Zeckendorf family identities generalized

Darij Grinberg

June 11, 2018, long version

Abstract. In [WooZei09], Philip Matchett Wood and Doron Zeilberger have constructed identities for the Fibonacci numbers f_n of the form

$$\begin{aligned}
&1f_n = f_n \text{ for all } n \geq 1;\\
&2f_n = f_{n-2} + f_{n+1} \text{ for all } n \geq 3;\\
&3f_n = f_{n-2} + f_{n+2} \text{ for all } n \geq 3;\\
&4f_n = f_{n-2} + f_n + f_{n+2} \text{ for all } n \geq 3;\\
&\quad \text{etc.};\\
&kf_n = \sum_{s \in S_k} f_{n+s} \text{ for all } n > \max\{-s \mid s \in S_k\},
\end{aligned}$$

where S_k is a fixed "lacunar" set of integers ("lacunar" means that no two elements of this set are consecutive integers) depending only on k. (The condition $n > \max\{-s \mid s \in S_k\}$ is only to make sure that all addends f_{n+s} are well-defined. If the Fibonacci sequence is properly continued to the negative, this condition drops out.)

In this note we prove a generalization of these identities: For any family $(a_1, a_2, \ldots, a_p)$ of integers, there exists one and only one finite lacunar set S of integers such that every n (high enough to make the Fibonacci numbers in the equation below well-defined) satisfies

$$f_{n+a_1} + f_{n+a_2} + \cdots + f_{n+a_p} = \sum_{s \in S} f_{n+s}.$$

The proof uses the Fibonacci-approximating properties of the golden ratio. It would be interesting to find a purely combinatorial proof.

This is a detailed version of my note [Grinbe11]. It contains the proof outlined in [Grinbe11] in much more detail and was written for the purpose of persuading myself that my proofs are correct.

---

1. The main result

The purpose of this note is to establish a generalization of the so-called *Zeckendorf family identities* which were discussed in [WooZei09]. Before we can state it, we need a few definitions:

Definition 1.1. A subset S of $\mathbb{Z}$ is called *lacunar* if it satisfies $(s+1 \notin S$ for every $s \in S)$.

In other words, a subset S of $\mathbb{Z}$ is lacunar if and only if it contains no two consecutive integers.

Definition 1.2. The *Fibonacci sequence* $(f_1, f_2, f_3, \ldots)$ is a sequence of positive integers defined recursively by the initial values $f_1 = 1$ and $f_2 = 1$ and the recurrence relation $(f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{N}$ satisfying $n \geq 3)$.

(Here and in the following, $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$.)

Remark 1.3. Many authors define the Fibonacci sequence slightly differently: They define it as a sequence $(f_0, f_1, f_2, \ldots)$ of nonnegative integers defined recursively by the initial values $f_0 = 0$ and $f_1 = 1$ and the recurrence relation $(f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{N}$ satisfying $n \geq 2)$. Thus, this sequence begins with a 0, unlike the Fibonacci sequence defined in our Definition 1.2. However, starting at its second term $f_1 = 1$, this sequence takes precisely the same values as the Fibonacci sequence defined in our Definition 1.2 (because both sequences satisfy $f_1 = 1$ and $f_2 = 1$, and from here on the recurrence relation ensures that their values remain equal). So it differs from the latter sequence only in the presence of one extra term $f_0 = 0$ at the front.

The Fibonacci sequence is one of the best known integer sequences from combinatorics. It has had conferences, books and a journal devoted to it. By way of example, let us only mention Vorobiev's book [Vorobi02], which is entirely concerned with Fibonacci numbers, and Benjamin's and Quinn's text [BenQui03] on bijective proofs, which includes many identities for Fibonacci numbers.

In [WooZei09], Wood and Zeilberger discuss bijective proofs of the so-called *Zeckendorf family identities*. These identities are a family of identities for Fibonacci numbers (one for each positive integer); the first seven of these identities are

$$\begin{aligned}
1f_n &= f_n \text{ for all } n \geq 1;\\
2f_n &= f_{n-2} + f_{n+1} \text{ for all } n \geq 3;\\
3f_n &= f_{n-2} + f_{n+2} \text{ for all } n \geq 3;\\
4f_n &= f_{n-2} + f_n + f_{n+2} \text{ for all } n \geq 3;\\
5f_n &= f_{n-4} + f_{n-1} + f_{n+3} \text{ for all } n \geq 5;\\
6f_n &= f_{n-4} + f_{n+1} + f_{n+3} \text{ for all } n \geq 5;\\
7f_n &= f_{n-4} + f_{n+4} \text{ for all } n \geq 5.
\end{aligned}$$

In general, for each positive integer k, the k-th Zeckendorf family identity expresses kf_n (for each sufficiently large integer n) as a sum of the form $\sum_{s \in S} f_{n+s}$, where S is a finite lacunar subset of $\mathbb{Z}$. Of course, the subset S does not depend on n.

Our main theorem is the following:

Theorem 1.4 (generalized Zeckendorf family identities)**.** Let T be a finite set, and let a_t be an integer for every $t \in T$.

Then, there exists one and only one finite lacunar subset S of $\mathbb{Z}$ such that[1]

$$\left(\begin{array}{c} \sum_{t \in T} f_{n+a_t} = \sum_{s \in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right) \end{array} \right).$$

Remark 1.5.

1. Theorem 1.4 generalizes the Zeckendorf family identities (which correspond to the case when all a_t are $= 0$).
2. The condition $n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$ in Theorem 1.4 is just a technical condition made in order to ensure that the Fibonacci numbers f_{n+a_t} for all $t \in T$ and f_{n+s} for all $s \in S$ are well-defined. (If we would define the Fibonacci numbers f_n for integers $n \leq 0$ by extending the recurrence relation $f_n = f_{n-1} + f_{n-2}$ "to the left", then we could drop this condition.)

The proof we shall give for Theorem 1.4 is not combinatorial. It will use inequalities and the properties of the golden ratio; in a sense, its underlying ideas come from analysis (although it will not actually use any results from analysis).

2. Basics on the Fibonacci sequence

We begin with some lemmas and notations:

We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ (and not the set $\{1, 2, 3, \ldots\}$, like some other authors do). Also, we denote by $\mathbb{N}_2$ the set $\{2, 3, 4, \ldots\} = \mathbb{N} \setminus \{0, 1\}$.

Also, let $\phi = \dfrac{1+\sqrt{5}}{2}$. This number ϕ is known as the *golden ratio*. We notice that $\phi \approx 1.618\ldots$ and that $\phi^2 = \phi + 1$. Binet's formula states that $f_n = \dfrac{\phi^n - \phi^{-n}}{\sqrt{5}}$ for every positive integer n. (See, e.g., [BenQui03, Identity 240] or [Vorobi02, (1.20)] for proofs of Binet's formula.)

We observe that the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ consists of positive integers (indeed, its two starting values $f_1 = 1$ and $f_2 = 1$ are positive integers, and thus the recurrence relation $f_n = f_{n-1} + f_{n-2}$ clearly ensures that all the following values

[1] Here and in the following, $\max \varnothing$ is understood to be 0.

are also positive integers). Thus, $f_n > 0$ for each positive integer n. Now, for each integer $n \geq 2$, we have $f_{n-1} > 0$ (since the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ consists of positive integers). The recurrence relation of the Fibonacci sequence shows that for each integer $n \geq 2$, we have $f_{n+1} = f_n + f_{n-1} > f_n$ (because $f_{n-1} > 0$), so that $f_n < f_{n+1}$. In other words, $f_2 < f_3 < f_4 < \cdots$. In other words, the Fibonacci sequence is strictly increasing beginning with its second term f_2. Furthermore, $f_1 = 1 = f_2$, so that $f_1 = f_2 < f_2 < f_3 < f_4 < \cdots$. Hence, the Fibonacci sequence is weakly increasing.

