Project: Annihilating polynomials of Hopf algebra endomorphisms

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This is an overview of some interrelated questions I'm interested in answering or seeing answered. All of them concern connected graded Hopf algebras over a commutative ring $\mathbf{k}$. I will use standard notations for Hopf algebras ([GriRei20, Chapter 1]). In particular:

- We fix a commutative ring $\mathbf{k}$. All tensor products, algebras, Hom spaces, etc. will be over $\mathbf{k}$.
- The comultiplication, the counit and the antipode of a Hopf algebra are denoted by Δ, ϵ and S. Sometimes, for disambiguation, we denote them by Δ_H, ϵ_H and S_H, where H is the relevant Hopf algebra.
- "Graded" (as in "graded $\mathbf{k}$-modules", "graded $\mathbf{k}$-bialgebras, etc.") will always mean "$\mathbb{N}$-graded". We do **not** twist our tensor products by the grading (i.e., we do not do any superalgebra here).
- The n-th graded component of a graded $\mathbf{k}$-module V will be called V_n. If $n < 0$, then this is the zero submodule 0.
- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The overarching thread of the first two questions is to find polynomials that annihilate certain operators defined on any connected graded Hopf algebra. The third question is an attempt to make questions like this brute-forceable (for any given degree).

1. Specific operators

1.1. Annihilating polynomials for $\operatorname{id} - S^2$

In [AguLau14, Proposition 7], Aguiar and Lauve showed that

$$\left(\operatorname{id} - S^2\right)^n (H_n) = 0$$

whenever H is a connected graded Hopf algebra over a field and n is a positive integer. Later, Aguiar [Aguiar17, Lemma 12.50] strengthened this equality to

$$\left((\mathrm{id}+S)\circ\left(\mathrm{id}-S^2\right)^{n-1}\right)(H_n)=0 \qquad \text{for each } n>0.$$

For specific combinatorially interesting Hopf algebras, even stronger results hold; in particular,

$$\left(\mathrm{id}-S^2\right)^{n-1}(H_n)=0 \qquad \text{holds for each } n>1$$

when H is the Malvenuto–Reutenauer Hopf algebra (see [AguLau14, Example 8]).

In [Grinbe21], I generalize this all to connected filtered (rather than graded) Hopf algebras over commutative rings (rather than fields), and even further. Here I am not interested in the filtered case, so let me only state my results for graded Hopf algebras ([Grinbe21, §2.4]):[1]

Corollary 1.1. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S. Then, for any positive integer u, we have

$$\left(\mathrm{id}-S^2\right)^{u-1}(H_u)\subseteq \operatorname{Prim} H \tag{1}$$

and

$$\left((\mathrm{id}+S)\circ\left(\mathrm{id}-S^2\right)^{u-1}\right)(H_u)=0 \tag{2}$$

and

$$\left(\mathrm{id}-S^2\right)^{u}(H_u)=0. \tag{3}$$

Corollary 1.2. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S.

Let p be a positive integer such that all $i\in\{2,3,\ldots,p\}$ satisfy

$$\left(\mathrm{id}-S^2\right)(H_i)=0. \tag{4}$$

Then:

(a) For any integer $u>p$, we have

$$\left(\mathrm{id}-S^2\right)^{u-p}(H_{\leq u})\subseteq \operatorname{Prim} H \tag{5}$$

and

$$\left((\mathrm{id}+S)\circ\left(\mathrm{id}-S^2\right)^{u-p}\right)(H_{\leq u})=0. \tag{6}$$

[1] If V is a graded $\mathbf{k}$-module, and if $n\in\mathbb{N}$, then $V_{\leq n}$ denotes the $\mathbf{k}$-submodule $V_0\oplus V_1\oplus\cdots\oplus V_n$ of V.

(b) For any integer $u \geq p$, we have

$$\left(\operatorname{id} - S^2\right)^{u-p+1} (H_{\leq u}) = 0. \tag{7}$$

Corollary 1.3. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S. Assume that

$$ab = ba \qquad \text{for every } a, b \in H_1. \tag{8}$$

Then:

(a) We have

$$\left(\operatorname{id} - S^2\right) (H_2) = 0.$$

(b) For any integer $u > 2$, we have

$$\left(\operatorname{id} - S^2\right)^{u-2} (H_{\leq u}) \subseteq \operatorname{Prim} H \tag{9}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-2}\right) (H_{\leq u}) = 0. \tag{10}$$

(c) For any integer $u > 1$, we have

$$\left(\operatorname{id} - S^2\right)^{u-1} (H_{\leq u}) = 0. \tag{11}$$

1.2. The random-to-top operator

Now, fix a connected graded Hopf algebra H over the commutative ring $\mathbf{k}$.

For each $n \in \mathbb{N}$, let $p_n : H \to H$ be the projection onto the n-th graded component H_n of H (viewed as an endomorphism of H). Note that $p_0(x) = \epsilon(x) \cdot 1_H$ for each $x \in H$, since H is connected. Also, each element of H_1 is primitive (again since H is connected).

Let $f : H \to H$ denote the map $p_1 \star \operatorname{id}$, where the asterisk "$\star$" means convolution of linear maps (so that we have $f(x) = \sum_{(x)} p_1\left(x_{(1)}\right) x_{(2)}$ for any $x \in H$, using Sweedler notation). The map f can be viewed as a generalized random-to-top or top-to-random shuffling operator: Indeed, the latter two operators are obtained by specializing H to be either the tensor algebra or the shuffle algebra of a $\mathbf{k}$-module.

We claim the following:[2]

[2]In the following, the $\prod$ sign means a product with respect to composition (not convolution). Also, the expression "$f - k$" for an integer k always means the map $f - k \cdot \operatorname{id} \in \operatorname{End} H$.

Theorem 1.4. **(a)** We have $f = 0$ on H_0, and we have $f = \operatorname{id}$ on H_1.

(b) For each $n \geq 2$, we have

$$(f - n) \circ (f - (n-2))^2 \circ \prod_{i=0}^{n-3} (f - i)^{n-1-i} = 0 \qquad \text{on } H_n.$$

For example, Theorem 1.4 **(b)** entails that

$$\begin{aligned} (f-2) \circ f^2 &= 0 && \text{on } H_2, && \text{and} \\ (f-3) \circ (f-1)^2 \circ f^2 &= 0 && \text{on } H_3, && \text{and} \\ (f-4) \circ (f-2)^2 \circ (f-1)^2 \circ f^3 &= 0 && \text{on } H_4. \end{aligned}$$

Theorem 1.4 should be compared with the classical fact (see, e.g., [Grinbe18, Theorem 3] and references therein) that $(f - n) \circ \prod_{i=0}^{n-2} (f - i) = 0$ on the n-th graded component of a tensor algebra. When passing to a general connected graded Hopf algebra, we have only been able to salvage this equality "up to exponents".

To prove Theorem 1.4, I will introduce some notations:

- We set $[p] := \{1, 2, \ldots, p\}$ for any $p \in \mathbb{N}$.
- Given two integers k and ℓ, we define a (k, ℓ)*-primitive product* to be a product $a_1 a_2 \cdots a_p$ of finitely many homogeneous primitive elements of H such that $\deg(a_1) + \deg(a_2) + \cdots + \deg(a_p) = k$ and such that there are precisely ℓ many $i \in [p]$ satisfying $\deg(a_i) = 1$.

  For example, if a, b, c, d are four homogeneous primitive elements of H such that $\deg a = 2$ and $\deg b = 1$ and $\deg c = 3$ and $\deg d = 1$, then $abcd$ is a $(7, 2)$-primitive product.
- Given two integers k and ℓ, we let $P_{k,\ell}$ be the $\mathbf{k}$-linear span of all (k, ℓ)-primitive products in H. This is a $\mathbf{k}$-submodule of H_k.

  Note that
  $$P_{0,0} = \operatorname{span}\{1_H\} = H_0.$$
  Furthermore,
  $$P_{k,\ell} = 0 \qquad \text{whenever } \ell > k.$$
  Moreover, since $k - (k-1) = 1$, we have
  $$P_{k,k-1} = 0 \qquad \text{for any } k \in \mathbb{Z}.$$
  Also, obviously,
  $$P_{k,\ell} = 0 \qquad \text{whenever } \ell < 0.$$

- If U and V are two $\mathbf{k}$-submodules of H, then UV shall mean the $\mathbf{k}$-linear span of all products uv with $u \in U$ and $v \in V$.

We will prove Theorem 1.4 by constructing a filtration of H_n and a sequence of maps $g_0, g_1, \ldots, g_{n-2}, g_n$ (sic!) that map larger to smaller parts in this filtration:

Theorem 1.5. Let $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define the $\mathbf{k}$-linear map

$$g_k := (f - k) \circ \prod_{i=0}^{k-2} (f - i) : H \to H.$$

For each $k \in \mathbb{N}$, we set

$$R_k := \sum_{\ell \in \mathbb{N}} P_{k,\ell} H_{n-k};$$

this is clearly a $\mathbf{k}$-submodule of $H_k H_{n-k} \subseteq H_n$.

For each $k \in \mathbb{N}$, we set

$$Q_k := R_k + R_{k+1} + R_{k+2} + \cdots = \sum_{\ell \geq k} R_\ell;$$

again, this is a $\mathbf{k}$-submodule of H_n.

The following holds:

(a) We have $g_k (Q_k) \subseteq Q_{k+1}$ for each $k \in \mathbb{N}$.

(b) We have $Q_{n-1} = Q_n$.

(c) We have $Q_{n+1} = 0$.

Proof of Theorem 1.5 (sketched). This is mostly an aide-memoire for myself.[3]

We shall use Sweedler notation. First, observe that the map p_1 is an (ϵ, ϵ)-derivation – i.e., that we have

$$p_1 (ab) = \epsilon (a) p_1 (b) + \epsilon (b) p_1 (a) \tag{12}$$

for all $a, b \in H$. (This is straightforward to check.) Using this fact, it is easy to see that

$$f (ab) = \sum_{(b)} p_1 \left(b_{(1)} \right) ab_{(2)} + f (a) b \tag{13}$$

[3] Note to self: In the final version, use the notation $b \rightharpoonup a$ for $\sum_{(b)} p_1 \left(b_{(1)} \right) ab_{(2)}$. This should simplify the argument below, possibly even getting rid of Sweedler notation altogether.

for all $a, b \in H$. (Indeed, if $a, b \in H$, then

$$f(ab) = (p_1 \star \mathrm{id})(ab) = \sum_{(a)(b)} \underbrace{p_1\left(a_{(1)}b_{(1)}\right)}_{\substack{=\epsilon\left(a_{(1)}\right)p_1\left(b_{(1)}\right)+\epsilon\left(b_{(1)}\right)p_1\left(a_{(2)}\right) \\ \text{(by (12))}}} a_{(2)}b_{(2)}$$

$$= \sum_{(a)(b)} \left(\epsilon\left(a_{(1)}\right)p_1\left(b_{(1)}\right) + \epsilon\left(b_{(1)}\right)p_1\left(a_{(1)}\right)\right)a_{(2)}b_{(2)}$$

$$= \underbrace{\sum_{(a)(b)} \epsilon\left(a_{(1)}\right)p_1\left(b_{(1)}\right)a_{(2)}b_{(2)}}_{=\sum_{(b)} p_1\left(b_{(1)}\right)ab_{(2)}} + \underbrace{\sum_{(a)(b)} \epsilon\left(b_{(1)}\right)p_1\left(a_{(1)}\right)a_{(2)}b_{(2)}}_{=\sum_{(a)} p_1\left(a_{(1)}\right)a_{(2)}b}$$

$$= \sum_{(b)} p_1\left(b_{(1)}\right)ab_{(2)} + \underbrace{\sum_{(a)} p_1\left(a_{(1)}\right)a_{(2)}}_{=(p_1\star\mathrm{id})(a)=f(a)}b = \sum_{(b)} p_1\left(b_{(1)}\right)ab_{(2)} + f(a)\,b,$$

so that (13) is proven.)

