Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2025: Homework 8 due 2025-05-30 at 11:59 PM

Please solve 3 of the 7 problems!

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1 Exercise 1

1.1 Problem

Let G be a multigraph such that every vertex of G has even degree. Let u and v be two distinct vertices of G. Prove that the number of paths from u to v is even.

1.2 Hint

When you add an edge joining u to v, the graph G becomes a graph with exactly two odd-degree vertices u and v, and the claim becomes "the number of paths from u to v is odd" (why?). In this form, the claim turns out to be easier to prove. Indeed, any path must start with some edge...

Keep in mind that paths can be replaced by trails, by Exercise 3 on homework set #3.

1.3 Solution

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2 Exercise 2

2.1 Problem

Let G be a simple graph with n vertices. Let k be a positive integer.

Prove the following:

(a) If G has a proper k-coloring, then G has no subgraph isomorphic to K_{k+1}.

(b) If $k \geq n-2$, then the converse to part **(a)** also holds: If G has no subgraph isomorphic to K_{k+1}, then G has a proper k-coloring.

(c) Does the converse to part **(a)** hold for $k < n-2$ as well? Specifically, does it hold for $n = 5$ and $k = 2$?

2.2 Solution

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3 Exercise 3

3.1 Problem

Fix two positive integers n and k with $n \geq 2k > 0$. Let $S = \{1, 2, \ldots, n\}$. Consider the k-Kneser graph $K_{S,k}$ as defined in Example 2.6.4. Prove that $K_{S,k}$ has a proper $(n-2k+2)$-coloring.

3.2 Hint

What can you say about the minima (i.e., smallest elements) of two disjoint subsets of S? (Being distinct is a good first step.)

3.3 Remark

Lóvasz has proved in 1978 (using topology!) that this result is optimal – in the sense that $n-2k+2$ is the smallest integer q such that $K_{S,k}$ has a proper q-coloring.

3.4 Solution

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4 Exercise 4

4.1 Problem

Let n and k be two positive integers. Let K be a set of size k. Let D be the de Bruijn digraph – i.e., the multidigraph constructed in the proof of Theorem 5.16.2. Let G be the result of removing all loops from the undirected graph D^{und}. Prove that G has a proper $(k+1)$-coloring.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let G be a connected loopless multigraph. Prove that G has a proper 2-coloring if and only if every three vertices u, v, w of G satisfy

$$2 \mid d(u,v) + d(v,w) + d(w,u).$$

5.2 Solution

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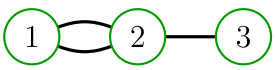
6 Exercise 6

6.1 Problem

Let $G = (V, E)$ be a simple graph such that each vertex of G has degree ≥ 1. Prove that there exists a subset S of V having size $\geq \sum_{v \in V} \dfrac{2}{1 + \deg v}$ and with the property that the induced subgraph $G[S]$ is a forest.

6.2 Remark

As the example of 

shows, this claim is not true for loopless multigraphs (unlike the similar Theorem 1.1.3 in Lecture 23).

6.3 Solution

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7 Exercise 7

7.1 Problem

Let n be a positive integer. Prove that there exists a tournament with n vertices and at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

7.2 Solution

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