Zeckendorf family identities generalized
Darij Grinberg, version 7, 18 March 2011
brief version

This is a brief version of my text [2]. For more detailed proofs, see [2] (but beware that [2] is sometimes over-precise and very boring).
This note has never been proofread by myself or anyone else. If you find any mistakes or typos, please inform me at $\Delta\Gamma$@gmail.com where $\Delta =$darij and $\Gamma =$grinberg
Thank you!

The purpose of this note is to establish a generalization of the so-called *Zeckendorf family identities* which were discussed in [1]. First, some definitions:

Definitions. 1) A subset S of $\mathbb{Z}$ is called *holey* if it satisfies ($s+1 \notin S$ for every $s \in S$).

2) Let $(f_1, f_2, f_3, ...)$ be the Fibonacci sequence (defined by $f_1 = f_2 = 1$ and the recurrence relation ($f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{N}$ satisfying $n \geq 3$)).

Our main theorem is:

Theorem 1 (generalized Zeckendorf family identities). Let T be a finite set, and a_t be an integer for every $t \in T$.

Then, there exists one and only one finite holey subset S of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)\right).$$

Remarks.

1) The *Zeckendorf family identities*[1] from [1] are the result of applying Theorem 1 to the case when all a_t are $= 0$.

2) The condition $n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$ in Theorem 1 is just a technical condition made in order to ensure that the Fibonacci numbers f_{n+a_t} for all $t \in T$ and f_{n+s} for all $s \in S$ are well-defined. (If we would define the Fibonacci numbers f_n for integers $n \leq 0$ by extending the recurrence relation $f_n = f_{n-1} + f_{n-2}$ "to the left", then we could drop this condition.)

The following is my proof of Theorem 1. It does not even try to be combinatorial - it is pretty much the opposite. Technically, it is completely elementary and does not

[1]The first seven of these identities are

$$1f_n = f_n \text{ for all } n \geq 1;$$
$$2f_n = f_{n-2} + f_{n+1} \text{ for all } n \geq 3;$$
$$3f_n = f_{n-2} + f_{n+2} \text{ for all } n \geq 3;$$
$$4f_n = f_{n-2} + f_n + f_{n+2} \text{ for all } n \geq 3;$$
$$5f_n = f_{n-4} + f_{n-1} + f_{n+3} \text{ for all } n \geq 5;$$
$$6f_n = f_{n-4} + f_{n+1} + f_{n+3} \text{ for all } n \geq 5;$$
$$7f_n = f_{n-4} + f_{n+4} \text{ for all } n \geq 5.$$

resort to any theorems from analysis; but the method used (choosing a "large enough" N to make an estimate work) is an analytic one.

First, some lemmas and notations:

We denote by $\mathbb{N}$ the set $\{0, 1, 2, ...\}$ (and not the set $\{1, 2, 3, ...\}$, like some other authors do). Also, we denote by $\mathbb{N}_2$ the set $\{2, 3, 4, ...\} = \mathbb{N} \setminus \{0, 1\}$.

Also, let $\phi = \dfrac{1+\sqrt{5}}{2}$. We notice that $\phi \approx 1.618...$ and that $\phi^2 = \phi + 1$.

First, some basic (and known) lemmas on the Fibonacci sequence:

Lemma 2. If S is a finite holey subset of $\mathbb{N}_2$, then $\sum\limits_{t \in S} f_t < f_{\max S + 1}$.

Proof. This is rather clear either by a telescoping sum argument (write the set S in the form $\{s_1, s_2, ..., s_k\}$ with $s_1 < s_2 < ... < s_k$, notice that

$$\sum_{t \in S} f_t = \sum_{i=1}^{k} f_{s_i} = \sum_{i=1}^{k} \left(f_{s_i+1} - f_{s_i-1}\right) = \sum_{i=1}^{k-1} \left(f_{s_i+1} - f_{s_{i+1}-1}\right) + \underbrace{f_{s_k+1}}_{=f_{\max S+1}} - \underbrace{f_{s_1-1}}_{>0},$$

and use $s_i + 1 \leq s_{i+1} - 1$ since the set S is holey) or by induction over $\max S$ (use $f_{\max S+1} = f_{\max S} + f_{\max S-1}$ here).

Lemma 3 (existence part of the Zeckendorf theorem). For every nonnegative integer n, there exists a finite holey subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$.

Proof. Induction over n. The main idea here is to let t_1 be the maximal $\tau \in \mathbb{N}_2$ satisfying $f_\tau \leq n$, and apply Lemma 3 to $n - t_1$ instead of n. The details are left to the reader (and can be found in [2]).

