Littlewood–Richardson coefficients and birational combinatorics

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Algebraic and Combinatorial Perspectives in the Mathematical Sciences

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/acpms2020.pdf
paper: arXiv:2008.06128 **aka** http://www.cip.ifi.lmu.de/~grinberg/algebra/lrhspr.pdf

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- I will then state a "hidden symmetry" conjectured by Pelletier and Ressayre (`arXiv:2005.09877`) and outline how I proved it.
- The proof is a nice example of birational combinatorics: the use of birational transformations in elementary combinatorics (specifically, here, in finding and proving a bijection).

Chapter 1

Littlewood–Richardson coefficients

References (among many):

- Richard Stanley, *Enumerative Combinatorics, vol. 2*, Chapter 7.
- Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*, arXiv:1409.8356.
- Emmanuel Briand, Mercedes Rosas, *The 144 symmetries of the Littlewood-Richardson coefficients of* SL_3, arXiv:2004.04995.
- Igor Pak, Ernesto Vallejo, *Combinatorics and geometry of Littlewood-Richardson cones*, arXiv:math/0407170.
- Emmanuel Briand, Rosa Orellana, Mercedes Rosas, *Rectangular symmetries for coefficients of symmetric functions*, arXiv:1410.8017.

Reminder on symmetric functions

- Fix a commutative ring $\mathbf{k}$ with unity. We shall do everything over $\mathbf{k}$.
- Consider the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.

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- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series f is said to be *symmetric* if it is invariant under permutations of the indeterminates.
- For example:
  - $1 + x_1 + x_2^3$ is bounded-degree but not symmetric.
  - $(1 + x_1)(1 + x_2)(1 + x_3) \cdots$ is symmetric but not bounded-degree.

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- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series f is said to be *symmetric* if it is invariant under permutations of the indeterminates.
- Let Λ be the set of all symmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbf{k}$-subalgebra, called the *ring of symmetric functions* over $\mathbf{k}$.

  It is also known as Sym.

- Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a *partition* (i.e., a weakly decreasing sequence of nonnegative integers such that $\lambda_i = 0$ for all $i \gg 0$).
  We commonly omit trailing zeroes: e.g., the partition $(4, 2, 2, 1, 0, 0, 0, 0, \ldots)$ is identified with the tuple $(4, 2, 2, 1)$.

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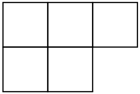
  The *Young diagram* of λ is like a matrix, but the rows have different lengths, and are left-aligned; the i-th row has λ_i cells.
  Examples:
  - The Young diagram of $(3, 2)$ has the form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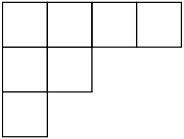
  - The Young diagram of $(4, 2, 1)$ has the form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- A *semistandard tableau* of shape λ is the Young diagram of λ, filled with positive integers, such that
  - the entries in each **row** are **weakly** increasing;
  - the entries in each **column** are **strictly** increasing.

Examples:

- A semistandard tableau of shape $(3, 2)$ is

2	3	3
3	5	

.

- A semistandard tableau of shape $(4, 2, 1)$ is

2	2	3	4
3	4		
5			

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Schur functions, part 2: semistandard tableaux

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- The semistandard tableaux of shape $(3, 2)$ are the arrays of the form

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2}\end{array}$$

with $a \leq b \leq c$ and $d \leq e$ and $a < d$ and $b < e$.

- Given a partition λ, we define the *Schur function s_λ* as the power series

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard} \\ \text{tableau of shape } \lambda}} x_T, \qquad \text{where } x_T = \prod_{p \text{ is a cell of } T} x_{T(p)}$$

(where $T(p)$ denotes the entry of T in p).

- **Examples:**
    - $$s_{(3,2)} = \sum_{\substack{a \leq b \leq c,\ d \leq e, \\ a < d,\ b < e}} x_a x_b x_c x_d x_e,$$

      because the semistandard tableau

      $T =$

a	b	c
d	e	

      contributes the addend $x_T = x_a x_b x_c x_d x_e$.

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- **Examples:**
  - For any $n \geq 0$, we have

$$s_{(n)} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

since the semistandard tableaux of shape (n) are the fillings

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with $i_1 \leq i_2 \leq \cdots \leq i_n$.
This symmetric function $s_{(n)}$ is commonly called h_n.

- **Theorem:** The Schur function s_λ is a symmetric function ($=$ an element of Λ) for any partition λ.
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- **Theorem:** Fix $n \geq 0$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition with at most n nonzero entries. Then,

$$s_\lambda(x_1, x_2, \ldots, x_n) = \underbrace{\det\left(\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}\right)}_{\text{this is called an } \textit{alternant}} \Big/ \underbrace{\det\left(\left(x_i^{n-j}\right)_{1\leq i,j\leq n}\right)}_{\substack{=\prod_{1\leq i<j\leq n}(x_i-x_j) \\ (=\text{ the Vandermonde determinant})}}.$$

Here, for any $f \in \Lambda$, we let $f(x_1, x_2, \ldots, x_n)$ denote the result of substituting 0 for $x_{n+1}, x_{n+2}, x_{n+3}, \ldots$ in f; this is a symmetric **polynomial** in $x_1, x_2, \ldots, x_n$.

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- For proofs, see any text on symmetric functions (e.g., Stanley's EC2, or Grinberg-Reiner, or Mark Wildon's notes).

