Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2025: Homework 9
due 2025-06-07 at 11:59 PM
Please solve 3 of the 9 problems!

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1 Exercise 1

1.1 Problem

Let $D = (V, A, \psi)$ be a strongly connected multidigraph.

A *wealth distribution* on D shall mean a family $(k_v)_{v \in V}$ of integers (one for each vertex $v \in V$). If $k = (k_v)_{v \in V}$ is a wealth distribution, then we refer to each value k_v as the *wealth* of the vertex v, and we define the *total wealth* of k to be the sum $\sum_{v \in V} k_v$. We say that a vertex v is *in debt* in a given wealth distribution $k = (k_v)_{v \in V}$ if its wealth k_v is negative.

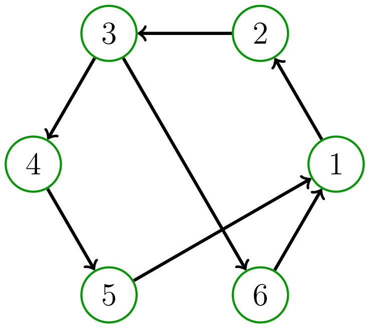
For any vertices v and w, we let $a_{v,w}$ denote the number of arcs that have source v and w.

A *donation* is an operation that transforms a wealth distribution as follows: We choose a vertex v, and we decrease its wealth by its outdegree $\deg^+ v$, and then increase the wealth of each vertex $w \in V$ (including v itself) by $a_{v,w}$. (You can think of v as donating a unit of wealth for each arc that has source v. This unit flows to the target to this arc. Note that a donation does not change the total wealth.)

Let k be a wealth distribution on D whose total wealth is larger than $|A| - |V|$. Prove that by an appropriately chosen finite sequence of donations, we can ensure that no vertex is in debt.

1.2 Remark

For instance, consider the digraph



with wealth distribution $(k_1, k_2, k_3, k_4, k_5, k_6) = (-1, -1, 1, 2, 0, 1)$. The vertices 1 and 2 are in debt here, but it is possible to get all vertices out of debt by having the vertices $4, 5, 6, 1$ donate in some order (the order clearly does not matter for the result[1]).

Note that vertices are allowed to donate multiple times (although in the above example, this was unnecessary).

1.3 Hint

Show first that if the total wealth is larger than $|A| - |V|$, then at least one vertex v has wealth $\geq \deg^+ v$.

1.4 Solution

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2 Exercise 2

2.1 Problem

We continue with the setting and terminology of Exercise 1.

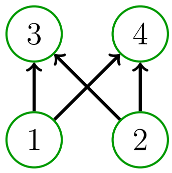
A *clawback* is an operation that transforms a wealth distribution as follows: We choose a vertex v, and we increase its wealth by its outdegree $\deg^+ v$, and then decrease the wealth of each vertex $w \in V$ (including v itself) by $a_{v,w}$. (Thus, a clawback is the inverse of a donation.)

Let k be a wealth distribution on D whose total wealth is larger than $|A| - |V|$. Prove that by an appropriately chosen finite sequence of clawbacks, we can ensure that no vertex is in debt.

[1] Depending on the order, some vertices will go into debt in the process, but this is okay as long as they ultimately end up debt-free.

2.2 Remark

Note that we are still assuming D to be strongly connected. Otherwise, the truth of the claim is not guaranteed. For instance, for the digraph



with wealth distribution $(k_1, k_2, k_3, k_4) = (0, 0, -1, 2)$, no sequence of donations and clawbacks will result in every vertex being out of debt (since the wealth difference $k_4 - k_3$ is preserved under any donation or clawback, but this difference is too large to come from a debt-free distribution with total weight 1).

2.3 Hint

Show that any donation is equivalent to an appropriately chosen composition of clawbacks.

2.4 Solution

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3 Exercise 3

3.1 Problem

Let X and Y be two finite sets such that $|X| \leq |Y|$. Let $f : X \to Y$ be a map that is not constant. (A map is said to be *constant* if all its values are equal.) Prove that there exists an injective map $g : X \to Y$ such that each $x \in X$ satisfies $g(x) \neq f(x)$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let (G, X, Y) be a bipartite graph with $X \neq \varnothing$. Assume that G has an X-complete matching.

An edge e of G will be called *useless* if G has no X-complete matching that contains e.

Prove that there exists a vertex $x \in X$ such that no edge that contains x is useless.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let $G = (V, E, \varphi)$ be a multigraph. Let M be a matching of G.

An *augmenting path* for M shall mean a path $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ of G such that k is odd[2] and such that

- the even-indexed edges $e_2, e_4, \ldots, e_{k-1}$ belong to M (note that this condition is vacuously true if $k = 1$);
- the odd-indexed edges $e_1, e_3, \ldots, e_k$ belong to $E \setminus M$;
- neither the starting point v_0 nor the ending point v_k is matched in M.

Prove that M has maximum size among all matchings of G if and only if there exists no augmenting path for M.

5.2 Hint

If M and M' are two matchings of G, what can you say about the symmetric difference $(M \cup M') \setminus (M \cap M')$?

5.3 Solution

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6 Exercise 6

6.1 Problem

Let (G, X, Y) be a bipartite graph. Prove that

$$\sum_{A \subseteq X} (-1)^{|A|} [N(A) = Y] = \sum_{B \subseteq Y} (-1)^{|B|} [N(B) = X]$$

(where we are using the Iverson bracket notation).

[2] Note that $k = 1$ is allowed.

6.2 Solution

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7 Exercise 7

7.1 Problem

Let A and B be two finite sets such that $|B| \geq |A|$. Let $d_{i,j}$ be a real number for each $(i,j) \in A \times B$. Let[3]

$$m_1 = \min_{\sigma: A \to B \text{ injective}} \max_{i \in A} d_{i,\sigma(i)}$$

and

$$m_2 = \max_{\substack{I \subseteq A;\ J \subseteq B; \\ |I|+|J|=|B|+1}} \min_{(i,j) \in I \times J} d_{i,j}.$$

Prove that $m_1 = m_2$.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let c and r be two positive integers. Let T be a tournament with more than r^c vertices. Each arc of T is colored with one of the c colors $1, 2, \ldots, c$. Prove that T has a monochromatic path of length r.

(A path is said to be *monochromatic* if all its arcs have the same color.)

8.2 Hint

Induct on c, and apply Gallai-Milgram to a certain digraph in the induction step. (Note that the case $c = 1$ recovers the Easy Redei Theorem.)

8.3 Solution

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[3] The notation "$\min_{\text{some kind of objects}}$ some kind of value" means the minimum of the given value over all objects of the given kind. An analogous notation is used for a maximum.

9 Exercise 9

9.1 Problem

Let D be a balanced multidigraph. Let s and t be two vertices of D. Let $k \in \mathbb{N}$. Assume that D has k pairwise arc-disjoint paths from s to t. Show that D has k pairwise arc-disjoint paths from t to s.

[We say that k paths $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$ are *pairwise arc-disjoint* if and only if no arc appears in more than one of these k paths.]

9.2 Solution

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