We recall some basic and well-known facts about the Fibonacci sequence:

Lemma 2.1. Let S be a finite lacunar subset of $\mathbb{N}_2$. Then, $\sum\limits_{t \in S} f_t < f_{\max S + 1}$.

Proof of Lemma 2.1. We WLOG assume that S is nonempty (since otherwise, Lemma 2.1 follows easily from our convention that $\max \varnothing = 0$).

Every $t \in \mathbb{N}_2$ satisfies $f_{t+1} = f_t + f_{t-1}$ (due to the relation $f_n = f_{n-1} + f_{n-2}$, applied to $n = t+1$), so that

$$f_t = f_{t+1} - f_{t-1}. \qquad (1)$$

Let us write the set S in the form $\{s_1, s_2, \ldots, s_k\}$, where $s_1 < s_2 < \cdots < s_k$. Then, $\sum\limits_{t \in S} f_t = \sum\limits_{i=1}^{k} f_{s_i}$ and $s_k = \max S$.

Notice that $k \geq 1$ (since S is nonempty). From $s_1 \in \{s_1, s_2, \ldots, s_k\} = S \subseteq \mathbb{N}_2$, we obtain $s_1 \geq 2$ and thus $s_1 - 1 \geq 1$. Hence, $f_{s_1 - 1} > 0$.

On the other hand, every $i \in \{1, 2, \ldots, k-1\}$ satisfies $s_i + 1 \leq s_{i+1} - 1$ [2], so that

$$f_{s_i + 1} \leq f_{s_{i+1} - 1} \qquad (2)$$

[2]*Proof.* Let $i \in \{1, 2, \ldots, k-1\}$. Thus, both i and $i+1$ belong to $\{1, 2, \ldots, k\}$.

The set S is lacunar, and thus $s + 1 \notin S$ for every $s \in S$. Applying this to $s = s_i$, we get $s_i + 1 \notin S$ (since $s_i \in \{s_1, s_2, \ldots, s_k\} = S$), so that $s_i + 1 \neq s_{i+1}$ (since $s_{i+1} \in \{s_1, s_2, \ldots, s_k\} = S$).

Since $s_1 < s_2 < \cdots < s_k$, we have $s_i < s_{i+1}$, so that $s_i + 1 \leq s_{i+1}$ (because s_i and s_{i+1} are integers). Since $s_i + 1 \neq s_{i+1}$, this becomes $s_i + 1 < s_{i+1}$, so that $s_i + 1 \leq s_{i+1} - 1$ (because $s_i + 1$ and s_{i+1} are integers).

(since the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is weakly increasing). Thus,

$$\sum_{t\in S} f_t = \sum_{i=1}^{k} \underbrace{f_{s_i}}_{\substack{=f_{s_i+1}-f_{s_i-1} \\ \text{(by (1), applied to } t=s_i)}} = \sum_{i=1}^{k} \left(f_{s_i+1} - f_{s_i-1}\right) = \underbrace{\sum_{i=1}^{k} f_{s_i+1}}_{=\sum\limits_{i=1}^{k-1} f_{s_i+1}+f_{s_k+1}} - \underbrace{\sum_{i=1}^{k} f_{s_i-1}}_{=f_{s_1-1}+\sum\limits_{i=2}^{k} f_{s_i-1}}$$

$$= \left(\sum_{i=1}^{k-1} \underbrace{f_{s_i+1}}_{\substack{\leq f_{s_{i+1}-1} \\ \text{(by (2))}}} + f_{s_k+1}\right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1}\right)$$

$$\leq \left(\sum_{i=1}^{k-1} f_{s_{i+1}-1} + f_{s_k+1}\right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1}\right)$$

$$= \left(\sum_{i=2}^{k} f_{s_i-1} + f_{s_k+1}\right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1}\right)$$

(here, we substituted i for $i+1$ in the first sum)

$$= f_{s_k+1} - f_{s_1-1} < f_{s_k+1} \qquad \left(\text{since } f_{s_1-1} > 0\right)$$
$$= f_{\max S+1}$$

(since $s_k = \max S$). This proves Lemma 2.1. $\square$

Lemma 2.2 (existence part of the Zeckendorf theorem)**.** Let $n \in \mathbb{N}$. Then, there exists a finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t\in T} f_t$.

Proof of Lemma 2.2. We are going to prove Lemma 2.2 by strong induction over n:

Induction base: Let $n = 0$. Then, there exists a finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t\in T} f_t$ (namely, $T = \varnothing$), and thus Lemma 2.2 holds for $n = 0$, and the induction base is completed.

Induction step: Let $\nu \in \mathbb{N}$ be such that $\nu > 0$. Assume that Lemma 2.2 holds for every nonnegative integer $n < \nu$. We must now prove that Lemma 2.2 holds for $n = \nu$.

In fact, we have $\nu > 0$, so that $\nu \geq 1$ (since ν is an integer). Thus, $f_2 = 1 \leq \nu$.

Let t_1 be the maximal integer τ from $\mathbb{N}_2$ satisfying $f_\tau \leq \nu$ [3]. Then, $f_{t_1} \leq \nu$

[3]Such an integer t_1 exists, because of the following:

The Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is strictly increasing beginning with f_2 and therefore unbounded from above (because every strictly increasing sequence of integers is unbounded from above). Hence, "sooner or later" this sequence will surpass any given integer. Thus, in particular, there are **only finitely many** integers τ from $\mathbb{N}_2$ satisfying $f_\tau \leq \nu$.

On the other hand, 2 is an integer τ from $\mathbb{N}_2$ satisfying $f_\tau \leq \nu$ (since $f_2 \leq \nu$). Hence, there exists **at least one** integer τ from $\mathbb{N}_2$ satisfying $f_\tau \leq \nu$. Thus, there exists a **maximal integer** τ from $\mathbb{N}_2$ satisfying $f_\tau \leq \nu$ (because we have already shown that there are **only finitely many** integers τ from $\mathbb{N}_2$ satisfying $f_\tau \leq \nu$). This is what we wanted to prove.

but $f_{t_1+1} > \nu$ (since t_1 is maximal). Hence, $\nu - f_{t_1}$ is a nonnegative integer (since $f_{t_1} \leq \nu$) and $< \nu$ (since $f_{t_1} > 0$). Thus, Lemma 2.2 holds for $n = \nu - f_{t_1}$ (since we assumed that Lemma 2.2 holds for every nonnegative integer $n < \nu$). In other words, there exists a finite lacunar subset T of $\mathbb{N}_2$ such that $\nu - f_{t_1} = \sum\limits_{t \in T} f_t$. We rename this subset T as S (so as not to confuse it with the set T that we want to construct for $n = \nu$). Thus, we have a finite lacunar subset S of $\mathbb{N}_2$ such that $\nu - f_{t_1} = \sum\limits_{t \in S} f_t$.

The relation $f_n = f_{n-1} + f_{n-2}$ (applied to $n = t_1 + 1$) yields $f_{t_1+1} = f_{t_1} + f_{t_1-1}$, so that $f_{t_1+1} - f_{t_1} = f_{t_1-1}$. Now, from $\nu < f_{t_1+1}$, we obtain $\nu - f_{t_1} < f_{t_1+1} - f_{t_1} = f_{t_1-1}$.