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- **Example:**

$$\begin{aligned} s_{(2,1)} s_{(3,1)} = s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(3,3,1)} \\ + s_{(4,1,1,1)} + 2 s_{(4,2,1)} + s_{(4,3)} \\ + s_{(5,1,1)} + s_{(5,2)}, \end{aligned}$$

so $c_{(2,1),(3,1)}^{(4,2,1)} = 2$ and $c_{(2,1),(3,1)}^{(3,3,1)} = 1$.

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- **Theorem:** The coefficients $c_{\mu,\nu}^{\lambda}$ are **nonnegative integers**. Various combinatorial interpretations ("*Littlewood–Richardson rules*") for them are known.

- In order to formulate the classic (or, at least, best known) Littlewood–Richardson rule, we need a
- **Definition:**
  - Two partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ are said to satisfy $\mu \subseteq \lambda$ if each $i \geq 1$ satisfies $\mu_i \leq \lambda_i$.
    (Equivalently: if the Young diagram of μ is contained in that of λ.)

Skew semistandard tableaux

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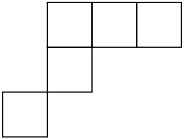
  - If λ / μ is a skew partition, then the *Young diagram* of λ / μ is obtained from the Young diagram λ when all cells of the Young diagram of μ are removed.

    Example: The Young diagram of $(4, 2, 1) / (1, 1)$ is



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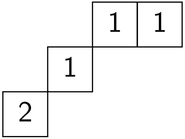
  - *Semistandard tableaux* of shape λ/μ are defined just as ones of shape λ, except that we are now only filling the cells of λ/μ.

Littlewood–Richardson rule: the classical version

- **Littlewood–Richardson rule:** Let λ, μ and ν be three partitions. Then, $c^{\lambda}_{\mu,\nu}$ is the number of semistandard tableaux T of shape λ / μ such that $\operatorname{cont} T = \nu$ and such that $\operatorname{cont}\left(T \mid_{\text{cols} \geq j}\right)$ is a partition for each j. Here,
  - $\operatorname{cont} T$ denotes the sequence $(c_1, c_2, c_3, \ldots)$, where c_i is the number of entries equal to i in T;
  - $T \mid_{\text{cols} \geq j}$ is what obtained from T when the first $j - 1$ columns are deleted.
- **Example:** $c^{(4,2,1)}_{(2,1),(3,1)} = 2$ due to the two tableaux



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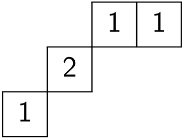

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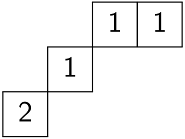

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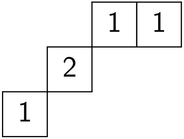

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- The shortest proof is due to Stembridge (using ideas by Gasharov); see John R. Stembridge, *A Concise Proof of the Littlewood-Richardson Rule*, 2002, or Section 2.6 in Grinberg-Reiner.

- **Gradedness:** $c^{\lambda}_{\mu,\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$, where $|\kappa|$ denotes the *size* (i.e., the sum of the entries) of a partition κ. (This is because Λ is a graded ring and the s_λ are homogeneous.)

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- **Transposition symmetry:** $c^\lambda_{\mu,\nu} = c^{\lambda^t}_{\mu^t,\nu^t}$, where κ^t denotes the *transpose* of a partition κ (i.e., the partition whose Young diagram is obtained from that of κ by flipping across the main diagonal).

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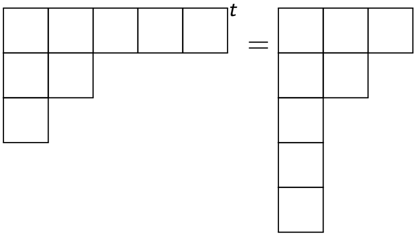

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- **Commutativity:** $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$.
  (Obvious from the definition, but hard to prove combinatorially using the Littlewood–Richardson rule.)

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- If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \operatorname{Par}[n]$, and if $k \geq 0$ is such that all entries of λ are $\leq k$, then $\lambda^{\vee k}$ shall denote the partition

$$(k - \lambda_n, k - \lambda_{n-1}, \ldots, k - \lambda_1) \in \operatorname{Par}[n].$$

This is called the *k-complement* of λ.

Example: If $n = 5$, then

$$\begin{aligned}(3,1,1)^{\vee 7} &= (3,1,1,0,0)^{\vee 7} = (7-0, 7-0, 7-1, 7-1, 7-3) \\ &= (7,7,6,6,4).\end{aligned}$$

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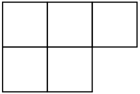
  $(3,2)^{\vee 4} =$  $^{\vee 4}$

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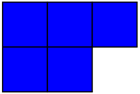
$(3, 2)^{\vee 4} =$  $^{\vee 4}$

Littlewood–Richardson coefficients: more symmetries

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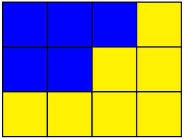

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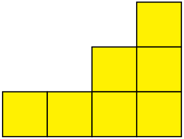

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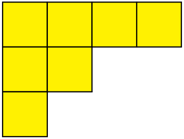
  Illustration: If $n = 3$, then

  $(3,2)^{\vee 4} =$ 

- Fix $n \in \mathbb{N}$. Let $\operatorname{Par}[n]$ be the set of all partitions having at most n nonzero entries.
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(This can be proved by applying skew Schur functions to $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$, or by interpreting Schur functions as fundamental classes in the cohomology of the Grassmannian. See Exercise 2.9.15 in Grinberg-Reiner for the former proof.)

