A representation-theoretical solution to MathOverflow question #88399

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version 2 (4 December 2013)

Sorry for hasty writing. Please let me know about any mistakes or unclarities (A@B.com with A=darijgrinberg and B=gmail).

§1. Statement of the problem

Let $n \in \mathbb{N}$.

For every $w \in S_n$, let $\sigma(w)$ denote the number of cycles in the cycle decomposition of the permutation w (this includes cycles consisting of one element).

We can consider the matrix $\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}$; this is a matrix over the polynomial ring $\mathbb{Q}[x]$, whose rows and whose columns are indexed by the elements of S_n. (So this is a matrix with $n!$ rows and $n!$ columns, although there is no explicit ordering on the set of rows/columns given.)

The claim of MathOverflow question #88399 is:

Theorem 1. The polynomial

$$\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right) \in \mathbb{Q}[x]$$

factors into linear factors of the form $x-\ell$ with $\ell \in \{-n+1, -n+2, ..., n-1\}$.

Before we head to the proof of this theorem, let us show some examples:

Example. If $n = 1$, then the matrix $\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}$ has only one row and one column, and its only entry is x. Its determinant thus is x, which is in agreement with Theorem 1.

If $n = 2$, then the matrix $\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}$ has two rows and two columns. Picking a reasonable ordering on S_n, we can represent it as the 2×2-matrix $\begin{pmatrix} x^2 & x \\ x & x^2 \end{pmatrix}$, which has determinant $x^2(x-1)(x+1)$.

If $n = 3$, then the matrix $\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}$ can be represented (by picking an ordering on S_n) by the 6×6-matrix

$$\begin{pmatrix} x^3 & x^2 & x^2 & x & x & x^2 \\ x^2 & x^3 & x & x^2 & x^2 & x \\ x^2 & x & x^3 & x^2 & x^2 & x \\ x & x^2 & x^2 & x^3 & x & x^2 \\ x & x^2 & x^2 & x & x^3 & x^2 \\ x^2 & x & x & x^2 & x^2 & x^3 \end{pmatrix},$$

and thus has determinant $x^6 (x-2)(x+2)(x-1)^5 (x+1)^5$. This, again, matches the claim of Theorem 1.

For $n = 4$, we have
$\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right) = (x-3)(x+3)(x-2)^{10}(x+2)^{10}(x-1)^{23}(x+1)^{23}x^{28}$.

Exercise 1. Prove that the polynomial $\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right)$ is even (that is, a polynomial in x^2) for every $n \geq 2$. (See the end of this note for a hint.)

§2. Reduction to representation theory

Let us first reduce Theorem 1 to a representation-theoretical statement:

For any finite group G, let $\operatorname{Irrep} G$ denote a set of representatives of all irreducible representations of G over $\mathbb{C}$ modulo isomorphism.[1]

From the theory of group determinants (more precisely, the results of [1], or the proof of Theorem 4.7 in [2]), we know that if G is a finite group, and X_g is an indeterminate[2] for every $g \in G$, then the matrix $\left(X_{gh^{-1}}\right)_{g,h\in G}$ (both rows and columns of this matrix are indexed by elements of G) has determinant

$$\det\left(\left(X_{gh^{-1}}\right)_{g,h\in G}\right) = \prod_{\rho\in\operatorname{Irrep} G}\left(\det\left(\sum_{g\in G}\rho(g)\,X_g\right)^{\dim\rho}\right).$$

Applying this to $G = S_n$ and evaluating this polynomial identity at $X_g = x^{\sigma(g)}$, we obtain

$$\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right) = \prod_{\rho\in\operatorname{Irrep} S_n}\left(\det\left(\sum_{g\in S_n}\rho(g)\,x^{\sigma(g)}\right)^{\dim\rho}\right). \qquad (1)$$

Hence, in order to show that the polynomial $\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right) \in \mathbb{Q}[x]$ factors into linear factors of the form $x-\ell$ with $\ell \in \{-n+1, -n+2, ..., n-1\}$, it is enough to prove that, for every irreducible representation ρ of S_n over $\mathbb{C}$,

[1] *Remark.* We are considering irreducible representations over $\mathbb{C}$ here for simplicity, but actually the argument works more generally: We can replace $\mathbb{C}$ by any field $\mathbb{K}$ of characteristic 0 such that the group algebra $\mathbb{K}[G]$ factors into a direct product of matrix rings over $\mathbb{K}$. In particular, the algebraic closure of $\mathbb{Q}$ does the trick. In the case $G = S_n$ (this is the case we are going to consider!), it is known that **any** field of characteristic 0 can be taken as $\mathbb{K}$, because the Specht modules are defined over $\mathbb{Q}$ and thus provide a factorization of the group algebra $\mathbb{K}[G]$ into a direct product of matrix rings over $\mathbb{K}$ for any field $\mathbb{K}$ of characteristic 0. See any good text on representation theory of S_n for details (the main reason for this to work is Corollary 4.38 of [2]).

