \Lambda_{\deg \leq \alpha}$ and $\Lambda_{\deg \leq \beta - 1} \subseteq \Lambda_{\deg \leq \beta}$.

We have $u \in a + \Lambda_{\deg \leq \alpha - 1}$. In other words, $u = a + x$ for some $x \in \Lambda_{\deg \leq \alpha - 1}$. Consider this x.

We have $v \in b + \Lambda_{\deg \leq \beta - 1}$. In other words, $v = b + y$ for some $y \in \Lambda_{\deg \leq \beta - 1}$. Consider this y.

We have $v \in \underbrace{b}_{\in \Lambda_{\deg \leq \beta}} + \underbrace{\Lambda_{\deg \leq \beta - 1}}_{\subseteq \Lambda_{\deg \leq \beta}} \subseteq \Lambda_{\deg \leq \beta} + \Lambda_{\deg \leq \beta} \subseteq \Lambda_{\deg \leq \beta}$ (since $\Lambda_{\deg \leq \beta}$ is a $\mathbf{k}$-module).

Now,

$$\begin{aligned}
\underbrace{x}_{\in \Lambda_{\deg \leq \alpha - 1}} \underbrace{v}_{\in \Lambda_{\deg \leq \beta}} &\in \Lambda_{\deg \leq \alpha - 1} \Lambda_{\deg \leq \beta} \subseteq \Lambda_{\deg \leq (\alpha - 1) + \beta} \\
&\qquad \left(\text{by Claim 1, applied to } i = \alpha - 1 \text{ and } j = \beta\right) \\
&= \Lambda_{\deg \leq \alpha + \beta - 1} \qquad \left(\text{since } (\alpha - 1) + \beta = \alpha + \beta - 1\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\underbrace{a}_{\in \Lambda_{\deg \leq \alpha}} \underbrace{y}_{\in \Lambda_{\deg \leq \beta - 1}} &\in \Lambda_{\deg \leq \alpha} \Lambda_{\deg \leq \beta - 1} \subseteq \Lambda_{\deg \leq \alpha + (\beta - 1)} \\
&\qquad \left(\text{by Claim 1, applied to } i = \alpha \text{ and } j = \beta - 1\right) \\
&= \Lambda_{\deg \leq \alpha + \beta - 1} \qquad \left(\text{since } \alpha + (\beta - 1) = \alpha + \beta - 1\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\underbrace{u}_{= a + x} v &= (a + x) v = a \underbrace{v}_{= b + y} + xv = a (b + y) + xv \\
&= ab + \underbrace{ay}_{\in \Lambda_{\deg \leq \alpha + \beta - 1}} + \underbrace{xv}_{\in \Lambda_{\deg \leq \alpha + \beta - 1}} \in ab + \underbrace{\Lambda_{\deg \leq \alpha + \beta - 1} + \Lambda_{\deg \leq \alpha + \beta - 1}}_{\substack{\subseteq \Lambda_{\deg \leq \alpha + \beta - 1} \\ (\text{since } \Lambda_{\deg \leq \alpha + \beta - 1} \text{ is a } \mathbf{k}\text{-module})}} \\
&\subseteq ab + \Lambda_{\deg \leq \alpha + \beta - 1}.
\end{aligned}$$

This proves Claim 2.]

Claim 3: We have $\varphi(\mathbf{e}_\lambda) \in \mathbf{e}_\lambda + \Lambda_{\deg \leq |\lambda| - 1}$ for each partition λ.

[*Proof of Claim 3:* We shall prove Claim 3 by induction on $\ell(\lambda)$.

Induction base: Claim 3 is clearly true when $\ell(\lambda) = 0$ [33]. This completes the induction base.

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[33] *Proof.* Let λ be a partition such that $\ell(\lambda) = 0$. We must show that $\varphi(\mathbf{e}_\lambda) \in \mathbf{e}_\lambda + \Lambda_{\deg \leq |\lambda| - 1}$.

Induction step: Let r be a positive integer. Assume (as the induction hypothesis) that Claim 3 is true whenever $\ell(\lambda) = r-1$. We must prove that Claim 3 is true whenever $\ell(\lambda) = r$.

So let λ be a partition such that $\ell(\lambda) = r$. We must prove that $\varphi(\mathbf{e}_\lambda) \in \mathbf{e}_\lambda + \Lambda_{\deg \leq |\lambda|-1}$.