Using (13) (and induction on p), we can easily prove the following: If $a_1, a_2, \ldots, a_p$ are finitely many primitive elements of H, then

$$f\left(a_1a_2\cdots a_p\right) = \sum_{i=1}^{p} p_1(a_i)\cdot a_1a_2\cdots\widehat{a_i}\cdots a_p \tag{14}$$

(where the hat over the a_i means "omit a_i from this product"). Using this equality and (13) again, we can conclude the following: If $a_1, a_2, \ldots, a_p$ are finitely many primitive elements of H, and if $b \in H$ is arbitrary, then

$$f\left(a_1a_2\cdots a_pb\right) = \sum_{(b)} p_1\left(b_{(1)}\right)a_1a_2\cdots a_pb_{(2)} + \sum_{i=1}^{p} p_1(a_i)\cdot a_1a_2\cdots\widehat{a_i}\cdots a_pb \tag{15}$$

(where the hat over the a_i means "omit a_i from this product"). Hence, in particular, if $a_1, a_2, \ldots, a_p$ are finitely many homogeneous primitive elements of H, and if $b \in H$ is arbitrary, then

$$f\left(a_1a_2\cdots a_pb\right) = \sum_{(b)} p_1\left(b_{(1)}\right)a_1a_2\cdots a_pb_{(2)} + \sum_{\substack{i\in[p];\\ \deg(a_i)=1}} a_i\cdot a_1a_2\cdots\widehat{a_i}\cdots a_pb. \tag{16}$$

Next, we claim the following: For each $k, \ell \in \mathbb{Z}$, we have

$$(f-\ell)\left(P_{k,\ell}H_{n-k}\right) \subseteq P_{k+1,\ell+1}H_{n-k-1} + \left(P_{k,\ell-1} + P_{k,\ell-2}\right)H_{n-k}. \tag{17}$$

[*Proof of (17):* Let $k, \ell \in \mathbb{Z}$. We must prove (17). We know that $P_{k,\ell} H_{n-k}$ is the $\mathbf{k}$-linear span of all products of the form $a_1 a_2 \cdots a_p b$, where $b \in H_{n-k}$ and where $a_1, a_2, \ldots, a_p$ are finitely many homogeneous primitive elements of H such that $\deg(a_1) + \deg(a_2) + \cdots + \deg(a_p) = k$ and such that there are precisely ℓ many $i \in [p]$ satisfying $\deg(a_i) = 1$. Thus, it will suffice to show that

$$(f - \ell)\left(a_1 a_2 \cdots a_p b\right) \in P_{k+1,\ell+1} H_{n-k-1} + \left(P_{k,\ell-1} + P_{k,\ell-2}\right) H_{n-k} \qquad (18)$$

for any such b and such $a_1, a_2, \ldots, a_p$.

So let us fix such b and such $a_1, a_2, \ldots, a_p$, and try to prove (18). From (16), we see that

$$f\left(a_1 a_2 \cdots a_p b\right) = \sum_{(b)} p_1\left(b_{(1)}\right) a_1 a_2 \cdots a_p b_{(2)} + \underbrace{\sum_{\substack{i \in [p];\\ \deg(a_i) = 1}} a_i \cdot a_1 a_2 \cdots \widehat{a_i} \cdots a_p b}_{\text{This sum has } \ell \text{ addends}},$$

so that[4]

$$\begin{aligned}
&(f - \ell)\left(a_1 a_2 \cdots a_p b\right) \\
&= \sum_{(b)} p_1\left(b_{(1)}\right) a_1 a_2 \cdots a_p b_{(2)} + \sum_{\substack{i \in [p];\\ \deg(a_i) = 1}} \underbrace{\left(a_i \cdot a_1 a_2 \cdots \widehat{a_i} \cdots a_p b - a_1 a_2 \cdots a_p b\right)}_{\substack{= \sum_{j=1}^{i-1} a_1 a_2 \cdots a_{j-1} \left[a_i, a_j\right] a_{j+1} a_{j+2} \cdots \widehat{a_i} \cdots a_p b \\ \text{(by a standard manipulation with commutators)}}} \\
&= \sum_{(b)} \underbrace{p_1\left(b_{(1)}\right) a_1 a_2 \cdots a_p b_{(2)}}_{\substack{\in P_{k+1,\ell+1} H_{n-k-1} \\ \text{(since } p_1\left(b_{(1)}\right) \text{ is a homogeneous} \\ \text{primitive element of degree 1)}}} + \sum_{\substack{i \in [p];\\ \deg(a_i) = 1}} \sum_{j=1}^{i-1} \underbrace{a_1 a_2 \cdots a_{j-1} \left[a_i, a_j\right] a_{j+1} a_{j+2} \cdots \widehat{a_i} \cdots a_p b}_{\substack{\in \left(P_{k,\ell-1} + P_{k,\ell-2}\right) H_{n-k} \\ \text{(since } a_1, a_2, \ldots, a_{j-1}, \left[a_i, a_j\right], a_{j+1}, a_{j+2}, \ldots, \widehat{a_i}, \ldots, a_p \\ \text{are homogeneous primitive elements,} \\ \text{and since } \deg\left[a_i, a_j\right] > 1)}} \\
&\in P_{k+1,\ell+1} H_{n-k-1} + \left(P_{k,\ell-1} + P_{k,\ell-2}\right) H_{n-k}.
\end{aligned}$$

This proves (18) and therefore (17).]

Now, it is not hard to check all parts of Theorem 1.5:

(a) Using (17), it is easy to see that $f(R_k) \subseteq R_k + R_{k+1} \subseteq Q_k$ for each $k \in \mathbb{N}$. Hence, it follows easily that

$$f(Q_k) \subseteq Q_k \qquad \text{for each } k \in \mathbb{N}. \qquad (19)$$

Fix $k \in \mathbb{N}$. Then, (19) yields $f(Q_{k+1}) \subseteq Q_{k+1}$. Hence, Q_{k+1} is not just a $\mathbf{k}$-module, but also a $\mathbf{k}[f]$-module (where f acts in the obvious way).

[4] We use the notation $[x, y]$ for the commutator $xy - yx$ of two elements $x, y \in H$.

Next, we claim that each integer $\ell \geq -1$ satisfies

$$\left(\prod_{i=0}^{\ell}(f-i)\right)(P_{k,\ell}H_{n-k}) \subseteq Q_{k+1}. \qquad (20)$$

Indeed, (20) easily follows by strong induction on ℓ (using $P_{k,-1} = 0$ for the case $\ell = -1$, and using (17) for the induction step[5]). Thus, each integer $\ell \in \{-1, 0, \ldots, k-2\}$ satisfies

$$g_k\left(P_{k,\ell}H_{n-k}\right) \subseteq Q_{k+1} \qquad (21)$$

(since $g_k = (f-k) \circ \prod_{i=0}^{k-2}(f-i)$ can be written as a composition $u \circ \prod_{i=0}^{\ell}(f-i)$ for

[5]Here is the *induction step*: Let $\ell \geq 0$. Assume (as the induction hypothesis) that

$$\left(\prod_{i=0}^{\ell'}(f-i)\right)\left(P_{k,\ell'}H_{n-k}\right) \subseteq Q_{k+1}$$

for every $\ell' < \ell$. Hence, in particular,

$$\left(\prod_{i=0}^{\ell-1}(f-i)\right)\left(P_{k,\ell-1}H_{n-k}\right) \subseteq Q_{k+1} \qquad \text{and}$$
$$\left(\prod_{i=0}^{\ell-2}(f-i)\right)\left(P_{k,\ell-2}H_{n-k}\right) \subseteq Q_{k+1} \qquad (\text{if } \ell \geq 1).$$

The latter of these two relations quickly entails

$$\left(\prod_{i=0}^{\ell-1}(f-i)\right)\left(P_{k,\ell-2}H_{n-k}\right) \subseteq Q_{k+1}$$

(in fact, this is obvious if $\ell < 2$, and otherwise follows from $\left(\prod_{i=0}^{\ell-1}(f-i)\right)\left(P_{k,\ell-2}H_{n-k}\right) = (f-(\ell-1))\underbrace{\left(\left(\prod_{i=0}^{\ell-2}(f-i)\right)\left(P_{k,\ell-2}H_{n-k}\right)\right)}_{\subseteq Q_{k+1}} \subseteq (f-(\ell-1))\left(Q_{k+1}\right) \subseteq Q_{k+1}$ (since Q_{k+1} is a $\mathbf{k}[f]$-module)). Now,

$$\begin{aligned}
&\left(\prod_{i=0}^{\ell}(f-i)\right)\left(P_{k,\ell}H_{n-k}\right) \\
&= \left(\prod_{i=0}^{\ell-1}(f-i)\right)\underbrace{\left((f-\ell)\left(P_{k,\ell}H_{n-k}\right)\right)}_{\substack{\subseteq P_{k+1,\ell+1}H_{n-k-1}+\left(P_{k,\ell-1}+P_{k,\ell-2}\right)H_{n-k} \\ \text{(by (17))}}} \\
&\subseteq \left(\prod_{i=0}^{\ell-1}(f-i)\right)\left(\underbrace{P_{k+1,\ell+1}H_{n-k-1}}_{\subseteq R_{k+1}\subseteq Q_{k+1}} + \underbrace{\left(P_{k,\ell-1}+P_{k,\ell-2}\right)H_{n-k}}_{=P_{k,\ell-1}H_{n-k}+P_{k,\ell-2}H_{n-k}}\right) \\
&\subseteq \underbrace{\left(\prod_{i=0}^{\ell-1}(f-i)\right)\left(Q_{k+1}\right)}_{\substack{\subseteq Q_{k+1} \\ \text{(since } Q_{k+1} \text{ is a } \mathbf{k}[f]\text{-module)}}} + \underbrace{\left(\prod_{i=0}^{\ell-1}(f-i)\right)\left(P_{k,\ell-1}H_{n-k}\right)}_{\substack{\subseteq Q_{k+1} \\ \text{(as shown above)}}} + \underbrace{\left(\prod_{i=0}^{\ell-1}(f-i)\right)\left(P_{k,\ell-2}H_{n-k}\right)}_{\substack{\subseteq Q_{k+1} \\ \text{(as shown above)}}} \\
&\subseteq Q_{k+1} + Q_{k+1} + Q_{k+1} \subseteq Q_{k+1}.
\end{aligned}$$

This completes the induction step.

some $u \in \mathbf{k}[f]$, and since Q_{k+1} is a $\mathbf{k}[f]$-module).

Moreover,

$$g_k(P_{k,k}H_{n-k}) \subseteq Q_{k+1} \tag{22}$$

(this is proved rather similarly to (20), but additionally using the fact that $P_{k,k-1} = 0$).

Now,

$$\begin{aligned} R_k &= \sum_{\ell \in \mathbb{N}} P_{k,\ell}H_{n-k} \\ &= \sum_{\ell=0}^{k} P_{k,\ell}H_{n-k} \qquad \text{(since } P_{k,\ell} = 0 \text{ whenever } \ell > k\text{)} \\ &= \sum_{\ell=0}^{k-2} P_{k,\ell}H_{n-k} + \underbrace{P_{k,k-1}}_{=0} H_{n-k} + P_{k,k}H_{n-k} \\ &= \sum_{\ell=0}^{k-2} P_{k,\ell}H_{n-k} + P_{k,k}H_{n-k}. \end{aligned}$$

Applying the linear map g_k to both sides of this equality, we obtain

$$g_k(R_k) = \sum_{\ell=0}^{k-2} g_k(P_{k,\ell}H_{n-k}) + g_k(P_{k,k}H_{n-k}) \subseteq Q_{k+1}$$

(by (21) and (22)). On the other hand, $g_k(Q_{k+1}) \subseteq Q_{k+1}$ (since Q_{k+1} is a $\mathbf{k}[f]$-module, whereas $g_k \in \mathbf{k}[f]$). Combining these two facts, we obtain $g_k(Q_k) \subseteq Q_{k+1}$ (since $Q_k = R_k + Q_{k+1}$). This proves Theorem 1.5 **(a)**.

(b) We have $P_{n-1,\ell}H_{n-(n-1)} \subseteq P_{n,\ell+1}$ for each $\ell \in \mathbb{Z}$, since $H_{n-(n-1)} = H_1$ consists of homogeneous primitive elements of degree 1. Thus, $R_{n-1} \subseteq R_n$ follows from the definitions of R_{n-1} and R_n. This entails the claim of Theorem 1.5 **(b)**.

(c) Each $k > n$ satisfies $R_k = \sum_{\ell \in \mathbb{N}} P_{k,\ell} \underbrace{H_{n-k}}_{\substack{=0 \\ (\text{since } k>n)}} = 0$. Thus, $Q_{n+1} = 0$. This proves Theorem 1.5 **(c)**. □

Proof of Theorem 1.4 (sketched). **(a)** Easy and left to the reader.