[2] Distinct indeterminates are presumed to commute.

the polynomial $\det\left(\sum_{g\in S_n}\rho(g)\,x^{\sigma(g)}\right)$ factors into linear factors of the form $x-\ell$ with $\ell\in\{-n+1,-n+2,...,n-1\}$.

We are going to show something better:

Theorem 2. Let $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)$ be a partition of n. Let m_λ be the number of nonzero parts of the partition λ. Let ρ_λ be the irreducible representation of S_n over $\mathbb{C}$ corresponding to the partition λ. Then,

$$\sum_{g\in S_n}\rho_\lambda(g)\,x^{\sigma(g)}=\frac{n!}{\dim\rho}\prod_{1\leq i<j\leq m_\lambda}\frac{\lambda_i-\lambda_j+j-i}{j-i}\cdot\prod_{i=1}^{m_\lambda}\frac{\dbinom{x+\lambda_i-i}{\lambda_i}}{\dbinom{\lambda_i+m_\lambda-i}{m_\lambda-i}}\cdot\operatorname{id}_{\rho_\lambda}. \tag{2}$$

Let us first see how Theorem 1 follows from Theorem 2:

Proof of Theorem 1. For every partition λ of n, let us denote by ρ_λ the irreducible representation of S_n over $\mathbb{C}$ corresponding to λ, and let us denote by m_λ the number of nonzero parts of the partition λ. It is known that the isomorphism classes of irreducible representations of S_n over $\mathbb{C}$ are in 1-to-1 correspondence with the partitions of n, and this correspondence sends every partition λ to the representation ρ_λ. Thus,

$$\begin{aligned}
&\prod_{\rho\in\operatorname{Irrep}S_n}\left(\det\left(\sum_{g\in S_n}\rho(g)\,x^{\sigma(g)}\right)^{\dim\rho}\right)\\
&=\prod_{\lambda\text{ partition of }n}\left(\det\left(\sum_{g\in S_n}\rho_\lambda(g)\,x^{\sigma(g)}\right)^{\dim\rho_\lambda}\right)\\
&=\prod_{\lambda\text{ partition of }n}\left(\left(\frac{n!}{\dim\rho}\prod_{1\leq i<j\leq m_\lambda}\frac{\lambda_i-\lambda_j+j-i}{j-i}\cdot\prod_{i=1}^{m_\lambda}\frac{\dbinom{x+\lambda_i-i}{\lambda_i}}{\dbinom{\lambda_i+m_\lambda-i}{m_\lambda-i}}\cdot\operatorname{id}_{\rho_\lambda}\right)^{\dim\rho_\lambda}\right)\\
&\qquad(\text{by }(2)).
\end{aligned}$$

Combined with (1), this yields

$$\begin{aligned}
&\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right)\\
&=\prod_{\lambda\text{ partition of }n}\left(\left(\frac{n!}{\dim\rho}\prod_{1\leq i<j\leq m_\lambda}\frac{\lambda_i-\lambda_j+j-i}{j-i}\cdot\prod_{i=1}^{m_\lambda}\frac{\dbinom{x+\lambda_i-i}{\lambda_i}}{\dbinom{\lambda_i+m_\lambda-i}{m_\lambda-i}}\cdot\operatorname{id}_{\rho_\lambda}\right)^{\dim\rho_\lambda}\right).
\end{aligned}$$

Now, the right hand side of this equation is clearly a polynomial in x which factors into a product of a constant and linear factors. All of the linear factors have the form $x + \lambda_i - i - \alpha$ for $\alpha \in \{0, 1, ..., \lambda_i - 1\}$ for various partitions λ of n and various $i \in \{1, 2, ..., m_\lambda\}$. [3] By very simple combinatorics, it is easy to see that each of these factors has the form $x - \ell$ for some $\ell \in \{-n+1, -n+2, ..., n-1\}$. Thus, the polynomial $\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right) \in \mathbb{Q}[x]$ factors into a product of a constant and linear factors of the form $x-\ell$ with $\ell \in \{-n+1, -n+2, ..., n-1\}$. Moreover, the constant is 1 because the polynomial $\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right)$ is monic[4]. Hence, the polynomial $\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right) \in \mathbb{Q}[x]$ factors into linear factors of the form $x-\ell$ with $\ell \in \{-n+1, -n+2, ..., n-1\}$. Thus, Theorem 1 is proven (using Theorem 2).

