Math 221 Winter 2025 (Darij Grinberg): midterm 3
due date: Monday 2025-03-17 at 11:59PM on gradescope (`https://www.gradescope.com/courses/930212`).
Please solve only **3 of the 8 exercises**.
NO collaboration allowed – this is a midterm!
(But you can still ask me questions.)

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$. Recall also that $[n]$ denotes the set $\{1, 2, \ldots, n\}$ whenever $n \in \mathbb{N}$.

Exercise 1. For each of the following functions, determine whether it is injective, surjective and/or bijective:

(Proofs are not required in this exercise!)

(a) The function

$$f : \mathbb{Q} \to \mathbb{Q}, \quad x \mapsto \frac{x}{1+x^2}.$$

(b) The function

$$f : \mathbb{Z} \to \mathbb{Q}, \quad x \mapsto \frac{x}{1+x^2}.$$

(c) The function

$$f : \{\text{finite nonempty subsets of } \mathbb{Z}\} \to \mathbb{Z}, \quad S \mapsto \min S$$

(which sends each set to its smallest element).

(d) The function

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (x, y) \mapsto 2x + 3y.$$

(e) The function

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (x, y) \mapsto 2x + 3y.$$

Exercise 2. Let $n \in \mathbb{N}$. Consider the set $[2n] = \{1, 2, \ldots, 2n\}$.

A set of integers will be called **parity-ambivalent** if it contains at least one even element and at least one odd element. (For instance, $\{2, 4, 5\}$ is parity-ambivalent, but $\{2, 4, 10\}$ is not.)

Compute the # of all parity-ambivalent subsets of $[2n]$.

[**Hint:** One way to solve it is by negation: How many subsets of $[2n]$ contain **no** even element? How many contain **no** odd element? How many contain neither?]

Exercise 3. **(a)** How many 7-digit numbers are there? (A "k-digit number" means a nonnegative integer that has k digits when written in the decimal system (without leading zeroes). For example, 3902 is a 4-digit number, not a 5-digit number.)

(b) How many 7-digit numbers are there that have no two equal digits?

(c) How many 7-digit numbers have an even sum of digits?

(d) How many 7-digit numbers are palindromes? (A "**palindrome**" is a number such that reading its digits from right to left yields the same number. For example, 5 and 1331 and 49094 are palindromes.)

[If your answer is a product or power, **you do not need to simplify it to a number**.]

Recall the notion of a "left inverse" of a map (as defined in Exercise 3 on homework set #6).

Exercise 4. Let $n, m \in \mathbb{N}$. Let X be an n-element set. Let Y be an m-element set. Let $f : X \to Y$ be an injective map. Prove that f has exactly n^{m-n} many left inverses.

If S is any set, and n is any nonnegative integer, then the Cartesian product $\underbrace{S \times S \times \cdots \times S}_{n \text{ times}}$ is denoted by S^n. For example, $S^3 = S \times S \times S$.

Recall that a k-tuple $(i_1, i_2, \ldots, i_k)$ is called **injective** if its k entries $i_1, i_2, \ldots, i_k$ are all distinct (i.e., if $i_a \neq i_b$ for all $a \neq b$).

Exercise 5. Let $n \in \mathbb{N}$. How many injective $(2n)$-tuples $(i_1, i_2, \ldots, i_{2n}) \in [2n]^{2n}$ are there such that all of the first n entries $i_1, i_2, \ldots, i_n$ are even?

(For instance, for $n = 2$, there are 4 such tuples: $(2, 4, 1, 3)$, $(2, 4, 3, 1)$, $(4, 2, 1, 3)$ and $(4, 2, 3, 1)$.)

Exercise 6. Let $n \geq 2$ be an integer.

(a) How many injective n-tuples $(i_1, i_2, \ldots, i_n) \in [n]^n$ begin with the entry 2 ?

(b) How many injective n-tuples $(i_1, i_2, \ldots, i_n) \in [n]^n$ contain the entry 1 before the entry 2 ? ("Before" means "somewhere to the left of", not necessarily "immediately before". For instance, for $n = 4$, the 4-tuple $(1, 3, 2, 4)$ qualifies, but the 4-tuple $(2, 3, 1, 4)$ does not.)

(c) How many injective n-tuples $(i_1, i_2, \ldots, i_n) \in [n]^n$ contain the entry 1 immediately preceding the entry 2 ? (Here, $(1, 3, 2, 4)$ no longer qualifies, but $(4, 1, 2, 3)$ does.)

If $h : S \to S$ is any map from a set to itself, then a **fixed point** of h means an element $s \in S$ satisfying $h(s) = s$. The set of all fixed points of h will be called $\operatorname{Fix} h$.

Exercise 7. Let X and Y be two finite sets (not necessarily of the same size).

Let $f : X \to Y$ and $g : Y \to X$ be two maps. Prove that

$$|\operatorname{Fix}(f \circ g)| = |\operatorname{Fix}(g \circ f)|.$$

[**Hint:** Show that $f(x) \in \operatorname{Fix}(f \circ g)$ for each $x \in \operatorname{Fix}(g \circ f)$. Thus, there is a map

$$f' : \operatorname{Fix}(g \circ f) \to \operatorname{Fix}(f \circ g), \quad x \mapsto f(x).$$

Construct a similar map g' in the opposite direction. Prove that these two maps f' and g' are inverse to each other.]

Now, recall Exercise 6 on homework set #6. In that exercise, we decided to call a set S of integers **pseudolacunar** if no two elements s, t of S satisfy $|s - t| = 2$. We denoted the # of pseudolacunar subsets of $[n]$ (for a given $n \in \mathbb{N}$) by p_n.

Recall also the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ that we introduced in §1.5. It is defined recursively by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$.

Finally, recall the floor notation (see Definition 3.3.13).

Exercise 8. Prove that

$$p_n = f_{\lfloor (n+1)/2 \rfloor + 2} \cdot f_{\lfloor n/2 \rfloor + 2} \qquad \text{for each } n \geq 2.$$

[**Hint:** What does the pseudolacunarity of a set S mean for the even elements of S ? What does it mean for the odd elements of S ?]