Math 221 Winter 2025 (Darij Grinberg): homework set 6

due date: Saturday 2025-03-08 at 11:59PM on gradescope (`https://www.gradescope.com/courses/930212`).

Please solve only **3 of the 7 exercises**.

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The first three exercises are about functions and their properties:

Exercise 1. For each of the following functions, determine whether it is injective, surjective and/or bijective:

(Proofs are not required in this exercise!)

(a) The function

$$f : \mathbb{Z} \to \mathbb{Z},\quad x \mapsto x^2.$$

(b) The function

$$f : \mathbb{Z} \to \mathbb{Z},\quad x \mapsto x^3.$$

(c) The function

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z},\quad (x, y) \mapsto x^2 + y^2.$$

(d) The function

$$f : \mathbb{Z} \to \mathbb{Z},\quad x \mapsto 3 - x.$$

(e) The function

$$f : \mathbb{Z} \to \mathbb{Z},\quad x \mapsto 3 - 2x.$$

(f) The function

$$f : \mathbb{N} \to \mathbb{N},\quad x \mapsto x!.$$

(Keep in mind that $0 \in \mathbb{N}$.)

(g) The function

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},\quad (x, y) \mapsto (x + y,\ x - y).$$

(h) The function

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},$$
$$(x, y) \mapsto (x - y,\ y - x).$$

(i) The function

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},$$
$$(x, y) \mapsto (x + 2y,\ x + y).$$

(j) The function

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},$$
$$(x, y) \mapsto (x + 2y,\ 2x + y).$$

Exercise 2. Let A, B, C, D be four sets. Let $f : A \to C$ and $g : B \to D$ be two maps. Define a new map $f * g : A \times B \to C \times D$ by setting

$$(f * g)(a, b) = (f(a),\ g(b)) \qquad \text{for every pair } (a, b) \in A \times B.$$

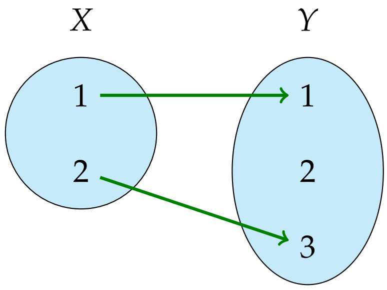
Prove the following:
(a) If f and g are injective, then $f * g$ is injective.
(b) If f and g are surjective, then $f * g$ is surjective.

Exercise 3. Let X and Y be two sets. Let $f : X \to Y$ be a map.

A **left inverse** of f means a map $g : Y \to X$ that satisfies $g \circ f = \mathrm{id}_X$ (but not necessarily $f \circ g = \mathrm{id}_Y$).

A **right inverse** of f means a map $g : Y \to X$ that satisfies $f \circ g = \mathrm{id}_Y$ (but not necessarily $g \circ f = \mathrm{id}_X$).

(a) Prove that f has a right inverse if and only if f is surjective.
(b) Assume that $X \neq \varnothing$. Prove that f has a left inverse if and only if f is injective.
(c) Find two distinct left inverses of the map



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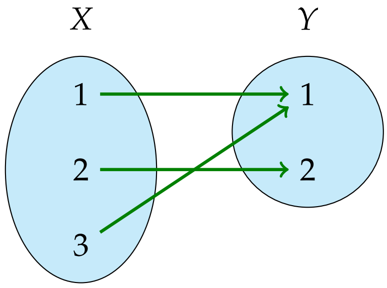
(d) Find two distinct right inverses of the map



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Some exercises on counting will follow now. Recall that $[n]$ denotes the set $\{1, 2, \ldots, n\}$ whenever $n \in \mathbb{N}$.

Exercise 4. Let $n \in \mathbb{N}$. Compute the # of 4-tuples $(a, b, c, d) \in [n]^4$ that satisfy $a \leq b < c \leq d$. (Not a typo: the second sign is a $<$, not a $\leq$.)

(Recall that $[n]^4 = [n] \times [n] \times [n] \times [n]$, so that a 4-tuple $(a, b, c, d) \in [n]^4$ means a 4-tuple of integers $a, b, c, d \in \{1, 2, \ldots, n\}$.)

Exercise 5. Let $n \in \mathbb{N}$. Compute the # of pairs (A, B) of subsets of $[n]$ that satisfy $A \cap B = \varnothing$.

(For example, if $n = 2$, then this # is 9, since there are 9 such pairs:

$$(\varnothing, \varnothing), \quad (\varnothing, \{1\}), \quad (\varnothing, \{2\}), \quad (\varnothing, \{1,2\}),$$
$$(\{1\}, \varnothing), \quad (\{1\}, \{2\}), \quad (\{2\}, \varnothing), \quad (\{2\}, \{1\}), \quad (\{1,2\}, \varnothing).$$

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Exercise 6. A set S of integers will be called **pseudolacunar** if it has the property that no two elements s, t of S satisfy $|s - t| = 2$. For instance, the set $\{2, 5, 6\}$ is pseudolacunar, but the set $\{2, 5, 7\}$ is not (since $|5 - 7| = 2$).

For each $n \in \mathbb{N}$, let p_n be the # of pseudolacunar subsets of $[n]$.

Prove that

$$p_n = p_{n-1} + p_{n-3} + p_{n-4} \qquad \text{for each } n \geq 4.$$

[**Hint:** To each pseudolacunar subset, assign one of three colors.]

Exercise 7. A set S of integers shall be called **self-starting** if its size $|S|$ is also its smallest element. (For example, $\{3, 5, 6\}$ is self-starting, while $\{2, 3, 4\}$ and $\{3\}$ are not.)

Let $n \in \mathbb{N}$.

(a) For any $k \in [n]$, find the number of self-starting subsets of $[n]$ having size k.

(b) Find the number of all self-starting subsets of $[n]$.