(b) Let $n \geq 2$. Define the maps g_k and the $\mathbf{k}$-submodules R_k and Q_k as in Theorem 1.5. Then, it is easy to see (by counting factors) that

$$(f-n) \circ (f-(n-2))^2 \circ \prod_{i=0}^{n-3} (f-i)^{n-1-i} = g_n \circ \prod_{k=0}^{n-2} g_k$$

(note that all maps in this equality commute, since they all belong to the commutative ring $\mathbf{k}[f]$). Hence, it remains to show that

$$g_n \circ \prod_{k=0}^{n-2} g_k = 0 \qquad \text{on } H_n. \tag{23}$$

To do so, we observe that $1_H \in P_{0,0}$, so that $H_n \subseteq P_{0,0}H_n = P_{0,0}H_{n-0} \subseteq R_0 \subseteq Q_0$. Therefore,

$$\begin{aligned}
&\left(g_n \circ \prod_{k=0}^{n-2} g_k\right)(H_n) \\
&\subseteq \left(g_n \circ \prod_{k=0}^{n-2} g_k\right)(Q_0) \\
&= g_n\left(g_{n-2}\left(g_{n-3}\left(g_{n-4}\left(\cdots g_2\left(g_1\left(\underbrace{g_0(Q_0)}_{\substack{\subseteq Q_1 \\ \text{(by Theorem 1.5 \textbf{(a)})}}}\right)\right)\right)\right)\right)\right) \\
&\subseteq g_n\left(g_{n-2}\left(g_{n-3}\left(g_{n-4}\left(\cdots g_2\left(\underbrace{g_1(Q_1)}_{\substack{\subseteq Q_2 \\ \text{(by Theorem 1.5 \textbf{(a)})}}}\right)\right)\right)\right)\right) \\
&\subseteq g_n\left(g_{n-2}\left(g_{n-3}\left(g_{n-4}\left(\cdots \underbrace{g_2(Q_2)}_{\substack{\subseteq Q_3 \\ \text{(by Theorem 1.5 \textbf{(a)})}}}\right)\right)\right)\right) \\
&\subseteq \cdots \\
&\subseteq g_n(Q_{n-1}) = g_n(Q_n) \qquad \text{(by Theorem 1.5 \textbf{(b)})} \\
&\subseteq Q_{n+1} \qquad \text{(by Theorem 1.5 \textbf{(a)})} \\
&= 0 \qquad \text{(by Theorem 1.5 \textbf{(c)})}.
\end{aligned}$$

This proves (23) and thus Theorem 1.4 **(b)**. □

Question 1.6. Can the exponents in Theorem 1.4 **(b)** be improved (i.e., made smaller)?

It is easy to see that they cannot be improved for $n = 2$ (consider the free algebra with three generators x, y_1, z_1 of degrees $2, 1, 1$, respectively, with $\Delta(x) = x \otimes 1 + 1 \otimes x + y_1 \otimes y_2$). Some more complicated computations (using the Hopf algebra H from the proof of Theorem 2.10 below) show that they cannot be improved for $n = 3$ and for $n = 4$ either. I don't know about higher values of n, but I wouldn't be surprised if the exponents are optimal.

It sounds like a good idea to test this on well-known combinatorial Hopf algebras such as FQSym and WQSym (and in fact, the f for $H =$ FQSym was studied in [Pang19, §3.4.1] and [Pang18, Theorem 6.6?]). However, they are too well-behaved to be of much use in this question. Indeed, if $H = \text{FQSym}$ or $H = \text{WQSym}$, then $(f - n) \circ \prod_{i=0}^{n-2} (f - i) = 0$ on H_n. More generally:

Theorem 1.7. Assume that every two elements of H_1 commute. Then, each $n \in \mathbb{N}$ satisfies

$$(f - n) \circ \prod_{i=0}^{n-2} (f - i) = 0 \qquad \text{on } H_n.$$

Note that this entails that f is diagonalizable (assuming that every two elements of H_1 commute) when $\mathbf{k}$ is a field of characteristic 0.

Here is a way of proving Theorem 1.7. We define minor variations of primitive elements and (k, ℓ)-primitive products:

- We consider the Lie subalgebra of H generated by H_1 (via the commutator). Its elements are $\mathbf{k}$-linear combinations of nested commutators of elements of H_1 (including the depth-1 commutators, which are just the elements of H_1 themselves). The elements of this Lie subalgebra will be called *ultraprimitive* elements of H. It is well-known (and easy to check) that all ultraprimitive elements of H are primitive.

  If every two elements of H_1 commute, then the ultraprimitive elements of H are precisely the elements of H_1.

- Given two integers k and ℓ, we define a *(k, ℓ)-ultraprimitive product* to be a product $a_1 a_2 \cdots a_p$ of finitely many homogeneous ultraprimitive elements of H such that $\deg(a_1) + \deg(a_2) + \cdots + \deg(a_p) = k$ and such that there are precisely ℓ many $i \in [p]$ satisfying $\deg(a_i) = 1$.

- Given two integers k and ℓ, we let $P'_{k,\ell}$ be the $\mathbf{k}$-linear span of all (k, ℓ)-ultraprimitive products in H. This is a $\mathbf{k}$-submodule of H_k.

  Note that if every two elements of H_1 commute, then $P'_{k,\ell} = 0$ whenever $k \neq \ell$ (because all nonzero ultraprimitive elements have degree 1 in this case).

Now, the following analogue of Theorem 1.5 holds:

Theorem 1.8. Let $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, we set

$$R'_k := \sum_{\ell \in \mathbb{N}} P'_{k,\ell} H_{n-k};$$

this is clearly a $\mathbf{k}$-submodule of $H_k H_{n-k} \subseteq H_n$.

For each $k \in \mathbb{N}$, we set

$$Q'_k := R'_k + R'_{k+1} + R'_{k+2} + \cdots = \sum_{\ell \geq k} R'_\ell;$$

again, this is a $\mathbf{k}$-submodule of H_n.

The following holds:

(a) If every two elements of H_1 commute, then $(f-k)\left(Q'_k\right) \subseteq Q'_{k+1}$ for each $k \in \mathbb{N}$.

(b) We have $Q'_{n-1} = Q'_n$.

(c) We have $Q'_{n+1} = 0$.

Proof of Theorem 1.8 (sketched). **(a)** Similar to the proof of Theorem 1.5 **(a)**. More precisely: Use an argument analogous to the proof of (17) to prove the equality

$$(f-\ell)\left(P'_{k,\ell}H_{n-k}\right) \subseteq P'_{k+1,\ell+1}H_{n-k-1} + \left(P'_{k,\ell-1} + P'_{k,\ell-2}\right)H_{n-k} \tag{24}$$

for each $k, \ell \in \mathbb{Z}$. Applying this to $\ell = k$, we obtain

$$(f-k)\left(P'_{k,k}H_{n-k}\right) \subseteq P'_{k+1,k+1}H_{n-k-1} + \left(P'_{k,k-1} + P'_{k,k-2}\right)H_{n-k}. \tag{25}$$

However, if every two elements of H_1 commute, then we have $P'_{k,\ell} = 0$ whenever $k \neq \ell$, and thus we can simplify (25) to

$$(f-k)\left(P'_{k,k}H_{n-k}\right) \subseteq P'_{k+1,k+1}H_{n-k-1}.$$

This rewrites further as $(f-k)\left(Q'_k\right) \subseteq Q'_{k+1}$, since we have $Q'_k = P'_{k,k}H_{n-k}$ (again because we have $P'_{k,\ell} = 0$ whenever $k \neq \ell$) and $Q'_{k+1} = P'_{k+1,k+1}H_{n-k-1}$ (similarly). Thus, Theorem 1.8 **(a)** is proved.

(b), **(c)** Similar to the proof of Theorem 1.5. □

Proof of Theorem 1.7 (sketched). This follows easily from Theorem 1.8 (similarly to how we proved Theorem 1.4 using Theorem 1.5). □

We remark the following:

Theorem 1.9. Assume that every two elements of H_1 commute. Then, each $x \in H$ and $k \in \mathbb{N}$ satisfy

$$\left(\prod_{i=0}^{k-1}(f-i)\right)(x) = \sum_{(x)} p_1\left(x_{(k)}\right) p_1\left(x_{(k-1)}\right) \cdots p_1\left(x_{(1)}\right) \cdot x_{(k+1)}$$

(where we use Sweedler notation).

This theorem (which is not hard to prove by induction on k) can be used to prove Theorem 1.7 again.

Question 1.10. What happens if we generalize $f = p_1 \star \mathrm{id}$ to $f_k = p_k \star \mathrm{id}$? Is f_k still annihilated by a reasonably nice polynomial on each H_n (in analogy to Theorem 1.4)?

Here is a partial answer for $n = 4$ and $k = 2$. If $a \in H_4$, then

$$f_2(a) = \sum_{(a)} p_2\left(a_{(1)}\right) a_{(2)} \in H_2 H_2.$$

Hence, $f_2(H_4) \subseteq H_2 H_2$. Furthermore, for any $a, b \in H$, we have

$$\begin{aligned} f_2(ab) &= \sum_{(a),(b)} p_2\left(a_{(1)} b_{(1)}\right) a_{(2)} b_{(2)} \\ &= \sum_{(b)} p_2\left(b_{(1)}\right) a b_{(2)} + \sum_{(a),(b)} p_1\left(a_{(1)}\right) p_1\left(b_{(1)}\right) a_{(2)} b_{(2)} + \sum_{(a)} p_2\left(a_{(1)}\right) a_{(2)} b. \end{aligned}$$

Thus, if $a, b \in H_2$, then

$$\begin{aligned} f_2(ab) &= \underbrace{\sum_{(b)} p_2\left(b_{(1)}\right) a b_{(2)}}_{=ba} + \underbrace{\sum_{(a),(b)} p_1\left(a_{(1)}\right) p_1\left(b_{(1)}\right) a_{(2)} b_{(2)}}_{\in H_1^4} + \underbrace{\sum_{(a)} p_2\left(a_{(1)}\right) a_{(2)} b}_{=a} \\ &\in ba + H_1^4 + ab, \end{aligned} \tag{26}$$

so that

$$(f_2 - 2)(ab) \in \underbrace{ab - ba}_{=[a,b] \in [H_2, H_2]} + H_1^4 \subseteq [H_2, H_2] + H_1^4.$$

Thus, $(f_2 - 2)(H_2 H_2) \subseteq [H_2, H_2] + H_1^4$. Furthermore, if $a, b \in H_2$, then

$$f_2([a, b]) \in H_1^4$$

(by applying (26) to b and a instead of a and b, and subtracting the result from the original (26)). Thus, $f_2([H_2, H_2]) \subseteq H_1^4$. Finally, on H_1^4, the map f_2 acts as the sum of all $(2,2)$-shuffles, so its eigenvalues are (a subset of) $0, 2, 6$ (this is a particular case of Proposition 1.11 below). Thus, $(f_2(f_2 - 2)(f_2 - 6))\left(H_1^4\right) = 0$. Combining this all, we see that

$$(f_2(f_2 - 2)(f_2 - 6)) \circ f_2 \circ (f_2 - 2) \circ f_2 = 0.$$

The last part of this argument (the part after falling down into H_1^4) is easy to generalize, and the result is known:

Proposition 1.11. Let $n \in \mathbb{N}$. In the group ring $\mathbb{Z}[\mathfrak{S}_n]$, let sh_k denote the sum of all permutations $\sigma \in \mathfrak{S}_n$ that satisfy $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(n)$. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ of n (with all i satisfying $\lambda_i > 0$), we define the *index* of λ to be the number of all subsets T of $[q]$ that satisfy $\sum_{i \in T} \lambda_i = k$. (Roughly speaking, the index of λ is thus the number of ways to choose a subsequence of λ that sums to k.) Let $G(n, k)$ denote the set of the indices of all partitions of n. Then,

$$\prod_{i \in G(n,k)} (\operatorname{sh}_k - i) = 0.$$

This is a particular case of [Pang18, Theorem 3.5] (since sh_k acting on a tensor algebra is exactly the f_k of that tensor algebra), but it also follows from [Schock04, (4.2) and Theorem 4.1].

Question 1.12. Can we prove Proposition 1.11 again using our filtrations?

The next question is surprisingly doable (apparently easier than Question 1.10):

Question 1.13. Is there an analogue of Theorem 1.7 for f_k ? For instance, is f_k diagonalizable (in characteristic 0) under the assumption that every two elements of $H_1 + H_2 + \cdots + H_k$ commute?

This has a positive and rather explicit answer:

For each $q \in \mathbb{Z}$, we let $[q]$ denote the set $\{1, 2, \ldots, q\}$. (This set is empty when $q \leq 0$.)

Fix a positive integer k.

We define an (n,k)-*snail* to be a tuple $(a_1, a_2, \ldots, a_q)$ (with arbitrary $q \in \mathbb{N}$) of homogeneous elements of H such that

$$0 < \deg(a_i) \qquad \text{for each } i \in [q]$$

and

$$\deg(a_i) \leq k \qquad \text{for each } i \in [q-1]$$

and

$$\deg(a_1) + \deg(a_2) + \cdots + \deg(a_q) = n.$$

(Note that $\deg(a_q)$ can be any positive integer if $q > 0$.)

