Zeckendorf family identities generalized
Darij Grinberg, version 7, 18 March 2011
long version

This is a detailed version of my text [2]. It contains the proof outlined in [2] in much more detail and was written for the purpose of persuading myself that my proofs are correct.
This note has never been proofread by myself or anyone else. If you find any mistakes or typos, please inform me at $\Delta\Gamma$`@gmail.com` where $\Delta =$`darij` and $\Gamma =$`grinberg`
Thank you!

Definitions. 1) A subset S of $\mathbb{Z}$ is called *holey* if it satisfies ($s+1 \notin S$ for every $s \in S$).

2) Let $(f_1, f_2, f_3, ...)$ be the Fibonacci sequence (defined by $f_1 = f_2 = 1$ and the recurrence relation ($f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{N}$ satisfying $n \geq 3$)).

Theorem 1 (generalized Zeckendorf family identities). Let T be a finite set, and a_t be an integer for every $t \in T$.

Then, there exists one and only one finite holey subset S of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)\right).$$

Remarks.

1) Theorem 1 generalizes the so-called *Zeckendorf family identities* (which correspond to the case when all a_t are $= 0$), which were discussed in [1].[1]

2) The condition $n > \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})$ in Theorem 1 is just a technical condition made in order to ensure that the Fibonacci numbers f_{n+a_t} for all $t \in T$ and f_{n+s} for all $s \in S$ are well-defined. (If we would define the Fibonacci numbers f_n for integers $n \leq 0$ by extending the recurrence relation $f_n = f_{n-1} + f_{n-2}$ "to the left", then we could drop this condition.)

The following is my proof of Theorem 1. It does not even try to be combinatorial - it is pretty much the opposite. While I won't use any results from analysis, the proof below has a distinctively analytic flavor.

First, some lemmas and notations:

We denote by $\mathbb{N}$ the set $\{0, 1, 2, ...\}$ (and not the set $\{1, 2, 3, ...\}$, like some other authors do). Also, we denote by $\mathbb{N}_2$ the set $\{2, 3, 4, ...\} = \mathbb{N} \setminus \{0, 1\}$.

[1] The first seven of these identities are

$$\begin{aligned}
&1f_n = f_n \text{ for all } n \geq 1;\\
&2f_n = f_{n-2} + f_{n+1} \text{ for all } n \geq 3;\\
&3f_n = f_{n-2} + f_{n+2} \text{ for all } n \geq 3;\\
&4f_n = f_{n-2} + f_n + f_{n+2} \text{ for all } n \geq 3;\\
&5f_n = f_{n-4} + f_{n-1} + f_{n+3} \text{ for all } n \geq 5;\\
&6f_n = f_{n-4} + f_{n+1} + f_{n+3} \text{ for all } n \geq 5;\\
&7f_n = f_{n-4} + f_{n+4} \text{ for all } n \geq 5.
\end{aligned}$$

Also, let $\phi = \dfrac{1+\sqrt{5}}{2}$. We notice that $\phi \approx 1.618...$ and that $\phi^2 = \phi + 1$.

We recall some known facts about the Fibonacci sequence:

Lemma 2. If S is a finite holey subset of $\mathbb{N}_2$, then $\sum\limits_{t\in S} f_t < f_{\max S+1}$.

Proof of Lemma 2. Every $t \in \mathbb{N}_2$ satisfies $f_{t+1} = f_t + f_{t-1}$ (due to the relation $f_n = f_{n-1} + f_{n-2}$, applied to $n = t+1$), so that $f_t = f_{t+1} - f_{t-1}$.

Let us write the set S in the form $\{s_1, s_2, ..., s_k\}$, where $s_1 < s_2 < ... < s_k$. Then, $\sum\limits_{t\in S} f_t = \sum\limits_{i=1}^{k} f_{s_i}$ and $s_k = \max S$. On the other hand, every $i \in \{1, 2, ..., k-1\}$ satisfies $s_i + 2 \le s_{i+1}$ [2] and thus $s_i + 1 \le s_{i+1} - 1$, so that $f_{s_i+1} \le f_{s_{i+1}-1}$ (since the Fibonacci sequence $(f_1, f_2, f_3, ...)$ is monotonically increasing). Thus,

$$\sum_{t\in S} f_t = \sum_{i=1}^{k} f_{s_i} = \sum_{i=1}^{k} \left(f_{s_i+1} - f_{s_i-1}\right) \qquad \left(\begin{array}{c}\text{since } f_{s_i} = f_{s_i+1} - f_{s_i-1} \text{ (due to } f_t = f_{t+1} - f_{t-1}, \\ \text{applied to } t = s_i)\end{array}\right)$$

$$= \underbrace{\sum_{i=1}^{k} f_{s_i+1}}_{=\sum_{i=1}^{k-1} f_{s_i+1} + f_{s_k+1}} - \underbrace{\sum_{i=1}^{k} f_{s_i-1}}_{=f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1}}$$

$$= \left(\sum_{i=1}^{k-1} \underbrace{f_{s_i+1}}_{\le f_{s_{i+1}-1}} + f_{s_k+1}\right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1}\right) \le \left(\sum_{i=1}^{k-1} f_{s_{i+1}-1} + f_{s_k+1}\right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1}\right)$$

$$= \left(\sum_{i=2}^{k} f_{s_i-1} + f_{s_k+1}\right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1}\right)$$

(here, we substituted i for $i+1$ in the first sum)

$$= f_{s_k+1} - f_{s_1-1} < f_{s_k+1} \qquad \left(\text{since } f_{s_1-1} > 0, \text{ because } s_1 - 1 > 0, \text{ since } s_1 \in S \subseteq \mathbb{N}_2\right)$$

$$= f_{\max S+1}$$

(since $s_k = \max S$). This proves Lemma 2.