Lemma 4 (uniqueness part of the Zeckendorf theorem). Let $n \in \mathbb{N}$, and let T and T' be two finite holey subsets of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$ and $n = \sum\limits_{t \in T'} f_t$. Then, $T = T'$.

Proof. Induction over n. Use Lemma 2 to show that $\max T < \max T' + 1$ and $\max T' < \max T + 1$, resulting in $\max T = \max T'$. Hence, the sets T and T' have an element in common, and we can reduce the situation to one with a smaller n by removing this common element from both sets.

Lemmata 3 and 4 together yield:

Theorem 5 (Zeckendorf theorem). For every nonnegative integer n, there exists one and only one finite holey subset T of $\mathbb{N}_2$ such that $n = \sum\limits_{t \in T} f_t$.

We will denote this set T by Z_n. Thus, $n = \sum\limits_{t \in Z_n} f_t$.

Now for something completely trivial:

Theorem 6. For every $n \in \mathbb{N}_2$, we have $\left|f_{n+1} - \phi f_n\right| = \dfrac{1}{\sqrt{5}} (\phi - 1)^n$.

Proof. Binet's formula yields $f_n = \dfrac{\phi^n - \phi^{-n}}{\sqrt{5}}$ and $f_{n+1} = \dfrac{\phi^{n+1} - \phi^{-(n+1)}}{\sqrt{5}}$; the rest is computation.

Yet another lemma:

Theorem 7. If S is a finite holey subset of $\mathbb{N}_2$, then $\sum_{s\in S} (\phi-1)^s \leq \phi - 1$.

Proof of Theorem 7. Since S is a holey subset of $\mathbb{N}_2$, the smallest element of S is at least 2, the second smallest element of S is at least 4 (since it is larger than the smallest element by at least 2), the third smallest element of S is at least 6 (since it is larger than the second smallest element by at least 2), and so on. Since $\mathbb{N} \to \mathbb{R}$, $s \mapsto (\phi-1)^s$ is a monotonically decreasing function (as $0 \leq \phi - 1 \leq 1$), we thus have

$$\sum_{s\in S} (\phi-1)^s \leq \sum_{s\in\{2,4,6,\ldots\}} (\phi-1)^s = \sum_{t\in\{1,2,3,\ldots\}} (\phi-1)^{2t} = \phi - 1$$

(by the formula for the sum of the geometric series, along with some computations). This proves Theorem 7.

Let us now come to the proof of Theorem 1. First, we formulate the existence part of this theorem:

Theorem 8 (existence part of the generalized Zeckendorf family identities). Let T be a finite set, and a_t be an integer for every $t \in T$.

Then, there exists a finite holey subset S of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)\right).$$

Before we start proving this, we need a new notation:

Definition. Let K be a subset of $\mathbb{Z}$, and $a \in \mathbb{Z}$. Then, $K + a$ will denote the subset $\{k + a \mid k \in K\}$ of $\mathbb{Z}$.

Clearly, $(K + a) + b = K + (a + b)$ for any two integers a and b. Also, $K + 0 = K$. Finally, if K is a holey subset of $\mathbb{Z}$, and if $a \in \mathbb{Z}$, then $K + a$ is holey as well.

Proof of Theorem 8. Choose a high enough integer N. What exactly "high enough" means we will see later; at the moment, we only require $N \in \mathbb{N}_2$ and $N > \max\{-a_t \mid t \in T\}$. We might later want N to be even higher, however.

Let $\nu = \sum_{t\in T} f_{N+a_t}$. Then, Lemma 3 yields $\nu = \sum_{t\in Z_\nu} f_t$ for a finite holey subset Z_ν of $\mathbb{N}_2$. Let $S = \{t - N \mid t \in Z_\nu\}$. Then, $S = Z_\nu + (-N)$ is a finite holey subset of $\mathbb{Z}$, and $\nu = \sum_{t\in Z_\nu} f_t$ becomes $\nu = \sum_{s\in S} f_{N+s}$. So now we know that $\sum_{t\in T} f_{N+a_t} = \sum_{s\in S} f_{N+s}$ (because both sides of this equation equal ν).

So, we have chosen a high N and found a finite holey subset S of $\mathbb{Z}$ which satisfies $\sum_{t\in T} f_{N+a_t} = \sum_{s\in S} f_{N+s}$. But Theorem 8 is not proven yet: Theorem 8 requires us to prove that there exists *one* finite holey subset S of $\mathbb{Z}$ which works for *every* N, while at the moment we cannot be sure yet whether different N's wouldn't produce *different* sets

S. And, in fact, different N's *can* produce different sets S, but (fortunately!) only if the N's are too small. If we take N high enough, the set S that we obtained turns out to be *universal*, i. e. it satisfies

$$\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \qquad \text{for every } n\in\mathbb{Z} \text{ which satisfies } n > \max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S\}\right). \tag{1}$$

We are now going to prove this.

