Drexel University, Department of Mathematics

# Math 530: Graph Theory, Spring 2025: Homework 4 due 2025-05-02 at 11:59 PM **Please solve 3 of the 6 problems!**

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## 1 Exercise 1

### 1.1 Problem

**(a)** Let $G = (V, E)$ be a simple graph, and let $u$ and $v$ be two distinct vertices of $G$ that are not adjacent. Let $n = |V|$. Assume that $\deg u + \deg v \geq n$. Let $G' = (V, E \cup \{uv\})$ be the simple graph obtained from $G$ by adding a new edge $uv$. Assume that $G'$ has a hamc. Prove that $G$ has a hamc.

**(b)** Does this remain true if we replace "hamc" by "hamp"?

### 1.2 Solution

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# 2 Exercise 2

## 2.1 Problem

Let $D$ be a multidigraph with at least one vertex. Prove the following:

**(a)** If each vertex $v$ of $D$ satisfies $\deg^+ v > 0$, then $D$ has a cycle.

**(b)** If each vertex $v$ of $D$ satisfies $\deg^+ v = \deg^- v = 1$, then each vertex of $D$ belongs to exactly one cycle of $D$. Here, two cycles are considered to be identical if one can be obtained from the other by cyclic rotation.

## 2.2 Solution

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# 3 Exercise 3

## 3.1 Problem

Prove the directed Euler–Hierholzer theorem:

Let $D$ be a weakly connected multidigraph. Then:

**(a)** The multidigraph $D$ has an Eulerian circuit if and only if each vertex $v$ of $D$ satisfies $\deg^+ v = \deg^- v$.

**(b)** The multidigraph $D$ has an Eulerian walk if and only if all but two vertices $v$ of $D$ satisfy $\deg^+ v = \deg^- v$, and the remaining two vertices $v$ satisfy $\left|\deg^+ v - \deg^- v\right| \leq 1$.

## 3.2 Solution

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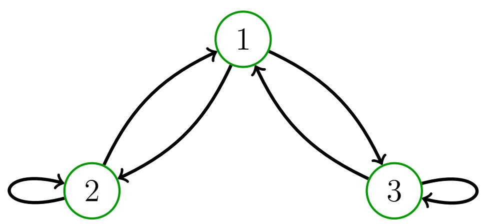
# 4 Exercise 4

## 4.1 Problem

Let $E$ be the following multidigraph:

$E =$ 


Let $n \in \mathbb{N}$. Compute the number of walks from 1 to 1 having length $n$.

### 4.2 Solution

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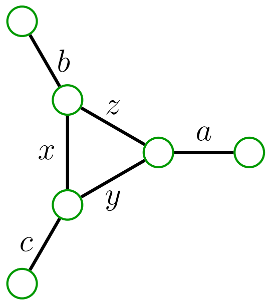
# 5 Exercise 5

## 5.1 Problem

Let $G = (V, E, \varphi)$ be a connected multigraph with $2m$ edges, where $m \in \mathbb{N}$. A set $\{e, f\}$ of two distinct edges will be called a *friendly couple* if $e$ and $f$ have at least one endpoint in common. Prove that the edge set of $G$ can be decomposed into $m$ disjoint[1] friendly couples (i.e., there exist $m$ disjoint friendly couples $\{e_1, f_1\}, \{e_2, f_2\}, \ldots, \{e_m, f_m\}$ such that $E = \{e_1, f_1, e_2, f_2, \ldots, e_m, f_m\}$).

[**Example:** Here is a graph with an even number of edges:




One possible decomposition into disjoint friendly pairs is $\{a, y\}, \{b, z\}, \{c, x\}$.]

## 5.2 Hint

Induct on $|E|$. Pick a vertex $v$ of degree $> 1$ and consider the components of $G \setminus v$.

## 5.3 Solution

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# 6 Exercise 6

## 6.1 Problem

Let $D$ be a simple digraph with $n$ vertices and $a$ arcs. Assume that $D$ has no loops, and that we have $a > n^2/2$. We define an *enhanced 3-cycle* to be a triple $(u, v, w)$ of distinct vertices of $D$ such that all four pairs $(u, v)$, $(v, w)$, $(w, u)$ and $(u, w)$ are arcs of $D$.

Prove that the digraph $D$ has an enhanced 3-cycle.

[1]"Disjoint" means "disjoint as sets" – i.e., having no edges in common.

## 6.2 Remark

Note that this is both an analogue and a generalization (why?) of Mantel's theorem.

## 6.3 Solution

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