The *index* of an (n,k)-snail $(a_1, a_2, \ldots, a_q)$ is defined as the number of all subsets T of $[q]$ satisfying $\sum_{i \in T} \deg(a_i) = k$. (For example, if $k = 1$, then the index of any (n,k)-snail $(a_1, a_2, \ldots, a_q)$ is either $q-1$ or q, since the only subsets $T \subseteq [q]$ satisfying $\sum_{i \in T} \deg(a_i) = 1$ are $\{1\}, \{2\}, \ldots, \{q-1\}$ and possibly $\{q\}$.)

Intuitively, you can think of the index of an (n,k)-snail $(a_1, a_2, \ldots, a_q)$ as the number of all ways to assemble a degree-k product by striking some factors from the product $a_1 a_2 \cdots a_q$.

We say that an integer $i \in \mathbb{Z}$ is (n,k)-*friendly* if there is an (n,k)-snail $(a_1, a_2, \ldots, a_q)$ whose index is i.

Let $F(n,k)$ denote the set of all (n,k)-friendly integers.

Clearly, this set $F(n,k)$ is finite (since the number of all possible degree profiles of (n,k)-snails is finite). Now I claim:

Theorem 1.14. Let k be a positive integer. Assume that every two elements of $H_1 + H_2 + \cdots + H_k$ commute. Let $f_k = p_k \star \operatorname{id}$. Let n be a positive integer. Then,

$$\prod_{i \in F(n,k)} (f_k - i) = 0 \qquad \text{on } H_n.$$

Example 1.15. Let us see what this says for $n = 4$ and $k = 2$. In this case, the (n,k)-snails have the following forms:

- (a_1) with $\deg a_1 = 4$; this has index 0.
- (a_1, a_2) with $\deg a_1 = 1$ and $\deg a_2 = 3$; this has index 0.
- (a_1, a_2) with $\deg a_1 = 2$ and $\deg a_2 = 2$; this has index 2.
- (a_1, a_2, a_3) with $\deg a_1 = 1$ and $\deg a_2 = 1$ and $\deg a_3 = 2$; this has index 2.
- (a_1, a_2, a_3) with $\deg a_1 = 1$ and $\deg a_2 = 2$ and $\deg a_3 = 1$; this has index 2.
- (a_1, a_2, a_3) with $\deg a_1 = 2$ and $\deg a_2 = 1$ and $\deg a_3 = 1$; this has index 2.
- (a_1, a_2, a_3, a_4) with $\deg a_1 = 1$ and $\deg a_2 = 1$ and $\deg a_3 = 1$ and $\deg a_4 = 1$; this has index 6.

Thus, $F(n,k) = \{0, 2, 6\}$ here. Hence, Theorem 1.14 yields

$$f_2 (f_2 - 2)(f_2 - 6) = 0 \qquad \text{on } H_4.$$

We notice that Theorem 1.7 can easily be obtained by applying Theorem 1.14 to $k = 1$ (since $F(n,1) = \{0, 1, \ldots, n-2, n\}$).

Proof of Theorem 1.14 (sketched). Very rough outline so far:

For each $i \in \mathbb{Z}$, we define V_i to be the $\mathbf{k}$-linear span of the products of all (n,k)-snails with index $\geq i$. Thus, $V_0 = H_n$, since each $h \in H_n$ is the product of the (n,k)-snail (h) whose index is ≥ 0. On the other hand, $V_i = 0$ for all sufficiently large i. Furthermore, $V_i = V_{i+1}$ for any $i \in \mathbb{N}$ that is not (n,k)-friendly.

Now, we claim that

$$(f_k - i)(V_i) \subseteq V_{i+1} \qquad \text{for each } i \in \mathbb{N}. \tag{27}$$

[*Proof of (27):* Here is just the main idea.

Let $i \in \mathbb{N}$. The typical generator of V_i is a product $a_1 a_2 \cdots a_q$ of an (n,k)-snail $(a_1, a_2, \ldots, a_q)$ with index $\geq i$. Consider this (n,k)-snail. We have

$$\begin{aligned}
& f_k (a_1 a_2 \cdots a_q) \\
&= (p_k \star \mathrm{id})(a_1 a_2 \cdots a_q) \\
&= \sum_{(a_1),(a_2),\ldots,(a_q)} p_k \left((a_1)_{(1)} (a_2)_{(1)} \cdots (a_q)_{(1)} \right) (a_1)_{(2)} (a_2)_{(2)} \cdots (a_q)_{(2)} \\
&= \sum_{j_1 + j_2 + \cdots + j_q = k} \ \sum_{(a_1),(a_2),\ldots,(a_q)} p_{j_1} \left((a_1)_{(1)} \right) p_{j_2} \left((a_2)_{(1)} \right) \cdots p_{j_q} \left((a_q)_{(1)} \right) \\
&\qquad (a_1)_{(2)} (a_2)_{(2)} \cdots (a_q)_{(2)}
\end{aligned}$$

(since $p_k(u_1u_2\cdots u_q) = \sum_{j_1+j_2+\cdots+j_q=k} p_{j_1}(u_1)\, p_{j_2}(u_2)\cdots p_{j_q}(u_q)$ for any $u_1, u_2, \ldots, u_q$ in any graded algebra).

The addends on the right hand side of this equality are again products of (n,k)-snails[6]. We must prove that

- **(a)** their indices are $\geq i$, and
- **(b)** when their indices are i, the original snail $(a_1, a_2, \ldots, a_q)$ must itself have index i, and furthermore these addends all equal $a_1a_2\cdots a_q$ and there are i many of them.

This is not particularly hard, but annoying and finicky. Each addend on the right hand side is a product of the form

$$p_{j_1}\left((a_1)_{(1)}\right) p_{j_2}\left((a_2)_{(1)}\right)\cdots p_{j_q}\left((a_q)_{(1)}\right)(a_1)_{(2)}(a_2)_{(2)}\cdots(a_q)_{(2)}.$$

After we remove the degree-0 factors from this product, we are left with a product of $\leq 2q$ factors. For each way of assembling a degree-k product from the original $a_1a_2\cdots a_q$ product, there is a way to do so from the new product (just replace each a_m by $p_{j_m}\left((a_m)_{(1)}\right)(a_m)_{(2)}$). Moreover, there is at least one new way of assembling a degree-k product, namely by taking $p_{j_1}\left((a_1)_{(1)}\right) p_{j_2}\left((a_2)_{(1)}\right)\cdots p_{j_q}\left((a_q)_{(1)}\right)$. This way is indeed new, unless each of the q indices j_s equals either 0 or $\deg a_s$ [7]. So the index of our addend is greater or equal to the index of $(a_1, a_2, \ldots, a_q)$, and is strictly greater unless each of the q indices j_s equals either 0 or $\deg a_s$. This proves part **(a)**, and more or less proves part **(b)**. All that remains to be done for part **(b)** is showing that if each j_s equals either 0 or $\deg a_s$, then

$$p_{j_1}\left((a_1)_{(1)}\right) p_{j_2}\left((a_2)_{(1)}\right)\cdots p_{j_q}\left((a_q)_{(1)}\right)(a_1)_{(2)}(a_2)_{(2)}\cdots(a_q)_{(2)} = a_1a_2\cdots a_q.$$

This is easy: All but perhaps the last factor $(a_q)_{(2)}$ in the product belong to $H_1 + H_2 + \cdots + H_k$, and thus commute with each other.] □

[6]Here we are using the fact that $p_{j_q}\left((a_q)_{(1)}\right)$ is homogeneous of degree $j_q \leq j_1 + j_2 + \cdots + j_q = k$.

[7]Why? Well, assume that some s satisfies $j_s \neq 0$ and $j_s \neq \deg a_s$. We must show that the product

$$p_{j_1}\left((a_1)_{(1)}\right) p_{j_2}\left((a_2)_{(1)}\right)\cdots p_{j_q}\left((a_q)_{(1)}\right)$$

equals none of the $\prod_{t\in T} p_{j_t}\left((a_t)_{(1)}\right)(a_t)_{(2)}$ products (as formal product, not just as elements of H). But this is easy: This product contains the non-constant factor $p_{j_s}\left((a_s)_{(1)}\right)$, but does not contain the non-constant factor $(a_s)_{(2)}$; however, any product of the form $\prod_{t\in T} p_{j_t}\left((a_t)_{(1)}\right)(a_t)_{(2)}$ must contain either none or both of these two factors.

Some thoughts on Question 1.10. What follows is handwaving that probably makes no sense. I am just storing it here so I don't forget it.

Again, argue by

$$f_k(abcd) = \sum_{u+v+w+x=k} \sum_{(a),\ (b),\ (c),\ (d)} f_u\left(a_{(1)}\right) f_v\left(b_{(1)}\right) f_w\left(c_{(1)}\right) f_x\left(d_{(1)}\right) a_{(2)} b_{(2)} c_{(2)} d_{(2)}.$$

Thus, factors of degree $> k$ get cut into pieces. Other factors get either cut or moved around. Either at least one factor gets cut, or all factors get moved (i.e., the factors get permuted). Thus, we obtain a filtration of H_n by the number of factors. Better, we obtain a "poset filtration" by the partitions of n (partially ordered by length). For each composition $\alpha = \left(\alpha_1, \alpha_2, \ldots, \alpha_p\right)$ of n, we let

$$W_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_p} \subseteq H_n.$$

For each partition λ of n, we let

$$V_\lambda := \sum_{\substack{\alpha \in \operatorname{Comp}_n; \\ \lambda = \operatorname{sort} \alpha}} W_\alpha$$

where Comp_n is the set of all compositions of n and where $\operatorname{sort} \alpha$ means the result of sorting α in weakly decreasing order. Then, $W_\alpha \subseteq W_\beta$ whenever α is a refinement of β. Moreover, f_k preserves $V_{\geq \lambda} := V_\lambda + \sum_{\substack{\mu \in \operatorname{Par}_n; \\ \ell(\mu) > \ell(\lambda)}} V_\mu$, and in fact acts by permutation of factors on $V_{\geq \lambda} / V_{> \lambda}$. This permutation action is some sort of Schocker sum like the sh_k in Proposition 1.11.

Better yet, we can replace f_k by any $p_{\alpha,\sigma}$ (see below) and the map still preserves the filtration and acts on each factor as a quotient of an $\mathbb{Z}\left[S_n\right]$-element. □

Another question, less fitting into this project but related (possibly the above results can be of some use):

Question 1.16. Can we find $\operatorname{Ker} f$ if $\mathbf{k}$ has characteristic 0 ? This would generalize [Grinbe16, Theorem 7.15].

To be fair, there are two ways to generalize [Grinbe16, Theorem 7.15], and it is not clear which one is better. Indeed, if H is the tensor algebra of a $\mathbf{k}$-module L, then the map $\mathbf{t}'$ from [Grinbe16, §7] can be viewed as either $f = p_1 \star \operatorname{id}$ or $(p_1 \otimes \operatorname{id}) \circ \Delta$; the two operators have the same kernel since the multiplication map $H \otimes H \to H$ is injective on each single $H_i \otimes H_j$. More generally, I believe I can show that $\operatorname{Ker} f = \operatorname{Ker}((p_1 \otimes \operatorname{id}) \circ \Delta)$ whenever H is cocommutative and $\operatorname{char} \mathbf{k} = 0$. However, if H is not cocommutative (e.g., the shuffle algebra) or if $\operatorname{char} \mathbf{k} \neq 0$, then this is generally false.

2. A Hopf algebra of natural Hopf algebra endomorphisms

This section is obsolete! A more up-to-date version can be found in [Grinbe24].

Let us now take a step back and look at both the Aguiar–Lauve $\left(\operatorname{id} - S^2\right)^n (H_n) = 0$ result and Theorem 1.4 from a bird's eye view: Both of them are statements of the form "A universally defined linear map on any connected graded Hopf algebra is 0 on the n-th homogeneous component". In the Aguiar–Lauve case, this map is $\left(\operatorname{id} - S^2\right)^n$, while in Theorem 1.4 it is $(f-n) \circ (f-(n-2))^2 \circ \prod_{i=0}^{n-3} (f-i)^{n-1-i}$ (for $n \geq 2$). Many other such maps can be identified, such as the Adams operators $\operatorname{id}^{\star k}$ (thanks, Amy, for reminding me about them), or composition powers S^k of the antipode, or various compositions or convolutions of projections $p_0, p_1, p_2, \ldots$.

This makes the following question rather natural:

Question 2.1. Is there a mechanical way to prove such statements?

Let me try to concretize this, following the lead of Patras and Reutenauer [PatReu98], which have studied the case of a connected graded cocommutative Hopf algebra.