In order to prove (1), we need two assertions:

Assertion 1: If some $n\in\mathbb{Z}$ satisfies $n\geq N$ and $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{s\in S} f_{n+s}$, then $\sum\limits_{t\in T} f_{(n+1)+a_t} = \sum\limits_{s\in S} f_{(n+1)+s}$.

Assertion 2: If some $n\in\mathbb{Z}$ satisfies $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{s\in S} f_{n+s}$ and $\sum\limits_{t\in T} f_{(n+1)+a_t} = \sum\limits_{s\in S} f_{(n+1)+s}$, then $\sum\limits_{t\in T} f_{(n-1)+a_t} = \sum\limits_{s\in S} f_{(n-1)+s}$ (if $n-1 > \max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S\}\right)$).

Obviously, Assertion 1 yields (by induction) that $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{s\in S} f_{n+s}$ for every $n\geq N$, and Assertion 2 then finishes off the remaining n's (by backwards induction, or, to be more precise, by an induction step from $n+1$ and n to $n-1$). Thus, once both Assertions 1 and 2 are proven, (1) will follow and thus Theorem 8 will be proven.

Assertion 2 is almost trivial (just notice that

$$\sum_{t\in T} f_{(n-1)+a_t} = \sum_{t\in T} \underbrace{f_{n+a_t-1}}_{=f_{n+a_t+1}-f_{n+a_t}} = \sum_{t\in T} f_{n+a_t+1} - \sum_{t\in T} f_{n+a_t} = \sum_{t\in T} f_{(n+1)+a_t} - \sum_{t\in T} f_{n+a_t}$$

and

$$\sum_{s\in S} f_{(n-1)+s} = \sum_{s\in S} \underbrace{f_{n+s-1}}_{=f_{n+s+1}-f_{n+s}} = \sum_{s\in S} f_{n+s+1} - \sum_{s\in S} f_{n+s} = \sum_{s\in S} f_{(n+1)+s} - \sum_{s\in S} f_{n+s}$$

), so it only remains to prove Assertion 1.

So let us prove Assertion 1. Here we are going to use that N is high enough (because otherwise, Assertion 1 wouldn't hold). We have $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{s\in S} f_{n+s}$ by assumption, so that $\sum\limits_{t\in T} f_{n+a_t} - \sum\limits_{s\in S} f_{n+s} = 0$. Thus,

$$\begin{aligned}
\sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s} &= \sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s} - \phi\left(\sum_{t\in T} f_{n+a_t} - \sum_{s\in S} f_{n+s}\right) \\
&= \sum_{t\in T}\left(f_{(n+1)+a_t} - \phi f_{n+a_t}\right) - \sum_{s\in S}\left(f_{(n+1)+s} - \phi f_{n+s}\right) \\
&= \sum_{t\in T}\left(f_{n+a_t+1} - \phi f_{n+a_t}\right) - \sum_{s\in S}\left(f_{n+s+1} - \phi f_{n+s}\right),
\end{aligned}$$

so that

$$\begin{aligned}
&\left|\sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s}\right| = \left|\sum_{t\in T}\left(f_{n+a_t+1} - \phi f_{n+a_t}\right) - \sum_{s\in S}\left(f_{n+s+1} - \phi f_{n+s}\right)\right| \\
&\leq \sum_{t\in T}\left|f_{n+a_t+1} - \phi f_{n+a_t}\right| + \sum_{s\in S}\left|f_{n+s+1} - \phi f_{n+s}\right| \qquad \text{(by the triangle inequality)} \\
&= \frac{1}{\sqrt{5}}\sum_{t\in T}(\phi-1)^{n+a_t} + \frac{1}{\sqrt{5}}\sum_{s\in S}(\phi-1)^{n+s} \qquad \text{(by Theorem 6)} \\
&< \sum_{t\in T}(\phi-1)^{n+a_t} + \sum_{s\in S}(\phi-1)^{n+s} \qquad \left(\text{since } \frac{1}{\sqrt{5}} < 1\right) \\
&\leq \sum_{t\in T}(\phi-1)^{N+a_t} + \sum_{s\in S}(\phi-1)^{N+s} \qquad \left(\begin{array}{c}\text{since } (\phi-1)^{n+a_t} \leq (\phi-1)^{N+a_t} \text{ and} \\ (\phi-1)^{n+s} \leq (\phi-1)^{N+s}\text{, because} \\ n \geq N \text{ and } 0 \leq \phi-1 \leq 1\end{array}\right) \\
&= \sum_{t\in T}(\phi-1)^{N+a_t} + \sum_{t\in Z_\nu}(\phi-1)^{t} \qquad (\text{since } S = \{t-N \mid t\in Z_\nu\}) \\
&= \sum_{t\in T}(\phi-1)^{N+a_t} + \sum_{s\in Z_\nu}(\phi-1)^{s} = (\phi-1)^N\sum_{t\in T}(\phi-1)^{a_t} + \sum_{s\in Z_\nu}(\phi-1)^{s} \\
&\leq (\phi-1)^N\sum_{t\in T}(\phi-1)^{a_t} + (\phi-1) \qquad (2)
\end{aligned}$$