Since the set S is lacunar, we know that

$$s + 1 \notin S \qquad \text{for every } s \in S. \tag{3}$$

Now, let $s \in S$. Then, f_s is an addend of the sum $\sum\limits_{t \in S} f_t$. Since f_t is nonnegative for every $t \in S$, we thus have

$$f_s \leq \sum_{t \in S} f_t = \nu - f_{t_1} < f_{t_1-1}$$

and thus $s < t_1 - 1$ (since the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is weakly increasing). This rewrites as $s + 1 < t_1$.

Now, forget that we fixed s. We thus have proven that

$$s + 1 < t_1 \qquad \text{for each } s \in S. \tag{4}$$

Applying this to $s = \max S$, we get $\max S + 1 < t_1$ (since $\max S \in S$) [4]. Hence, $t_1 > \max S + 1 > \max S$, and thus $t_1 \notin S$ (because if an integer x satisfies $x > \max S$, then $x \notin S$). Also, $t_1 + 1 > t_1 > \max S$, so that $t_1 + 1 \notin S$ (because if an integer x satisfies $x > \max S$, then $x \notin S$). Combining $t_1 + 1 \notin S$ with $t_1 + 1 \notin \{t_1\}$ (which is obvious), we obtain $t_1 + 1 \notin S \cup \{t_1\}$.

Again, let $s \in S$. From (4), we obtain $s + 1 < t_1$. Thus, $s + 1 \neq t_1$; in other words $s + 1 \notin \{t_1\}$. But (3) yields $s + 1 \notin S$. Combining this with $s + 1 \notin \{t_1\}$, we get $s + 1 \notin S \cup \{t_1\}$.

Now, forget that we fixed s. We thus have proven that

$$s + 1 \notin S \cup \{t_1\} \qquad \text{for every } s \in S. \tag{5}$$

Hence, $s + 1 \notin S \cup \{t_1\}$ for every $s \in S \cup \{t_1\}$ [5]. In other words, the set $S \cup \{t_1\}$ is lacunar. Denoting this set $S \cup \{t_1\}$ by Q, we thus have shown that the set Q is

[4] Strictly speaking, this argument only works when S is nonempty. But when S is empty, the inequality $\max S + 1 < t_1$ is obvious for a different reason: Namely, in this case, we have $\max S = \max \varnothing = 0$, so that $\max S + 1 = 1 < 2 \leq t_1$ (since $t_1 \in \mathbb{N}_2$).

[5] *Proof.* Let $s \in S \cup \{t_1\}$. We must prove that $s + 1 \notin S \cup \{t_1\}$.

If $s \in S$, then this follows immediately from (5). Thus, we can WLOG assume that $s \notin S$. Assume this. Combining $s \in S \cup \{t_1\}$ with $s \notin S$, we obtain $s \in (S \cup \{t_1\}) \setminus S \subseteq \{t_1\}$. Therefore, $s = t_1$. Hence, $s + 1 = t_1 + 1 \notin S \cup \{t_1\}$. This proves $s + 1 \notin S \cup \{t_1\}$.

lacunar. Clearly, Q is a finite set (since S is finite). Moreover, $\{t_1\} \subseteq \mathbb{N}_2$ (since $t_1 \in \mathbb{N}_2$), so that $Q = \underbrace{S}_{\subseteq \mathbb{N}_2} \cup \underbrace{\{t_1\}}_{\subseteq \mathbb{N}_2} \subseteq \mathbb{N}_2 \cup \mathbb{N}_2 = \mathbb{N}_2$. From $Q = S \cup \{t_1\}$, we obtain $Q \setminus \{t_1\} = (S \cup \{t_1\}) \setminus \{t_1\} = S$ (since $t_1 \notin S$). Also, $t_1 \in \{t_1\} \subseteq S \cup \{t_1\} = Q$. Therefore,

$$\sum_{t \in Q} f_t = \sum_{t \in Q \setminus \{t_1\}} f_t + f_{t_1} = \underbrace{\sum_{t \in S} f_t}_{= \nu - f_{t_1}} + f_{t_1} \qquad (\text{since } Q \setminus \{t_1\} = S)$$
$$= (\nu - f_{t_1}) + f_{t_1} = \nu.$$

Hence, we have found a finite lacunar subset Q of $\mathbb{N}_2$ such that $\nu = \sum_{t \in Q} f_t$. Hence, there exists a finite lacunar subset T of $\mathbb{N}_2$ such that $\nu = \sum_{t \in T} f_t$ (namely, $T = Q$). This proves Lemma 2.2 for the case $n = \nu$. This completes the induction step, and thus the induction proof of Lemma 2.2 is complete. □

Lemma 2.3 (uniqueness part of the Zeckendorf theorem)**.** Let $n \in \mathbb{N}$, and let T and T' be two finite lacunar subsets of $\mathbb{N}_2$ such that $n = \sum_{t \in T} f_t$ and $n = \sum_{t \in T'} f_t$. Then, $T = T'$.

Proof of Lemma 2.3. We are going to prove Lemma 2.3 by strong induction over n:

Induction base: Let $n = 0$. Then, $n = \sum_{t \in T} f_t$ yields $T = \varnothing$ [6]. Similarly, $T' = \varnothing$. Comparing this with $T = \varnothing$, we obtain $T = T'$. Hence, we have shown that Lemma 2.3 holds for $n = 0$, and the induction base is completed.

Induction step: Let $\nu \in \mathbb{N}$ be such that $\nu > 0$. Assume that Lemma 2.3 holds for every nonnegative integer $n < \nu$. We must now prove that Lemma 2.3 holds for $n = \nu$.

So let T and T' be two finite lacunar subsets of $\mathbb{N}_2$ such that $\nu = \sum_{t \in T} f_t$ and $\nu = \sum_{t \in T'} f_t$. Then, we want to prove that $T = T'$.

Since $\sum_{t \in T} f_t = \nu > 0$, we have $T \neq \varnothing$. Thus, $\max T \in T$. Similarly, $\max T' \in T'$.

But $f_{\max T}$ is an addend in the sum $\sum_{t \in T} f_t$ (since $\max T \in T$). Since the Fibonacci numbers f_t are all nonnegative, we thus obtain $f_{\max T} \leq \sum_{t \in T} f_t = \nu = \sum_{t \in T'} f_t < f_{\max T' + 1}$ (by Lemma 2.1, applied to $S = T'$). Hence, $\max T < \max T' + 1$ (since the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is weakly increasing), so that $\max T \leq \max T'$ (since $\max T$ and $\max T'$ are integers). The same argument (with the roles of T

[6] *Proof.* Assume the contrary. Thus, $T \neq \varnothing$. Hence, $\sum_{t \in T} f_t$ is a nonempty sum of positive integers (since the Fibonacci numbers f_t are positive), and thus itself is a positive integer. Thus, $\sum_{t \in T} f_t > 0$. This contradicts $\sum_{t \in T} f_t = n = 0$. This contradiction shows that our assumption was wrong, qed.

and T' interchanged) shows that $\max T' \leq \max T$. Combining this with $\max T \leq \max T'$, we get $\max T = \max T'$. Let μ denote the number $\max T = \max T'$. Then, $\mu = \max T \in T$ and $\mu = \max T' \in T'$. Let $S = T \setminus \{\mu\}$ and $S' = T' \setminus \{\mu\}$. Clearly, S is a finite subset of $\mathbb{N}_2$. Furthermore, S is a subset of the lacunar subset T of $\mathbb{Z}$ (because $S = T \setminus \{\mu\} \subseteq T$), and thus itself is lacunar (since every subset of a lacunar subset of $\mathbb{Z}$ is lacunar). Thus, S is a finite lacunar subset of $\mathbb{N}_2$. Similarly, S' is a finite lacunar subset of $\mathbb{N}_2$. Obviously, $\sum\limits_{t \in S} f_t \geq 0$ (since all f_t are nonnegative). Now,

$$\begin{aligned} \nu - f_\mu &= \sum_{t \in T} f_t - f_\mu = \sum_{t \in T \setminus \{\mu\}} f_t \qquad \text{(since } \mu \in T\text{)} \\ &= \sum_{t \in S} f_t \end{aligned}$$