Lemma 3 (existence part of the Zeckendorf theorem). For every nonnegative integer n, there exists a finite holey subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t\in T} f_t$.

Proof of Lemma 3. We are going to prove Lemma 3 by strong induction over n:

Induction base: Let $n = 0$. Then, there exists a finite holey subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t\in T} f_t$ (namely, $T = \varnothing$), and thus Lemma 3 holds for $n = 0$, and the induction base is completed.

[2] *Proof.* The set S is holey, and thus $s + 1 \notin S$ for every $s \in S$. Applying this to $s = s_i$, we get $s_i + 1 \notin S$, so that $s_i + 1 \ne s_{i+1}$ (since $s_{i+1} \in S$).

Since $s_1 < s_2 < ... < s_k$, we have $s_i < s_{i+1}$, so that $s_i + 1 \le s_{i+1}$ (because s_i and s_{i+1} are integers). Since $s_i + 1 \ne s_{i+1}$, this becomes $s_i + 1 < s_{i+1}$, so that $s_i + 2 \le s_{i+1}$ (because s_i and s_{i+1} are integers).

Induction step: Let $\nu \in \mathbb{N}$ be such that $\nu > 0$. Assume that Lemma 3 holds for every nonnegative integer $n < \nu$. We must now prove that Lemma 3 holds for $n = \nu$.

In fact, let t_1 be the maximal integer τ from $\mathbb{N}_2$ satisfying $f_\tau \leq \nu$ [3]. Then, $f_{t_1} \leq \nu$ but $f_{t_1+1} > \nu$. Then, $\nu - f_{t_1}$ is a nonnegative integer (since $f_{t_1} \leq \nu$) and $< \nu$ (since $f_{t_1} > 0$). Thus, Lemma 3 holds for $n = \nu - f_{t_1}$ (since we assumed that Lemma 3 holds for every nonnegative integer $n < \nu$). In other words, there exists a finite holey subset T of $\mathbb{N}_2$ such that $\nu - f_{t_1} = \sum\limits_{t \in T} f_t$. We rename this subset T as S (so as not to confuse it with the set T that we want to construct for $n = \nu$). Thus, we have a finite holey subset S of $\mathbb{N}_2$ such that $\nu - f_{t_1} = \sum\limits_{t \in S} f_t$.

Since f_t is nonnegative for every $t \in S$, it is clear that every $s \in S$ satisfies

$$\begin{aligned} f_s &\leq \sum_{t \in S} f_t = \nu - f_{t_1} < f_{t_1+1} - f_{t_1} && (\text{since } \nu < f_{t_1+1}) \\ &= f_{t_1-1} && \left(\begin{array}{c} \text{because the relation } f_n = f_{n-1} + f_{n-2} \text{ (applied to } n = t_1 + 1) \text{ yields} \\ f_{t_1+1} = f_{t_1} + f_{t_1-1}, \text{ so that } f_{t_1+1} - f_{t_1} = f_{t_1-1} \end{array} \right) \end{aligned}$$

and thus $s < t_1 - 1$ (since the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is monotonically increasing), which rewrites as $s + 1 < t_1$. Thus, $s + 1 \neq t_1$; in other words $s + 1 \notin \{t_1\}$. Applying $s + 1 < t_1$ to $s = \max S$, we get $\max S + 1 < t_1$.

Since the set S is holey, we know that $s + 1 \notin S$ for every $s \in S$. Combining this with $s + 1 \notin \{t_1\}$, we get $s + 1 \notin S \cup \{t_1\}$ for every $s \in S$.

On the other hand, $s + 1 \notin S \cup \{t_1\}$ for every $s \in \{t_1\}$. [4] Thus, $s + 1 \notin S \cup \{t_1\}$ for every $s \in S \cup \{t_1\}$ (because $s \in S \cup \{t_1\}$ yields that one of the two cases $s \in S$ and $s \in \{t_1\}$ must hold, but in each of these two cases we have proved that $s + 1 \notin S \cup \{t_1\}$). In other words, the set $S \cup \{t_1\}$ is holey. Denoting this set $S \cup \{t_1\}$ by T, we thus have shown that T is a holey set. Clearly, T is a finite set (since S is finite) and satisfies

$$\begin{aligned} \sum_{t \in T} f_t &= \sum_{t \in S \cup \{t_1\}} f_t = \underbrace{\sum_{t \in S} f_t}_{= \nu - f_{t_1}} + f_{t_1} && (\text{since } t_1 \notin S, \text{ because } t_1 > \max S + 1 > \max S) \\ &= \nu. \end{aligned}$$

Thus, Lemma 3 is proven for the case $n = \nu$. This completes the induction step, and thus the induction proof of Lemma 3 is complete.

> **Lemma 4 (uniqueness part of the Zeckendorf theorem).** Let $n \in \mathbb{N}$, and let T and T' be two finite holey subsets of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$ and $n = \sum\limits_{t \in T'} f_t$. Then, $T = T'$.

Proof of Lemma 4. We are going to prove Lemma 4 by strong induction over n:

[3] Such an integer t_1 exists, since the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is strictly monotonically increasing beginning with f_2 and therefore unbounded from above (because every strictly monotonically increasing sequence of integers is unbounded from above).

[4] In fact, this is just a complicated way to state that $t_1 + 1 \notin S \cup \{t_1\}$, which is rather obvious (because $t_1 + 1 \notin S$ (since $t_1 + 1 > t_1 > \max S + 1 > \max S$) and $t_1 + 1 \notin \{t_1\}$).