2.1. The maps $p_{\alpha,\sigma}$ for a graded bialgebra H

Consider a graded bialgebra H over a commutative ring $\mathbf{k}$. The $\mathbf{k}$-module $\operatorname{End}_{\operatorname{gr}} H$ of graded $\mathbf{k}$-module endomorphisms of H is itself a $\mathbf{k}$-algebra in two different ways: It has an "internal multiplication" that corresponds to composition of endomorphisms, and an "external multiplication" that corresponds to convolution of endomorphisms. We consider the $\mathbf{k}$-submodule $\mathbf{E}(H)$ of $\operatorname{End}_{\operatorname{gr}} H$ that consists only of those $f \in \operatorname{End}_{\operatorname{gr}} H$ that annihilate all but finitely many graded components of H (that is, that satisfy $f(H_n) = 0$ for all sufficiently high n). This submodule $\mathbf{E}(H)$ is itself graded, with the n-th graded component being canonically isomorphic to $\operatorname{End}_{\mathbf{k}}(H_n)$. This submodule $\mathbf{E}(H)$ is preserved under both internal and external multiplication.

Unlike Hazewinkel in [Hazewi04], we shall however not consider $\mathbf{E}(H)$ for a specific H, but we shall use it as inspiration to study the functorial endomorphisms of H defined for all connected graded Hopf algebras. Such endomorphisms include

- the projections $p_0, p_1, p_2, \ldots$;
- their convolutions $p_{(i_1, i_2, \ldots, i_k)} := p_{i_1} \star p_{i_2} \star \cdots \star p_{i_k}$ with $i_1, i_2, \ldots, i_k \geq 0$;
- the compositions $p_\alpha \circ p_\beta \circ \cdots \circ p_\kappa$ of such convolutions.

However, we can define a broader class of such endomorphisms. To do so, we need some notations:

Definition 2.2. Let H be any $\mathbf{k}$-module. Then, we let the group $\mathfrak{S}_k$ act on $H^{\otimes k}$ from the left by permuting tensorands, according to the rule

$$\sigma \cdot (h_1 \otimes h_2 \otimes \cdots \otimes h_k) = h_{\sigma^{-1}(1)} \otimes h_{\sigma^{-1}(2)} \otimes \cdots \otimes h_{\sigma^{-1}(k)} \qquad \text{for all } \sigma \in \mathfrak{S}_k \text{ and } h_1, h_2, \ldots, h_k \in H.$$

Definition 2.3. Let H be a $\mathbf{k}$-bialgebra. For any $k \in \mathbb{N}$, we let $m^{[k]} : H^{\otimes k} \to H$ and $\Delta^{[k]} : H \to H^{\otimes k}$ be the iterated multiplication and iterated comultiplication maps (denoted $m^{(k-1)}$ and $\Delta^{(k-1)}$ in [GriRei20]). Note that $m^{[k]}$ sends each pure tensor $a_1 \otimes a_2 \otimes \cdots \otimes a_k$ to the product $a_1 a_2 \cdots a_k$, whereas $\Delta^{[k]}$ sends each element $x \in H$ to $\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(k)}$ (using Sweedler notation).

Definition 2.4. **(a)** A *weak composition* means a finite tuple of nonnegative integers.

(b) If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a weak composition, then its *size* $|\alpha|$ is defined to be the number $\alpha_1 + \alpha_2 + \cdots + \alpha_k \in \mathbb{N}$.

Definition 2.5. Let H be a graded $\mathbf{k}$-module.

For any weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we define the projection map $P_\alpha : H^{\otimes k} \to H^{\otimes k}$ to be the tensor product $p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}$. (Thus, if we regard $H^{\otimes k}$ as an $\mathbb{N}^k$-graded $\mathbf{k}$-module, then P_α is its projection on its $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-degree component.)

Definition 2.6. Let H be a graded $\mathbf{k}$-bialgebra.

For any weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and any permutation $\sigma \in \mathfrak{S}_k$, we define a map $p_{\alpha,\sigma} : H \to H$ as follows:

- We can define it by the formula

$$p_{\alpha,\sigma}(x) = \sum_{(x)} p_{\alpha_1}\left(x_{(\sigma(1))}\right) p_{\alpha_2}\left(x_{(\sigma(2))}\right) \cdots p_{\alpha_k}\left(x_{(\sigma(k))}\right)$$

  for every $x \in H$, where we are using the Sweedler notation $\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(k)}$ for the iterated coproduct $\Delta^{[k]}(x) \in H^{\otimes k}$.

- More formally, we can define it as follows:

$$p_{\alpha,\sigma} := m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]}, \tag{28}$$

  where the σ^{-1} really means the action of $\sigma^{-1} \in \mathfrak{S}_k$ on $H^{\otimes k}$ (as in Definition 2.2).

This map $p_{\alpha,\sigma}$ is a graded $\mathbf{k}$-module endomorphism of H that sends $H_{|\alpha|}$ to $H_{|\alpha|}$ and sends each other H_n to 0. Thus, $p_{\alpha,\sigma}$ lies in the $|\alpha|$-th graded component of $\mathbf{E}(H)$.

If H is commutative or cocommutative, then we can bring $p_{\alpha,\sigma}$ to the form p_β for some weak composition β; indeed, we have

$$p_{\alpha,\sigma} = p_{\sigma \cdot \alpha} \qquad \text{if } H \text{ is commutative,}$$

where $\sigma \cdot (\alpha_1, \alpha_2, \ldots, \alpha_k) := \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \ldots, \alpha_{\sigma^{-1}(k)}\right)$, and we have

$$p_{\alpha,\sigma} = p_\alpha \qquad \text{if } H \text{ is cocommutative.}$$

However, in general, when H is neither commutative nor cocommutative, we cannot "simplify" $p_{\alpha,\sigma}$.

Furthermore, when H is connected, each $p_{\alpha,\sigma}$ for a weak composition α and a permutation σ can be rewritten in the form $p_{\beta,\tau}$ for a composition (not just weak composition) β and a permutation τ. (Indeed, since H is connected, we can remove all $p_0\left(x_{(i)}\right)$ factors from the product in the definition of $p_{\alpha,\sigma}$.) See Proposition 2.16 below for an explicit statement of this claim.

It is easy to see that any convolution of two maps of the form $p_{\alpha,\sigma}$ is again a map of such form:

Proposition 2.7. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be a weak composition, and let $\tau \in \mathfrak{S}_\ell$ be a permutation.

Then,

$$p_{\alpha,\sigma} \star p_{\beta,\tau} = p_{\alpha\beta,\sigma\oplus\tau}, \tag{29}$$

where $\alpha\beta$ is the concatenation of α and β (that is, the weak composition $(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_\ell)$), whereas $\sigma \oplus \tau$ is the image of (σ, τ) under the obvious map $\mathfrak{S}_k \times \mathfrak{S}_\ell \to \mathfrak{S}_{k+\ell}$.

It is more interesting to compute the composition of two maps of the form $p_{\alpha,\sigma}$. It turns out that this is again a **k**-linear combination of maps of such form, and the explicit formula is similar to Solomon's Mackey formula for the descent algebra (or, even more closely related, [Reuten93, Theorem 9.2 and §9.5.1]):

Theorem 2.8. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be a weak composition, and let $\tau \in \mathfrak{S}_\ell$ be a permutation.

Then,

$$p_{\alpha,\sigma} \circ p_{\beta,\tau} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} p_{\left(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell}\right), \tau[\sigma]},$$

where $\tau[\sigma] \in \mathfrak{S}_{k\ell}$ is the permutation that sends each $\ell(i-1)+j$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\tau(j)-1)+\sigma(i)$.

Example 2.9. For example, let us try to compute $p_{(a,b),s_1} \circ p_{(c,d),s_1}$, where $s_1 \in \mathfrak{S}_2$ is the transposition that swaps 1 with 2. We have (using sumfree Sweedler notation)

$$\begin{aligned}
&\left(p_{(a,b),s_1} \circ p_{(c,d),s_1}\right)(x) \\
&= p_{(a,b),s_1} \underbrace{\left(p_{(c,d),s_1}(x)\right)}_{= p_c\left(x_{(2)}\right) p_d\left(x_{(1)}\right)} \\
&= p_{(a,b),s_1}\left(p_c\left(x_{(2)}\right) p_d\left(x_{(1)}\right)\right) \\
&= p_a\left(\left(p_c\left(x_{(2)}\right) p_d\left(x_{(1)}\right)\right)_{(2)}\right) \cdot p_b\left(\left(p_c\left(x_{(2)}\right) p_d\left(x_{(1)}\right)\right)_{(1)}\right) \\
&= p_a\left(\left(p_c\left(x_{(2)}\right)\right)_{(2)} \left(p_d\left(x_{(1)}\right)\right)_{(2)}\right) \cdot p_b\left(\left(p_c\left(x_{(2)}\right)\right)_{(1)} \left(p_d\left(x_{(1)}\right)\right)_{(1)}\right) \\
&= \sum_{\substack{c_1+c_2=c;\\ d_1+d_2=d}} p_a\left(p_{c_2}\left(\left(x_{(2)}\right)_{(2)}\right) p_{d_2}\left(\left(x_{(1)}\right)_{(2)}\right)\right) \\
&\qquad \cdot p_b\left(p_{c_1}\left(\left(x_{(2)}\right)_{(1)}\right) p_{d_1}\left(\left(x_{(1)}\right)_{(1)}\right)\right) \\
&\qquad\qquad \left(\text{since } \Delta \circ p_c = \left(\sum_{c_1+c_2=c} p_{c_1} \otimes p_{c_2}\right) \circ \Delta \text{ and likewise for } p_d\right) \\
&= \sum_{\substack{c_1+c_2=c;\\ d_1+d_2=d;\\ c_2+d_2=a;\\ c_1+d_1=b}} p_{c_2}\left(\left(x_{(2)}\right)_{(2)}\right) p_{d_2}\left(\left(x_{(1)}\right)_{(2)}\right) \cdot p_{c_1}\left(\left(x_{(2)}\right)_{(1)}\right) p_{d_1}\left(\left(x_{(1)}\right)_{(1)}\right) \\
&\qquad\qquad \left(\text{since } p_a\left(p_i(y)\, p_j(z)\right) = \begin{cases} 0, & \text{if } i+j \neq a; \\ p_i(y)\, p_j(z), & \text{if } i+j = a \end{cases}\right) \\
&= \sum_{\substack{c_1+c_2=c;\\ d_1+d_2=d;\\ c_2+d_2=a;\\ c_1+d_1=b}} p_{c_2}\left(x_{(4)}\right) p_{d_2}\left(x_{(2)}\right) \cdot p_{c_1}\left(x_{(3)}\right) p_{d_1}\left(x_{(1)}\right) = \sum_{\substack{c_1+c_2=c;\\ d_1+d_2=d;\\ c_2+d_2=a;\\ c_1+d_1=b}} p_{(c_2,d_2,c_1,d_1),\sigma}(x),
\end{aligned}$$

where $\sigma \in \mathfrak{S}_4$ is the permutation that sends $1,2,3,4$ to $4,2,3,1$, respectively. This is again a sum of $p_{\alpha,\sigma}$'s, with $\ell(\alpha)$ varying between 2 and 4 (keep in mind that some of c_1, c_2, d_1, d_2 can be 0, in which case we need to remove them from our compositions).

Proof of Theorem 2.8 (sketched). The formal definition (28) yields

$$p_{\alpha,\sigma} = m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \qquad \text{and} \tag{30}$$

$$p_{\beta,\tau} = m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]}. \tag{31}$$

We shall next state some rules for "commuting" the operators in these equalities past each other.

Let $\zeta \in \mathfrak{S}_{k\ell}$ be the permutation that sends each $k(j-1)+i$ (with $i \in [k]$ and $j \in [\ell]$) to $\ell(i-1)+j$. This is called the *Zolotarev shuffle* (and appears, e.g., as $\nu^{-1}\mu$ in [Rousse94]).

From [GriRei20, Exercise 1.4.22(c)] (applied to $k-1$ and $\ell-1$ instead of k and ℓ), we obtain

$$m^{[\ell]}_{H^{\otimes k}} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}, \tag{32}$$

where $m^{[\ell]}_{H^{\otimes k}}$ is the map defined just as $m^{[\ell]}$ but for the algebra $H^{\otimes k}$ instead of H. However, it is easy to see (and should be known) that

$$m^{[\ell]}_{H^{\otimes k}} = \left(m^{[\ell]}\right)^{\otimes k} \circ \zeta. \tag{33}$$

Indeed, this can be checked on pure tensors, using the combinatorial observation that

$$\begin{aligned}
&\zeta \left(h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k}\right.\\
&\qquad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k}\\
&\qquad \otimes \ \cdots\\
&\qquad \left.\otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}\right)\\
&= h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1}\\
&\qquad \otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2}\\
&\qquad \otimes \cdots\\
&\qquad \otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}
\end{aligned} \tag{34}$$

for all matrices $\left(h_{j,i}\right)_{i\in[k],\ j\in[\ell]} \in H^{k\times\ell}$ (sorry for the idiotic indexing)[8].