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- **Complementation symmetry II:** Let $\lambda, \mu, \nu \in \operatorname{Par}[n]$ and $q, r \geq 0$ be such that all entries of μ are $\leq q$, and all entries of ν are $\leq r$. Then:
  - If all entries of λ are $\leq q + r$, then $c^{\lambda}_{\mu,\nu} = c^{\lambda^{\vee(q+r)}}_{\mu^{\vee q},\nu^{\vee r}}$.
  - If not, then $c^{\lambda}_{\mu,\nu} = 0$.

(See, e.g., Exercise 2.9.16 in Grinberg-Reiner.)

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- For $n \in \{4, 5, \ldots, 7\}$, they only found the complementation symmetries above, as well as the trivial translation symmetries (adding 1 to each entry of λ and ν does not change $c^{\lambda}_{\mu,\nu}$; nor does adding 1 to each entry of λ and μ).

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- For $n = 3$, they found an extra symmetry:

$$c^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2),(\nu_1,\nu_2)} = c^{(\lambda_1,\nu_1,\lambda_3)}_{(\mu_1+\nu_1-\lambda_2,\mu_2+\nu_1-\lambda_2),(\lambda_2,\nu_2)} .$$

(Read the right hand side as 0 if the tuples are not partitions.)

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  Question: Is there a non-computer proof? What is the meaning of this identity?

Chapter 2

The Pelletier–Ressayre symmetry

References (among many):

- Darij Grinberg, *The Pelletier–Ressayre hidden symmetry for Littlewood–Richardson coefficients*, `arXiv:2008.06128`.
- Maxime Pelletier, Nicolas Ressayre, *Some unexpected properties of Littlewood-Richardson coefficients*, arXiv:2005.09877.
- Robert Coquereaux, Jean-Bernard Zuber, *On sums of tensor and fusion multiplicities*, 2011.

Inspiration: The Coquereaux–Zuber sum identity, 1

- **Theorem (Coquereaux and Zuber, 2011):** Let $n \geq 0$ and $\mu, \nu \in \mathsf{Par}[n]$. Let $k \geq 0$ be such that all entries of μ are $\leq k$. Then,
$$\sum_{\lambda \in \mathsf{Par}[n]} c^{\lambda}_{\mu,\nu} = \sum_{\lambda \in \mathsf{Par}[n]} c^{\lambda}_{\mu^{\vee k},\nu}.$$
(See https://mathoverflow.net/a/236220/ for a hint at a combinatorial proof.)

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$$\sum_{\lambda \in \operatorname{Par}[n]} c_{\mu,\nu}^{\lambda} = \sum_{\lambda \in \operatorname{Par}[n]} c_{\mu^{\vee k},\nu}^{\lambda}.$$
- This can be interpreted in terms of Schur **polynomials**. For any $\lambda \in \operatorname{Par}[n]$, the *Schur polynomial* $s_\lambda(x_1, x_2, \ldots, x_n)$ is the symmetric polynomial
$$s_\lambda(x_1, x_2, \ldots, x_n) = \underbrace{\det\left(\left(x_i^{\lambda_j + n - j}\right)_{1 \leq i,j \leq n}\right)}_{\text{this is called an } \textit{alternant}} \Big/ \underbrace{\det\left(\left(x_i^{n-j}\right)_{1 \leq i,j \leq n}\right)}_{= \prod_{1 \leq i < j \leq n} (x_i - x_j)}$$
(= the Vandermonde determinant)

  in $x_1, x_2, \ldots, x_n$ obtained by setting $x_{n+1} = x_{n+2} = x_{n+3} = \cdots = 0$ in s_λ.

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- The family $(s_\lambda(x_1, x_2, \ldots, x_n))_{\lambda \in \operatorname{Par}[n]}$ is a basis of the **k**-module of symmetric polynomials in $x_1, x_2, \ldots, x_n$. We call it the *Schur basis*.

Inspiration: The Coquereaux–Zuber sum identity, 2

- The theorem of Coquereaux and Zuber says that

$$\begin{aligned} & \operatorname{coeffsum}\left(s_\mu\left(x_1, x_2, \ldots, x_n\right) s_\nu\left(x_1, x_2, \ldots, x_n\right)\right) \\ & = \operatorname{coeffsum}\left(s_{\mu^{\vee k}}\left(x_1, x_2, \ldots, x_n\right) s_\nu\left(x_1, x_2, \ldots, x_n\right)\right), \end{aligned}$$

where coeffsum f denotes the sum of all coefficients in the expansion of a symmetric polynomial f in the Schur basis.

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- So the products

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  Question: Does this hold for $n \leq 4$? (Proved for $n = 3$.)

- **Conjecture (Pelletier and Ressayre, 2020):** It does hold when μ is *near-rectangular* – i.e., when $\mu = \left(a+b, a^{n-2}\right)$ for some $a, b \geq 0$. Here, a^{n-2} means $\underbrace{a, a, \ldots, a}_{n-2 \text{ times}}$.
  In this case, for $k = a + b$, we have $\mu^{\vee k} = \left(a+b, b^{n-2}\right)$.
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- This means that there should be a bijection $\varphi : \operatorname{Par}[n] \to \operatorname{Par}[n]$ such that
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- **Theorem (G., 2020):** This is true. Moreover, this bijection φ can more or less be defined explicitly in terms of maxima of sums of entries of λ and ν.
(“More or less” means that we find a bijection $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$, not $\varphi : \operatorname{Par}[n] \to \operatorname{Par}[n]$, where we set $c_{\alpha,\nu}^{\lambda} = c_{\beta,\nu}^{\lambda} = 0$ for all $\lambda \in \mathbb{Z}^n \setminus \operatorname{Par}[n]$.)