Induction base: Let $n = 0$. Then, $n = \sum_{t \in T} f_t$ becomes $0 = \sum_{t \in T} f_t$, and this yields $T = \varnothing$ (since otherwise, $T \neq \varnothing$ and thus $\sum_{t \in T} f_t > 0$ because the Fibonacci numbers f_t are positive). Similarly, $T' = \varnothing$. Hence, $T = T'$, and thus Lemma 4 holds for $n = 0$, and the induction base is completed.

Induction step: Let $\nu \in \mathbb{N}$ be such that $\nu > 0$. Assume that Lemma 4 holds for every nonnegative integer $n < \nu$. We must now prove that Lemma 4 holds for $n = \nu$.

So let T and T' be two holey subsets of $\mathbb{N}_2$ such that $\nu = \sum_{t \in T} f_t$ and $\nu = \sum_{t \in T'} f_t$. Then, we want to prove that $T = T'$.

Since $\sum_{t \in T} f_t = \nu > 0$, we have $T \neq \varnothing$. Thus, $\max T \in T$. Thus, since the Fibonacci numbers f_t are all nonnegative, we have $f_{\max T} \leq \sum_{t \in T} f_t = \nu = \sum_{t \in T'} f_t < f_{\max T' + 1}$ (by Lemma 2, applied to $S = T'$). Hence, $\max T < \max T' + 1$ (since the Fibonacci sequence $(f_1, f_2, f_3, ...)$ is monotonically increasing), so that $\max T \leq \max T'$ (since $\max T$ and $\max T'$ are integers). Similarly, $\max T' \leq \max T$ (since the sets T and T' are completely equal in rights). Combining this with $\max T \leq \max T'$, we get $\max T = \max T'$. Let μ denote the number $\max T = \max T'$. Then, $\mu = \max T \in T$ and $\mu = \max T' \in T'$. Let $S = T \setminus \{\mu\}$ and $S' = T' \setminus \{\mu\}$. Clearly, S is a finite holey subset of $\mathbb{N}_2$ (because $S = T \setminus \{\mu\}$ is a subset of T, and every subset of a finite holey set is finite holey). Similarly, S' is a finite holey subset of $\mathbb{N}_2$. Now,

$$\begin{aligned} \nu - f_\mu &= \sum_{t \in T} f_t - f_\mu = \sum_{t \in T \setminus \{\mu\}} f_t \qquad \text{(since } \mu \in T\text{)} \\ &= \sum_{t \in S} f_t \end{aligned}$$

(since $T \setminus \{\mu\} = S$) and similarly $\nu - f_\mu = \sum_{t \in S'} f_t$. Since $\nu - f_\mu$ is a nonnegative integer (because $\nu - f_\mu = \sum_{t \in S'} f_t$, and since all f_t are nonnegative) and satisfies $\nu - f_\mu < \nu$, we can thus apply Lemma 4 to $\nu - f_\mu$ instead of n and to the holey subsets S and S' instead of T and T' (since we assumed that Lemma 4 holds for every nonnegative integer $n < \nu$), and we obtain $S = S'$. Now, $S = T \setminus \{\mu\}$ yields $T = S \cup \{\mu\}$ (since $\mu \in T$), and similarly $T' = S' \cup \{\mu\}$, so that $T = \underbrace{S}_{=S'} \cup \{\mu\} = S' \cup \{\mu\} = T'$. This proves Lemma 4 for the case $n = \nu$. Thus, the induction step is completed, and the induction proof of Lemma 4 is done.

Theorem 5 (Zeckendorf theorem). For every nonnegative integer n, there exists one and only one finite holey subset T of $\mathbb{N}_2$ such that $n = \sum_{t \in T} f_t$.

We will denote this set T by Z_n. Thus, $n = \sum_{t \in Z_n} f_t$.

Proof of Theorem 5. Fix some $n \in \mathbb{N}$. Then, there exists a finite holey subset T of $\mathbb{N}_2$ such that $n = \sum_{t \in T} f_t$ (according to Lemma 3), and such a subset is unique (because any two such subsets are equal, according to Lemma 4). This proves Theorem 5.

Several books should have this and similar proofs of Theorem 5.

Now for something completely straightforward:

Theorem 6. For every $n \in \mathbb{N}_2$, we have $|f_{n+1} - \phi f_n| = \frac{1}{\sqrt{5}} (\phi - 1)^n$.

Proof of Theorem 6. By Binet's formula,

$$f_n = \frac{\phi^n - \phi^{-n}}{\sqrt{5}} = \frac{\phi^n (1 - \phi^{-2n})}{\sqrt{5}} = \frac{1}{\sqrt{5}} \phi^n \left(1 - \phi^{-2n}\right).$$