We have $\ell(\lambda) = r$. Thus, the entries $\lambda_1, \lambda_2, \ldots, \lambda_r$ of λ are positive, while $\lambda_{r+1} = \lambda_{r+2} = \lambda_{r+3} = \cdots = 0$. Hence, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.

We have $1 \in \{1, 2, \ldots, r\}$ (since r is positive). Hence, λ_1 is positive (since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive).

Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then, $\overline{\lambda} = (\lambda_2, \lambda_3, \lambda_4, \ldots) = (\lambda_2, \lambda_3, \ldots, \lambda_r)$ (since $\lambda_{r+1} = \lambda_{r+2} = \lambda_{r+3} = \cdots = 0$), so that $\ell\left(\overline{\lambda}\right) = r-1$ (since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive). Hence, our induction hypothesis shows that Claim 3 holds for $\overline{\lambda}$ instead of λ. In other words, we have $\varphi\left(\mathbf{e}_{\overline{\lambda}}\right) \in \mathbf{e}_{\overline{\lambda}} + \Lambda_{\deg \leq |\overline{\lambda}|-1}$.

But from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_r)$, we see easily that $|\lambda| = \lambda_1 + \left|\overline{\lambda}\right|$. Furthermore, $\lambda_1 \in \{1, 2, 3, \ldots\}$ (since λ_1 is positive). Hence, (163) (applied to $i = \lambda_1$) yields $\varphi\left(\mathbf{e}_{\lambda_1}\right) \in \mathbf{e}_{\lambda_1} + \Lambda_{\deg \leq \lambda_1 - 1}$.

The symmetric function $\mathbf{e}_{\lambda_1}$ is homogeneous of degree λ_1. Thus, $\mathbf{e}_{\lambda_1} \in \Lambda_{\deg \leq \lambda_1}$.

The symmetric function $\mathbf{e}_{\overline{\lambda}}$ is homogeneous of degree $\left|\overline{\lambda}\right|$. Thus, $\mathbf{e}_{\overline{\lambda}} \in \Lambda_{\deg \leq |\overline{\lambda}|}$.

We have now shown that $\mathbf{e}_{\lambda_1} \in \Lambda_{\deg \leq \lambda_1}$ and $\mathbf{e}_{\overline{\lambda}} \in \Lambda_{\deg \leq |\overline{\lambda}|}$ and $\varphi\left(\mathbf{e}_{\lambda_1}\right) \in \mathbf{e}_{\lambda_1} + \Lambda_{\deg \leq \lambda_1 - 1}$ and $\varphi\left(\mathbf{e}_{\overline{\lambda}}\right) \in \mathbf{e}_{\overline{\lambda}} + \Lambda_{\deg \leq |\overline{\lambda}|-1}$. Thus, Claim 2 (applied to $\alpha = \lambda_1$, $\beta = \left|\overline{\lambda}\right|$, $a = \mathbf{e}_{\lambda_1}$, $b = \mathbf{e}_{\overline{\lambda}}$, $u = \varphi\left(\mathbf{e}_{\lambda_1}\right)$ and $v = \varphi\left(\mathbf{e}_{\overline{\lambda}}\right)$) yields that

$$\begin{aligned} \varphi\left(\mathbf{e}_{\lambda_1}\right) \varphi\left(\mathbf{e}_{\overline{\lambda}}\right) &\in \mathbf{e}_{\lambda_1} \mathbf{e}_{\overline{\lambda}} + \Lambda_{\deg \leq \lambda_1 + |\overline{\lambda}| - 1} \\ &= \mathbf{e}_{\lambda_1} \mathbf{e}_{\overline{\lambda}} + \Lambda_{\deg \leq |\lambda| - 1} \end{aligned} \tag{164}$$

(since $\lambda_1 + \left|\overline{\lambda}\right| = |\lambda|$).

