Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2025: Homework 2
due 2025-04-19 at 11:59 PM
Please solve 4 of the 6 problems!

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1 Exercise 1

1.1 Problem

Let G be a simple graph.

(a) Prove that if G has a closed walk of odd length, then G has a cycle of odd length.

(b) Is it true that if G has a closed walk of length not divisible by 3, then G has a cycle of length not divisible by 3 ?

(c) Does the answer to part **(b)** change if we replace "walk" by "non-backtracking walk"? (A walk $\mathbf{w}$ with edges $e_1, e_2, \ldots, e_k$ (in this order) is said to be *non-backtracking* if each $i \in \{1, 2, \ldots, k-1\}$ satisfies $e_i \neq e_{i+1}$.)

(d) Does the answer to part **(b)** change if we replace "walk" by "trail"? (A *trail* means a walk whose edges are distinct.)

(Proofs and counterexamples should be given.)

1.2 Solution

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2 Exercise 2

2.1 Problem

Let $n \geq 3$ be an integer. Find[1] a formula for the smallest size of a dominating set of the cycle graph C_n. You can use the *ceiling function* $x \mapsto \lceil x \rceil$, which sends a real number x to the smallest integer that is $\geq x$.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let n and k be positive integers such that $n \geq k(k+1)$ and $k > 1$. Recall the Kneser graph $KG_{n,k}$, whose vertices are the k-element subsets of $\{1, 2, \ldots, n\}$, and whose edges are the unordered pairs $\{A, B\}$ of such subsets with $A \cap B = \varnothing$.

Prove that the minimum size of a dominating set of $KG_{n,k}$ is $k+1$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $G = (V, E)$ be a connected simple graph with at least two vertices.

The *distance* $d(v, w)$ between two vertices v and w of G is defined to be the smallest length of a path from v to w. (In particular, $d(v, v) = 0$ for each $v \in V$.)

Fix a vertex $v \in V$. Define two subsets

$$A = \{w \in V \mid d(v, w) \text{ is even}\} \qquad \text{and} \qquad B = \{w \in V \mid d(v, w) \text{ is odd}\}$$

of V.

[1] Here and in what follows, "finding" includes proving your finds.

(a) Prove that A is dominating.

(b) Prove that B is dominating.

(c) Prove that there exists a dominating set of G that has size $\leq |V|/2$.

(d) Prove that the claim of part **(c)** holds even if we don't assume that G is connected, as long as we assume that each vertex of G has at least one neighbor.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let $G = (V, E)$ be a simple graph with at least one vertex. Let $n = |V|$. A *detached pair* means a pair (A, B) of two disjoint subsets A and B of V such that there exists no edge $ab \in E$ with $a \in A$ and $b \in B$.

Prove the following generalization of the Heinrich–Tittmann formula:

$$\sum_{\substack{S \text{ is a dominating} \\ \text{set of } G}} x^{|S|} = (1+x)^n - 1 + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|A|} x^{|B|}.$$

(Here, both sides are polynomials in a single indeterminate x with coefficients in $\mathbb{Z}$.)

5.2 Remark

This is a generalization of the Heinrich–Tittmann formula for the number of dominating sets. (The latter formula can be obtained fairly easily by substituting $x = 1$ into the above and subsequently cancelling the addends with $|A| \not\equiv |B| \mod 2$ against each other[2].) You are free to copy arguments from [Grinbe17] and change whatever needs to be changed. (Some lemmas can even be used without any changes – they can then be cited without proof.)

5.3 Solution

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[2]The addend for (A, B) cancels the addend for (B, A).

6 Exercise 6

6.1 Problem

Let $G = (V, E)$ be a simple graph. Let S be a subset of V, and let $k = |S|$. Prove that

$$\sum_{v \in S} \deg v \leq k(k-1) + \sum_{v \in V \setminus S} \min\{\deg v, k\}.$$

6.2 Solution

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References

[Grinbe17] Darij Grinberg, *Notes on graph theory*, draft of two chapters, 4th April 2022. `https://www.cip.ifi.lmu.de/~grinberg/t/17s/nogra.pdf`