Using (33), we can rewrite (32) as

$$\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}. \tag{35}$$

[8]Let me sketch a *proof of (34)* because it is far too easy to get such things wrong.

Define two maps λ and ρ from $[k] \times [\ell]$ to $[k\ell]$ by setting

$$\lambda(i,j) := k(j-1)+i \in [k\ell] \qquad \text{and} \qquad \rho(i,j) := \ell(i-1)+j \in [k\ell]$$

for all $(i,j) \in [k] \times [\ell]$. These maps λ and ρ are bijections.

Let $\left(h_{j,i}\right)_{i\in[k],\ j\in[\ell]} \in H^{k\times\ell}$ be a matrix. Set $g_{(i,j)} := h_{j,i}$ for all $(i,j) \in [k] \times [\ell]$. Then,

$$\begin{aligned}
g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)} &= h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k}\\
&\quad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k}\\
&\quad \otimes \cdots\\
&\quad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}
\end{aligned}$$

On the other hand, $\mathbb{N}^k$ is the set of all weak compositions of length k. The symmetric group $\mathfrak{S}_k$ acts on the right on this set $\mathbb{N}^k$ by permuting the entries: For any $(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$ and $\pi \in \mathfrak{S}_k$, we have

$$(\gamma_1, \gamma_2, \ldots, \gamma_k) \cdot \pi = \left(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \ldots, \gamma_{\pi(k)}\right).$$

This action has the property that

$$P_\gamma \circ \pi = \pi \circ P_{\gamma \cdot \pi} \tag{36}$$

for any $\pi \in \mathfrak{S}_k$ and $\gamma \in \mathbb{N}^k$. (This is easy to check.) Hence,

$$\pi \circ P_\gamma = P_{\gamma \cdot \pi^{-1}} \circ \pi \tag{37}$$

for any $\pi \in \mathfrak{S}_k$ and $\gamma \in \mathbb{N}^k$.

The map $m^{[\ell]} : H^{\otimes \ell} \to H$ is graded; thus, each $i \in \mathbb{N}$ satisfies

$$p_i \circ m^{[\ell]} = \sum_{\substack{(i_1, i_2, \ldots, i_\ell) \in \mathbb{N}^\ell; \\ i_1 + i_2 + \cdots + i_\ell = i}} m^{[\ell]} \circ \left(p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_\ell}\right).$$

By tensoring together k such equalities, we obtain

$$P_\gamma \circ \left(m^{[\ell]}\right)^{\otimes k} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \cdots + \gamma_{i,\ell} = \gamma_i \text{ for all } i \in [k]}} \left(m^{[\ell]}\right)^{\otimes k} \circ P_{\left(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell}\right)} \tag{38}$$

---

and

$$\begin{aligned} g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)} = h_{1,1} &\otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1} \\ &\otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2} \\ &\otimes \cdots \\ &\otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}. \end{aligned}$$

Hence, we must show that

$$\zeta\left(g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)}\right) = g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)}.$$

Since

$$\zeta\left(g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)}\right) = g_{\lambda^{-1}\left(\zeta^{-1}(1)\right)} \otimes g_{\lambda^{-1}\left(\zeta^{-1}(2)\right)} \otimes \cdots \otimes g_{\lambda^{-1}\left(\zeta^{-1}(k\ell)\right)},$$

this is equivalent to showing that

$$g_{\lambda^{-1}\left(\zeta^{-1}(1)\right)} \otimes g_{\lambda^{-1}\left(\zeta^{-1}(2)\right)} \otimes \cdots \otimes g_{\lambda^{-1}\left(\zeta^{-1}(k\ell)\right)} = g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)}.$$

Thus, we need to check that $\lambda^{-1}\left(\zeta^{-1}(q)\right) = \rho^{-1}(q)$ for each $q \in [k\ell]$. In other words, we need to check that $\lambda^{-1} \circ \zeta^{-1} = \rho^{-1}$. In other words, we need to check that $\zeta \circ \lambda = \rho$. However, this is easy: The definition of ζ yields $\zeta\left(k(j-1)+i\right) = \ell(i-1)+j$ for all i, j; but this rewrites as $\zeta(\lambda(i,j)) = \rho(i,j)$. So (34) is proven.

for any $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$. A dual argument shows that

$$\left(\Delta^{[\ell]}\right)^{\otimes k} \circ P_\gamma = \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\gamma_i \text{ for all } i\in[k]}} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ \left(\Delta^{[\ell]}\right)^{\otimes k} \tag{39}$$

for any $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$. Switching the roles of k and ℓ in this equality, we obtain the following: For any $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell) \in \mathbb{N}^\ell$, we have

$$\begin{aligned}
&\left(\Delta^{[k]}\right)^{\otimes \ell} \circ P_\gamma \\
&= \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[\ell] \text{ and } j\in[k];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,k}=\gamma_i \text{ for all } i\in[\ell]}} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{\ell,k}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
&= \sum_{\substack{\delta_{j,i}\in\mathbb{N} \text{ for all } i\in[\ell] \text{ and } j\in[k];\\ \delta_{1,i}+\delta_{2,i}+\cdots+\delta_{k,i}=\gamma_i \text{ for all } i\in[\ell]}} P_{\left(\delta_{1,1},\delta_{2,1},\ldots,\delta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
&\qquad \left(\text{here, we have renamed the } \gamma_{i,j} \text{ as } \delta_{j,i}\right) \\
&= \sum_{\substack{\delta_{i,j}\in\mathbb{N} \text{ for all } j\in[\ell] \text{ and } i\in[k];\\ \delta_{1,j}+\delta_{2,j}+\cdots+\delta_{k,j}=\gamma_j \text{ for all } j\in[\ell]}} P_{\left(\delta_{1,1},\delta_{2,1},\ldots,\delta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
&\qquad \left(\text{here, we have renamed the indices } i \text{ and } j \text{ as } j \text{ and } i\right) \\
&= \sum_{\substack{\delta_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \delta_{1,j}+\delta_{2,j}+\cdots+\delta_{k,j}=\gamma_j \text{ for all } j\in[\ell]}} P_{\left(\delta_{1,1},\delta_{2,1},\ldots,\delta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell}
\end{aligned} \tag{40}$$

(here, we have just rewritten the "$\delta_{i,j} \in \mathbb{N}$ for all $j \in [\ell]$ and $i \in [k]$" under the summation sign as "$\delta_{i,j} \in \mathbb{N}$ for all $i \in [k]$ and $j \in [\ell]$").

Any two weak compositions γ and δ of length $k\ell$ satisfy

$$P_\gamma \circ P_\delta = \begin{cases} P_\gamma, & \text{if } \gamma = \delta; \\ 0, & \text{if } \gamma \neq \delta. \end{cases} \tag{41}$$

It is easy to see that

$$\pi \circ \left(m^{[\ell]}\right)^{\otimes k} = \left(m^{[\ell]}\right)^{\otimes k} \circ \pi^{\times \ell} \tag{42}$$

for each $\pi \in \mathfrak{S}_k$, where $\pi^{\times \ell}$ denotes the permutation in $\mathfrak{S}_{k\ell}$ that sends each

$\ell(i-1)+j$ (with $i \in [k]$ and $j \in [\ell]$) to $\ell(\pi(i)-1)+j$ [9]. Dually, it is easy to see that

$$\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \pi = \pi^{k\times} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \tag{43}$$

for each $\pi \in \mathfrak{S}_\ell$, where $\pi^{k\times}$ denotes the permutation in $\mathfrak{S}_{k\ell}$ that sends each $k(j-1)+i$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\pi(j)-1)+i$.

[9] Actually, this equality has nothing to do with $m^{[\ell]}$. More generally, any permutation $\pi \in \mathfrak{S}_k$ and any **k**-linear map $f : H^{\otimes \ell} \to H$ satisfy

$$\pi \circ f^{\otimes k} = f^{\otimes k} \circ \pi^{\times \ell}.$$

Applying this to $f = m^{[\ell]}$, we obtain (42).

Now, (30) and (31) yield

$$
\begin{aligned}
& p_{\alpha,\sigma} \circ p_{\beta,\tau} \\
&= m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \underbrace{\Delta^{[k]} \circ m^{[\ell]}}_{\substack{= \left(m^{[\ell]}\right)^{\otimes k} \circ \zeta \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\ \text{(by (35))}}} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]} \\
&= m^{[k]} \circ P_\alpha \circ \underbrace{\sigma^{-1} \circ \left(m^{[\ell]}\right)^{\otimes k}}_{\substack{= \left(m^{[\ell]}\right)^{\otimes k} \circ \left(\sigma^{-1}\right)^{\times \ell} \\ \text{(by (42))}}} \circ \zeta \\
&\qquad \circ \underbrace{\left(\Delta^{[k]}\right)^{\otimes \ell} \circ P_\beta}_{\substack{= \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \cdots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} P_{\left(\delta_{1,1}, \delta_{2,1}, \ldots, \delta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\ \text{(by (40))}}} \circ \tau^{-1} \circ \Delta^{[\ell]} \\
&= \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \cdots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k]} \circ \underbrace{P_\alpha \circ \left(m^{[\ell]}\right)^{\otimes k}}_{\substack{= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \cdots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \left(m^{[\ell]}\right)^{\otimes k} \circ P_{\left(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell}\right)} \\ \text{(by (38))}}} \\
&\qquad \circ \left(\sigma^{-1}\right)^{\times \ell} \circ \zeta \circ P_{\left(\delta_{1,1}, \delta_{2,1}, \ldots, \delta_{k,\ell}\right)} \circ \underbrace{\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \tau^{-1}}_{\substack{= \left(\tau^{-1}\right)^{k \times} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\ \text{(by (43))}}} \circ \Delta^{[\ell]} \\
&= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \cdots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\delta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \delta_{1,j} + \delta_{2,j} + \cdots + \delta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} \\
&\qquad \underbrace{m^{[k]} \circ \left(m^{[\ell]}\right)^{\otimes k}}_{= m^{[k\ell]}} \circ P_{\left(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell}\right)} \circ \left(\sigma^{-1}\right)^{\times \ell} \\
&\qquad \circ \underbrace{\zeta \circ P_{\left(\delta_{1,1}, \delta_{2,1}, \ldots, \delta_{k,\ell}\right)}}_{\substack{= P_{\left(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{k,\ell}\right)} \circ \zeta \\ \text{(by (36),} \\ \text{since } \left(\delta_{1,1}, \delta_{2,1}, \ldots, \delta_{k,\ell}\right) \cdot \zeta^{-1} \\ = \left(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{k,\ell}\right))}} \circ \left(\tau^{-1}\right)^{k \times} \circ \underbrace{\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \Delta^{[\ell]}}_{= \Delta^{[k\ell]}}
\end{aligned}
$$

$$= \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k]}} \sum_{\substack{\delta_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \delta_{1,j}+\delta_{2,j}+\cdots+\delta_{k,j}=\beta_j \text{ for all } j\in[\ell]}} m^{[k\ell]} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ \underbrace{\left(\sigma^{-1}\right)^{\times\ell} \circ P_{\left(\delta_{1,1},\delta_{1,2},\ldots,\delta_{k,\ell}\right)}}_{\substack{=P_{\left(\delta_{\sigma(1),1},\delta_{\sigma(1),2},\ldots,\delta_{\sigma(k),\ell}\right)}\circ\left(\sigma^{-1}\right)^{\times\ell} \\ \text{(by (36),} \\ \text{since } \left(\delta_{1,1},\delta_{1,2},\ldots,\delta_{k,\ell}\right)\cdot\left(\left(\sigma^{-1}\right)^{\times\ell}\right)^{-1} \\ =\left(\delta_{1,1},\delta_{1,2},\ldots,\delta_{k,\ell}\right)\cdot\sigma^{\times\ell} \\ =\left(\delta_{\sigma(1),1},\delta_{\sigma(1),2},\ldots,\delta_{\sigma(k),\ell}\right))}} \circ \zeta \circ \left(\tau^{-1}\right)^{k\times} \circ \Delta^{[k\ell]}$$

$$= \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k]}} \sum_{\substack{\delta_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \delta_{1,j}+\delta_{2,j}+\cdots+\delta_{k,j}=\beta_j \text{ for all } j\in[\ell]}} m^{[k\ell]} \circ \underbrace{P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ P_{\left(\delta_{\sigma(1),1},\delta_{\sigma(1),2},\ldots,\delta_{\sigma(k),\ell}\right)}}_{\substack{=\begin{cases} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}, & \text{if } \gamma_{i,j}=\delta_{\sigma(i),j} \text{ for all } i,j; \\ 0, & \text{otherwise} \end{cases} \\ \text{(by (41))}}} \circ \underbrace{\left(\sigma^{-1}\right)^{\times\ell} \circ \zeta \circ \left(\tau^{-1}\right)^{k\times}}_{=(\tau[\sigma])^{-1}} \circ \Delta^{[k\ell]}$$