But $\overline{\lambda} = (\lambda_2, \lambda_3, \lambda_4, \ldots)$; thus, the definition of $\mathbf{e}_{\overline{\lambda}}$ yields

$$\mathbf{e}_{\overline{\lambda}} = \mathbf{e}_{\lambda_2} \mathbf{e}_{\lambda_3} \mathbf{e}_{\lambda_4} \cdots . \tag{165}$$

But the definition of $\mathbf{e}_\lambda$ yields

$$\mathbf{e}_\lambda = \mathbf{e}_{\lambda_1} \mathbf{e}_{\lambda_2} \mathbf{e}_{\lambda_3} \cdots = \mathbf{e}_{\lambda_1} \underbrace{\left(\mathbf{e}_{\lambda_2} \mathbf{e}_{\lambda_3} \mathbf{e}_{\lambda_4} \cdots\right)}_{\substack{= \mathbf{e}_{\overline{\lambda}} \\ \text{(by (165))}}} = \mathbf{e}_{\lambda_1} \mathbf{e}_{\overline{\lambda}}. \tag{166}$$

We have $\lambda = \varnothing$ (since $\ell(\lambda) = 0$) and thus $\mathbf{e}_\lambda = \mathbf{e}_\varnothing = 1$. Hence, $\varphi(\mathbf{e}_\lambda) = \varphi(1) = 1$ (since φ is a $\mathbf{k}$-algebra homomorphism). Thus, $\underbrace{\varphi(\mathbf{e}_\lambda)}_{=1} - \underbrace{\mathbf{e}_\lambda}_{=1} = 1 - 1 = 0 \in \Lambda_{\deg \leq |\lambda|-1}$ (since $\Lambda_{\deg \leq |\lambda|-1}$ is a $\mathbf{k}$-module), so that $\varphi(\mathbf{e}_\lambda) \in \mathbf{e}_\lambda + \Lambda_{\deg \leq |\lambda|-1}$. This is precisely what we needed to show; qed.

Applying the map φ to both sides of this equality, we obtain

$$\begin{aligned}\varphi\left(\mathbf{e}_{\lambda}\right) &= \varphi\left(\mathbf{e}_{\lambda_1}\mathbf{e}_{\overline{\lambda}}\right) = \varphi\left(\mathbf{e}_{\lambda_1}\right)\varphi\left(\mathbf{e}_{\overline{\lambda}}\right) && \text{(since φ is a $\mathbf{k}$-algebra homomorphism)}\\ &\in \underbrace{\mathbf{e}_{\lambda_1}\mathbf{e}_{\overline{\lambda}}}_{\substack{=\mathbf{e}_{\lambda}\\ \text{(by (166))}}} + \Lambda_{\deg\leq|\lambda|-1} && \text{(by (164))}\\ &= \mathbf{e}_{\lambda} + \Lambda_{\deg\leq|\lambda|-1}.\end{aligned}$$

Now, forget that we fixed λ. We thus have proven that $\varphi\left(\mathbf{e}_{\lambda}\right) \in \mathbf{e}_{\lambda} + \Lambda_{\deg\leq|\lambda|-1}$ for each partition λ satisfying $\ell\left(\lambda\right) = r$. In other words, Claim 3 is true whenever $\ell\left(\lambda\right) = r$. This completes the induction step. Thus, Claim 3 is proven.]

We also notice that

$$\Lambda_{\deg\leq-1} \subseteq \Lambda_{\deg\leq0} \subseteq \Lambda_{\deg\leq1} \subseteq \Lambda_{\deg\leq2} \subseteq \cdots \tag{167}$$

(by the definition of the $\Lambda_{\deg\leq m}$).

(a) Let $m \in \mathbb{N}$. Let $v \in \Lambda_{\deg\leq m}$. We must prove that $\varphi\left(v\right) \in v + \Lambda_{\deg\leq m-1}$.

We know that v is a symmetric function of degree $\leq m$ (since $v \in \Lambda_{\deg\leq m}$). Thus, v is a symmetric function of degree $< m+1$. Hence, Lemma 12.16 (applied to $N = m+1$ and $f = v$) yields that there exists a family $\left(c_{\kappa}\right)_{\kappa\in\operatorname{Par};\ |\kappa|<m+1}$ of elements of $\mathbf{k}$ such that

$$v = \sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<m+1}} c_{\kappa}\mathbf{e}_{\kappa}. \tag{168}$$

Consider this $\left(c_{\kappa}\right)_{\kappa\in\operatorname{Par};\ |\kappa|<m+1}$.