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- The rest of this talk will sketch how this bijection φ was found.

- First, we notice that

$$\alpha = \left(a+b, a^{n-2}\right) = \left(a+b, a^{n-2}, 0\right) \qquad \text{(as } n\text{-tuple)}$$
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- This suggest allowing "partitions with negative entries". We call them **snakes**.
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- Snakes index rational representations of $\operatorname{GL}(n)$: See John R. Stembridge, *Rational tableaux and the tensor algebra of $\mathfrak{gl}_n$*, 1987.

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$$\operatorname{Par}[n] \subseteq \{\text{snakes}\} \subseteq \mathbb{Z}^n.$$
- If $\lambda \in \mathbb{Z}^n$ is any n-tuple, then
  - we let λ_i denote the i-th entry of λ (for any i);
  - we let $\lambda + a$ denote the n-tuple $(\lambda_1 + a, \lambda_2 + a, \ldots, \lambda_n + a)$;
  - we let $\lambda - a$ denote the n-tuple $(\lambda_1 - a, \lambda_2 - a, \ldots, \lambda_n - a)$.

Schur Laurent polynomials

- We have defined a Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n) \in \mathbf{k}[x_1, x_2, \ldots, x_n]$ for any $\lambda \in \operatorname{Par}[n]$. We now denote it by $\overline{s}_\lambda$.

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- This allows us to extend the definition of $\overline{s}_\lambda$ from the case $\lambda \in \operatorname{Par}[n]$ to the more general case $\lambda \in \{\text{snakes}\}$:
  If λ is a snake, then we choose some $a \geq 0$ such that $\lambda + a \in \operatorname{Par}[n]$, and define

$$\overline{s}_\lambda = (x_1 x_2 \cdots x_n)^{-a} \, \overline{s}_{\lambda+a}.$$

  This is a Laurent polynomial in $\mathbf{k}\left[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\right]$.

- We have defined a Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n) \in \mathbf{k}[x_1, x_2, \ldots, x_n]$ for any $\lambda \in \operatorname{Par}[n]$. We now denote it by $\overline{s}_\lambda$.
- It is easy to see that

$$\overline{s}_{\lambda+a} = (x_1 x_2 \cdots x_n)^a \, \overline{s}_\lambda \qquad \text{for any } \lambda \in \operatorname{Par}[n] \text{ and } a \geq 0.$$

- This allows us to extend the definition of $\overline{s}_\lambda$ from the case $\lambda \in \operatorname{Par}[n]$ to the more general case $\lambda \in \{\text{snakes}\}$:
  If λ is a snake, then we choose some $a \geq 0$ such that $\lambda + a \in \operatorname{Par}[n]$, and define

$$\overline{s}_\lambda = (x_1 x_2 \cdots x_n)^{-a} \, \overline{s}_{\lambda+a}.$$

  This is a Laurent polynomial in $\mathbf{k}\left[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\right]$.
- Alternatively, we can define $\overline{s}_\lambda$ explicitly by

$$\overline{s}_\lambda = \det\left(\left(x_i^{\lambda_j + n - j}\right)_{1 \leq i,j \leq n}\right) \Big/ \det\left(\left(x_i^{n-j}\right)_{1 \leq i,j \leq n}\right)$$

  (same formula as before).

- For any $k \geq 0$, define the two Laurent polynomials

$$h_k^+ = h_k\left(x_1, x_2, \ldots, x_n\right),$$
$$h_k^- = h_k\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right).$$

(Recall: $h_k = s_{(k)} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$.)

- For any $k \geq 0$, define the two Laurent polynomials

$$h_k^+ = h_k\left(x_1, x_2, \ldots, x_n\right) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

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- **Proposition:** Let $a, b \geq 0$. Then,

$$\overline{s}_{(b,0^{n-2},-a)} = h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+.$$

- For any $k \geq 0$, define the two Laurent polynomials

$$h_k^+ = h_k\left(x_1, x_2, \ldots, x_n\right) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

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- **Proposition:** Let $a, b \geq 0$. Then,

$$\overline{s}_{(b, 0^{n-2}, -a)} = h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+.$$

- **Corollary:** Let $a, b \geq 0$. Let $\alpha = \left(a + b, a^{n-2}\right)$ and $\beta = \left(a + b, b^{n-2}\right)$. Then,

$$\overline{s}_\alpha = \left(x_1 x_2 \cdots x_n\right)^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right);$$

$$\overline{s}_\beta = \left(x_1 x_2 \cdots x_n\right)^b \cdot \left(h_b^- h_a^+ - h_{b-1}^- h_{a-1}^+\right).$$

- For any $k \geq 0$, define the two Laurent polynomials

$$h_k^+ = h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

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- **Corollary:** Let $a, b \geq 0$. Let $\alpha = \left(a + b, a^{n-2}\right)$ and $\beta = \left(a + b, b^{n-2}\right)$. Then,

$$\overline{s}_\alpha = (x_1 x_2 \cdots x_n)^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right);$$

$$\overline{s}_\beta = (x_1 x_2 \cdots x_n)^b \cdot \left(h_b^- h_a^+ - h_{b-1}^- h_{a-1}^+\right).$$

- Thus, if we "know how to multiply by" h_k^- and h_k^+, then we "know how to multiply by" $\overline{s}_\alpha$ and $\overline{s}_\beta$.

