Math 221 Winter 2025 (Darij Grinberg): midterm 2
due date: Saturday 2025-03-01 at 11:59PM on gradescope (`https://www.gradescope.com/courses/930212`).
Please solve only **3 of the 6 exercises**.
NO collaboration allowed – this is a midterm!
(But you can still ask me questions.)

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Theorem 3.6.3 shows that if p is a prime, then all the binomial coefficients $\dbinom{p}{i}$ in the p-th row of Pascal's triangle are divisible by p (except for the two 1's on the borders of the triangle). The following exercise, in contrast, claims that the binomial coefficients in the $(p-1)$-st row are alternatingly congruent to 1 and to -1 modulo p:

Exercise 1. Let p be a prime. Prove that

$$\binom{p-1}{i} \equiv (-1)^i \bmod p \qquad \text{for each } i \in \{0, 1, \ldots, p-1\}.$$

[**Hint:** What connects the three binomial coefficients $\dbinom{p-1}{i}$, $\dbinom{p-1}{i-1}$ and $\dbinom{p}{i}$?]

Here is another divisibility related to prime numbers:

Exercise 2. Let p be a prime larger than 3. Prove that $p^2 \equiv 1 \bmod 24$.

[**Hint:** Recall some older problems. Also note that the integers 3 and 8 are coprime.]

Now to some counting problems. The symbol "#" means "number".

To "compute" a number means to find a closed-form expression for this number (with no summation signs) and to prove this formula. I expect proofs to be given at the level of detail and rigor seen in Chapter 4.

Exercise 3. Let $n \geq 2$ be an integer.

(a) Compute the # of subsets of $\{1, 2, \ldots, n\}$ that contain both 1 and 2.

(b) Compute the # of 3-element subsets of $\{1, 2, \ldots, n\}$ that contain both 1 and 2.

Exercise 4. Let $n \in \mathbb{N}$. Compute the # of all pairs $(a, b) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ satisfying $a \equiv b \bmod 2$.

(The answer will depend on whether n is even or odd. You can find a unified formula using the floor of a number, but you don't have to.)

Exercise 5. Let $n \in \mathbb{N}$. Compute the # of all pairs $(a, b) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ satisfying $2a \leq b$.

(Again, the answer will depend on whether n is even or odd.)

Finally, an exercise combining functions with number theory:

Exercise 6. For any positive integer d, let us define the function

$$r_d : \mathbb{Z} \to \mathbb{Z},$$
$$n \mapsto n\%d$$

(which sends each integer n to the remainder of the division of n by d). For example, $r_5(18) = 18\%5 = 3$ and $r_6(18) = 18\%6 = 0$.

(a) Make a table of the values of the function $r_2 \circ r_3$ on the inputs $0, 1, 2, 3, 4, 5$.

(b) Prove that $r_2 \circ r_3 \neq r_2$.

(c) Let d and e be two positive integers such that $d \mid e$. Prove that $r_d \circ r_e = r_d$.

Appendix: Rows 0 to 15 of Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

Rows corresponding to prime numbers n are printed in bold. LaTeX/tikz source code courtesy of Caramdir on tex.stackexchange (post #17527).