For every $\kappa \in \operatorname{Par}$ satisfying $|\kappa| < m+1$, we have

$$\varphi\left(\mathbf{e}_{\kappa}\right) \in \mathbf{e}_{\kappa} + \Lambda_{\deg\leq m-1}. \tag{169}$$

[*Proof of (169):* Let $\kappa \in \operatorname{Par}$ be such that $|\kappa| < m+1$. From $|\kappa| < m+1$, we obtain $|\kappa| - 1 < m$ and thus $|\kappa| - 1 \leq m-1$ (since $|\kappa| - 1$ and m are integers). Hence, $\Lambda_{\deg\leq|\kappa|-1} \subseteq \Lambda_{\deg\leq m-1}$ (by (167)).

But Claim 3 (applied to $\lambda = \kappa$) yields $\varphi\left(\mathbf{e}_{\kappa}\right) \in \mathbf{e}_{\kappa} + \underbrace{\Lambda_{\deg\leq|\kappa|-1}}_{\subseteq\Lambda_{\deg\leq m-1}} \subseteq \mathbf{e}_{\kappa} + \Lambda_{\deg\leq m-1}$. This proves (169).]

Now, applying the map φ to both sides of the equality (168), we obtain

$$\begin{aligned}
\varphi(v) &= \varphi\left(\sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<m+1}} c_\kappa \mathbf{e}_\kappa\right) = \sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<m+1}} c_\kappa \underbrace{\varphi(\mathbf{e}_\kappa)}_{\substack{\in \mathbf{e}_\kappa + \Lambda_{\deg\leq m-1}\\ \text{(by (169))}}} \qquad \text{(since the map } \varphi \text{ is } \mathbf{k}\text{-linear)}\\
&\in \sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<m+1}} \underbrace{c_\kappa\left(\mathbf{e}_\kappa + \Lambda_{\deg\leq m-1}\right)}_{= c_\kappa \mathbf{e}_\kappa + c_\kappa \Lambda_{\deg\leq m-1}} = \sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<m+1}} \left(c_\kappa \mathbf{e}_\kappa + c_\kappa \Lambda_{\deg\leq m-1}\right)\\
&= \underbrace{\sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<m+1}} c_\kappa \mathbf{e}_\kappa}_{\substack{=v\\ \text{(by (168))}}} + \underbrace{\sum_{\substack{\kappa\in\operatorname{Par};\\ |\kappa|<m+1}} c_\kappa \Lambda_{\deg\leq m-1}}_{\substack{\subseteq \Lambda_{\deg\leq m-1}\\ \text{(since } \Lambda_{\deg\leq m-1} \text{ is a } \mathbf{k}\text{-module)}}} \subseteq v + \Lambda_{\deg\leq m-1}.
\end{aligned}$$

This proves Lemma 12.17 **(a)**.

(b) The map $\varphi : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra homomorphism, thus a $\mathbf{k}$-module homomorphism. Lemma 12.17 **(a)** shows that $\varphi(v) \in v + \Lambda_{\deg\leq m-1}$ for each $m \in \mathbb{N}$ and $v \in \Lambda_{\deg\leq m}$, where $\Lambda_{\deg\leq -1}$ denotes the $\mathbf{k}$-submodule 0 of Λ. Hence, Proposition 12.15 **(a)** (applied to $V = \Lambda$, $V_m = \Lambda_{\deg\leq m}$ and $g = \varphi$) yields that the $\mathbf{k}$-module homomorphism φ is an isomorphism. Hence, this homomorphism φ is bijective and thus a $\mathbf{k}$-algebra isomorphism (since it is a $\mathbf{k}$-algebra homomorphism). This proves Lemma 12.17 **(b)**.

(c) The map $\varphi : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra homomorphism, thus a $\mathbf{k}$-module homomorphism. Lemma 12.17 **(a)** shows that $\varphi(v) \in v + \Lambda_{\deg\leq m-1}$ for each $m \in \mathbb{N}$ and $v \in \Lambda_{\deg\leq m}$, where $\Lambda_{\deg\leq -1}$ denotes the $\mathbf{k}$-submodule 0 of Λ. Hence, Proposition 12.15 **(b)** (applied to $V = \Lambda$, $V_m = \Lambda_{\deg\leq m}$ and $g = \varphi$) yields that each of the maps φ and φ^{-1} respects the filtration. This proves Lemma 12.17 **(c)**. ☐

12.4. Linear independence

Proof of Theorem 12.3. Corollary 12.11 shows that the family $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda\in P_{k,n}}$ spans the $\mathbf{k}$-module Λ/K. We need to prove that it is a basis of Λ/K.