Applying this to $n+1$ instead of n, we get

$$f_{n+1} = \frac{1}{\sqrt{5}} \phi^{n+1} \left(1 - \phi^{-2(n+1)}\right).$$

These two equalities yield

$$\begin{aligned}
f_{n+1} - \phi f_n &= \frac{1}{\sqrt{5}} \phi^{n+1} \left(1 - \phi^{-2(n+1)}\right) - \underbrace{\phi \cdot \frac{1}{\sqrt{5}} \phi^n}_{= \frac{1}{\sqrt{5}} \phi \phi^n} \left(1 - \phi^{-2n}\right) \\
&= \frac{1}{\sqrt{5}} \phi^{n+1} \left(1 - \phi^{-2(n+1)}\right) - \frac{1}{\sqrt{5}} \underbrace{\phi \phi^n}_{= \phi^{n+1}} \left(1 - \phi^{-2n}\right) = \frac{1}{\sqrt{5}} \phi^{n+1} \left(1 - \phi^{-2(n+1)}\right) - \frac{1}{\sqrt{5}} \phi^{n+1} \left(1 - \phi^{-2n}\right) \\
&= \frac{1}{\sqrt{5}} \phi^{n+1} \left(\underbrace{\left(1 - \phi^{-2(n+1)}\right) - \left(1 - \phi^{-2n}\right)}_{= \phi^{-2n} - \phi^{-2(n+1)}} \right) = \frac{1}{\sqrt{5}} \phi^{n+1} \left(\phi^{-2n} - \underbrace{\phi^{-2(n+1)}}_{= \phi^{-2n-2} = \phi^{-2n} \phi^{-2}} \right) \\
&= \frac{1}{\sqrt{5}} \phi^{n+1} \left(\phi^{-2n} - \phi^{-2n} \phi^{-2}\right) = \frac{1}{\sqrt{5}} \underbrace{\phi^{n+1} \phi^{-2n}}_{= \phi^{(n+1)+(-2n)} = \phi^{-(n-1)} = (\phi^{-1})^{n-1}} \left(1 - \phi^{-2}\right) \\
&= \frac{1}{\sqrt{5}} \left(\underbrace{\phi^{-1}}_{= \phi - 1 \text{ (as can be easily seen)}} \right)^{n-1} \underbrace{\left(1 - \phi^{-2}\right)}_{= \phi - 1 \text{ (as can be easily seen)}} = \frac{1}{\sqrt{5}} \underbrace{(\phi - 1)^{n-1} (\phi - 1)}_{= (\phi - 1)^n} = \frac{1}{\sqrt{5}} (\phi - 1)^n,
\end{aligned}$$

so that

$$|f_{n+1} - \phi f_n| = \left| \frac{1}{\sqrt{5}} (\phi - 1)^n \right| = \frac{1}{\sqrt{5}} (\phi - 1)^n \qquad \left(\text{since } \frac{1}{\sqrt{5}} (\phi - 1)^n \geq 0\right),$$

and Theorem 6 is proven.

Yet another lemma:

Theorem 7. If S is a finite holey subset of $\mathbb{N}_2$, then $\sum_{s \in S} (\phi - 1)^s \leq \phi - 1$.

Proof of Theorem 7. Let $\psi = \phi - 1$. It is easily seen that $0 < \psi < 1$. Also, $\psi = \phi - 1$ yields $\psi^2 = (\phi - 1)^2 = \underbrace{\phi^2}_{= \phi + 1} - 2\phi + 1 = (\phi + 1) - 2\phi + 1 = 2 - \phi$ and thus $1 - \psi^2 = 1 - (2 - \phi) = \phi - 1 = \psi$, so that $\frac{\psi^2}{1 - \psi^2} = \frac{\psi^2}{\psi} = \psi$.

Let us write the set S in the form $\{s_1, s_2, ..., s_k\}$, where $s_1 < s_2 < ... < s_k$. Then, $\sum\limits_{s\in S} \psi^s = \sum\limits_{i=1}^{k} \psi^{s_k}$. On the other hand, every $i \in \{1, 2, ..., k-1\}$ satisfies $s_i+2 \leq s_{i+1}$ (this was proven during the proof of Lemma 2) and thus $\psi^{s_i+2} \geq \psi^{s_i+1}$ (since $0 < \psi < 1$). Besides, $s_1 \geq 2$ (since $s_1 \in S \subseteq \mathbb{N}_2$) and thus $\psi^{s_1} \leq \psi^2$ (since $0 < \psi < 1$). Now, $\sum\limits_{s\in S} \psi^s = \sum\limits_{i=1}^{k} \psi^{s_k}$ yields

$$\begin{aligned}
\left(1-\psi^2\right)\sum_{s\in S}\psi^s &= \left(1-\psi^2\right)\sum_{i=1}^{k}\psi^{s_i} = \underbrace{\sum_{i=1}^{k}\psi^{s_i}}_{=\psi^{s_1}+\sum\limits_{i=2}^{k}\psi^{s_i}} - \underbrace{\psi^2\sum_{i=1}^{k}\psi^{s_i}}_{=\sum\limits_{i=1}^{k}\psi^2\psi^{s_i}=\sum\limits_{i=1}^{k}\psi^{s_i+2}=\sum\limits_{i=1}^{k-1}\psi^{s_i+2}+\psi^{s_k+2}} \\
&= \left(\psi^{s_1}+\sum_{i=2}^{k}\psi^{s_i}\right) - \left(\sum_{i=1}^{k-1}\underbrace{\psi^{s_i+2}}_{\geq\psi^{s_i+1}}+\psi^{s_k+2}\right) \\
&\leq \psi^{s_1}+\sum_{i=2}^{k}\psi^{s_i}-\sum_{i=1}^{k-1}\psi^{s_{i+1}}-\psi^{s_k+2} \\
&= \psi^{s_1}+\sum_{i=2}^{k}\psi^{s_i}-\sum_{i=2}^{k}\psi^{s_i}-\psi^{s_k+2} \qquad \text{(here, we substituted } i \text{ for } i+1 \text{ in the second sum)} \\
&= \underbrace{\psi^{s_1}}_{\leq\psi^2}-\underbrace{\psi^{s_k+2}}_{\geq 0} \leq \psi^2 - 0 = \psi^2.
\end{aligned}$$

Dividing this inequality by $1-\psi^2$ (here we are using $1-\psi^2 > 0$, which follows from $0 < \psi < 1$), we get $\sum\limits_{s\in S}\psi^s \leq \dfrac{\psi^2}{1-\psi^2} = \psi$. Replacing ψ by $\phi - 1$ in this inequality (since $\psi = \phi - 1$), we rewrite it as $\sum\limits_{s\in S}(\phi-1)^s \leq \phi - 1$. This proves Theorem 7.