Multiplying by h_k^+: the h-Pieri rule, 1

- **Theorem (h-Pieri rule):** Let λ be a partition. Let $k \in \mathbb{Z}$. Then,

$$h_k \cdot s_\lambda = \sum_{\substack{\mu \text{ is a partition;} \\ |\mu|-|\lambda|=k; \\ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots}} s_\mu .$$

Here:

  - We let $h_k = 0$ if $k < 0$. (And we recall that $h_0 = 1$.)
  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any partition κ.
  - The i-th entry of a partition κ is denoted by κ_i.

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- **Theorem (h-Pieri rule):** Let λ be a partition. Let $k \in \mathbb{Z}$. Then,

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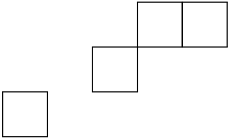
  Here:
  - We let $h_k = 0$ if $k < 0$. (And we recall that $h_0 = 1$.)
  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any partition κ.
  - The i-th entry of a partition κ is denoted by κ_i.
- Note that the chain of inequalities $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots$ is saying that the diagram μ/λ is a *horizontal strip* (i.e., has no two cells in the same column). For example,



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  - We let $h_k = 0$ if $k < 0$. (And we recall that $h_0 = 1$.)
  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any partition κ.
  - The i-th entry of a partition κ is denoted by κ_i.
- The Pieri rule is actually a particular case of the Littlewood–Richardson rule (exercise!).

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  Here:
  - We let $h_k = 0$ if $k < 0$. (And we recall that $h_0 = 1$.)
  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any partition κ.
  - The i-th entry of a partition κ is denoted by κ_i.
- By evaluating both sides at $x_1, x_2, \ldots, x_n$ (and recalling that $s_\mu(x_1, x_2, \ldots, x_n) = 0$ whenever μ is a partition with more than n nonzero entries), we obtain:

Multiplying by h_k^+: the h-Pieri rule, 2

- **Theorem (h^+-Pieri rule for symmetric polynomials):** Let $\lambda \in \operatorname{Par}[n]$. Let $k \in \mathbb{Z}$. Then,

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \operatorname{Par}[n]; \\ |\mu| - |\lambda| = k; \\ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n}} \overline{s}_\mu .$$

  Here:
  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any n-tuple κ.
  - The i-th entry of an n-tuple κ is denoted by κ_i.

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  Here:
  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any n-tuple κ.
  - The i-th entry of an n-tuple κ is denoted by κ_i.
- We can easily extend this from $\operatorname{Par}[n]$ to $\{\text{snakes}\}$, and obtain the following:

- **Theorem (h^+-Pieri rule for Laurent polynomials):** Let $\lambda \in \{\text{snakes}\}$. Let $k \in \mathbb{Z}$. Then,

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \{\text{snakes}\}; \\ |\mu| - |\lambda| = k; \\ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n}} \overline{s}_\mu .$$

  Here:

  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any n-tuple κ.
  - The i-th entry of an n-tuple κ is denoted by κ_i.

Multiplying by h_k^+: the h-Pieri rule, 3

- **Theorem (h^+-Pieri rule for Laurent polynomials):** Let $\lambda \in \{\text{snakes}\}$. Let $k \in \mathbb{Z}$. Then,

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \{\text{snakes}\}; \\ |\mu| - |\lambda| = k; \\ \mu \rightharpoonup \lambda}} \overline{s}_\mu .$$

Here:

  - We let $|\kappa|$ denote the *size* (i.e., the sum of the entries) of any n-tuple κ.
  - The i-th entry of an n-tuple κ is denoted by κ_i.
  - The notation $\mu \rightharpoonup \lambda$ stands for $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n$.
    (Note that if $\lambda, \mu \in \mathbb{Z}^n$ satisfy $\mu \rightharpoonup \lambda$, then λ and μ are snakes automatically.)

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    (Note that if $\lambda, \mu \in \mathbb{Z}^n$ satisfy $\mu \rightharpoonup \lambda$, then λ and μ are snakes automatically.)
- So we know how to multiply $\overline{s}_\lambda$ by h_k^+. What about h_k^- ?

Multiplying by h_k^-: the reversed h-Pieri rule

- **Theorem (h^--Pieri rule for Laurent polynomials):** Let $\lambda \in \{\text{snakes}\}$. Let $k \in \mathbb{Z}$. Then,

$$h_k^- \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \{\text{snakes}\};\\ |\lambda| - |\mu| = k;\\ \lambda \rightharpoonup \mu}} \overline{s}_\mu .$$

- **Theorem (h^--Pieri rule for Laurent polynomials):** Let $\lambda \in \{\text{snakes}\}$. Let $k \in \mathbb{Z}$. Then,

$$h_k^- \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \{\text{snakes}\};\\ |\lambda| - |\mu| = k;\\ \lambda \rightharpoonup \mu}} \overline{s}_\mu .$$

- This follows from the h^+-Pieri rule by substituting $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$ for $x_1, x_2, \ldots, x_n$, using the following fact:
  Proposition: For any snake λ, we have

$$\overline{s}_{\lambda^\vee} = \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right).$$

  Here, $\lambda^\vee$ denotes the snake $(-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)$ (formerly denoted by $\lambda^{\vee 0}$, but now defined for any snake λ).

