Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2025: Homework 6
due 2025-05-16 at 11:59 PM
Please solve 3 of the 6 problems!

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April 18, 2025

1 Exercise 1

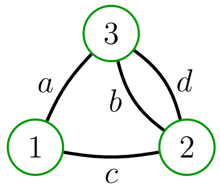
1.1 Problem

Let G be a connected multigraph. Let $\mathcal{S}$ be the simple graph whose vertices are the spanning trees of G, and whose edges are defined as follows: Two spanning trees T_1 and T_2 of G are adjacent (as vertices of $\mathcal{S}$) if and only if T_2 can be obtained from T_1 by removing an edge and adding another (i.e., if and only if there exist an edge e_1 of T_1 and an edge e_2 of T_2 such that $e_2 \neq e_1$ and $T_2 \setminus e_2 = T_1 \setminus e_1$).

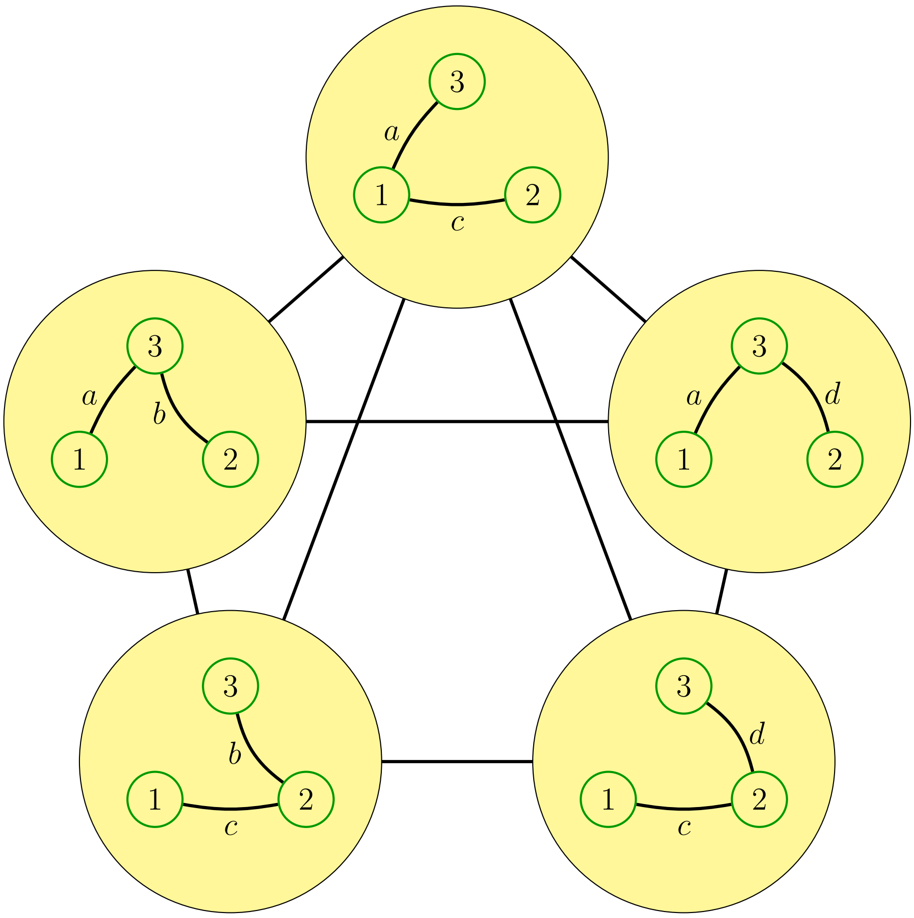
Prove that the simple graph $\mathcal{S}$ is itself connected. (In simpler language: Prove that any spanning tree of G can be transformed into any other spanning tree of G by a sequence of legal "remove an edge and add another" operations, where such an operation is called *legal* if its result is a spanning tree of G.)

1.2 Remark

Let G be the following multigraph:



Then, the graph $\mathcal{S}$ looks as follows:



1.3 Hint

You must show that any two spanning trees T_1 and T_2 are path-connected in $\mathcal{S}$. Induct on the number of edges of T_1 that are not edges of T_2.