$$= \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k]}} \sum_{\substack{\delta_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \delta_{1,j}+\delta_{2,j}+\cdots+\delta_{k,j}=\beta_j \text{ for all } j\in[\ell]}} m^{[k\ell]} \circ \begin{cases} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}, & \text{if } \gamma_{i,j}=\delta_{\sigma(i),j} \text{ for all } i,j; \\ 0, & \text{otherwise} \end{cases} \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]}$$

$$= \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k];\\ \gamma_{\sigma^{-1}(1),j}+\gamma_{\sigma^{-1}(2),j}+\cdots+\gamma_{\sigma^{-1}(k),j}=\beta_j \text{ for all } j\in[\ell]}} m^{[k\ell]} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]}$$

$$\left(\begin{array}{c}\text{here, we have discarded all the addends that do } \textbf{not} \\ \text{satisfy } \left(\gamma_{i,j} = \delta_{\sigma(i),j} \text{ for all } i,j\right) \text{, because these addends} \\ \text{are 0; the remaining entries have been combined} \\ \text{into a single sum, with } \delta_{i,j} \text{ rewritten as } \gamma_{\sigma^{-1}(i),j}\end{array}\right)$$

$$= \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k];\\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j\in[\ell]}} \underbrace{m^{[k\ell]} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]}}_{\substack{=p_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right),\tau[\sigma]}\\ \text{(by (28))}}}$$

$$\left(\begin{array}{c}\text{here, we have rewritten the ``}\gamma_{\sigma^{-1}(1),j} + \gamma_{\sigma^{-1}(2),j} + \cdots + \gamma_{\sigma^{-1}(k),j}\text{''} \\ \text{under the summation sign as ``}\gamma_{1,j} + \gamma_{2,j} + \cdots + \gamma_{k,j}\text{''}\end{array}\right)$$

$$= \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k];\\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j\in[\ell]}} p_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right),\tau[\sigma]}.$$

This proves Theorem 2.8. □

Note that both Theorem 2.8 and Proposition 2.7 work for arbitrary graded (not necessarily connected) bialgebras.

Using Theorem 2.8 and (29), we can expand any nested convolution-and-composition of $p_{\alpha,\sigma}$'s as a **k**-linear combination of single $p_{\alpha,\sigma}$'s. This allows us to mechnically prove equalities for $p_{\alpha,\sigma}$'s that involve only convolution, composition and **k**-linear combination and that are supposed to be valid for any connected graded Hopf algebra H. The reason why this works is the following "generic linear independence" theorem:

Theorem 2.10. (a) There is a connected graded Hopf algebra H such that the family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k\in\mathbb{N};\\ \alpha \text{ is a composition of length } k;\\ \sigma\in\mathfrak{S}_k}}$$

(of endomorphisms of H) is **k**-linearly independent.

(b) Let $n \in \mathbb{N}$. Then, there is a connected graded Hopf algebra H such that the family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k\in\mathbb{N};\\ \alpha \text{ is a composition of length } k \text{ and size } <n;\\ \sigma\in\mathfrak{S}_k}}$$

(of endomorphisms of H) is **k**-linearly independent, and such that each H_n is a free **k**-module with a finite basis.

Proof of Theorem 2.10 (sketched). **(b)** Let H be the free $\mathbf{k}$-algebra with generators

$$x_{i,j} \qquad \text{with } i,j \in \mathbb{Z} \text{ satisfying } 1 \leq i < j \leq n.$$

We also set

$$x_{k,k} := 1_H \qquad \text{for each } k \in \{1,2,\ldots,n\}.$$

We make the $\mathbf{k}$-algebra H graded by declaring each $x_{i,j}$ to be homogeneous of degree $j-i$. We define a comultiplication $\Delta : H \to H \otimes H$ on H to be the $\mathbf{k}$-algebra homomorphism that satisfies

$$\Delta\left(x_{i,j}\right) = \sum_{k=i}^{j} x_{i,k} \otimes x_{k,j} \qquad \text{for each } i,j \in \mathbb{Z} \text{ satisfying } 1 \leq i < j \leq n.$$

We define a counit $\epsilon : H \to \mathbf{k}$ in the obvious way to preserve the grading (so that $\epsilon\left(x_{i,j}\right) = 0$ whenever $i < j$). It is easy to see that H is a connected graded $\mathbf{k}$-bialgebra, thus a connected graded Hopf algebra.

(You can think of H either as a noncommutative version of a reduced incidence algebra of the chain poset with n elements, or as a noncommutative variant of the subalgebra of the Schur algebra corresponding to the upper-triangular matrices with equal numbers on the diagonal. But we won't need all this intuition.)

It is easy to see that

$$\Delta^{(k-1)}\left(x_{i,j}\right) = \sum_{i=u_0 \leq u_1 \leq \cdots \leq u_k = j} x_{u_0,u_1} \otimes x_{u_1,u_2} \otimes \cdots \otimes x_{u_{k-1},u_k}$$

for any $i \leq j$ and any $k \in \mathbb{N}$. Thus, for any weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and any $\sigma \in \mathfrak{S}_k$ and any $1 \leq i < j \leq n$ satisfying $j - i = |\alpha|$, we have

$$p_{\alpha,\sigma}\left(x_{i,j}\right) = x_{u_{\sigma(1)-1},u_{\sigma(1)}} x_{u_{\sigma(2)-1},u_{\sigma(2)}} \cdots x_{u_{\sigma(k)-1},u_{\sigma(k)}},$$

where $(u_0 \leq u_1 \leq \cdots \leq u_k)$ is the unique weakly increasing sequence of integers satisfying $u_0 = i$ and $u_k = j$ and $u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i$ for all $i \in \{1,2,\ldots,k\}$. Hence, for any choice of $1 \leq i < j \leq n$, the images $p_{\alpha,\sigma}\left(x_{i,j}\right)$ as α runs over all compositions of $j-i$ and σ runs over all permutations of $[\ell(\alpha)]$ are distinct monomials and therefore are $\mathbf{k}$-linearly independent. This yields the $\mathbf{k}$-linear independence of the family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k \in \mathbb{N}; \\ \alpha \text{ is a composition of length } k \text{ and size } m; \\ \sigma \in \mathfrak{S}_k}}$$

for any given $m \in \{0,1,\ldots,n-1\}$. Since each $p_{\alpha,\sigma}$ lies in $\operatorname{End}_{\mathbf{k}}\left(H_{|\alpha|}\right)$, we thus obtain the $\mathbf{k}$-linear independence of the entire family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k \in \mathbb{N}; \\ \alpha \text{ is a composition of length } k \text{ and size } <n; \\ \sigma \in \mathfrak{S}_k}}$$

(since the sum $\sum_{m=0}^{n-1} \operatorname{End}_{\mathbf{k}}(H_m)$ is a direct sum). This proves Theorem 2.10 **(b)**.

(a) As for part **(b)**, but remove "$\leq n$" everywhere. ∎

Theorem 2.8 suggests that we should consider the algebra of $p_{\alpha,\sigma}$'s as some twisted (Hopf-monoid-like?) version of NSym, and the twist appears rather simple since all addends on the right hand side have the same permutation $\tau[\sigma]$. However, one should keep in mind that some of the $\gamma_{i,j}$ can be 0, in which case some entries have to be "collapsed" to normalize the right hand side.

How all-encompassing are our $p_{\alpha,\sigma}$'s? Do they span all the natural endomorphisms?

Question 2.11. Let g be a natural graded $\mathbf{k}$-module endomorphism on the category of connected graded $\mathbf{k}$-Hopf algebras. (That is, for each connected graded Hopf algebra H, we have a graded $\mathbf{k}$-module endomorphism g_H, and each graded Hopf algebra morphism $\varphi : H \to H'$ gives a commutative diagram.) Is it true that g is an infinite $\mathbf{k}$-linear combination of $p_{\alpha,\sigma}$'s?

We observe one more property of $p_{\alpha,\sigma}$'s:

Proposition 2.12. Let H and G be two graded bialgebras. Let α be a weak composition of length k, and let $\sigma \in \mathfrak{S}_k$ be a permutation. Then,

$$(p_{\alpha,\sigma} \text{ for } H \otimes G) = \sum_{\substack{\beta,\gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} \left(p_{\beta,\sigma} \text{ for } H\right) \otimes \left(p_{\gamma,\sigma} \text{ for } G\right)$$

as endomorphisms of $H \otimes G$.

Proof. Straightforward. □

2.2. A combinatorial Hopf algebra of "formal $p_{\alpha,\sigma}$"s

Next, let us try to find a combinatorial algebra in which the $p_{\alpha,\sigma}$'s themselves live. This requires us to make some implicit things explicit and introduce some more notation:

Definition 2.13. A *mopiscotion* (please find a better name for this!) is a pair (α,σ), where α is a composition of length k (for some $k \in \mathbb{N}$) and σ is a permutation in $\mathfrak{S}_k$.

Definition 2.14. A *weak mopiscotion* is a pair (α,σ), where α is a weak composition of length k (for some $k \in \mathbb{N}$) and σ is a permutation in $\mathfrak{S}_k$.

Definition 2.15. Let (α,σ) be a weak mopiscotion, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\sigma \in \mathfrak{S}_k$. Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements i of $\{1, 2, \ldots, k\}$ satisfying $\alpha_i \neq 0$, in increasing order. Let $\tau \in \mathfrak{S}_h$ be the standardization of the list $(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_h))$. (See [GriRei20, Definition 5.3.3] for the meaning of "standardization".) Let α^{red} denote the composition $\left(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_h}\right)$ (which consists of all nonzero entries of α). Then, we define $(\alpha,\sigma)^{\text{red}}$ to be the mopiscotion $\left(\alpha^{\text{red}}, \tau\right)$.

For example,

$$((3,0,1,2,0)\,,\ [4,5,1,3,2])^{\text{red}} = ((3,1,2)\,,\ [3,1,2])\,,$$

where the square brackets indicate a permutation written in one-line notation.

Clearly, if (α,σ) is a mopiscotion, then $(\alpha,\sigma)^{\text{red}} = (\alpha,\sigma)$.

The following is easy to see:

Proposition 2.16. Let H be a connected graded bialgebra. Let (α,σ) be a weak mopiscotion, and let $(\beta,\tau) = (\alpha,\sigma)^{\text{red}}$. Then,

$$p_{\alpha,\sigma} = p_{\beta,\tau}.$$

Definition 2.17. Let PNSym be the free $\mathbf{k}$-module with basis $(F_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a mopiscotion}}$.

For any weak mopiscotion (α,σ), we set

$$F_{\alpha,\sigma} := F_{\beta,\tau},$$

where $(\beta,\tau) = (\alpha,\sigma)^{\text{red}}$.

Define two multiplications on PNSym: one "external multiplication" (which mirrors convolution of $p_{\alpha,\sigma}$'s as expressed in Proposition 2.7) given by

$$F_{\alpha,\sigma} \cdot F_{\beta,\tau} = F_{\alpha\beta,\sigma\oplus\tau};$$

and another "internal multiplication" (which mirrors composition of $p_{\alpha,\sigma}$'s as expressed in Theorem 2.8) given by

$$F_{\alpha,\sigma} * F_{\beta,\tau} = \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k];\\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j\in[\ell]}} F_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}),\tau[\sigma]}$$

(where $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^\ell$). Also, we define a comultiplication $\Delta : \text{PNSym} \to \text{PNSym} \otimes \text{PNSym}$ on PNSym by

$$\Delta\left(F_{\alpha,\sigma}\right) = \sum_{\substack{\beta,\gamma \text{ weak compositions;}\\ \text{entrywise sum } \beta+\gamma=\alpha}} F_{\beta,\sigma} \otimes F_{\gamma,\sigma}$$

(mirroring the formula from Proposition 2.12).

We also equip the $\mathbf{k}$-module PNSym with a grading by letting each $F_{\alpha,\sigma}$ be homogeneous of degree $|\alpha|$.

If I have not made any mistakes, then:

Theorem 2.18. The **k**-module PNSym becomes a connected graded Hopf algebra when equipped with the external multiplication $\cdot$, and a (non-graded) non-unital bialgebra when equipped with the internal multiplication $*$. In particular, both multiplications are associative.

There are two ways to prove this. I shall very briefly outline both:

First proof idea for Theorem 2.18. Most claims can be derived from properties of the operators $p_{\alpha,\sigma}$, using the H from Theorem 2.10 **(a)** as a faithful representation.

For an example, let us prove that the internal multiplication $*$ on PNSym is associative.