Multiplying by h_k^-: the reversed h-Pieri rule

- **Theorem (h^--Pieri rule for Laurent polynomials):** Let $\lambda \in \{\text{snakes}\}$. Let $k \in \mathbb{Z}$. Then,

$$h_k^- \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \{\text{snakes}\}; \\ |\lambda| - |\mu| = k; \\ \lambda \rightharpoonup \mu}} \overline{s}_\mu .$$

- This follows from the h^+-Pieri rule by substituting $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$ for $x_1, x_2, \ldots, x_n$, using the following fact:
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Here, $\lambda^\vee$ denotes the snake $(-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)$ (formerly denoted by $\lambda^{\vee 0}$, but now defined for any snake λ).
- So we now know how to multiply $\overline{s}_\lambda$ by h_k^-.

- A consequence of the above:
  Corollary: Let μ be a snake. Let $a, b \in \mathbb{Z}$. Then,
  $$h_a^- h_b^+ \overline{s}_\mu = \sum_{\gamma \text{ is a snake}} |R_{\mu,a,b}(\gamma)| \, \overline{s}_\gamma,$$
  where $R_{\mu,a,b}(\gamma)$ is the set of all snakes ν satisfying
  $$\mu \rightharpoonup \nu \quad \text{and} \quad |\mu| - |\nu| = a \quad \text{and} \quad \gamma \rightharpoonup \nu \quad \text{and} \quad |\gamma| - |\nu| = b.$$

Back to the conjecture

- A consequence of the above:
  Corollary: Let μ be a snake. Let $a, b \in \mathbb{Z}$. Then,

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- **Corollary:** Let $\nu \in \operatorname{Par}[n]$. Let $a, b \geq 0$. Define the partition $\alpha = \left(a + b, a^{n-2}\right)$. Then, every $\lambda \in \mathbb{Z}^n$ satisfies

$$c_{\alpha,\nu}^{\lambda} = \left| R_{\nu,a,b}(\lambda - a) \right| - \left| R_{\nu,a-1,b-1}(\lambda - a) \right|.$$

  Here, we understand $c_{\alpha,\nu}^{\lambda}$ to mean 0 if λ is not a partition (i.e., not a snake with all entries nonnegative).

- A consequence of the above:
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- **Corollary:** Let $\nu \in \operatorname{Par}[n]$. Let $a, b \geq 0$. Define the partition $\alpha = \left(a+b, a^{n-2}\right)$. Then, every $\lambda \in \mathbb{Z}^n$ satisfies
  $$c_{\alpha,\nu}^{\lambda} = |R_{\nu,a,b}(\lambda - a)| - |R_{\nu,a-1,b-1}(\lambda - a)|.$$
  Here, we understand $c_{\alpha,\nu}^{\lambda}$ to mean 0 if λ is not a partition (i.e., not a snake with all entries nonnegative).
- Recall that we want a bijection $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ such that
  $$c_{\alpha,\mu}^{\lambda} = c_{\beta,\mu}^{\varphi(\lambda)} \qquad \text{for each } \lambda \in \operatorname{Par}[n].$$

Closing in on the bijection, 1

- So we want a bijection $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

$$|R_{\mu,a,b}(\lambda - a)| - |R_{\mu,a-1,b-1}(\lambda - a)|$$
$$= |R_{\mu,b,a}(\varphi(\lambda) - b)| - |R_{\mu,b-1,a-1}(\varphi(\lambda) - b)|$$

for all $\lambda \in \mathbb{Z}^n$.

- So we want a bijection $\mathbf{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

$$\begin{aligned} &|R_{\mu,a,b}(\gamma)| - |R_{\mu,a-1,b-1}(\gamma)| \\ &= |R_{\mu,b,a}(\mathbf{f}(\gamma))| - |R_{\mu,b-1,a-1}(\mathbf{f}(\gamma))| \end{aligned}$$

for all $\gamma \in \mathbb{Z}^n$.

Closing in on the bijection, 1

- So we want a bijection $\mathbf{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

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  for all $\gamma \in \mathbb{Z}^n$.
- It clearly suffices to find a bijection $\mathbf{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

$$|R_{\mu,a,b}(\gamma)| = |R_{\mu,b,a}(\mathbf{f}(\gamma))| \qquad \text{for all } \gamma \in \mathbb{Z}^n,$$

  as long as this $\mathbf{f}$ is independent on a and b.

Closing in on the bijection, 1

- So we want a bijection $\mathbf{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

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- It clearly suffices to find a bijection $\mathbf{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

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  as long as this $\mathbf{f}$ is independent on a and b.
- In other words, if $\mathbf{f}(\gamma) = \eta$, then we want

$$\left|R_{\mu,a,b}(\gamma)\right| = \left|R_{\mu,b,a}(\eta)\right|.$$

Closing in on the bijection, 2

- In other words, if $\mathbf{f}(\gamma) = \eta$, then we want there to be a bijection from the snakes ν satisfying

  $$\mu \rightharpoonup \nu \quad \text{and} \quad |\mu| - |\nu| = a \quad \text{and} \quad \gamma \rightharpoonup \nu \quad \text{and} \quad |\gamma| - |\nu| = b$$

  to the snakes ζ satisfying

  $$\mu \rightharpoonup \zeta \quad \text{and} \quad |\mu| - |\zeta| = b \quad \text{and} \quad \eta \rightharpoonup \zeta \quad \text{and} \quad |\eta| - |\zeta| = a.$$