§3. Proof of Theorem 2

Proof of Theorem 2. First of all, (2) is a polynomial identity in x. Hence, we can WLOG assume that x is not a polynomial indeterminate in $\mathbb{Q}[x]$, but an integer greater than n (because if a polynomial identity over $\mathbb{Q}$ holds for infinitely many integers, then it must always hold). Assume this.

Since x is an integer greater than n, we have $x \in \mathbb{N}$. This allows us to find a $\mathbb{Q}$-vector space of dimension x. Let V be such a vector space.

For every S_n-module P, let χ_P denote the character of this module P. Note that every $h \in S_n$ satisfies

$$\chi_{V^{\otimes n}}(h) = x^{\sigma(h)}. \tag{3}$$

[5]

Let L_λ be the representation of $\mathrm{GL}(V)$ corresponding to the partition λ of n. In other words, let L_λ be the image of V under the λ-th Schur functor.

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[3] In fact, the only place where x occurs on the right hand side of this equation is $\binom{x+\lambda_i-i}{\lambda_i}$, and this factors as $\binom{x+\lambda_i-i}{\lambda_i} = \dfrac{(x+\lambda_i-i)(x+\lambda_i-i-1)\ldots(x+\lambda_i-i-(\lambda_i-1))}{\lambda_i!}$.

[4] *Proof.* In order to see this, it is enough to show that when the determinant $\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right)$ is written as a sum over all permutations of the set S_n (nota bene: permutations of S_n, not permutations in S_n), the highest degree of x is contributed by the product of the main diagonal. But this is clear, because the main diagonal of the matrix $\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}$ is filled with $x^{\sigma(\mathrm{id})} = x^n$ terms, while all other entries of the matrix are lower powers of x.

[5] *Proof.* Let $h \in S_n$. Denote the action of h on $V^{\otimes n}$ by $h\mid_{V^{\otimes n}}$. Then, by the definition of a character, $\chi_{V^{\otimes n}}(h) = \mathrm{Tr}\left(h\mid_{V^{\otimes n}}\right)$.

Pick a basis $(e_1, e_2, ..., e_x)$ of V. This basis induces a basis $(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_n})_{(i_1,i_2,...,i_n)\in\{1,2,...,x\}^n}$ of $V^{\otimes n}$. By the definition of the action of

Then, $L_\lambda = \operatorname{Hom}_{\mathbb{Q}[S_n]}(\rho_\lambda, V^{\otimes n})$ (by one of the definitions of Schur functors), so that

$$\begin{aligned}
\dim L_\lambda &= \dim\left(\operatorname{Hom}_{\mathbb{Q}[S_n]}\left(\rho_\lambda, V^{\otimes n}\right)\right) = \left\langle \chi_{V^{\otimes n}}, \chi_{\rho_\lambda}\right\rangle \\
&\qquad \left(\text{by Theorem 3.8 of [2], applied to } V = V^{\otimes n} \text{ and } W = \rho_\lambda\right) \\
&= \underbrace{\frac{1}{|S_n|}}_{=\frac{1}{n!}} \sum_{g\in S_n} \underbrace{\chi_{\rho_\lambda}(g)}_{=\operatorname{Tr}(\rho_\lambda(g))} \underbrace{\chi_{V^{\otimes n}}\left(g^{-1}\right)}_{\substack{=x^{\sigma\left(g^{-1}\right)} \\ (\text{by } (3))}} \\
&\qquad (\text{by one of the definitions of the inner product of characters}) \\
&= \frac{1}{n!}\sum_{g\in S_n} \operatorname{Tr}(\rho_\lambda(g))\, x^{\sigma\left(g^{-1}\right)} = \frac{1}{n!}\operatorname{Tr}\left(\sum_{g\in S_n}\rho_\lambda(g)\, x^{\sigma\left(g^{-1}\right)}\right) \\
&= \frac{1}{n!}\operatorname{Tr}\left(\sum_{g\in S_n}\rho_\lambda(g)\, x^{\sigma(g)}\right) \qquad \left(\text{since every } g\in S_n \text{ satisfies } \sigma\left(g^{-1}\right) = \sigma(g)\right).
\end{aligned} \tag{4}$$

S_n on $V^{\otimes n}$, every $(i_1, i_2, ..., i_n) \in \{1, 2, ..., x\}^n$ satisfies

$$h\left(e_{i_1}\otimes e_{i_2}\otimes ...\otimes e_{i_n}\right) = e_{h^{-1}(i_1)}\otimes e_{h^{-1}(i_2)}\otimes ...\otimes e_{h^{-1}(i_n)}.$$