(since $\sum\limits_{s\in Z_\nu}(\phi-1)^s \leq \phi - 1$ by Theorem 7, because Z_ν is a holey subset of $\mathbb{N}_2$).

Now, $\sum\limits_{t\in T}(\phi-1)^{a_t}$ is a constant, while $(\phi-1)^N \to 0$ for $N \to \infty$. Hence, we can make the product $(\phi-1)^N\sum\limits_{t\in T}(\phi-1)^{a_t}$ arbitrarily close to 0 if we choose N high enough. Since $\phi - 1 < 1$, we have

$$(\phi-1)^N\sum_{t\in T}(\phi-1)^{a_t} + (\phi-1) < 1 \qquad (3)$$

if $(\phi-1)^N\sum\limits_{t\in T}(\phi-1)^{a_t}$ is sufficiently close to 0, what we can enforce by taking a high enough N. This is exactly the point where we require N to be high enough.

So let us take N high enough so that (3) holds. Combined with (2), it then yields

$$\left|\sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s}\right| < 1,$$

which leads to $\left|\sum\limits_{t\in T} f_{(n+1)+a_t} - \sum\limits_{s\in S} f_{(n+1)+s}\right| = 0$ (since $\left|\sum\limits_{t\in T} f_{(n+1)+a_t} - \sum\limits_{s\in S} f_{(n+1)+s}\right|$ is a nonnegative integer). In other words, $\sum\limits_{t\in T} f_{(n+1)+a_t} = \sum\limits_{s\in S} f_{(n+1)+s}$. This completes the proof of Assertion 1, and, with it, the proof of Theorem 8.

All that remains now is the (rather trivial) uniqueness part of Theorem 1:

Lemma 9 (uniqueness part of the generalized Zeckendorf family identities). Let T be a finite set, and a_t be an integer for every $t \in T$.

Let S be a finite holey subset of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n\in\mathbb{Z} \text{ which satisfies } n > \max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S\}\right)\right).$$

Let S' be a finite holey subset of $\mathbb{Z}$ such that

$$\left(\sum_{t\in T} f_{n+a_t} = \sum_{s\in S'} f_{n+s} \text{ for every } n\in\mathbb{Z} \text{ which satisfies } n > \max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S'\}\right)\right).$$

Then, $S = S'$.

Proof of Lemma 9. Let

$$n = \max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S\}\cup\{-s \mid s\in S'\}\right) + 2. \tag{4}$$

Then, n satisfies $n > \max\left(\{-a_t \mid t\in T\}\cup\{-s \mid s\in S\}\right)$, so that

$$\begin{aligned}\sum_{t\in T} f_{n+a_t} &= \sum_{s\in S} f_{n+s} && \text{(by the condition of Lemma 9)}\\ &= \sum_{t\in S+n} f_t && \left(\begin{array}{c}\text{here, we substituted } t \text{ for } n+s\text{, since the map}\\ S\to S+n,\ s\mapsto n+s \text{ is a bijection}\end{array}\right).\end{aligned}$$

Similarly, $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{t\in S'+n} f_t$. Hence, $\sum\limits_{t\in S+n} f_t = \sum\limits_{t\in S'+n} f_t$. Since the sets $S+n$ and $S'+n$ are both holey (since so are S and S') and finite (since so are S and S'), and are subsets of $\mathbb{N}_2$ (here is where we use (4)), we can now conclude from Lemma 4 that $S+n = S'+n$, so that $S = S'$, proving Lemma 9.

Now, Theorem 1 is clear, since the existence follows from Theorem 8 and the uniqueness from Lemma 9.

References

[1] Philip Matchett Wood, Doron Zeilberger, *A translation method for finding combinatorial bijections*, to appear in Annals of Combinatorics.
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trans-method.html`

[2] Darij Grinberg, *Zeckendorf family identities generalized*, version 7, 18 March 2011 *long version*.
`http://www.cip.ifi.lmu.de/~grinberg/zeckendorfLONG.pdf`