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- Forget at first about the size conditions ($|\mu| - |\nu| = a$, etc.). Then the former snakes satisfy

$$\begin{aligned}
& \mu \rightharpoonup \nu \quad \text{and} \quad \gamma \rightharpoonup \nu \\
\iff \quad & (\mu_i \geq \nu_i \text{ for all } i \leq n) \wedge (\nu_i \geq \mu_{i+1} \text{ for all } i < n) \\
& \wedge (\gamma_i \geq \nu_i \text{ for all } i \leq n) \wedge (\gamma_i \geq \gamma_{i+1} \text{ for all } i < n) \\
\iff \quad & (\min\{\mu_i, \gamma_i\} \geq \nu_i \text{ for all } i \leq n) \\
& \wedge (\nu_i \geq \max\{\mu_{i+1}, \gamma_{i+1}\} \text{ for all } i < n) \\
\iff \quad & (\nu_i \in [\max\{\mu_{i+1}, \gamma_{i+1}\}, \min\{\mu_i, \gamma_i\}] \text{ for all } i < n) \\
& \wedge (\min\{\mu_n, \gamma_n\} \geq \nu_n).
\end{aligned}$$

Closing in on the bijection, 3

- Compare the condition

$$\nu_i \in [\max\{\mu_{i+1}, \gamma_{i+1}\}, \min\{\mu_i, \gamma_i\}] \text{ for all } i < n$$

with the analogous condition

$$\zeta_i \in [\max\{\mu_{i+1}, \eta_{i+1}\}, \min\{\mu_i, \eta_i\}] \text{ for all } i < n$$

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on ζ.

- It is thus reasonable to hope for

$$\min\left\{\mu_i, \gamma_i\right\} - \max\left\{\mu_{i+1}, \gamma_{i+1}\right\} = \min\left\{\mu_i, \eta_i\right\} - \max\left\{\mu_{i+1}, \eta_{i+1}\right\}$$

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for all $i < n$.

- Size conditions also suggest that we should have

$$|\eta| - |\mu| = |\mu| - |\gamma| .$$

Closing in on the bijection, 3

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- It is thus reasonable to hope for

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for all $i < n$.

- Size conditions also suggest that we should have

$$|\eta| - |\mu| = |\mu| - |\gamma|.$$

- These conditions do not suffice to determine $\mathbf{f}(\gamma) = \eta$ (nor probably to guarantee $|R_{\mu,a,b}(\gamma)| = |R_{\mu,b,a}(\eta)|$), but let's see what they tell us.

Closing in on the bijection: the case $n = 3$

- Let $n = 3$. We want $\mathbf{f}(\gamma) = \eta$ to satisfy

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- This is a system of equations that only involves the operations $+$, $-$ and min. (Recall: $2a = a + a$.)
- There is a trick for studying such systems: **detropicalization**.

Detropicalization in a nutshell

- A *semifield* is defined in the same way as a field, but
  - additive inverses and a zero element are **not** required, and
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  The same construction works for any totally ordered abelian group instead of $\mathbb{Z}$.

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- If you see a system of equations using only $+$ and min, you can thus
  - view it as a system of **polynomial** equations over $\mathbb{Z}_{\text{trop}}$;
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  This strategy is known as *detropicalization*.
- It is particularly useful if you just want **one** solution (rather than all of them). Often, solutions over $\mathbb{Q}_+$ are unique, while those over the min tropical semifield are not.

- Recall our system

$$\min\{\mu_1, \gamma_1\} + \min\{-\mu_2, -\gamma_2\} = \min\{\mu_1, \eta_1\} + \min\{-\mu_2, -\eta_2\};$$
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- Detropicalization transforms this into

$$(\mu_1 + \gamma_1)\left(\frac{1}{\mu_2} + \frac{1}{\gamma_2}\right) = (\mu_1 + \eta_1)\left(\frac{1}{\mu_2} + \frac{1}{\eta_2}\right);$$
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$$(\gamma_1\gamma_2\gamma_3)(\eta_1\eta_2\eta_3) = (\mu_1\mu_2\mu_3)^2.$$

- So we now need to solve the system

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Detropicalizing our system ($n = 3$), 2

- Let us rename μ, γ, η as u, x, y. Then, this becomes

$$(u_1 + x_1)\left(\frac{1}{u_2} + \frac{1}{x_2}\right) = (u_1 + y_1)\left(\frac{1}{u_2} + \frac{1}{y_2}\right);$$
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- This is a system of polynomial equations, so we can give it to a computer. The answer is:

Solving the detropicalized system ($n = 3$)

- *Solution 1:*

$$y_1 = \frac{u_1\left(u_1u_2u_3 + x_1u_2u_3 + x_1x_2u_3 + x_1x_2x_3\right)}{u_1x_2u_3 - x_1x_2x_3},$$

$$y_2 = \frac{-u_1u_2u_3}{x_1x_3},$$

$$y_3 = \frac{u_2u_3\left(x_1x_3 - u_1u_3\right)}{u_1u_2u_3 + x_1u_2u_3 + x_1x_2u_3 + x_1x_2x_3}.$$

- *Solution 2:*

$$y_1 = \frac{u_1u_3\left(u_1u_2 + x_1u_2 + x_1x_2\right)}{x_2\left(u_1u_3 + u_1x_3 + x_1x_3\right)},$$

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- Solution 1 is useless, since we want $y_1, y_2, y_3 \in \mathbb{Q}_+$.

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- But Solution 2 looks promising. Note in particular the (unexpected) **cyclic symmetry**!

