Chromatic symmetric functions and broken circuits [talk slides]

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In his 1995 paper introducing the chromatic symmetric function of a graph, Richard Stanley proved an expression for it as a sign-alternating sum of p-functions over subsets that contain no broken circuit of the graph. This (partly expository) talk will generalize this formula to some degree, in that the exclusion of all broken circuits will be replaced by the exclusion of some (arbitrarily chosen) set of broken circuits. This generality comes for cheap, but has a curious application to bipartite graphs and to an isotropic analogue of hyperplane arrangements. An analogous generalization of the broken-circuit formula for the characteristic polynomial of a matroid will also be given.

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Preprint:

- Darij Grinberg, *Generalized Whitney formulas for broken circuits in ambigraphs and matroids*, preprint,
  `https://www.cip.ifi.lmu.de/~grinberg/algebra/chromatic.pdf`

  (Previously called *"A note on non-broken-circuit sets and the chromatic polynomial"*.)

Slides of this talk:

- `https://www.cip.ifi.lmu.de/~grinberg/algebra/acpms2023.pdf`

- This talk is about a project started back in 2016, still in a somewhat larval stage.
- I hope it gives some food for thought even if there is a lack of deep and striking theorems.

1. Motivation: Hyperplane arrangements

- Recall: A *linear hyperplane arrangement* in a vector space V is a finite set $\mathcal{A}$ of linear hyperplanes in V.

  ("Linear" means "passing through 0".)
- A classical example of such an arrangement:
- **Definition.** Let G be a finite graph with vertex set $\{1, 2, \ldots, n\}$. Then, its *graphical arrangement* $\mathcal{A}_G$ (over a field K) is the hyperplane arrangement in K^n defined by

$$\{\{x = (x_1, x_2, \ldots, x_n) \in K^n \mid x_i = x_j\}\}_{\{i,j\} \text{ is an edge of } G}.$$

- **Definition.** Let $\mathcal{A}$ be a linear hyperplane arrangement in a vector space V. Then, its *characteristic polynomial* $\chi_{\mathcal{A}}$ is defined by

$$\chi_{\mathcal{A}} = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} x^{\dim(\bigcap \mathcal{B})} \in \mathbb{Z}[x].$$

  Here, $\bigcap \mathcal{B}$ denotes the intersection of the hyperplanes in $\mathcal{B}$.
- For a graphical arrangement, this turns out to recover the *chromatic polynomial*:
- **Definition.** Let G be a finite graph with vertex set V. A *proper coloring* of G means a map $f : V \to \{1, 2, 3, \ldots\}$ such that

$$f(v) \neq f(w) \text{ for each edge } \{v, w\} \text{ of } G.$$

  The values of such a map are called its *colors*.

  The *chromatic polynomial* χ_G of G is the polynomial in $\mathbb{Z}[x]$ such that

$$\chi_G(q) = (\text{\# of proper colorings of } G \text{ with colors in } \{1, 2, \ldots, q\}) \quad \text{for all } q \in \mathbb{N}.$$

- **Theorem (folklore; see Theorem 2.7 in Stanley's Introduction).** Let G be a finite graph with vertex set $\{1, 2, \ldots, n\}$. Then, the characteristic polynomial $\chi_{\mathcal{A}_G}$ of its graphical arrangement $\mathcal{A}_G$ equals the chromatic polynomial of G:

$$\chi_{\mathcal{A}_G} = \chi_G.$$

2. Coisotropic hyperplane arrangements

- Now, consider a vector space V (over a field K) equipped with a bilinear form $f : V \times V \to K$.
- Assume that this form f is symmetric or alternating.
- Then, for each vector subspace W of V, there is an *orthogonal space*

$$\begin{aligned} W^{\perp} &:= \{v \in V \mid f(v,w) = 0 \text{ for all } w \in W\} \\ &= \{v \in V \mid f(w,v) = 0 \text{ for all } w \in W\} \end{aligned}$$