Let us now come to the proof of Theorem 1. First, we formulate the existence part of this theorem:

Theorem 8 (existence part of the generalized Zeckendorf family identities). Let T be a finite set, and a_t be an integer for every $t \in T$.

Then, there exists a finite holey subset S of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n\in\mathbb{Z} \text{ which satisfies } n > \max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S\}\right)\right).$$

Before we start proving this, let us introduce a notation:

Definition. Let K be a subset of $\mathbb{Z}$, and $a \in \mathbb{Z}$. Then, $K + a$ will denote the subset $\{k + a \mid k \in K\}$ of $\mathbb{Z}$.

Clearly, $(K + a) + b = K + (a + b)$ for any two integers a and b. Also, $K + 0 = K$. Finally,

if K is a holey subset of $\mathbb{Z}$, and if $a \in \mathbb{Z}$, then $K + a$ is holey as well (1) (because every $s \in K + a$ satisfies $s + 1 \notin K + a$ [5]).

Proof of Theorem 8. Let us define a real constant C by $C = \sum\limits_{t \in T} \dfrac{1}{\sqrt{5}} (\phi - 1)^{a_t}$. Clearly, $C \geq 0$ (since $\phi - 1 > 0$).

First, notice that $0 < \phi - 1 < 1$ yields $\lim\limits_{n \to \infty} (\phi - 1)^n = 0$, so that $\lim\limits_{n \to \infty} ((\phi - 1)^n C) = \underbrace{\lim\limits_{n \to \infty} (\phi - 1)^n}_{=0} \cdot C = 0$ as well. Therefore, for every $\varepsilon > 0$ every sufficiently high integer N satisfies $(\phi - 1)^N C < \varepsilon$. In particular, taking $\varepsilon = 2 - \phi$ (here we are using that $2 - \phi > 0$), we see that every sufficiently high integer N satisfies $(\phi - 1)^N C < 2 - \phi$. Also, obviously, every sufficiently high integer N satisfies $N > \max \{-a_t \mid t \in T\}$. Hence, if we take our integer N high enough, we can ensure that it will satisfy *both* $(\phi - 1)^N C < 2 - \phi$ and $N > \max \{-a_t \mid t \in T\}$. So let us fix some integer $N \in \mathbb{N}_2$ high enough that it satisfies both $(\phi - 1)^N C < 2 - \phi$ and $N > \max \{-a_t \mid t \in T\}$.

Since $N > \max \{-a_t \mid t \in T\}$, we have $N > -a_t$ for every $t \in T$, and thus $N + a_t > 0$ for every $t \in T$. This shows that the Fibonacci number f_{N+a_t} is well-defined for every $t \in T$. (This was exactly the reason why we have required $N > \max \{-a_t \mid t \in T\}$. The reason for the second condition $(\phi - 1)^N C < 2 - \phi$ will become clear later in the proof.)

Let $\nu = \sum\limits_{t \in T} f_{N+a_t}$. Then, Lemma 3 (applied to ν instead of n) yields that there exists a finite holey subset Z_ν of $\mathbb{N}_2$ such that $\nu = \sum\limits_{t \in Z_\nu} f_t$. Let $S = Z_\nu + (-N)$. Then, the set $S = Z_\nu + (-N)$ is holey (by (1) (applied to $K = Z_\nu$ and $a = -N$), because Z_ν is a holey subset of $\mathbb{Z}$) and finite (since Z_ν is finite), and satisfies

$$\nu = \sum_{t \in Z_\nu} f_t = \sum_{s \in Z_\nu + (-N)} f_{N+s} \qquad \left(\begin{array}{c} \text{here, we substituted } N + s \text{ for } t \text{, because the map} \\ Z_\nu + (-N) \to Z_\nu,\ s \mapsto N + s \text{ is a bijection} \end{array} \right)$$
$$= \sum_{s \in S} f_{N+s} \qquad \left(\text{since } Z_\nu + (-N) = S\right).$$

So now we know that $\sum\limits_{t \in T} f_{N+a_t} = \sum\limits_{s \in S} f_{N+s}$ (because both sides of this equation equal ν).

So, we have chosen a large N and found a finite holey subset S of $\mathbb{Z}$ which satisfies $\sum\limits_{t \in T} f_{N+a_t} = \sum\limits_{s \in S} f_{N+s}$. In other words, we have showed that the equation

$$\sum_{t \in T} f_{n+a_t} = \sum_{s \in S} f_{n+s} \tag{2}$$

holds for $n = N$. But we must show that this equation holds for *every* $n > \max (\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})$. In order to do this, first let us prove that (2) holds for $n = N + 1$. Actually, we are going to show a bit more:

[5] *Proof.* Assume the contrary. In other words, assume that $s + 1 \in K + a$. Then, $s + 1 \in K + a = \{k + a \mid k \in K\}$, so there exists some $\ell \in K$ such that $s + 1 = \ell + a$.

Now, $s \in K + a = \{k + a \mid k \in K\}$ yields that there exists some $k \in K$ such that $s = k + a$. Since K is a holey set, we have ($s + 1 \notin K$ for every $s \in K$). Applying this to $s = k$, we get $k + 1 \notin K$. But $\ell + a = \underbrace{s}_{=k+a} + 1 = k + a + 1$ yields $\ell = (k + a + 1) - a = k + 1 \notin K$, contradicting $\ell \in K$. This contradiction shows that our assumption $s + 1 \in K + a$ was wrong. Hence, $s + 1 \notin K + a$ is proven.