Let H be any connected graded **k**-bialgebra. Let $\operatorname{ev}_H : \operatorname{PNSym} \to \operatorname{End} H$ be the **k**-linear map that sends any $F_{\alpha,\sigma}$ to the operator $p_{\alpha,\sigma} \in \operatorname{End} H$ for any mopiscotion (α, σ). Note that

$$\operatorname{ev}_H (F_{\alpha,\sigma}) = p_{\alpha,\sigma} \tag{44}$$

is true not only for all mopiscotions (α, σ), but also for all weak mopiscotions (α, σ) (because if (α, σ) is any weak mopiscotion, and if $(\beta, \tau) = (\alpha, \sigma)^{\text{red}}$, then $p_{\alpha,\sigma} = p_{\beta,\tau}$ and $F_{\alpha,\sigma} = F_{\beta,\tau}$).

Now, let H be the connected graded Hopf algebra H from Theorem 2.10 **(a)**. Then, Theorem 2.10 **(a)** says that the family $(p_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a mopiscotion}}$ is **k**-linearly independent. Hence, the linear map ev_H is injective.

The formula for $F_{\alpha,\sigma} * F_{\beta,\tau}$ that we used to define the internal multiplication $*$ is very similar to the formula for $p_{\alpha,\sigma} \circ p_{\beta,\tau}$ in Theorem 2.8. In view of (44), this entails that

$$\operatorname{ev}_H \left(F_{\alpha,\sigma} * F_{\beta,\tau}\right) = p_{\alpha,\sigma} \circ p_{\beta,\tau} = \operatorname{ev}_H (F_{\alpha,\sigma}) \circ \operatorname{ev}_H \left(F_{\beta,\tau}\right)$$

for any two mopiscotions (α, σ) and (β, τ). By bilinearity, this entails that

$$\operatorname{ev}_H (f * g) = (\operatorname{ev}_H f) \circ (\operatorname{ev}_H g)$$

for any $f, g \in \operatorname{PNSym}$. Thus, the injective **k**-linear map $\operatorname{ev}_H : \operatorname{PNSym} \to \operatorname{End} H$ embeds the **k**-module PNSym with its binary operation $*$ into the algebra $\operatorname{End} H$ with its binary operation $\circ$. Since the latter operation $\circ$ is associative, it thus follows that the former operation $*$ is associative as well.

Similarly, we can show that the operation $\cdot$ on PNSym is associative and unital with the unity $1 = F_{\varnothing,\varnothing}$ (although this is pretty obvious).

It is very easy to see that the cooperation Δ is coassociative and counital. It is also clear that both the multiplication $\cdot$ and the comultiplication Δ on PNSym are graded.

The next difficulty is to prove that Δ is a **k**-algebra homomorphism, i.e., that $\Delta (fg) = \Delta (f) \cdot \Delta (g)$ for all $f, g \in \operatorname{PNSym}$ (where the "$\cdot$" on the right hand side is the extension of the external multiplication $\cdot$ to $\operatorname{PNSym} \otimes \operatorname{PNSym}$). Here, we can argue as above, using the fact (a consequence of Theorem 2.12) that

$$\operatorname{ev}_{H\otimes H} (f) = (\operatorname{ev}_H \otimes \operatorname{ev}_H) (\Delta (f)) \in \operatorname{End} (H \otimes H)$$
$$\text{for every } f \in \operatorname{PNSym}$$

to make sense of Δ), and using the fact that the map

$$\operatorname{ev}_H \otimes \operatorname{ev}_H : \text{PNSym} \otimes \text{PNSym} \to \text{End}\, H \otimes \text{End}\, H \to \text{End}\,(H \otimes H)$$

is injective (this is not hard to show using the argument used in the proof of Theorem 2.10).

What we have shown so far yields that PNSym (equipped with $\cdot$ and Δ) is a connected graded **k**-bialgebra. Thus, PNSym is a Hopf algebra (since any connected graded **k**-bialgebra is a Hopf algebra).

It remains to show that PNSym (equipped with $*$ and Δ) is a non-unital **k**-bialgebra. Having already verified that $*$ is associative, we only need to show that $\Delta(f * g) = \Delta(f) * \Delta(g)$ for all $f, g \in \text{PNSym}$. But this is similar to the proof of $\Delta(fg) = \Delta(f) \cdot \Delta(g)$ above. Thus, the proof of Theorem 2.18 is complete. □

Second proof idea for Theorem 2.18. There is also a more direct combinatorial approach to this theorem. First, we shall define two smaller bialgebras NNSym and Perm, and then present PNSym as a quotient of their tensor product NNSym $\otimes$ Perm.

Here are some details:

We define NNSym to be the free **k**-module with basis $(C_\alpha)_{\alpha \text{ is a weak composition}}$. We equip this **k**-module NNSym with an "external multiplication" defined by

$$C_\alpha \cdot C_\beta = C_{\alpha\beta}$$

(where $\alpha\beta$ is the concatenation of α and β), and an "internal multiplication" defined by

$$C_\alpha * C_\beta = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k];\\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j \in [\ell]}} C_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell})}$$

(where $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^\ell$), and a comultiplication $\Delta : \text{NNSym} \to \text{NNSym} \otimes \text{NNSym}$ defined by

$$\Delta(C_\alpha) = \sum_{\substack{\beta,\gamma \in \mathbb{N}^k;\\ \text{entrywise sum } \beta+\gamma=\alpha}} C_\beta \otimes C_\gamma \qquad \text{for any } \alpha \in \mathbb{N}^k.$$

It is not too hard to show that NNSym thus becomes a graded (but not connected!) **k**-bialgebra when equipped with the external multiplication $\cdot$, and a (non-graded) non-unital bialgebra when equipped with the internal multiplication $*$. (Indeed, this NNSym is a mild variation on the Hopf algebra NSym of noncommutative symmetric functions, which is studied (e.g.) in [GKLLRT94] or [GriRei20, §5.4]; the only difference is that compositions have been replaced by weak compositions.)

We define $\mathfrak{S}$ to be the disjoint union $\bigsqcup_{k \in \mathbb{N}} \mathfrak{S}_k$ of all symmetric groups $\mathfrak{S}_k$ for all $k \in \mathbb{N}$. We define Perm to be the free **k**-module with basis $(P_\sigma)_{\sigma \in \mathfrak{S}}$. We equip this **k**-module Perm with an "external multiplication" defined by

$$P_\sigma \cdot P_\tau = P_{\sigma \oplus \tau},$$

and an "internal multiplication" defined by

$$P_\sigma * P_\tau = P_{\tau[\sigma]},$$

and a comultiplication $\Delta : \operatorname{Perm} \to \operatorname{Perm} \otimes \operatorname{Perm}$ defined by

$$\Delta\left(P_\sigma\right) = P_\sigma \otimes P_\sigma.$$

It is not too hard to show that Perm becomes a $\mathbf{k}$-bialgebra when equipped with either of the two multiplications (but not a graded one). Indeed, in both cases, it becomes the monoid algebra of an appropriate monoid on the set Perm. (Checking the associativity of the internal multiplication is a neat exercise in combinatorial Yang–Baxter matrices.)

Now, let PNNSym be the tensor product $\operatorname{NNSym} \otimes \operatorname{Perm}$. We equip this tensor product PNNSym with an "external multiplication" (obtained by tensoring the external multiplications of NNSym and of Perm), an "internal multiplication" (similarly) and a comultiplication (similarly). We furthermore set

$$\widehat{F}_{\alpha,\sigma} := C_\alpha \otimes P_\sigma \in \operatorname{PNNSym} \qquad \text{for any weak mopiscotion } (\alpha,\sigma).$$

Then, $\left(\widehat{F}_{\alpha,\sigma}\right)_{(\alpha,\sigma) \text{ is a weak mopiscotion}}$ is a basis of the $\mathbf{k}$-module PNNSym, and our operations $\cdot$, $*$ and Δ on PNNSym satisfy the same relations for this basis as the analogous operations on PNSym do for the basis $\left(F_{\alpha,\sigma}\right)_{(\alpha,\sigma) \text{ is a mopiscotion}}$ (we just have to replace each "F" by "$\widehat{F}$"). It is thus easy to show that PNNSym is a graded (but not connected) $\mathbf{k}$-bialgebra when equipped with $\cdot$, and a (non-graded) non-unital $\mathbf{k}$-bialgebra when equipped with $*$. (Here we use the facts that the tensor product of two bialgebras is a bialgebra, and that the tensor product of two non-unital bialgebras is a non-unital bialgebra.)

As we said, PNNSym is almost the PNSym that we care about. But PNNSym does not satisfy the rule

$$F_{\alpha,\sigma} = F_{\beta,\tau} \qquad \text{for } (\beta,\tau) = (\alpha,\sigma)^{\text{red}}$$

that is fundamental to the definition of PNSym. Hence, PNSym is not quite PNNSym but rather a quotient of PNNSym. To be specific, we define a $\mathbf{k}$-submodule I_{red} of PNNSym by

$$\begin{aligned} I_{\text{red}} &:= \operatorname{span}_{\mathbf{k}}\left(\widehat{F}_{\alpha,\sigma} - \widehat{F}_{\beta,\tau} \mid (\beta,\tau) = (\alpha,\sigma)^{\text{red}}\right) \\ &= \operatorname{span}_{\mathbf{k}}\left(\widehat{F}_{\alpha,\sigma} - \widehat{F}_{\beta,\tau} \mid (\beta,\tau)^{\text{red}} = (\alpha,\sigma)^{\text{red}}\right). \end{aligned}$$

It is not too hard to show that this I_{red} is an ideal of PNNSym with respect to both $\cdot$ and $*$ and a coideal with respect to Δ. Hence, the quotient $\operatorname{PNNSym}/I_{\text{red}}$ inherits all operations of PNNSym, thus becoming a graded bialgebra under $\cdot$ and Δ and a non-unital bialgebra under $*$ and Δ. Moreover, the graded bialgebra $\operatorname{PNNSym}/I_{\text{red}}$ is connected (since $(\alpha,\sigma)^{\text{red}} = (\varnothing,\varnothing)$ whenever $|\alpha| = 0$), and thus is a Hopf algebra. This completes the proof of Theorem 2.18 again. □

Theorem 2.19. Let $\operatorname{PNSym}^{(2)}$ be the non-unital algebra PNSym with multiplication $*$. Then, every connected graded bialgebra H becomes a $\operatorname{PNSym}^{(2)}$-module, with $F_{\alpha,\sigma}$ acting as $p_{\alpha,\sigma}$.

Proof idea. This follows from Theorem 2.8. □

There is also an analogue of the "splitting formula" ([GKLLRT94, Proposition 5.2]) for PNSym, connecting the two products (internal and external) with the comultiplication:

Theorem 2.20. Let $f, g, h \in \operatorname{PNSym}$. Write $\Delta(h)$ as $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ (using Sweedler notation). Then,

$$(fg) * h = \sum_{(h)} \left(f * h_{(1)}\right) \left(g * h_{(2)}\right).$$

(To be more precise, the analogue of [GKLLRT94, Proposition 5.2] would be the generalization of this formula to iterated coproducts $\Delta^{[r]}(h)$, but this generalization follows from Theorem 2.20 by a straightforward induction on r.)

Proof idea for Theorem 2.20. Fairly easy using Definition 2.17. (An important first step is to show that the formulas for $F_{\alpha,\sigma} \cdot F_{\beta,\tau}$ and $F_{\alpha,\sigma} * F_{\beta,\tau}$ hold not only for mopiscotions (α, σ) and (β, τ) but also for weak mopiscotions (α, σ) and (β, τ).) □

Question 2.21. What is the combinatorial meaning of PNSym ?

Question 2.22. Is there a cancellation-free formula for the antipode of PNSym ?

Question 2.23. Does PNSym embed into any kind of noncommutative formal power series?

Remark 2.24. An analogue of $\operatorname{PNSym}^{(2)}$ for Hopf monoids is the Janus algebra of Aguiar and Mahajan [AguMah20, §11.3]. Is there a way to translate results? Could PNSym be the result of transforming NSym into a Hopf monoid using one of Aguiar's functors, and then transforming it back into an algebra using another? My impression is that this procedure could give us the permutations σ in the $F_{\alpha,\sigma}$'s.

Note that there is a canonical bijection between mopiscotions and set compositions into intervals. Specifically, if (α, σ) is a mopiscotion with $|\alpha| = n$, then we can consider the set composition of the set $\{1, 2, \ldots, n\}$ into the intervals whose lengths are the parts of α and whose order is determined by σ. (Unfortunately, there are some choices involved here, and I'm not sure which is the right one.) This might be helpful for embedding PNSym in some Hopf algebra of set compositions, such as (the dual of) WQSym.

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