  (also a subspace of V).
- We say that a vector subspace W of V is *coisotropic* if $W^{\perp} \subseteq W$.
- A coisotropic subspace W always has dimension $\geq \dfrac{\dim W}{2}$.
- A linear hyperplane arrangement $\mathcal{A}$ in V will be called *coisotropic* if each hyperplane $H \in \mathcal{A}$ is coisotropic.
- This does not mean that every intersection $\bigcap \mathcal{B}$ for $\mathcal{B} \subseteq \mathcal{A}$ is coisotropic!
- **Definition.** Let $\mathcal{A}$ be a coisotropic linear hyperplane arrangement in V. Then, its *coisotropic characteristic polynomial* $\chi_{\mathcal{A},\perp}$ is defined by

$$\chi_{\mathcal{A},\perp} = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A}; \\ \bigcap \mathcal{B} \text{ is a coisotropic} \\ \text{subspace of } V}} (-1)^{|\mathcal{B}|} x^{\dim(\bigcap \mathcal{B})} \in \mathbb{Z}[x].$$

- We have an analogue of the graphical arrangement:
- **Definition.** Let D be a **digraph** (= directed graph) with vertex set $\{1, 2, \ldots, n\}$ and with no loops (i.e., no arcs of the form (v, v)). Let V be the K-vector space $K^n \oplus K^n$. A typical vector in V has the form $(x; y) = (x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$.

  Define a bilinear form

$$\begin{aligned} f : V \times V &\to K, \\ ((x;y), (x';y')) &\mapsto \sum_{i=1}^{n} x_i y'_i + \sum_{i=1}^{n} y_i x'_i. \end{aligned}$$

  Let $\mathcal{A}_D^{\perp}$ be the coisotropic hyperplane arrangement

$$\left\{ \{(x;y) \in V \mid x_i = y_j\}_{(i,j) \text{ is an arc of } D} \right\}.$$

  Assume that the field K has characteristic $\neq 2$.

- **Proposition.** In this situation,

$$\chi_{\mathcal{A}_D^{\perp},\perp} = \sum_{\substack{F \text{ is a set of arcs of } D; \\ F \text{ is 2-step-free}}} (-1)^{|F|} x^{n+\kappa(F)}.$$

  Here,

  - a set F of arcs of D is said to be 2-*step-free* if two arcs of the form (i,j) and (j,k) cannot simultaneously belong to F.
  - the number $\kappa(F)$ is the number of connected components of the graph whose vertices are $1, 2, \ldots, n$ and whose edges are the undirected versions of the arcs in F.

- **Theorem (suggested by Postnikov 2016).** Assume that this digraph D is *transitive* – i.e., if (i,j) and (j,k) are arcs of D, then so is (i,k). Then,

$$\chi_{\mathcal{A}_D^{\perp},\perp} = x^n \chi_{\underline{D}}.$$

  Here, $\underline{D}$ denotes the underlying undirected graph of D (that is, replace all arcs by edges in D), and $\chi_{\underline{D}}$ is its chromatic polynomial.

- Proving this theorem has motivated the following combinatorial considerations.

3. Chromatic symmetric functions

- In 1995, Richard P. Stanley generalized chromatic polynomials to "chromatic symmetric functions": a finer invariant of a graph.

- **Definition.** If V is a set, then $\mathcal{P}_2(V)$ is the set of all 2-element subsets of V.

- **Definition.** A *graph* (or *simple graph*) means a pair (V, E) of a finite set V and a subset E of $\mathcal{P}_2(V)$ (that is, each edge is a 2-element set of vertices).

  Of course, the elements of V are called *vertices*, and the elements of E are called *edges* of the graph.

- **Definition.** If $G = (V, E)$ is a graph, then

  - a *coloring* of G means a map $f : V \to \mathbb{N}_+$;
  - a *proper coloring* of G means a coloring f of G such that

  $$f(v) \neq f(w) \quad \text{for each edge } \{v, w\} \in E.$$

  Here and in the following, $\mathbb{N}_+ = \{1, 2, 3, \ldots\} = \mathbb{N} \setminus \{0\}$.