(since $T \setminus \{\mu\} = S$) and similarly $\nu - f_\mu = \sum\limits_{t \in S'} f_t$. Since $\nu - f_\mu$ is a nonnegative integer (because $\nu - f_\mu = \sum\limits_{t \in S} f_t \geq 0$) and satisfies $\nu - f_\mu < \nu$ (because $f_\mu > 0$), we can thus apply Lemma 2.3 to $\nu - f_\mu$ instead of n and to the lacunar subsets S and S' instead of T and T' (since we assumed that Lemma 2.3 holds for every nonnegative integer $n < \nu$), and we obtain $S = S'$. Now, $S = T \setminus \{\mu\}$ yields $T = S \cup \{\mu\}$ (since $\mu \in T$), and similarly $T' = S' \cup \{\mu\}$, so that $T = \underbrace{S}_{=S'} \cup \{\mu\} = S' \cup \{\mu\} = T'$. This proves Lemma 2.3 for the case $n = \nu$. Thus, the induction step is completed, and the induction proof of Lemma 2.3 is done. □

Theorem 2.4 (Zeckendorf theorem)**.** Let $n \in \mathbb{N}$. Then, there exists one and only one finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$.

Proof of Theorem 2.4. There exists a finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$ (according to Lemma 2.2), and such a subset is unique (because any two such subsets are equal (according to Lemma 2.3)). Thus, there exists **one and only one** finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$. This proves Theorem 2.4. □

Theorem 2.4 is a classical result known as the *Zeckendorf theorem*; it can be found in various places. In particular, the proof given in [Hender16] is rather close to ours. Hoggatt proved a generalization of Theorem 2.4 in [Hoggat72].

Definition 2.5. Let $n \in \mathbb{N}$. Theorem 2.4 shows that there exists one and only one finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$. We will denote this set T by Z_n. Thus, $n = \sum\limits_{t \in Z_n} f_t$.

3. Inequalities for the golden ratio

Next, we state a completely straightforward (and well-known) theorem, which shows that the Fibonacci sequence grows roughly exponentially (with the exponent being the golden ratio ϕ):

Theorem 3.1. For every positive integer n, we have $|f_{n+1} - \phi f_n| = \frac{1}{\sqrt{5}} (\phi - 1)^n$.

Theorem 3.1 can easily be derived from [BenQui03, Chapter 9, Corollary 34]. For the sake of self-containedness, let us nevertheless give a proof of it here.

Proof of Theorem 3.1. It is easy to see that $\phi^{-1} = \phi - 1$ and $1 - \phi^{-2} = \phi - 1$. Also, $(\phi - 1)^n \geq 0$ (since $\phi - 1 \geq 0$) and thus $\frac{1}{\sqrt{5}} (\phi - 1)^n \geq 0$.

Let n be a positive integer. By Binet's formula, we have

$$f_n = \frac{\phi^n - \phi^{-n}}{\sqrt{5}} = \frac{\phi^n \left(1 - \phi^{-2n}\right)}{\sqrt{5}} = \frac{1}{\sqrt{5}} \phi^n \left(1 - \phi^{-2n}\right).$$

Applying this to $n + 1$ instead of n, we get

$$f_{n+1} = \frac{1}{\sqrt{5}} \phi^{n+1} \left(1 - \phi^{-2(n+1)}\right).$$

These two equalities yield

$$\begin{aligned}
f_{n+1} - \phi f_n &= \frac{1}{\sqrt{5}}\phi^{n+1}\left(1-\phi^{-2(n+1)}\right) - \underbrace{\phi \cdot \frac{1}{\sqrt{5}}\phi^n}_{=\frac{1}{\sqrt{5}}\phi\phi^n}\left(1-\phi^{-2n}\right) \\
&= \frac{1}{\sqrt{5}}\phi^{n+1}\left(1-\phi^{-2(n+1)}\right) - \frac{1}{\sqrt{5}}\underbrace{\phi\phi^n}_{=\phi^{n+1}}\left(1-\phi^{-2n}\right) \\
&= \frac{1}{\sqrt{5}}\phi^{n+1}\left(1-\phi^{-2(n+1)}\right) - \frac{1}{\sqrt{5}}\phi^{n+1}\left(1-\phi^{-2n}\right) \\
&= \frac{1}{\sqrt{5}}\phi^{n+1}\left(\underbrace{\left(1-\phi^{-2(n+1)}\right) - \left(1-\phi^{-2n}\right)}_{=\phi^{-2n}-\phi^{-2(n+1)}}\right) \\
&= \frac{1}{\sqrt{5}}\phi^{n+1}\left(\phi^{-2n} - \underbrace{\phi^{-2(n+1)}}_{=\phi^{-2n-2}=\phi^{-2n}\phi^{-2}}\right) \\
&= \frac{1}{\sqrt{5}}\phi^{n+1}\left(\phi^{-2n} - \phi^{-2n}\phi^{-2}\right) = \frac{1}{\sqrt{5}}\underbrace{\phi^{n+1}\phi^{-2n}}_{=\phi^{(n+1)+(-2n)}=\phi^{-(n-1)}=\left(\phi^{-1}\right)^{n-1}}\left(1-\phi^{-2}\right) \\
&= \frac{1}{\sqrt{5}}\left(\underbrace{\phi^{-1}}_{=\phi-1}\right)^{n-1}\underbrace{\left(1-\phi^{-2}\right)}_{=\phi-1} = \frac{1}{\sqrt{5}}\underbrace{(\phi-1)^{n-1}(\phi-1)}_{=(\phi-1)^n} = \frac{1}{\sqrt{5}}(\phi-1)^n,
\end{aligned}$$

so that

$$|f_{n+1} - \phi f_n| = \left|\frac{1}{\sqrt{5}}(\phi-1)^n\right| = \frac{1}{\sqrt{5}}(\phi-1)^n \qquad \left(\text{since } \frac{1}{\sqrt{5}}(\phi-1)^n \geq 0\right),$$

and Theorem 3.1 is proven. □

Let us show yet another lemma for later use:

Lemma 3.2. Let S be a finite lacunar subset of $\mathbb{N}_2$. Then, $\sum_{s \in S}(\phi-1)^s \leq \phi - 1$.

Proof of Lemma 3.2. We WLOG assume that S is nonempty (since otherwise, Lemma 3.2 follows easily from $0 \leq \phi - 1$).

Let $\psi = \phi - 1$. It is easily seen that $0 < \psi < 1$. Also, $\psi = \phi - 1$ yields $\psi^2 = (\phi-1)^2 = \underbrace{\phi^2}_{=\phi+1} - 2\phi + 1 = (\phi+1) - 2\phi + 1 = 2 - \phi$ and thus $1 - \psi^2 = 1 -$

$(2-\phi) = \phi - 1 = \psi$, so that $\dfrac{\psi^2}{1-\psi^2} = \dfrac{\psi^2}{\psi} = \psi$. Also, from $0 < \psi < 1$, we obtain $0 < \psi^2 < 1$, so that $1 - \psi^2 > 0$.