Let us first recall that $\Lambda/\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots\rangle \cong \mathcal{S}$. More precisely, there is a canonical surjective $\mathbf{k}$-algebra homomorphism $\pi : \Lambda \to \mathcal{S}$ which is given by substituting 0 for each of the variables $x_{k+1}, x_{k+2}, x_{k+3}, \ldots$; the kernel of this homomorphism is precisely the ideal $\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots\rangle$ of Λ. This homomorphism sends each $\mathbf{h}_m \in \Lambda$ to the polynomial $h_m \in \mathcal{S}$ defined in (1).

It is well-known that the commutative $\mathbf{k}$-algebra Λ is freely generated by its elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$. Hence, we can define an $\mathbf{k}$-algebra homomorphism $\varphi : \Lambda \to \Lambda$ by letting

$$\varphi(\mathbf{e}_i) = \mathbf{e}_i \qquad \text{for each } i \in \{1, 2, \ldots, k\}; \tag{170}$$

$$\varphi(\mathbf{e}_i) = \mathbf{e}_i - \mathbf{b}_{i-k} \qquad \text{for each } i \in \{k+1, k+2, k+3, \ldots\}. \tag{171}$$

Consider this φ. Then, we have $\varphi\left(\mathbf{e}_i\right) \in \mathbf{e}_i + \Lambda_{\deg \leq i-1}$ for each $i \in \{1,2,3,\ldots\}$ [34]. Hence, Lemma 12.17 **(a)** shows that we have

$$\varphi\left(v\right) \in v + \Lambda_{\deg \leq m-1} \qquad \text{for each } m \in \mathbb{N} \text{ and } v \in \Lambda_{\deg \leq m}. \tag{172}$$

(Here, $\Lambda_{\deg \leq -1}$ denotes the $\mathbf{k}$-submodule 0 of Λ.) Furthermore, Lemma 12.17 **(b)** shows that the map $\varphi : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra isomorphism. In other words, φ is an automorphism of the $\mathbf{k}$-algebra Λ. Finally, Lemma 12.17 **(c)** shows that each of the maps φ and φ^{-1} respects the filtration.

The map φ is a $\mathbf{k}$-algebra automorphism of Λ and sends the elements

$$\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \qquad \text{to} \qquad \mathbf{e}_{k+1} - \mathbf{b}_1, \mathbf{e}_{k+2} - \mathbf{b}_2, \mathbf{e}_{k+3} - \mathbf{b}_3, \ldots,$$

respectively (according to (171)). Hence, it sends the ideal $\left\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \right\rangle$ of Λ to the ideal $\left\langle \mathbf{e}_{k+1} - \mathbf{b}_1, \mathbf{e}_{k+2} - \mathbf{b}_2, \mathbf{e}_{k+3} - \mathbf{b}_3, \ldots \right\rangle$ of Λ. In other words,

$$\varphi\left(\left\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \right\rangle\right) = \left\langle \mathbf{e}_{k+1} - \mathbf{b}_1, \mathbf{e}_{k+2} - \mathbf{b}_2, \mathbf{e}_{k+3} - \mathbf{b}_3, \ldots \right\rangle. \tag{173}$$

For each $i \in \{1,2,\ldots,k\}$, define $\mathbf{c}_i \in \Lambda$ by

$$\mathbf{c}_i = \varphi^{-1}\left(\varphi\left(\mathbf{h}_{n-k+i}\right) - \mathbf{h}_{n-k+i} + \mathbf{a}_i\right). \tag{174}$$

This is well-defined, since φ is an isomorphism. For each $i \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned} \varphi\left(\mathbf{h}_{n-k+i} - \mathbf{c}_i\right) &= \varphi\left(\mathbf{h}_{n-k+i}\right) - \underbrace{\varphi\left(\mathbf{c}_i\right)}_{\substack{=\varphi\left(\mathbf{h}_{n-k+i}\right) - \mathbf{h}_{n-k+i} + \mathbf{a}_i \\ \text{(by (174))}}} \\ &\qquad \left(\text{since } \varphi \text{ is a } \mathbf{k}\text{-algebra homomorphism}\right) \\ &= \varphi\left(\mathbf{h}_{n-k+i}\right) - \left(\varphi\left(\mathbf{h}_{n-k+i}\right) - \mathbf{h}_{n-k+i} + \mathbf{a}_i\right) = \mathbf{h}_{n-k+i} - \mathbf{a}_i. \end{aligned}$$

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[34] *Proof.* Let $i \in \{1,2,3,\ldots\}$. We must prove that $\varphi\left(\mathbf{e}_i\right) \in \mathbf{e}_i + \Lambda_{\deg \leq i-1}$.