- **Definition.** Let $G = (V, E)$ be a graph. The *chromatic symmetric function* of G is defined to be the formal power series

  $$X_G := \sum_{f \text{ is a proper coloring of } G} \mathbf{x}_f, \qquad \text{where } \mathbf{x}_f = \prod_{v \in V} x_{f(v)}.$$

  This is a symmetric function in $x_1, x_2, x_3, \ldots$ over $\mathbb{Z}$.

- **Remark.** The chromatic polynomial χ_G of G is then determined by

  $$\chi_G(q) = X_G\left(\underbrace{1, 1, \ldots, 1}_{q \text{ times}}, 0, 0, 0, \ldots\right) \qquad \text{for all } q \in \mathbb{N}.$$

- **Theorem (all-subsets Whitney formula; Stanley 1995).** Let $G = (V, E)$ be a graph. Then,

  $$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V,F)}.$$

  Here, $\lambda(V, F)$ is the partition whose entries are the sizes of the connected components of the graph (V, F).

  And p_μ is the power-sum symmetric function for the partition μ.

- This formula has lots of cancellation, but it yields that X_G is p-integral.

- **Corollary (Whitney's original all-subsets formula; Whitney 1932).** Let $G = (V, E)$ be a graph. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)},$$

where conn means "number of connected components".

4. Broken circuits

- A way to manage this cancellation was found by Whitney (1932) and extended to symmetric functions by Stanley (1995). We now recall the definitions.

- **Definition.** Let $G = (V, E)$ be a graph. If $\mathbf{c}$ is a cycle of G, then the set of all edges of $\mathbf{c}$ is called a *circuit* of G.

- **Definition.** Let $G = (V, E)$ be a graph. A *labeling function* means a map $\ell : E \to X$ to some totally ordered set X. Its value $\ell(e)$ is called the *label* of an edge e.

- **Definition.** Let $G = (V, E)$ be a graph. Fix a labeling function $\ell : E \to X$. If C is a circuit of G, and if C contains a unique edge e of highest label (among all edges in C), then the set $C \setminus \{e\}$ is called a *broken circuit* of G.

- **Note.** If this e is not unique, then C generates no broken circuit.

- **Theorem (broken-circuits Whitney formula; Stanley 1995).** Let $G = (V, E)$ be a graph. Let $\ell : E \to X$ be a labeling function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

- Note that each (V, F) in this sum is a forest if ℓ is injective.

- **Corollary (Whitney's original broken-circuits formula).** Let $G = (V, E)$ be a graph. Let $\ell : E \to X$ be an injective labeling function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{|V|-|F|}.$$

5. Relaxing Whitney's formula

- For each of X_G and χ_G, we have seen two formulas: one sum over all subsets of E, and one that excludes all subsets that contain any broken circuit.
- Everything inbetween also works!
- **Theorem (relaxed Whitney formula; explicitly G. 2016+, but essentially Dohmen/Trinks 2014).** Let $G = (V, E)$ be a graph. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

  Here (and in the following), "is $\mathfrak{K}$-free" means "contains no $K \in \mathfrak{K}$ as a subset".

- *Proof idea.* For any $F \subseteq E$, we have

$$p_{\lambda(V,F)} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a map;} \\ f \text{ is constant on each} \\ \text{connected component of } (V,F)}} \mathbf{x}_f.$$

  Thus,

$$\sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)} = \sum_{f:V\to\mathbb{N}_+ \text{ is a map}} \left(\sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free;} \\ f \text{ is constant on each} \\ \text{connected component of } (V,F)}} (-1)^{|F|} \right) \mathbf{x}_f.$$

  It remains to argue that the inner sum is 1 whenever f is a proper coloring of G, and 0 otherwise.

  The former is trivial; the latter follows by a sign-reversing involution (pick an edge $\{s, t\} \in E$ with $f(s) = f(t)$ that has maximum possible label; toggle it in/out of F).