Let us write the set S in the form $\{s_1, s_2, \ldots, s_k\}$, where $s_1 < s_2 < \cdots < s_k$. Then, $\sum\limits_{s \in S} \psi^s = \sum\limits_{i=1}^{k} \psi^{s_i}$. Also, $k \geq 1$ (since the set S is nonempty). On the other hand, every $i \in \{1, 2, \ldots, k-1\}$ satisfies $s_i + 1 \leq s_{i+1} - 1$ (this was proven during the proof of Lemma 2.1) and thus $s_i + 2 \leq s_{i+1}$ and therefore

$$\psi^{s_i+2} \geq \psi^{s_{i+1}} \tag{6}$$

(since $0 < \psi < 1$). Besides, $s_1 \geq 2$ (since $s_1 \in S \subseteq \mathbb{N}_2$) and thus $\psi^{s_1} \leq \psi^2$ (since $0 < \psi < 1$). Now, $\sum\limits_{s \in S} \psi^s = \sum\limits_{i=1}^{k} \psi^{s_i}$ yields

$$\begin{aligned}
\left(1-\psi^2\right) \sum_{s \in S} \psi^s &= \left(1-\psi^2\right) \sum_{i=1}^{k} \psi^{s_i} = \underbrace{\sum_{i=1}^{k} \psi^{s_i}}_{= \psi^{s_1} + \sum_{i=2}^{k} \psi^{s_i}} - \underbrace{\psi^2 \sum_{i=1}^{k} \psi^{s_i}}_{= \sum_{i=1}^{k} \psi^2 \psi^{s_i} = \sum_{i=1}^{k} \psi^{s_i+2} = \sum_{i=1}^{k-1} \psi^{s_i+2} + \psi^{s_k+2}} \\
&= \left(\psi^{s_1} + \sum_{i=2}^{k} \psi^{s_i}\right) - \left(\sum_{i=1}^{k-1} \underbrace{\psi^{s_i+2}}_{\substack{\geq \psi^{s_{i+1}} \\ \text{(by (6))}}} + \psi^{s_k+2}\right) \\
&= \psi^{s_1} + \sum_{i=2}^{k} \psi^{s_i} - \sum_{i=1}^{k-1} \underbrace{\psi^{s_i+2}}_{\substack{\geq \psi^{s_{i+1}} \\ \text{(by (6))}}} - \psi^{s_k+2} \\
&\leq \psi^{s_1} + \sum_{i=2}^{k} \psi^{s_i} - \sum_{i=1}^{k-1} \psi^{s_{i+1}} - \psi^{s_k+2} = \psi^{s_1} + \sum_{i=2}^{k} \psi^{s_i} - \sum_{i=2}^{k} \psi^{s_i} - \psi^{s_k+2} \\
&\qquad \text{(here, we substituted } i \text{ for } i+1 \text{ in the second sum)} \\
&= \underbrace{\psi^{s_1}}_{\leq \psi^2} - \underbrace{\psi^{s_k+2}}_{\geq 0} \leq \psi^2 - 0 = \psi^2.
\end{aligned}$$

Dividing this inequality by $1 - \psi^2$ (here we are using $1 - \psi^2 > 0$), we get $\sum\limits_{s \in S} \psi^s \leq \dfrac{\psi^2}{1-\psi^2} = \psi$. Replacing ψ by $\phi - 1$ in this inequality (since $\psi = \phi - 1$), we rewrite it as $\sum\limits_{s \in S} (\phi - 1)^s \leq \phi - 1$. This proves Lemma 3.2. $\square$

4. Proving Theorem 1.4

Let us now come to the proof of Theorem 1.4. First, we formulate the existence part of this theorem:

Theorem 4.1 (existence part of the generalized Zeckendorf family identities)**.** Let T be a finite set, and let a_t be an integer for every $t \in T$.

Then, there exists a finite lacunar subset S of $\mathbb{Z}$ such that

$$\left(\begin{array}{c} \sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right) \end{array} \right).$$

Before we start proving this, let us introduce a notation:

Definition 4.2. Let K be a subset of $\mathbb{Z}$, and let $a \in \mathbb{Z}$. Then, $K + a$ will denote the subset $\{k + a \mid k \in K\}$ of $\mathbb{Z}$.

Clearly, $(K + a) + b = K + (a + b)$ for any two integers a and b. Also, $K + 0 = K$. Finally,

$$\text{if } K \text{ is a lacunar subset of } \mathbb{Z}, \text{ and if } a \in \mathbb{Z}, \text{ then } K + a \text{ is lacunar as well} \tag{7}$$

[7].

Let us furthermore make two basic observations:

- If u and v are two real numbers, then

$$|u - v| \leq |u| + |v|. \tag{8}$$

(Indeed, this is one of the forms of the triangle inequality.)

- If m is a positive integer, then

$$f_m = f_{m+2} - f_{m+1}. \tag{9}$$

[7]*Proof.* Let K be a lacunar subset of $\mathbb{Z}$. Let $a \in \mathbb{Z}$. Let $q \in K + a$. We shall prove that $q + 1 \notin K + a$.

Assume the contrary. In other words, assume that $q + 1 \in K + a$. Thus, $q + 1 \in K + a = \{k + a \mid k \in K\}$. Hence, there exists some $\ell \in K$ such that $q + 1 = \ell + a$. Consider this ℓ.

Now, $q \in K + a = \{k + a \mid k \in K\}$ yields that there exists some $k \in K$ such that $q = k + a$. Consider this k. Since K is a lacunar set, we have ($s + 1 \notin K$ for every $s \in K$). Applying this to $s = k$, we get $k + 1 \notin K$. But $\ell + a = \underbrace{q}_{=k+a} + 1 = k + a + 1$ yields $\ell = (k + a + 1) - a = k + 1 \notin K$, contradicting $\ell \in K$. This contradiction shows that our assumption was wrong. Hence, $q + 1 \notin K + a$ is proven.

Now, forget that we fixed q. We thus have shown that $q + 1 \notin K + a$ for every $q \in K + a$. Renaming the variable q as s in this statement, we obtain ($s + 1 \notin K + a$ for every $s \in K + a$). In other words, the subset $K + a$ of $\mathbb{Z}$ is lacunar. Qed.

[*Proof of (9):* Let m be a positive integer. Thus, $m \geq 1$, so that $m+2 \geq 1+2 = 3$. But recall that every integer $n \geq 3$ satisfies $f_n = f_{n-1} + f_{n-2}$, so that $f_{n-2} = f_n - f_{n-1}$. Applying this to $n = m+2$, we obtain $f_{(m+2)-2} = f_{m+2} - f_{(m+2)-1}$. This simplifies to $f_m = f_{m+2} - f_{m+1}$. Thus, (9) is proven.]

Proof of Theorem 4.1. Let us define a real constant C by $C = \sum\limits_{t \in T} \dfrac{1}{\sqrt{5}} (\phi - 1)^{a_t}$. Clearly, $C \geq 0$ (since $\phi - 1 > 0$).

First, notice that $0 < \phi - 1 < 1$ yields $\lim\limits_{n\to\infty} (\phi-1)^n = 0$, so that $\lim\limits_{n\to\infty} \left((\phi-1)^n C \right) = \underbrace{\lim\limits_{n\to\infty} (\phi-1)^n}_{=0} \cdot C = 0$ as well. Therefore, for every $\varepsilon > 0$, every sufficiently high integer N satisfies $(\phi-1)^N C < \varepsilon$. In particular, taking $\varepsilon = 2 - \phi$ (here we are using that $2 - \phi > 0$), we see that every sufficiently high integer N satisfies $(\phi-1)^N C < 2 - \phi$. Also, obviously, every sufficiently high integer N satisfies $N > \max \{-a_t \mid t \in T\}$. Hence, if we take our integer N high enough, we can ensure that it will satisfy *both* $(\phi-1)^N C < 2-\phi$ and $N > \max\{-a_t \mid t \in T\}$. So let us fix some integer $N \in \mathbb{N}_2$ high enough that it satisfies both $(\phi-1)^N C < 2-\phi$ and $N > \max\{-a_t \mid t \in T\}$.