- Reverse-engineering Solution 2, we come up with the following **Definition:** Let $\mathbb{K}$ be a semifield, let $n \geq 1$, and let $u \in \mathbb{K}^n$. We define a map $\mathbf{f} : \mathbb{K}^n \to \mathbb{K}^n$ as follows: Let $x \in \mathbb{K}^n$ be an n-tuple. For each $j \in \mathbb{Z}$ and $r \geq 0$, define an element $t_{r,j} \in \mathbb{K}$ by

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(Here and in the following, all indices are cyclic modulo n.)
Define $y \in \mathbb{K}^n$ by setting

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- Note that $\mathbf{f}$ depends on u (whence I call it $\mathbf{f}_u$ in the paper).

The map f: main properties

- **Theorem.** Let $\mathbb{K}$ be a semifield, $n \geq 1$ and $u \in \mathbb{K}^n$. Then:
  (a) The map $\mathbf{f}$ is an involution (i.e., we have $\mathbf{f} \circ \mathbf{f} = \mathrm{id}$).

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  (c) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}(x)$. Then,

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right)$$

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  (d) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}(x)$. Then,
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$$\prod_{i=1}^{n} \frac{u_i + x_i}{x_i} = \prod_{i=1}^{n} \frac{u_i + y_i}{u_i}.$$
- In short: $\mathbf{f}(x)$ solves our system and more. (Note that the $i = n$ case of part **(c)** is not part of our original system!)

- **Theorem.** Let $\mathbb{K}$ be a semifield, $n \geq 1$ and $u \in \mathbb{K}^n$. Then:
  (a) The map $\mathbf{f}$ is an involution (i.e., we have $\mathbf{f} \circ \mathbf{f} = \mathrm{id}$).
  (b) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}(x)$. Then,

$$(y_1 y_2 \cdots y_n) \cdot (x_1 x_2 \cdots x_n) = (u_1 u_2 \cdots u_n)^2 .$$

  (c) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}(x)$. Then,

$$(u_i + x_i) \left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}} \right) = (u_i + y_i) \left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}} \right)$$

  for each $i \in \mathbb{Z}$. (Recall that indices are cyclic modulo n.)
  (d) Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}(x)$. Then,

$$\prod_{i=1}^{n} \frac{u_i + x_i}{x_i} = \prod_{i=1}^{n} \frac{u_i + y_i}{u_i}.$$

- The proof is heavily computational but not too hard (various auxiliary identities had to be discovered).

- Recall that we were looking for a bijection $\mathbf{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ (independent on a and b) such that

$$|R_{\mu,a,b}(\gamma)| = |R_{\mu,b,a}(\mathbf{f}(\gamma))| \qquad \text{for all } \gamma \in \mathbb{Z}^n.$$

Back to snakes

- Recall that we were looking for a bijection $\mathbf{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ (independent on a and b) such that

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- The map $\mathbf{f}$ constructed above, applied to $\mathbb{K} = \mathbb{Z}_{\text{trop}}$ and $u = (\mu_1, \mu_2, \ldots, \mu_n)$, does the trick. (This is not hard to prove using the above Theorem.)

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- The map $\mathbf{f}$ constructed above, applied to $\mathbb{K} = \mathbb{Z}_{\text{trop}}$ and $u = (\mu_1, \mu_2, \ldots, \mu_n)$, does the trick. (This is not hard to prove using the above Theorem.)
- Shifting by a and b thus produces the bijection φ needed for the Pelletier–Ressayre conjecture. Explicitly:

- **Theorem (G., 2020):** Assume that $n \geq 2$. Let $a, b \geq 0$, and set $\alpha = \left(a+b, a^{n-2}\right)$ and $\beta = \left(a+b, b^{n-2}\right)$.
  Fix any partition $\mu \in \operatorname{Par}[n]$.
  Define a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ as follows:
  Let $\omega \in \mathbb{Z}^n$. Set $\nu = \omega - a \in \mathbb{Z}^n$. For each $j \in \mathbb{Z}$, set

$$\tau_j = \min \{ (\nu_{j+1} + \nu_{j+2} + \cdots + \nu_{j+k}) \\ + (\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+n-1}) \\ \mid k \in \{0, 1, \ldots, n-1\} \},$$

  where (unusually for partitions!) all indices are cyclic modulo n.
  Define an n-tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{Z}^n$ by setting

$$\eta_i = \mu_i + (\mu_{i-1} + \tau_{i-1}) - (\nu_{i+1} + \tau_{i+1}) \qquad \text{for each } i.$$

  Let $\varphi(\omega)$ be the n-tuple $\eta + b \in \mathbb{Z}^n$. Thus, we have defined a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$.

The Pelletier–Ressayre hidden symmetry, 2

- **Theorem (cont'd):** Then:
  (a) The map φ is a bijection.
  (b) We have

$$c^{\omega}_{\alpha,\mu} = c^{\varphi(\omega)}_{\beta,\mu} \qquad \text{for each } \omega \in \mathbb{Z}^n.$$

  Here, we are using the convention that every n-tuple $\omega \in \mathbb{Z}^n$ that is not a partition satisfies $c^{\omega}_{\alpha,\mu} = 0$ and $c^{\omega}_{\beta,\mu} = 0$.

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- **Question:** Can φ be written as a composition of "toggles" (i.e., "local" transformations, each affecting only one entry of the tuple)?

Uniqueness questions, 1

- **Question:** Given a semifield $\mathbb{K}$ and $n \geq 2$ and $u \in \mathbb{K}^n$. Assume that $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ satisfy

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- Yes if $\mathbb{K} = \mathbb{Q}_+$. (Nice exercise!)
- No if $\mathbb{K} = \mathbb{Z}_{\text{trop}}$.
- Thus, detropicalization has made the solution unique by removing the "extraneous" solutions.

Thank you

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- **Joscha Diehl** for the invitation.
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