- Both Whitney formulas for X_G follow.
- **Corollary (relaxed Whitney formula; explicitly G. 2016+, but essentially Dohmen/Trinks 2014).** Let $G = (V, E)$ be a graph. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

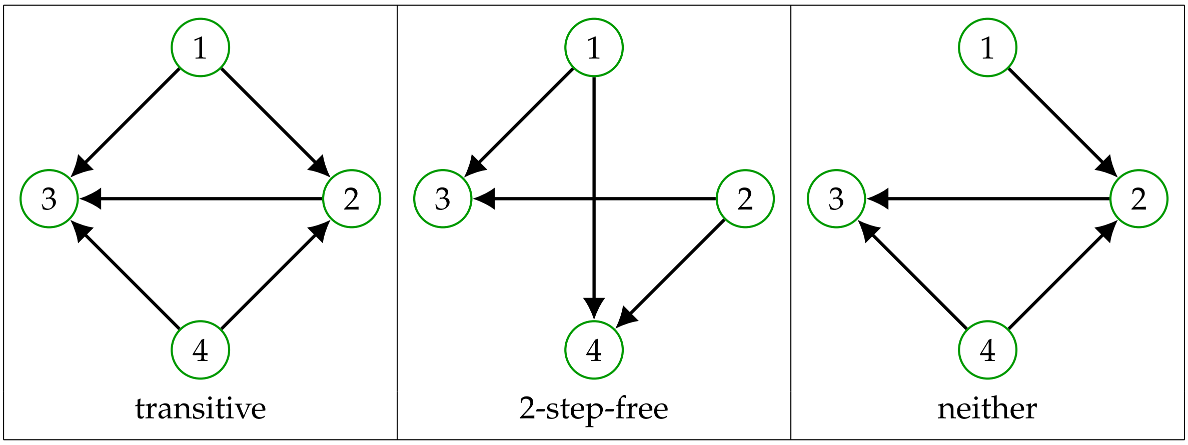
- *Proof idea.* Apply the previous theorem to $(1, 1, \ldots, 1, 0, 0, 0, \ldots)$.
- Both Whitney formulas for χ_G follow.
- Even a somewhat more general result holds:
- **Theorem (generalized Whitney formula; explicitly G. 2016+, but essentially Dohmen/Trinks 2014).** Let $G = (V, E)$ be a graph. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of the base ring for every $K \in \mathfrak{K}$. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda(V,F)}.$$

- *Proof.* Essentially the same argument.

6. Application to transitive digraphs

- The freedom to exclude some (rather than all) broken circuits has some applications. In particular, we can prove what we need for the hyperplane arrangement equality.
- **Definition.** A *digraph* (= *directed graph*) means a pair (V, A) of a finite set V (the set of *vertices*) and a subset A of $V \times V$ (the set of *arcs*). Thus, each arc is an ordered pair of vertices.
- **Definition.** A digraph (V, A) is said to be
  - *loopless* if $(v, v) \notin A$ for all $v \in V$;
  - *transitive* if $(u, v) \in A$ and $(v, w) \in A$ imply $(u, w) \in A$;
  - 2-*step-free* if $(u, v) \in A$ and $(v, w) \in A$ never happen together.
- **Definition.** If D is a loopless digraph, then $\underline{D}$ shall mean the underlying undirected graph (i.e., replace each arc by an edge).
- **Examples.** Here are three loopless digraphs:



- **Note.** Transitive loopless digraphs have no cycles.
- **Theorem (G. 2016+, suggested by Alex Postnikov).** Let $D = (V, A)$ be a transitive loopless digraph. Then,

$$\chi_{\underline{D}} = \sum_{\substack{F \subseteq A; \\ \text{the digraph } (V,F) \text{ is 2-step-free}}} (-1)^{|F|} x^{\operatorname{conn} \underline{(V,F)}},$$

  where conn means "number of connected components".
- *Proof idea.* Let E be the edge set of $\underline{D}$. The map

$$A \to E, \qquad (i, j) \mapsto \{i, j\}$$

is a bijection (since D has no 2-cycles).