1.4 Solution

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2 Exercise 2

2.1 Problem

Let D be a multidigraph. Prove that the strong components of D are the weak components of D if and only if each arc of D is contained in at least one cycle.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let G be a connected multigraph with an even number of vertices. Prove that there exists a spanning subgraph H of G such that each vertex of H has odd degree (in H).

3.2 Hint

One solution begins by reducing the problem to the case when G is a tree.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let $D = (V, A, \phi)$ be a multidigraph that has no cycles[1]. Let $r \in V$ be some vertex of D. Prove the following:

(a) If $\deg^- u > 0$ holds for all $u \in V \setminus \{r\}$, then r is a from-root of D.

(b) If $\deg^- u = 1$ holds for all $u \in V \setminus \{r\}$, then D is an arborescence rooted from r.

4.2 Solution

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[1] Recall that cycles in a digraph have to be directed cycles – i.e., each arc is traversed from its source to its target.

5 Exercise 5

5.1 Problem

Let $D = (V, A)$ be a simple digraph that has no cycles.

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a list of vertices of D (not necessarily a walk!), then a *back-cut* of $\mathbf{v}$ shall mean an arc $a \in A$ whose source is v_i and whose target is v_j for some $i, j \in \{1, 2, \ldots, n\}$ satisfying $i > j$. (Colloquially speaking, a back-cut of $\mathbf{v}$ is an arc of D that leads from some vertex of $\mathbf{v}$ to some earlier vertex of $\mathbf{v}$.)

A list $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ of vertices of D is said to be a *toposort*[2] of D if it contains each vertex of D exactly once and has no back-cuts.

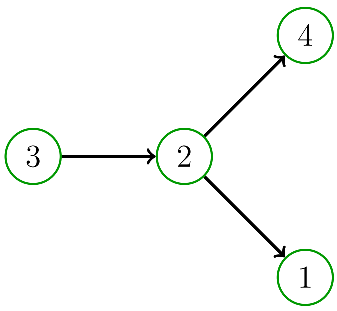
Prove the following:

(a) The digraph D has at least one toposort.

(b) If D has only one toposort, then this toposort is a Hamiltonian path of D.

5.2 Remark

For example, the digraph



has two toposorts: $(3, 2, 1, 4)$ and $(3, 2, 4, 1)$.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let n be a positive integer. Let $K_{n,2}$ be the simple graph with vertex set $\{1, 2, \ldots, n\} \cup \{-1, -2\}$ such that two vertices of $K_{n,2}$ are adjacent if and only if they have opposite signs (i.e., each positive vertex is adjacent to each negative vertex, but no two vertices of the same sign are adjacent). We regard $K_{n,2}$ as a multigraph in the usual way.

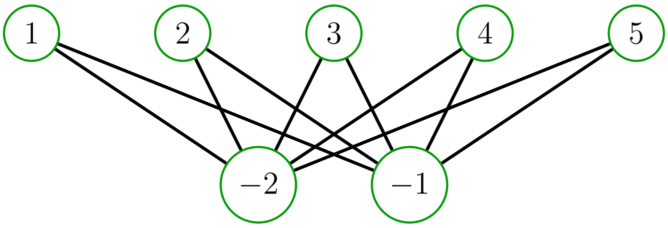
(a) Without using the matrix-tree theorem, prove that the number of spanning trees of $K_{n,2}$ is $n \cdot 2^{n-1}$.

(b) Let $K'_{n,2}$ be the graph obtained by adding a new edge $\{-1, -2\}$ to $K_{n,2}$. How many spanning trees does $K'_{n,2}$ have?

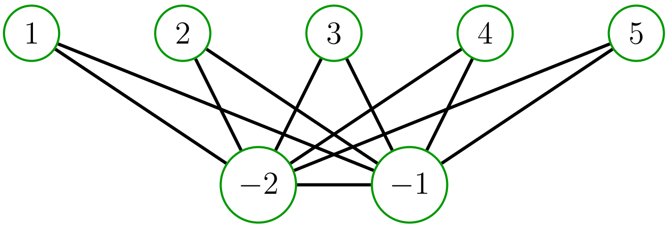
[2] This is short for "topological sorting". I don't know where this name comes from.

6.2 Remark

Here is the graph $K_{n,2}$ for $n = 5$:



And here is the corresponding graph $K'_{n,2}$:



6.3 Solution

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