Drexel University, Department of Mathematics

# Math 530: Graph Theory, Spring 2025: Homework 3
# due 2025-04-25 at 11:59 PM
# **Please solve 3 of the 6 problems!**

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## 1 Exercise 1

### 1.1 Problem

Which of the exercises 1, 2, 3, 4 **(a)**, 5 from homework set #1 remain true if "simple graph" is replaced by "multigraph"?

(For each exercise that becomes false, provide a counterexample. For each exercise that remains true, either provide a new solution that works for multigraphs, or argue that the solution we have seen applies verbatim to multigraphs, or derive the multigraph case from the simple graph case. For Exercise 4 **(a)**, "remains true" means that the descriptions of the graphs $G$ are still the same, i.e., no other possibilities appear for $G$ up to isomorphism.)

### 1.2 Solution

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# 2 Exercise 2

## 2.1 Problem

Let $G$ be a multigraph with at least one vertex. Let $d > 2$ be an integer. Assume that $\deg v > 2$ for each vertex $v$ of $G$. Prove that $G$ has a cycle whose length is not divisible by $d$.

## 2.2 Solution

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# 3 Exercise 3

## 3.1 Problem

Let $G$ be a loopless multigraph. Recall that a *trail* means a walk whose edges are distinct (but whose vertices are not necessarily distinct). Let $u$ and $v$ be two vertices of $G$. As usual, "trail from $u$ to $v$" means "trail that starts at $u$ and ends at $v$".

Prove that

$$\begin{aligned} & (\text{the number of trails from } u \text{ to } v \text{ in } G) \\ \equiv & (\text{the number of paths from } u \text{ to } v \text{ in } G) \mod 2. \end{aligned}$$

## 3.2 Hint

Try to pair up the non-path trails into pairs. Make sure to prove that this pairing is well-defined (i.e., each non-path trail $\mathbf{t}$ has exactly one partner, which is not itself, and that $\mathbf{t}$ is the designated partner of its partner!).

## 3.3 Solution

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# 4 Exercise 4

## 4.1 Problem

Let $n$ and $k$ be two integers such that $n > k > 0$. Define the simple graph $Q_{n,k}$ as follows: Its vertices are the bitstrings $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$; two such bitstrings are adjacent if and only if they differ in exactly $k$ bits[1]. (Thus, $Q_{n,1}$ is the $n$-hypercube graph $Q_n$.)

[1] In other words: Two vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are adjacent if and only if the number of $i \in \{1, 2, \ldots, n\}$ satisfying $a_i \neq b_i$ equals $k$.

**(a)** Does $Q_{n,k}$ have a hamc[2] when $k$ is even?

**(b)** Does $Q_{n,k}$ have a hamc when $k$ is odd?

### 4.2 Remark

One way to approach part **(b)** is by identifying the set $\{0,1\}$ with the field $\mathbb{F}_2$ with two elements. The bitstrings $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ thus become the size-$n$ row vectors in the $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$. Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors of $\mathbb{F}_2^n$ (so that $e_i$ has a 1 in its $i$-th position and zeroes everywhere else). Then, two vectors are adjacent in the $n$-hypercube graph $Q_n$ (resp. in the graph $Q_{n,k}$) if and only if their difference is one of the standard basis vectors (resp., a sum of $k$ distinct standard basis vectors). Try to use this to find a graph isomorphism from $Q_n$ to a subgraph of $Q_{n,k}$.

### 4.3 Solution

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## 5 Exercise 5

### 5.1 Problem

Let $k \in \mathbb{N}$. Let $S$ be a finite set. The *Kneser graph* $K_{S,k}$ is the simple graph whose vertices are the $k$-element subsets of $S$, and whose edges are the unordered pairs $\{A, B\}$ consisting of two such subsets $A$ and $B$ that satisfy $A \cap B = \varnothing$.

Prove that this Kneser graph $K_{S,k}$ is connected if $|S| \geq 2k+1$.

### 5.2 Remark

Can the "if" here be replaced by an "if and only if"? Not quite, because the graph $K_{S,k}$ is also connected if $|S| = 2$ and $k = 1$ (in which case it has two vertices and one edge), or if $|S| = k$ (in which case it has only one vertex). There might be more such "exceptions".

### 5.3 Solution

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## 6 Exercise 6

### 6.1 Problem

Let $n \geq 1$. Let $Q_n$ be the $n$-hypercube graph, as in Definition 2.14.7.

[2] Recall that "hamc" is short for "Hamiltonian cycle".

At what vertices can a hamp[3] of $Q_n$ end if it starts at the vertex $00\cdots 0$ ? (Find all possibilities, and prove that they are possible and all other vertices are impossible.)

## 6.2 Solution

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[3] Recall that "hamp" is short for "Hamiltonian path".