Hence, each triple $(i, j, k) \in V \times V \times V$ with $(i, j) \in A$ and $(j, k) \in A$ yields a 3-cycle (i, j, k, i) in $\underline{D}$, and thus a circuit

$$\{\{i, j\}, \{i, k\}, \{j, k\}\}$$

of $\underline{D}$. Define a labeling function $\ell : E \to \mathbb{N}$ in such a way that the edge $\{i, k\}$ of this circuit has a higher label than the other two (this is possible since D has no cycles). Thus,

$$\{\{i, j\}, \{j, k\}\}$$

is a broken circuit of $\underline{D}$. Let $\mathfrak{K}$ be the set of all such broken circuits.

Argue that the above bijection $A \to E$ transforms subsets $F \subseteq A$ for which the digraph (V, F) is 2-step-free into subsets $F' \subseteq E$ that are $\mathfrak{K}$-free. Now, apply the relaxed Whitney formula for χ_G to $G = \underline{D}$.

- Note that $\mathfrak{K}$ is not the set of all broken circuits of $\underline{D}$, only some of them. We don't want to remove too much!
- The above theorem can then be used to obtain the $\chi_{\mathcal{A}_{D}^{\perp}, \perp} = x^n \chi_{\underline{D}}$ theorem.

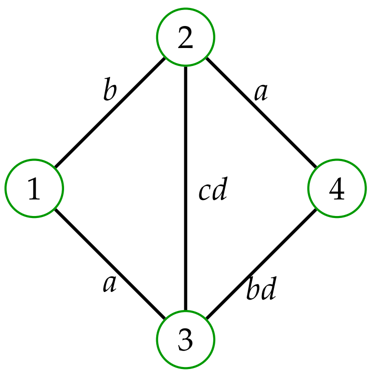
7. Ambigraphs

- Simple graphs are not the end-all. Chromatic polynomials have also been defined for
  - *multigraphs* (graphs with multiple edges);
  - *hypergraphs* ("graphs" where an edge can have more than 2 endpoints).

  They still satisfy Whitney formulas (folklore for multigraphs; Dohmen 1995 and Dohmen/Trinks 2014 for hypergraphs).
- For a multigraph, parallel edges don't affect X_G and χ_G but make the alternating sums bigger.
- We ignore loops, as they trivialize everything ($0 = 0$) but make proofs messier.
- For a hypergraph, a proper coloring has to ensure that no edge is monochromatic: i.e., if f is a coloring, and $\{v_1, v_2, \ldots, v_k\}$ is an edge, then **at least two** of the values $f(v_1), f(v_2), \ldots, f(v_k)$ must be distinct in order for f to be proper.
- **Observation (new?):** Hypergraphs aren't the right level of generality for this.
- **Definition.** An *ambigraph* shall mean a triple (V, E, φ), where V and E are two finite sets, and where $\varphi : E \to \mathcal{P}(\mathcal{P}_2(V))$ is a map (where $\mathcal{P}(S)$ means the set of all subsets of S). Thus, the map φ sends each $e \in E$ to a set of 2-element subsets of V.

  Some related terminology:
  - The elements of V are called *vertices*, and the elements of E are called *edgeries*.
  - The *edges* of an edgery $e \in E$ are the elements of $\varphi(e)$. These edges are "real" edges, i.e., sets of the form $\{s, t\}$ for $s \neq t$ in V.
  - An edgery $e \in E$ is called *singleton* if e has only 1 edge.

  Idea: An edgery is a collection of edges (in the simple-graph sense).
- **Example.** This here:



shows the ambigraph (V, E, φ) with $V = \{1,2,3,4\}$ and $E = \{a,b,c,d\}$ and

$$\varphi(a) = \{\{1,3\}, \{2,4\}\}, \qquad \varphi(b) = \{\{1,2\}, \{3,4\}\},$$
$$\varphi(c) = \{\{2,3\}\}, \qquad \varphi(d) = \{\{2,3\}, \{3,4\}\}.$$

The only singleton edgery here is c, whose only edge is $\{2,3\}$.