Thus, if $h^{(\times n)}$ denotes the permutation of the set $\{1, 2, ..., x\}^n$ which sends every $(i_1, i_2, ..., i_n) \in \{1, 2, ..., x\}^n$ to $\left(h^{-1}(i_1), h^{-1}(i_2), ..., h^{-1}(i_n)\right)$, then the linear map $h\mid_{V^{\otimes n}}$ is represented by the permutation matrix of the permutation $h^{(\times n)}$ with respect to the basis $\left(e_{i_1}\otimes e_{i_2}\otimes ...\otimes e_{i_n}\right)_{(i_1,i_2,...,i_n)\in\{1,2,...,x\}^n}$ of $V^{\otimes n}$. Hence,

$$\operatorname{Tr}\left(h\mid_{V^{\otimes n}}\right) = \operatorname{Tr}\left(\text{permutation matrix of the permutation } h^{(\times n)}\right) = \left(\text{number of fixed points of } h^{(\times n)}\right)$$

(because the trace of a permutation matrix always equals the number of fixed points of the corresponding permutation). Now, let us count the fixed points of $h^{(\times n)}$.

Clearly, an n-tuple $(i_1, i_2, ..., i_n) \in \{1, 2, ..., x\}^n$ is a fixed point of $h^{(\times n)}$ if and only if every $j \in \{1, 2, ..., n\}$ satisfies $i_j = i_{h^{-1}(j)}$. In other words, an n-tuple $(i_1, i_2, ..., i_n) \in \{1, 2, ..., x\}^n$ is a fixed point of $h^{(\times n)}$ if and only if each pair of elements j and k of $\{1, 2, ..., n\}$ which lie in the same cycle of h satisfies $i_j = i_k$. Hence, if we want to choose a fixed point of $h^{(\times n)}$, we need only to specify, for every cycle c of h, the value of i_j for some element j of this cycle c (which element j we choose doesn't matter). Thus, we have to choose one element of the set $\{1, 2, ..., x\}$ for each cycle of h; these choices are arbitrary and independent, but beside them we have no more freedom. Thus, there is a total of $x^{\sigma(h)}$ ways to choose a fixed point of $h^{(\times n)}$ (because there are $\sigma(h)$ cycles of h, and there are x elements of the set $\{1, 2, ..., x\}$). In other words,

$$x^{\sigma(h)} = \left(\text{number of fixed points of } h^{(\times n)}\right) = \operatorname{Tr}\left(h\mid_{V^{\otimes n}}\right) = \chi_{V^{\otimes n}}(h).$$

This proves (3).

On the other hand, Theorem 4.63 of [2] (the Weyl character formula) yields

$$\begin{aligned}
\dim L_\lambda &= \prod_{1\leq i<j\leq x} \frac{\lambda_i-\lambda_j+j-i}{j-i} \qquad \text{(where } \lambda_\ell \text{ denotes } 0 \text{ for all } \ell > m_\lambda\text{)} \\
&= \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i-\lambda_j+j-i}{j-i} \cdot \underbrace{\prod_{1\leq i\leq m_\lambda<j\leq x}}_{=\prod_{i=1}^{m_\lambda}\prod_{j=m_\lambda+1}^{x}} \underbrace{\frac{\lambda_i-\lambda_j+j-i}{j-i}}_{\substack{=\frac{\lambda_i+j-i}{j-i} \\ \text{(since } m_\lambda<j \text{ yields } \lambda_j=0\text{)}}} \cdot \prod_{m_\lambda\leq i<j\leq x} \underbrace{\frac{\lambda_i-\lambda_j+j-i}{j-i}}_{\substack{=1 \text{ (since } m_\lambda\leq i<j \\ \text{yields that both } \lambda_i \text{ and } \lambda_j \text{ are } 0\text{)}}} \\
&= \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i-\lambda_j+j-i}{j-i} \cdot \prod_{i=1}^{m_\lambda} \underbrace{\prod_{j=m_\lambda+1}^{x} \frac{\lambda_i+j-i}{j-i}}_{\substack{=\frac{\binom{x+\lambda_i-i}{\lambda_i}}{\binom{\lambda_i+m_\lambda-i}{m_\lambda-i}} \\ \text{(this is straightforward to check)}}} \cdot \underbrace{\prod_{m_\lambda\leq i<j\leq x} 1}_{=1} \\
&= \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i-\lambda_j+j-i}{j-i} \cdot \prod_{i=1}^{m_\lambda} \frac{\binom{x+\lambda_i-i}{\lambda_i}}{\binom{\lambda_i+m_\lambda-i}{m_\lambda-i}}.
\end{aligned}$$