If $i \in \{1,2,\ldots,k\}$, then this is obvious (since the definition of φ yields $\varphi\left(\mathbf{e}_i\right) = \mathbf{e}_i = \mathbf{e}_i + \underbrace{0}_{\in \Lambda_{\deg \leq i-1}} \in \mathbf{e}_i + \Lambda_{\deg \leq i-1}$ in this case). Hence, for the rest of this proof, we WLOG assume that we don't have $i \in \{1,2,\ldots,k\}$. Hence,

$$i \in \{1,2,3,\ldots\} \setminus \{1,2,\ldots,k\} = \{k+1, k+2, k+3, \ldots\}.$$

Thus, the definition of φ yields $\varphi\left(\mathbf{e}_i\right) = \mathbf{e}_i - \mathbf{b}_{i-k}$. But (141) (applied to $i-k$ instead of i) yields that

$$\begin{aligned} \mathbf{b}_{i-k} &= \left(\text{some symmetric function of degree } < \underbrace{k + (i-k)}_{=i}\right) \\ &= \left(\text{some symmetric function of degree } < i\right) \\ &= \left(\text{some symmetric function of degree } \leq i-1\right) \in \Lambda_{\deg \leq i-1}. \end{aligned}$$

Thus, $\varphi\left(\mathbf{e}_i\right) = \mathbf{e}_i - \underbrace{\mathbf{b}_{i-k}}_{\in \Lambda_{\deg \leq i-1}} \in \mathbf{e}_i - \Lambda_{\deg \leq i-1} = \mathbf{e}_i + \Lambda_{\deg \leq i-1}$ (since $\Lambda_{\deg \leq i-1}$ is a $\mathbf{k}$-module).

Qed.

In other words, the map φ sends the elements $\mathbf{h}_{n-k+1}-\mathbf{c}_1, \mathbf{h}_{n-k+2}-\mathbf{c}_2, \ldots, \mathbf{h}_n-\mathbf{c}_k$ to the elements $\mathbf{h}_{n-k+1}-\mathbf{a}_1, \mathbf{h}_{n-k+2}-\mathbf{a}_2, \ldots, \mathbf{h}_n-\mathbf{a}_k$, respectively. Thus, it sends the ideal $\langle \mathbf{h}_{n-k+1}-\mathbf{c}_1, \mathbf{h}_{n-k+2}-\mathbf{c}_2, \ldots, \mathbf{h}_n-\mathbf{c}_k \rangle$ of Λ to the ideal $\langle \mathbf{h}_{n-k+1}-\mathbf{a}_1, \mathbf{h}_{n-k+2}-\mathbf{a}_2, \ldots, \mathbf{h}_n-\mathbf{a}_k \rangle$ of Λ (since φ is a $\mathbf{k}$-algebra automorphism). In other words,

$$\begin{aligned} &\varphi\left(\langle \mathbf{h}_{n-k+1}-\mathbf{c}_1, \mathbf{h}_{n-k+2}-\mathbf{c}_2, \ldots, \mathbf{h}_n-\mathbf{c}_k \rangle\right) \\ &= \langle \mathbf{h}_{n-k+1}-\mathbf{a}_1, \mathbf{h}_{n-k+2}-\mathbf{a}_2, \ldots, \mathbf{h}_n-\mathbf{a}_k \rangle . \end{aligned} \tag{175}$$