Since $N > \max\{-a_t \mid t \in T\}$, we have $N > -a_t$ for every $t \in T$, and thus $N + a_t > 0$ for every $t \in T$. This shows that the Fibonacci number f_{N+a_t} is well-defined for every $t \in T$. (This was exactly the reason why we have required $N > \max\{-a_t \mid t \in T\}$. The reason for the second condition $(\phi-1)^N C < 2-\phi$ will become clear later in the proof.)

Let $\nu = \sum\limits_{t\in T} f_{N+a_t}$. Recall that Z_ν is a finite lacunar subset of $\mathbb{N}_2$ such that $\nu = \sum\limits_{t \in Z_\nu} f_t$ (by the definition of Z_ν). Let $S = Z_\nu + (-N)$. Then, the set $S = Z_\nu + (-N)$ is lacunar (by (7) (applied to $K = Z_\nu$ and $a = -N$), because Z_ν is a lacunar subset of $\mathbb{Z}$) and finite (since Z_ν is finite), and satisfies

$$\nu = \sum_{t \in Z_\nu} f_t = \sum_{s \in Z_\nu + (-N)} f_{N+s} \qquad \left(\begin{array}{c} \text{here, we substituted } N+s \text{ for } t \text{, because the map} \\ Z_\nu + (-N) \to Z_\nu,\ s \mapsto N+s \text{ is a bijection} \end{array} \right)$$
$$= \sum_{s \in S} f_{N+s} \qquad (\text{since } Z_\nu + (-N) = S).$$

So now we know that $\sum\limits_{t\in T} f_{N+a_t} = \sum\limits_{s \in S} f_{N+s}$ (because both sides of this equation equal ν).

So, we have chosen a large N and found a finite lacunar subset S of $\mathbb{Z}$ which satisfies $\sum\limits_{t\in T} f_{N+a_t} = \sum\limits_{s\in S} f_{N+s}$. In other words, we have showed that the equation

$$\sum_{t \in T} f_{n+a_t} = \sum_{s \in S} f_{n+s} \tag{10}$$

holds for $n = N$. But we must show that this equation holds for *every* $n > \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})$. In order to do this, first let us prove that (10) holds for $n = N+1$. Actually, we are going to show a bit more:

Assertion α: Let $m \geq N$ be an integer such that (10) holds for $n = m$. Then, (10) also holds for $n = m + 1$.

[*Proof of Assertion α:* Since (10) holds for $n = m$, we have $\sum\limits_{t \in T} f_{m+a_t} = \sum\limits_{s \in S} f_{m+s}$. Now,

$$\begin{aligned}
&\sum_{t \in T} \underbrace{f_{(m+1)+a_t}}_{\substack{=f_{m+a_t+1} \\ =\left(f_{m+a_t+1}-\phi f_{m+a_t}\right)+\phi f_{m+a_t}}} - \sum_{s \in S} \underbrace{f_{(m+1)+s}}_{=f_{m+s+1}} \\
&= \underbrace{\sum_{t \in T} \left(\left(f_{m+a_t+1} - \phi f_{m+a_t} \right) + \phi f_{m+a_t} \right)}_{=\sum\limits_{t \in T} \left(f_{m+a_t+1}-\phi f_{m+a_t}\right)+\phi \sum\limits_{t \in T} f_{m+a_t}} - \sum_{s \in S} f_{m+s+1} \\
&= \sum_{t \in T} \left(f_{m+a_t+1} - \phi f_{m+a_t} \right) + \phi \underbrace{\sum_{t \in T} f_{m+a_t}}_{=\sum\limits_{s \in S} f_{m+s}} - \sum_{s \in S} f_{m+s+1} \\
&= \sum_{t \in T} \left(f_{m+a_t+1} - \phi f_{m+a_t} \right) + \underbrace{\phi \sum_{s \in S} f_{m+s} - \sum_{s \in S} f_{m+s+1}}_{=\sum\limits_{s \in S} \left(\phi f_{m+s}-f_{m+s+1}\right)=-\sum\limits_{s \in S} \left(f_{m+s+1}-\phi f_{m+s}\right)} \\
&= \sum_{t \in T} \left(f_{m+a_t+1} - \phi f_{m+a_t} \right) - \sum_{s \in S} \left(f_{m+s+1} - \phi f_{m+s} \right),
\end{aligned}$$

so that

$$\begin{aligned}
&\left| \sum_{t \in T} f_{(m+1)+a_t} - \sum_{s \in S} f_{(m+1)+s} \right| = \left| \sum_{t \in T} \left(f_{m+a_t+1} - \phi f_{m+a_t} \right) - \sum_{s \in S} \left(f_{m+s+1} - \phi f_{m+s} \right) \right| \\
&\leq \left| \sum_{t \in T} \left(f_{m+a_t+1} - \phi f_{m+a_t} \right) \right| + \left| \sum_{s \in S} \left(f_{m+s+1} - \phi f_{m+s} \right) \right| \qquad (11)
\end{aligned}$$

(by (8), applied to $u = \sum\limits_{t \in T} \left(f_{m+a_t+1} - \phi f_{m+a_t} \right)$ and $v = \sum\limits_{s \in S} \left(f_{m+s+1} - \phi f_{m+s} \right)$).

Now, the triangle inequality yields

$$\begin{aligned}
&\left|\sum_{t\in T}\left(f_{m+a_t+1}-\phi f_{m+a_t}\right)\right| \\
&\leq \sum_{t\in T} \underbrace{\left|f_{m+a_t+1}-\phi f_{m+a_t}\right|}_{\substack{=\frac{1}{\sqrt{5}}(\phi-1)^{m+a_t} \\ \text{(by Theorem 3.1, applied to} \\ m+a_t \text{ instead of } n)}} = \sum_{t\in T}\frac{1}{\sqrt{5}} \underbrace{(\phi-1)^{m+a_t}}_{=(\phi-1)^m(\phi-1)^{a_t}} = (\phi-1)^m \underbrace{\sum_{t\in T}\frac{1}{\sqrt{5}}(\phi-1)^{a_t}}_{=C} \\
&= \underbrace{(\phi-1)^m}_{\substack{\leq(\phi-1)^N \text{ (since} \\ 0<\phi-1<1 \text{ and } m\geq N)}} C \leq (\phi-1)^N C \qquad (\text{since } C\geq 0) \\
&< 2-\phi.
\end{aligned}$$

On the other hand, S is a lacunar subset of $\mathbb{Z}$, and thus the set $S+m$ is lacunar as well (by (7), applied to S and m instead of K and a), and besides $\underbrace{S}_{=Z_\nu+(-N)}+m = (Z_\nu+(-N))+m = Z_\nu+((-N)+m) = \{z+((-N)+m) \mid z\in Z_\nu\} \subseteq \mathbb{N}_2$ [8]. Thus, $S+m$ is a finite lacunar subset of $\mathbb{N}_2$ (indeed, the set $S+m$ is finite since S is finite). Hence, Lemma 3.2 (applied to $S+m$ instead of S) yields $\sum\limits_{s\in S+m}(\phi-1)^s \leq \phi-1$.