- **Definition.** If $G = (V, E, \varphi)$ is an ambigraph, then
  - a *coloring* of G means a map $f : V \to \mathbb{N}_+$;
  - a *proper coloring* of G means a coloring f of G such that

$$\text{each edgery } e \in E \text{ has at least one edge } \{v, w\} \text{ satisfying } f(v) \neq f(w).$$

- **Example.** If G is as above, then a proper coloring of G is a map $f : V \to \mathbb{N}_+$ such that

$$(f(1) \neq f(3) \text{ or } f(2) \neq f(4)) \qquad \text{and} \qquad (f(1) \neq f(2) \text{ or } f(3) \neq f(4))$$
$$\text{and} \qquad f(2) \neq f(3) \qquad \text{and} \qquad (f(2) \neq f(3) \text{ or } f(3) \neq f(4)).$$

(Note: The last condition is redundant, since $\varphi(c) \subseteq \varphi(d)$.)

- **Remark.** If G has an edgery with no edges, then G has no proper colorings. (This is like having a loop.)

- Ambigraphs generalize multigraphs (all edgeries are singleton or empty) and hypergraphs (e.g., an edge $\{a,b,c,d\}$ becomes an edgery with 6 edges).

- **Definition.** Let $G = (V, E, \varphi)$ be an ambigraph. The *chromatic symmetric function* of G is defined to be the formal power series

$$X_G := \sum_{f \text{ is a proper coloring of } G} \mathbf{x}_f, \qquad \text{where } \mathbf{x}_f = \prod_{v \in V} x_{f(v)}.$$

This is a symmetric function in $x_1, x_2, x_3, \ldots$ over $\mathbb{Z}$.

- Again, we can define the chromatic polynomial χ_G accordingly:

$$\chi_G(q) = X_G \left(\underbrace{1, 1, \ldots, 1}_{q \text{ ones}}, 0, 0, 0, \ldots \right)$$
$$= (\# \text{ of proper colorings of } G \text{ with colors in } \{1, 2, \ldots, q\}).$$

- **Theorem (all-subsets Whitney formula; G. 2016+).** Let $G = (V, E, \varphi)$ be an ambigraph. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

Here, $\operatorname{union} F$ denotes the set $\bigcup_{e \in F} \varphi(e)$, which consists of all edges of all edgeries in F.

- We can get rid of some (not all) cancellation in this formula by introducing broken circuits.

8. Whitney formulas for ambigraphs

- **Definition.** Let $G = (V, E, \varphi)$ be an ambigraph. A *cycle* of G means a list

$$(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$$

  with the following properties:

  - $v_1, v_2, \ldots, v_{m+1} \in V$ and $e_1, e_2, \ldots, e_m \in E$.
  - $m \geq 1$.
  - $v_{m+1} = v_1$.
  - The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
  - The edgeries $e_1, e_2, \ldots, e_m$ are pairwise distinct.
  - $\{v_i, v_{i+1}\} \in \varphi(e_i)$ for every $i \in \{1, 2, \ldots, m\}$.

  If $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ is a cycle of G, then the set $\{e_1, e_2, \ldots, e_m\}$ is called a *circuit* of G.

- **Definition.** Let $G = (V, E, \varphi)$ be an ambigraph. A *labeling function* means a map $\ell : E \to X$ for some totally ordered set X. Its value $\ell(e)$ is called the *label* of an edgery e.

- **Definition.** Let $G = (V, E, \varphi)$ be an ambigraph. Fix a labeling function $\ell : E \to X$. If C is a circuit of G, and if C contains a unique **singleton** edgery e of highest label (among all singleton edgeries in C), then the set $C \setminus \{e\}$ is called a *broken circuit* of G.

- **Theorem (broken-circuits Whitney formula for ambigraphs; G. 2016+).** Let $G = (V, E, \varphi)$ be an ambigraph. Let $\ell : E \to X$ be a labeling function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

- Again, we get a formula for χ_G as a corollary. Note that we cannot simplify $\operatorname{conn}(V, \operatorname{union} F)$ this time, since $(V, \operatorname{union} F)$ needs not be a forest.

- More generally:

- **Theorem (relaxed Whitney formula for ambigraphs; G. 2016+).** Let $G = (V, E, \varphi)$ be an ambigraph. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

- *Proof.* Quite similar to the one for graphs.