Assertion α: Let $m \geq N$ be an integer such that (2) holds for $n = m$. Then, (2) also holds for $n = m + 1$.

Proof of Assertion α: Since (2) holds for $n = m$, we have $\sum\limits_{t\in T} f_{m+a_t} = \sum\limits_{s\in S} f_{m+s}$. Now,

$$\begin{aligned}
\sum_{t\in T} \underbrace{\underbrace{f_{(m+1)+a_t}}_{=f_{m+a_t+1}}}_{=f_{m+a_t+1}-\phi f_{m+a_t}+\phi f_{m+a_t}} - \sum_{s\in S} \underbrace{f_{(m+1)+s}}_{=f_{m+s+1}} &= \underbrace{\sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t} + \phi f_{m+a_t}\right)}_{=\sum\limits_{t\in T}\left(f_{m+a_t+1}-\phi f_{m+a_t}\right)+\phi\sum\limits_{t\in T} f_{m+a_t}} - \sum_{s\in S} f_{m+s+1} \\
&= \sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t}\right) + \phi \underbrace{\sum_{t\in T} f_{m+a_t}}_{=\sum\limits_{s\in S} f_{m+s}} - \sum_{s\in S} f_{m+s+1} \\
&= \sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t}\right) + \underbrace{\phi \sum_{s\in S} f_{m+s} - \sum_{s\in S} f_{m+s+1}}_{=\sum\limits_{s\in S}\left(\phi f_{m+s}-f_{m+s+1}\right)=-\sum\limits_{s\in S}\left(f_{m+s+1}-\phi f_{m+s}\right)} \\
&= \sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t}\right) - \sum_{s\in S} \left(f_{m+s+1} - \phi f_{m+s}\right),
\end{aligned}$$

so that

$$\begin{aligned}
&\left|\sum_{t\in T} f_{(m+1)+a_t} - \sum_{s\in S} f_{(m+1)+s}\right| = \left|\sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t}\right) - \sum_{s\in S} \left(f_{m+s+1} - \phi f_{m+s}\right)\right| \\
&\leq \left|\sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t}\right)\right| + \left|\sum_{s\in S} \left(f_{m+s+1} - \phi f_{m+s}\right)\right| \qquad \text{(by the triangle inequality)}. \qquad (3)
\end{aligned}$$

Now, the triangle inequality yields

$$\begin{aligned}
&\left|\sum_{t\in T} \left(f_{m+a_t+1} - \phi f_{m+a_t}\right)\right| \\
&\leq \sum_{t\in T} \underbrace{\left|f_{m+a_t+1} - \phi f_{m+a_t}\right|}_{\substack{=\frac{1}{\sqrt{5}}(\phi-1)^{m+a_t} \\ \text{(by Theorem 6, applied to} \\ m+a_t \text{ instead of } n)}} = \sum_{t\in T} \frac{1}{\sqrt{5}} \underbrace{(\phi - 1)^{m+a_t}}_{=(\phi-1)^m(\phi-1)^{a_t}} = (\phi - 1)^m \underbrace{\sum_{t\in T} \frac{1}{\sqrt{5}} (\phi - 1)^{a_t}}_{=C} \\
&= \underbrace{(\phi - 1)^m}_{\substack{\leq(\phi-1)^N \text{ (since} \\ 0<\phi-1<1 \text{ and } m\geq N)}} C \leq (\phi - 1)^N C \qquad \text{(since } C \geq 0) \\
&< 2 - \phi.
\end{aligned}$$

On the other hand, S is a holey subset of $\mathbb{Z}$, and thus the set $S + m$ is holey as well (by (1), applied to S and m instead of K and a), and besides $\underbrace{S}_{=Z_\nu+(-N)} + m = (Z_\nu + (-N)) + m = Z_\nu + ((-N) + m) = \{z + ((-N) + m) \mid z \in Z_\nu\} \subseteq \mathbb{N}_2$ (because every $z \in Z_\nu$ satisfies $\underbrace{z}_{\geq 2} + \left((-N) + \underbrace{m}_{\geq N}\right) \geq 2+((-N)+N) = 2$ and thus $z \in \mathbb{N}_2$).

Thus, $S+m$ is a holey finite subset of $\mathbb{N}_2$. Hence, Theorem 7 (applied to $S+m$ instead of S) yields $\sum_{s\in S+m} (\phi-1)^s \leq \phi - 1$.

The triangle inequality yields

$$\begin{aligned}
&\left|\sum_{s\in S} (f_{m+s+1} - \phi f_{m+s})\right| \\
&\leq \sum_{s\in S} \underbrace{|f_{m+s+1} - \phi f_{m+s}|}_{\substack{=\frac{1}{\sqrt{5}}(\phi-1)^{m+s} \\ \text{(by Theorem 6, applied to} \\ m+s \text{ instead of } n)}} = \sum_{s\in S} \frac{1}{\sqrt{5}} (\phi-1)^{m+s} = \frac{1}{\sqrt{5}} \sum_{s\in S} (\phi-1)^{m+s} \\
&= \frac{1}{\sqrt{5}} \underbrace{\sum_{s\in S+m} (\phi-1)^s}_{\leq \phi-1} \qquad \left(\begin{array}{c}\text{here, we substituted } s \text{ for } m+s\text{, since the map} \\ S \to S+m,\ s \mapsto m+s \text{ is a bijection}\end{array}\right) \\
&\leq \underbrace{\frac{1}{\sqrt{5}}}_{<1} \cdot (\phi-1) < \phi - 1.
\end{aligned}$$