The triangle inequality yields

$$\begin{aligned}
&\left|\sum_{s\in S}\left(f_{m+s+1}-\phi f_{m+s}\right)\right| \\
&\leq \sum_{s\in S} \underbrace{\left|f_{m+s+1}-\phi f_{m+s}\right|}_{\substack{=\frac{1}{\sqrt{5}}(\phi-1)^{m+s} \\ \text{(by Theorem 3.1, applied to} \\ m+s \text{ instead of } n)}} = \sum_{s\in S}\frac{1}{\sqrt{5}}(\phi-1)^{m+s} = \frac{1}{\sqrt{5}}\sum_{s\in S}(\phi-1)^{m+s} \\
&= \frac{1}{\sqrt{5}} \underbrace{\sum_{s\in S+m}(\phi-1)^s}_{\leq\phi-1} \qquad \left(\begin{array}{c}\text{here, we substituted } s \text{ for } m+s \text{, since the map} \\ S\to S+m,\ s\mapsto m+s \text{ is a bijection}\end{array}\right) \\
&\leq \underbrace{\frac{1}{\sqrt{5}}}_{<1}\cdot(\phi-1) < \phi-1 \qquad (\text{since } \phi-1>0).
\end{aligned}$$

[8] because every $z\in Z_\nu$ satisfies $\underbrace{z}_{\substack{\geq 2 \\ (\text{since } z\in Z_\nu\subseteq\mathbb{N}_2)}} + \left((-N)+\underbrace{m}_{\geq N}\right) \geq 2+((-N)+N) = 2$ and thus $z+((-N)+m)\in\mathbb{N}_2$

Thus, (11) becomes

$$\left|\sum_{t\in T} f_{(m+1)+a_t} - \sum_{s\in S} f_{(m+1)+s}\right| \leq \underbrace{\left|\sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t}\right)\right|}_{<2-\phi} + \underbrace{\left|\sum_{s\in S} \left(f_{m+s+1} - \phi f_{m+s}\right)\right|}_{<\phi-1}$$
$$< (2-\phi) + (\phi - 1) = 1.$$

Since $\sum\limits_{t\in T} f_{(m+1)+a_t} - \sum\limits_{s\in S} f_{(m+1)+s}$ is an integer, we thus conclude that $\sum\limits_{t\in T} f_{(m+1)+a_t} - \sum\limits_{s\in S} f_{(m+1)+s}$ is an integer with an absolute value < 1. But the only integer with an absolute value < 1 is 0. Thus, $\sum\limits_{t\in T} f_{(m+1)+a_t} - \sum\limits_{s\in S} f_{(m+1)+s} = 0$, so that $\sum\limits_{t\in T} f_{(m+1)+a_t} = \sum\limits_{s\in S} f_{(m+1)+s}$. In other words, (10) holds for $n = m+1$. This proves Assertion α.]

Assertion α almost immediately yields the following:

Assertion β: The equation (10) holds for every $n \geq N$.

[*Proof of Assertion β:* We are going to prove Assertion β by induction over n:

As the *induction base* we take the case $n = N$. In this case, the equation (10) holds (as we already know), so that Assertion β is proved for $n = N$, and thus the induction base is completed.

Induction step: Let $m \geq N$ be an integer. Assume that Assertion β holds for $n = m$. In other words, the equation (10) holds for $n = m$. Then, Assertion α yields that the equation (10) holds for $n = m+1$ as well. In other words, Assertion β holds for $n = m+1$ as well. This completes the induction step, and thus Assertion β is proven by induction over n.]

With Assertion β we now know that (10) holds for all sufficiently high n, namely for all $n \geq N$. But in order to prove Theorem 8, we must show that it also holds for all $n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$; usually, these n are not all $\geq N$. What remains to do is "backwards induction" or an application of the maximum principle. Here are the details:

Let $M = \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$. We thus must show that (10) holds for all $n > M$.

Define a subset $\mathcal{R}$ of $\mathbb{Z}$ by

$$\mathcal{R} = \{n \in \mathbb{Z} \mid \text{we have } n > M\text{, and (10) does not hold}\}. \tag{12}$$

This set $\mathcal{R}$ is bounded from above by N (in fact, it does not contain any $n \geq N$, because (10) does hold for all $n \geq N$ (according to Assertion β)), and bounded from below by M (because every element of this set is $> M$ by definition). Thus, this set $\mathcal{R}$ is finite (since any subset of $\mathbb{Z}$ that is bounded from above and bounded from below is finite).

Let us assume (for the sake of contradiction) that $\mathcal{R}$ is nonempty. Then, the set $\mathcal{R}$ is a nonempty finite set of integers, and thus has a maximum (since a nonempty finite set of integers always has a maximum). Let λ be this maximum. Then, $\lambda \in \mathcal{R} = \{n \in \mathbb{Z} \mid \text{ we have } n > M\text{, and (10) does not hold}\}$. In other words, λ is an element of $\mathbb{Z}$ satisfying $\lambda > M$, and (10) does not hold for $n = \lambda$. On the other hand,

$$\text{for every integer } \mu > \lambda\text{, the equation (10) holds for } n = \mu \tag{13}$$

[9]. In particular, applying (13) to $\mu = \lambda + 1$, we see that (10) holds for $n = \lambda + 1$; in other words, $\sum_{t \in T} f_{\lambda+1+a_t} = \sum_{s \in S} f_{\lambda+1+s}$. Besides, applying (13) to $\mu = \lambda + 2$, we see that (10) holds for $n = \lambda + 2$; in other words, $\sum_{t \in T} f_{\lambda+2+a_t} = \sum_{s \in S} f_{\lambda+2+s}$.

Now, we are going to prove the equation $\sum_{t \in T} f_{\lambda+a_t} = \sum_{s \in S} f_{\lambda+s}$. (We notice that this equation indeed makes sense, because the Fibonacci number $f_{\lambda+a_t}$ is well-defined for every $t \in T$ [10], and because the Fibonacci number $f_{\lambda+s}$ is well-defined for every $s \in S$ [11].) Applying (9) to $m = \lambda + a_t$, we obtain

$$f_{\lambda+a_t} = \underbrace{f_{\lambda+a_t+2}}_{=f_{\lambda+2+a_t}} - \underbrace{f_{\lambda+a_t+1}}_{=f_{\lambda+1+a_t}} = f_{\lambda+2+a_t} - f_{\lambda+1+a_t} \tag{14}$$

for every $t \in T$. On the other hand, applying (9) to $m = \lambda + s$, we obtain

$$f_{\lambda+s} = \underbrace{f_{\lambda+s+2}}_{=f_{\lambda+2+s}} - \underbrace{f_{\lambda+s+1}}_{=f_{\lambda+1+s}} = f_{\lambda+2+s} - f_{\lambda+1+s} \tag{15}$$

[9]*Proof.* Let $\mu > \lambda$ be an integer. We must prove that the equation (10) holds for $n = \mu$.

Assume the contrary. Thus, the equation (10) does not hold for $n = \mu$. Hence, μ is an integer with the property that $\mu > M$ (since $\mu > \lambda > M$), and (10) does not hold for $n = \mu$. In other words, $\mu \in \{n \in \mathbb{Z} \mid \text{ we have } n > M\text{, and (10) does not hold}\}$. In view of (12), this rewrites as $\mu \in \mathcal{R}$.

But λ is the maximum of $\mathcal{R}$. Thus, every $r \in \mathcal{R}$ satisfies $r \leq \lambda$. Applying this to $r = \mu$, we obtain $\mu \leq \lambda$ (since $\mu \in \mathcal{R}$). This contradicts $\mu > \lambda$. This contradiction shows that our assumption was wrong. Hence, the equation (10) holds for $n = \mu$. This proves (13).