Recall that φ is a $\mathbf{k}$-algebra homomorphism; thus,

$$\begin{aligned} &\varphi\left(\langle \mathbf{h}_{n-k+1}-\mathbf{c}_1, \mathbf{h}_{n-k+2}-\mathbf{c}_2, \ldots, \mathbf{h}_n-\mathbf{c}_k \rangle + \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle\right) \\ &= \underbrace{\varphi\left(\langle \mathbf{h}_{n-k+1}-\mathbf{c}_1, \mathbf{h}_{n-k+2}-\mathbf{c}_2, \ldots, \mathbf{h}_n-\mathbf{c}_k \rangle\right)}_{\substack{=\langle \mathbf{h}_{n-k+1}-\mathbf{a}_1, \mathbf{h}_{n-k+2}-\mathbf{a}_2, \ldots, \mathbf{h}_n-\mathbf{a}_k \rangle \\ \text{(by (175))}}} + \underbrace{\varphi\left(\langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle\right)}_{\substack{=\langle \mathbf{e}_{k+1}-\mathbf{b}_1, \mathbf{e}_{k+2}-\mathbf{b}_2, \mathbf{e}_{k+3}-\mathbf{b}_3, \ldots \rangle \\ \text{(by (173))}}} \\ &= \langle \mathbf{h}_{n-k+1}-\mathbf{a}_1, \mathbf{h}_{n-k+2}-\mathbf{a}_2, \ldots, \mathbf{h}_n-\mathbf{a}_k \rangle + \langle \mathbf{e}_{k+1}-\mathbf{b}_1, \mathbf{e}_{k+2}-\mathbf{b}_2, \mathbf{e}_{k+3}-\mathbf{b}_3, \ldots \rangle \\ &= K \end{aligned} \tag{176}$$

(by the definition of K).

For each $i \in \{1, 2, \ldots, k\}$, let us consider the projection $\overline{\mathbf{c}_i}$ of $\mathbf{c}_i \in \Lambda$ onto $\mathcal{S}$. Let $I_{\mathbf{c}}$ denote the ideal of $\mathcal{S}$ generated by the k differences

$$h_{n-k+1}-\overline{\mathbf{c}_1}, h_{n-k+2}-\overline{\mathbf{c}_2}, \ldots, h_n-\overline{\mathbf{c}_k}.$$

Moreover, for each $i \in \{1, 2, \ldots, k\}$, the element $\mathbf{c}_i$ is a symmetric function of degree $< n-k+i$ [35]. Hence, for each $i \in \{1, 2, \ldots, k\}$, the projection $\overline{\mathbf{c}_i}$

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[35] *Proof.* Let $i \in \{1, 2, \ldots, k\}$. Thus, $\mathbf{h}_{n-k+i}$ is a homogeneous symmetric function of degree $n-k+i$. Hence, $\mathbf{h}_{n-k+i} \in \Lambda_{\deg \leq n-k+i}$. Thus, (172) (applied to $m = n-k+i$ and $v = \mathbf{h}_{n-k+i}$) yields $\varphi(\mathbf{h}_{n-k+i}) \in \mathbf{h}_{n-k+i} + \Lambda_{\deg \leq n-k+i-1}$. In other words, $\varphi(\mathbf{h}_{n-k+i}) - \mathbf{h}_{n-k+i} \in \Lambda_{\deg \leq n-k+i-1}$.

Also, (140) yields

$$\begin{aligned} \mathbf{a}_i &= (\text{some symmetric function of degree } < n-k+i) \\ &= (\text{some symmetric function of degree } \leq n-k+i-1) \in \Lambda_{\deg \leq n-k+i-1}. \end{aligned}$$

Hence,

$$\underbrace{\varphi(\mathbf{h}_{n-k+i}) - \mathbf{h}_{n-k+i}}_{\in \Lambda_{\deg \leq n-k+i-1}} + \underbrace{\mathbf{a}_i}_{\in \Lambda_{\deg \leq n-k+i-1}} \in \Lambda_{\deg \leq n-k+i-1} + \Lambda_{\deg \leq n-k+i-1} \subseteq \Lambda_{\deg \leq n-k+i-1}$$