Thus, (3) becomes

$$\begin{aligned}
\left|\sum_{t\in T} f_{(m+1)+a_t} - \sum_{s\in S} f_{(m+1)+s}\right| &\leq \underbrace{\left|\sum_{t\in T} (f_{m+a_t+1} - \phi f_{m+a_t})\right|}_{\leq 2-\phi} + \underbrace{\left|\sum_{s\in S} (f_{m+s+1} - \phi f_{m+s})\right|}_{<\phi-1} \\
&< (2-\phi) + (\phi-1) = 1.
\end{aligned}$$

Since $\sum_{t\in T} f_{(m+1)+a_t} - \sum_{s\in S} f_{(m+1)+s}$ is an integer, we thus conclude that $\sum_{t\in T} f_{(m+1)+a_t} - \sum_{s\in S} f_{(m+1)+s}$ is an integer with an absolute value < 1. But the only integer with an absolute value < 1 is 0. Thus, $\sum_{t\in T} f_{(m+1)+a_t} - \sum_{s\in S} f_{(m+1)+s} = 0$, so that $\sum_{t\in T} f_{(m+1)+a_t} = \sum_{s\in S} f_{(m+1)+s}$. In other words, (2) holds for $n = m+1$. This proves Assertion α.

Assertion α almost immediately yields:

Assertion β: The equation (2) holds for every $n \geq N$.

Proof of Assertion β: We are going to prove Assertion β by induction over n:

As the *induction base* we take the case $n = N$. In this case, the equation (2) holds (as we already know), so that Assertion β is proved for $n = N$, and thus the induction base is completed.

Induction step: Let $m \geq N$ be an integer. Assume that Assertion α holds for $n = m$. In other words, the equation (2) holds for $n = m$. Then, Assertion α yields that the equation (2) holds for $n = m+1$ as well. In other words, Assertion α holds for $n = m+1$ as well. This completes the induction step, and thus Assertion β is proven by induction over n.

With Assertion β we now know that (2) holds for all sufficiently high n, namely for all $n \geq N$. But in order to prove Theorem 8, we must show that it also holds for all $n > \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})$; usually, these n are not all $\geq N$. What remains to do is "backwards induction" or an application of the maximum principle. Here are the details:

Let $M = \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})$. We want to show that (2) holds for all $n > M$. In order to prove this, we assume the contrary, i. e. we assume that there exists some $n > M$ such that (2) does not hold. In other words, the set $\{n \in \mathbb{Z} \mid \text{we have } n > M, \text{ and (2) doesn't hold}\}$ is nonempty. This set is bounded from above by N (in fact, it doesn't contain any $n \geq N$, because (2) does hold for $n \geq N$ according to Assertion β), and thus it has a maximum (since a nonempty set of integers which is bounded from above always has a maximum). Let λ be this maximum. Then, $\lambda \in \{n \in \mathbb{Z} \mid \text{we have } n > M, \text{ and (2) doesn't hold}\}$. Therefore, λ itself satisfies $\lambda > M$, and (2) doesn't hold for $n = \lambda$. On the other hand,

$$\text{for every integer } \mu > \lambda, \text{ the equation (2) holds for } n = \mu. \tag{4}$$

[6] In particular, applying (4) to $\mu = \lambda + 1$, we see that (2) holds for $n = \lambda + 1$; in other words, $\sum\limits_{t \in T} f_{\lambda+1+a_t} = \sum\limits_{s \in S} f_{\lambda+1+s}$. Besides, applying (4) to $\mu = \lambda + 2$, we see that (2) holds for $n = \lambda + 2$; in other words, $\sum\limits_{t \in T} f_{\lambda+2+a_t} = \sum\limits_{s \in S} f_{\lambda+2+s}$.

Now, we are going to prove the equation $\sum\limits_{t \in T} f_{\lambda+a_t} = \sum\limits_{s \in S} f_{\lambda+s}$. (We notice that this equation indeed makes sense, because the Fibonacci number $f_{\lambda+a_t}$ is well-defined for every $t \in T$ [7], and because the Fibonacci number $f_{\lambda+s}$ is well-defined for every $s \in S$ [8].) In fact, every integer $n \geq 3$ satisfies $f_n = f_{n-1} + f_{n-2}$, so that $f_{n-2} = f_n - f_{n-1}$. Applying this to $n = \lambda + a_t + 2$, we obtain $f_{\lambda+a_t} = \underbrace{f_{\lambda+a_t+2}}_{=f_{\lambda+2+a_t}} - \underbrace{f_{\lambda+a_t+1}}_{=f_{\lambda+1+a_t}} = f_{\lambda+2+a_t} - f_{\lambda+1+a_t}$ for every $t \in T$. On the other hand, applying $f_{n-2} = f_n - f_{n-1}$ to $n = \lambda + s + 2$, we obtain $f_{\lambda+s} = \underbrace{f_{\lambda+s+2}}_{=f_{\lambda+2+s}} - \underbrace{f_{\lambda+s+1}}_{=f_{\lambda+1+s}} = f_{\lambda+2+s} - f_{\lambda+1+s}$ for every $s \in S$. Thus,

$$\begin{aligned}
\sum_{t \in T} \underbrace{f_{\lambda+a_t}}_{=f_{\lambda+2+a_t} - f_{\lambda+1+a_t}} &= \sum_{t \in T} \left(f_{\lambda+2+a_t} - f_{\lambda+1+a_t}\right) = \underbrace{\sum_{t \in T} f_{\lambda+2+a_t}}_{=\sum\limits_{s \in S} f_{\lambda+2+s}} - \underbrace{\sum_{t \in T} f_{\lambda+1+a_t}}_{=\sum\limits_{s \in S} f_{\lambda+1+s}} = \sum_{s \in S} f_{\lambda+2+s} - \sum_{s \in S} f_{\lambda+1+s} \\
&= \sum_{s \in S} \underbrace{\left(f_{\lambda+2+s} - f_{\lambda+1+s}\right)}_{=f_{\lambda+s}} = \sum_{s \in S} f_{\lambda+s}.
\end{aligned}$$

