

Math 221 Winter 2025 (Darij Grinberg): homework set 3

due date: Sunday 2025-01-30 at 11:59PM on gradescope (

<https://www.gradescope.com/courses/930212>).

Please solve only 3 of the 6 exercises.

Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$.**Exercise 1.** Let $a, b \in \mathbb{Z}$. Prove the following:(a) If $a \mid b$, then $a^k \mid b^k$ for each $k \in \mathbb{N}$.(b) Let $n \in \mathbb{Z}$. If $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$ for each $k \in \mathbb{N}$.**Exercise 2.** Let $n, d, a, b \in \mathbb{Z}$, and assume that $d \neq 0$ and $da \equiv db \pmod{dn}$.(a) Prove that $a \equiv b \pmod{n}$.(b) Show by an example that $a \equiv b \pmod{dn}$ is not necessarily true (i.e., we cannot simply cancel the d from da and db while leaving the dn unchanged).**Exercise 3.** Let n be any integer. Prove the following:(a) If n is odd, then $8 \mid n^2 - 1$.(b) If $3 \nmid n$, then $3 \mid n^2 - 1$.[Hint: In part (a), write n as $2k + 1$. In part (b), write n as $q \cdot 3 + r$ and consider the possible values for r .]**Exercise 4.** Let p be a positive integer.Assume that you are given p -cent coins and $(p + 1)$ -cent coins (each in infinite supply).Prove that you can pay n cents using these coins for every integer $n \geq p^2 - p$.In other words, prove that each integer $n \geq p^2 - p$ can be written as $a(p + 1) + bp$ with $a, b \in \mathbb{N}$.

Now for two more properties of binomial coefficients:

Exercise 5. (a) Prove that $k \binom{n}{k} = n \binom{n-1}{k-1}$ for any two numbers n and k .(b) Prove that $\sum_{k=0}^n k \binom{n}{k} x^k = nx(x+1)^{n-1}$ for any positive integer n and any number x .[Hint: Don't forget about cases like $k = 0$ and $k \notin \mathbb{N}$.]**Exercise 6.** Let (f_0, f_1, f_2, \dots) be the Fibonacci sequence. Let $n \in \mathbb{N}$. Prove that

$$2^{n-1} \cdot f_n = \sum_{k=0}^n \binom{n}{2k+1} \cdot 5^k.$$

[Hint: The 5 on the right hand side looks suspiciously like the 5 in $\frac{1+\sqrt{5}}{2}$, whereas the binomial coefficients look like the binomial formula...]