Combined with (4), this yields

$$\frac{1}{n!}\operatorname{Tr}\left(\sum_{g\in S_n} \rho_\lambda(g)\, x^{\sigma(g)}\right) = \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i-\lambda_j+j-i}{j-i} \cdot \prod_{i=1}^{m_\lambda} \frac{\binom{x+\lambda_i-i}{\lambda_i}}{\binom{\lambda_i+m_\lambda-i}{m_\lambda-i}},$$

so that

$$\operatorname{Tr}\left(\sum_{g\in S_n} \rho_\lambda(g)\, x^{\sigma(g)}\right) = n! \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i-\lambda_j+j-i}{j-i} \cdot \prod_{i=1}^{m_\lambda} \frac{\binom{x+\lambda_i-i}{\lambda_i}}{\binom{\lambda_i+m_\lambda-i}{m_\lambda-i}}. \quad (5)$$

But $\sum_{g\in S_n} gx^{\sigma(g)}$ is a central element of $\mathbb{Q}[S_n]$ (since the map $S_n \to \mathbb{Q}$, $g \mapsto \sigma(g)$ is a class function), so that $\sum_{g\in S_n} gx^{\sigma(g)}$ acts on any irreducible representation of S_n as a scalar multiple of id (by Schur's lemma). In particular, this yields that $\rho_\lambda\left(\sum_{g\in S_n} gx^{\sigma(g)}\right) = \kappa \cdot \mathrm{id}_{\rho_\lambda}$ for some $\kappa \in \mathbb{C}$ (since ρ_λ is an irreducible representation of S_n). Consider this κ. Then,

$$\sum_{g\in S_n} \rho_\lambda(g)\, x^{\sigma(g)} = \rho_\lambda\left(\sum_{g\in S_n} g x^{\sigma(g)}\right) = \kappa \cdot \operatorname{id}_{\rho_\lambda}, \tag{6}$$

so that

$$\operatorname{Tr}\left(\sum_{g\in S_n} \rho_\lambda(g)\, x^{\sigma(g)}\right) = \operatorname{Tr}\left(\kappa \cdot \operatorname{id}_{\rho_\lambda}\right) = \kappa \cdot \dim \rho_\lambda.$$

Combined with (5), this yields

$$\kappa \cdot \dim \rho_\lambda = n! \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i - \lambda_j + j - i}{j-i} \cdot \prod_{i=1}^{m_\lambda} \frac{\dbinom{x+\lambda_i - i}{\lambda_i}}{\dbinom{\lambda_i + m_\lambda - i}{m_\lambda - i}},$$

so that

$$\kappa = \frac{n!}{\dim \rho} \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i - \lambda_j + j - i}{j-i} \cdot \prod_{i=1}^{m_\lambda} \frac{\dbinom{x+\lambda_i - i}{\lambda_i}}{\dbinom{\lambda_i + m_\lambda - i}{m_\lambda - i}}.$$

Thus, (6) becomes

$$\sum_{g\in S_n} \rho_\lambda(g)\, x^{\sigma(g)} = \frac{n!}{\dim \rho} \prod_{1\leq i<j\leq m_\lambda} \frac{\lambda_i - \lambda_j + j - i}{j-i} \cdot \prod_{i=1}^{m_\lambda} \frac{\dbinom{x+\lambda_i - i}{\lambda_i}}{\dbinom{\lambda_i + m_\lambda - i}{m_\lambda - i}} \cdot \operatorname{id}_{\rho_\lambda}.$$

This proves Theorem 2.

Hints to exercises

Hint to exercise 1: Let $n \geq 2$. Expand $\det\left(\left(x^{\sigma\left(gh^{-1}\right)}\right)_{g,h\in S_n}\right)$ as a product over all permutations of S_n (a total of $(n!)!$ permutations, but you don't have to actually do the computations...). It is clearly enough to show that every such permutation gives rise to a product which simplifies to x^m for some even m. To prove this, show that any permutation $\alpha \in S_n$ satisfies $\operatorname{sign} \alpha = (-1)^{n-\sigma(\alpha)}$.

References

[1] Keith Conrad, *The Origin of Representation Theory.*
`http://www.math.uconn.edu/~kconrad/articles/groupdet.pdf`

[2] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, Elena Yudovina, *Introduction to representation theory*, arXiv:0901.0827v5.
`http://arxiv.org/abs/0901.0827v5`