(since $\Lambda_{\deg \leq n-k+i-1}$ is a $\mathbf{k}$-module). But the map φ^{-1} respects the filtration; in other words, we have $\varphi^{-1}\left(\Lambda_{\deg \leq m}\right) \subseteq \Lambda_{\deg \leq m}$ for each $m \in \mathbb{N}$. Applying this to $m = n-k+i-1$, we obtain $\varphi^{-1}\left(\Lambda_{\deg \leq n-k+i-1}\right) \subseteq \Lambda_{\deg \leq n-k+i-1}$. Now, (174) becomes

$$\mathbf{c}_i = \varphi^{-1}\left(\underbrace{\varphi(\mathbf{h}_{n-k+i}) - \mathbf{h}_{n-k+i} + \mathbf{a}_i}_{\in \Lambda_{\deg \leq n-k+i-1}}\right) \in \varphi^{-1}\left(\Lambda_{\deg \leq n-k+i-1}\right) \subseteq \Lambda_{\deg \leq n-k+i-1}.$$

of $\mathbf{c}_i \in \Lambda$ onto $\mathcal{S}$ is a symmetric polynomial of degree $< n-k+i$ (because projecting a symmetric function from Λ onto $\mathcal{S}$ cannot raise the degree). Thus, Theorem 2.7 (applied to $\overline{\mathbf{c}_i}$ and $I_{\mathbf{c}}$ instead of a_i and I) yields that the $\mathbf{k}$-module $\mathcal{S}/I_{\mathbf{c}}$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$. Hence, this $\mathbf{k}$-module $S/I_{\mathbf{c}}$ is free and has a basis of size $|P_{k,n}|$.

But φ is a $\mathbf{k}$-algebra automorphism of Λ. Thus, we have a $\mathbf{k}$-module isomorphism

$$\begin{aligned}
&\Lambda / \left(\langle \mathbf{h}_{n-k+1} - \mathbf{c}_1, \mathbf{h}_{n-k+2} - \mathbf{c}_2, \ldots, \mathbf{h}_n - \mathbf{c}_k \rangle + \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \right) \\
&\cong \Lambda / \underbrace{\varphi \left(\langle \mathbf{h}_{n-k+1} - \mathbf{c}_1, \mathbf{h}_{n-k+2} - \mathbf{c}_2, \ldots, \mathbf{h}_n - \mathbf{c}_k \rangle + \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \right)}_{\substack{=K \\ \text{(by (176))}}} \\
&= \Lambda / K.
\end{aligned}$$

Hence, we have the following chain of $\mathbf{k}$-module isomorphisms:

$$\begin{aligned}
\Lambda / K &\cong \Lambda / \left(\langle \mathbf{h}_{n-k+1} - \mathbf{c}_1, \mathbf{h}_{n-k+2} - \mathbf{c}_2, \ldots, \mathbf{h}_n - \mathbf{c}_k \rangle + \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \right) \\
&\cong \underbrace{\left(\Lambda / \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \right)}_{\cong \mathcal{S}} / \left\langle \overline{\mathbf{h}_{n-k+1} - \mathbf{c}_1}, \overline{\mathbf{h}_{n-k+2} - \mathbf{c}_2}, \ldots, \overline{\mathbf{h}_n - \mathbf{c}_k} \right\rangle \\
&\qquad \left(\begin{array}{c} \text{where } \overline{\mathbf{h}_{n-k+1} - \mathbf{c}_1}, \overline{\mathbf{h}_{n-k+2} - \mathbf{c}_2}, \ldots, \overline{\mathbf{h}_n - \mathbf{c}_k} \text{ denote} \\ \text{the projections of } \mathbf{h}_{n-k+1} - \mathbf{c}_1, \mathbf{h}_{n-k+2} - \mathbf{c}_2, \ldots, \mathbf{h}_n - \mathbf{c}_k \\ \text{onto } \Lambda / \langle \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots \rangle \end{array} \right) \\
&\cong \mathcal{S} / \underbrace{\langle h_{n-k+1} - \overline{\mathbf{c}_1}, h_{n-k+2} - \overline{\mathbf{c}_2}, \ldots, h_n - \overline{\mathbf{c}_k} \rangle}_{\substack{=I_{\mathbf{c}} \\ \text{(by the definition of } I_{\mathbf{c}}\text{)}}} = \mathcal{S} / I_{\mathbf{c}}.
\end{aligned}$$

Hence, the $\mathbf{k}$-module Λ/K is free and has a basis of size $|P_{k,n}|$ (since the $\mathbf{k}$-module $S/I_{\mathbf{c}}$ is free and has a basis of size $|P_{k,n}|$).

Now, recall that the family $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda \in P_{k,n}}$ spans the $\mathbf{k}$-module Λ/K. Hence, Lemma 5.3 shows that this family must be a basis of Λ/K (since it has the same size as a basis of Λ/K). This proves Theorem 12.3. ☐

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