9. Weighted versions

- A weighted version of the chromatic symmetric function was defined for graphs by Crew and Spirkl in 2020. We can do the same for ambigraphs:

- **Definition.** A *weight function* on a set V means a map $w : V \to \mathbb{N}_+$.

- **Definition.** Let $G = (V, E, \varphi)$ be an ambigraph. Let w be a weight function on V. The *chromatic symmetric function* of (G, w) is defined to be the formal power series

$$X_{G,w} := \sum_{f \text{ is a proper coloring of } G} \mathbf{x}_{f,w}, \qquad \text{where } \mathbf{x}_{f,w} = \prod_{v \in V} x_{f(v)}^{w(v)}.$$

  This is a symmetric function in $x_1, x_2, x_3, \ldots$ over $\mathbb{Z}$.

- **Theorem (weighted relaxed Whitney formula for ambigraphs; G. 2016+).** Let $G = (V, E, \varphi)$ be an ambigraph. Let w be a weight function on V. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \, p_{\lambda((V, \text{union } F), w)}.$$

  Here, $\lambda(H, w)$ (for a graph H) means the partition whose parts are the **total weights** of the connected components of H.

- *Proof.* Analogous to the unweighted case.

- The advantage of $X_{G,w}$ (compared to X_G) is the existence of a deletion-contraction recurrence. Note:

  - To contract an edgery e means to contract every edge of e simultaneously. Loops are removed if they appear.
  - When an edge is contracted, the weights of its endpoints are added.

10. Matroids

- Some features of graphs are generalized by matroids.
- In particular, the chromatic polynomial χ_G of a graph G is generalized by the characteristic polynomial $\widetilde{\chi}_M$ of a matroid M.
- **Definition.** Let M be a matroid with ground set E, rank function r_M and rank m. The *characteristic polynomial* $\widetilde{\chi}_M$ of the matroid M is defined to be the polynomial

$$\begin{aligned}\widetilde{\chi}_M &= \sum_{F \subseteq E} (-1)^{|F|} x^{m - r_M(F)} \\ &= [\overline{\varnothing} = \varnothing] \sum_{F \in \text{Flats } M} \underbrace{\mu\left(\overline{\varnothing}, F\right)}_{\substack{\text{Möbius function of} \\ \text{the lattice of flats}}} x^{m - r_M(F)} \\ &\qquad\qquad (\text{where } \overline{\varnothing} \text{ is the rank-0 flat, i.e., the set of loops}) \\ &\in \mathbb{Z}[x].\end{aligned}$$

- There is also a slightly different definition, which has no $[\overline{\varnothing} = \varnothing]$ factor. These agree if M has no loops.
- Matroids also have circuits (= minimal dependent sets) and therefore, given a labeling function $\ell : E \to X$, also have broken circuits (= circuits minus their highest-label element).
- **Theorem (relaxed Whitney formula; explicitly G. 2016+, but essentially Dohmen/Trinks 2014).** Let M be a matroid with ground set E and rank m. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{m - r_M(F)}.$$

- **Theorem (generalized Whitney formula; explicitly G. 2016+, but essentially Dohmen/Trinks 2014).** Let M be a matroid with ground set E and rank m. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Let a_K be an element of the base ring for every $K \in \mathfrak{K}$. Then,

$$\widetilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{m - r_M(F)}.$$

- I am not aware of any generalization of X_G to matroids. Are you?

11. Questions

- I think ambigraphs and coisotropic hyperplane arrangements are worth exploring!
- **Question.** What can we say about $\chi_{\mathcal{A}_D^{\perp},\perp}$ for general D ?
- **Question.** Are there similar formulas for the Tutte polynomial?
- **Question.** Is there a Stanley (-1)-color theorem for ambigraphs?

  (The sign of $(-1)^{|V(G)|}\chi_G(-1)$ is inconsistent, so maybe not, or it is an alternating sum.)
- **Question.** Is there a Tutte polynomial for ambigraphs?
- **Question.** Is there a chromatic homology theory (categorifying χ_G) for ambigraphs?

12. I thank

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- **Kurusch Ebrahimi-Fard** for the invitation.
- **you** for your patience.