Math 221 Winter 2025 (Darij Grinberg): homework set 5
due date: Thursday 2025-02-20 at 11:59PM on gradescope (`https://www.gradescope.com/courses/930212`).
Please solve only **3 of the 6 exercises**.

Exercise 1. Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ be integers satisfying $a_1 \mid b_1$ and $a_2 \mid b_2$. Prove that $\gcd(a_1, a_2) \mid \gcd(b_1, b_2)$.

Exercise 2. Let $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$. Prove that

$$\gcd\left(a^k, b^k\right) = (\gcd(a, b))^k .$$

Exercise 3. Let a and b be two coprime positive integers. Prove that

$$a! \cdot b! \mid (a + b - 1)!.$$

[**Hint:** Let $g = \dfrac{(a+b-1)!}{a! \cdot b!}$. Prove that both ag and bg are integers. What then?]

The following exercise is a generalization of the prime divisor separation theorem (Theorem 3.6.5):

Exercise 4. Let p be a prime, and let $m \in \mathbb{N}$. Let a and b be two integers such that $p^m \mid ab$ and $p^m \nmid a$. Prove that $p \mid b$.

Next comes a generalization of the $p \mid \dbinom{p}{k}$ divisibility (Theorem 3.6.3):

Exercise 5. Let p be a prime. Let $m \in \mathbb{N}$, and let $k \in \{1, 2, \ldots, p^m - 1\}$. Prove that $p \mid \dbinom{p^m}{k}$.

Exercise 6. Let p and q be two distinct primes. Prove that $p^q + q^p \equiv p + q \bmod pq$ and $p^{q-1} + q^{p-1} \equiv 1 \bmod pq$.

[**Hint:** First show that p and q are coprime.]