[10]*Proof.* In fact, $\lambda > M = \max\left(\underbrace{\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}}_{\supseteq\{-a_t \mid t \in T\}}\right) \geq \max\{-a_t \mid t \in T\}$ yields that $\lambda > -a_t$ for every $t \in T$. Thus, for every $t \in T$, we have $\lambda + a_t > 0$, and thus $f_{\lambda+a_t}$ is well-defined.

[11]*Proof.* In fact, $\lambda > M = \max\left(\underbrace{\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}}_{\supseteq\{-s \mid s \in S\}}\right) \geq \max\{-s \mid s \in S\}$ yields that $\lambda > -s$ for every $s \in S$. Thus, for every $s \in S$, we have $\lambda + s > 0$, and thus $f_{\lambda+s}$ is well-defined.

for every $s \in S$. Thus,

$$\sum_{t\in T} \underbrace{f_{\lambda+a_t}}_{\substack{=f_{\lambda+2+a_t}-f_{\lambda+1+a_t} \\ \text{(by (14))}}} = \sum_{t\in T} \left(f_{\lambda+2+a_t} - f_{\lambda+1+a_t}\right) = \underbrace{\sum_{t\in T} f_{\lambda+2+a_t}}_{=\sum_{s\in S} f_{\lambda+2+s}} - \underbrace{\sum_{t\in T} f_{\lambda+1+a_t}}_{=\sum_{s\in S} f_{\lambda+1+s}}$$
$$= \sum_{s\in S} f_{\lambda+2+s} - \sum_{s\in S} f_{\lambda+1+s} = \sum_{s\in S} \underbrace{\left(f_{\lambda+2+s} - f_{\lambda+1+s}\right)}_{\substack{=f_{\lambda+s} \\ \text{(by (15))}}} = \sum_{s\in S} f_{\lambda+s}.$$

In other words, (10) holds for $n = \lambda$. This contradicts our knowledge that (10) does not hold for $n = \lambda$. This contradiction shows that our assumption (the assumption that the set $\mathcal{R}$ is nonempty) was wrong. Hence, the set $\mathcal{R}$ is empty. In other words, the set

$$\{n \in \mathbb{Z} \mid \text{we have } n > M\text{, and (10) does not hold}\}$$

is empty (since $\mathcal{R} = \{n \in \mathbb{Z} \mid \text{we have } n > M\text{, and (10) does not hold}\}$). In other words, there exists no $n \in \mathbb{Z}$ satisfying $n > M$ such that (10) does not hold. In other words, (10) holds for every $n \in \mathbb{Z}$ satisfying $n > M$. Since $M = \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$, this is equivalent to saying that (10) holds for every $n \in \mathbb{Z}$ satisfying $n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$. Consequently, Theorem 4.1 is proven. □

All that remains now is the (rather trivial) uniqueness part of Theorem 1.4:

Lemma 4.3 (uniqueness part of the generalized Zeckendorf family identities)**.** Let T be a finite set, and let a_t be an integer for every $t \in T$.

Let S be a finite lacunar subset of $\mathbb{Z}$ such that

$$\left(\begin{array}{c} \sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right) \end{array}\right). \tag{16}$$

Let S' be a finite lacunar subset of $\mathbb{Z}$ such that

$$\left(\begin{array}{c} \sum_{t\in T} f_{n+a_t} = \sum_{s\in S'} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S'\}\right) \end{array}\right). \tag{17}$$

Then, $S = S'$.

Proof of Lemma 4.3. Let

$$n = \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\} \cup \{-s \mid s \in S'\}\right) + 2.$$

Then,

$$n > \max\left(\underbrace{\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\} \cup \{-s \mid s \in S'\}}_{\supseteq\{-a_t|t\in T\}\cup\{-s|s\in S\}}\right)$$
$$\geq \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right),$$

so that

$$\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \qquad \text{(by (16))}$$
$$= \sum_{t\in S+n} f_t \qquad \left(\begin{array}{c}\text{here, we substituted } t \text{ for } n+s\text{, since the map}\\ S \to S+n,\ s \mapsto n+s \text{ is a bijection}\end{array}\right).$$

Similarly, $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{t\in S'+n} f_t$.

Now, $S+n$ is a lacunar set (by (7) (applied to $K = S$ and $a = n$), since S is a lacunar subset of $\mathbb{Z}$) and a subset of $\mathbb{N}_2$ [12]. In other words, $S+n$ is a finite lacunar subset of $\mathbb{N}_2$ (since $S+n$ is finite (because S is finite)). Similarly, $S'+n$ is a finite lacunar subset of $\mathbb{N}_2$. Applying Lemma 2.3 to $S+n$, $S'+n$ and $\sum\limits_{t\in T} f_{n+a_t}$ instead of T, T' and n yields that $S+n = S'+n$ (because $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{t\in S+n} f_t$ and $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{t\in S'+n} f_t$). Hence,

$$S = S + \underbrace{0}_{=(n+(-n))} = S + (n + (-n)) = \underbrace{(S+n)}_{=S'+n} + (-n)$$
$$= \left(S'+n\right) + (-n) = S' + \underbrace{(n+(-n))}_{=0} = S' + 0 = S'.$$

This proves Lemma 4.3. $\square$

Proof of Theorem 1.4. There exists a finite lacunar subset S of $\mathbb{Z}$ such that

$$\left(\begin{array}{c}\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which}\\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)\end{array}\right)$$

[12] *Proof.* Since

$$n = \underbrace{\max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\} \cup \{-s \mid s \in S'\}\right)}_{\substack{\geq \max\{-s|s\in S\} \\ (\text{since } \{-a_t|t\in T\}\cup\{-s|s\in S\}\cup\{-s|s\in S'\}\supseteq\{-s|s\in S\})}} + 2 \geq \max\{-s \mid s \in S\} + 2,$$

we have $n - 2 \geq \max\{-s \mid s \in S\}$, and thus $n - 2 \geq -s$ for every $s \in S$. Hence, every $s \in S$ satisfies $n - 2 + s \geq 0$, which rewrites as $s + n \geq 2$. Equivalently, $s + n \in \mathbb{N}_2$. Thus, we have shown that $s + n \in \mathbb{N}_2$ for each $s \in S$. In other words, $\{s + n \mid s \in S\} \subseteq \mathbb{N}_2$. Hence,

$$S + n = \{s + n \mid s \in S\} \subseteq \mathbb{N}_2.$$

(according to Theorem 4.1), and such a subset is unique (because any two such subsets are equal (according to Lemma 4.3)). Thus, there exists **one and only one** such subset. This proves Theorem 1.4. □

References

[BenQui03] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, The Mathematical Association of America, 2003.

[Grinbe11] Darij Grinberg, *Zeckendorf family identities generalized*, June 11, 2018, *brief version*.
`http://www.cip.ifi.lmu.de/~grinberg/zeckendorfBRIEF.pdf`
Also available as arXiv preprint arXiv:1103.4507v2.

[Hender16] Nik Henderson, *What is Zeckendorf's Theorem?*, 23 July 2016.
`https://math.osu.edu/sites/math.osu.edu/files/henderson_zeckendorf.pdf`

[Hoggat72] V. E. Hoggatt, Jr., *Generalized Zeckendorf theorem*, The Fibonacci Quarterly **10** (1972), Issue 1, pp. 89–94.
`https://www.fq.math.ca/Scanned/10-1/hoggatt2.pdf`

[Vorobi02] Nicolai N. Vorobiev, *Fibonacci numbers*, translated from the Russian by Mircea Martin, Springer 2002.

[WooZei09] Philip Matchett Wood, Doron Zeilberger, *A translation method for finding combinatorial bijections*, Annals of Combinatorics **13** (2009), pp. 383–402.
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trans-method.html`