[6] *Proof.* Assume the opposite. Then, not for every integer $\mu > \lambda$, the equation (2) holds for $n = \mu$. So there exists some integer $\mu > \lambda$ such that the equation (2) doesn't hold for $n = \mu$. Since this μ must satisfy $\mu > M$ (since $\mu > \lambda > M$), we conclude that μ lies in the set $\{n \in \mathbb{Z} \mid \text{we have } n > M, \text{ and (2) doesn't hold}\}$. Since λ is the maximum of this set, we thus get $\mu \leq \lambda$, contradicting $\mu > \lambda$. This contradiction shows that our assumption was wrong, and thus (4) is correct.

[7] In fact, $\lambda > M = \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}) \geq \max\{-a_t \mid t \in T\}$ yields that $\lambda > -a_t$ for every $t \in T$, so that $\lambda + a_t > 0$, and thus $f_{\lambda+a_t}$ is well-defined.

[8] In fact, $\lambda > M = \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}) \geq \max\{-s \mid s \in S\}$ yields that $\lambda > -s$ for every $s \in S$, so that $\lambda + s > 0$, and thus $f_{\lambda+s}$ is well-defined.

In other words, (2) holds for $n = \lambda$. This contradicts to our knowledge that (2) doesn't hold for $n = \lambda$. This contradiction shows that our assumption (the assumption that there exists some $n > M$ such that (2) does not hold) was wrong. Hence, there exists no $n > M$ such that (2) does not hold. In other words, (2) holds for every $n > M$. Since $M = \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})$, this is equivalent to saying that (2) holds for every $n > \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})$. Consequently, Theorem 8 is proven.

All that remains now is the (rather trivial) uniqueness part of Theorem 1:

Lemma 9 (uniqueness part of the generalized Zeckendorf family identities). Let T be a finite set, and a_t be an integer for every $t \in T$.

Let S be a finite holey subset of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which satisfies } n > \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\})\right).$$

Let S' be a finite holey subset of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S'} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which satisfies } n > \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S'\})\right).$$

Then, $S = S'$.

Proof of Lemma 9. Let

$$n = \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\} \cup \{-s \mid s \in S'\}) + 2.$$

Then,

$$n > \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\} \cup \{-s \mid s \in S'\}) \geq \max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}),$$

so that

$$\begin{aligned} \sum_{t\in T} f_{n+a_t} &= \sum_{s\in S} f_{n+s} && \text{(by the condition of Lemma 9)} \\ &= \sum_{t\in S+n} f_t && \left(\begin{array}{c}\text{here, we substituted } t \text{ for } n+s \text{, since the map} \\ S \to S+n,\ s \mapsto n+s \text{ is a bijection}\end{array}\right). \end{aligned}$$

Similarly, $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{t\in S'+n} f_t$.

Now, $S+n$ is a holey set (by (1) (applied to $K = S$ and $a = n$), since S is a holey subset of $\mathbb{Z}$) and a subset of $\mathbb{N}_2$ [9]. In other words, $S+n$ is a finite holey subset of $\mathbb{N}_2$.

[9] *Proof.* Since

$$n = \underbrace{\max(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\} \cup \{-s \mid s \in S'\})}_{\geq \max\{-s \mid s\in S\}} + 2 \geq \max\{-s \mid s \in S\} + 2,$$

we have $n - 2 \geq \max\{-s \mid s \in S\}$, and thus $n - 2 \geq -s$ for every $s \in S$. Hence, every $s \in S$ satisfies $n - 2 + s \geq 0$, which rewrites as $s + n \geq 2$. Equivalently, $s + n \in \mathbb{N}_2$. Thus,

$$S + n = \left\{ \underbrace{s+n}_{\in \mathbb{N}_2} \mid s \in S \right\} \subseteq \mathbb{N}_2.$$

Similarly, $S'+n$ is a finite holey subset of $\mathbb{N}_2$. Applying Lemma 4 to $S+n$, $S'+n$ and $\sum\limits_{t\in T} f_{n+a_t}$ instead of T, T' and n yields that $S+n=S'+n$ (because $\sum\limits_{t\in T} f_{n+a_t}=\sum\limits_{t\in S+n} f_t$ and $\sum\limits_{t\in T} f_{n+a_t}=\sum\limits_{t\in S'+n} f_t$). Hence,

$$S=S+\underbrace{0}_{=(n+(-n))}=S+(n+(-n))=\underbrace{(S+n)}_{=S'+n}+(-n)=(S'+n)+(-n)=S'+\underbrace{(n+(-n))}_{=0}=S'+0=S'.$$

This proves Lemma 9.

Proof of Theorem 1. Fix some $n\in\mathbb{N}$. Then, there exists a finite holey subset S of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t}=\sum_{s\in S} f_{n+s} \text{ for every } n\in\mathbb{Z} \text{ which satisfies } n>\max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S\}\right)\right)$$

(according to Theorem 8), and such a subset is unique (because any two such subsets are equal, according to Lemma 9). This proves Theorem 1.

References

[1] Philip Matchett Wood, Doron Zeilberger, *A translation method for finding combinatorial bijections*, to appear in Annals of Combinatorics.
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trans-method.html`

[2] Darij Grinberg, *Zeckendorf family identities generalized*, version 7, 18 March 2011 *brief version*.
`http://www.cip.ifi.lmu.de/~grinberg/zeckendorfBRIEF.pdf`